# Chaos in the double pendulum

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## I. INTRODUCTION

The purpose of this assignment is to find solutions to and solve the double pendulum system. And, beyond that analyze the chaotic behaviour that emerges from it.

## II. METHOD

### A. Lagrangian approach to the doubble pendulum

Using the coordinates shown in 1. We can observe that we have four degrees of freedom with two constraints. This leaves us with $4 - 2 = 2$ generalized coordinates required to describe the system. [1]

In the assignment, I took the liberty of assuming that the only objects in the systems that carry any mass is $m_1$ and $m_2$ marked in 1. This means that the only part of the system that can hold any energy is those masses. I now then proceed to write out the position vectors of those masses.

$$r_1 = (l_1 \sin(\phi), l_1 \cos(\phi))$$
$$r_2 = (l_1 \sin(\phi) + l_2 \sin(\theta), l_1 \cos(\phi) + l_2 \cos(\theta))$$

Now that we have the position vectors we are ready to try and write the Lagrangian ($\mathcal{L}$).

$$\mathcal{L} = \frac{1}{2} m\dot{r}_1^2 + \frac{1}{2} m\dot{r}_2^2 - V_1 - V_2 \tag{1}$$

$$V_1 = m_1 g l_1 (1 - \cos(\phi))$$
$$V_2 = m_2 g (l_1(1 - \cos(\phi)) + l_2(1 - \cos(\theta))) \tag{2}$$

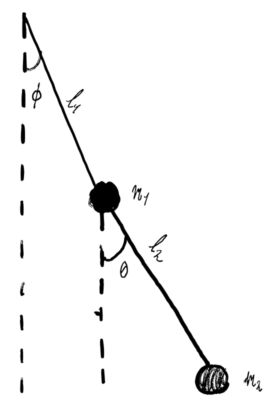

FIG. 1. Doubble pendulum

This expression carries terms that are just constants (If you would to multiply in the factors outside of the parentheses) which later on, in the Euler-Lagrange equations cancell. For this reason, any further refrence to 2 will be made without them. [2]

$$\dot{r}_1^2 = (l_1\dot{\phi})^2$$
$$\dot{r}_2^2 = l_1^2\dot{\phi}^2 + l_2\dot{\theta}^2 + ...$$
$$... + 2l_1l_2(\cos(\phi)\cos(\theta) + \sin(\phi)\sin(\theta))\dot{\phi}\dot{\theta}$$

Which turns 1 into;

$$\mathcal{L} = \frac{m_1}{2} l_1^2\dot{\phi}^2 + \frac{m_2}{2}\left(l_1^2\dot{\phi}^2 + l_2^2\dot{\phi}^2 + 2l_1l_2(\cos(\phi)\cos(\theta) + \sin(\phi)\sin(\theta))\dot{\phi}\dot{\theta}\right) + g(m_1+m_2)(\cos(\phi))l_1 + gm_2l_2(\cos(\theta)) \tag{3}$$

We can shorten this expression slightly by using the identity $\cos(s)\cos(t) + \sin(s)\sin(t) = \cos(s-t)$ But it doesn't get more compact than that.

Now turning to the Euler-Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \tag{4}$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m_2 l_1 l_2 \dot{\phi}\dot{\theta}\sin(\phi - \theta) - g(m_2 + m_1) l_1 \sin(\phi)$$
$$\frac{\partial \mathcal{L}}{\partial \theta} = m_2 l_1 l_2 \dot{\phi}\dot{\theta}\sin(\phi - \theta) - g m_2 l_1 \sin(\theta)$$
$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = l_1^2(m_1 + m_2)\ddot{\phi} + m_2 l_1 l_2 \ddot{\theta}\cos(\phi - \theta)$$
$$... - m_2 l_1 l_2 \dot{\theta}\sin(\phi - \theta)\dot{\phi} + m_2 l_1 l_2 \dot{\theta}\dot{\phi}\sin(\phi - \theta)$$
$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = l_2^2 m_2 \ddot{\theta} + m_2 l_1 l_2 \ddot{\phi}\cos(\phi - \theta)$$
$$... - m_2 l_1 l_2 \dot{\phi}^2 \sin(\phi - \theta) - m_2 l_1 l_2 \dot{\theta}\dot{\phi}\sin(\phi - \theta)$$

With these expressions we can now write 4 as:

$$-g(m_2+m_1)l_1\sin(\phi) = l_1^2(m_1+m_2)\ddot{\phi} + m_2l_1l_2\ddot{\theta}\cos(\phi-\theta) + m_2l_1l_2\dot{\theta}\dot{\phi}\sin(\phi-\theta) \tag{5}$$

$$-gm_2l_1\sin(\theta) = l_2^2m_2\ddot{\theta} + m_2l_1l_2\ddot{\phi}\cos(\phi-\theta) - m_2l_1l_2\dot{\phi}^2\sin(\phi-\theta) \tag{6}$$

These are nothing but a seccond degree differential equation. We could try to solve that but instead, the easier approach is to try and turn it into a first degree differential equation. We can do this by first using equations 5 and 6 as a linear systems of equations and solving for the seccond derrivatives.I'll show this derrivation in appendix A. Now defining new coordinates $\omega_1 = \dot{\phi}$ and $\omega_2 = \dot{\theta}$ allows us to write the following :

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \ddot{\phi} \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ g_1(\phi,\theta,\omega_1,\omega_2) \\ g_2(\phi,\theta,\omega_1,\omega_2) \end{pmatrix} \tag{7}$$

[3].

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[1] Here we note that since the only conserved quantity is energy, we do in fact have one more degrees of freedom than conserved quantities. Indicative of a chaotic system.

[2] From what I can tell, this is the standard way of treating the constants to parts of the lagrangian that ends up disapering. Typically, without any reference to doing so whatsoever. Which I personally find very confusing. So I decided to include a small paragraph notifying the reader of doing so.

[3] Definitions of $g_1$ and $g_2$ will be included in the appendix.