Chaos in the double pendulum

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I. INTRODUCTION

The purpose of this assignment is to find solutions to and solve the double pendulum system. And, beyond that analyze the chaotic behaviour that emerges from it.

II. METHOD

A. Lagrangian approach to the double pendulum

Using the coordinates shown in 1. We can observe that we have four degrees of freedom with two constraints. This leaves us with $4 - 2 = 2$ generalized coordinates required to describe the system. [1]

In the assignment, I took the liberty of assuming that the only objects in the systems that carry any mass is m_1 and m_2 marked in 1. This means that the only part of the system that can hold any energy is those masses. I now then proceed to write out the position vectors of those masses.

$$
\begin{aligned}
r_1 &= (l_1 \sin(\phi), l_1 \cos(\phi)) \\
r_2 &= (l_1 \sin(\phi) + l_2 \sin(\theta), l_1 \cos(\phi) + l_2 \cos(\theta))
\end{aligned}
$$

Now that we have the position vectors we are ready to try and write the Lagrangian ($\mathcal{L}$).

$$
\mathcal{L} = \frac{1}{2} m \dot{r}_1^2 + \frac{1}{2} m \dot{r}_2^2 - V_1 - V_2 \tag{1}
$$

$$
\begin{aligned}
V_1 &= m_1 g l_1 (1 - \cos(\phi)) \\
V_2 &= m_2 g (l_1 (1 - \cos(\phi)) + l_2 (1 - \cos(\theta)))
\end{aligned} \tag{2}
$$

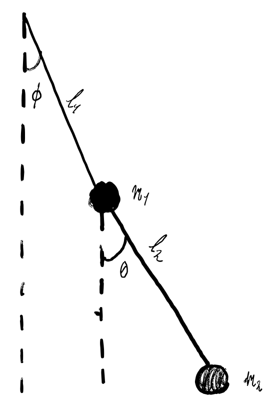

FIG. 1. Double pendulum

This expression carries terms that are just constants (If you would to multiply in the factors outside of the parentheses) which later on, in the Euler-Lagrange equations cancel. For this reason, any further reference to 2 will be made without them. [2]

$$
\begin{aligned}
\dot{r}_1^2 &= (l_1 \dot{\phi})^2 \\
\dot{r}_2^2 &= l_1^2 \dot{\phi}^2 + l_2 \dot{\theta}^2 + ... \\
&\quad ... + 2 l_1 l_2 (\cos(\phi)\cos(\theta) + \sin(\phi)\sin(\theta)) \dot{\phi}\dot{\theta}
\end{aligned}
$$

Which turns 1 into;

$$
\mathcal{L} = \frac{m_1}{2} l_1^2 \dot{\phi}^2 + \frac{m_2}{2} \left(l_1^2 \dot{\phi}^2 + l_2^2 \dot{\phi}^2 + 2 l_1 l_2 (\cos(\phi)\cos(\theta) + \sin(\phi)\sin(\theta)) \dot{\phi}\dot{\theta} \right) + g(m_1 + m_2)(\cos(\phi)) l_1 + g m_2 l_2 (\cos(\theta)) \tag{3}
$$

We can shorten this expression slightly by using the identity $\cos(s)\cos(t) + \sin(s)\sin(t) = \cos(s - t)$ But it doesn't get more compact than that.

Now turning to the Euler-Lagrange equations:

$$
\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \tag{4}
$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \phi} &= -m_2 l_1 l_2 \dot{\phi}\dot{\theta} \sin(\phi - \theta) - g(m_2 + m_1) l_1 \sin(\phi) \\
\frac{\partial \mathcal{L}}{\partial \theta} &= m_2 l_1 l_2 \dot{\phi}\dot{\theta} \sin(\phi - \theta) - g m_2 l_1 \sin(\theta) \\
\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) &= l_1^2 (m_1 + m_2) \ddot{\phi} + m_2 l_1 l_2 \ddot{\theta} \cos(\phi - \theta) \\
&\quad ... - m_2 l_1 l_2 \dot{\theta} \sin(\phi - \theta) \dot{\phi} + m_2 l_1 l_2 \dot{\theta}\dot{\phi} \sin(\phi - \theta) \\
\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) &= l_2^2 m_2 \ddot{\theta} + m_2 l_1 l_2 \ddot{\phi} \cos(\phi - \theta) \\
&\quad ... - m_2 l_1 l_2 \dot{\phi}^2 \sin(\phi - \theta) - m_2 l_1 l_2 \dot{\theta}\dot{\phi} \sin(\phi - \theta)
\end{aligned}
$$

With these expressions we can now write 4 as:

$$- g(m_2 + m_1)l_1 \sin(\phi) = l_1^2(m_1 + m_2)\ddot{\phi} + m_2 l_1 l_2 \ddot{\theta}\cos(\phi - \theta) + m_2 l_1 l_2 \dot{\theta}\dot{\phi}\sin(\phi - \theta) \tag{5}$$

$$- g m_2 l_1 \sin(\theta) = l_2^2 m_2 \ddot{\theta} + m_2 l_1 l_2 \ddot{\phi}\cos(\phi - \theta) - m_2 l_1 l_2 \dot{\phi}^2 \sin(\phi - \theta) \tag{6}$$

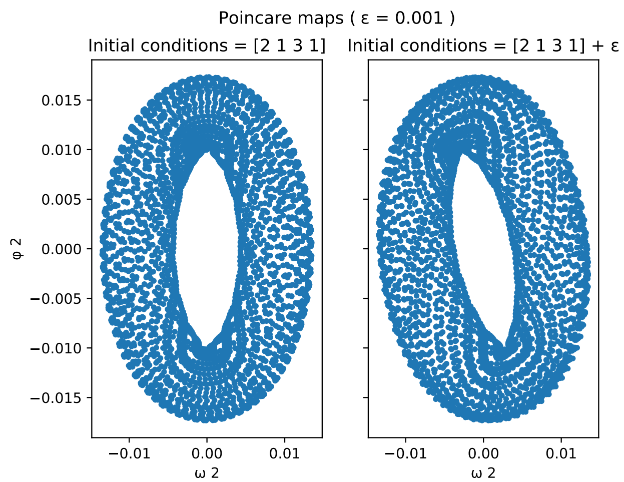

FIG. 2. Poincare map with 3000 points

These are nothing but a second degree differential equations. We could try to solve that but instead, the easier approach is to try and turn it into a first degree differential equation. We can do this by first using equations 5 and 6 as a linear systems of equations and solving for the second derivatives.I'll show this derivation in appendix A. Now defining new coordinates $\omega_1 = \dot{\phi}$ and $\omega_2 = \dot{\theta}$ allows us to write the following :

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \ddot{\phi} \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ g_1(\phi, \theta, \omega_1, \omega_2) \\ g_2(\phi, \theta, \omega_1, \omega_2) \end{pmatrix} \tag{7}$$

[3].

This is precisely the form we need the differential equation to apply Runge-Kutta 4 on, for details see [4].[5]

There are two different types of initial conditions that we can manipulate. We can either change the masses and lengths of the rods. Alternatively, we could change the initial displacement of the generalized coordinates. This would correspond to lifting either of the bobs in the pendulum slightly before releasing it. It's the former of these two that we will manipulate to see how small changes affects the chaos of the system. Both would of course be of interest, but we shall have to contain our analysis somewhat for the sake of brevity.

After solving the differential equation with the Runge-Kutta[4] method we can plot our results in the form of Poincare maps. See 2. Here we can see how drastic a difference a slight change in initial conditions (a change of ε in each of the parameters of the set of initial conditions) can have on the outcome of the total system. To underline this point even more one can observe what happens when the initial disruption is changed from $\varepsilon = 0.001$ to $\varepsilon = 0.1$. What previously caused a shift in the structure of the system, has now become a completely unrecognizable one.

III. SUMMARY

It seems that the pendulum is indeed chaotic. Closer analysis of how it's chaotic remains to be done. For further analysis one could look at Lyapunov exponents to try and quantify the chaotic behaviour somewhat. One could also look at how changing the initial conditions of the generalized coordinates changes the system.

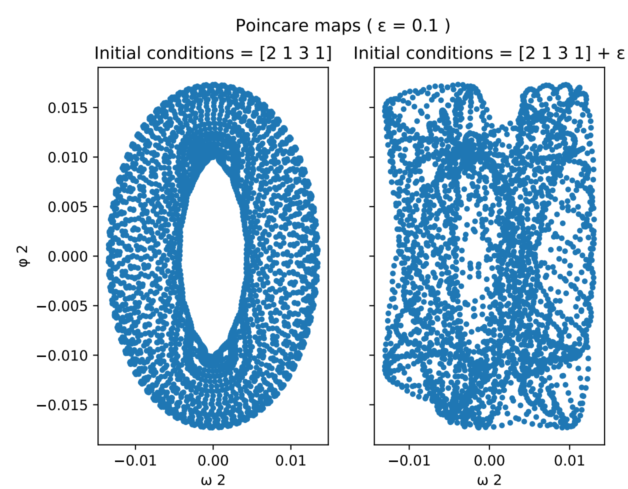

FIG. 3. Poincare map with 3000 points

Appendix A: Solution of system of linear equation for Euler-Lagrange equations

We can write equations 5 and 6 as:

$$\ddot{\phi} + \ddot{\theta}\frac{m_2 l_2 \cos(\phi-\theta)}{(m_1+m_2)l_1} = -\frac{g\sin(\phi)}{l_1} - \frac{m_2 l_2 \dot{\theta}^2}{(m_1+m_2)l_1} \tag{A1}$$

$$\ddot{\theta} + \ddot{\phi}\frac{l_1}{l_2}\cos(\phi-\theta) = \frac{l_1}{l_2}\dot{\phi}^2\sin(\phi-\theta) - \frac{g}{l_2}\sin(\theta) \tag{A2}$$

Or equivalently: [6]

$$\ddot{\phi} + \ddot{\theta}\alpha_1 = f_1 \tag{A3}$$

$$\ddot{\theta} + \ddot{\alpha}_2 = f_2 \tag{A4}$$

With,

$$\alpha_1 = \frac{m_2 l_2 \cos(\phi-\theta)}{(m_1+m_2)l_1} \tag{A5}$$

$$f_1 = -\frac{g\sin(\phi)}{l_1} - \frac{m_2 l_2 \dot{\theta}^2}{(m_1+m_2)l_1} \tag{A6}$$

$$\alpha_2 = \frac{l_1}{l_2}\cos(\phi-\theta) \tag{A7}$$

$$f_2 = \frac{l_1}{l_2}\dot{\phi}^2\sin(\phi-\theta) - \frac{g}{l_2}\sin(\theta) \tag{A8}$$

We can solve this like any other system of linear equations.

$$\left(\begin{array}{cc|c} 1 & \alpha_1 & f_1 \\ \alpha_2 & 1 & f_2 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & \frac{-f_1+f_2\alpha_1}{\alpha_1\alpha_2-1} \\ 0 & 1 & \frac{-f_2+f_1\alpha_2}{\alpha_1\alpha_2-1} \end{array}\right) \tag{A9}$$

So we have

$$g_1(\phi,\theta,\dot{\phi},\dot{\theta}) = \frac{-f_1+f_2\alpha_1}{\alpha_1\alpha_2-1} \tag{A10}$$

$$g_2(\phi,\theta,\dot{\phi},\dot{\theta}) = \frac{-f_2+f_1\alpha_2}{\alpha_1\alpha_2-1} \tag{A11}$$

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[1] Here we note that since the only conserved quantity is energy, we do in fact have one more degrees of freedom than conserved quantities. Indicative of a chaotic system.

[2] From what I can tell, this is the standard way of treating the constants to parts of the Lagrangian that ends up disappearing. Typically, without any reference to doing so whatsoever. Which I personally find very confusing. So I decided to include a small paragraph notifying the reader of doing so.

[3] Definitions of g_1 and g_2 will be included in the appendix.

[4] M. Newman, *Computational physics* (CrCreateSpace, 2013).

[5] Anyone interested in how this was done specifically or seeing any of the code I wrote. You can find all of it on `https://github.com/SixtenNordegren/DoubblePendulum`.

[6] The idea to write out the equations in different terms of the functions of the respective 2'nd derivatives was something I found on a blog, written by a guy named Diego Assencio. `https://diego.assencio.com/?index=1500c66ae7ab27bb0106467c68feebc6`.