5
Undersampling

The surface reconstruction algorithm in the previous chapter assumes that the sample is sufficiently dense, that is, ε is sufficiently small. However, the cases of undersampling where this density condition is not met are prevalent in practice. The input data may be dense only in parts of the sampled surface. Regions with small features such as high curvatures are often not well sampled. When sampled with scanners, occluded regions are not sampled at all. Nonsmooth surfaces such as the ones considered in CAD are bound to have undersampling since no finite point set can sample nonsmooth regions to satisfy the ε-sampling condition for a strictly positive ε. Even some surfaces with boundaries can be viewed as a case of undersampling. If Σ is a surface without boundary and $\Sigma' \subset \Sigma$ is a surface with boundary, a sample of Σ' is also a sample of Σ. This sample may be dense for Σ' and not for Σ.

In this chapter we describe an algorithm that detects the regions of undersampling. This detection helps in reconstructing surfaces with boundaries. Later, we will see that this detection also helps in repairing the unwanted holes created in the reconstructed surface due to undersampling.

5.1 Samples and Boundaries

Let P be an input point set that samples a surface Σ where Σ does not have any boundary. The set P does not necessarily sample Σ equally well everywhere, but it does so for a subset (patches) of Σ which we call Σ^ε. The complement $\Sigma \setminus \Sigma^\varepsilon$ are undersampled regions. The boundaries of Σ^ε coincide with those of the undersampled regions. The goal is to reconstruct these boundaries from the input sample P. Since only P is known, we have to define the notion of boundary also with respect to P.

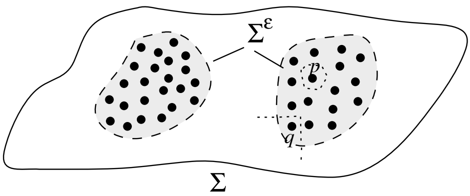

Figure 5.1. ε-sampled patches are shaded darker.

5.1.1 Boundary Sample Points

Definition 5.1. *For any $\varepsilon > 0$, an ε-sampled patch $\Sigma^\varepsilon \subseteq \Sigma$ is the closure of the set $\{x \mid B_{x,\varepsilon f(x)} \cap P \neq \emptyset\}$.*

In the above definition $f : \Sigma \to \mathbb{R}$ is the local feature size function of Σ and not of Σ^ε. Figure 5.1 illustrates the notion of ε-sampled patches for a small ε. Notice that Σ^ε is orientable as it is a subset of a surface $\Sigma \subset \mathbb{R}^3$ without boundary which must be orientable. Also, by definition, Σ^ε is compact.

In any compact surface, interior points are distinguished from boundary points by their neighborhoods. An interior point has a neighborhood homeomorphic to the plane $\mathbb{R}^2$. A boundary point, on the other hand, has a neighborhood homeomorphic to the half plane $\mathbb{H}^2_+ = \{(x_1, x_2) \in \mathbb{R}^2 \;\; x_1 \geq 0\}$. Even though all sample points in P may be interior points of the well-sampled patch Σ^ε, the existence of the nonempty boundary should be evident from the arrangement of points in P. We aim for a classification of *interior* and *boundary* sample points that capture the intuitive difference between interior and boundary points. We use the intersection of Σ^ε with the Voronoi diagrams to make this distinction. Let $F^\varepsilon_p = (\mathrm{Int} V_p) \cap \Sigma^\varepsilon$. The set F^ε_p consists of all points in Σ^ε that have p as their nearest sample point. In other words, p is a discrete representative of the surface patch F^ε_p. Or, conversely, F^ε_p can be taken as the neighborhood of p. Using this notion of neighborhood, we define the *interior* and *boundary* sample points.

Definition 5.2. *A sample point p from a sample P of Σ^ε is called* interior *if F^ε_p does not have a boundary point of Σ^ε. Points in P that are not interior are called* boundary *sample points.*

Observe that if p is a boundary sample point, the boundary of Σ^ε intersects the interior of V_p. In Figure 5.1, p is an interior sample point whereas q is a boundary sample point.

5.1.2 Flat Sample Points

The definitions of interior and boundary sample points are useless for computations since the restricted Voronoi diagram Vor $P|\Sigma^{\varepsilon}$ cannot be computed using only P. Therefore, we need a characterization of the sample points so that they can be distinguished algorithmically. To this end we define a *flatness* condition that can be checked with P while Σ^{ε} being unknown. It is shown that, under some mild assumptions, the boundary sample points cannot be flat whereas most of the interior sample points are flat.

The definition of flatness is motivated by the observation that the interior sample points have their Voronoi cells skinny and elongated along the normal, a property not satisfied by the boundary sample points. So, we need a measure to determine the "skinnyness" of the Voronoi cells. This motivates the following definitions of *radius* and *height*.

Definition 5.3. *The* radius r_p *of a Voronoi cell* V_p *is the radius of the cocone* C_p*, that is,* $r_p = \max\{\|y - p\| \mid y \in C_p\}$. *The* height h_p *is the distance* $\|p - p^-\|$ *where* p^- *is the negative pole defined in Section 4.1.*

The radius captures how "fat" the Voronoi cell is, whereas the height captures how "long" it is. The ratio of the radius over the height gives a measure how "skinny" the Voronoi cell is. It is important that the height be defined as the distance to the negative pole rather than to the positive one. Otherwise, a Voronoi cell only stretched toward the positive pole may qualify for a skinny cell, a structure not supported by interior sample points.

Not only do we want to capture the "skinnyness" of the Voronoi cells, but also the direction of their elongation. In case of an interior sample point p, the direction of elongation direction follows the direction of $\mathbf{n}_p$. This means that the pole vector $\mathbf{v}_p$ or its opposite vector match with those at the neighboring sample points in directions. We take the *cocone neighbors* for this check. The set of points in P whose Voronoi cells intersect the cocone of p are called the cocone neighbors of p. Formally, the set

$$N_p = \{q \in P : C_p \cap V_q \neq \emptyset\}$$

is the *cocone neighbors* of p.

The flatness condition is defined relative to two parameters ρ and α.

Definition 5.4. *A sample point* $p \in P$ *is called* (ρ, α)-flat *if the following two conditions hold:*

(i) Ratio condition: $r_p \leq \rho h_p$,
(ii) Normal condition: $\forall q$ *with* $p \in N_q$, $\angle_a(\mathbf{v}_p, \mathbf{v}_q) \leq \alpha$.

Ratio condition imposes that the Voronoi cell is long and thin in the direction of $\mathbf{v}_p$. The normal condition stipulates that the direction of elongation of V_p matches that of the Voronoi cell of any sample point whose cocone neighbor is p. For the theoretical guarantees, we use $\rho = 1.3\varepsilon$ and $\alpha = 0.14$.

We will need the Normal Lemma 3.2 for further analysis. Since we proved this lemma for surfaces without boundary, we cannot apply it to each sample point in P since P only samples Σ^ε well which may have boundaries. However, we can adopt the result for interior sample points as stated below. We can copy the entire proof for the Normal Lemma 3.2 since each point x of Σ^ε in V_p is within $\varepsilon f(x)$ distance from p.

Lemma 5.1 (Interior Normal). *Let p be an interior sample point in an ε-sampled patch Σ^ε with the surface normal $\mathbf{n}_p$ at p. Let y be any point in the Voronoi cell V_p such that $||y - p|| > \mu f(p)$ for some $\mu > 0$. For $\varepsilon < 1$, one has*

$$\angle_a(\overrightarrow{py}, \mathbf{n}_p) \leq \arcsin\left(\frac{\varepsilon}{\mu(1-\varepsilon)}\right) + \arcsin\left(\frac{\varepsilon}{1-\varepsilon}\right).$$

5.2 Flatness Analysis

Our goal is to exploit the definition of flat sample points in a boundary detection algorithm. We prove two theorems that form the basis of this algorithm. The Interior Sample Theorem 5.1 says that the interior sample points with well-sampled neighborhoods are flat and the Boundary Sample Theorem 5.2 says that the boundary sample points cannot be flat.

Lemma 5.2 (Ratio). *Interior sample points satisfy the ratio condition for $\rho = 1.3\varepsilon$ and $\varepsilon \leq 0.01$.*

Proof. Let p be any interior sample point. Letting $\mu = 1$ and y equal a pole of p in the Interior Normal Lemma 5.1 we get, for $\varepsilon \leq 0.01$,

$$\begin{aligned}\phi &= \angle_a(\mathbf{v}_p, \mathbf{n}_p)\\ &\leq 2\arcsin\left(\frac{\varepsilon}{1-\varepsilon}\right).\end{aligned}$$

Let y be any point in C_p. By definition $\angle_a(\mathbf{v}_p, \overrightarrow{yp}) \geq \frac{3\pi}{8}$. From the Interior Normal Lemma 5.1 (applying the contrapositive of the implication stated there) we get $||y - p|| \leq \mu f(p)$ where μ fulfills the inequality

$$\arcsin\left(\frac{\varepsilon}{\mu(1-\varepsilon)}\right) + \arcsin\left(\frac{\varepsilon}{1-\varepsilon}\right) + \phi \leq \frac{3\pi}{8}. \tag{5.1}$$

One can deduce that $\mu \leq 1.3\varepsilon$ satisfies Inequality 5.1 for $\varepsilon \leq 0.01$. That is, the radius of the Voronoi cell V_p is at most $1.3\varepsilon f(p)$, that is, $r_p \leq 1.3\varepsilon f(p)$.

Next, we show that the height $h_p = ||p^- - p||$ is at least $f(p)$. Recall that p^- is the farthest point in V_p from p so that $(p^- - p)^T \mathbf{v}_p < 0$. Since $\mathbf{n}_p$ makes a small angle up to orientation with $\mathbf{v}_p$, one of the two medial balls going through p has its center m such that the vector $\overrightarrow{mp}$ does not point in the same direction as $\mathbf{v}_p$, that is, $(m-p)^T\mathbf{v}_p < 0$. We know that $||m - p|| \geq f(p)$ and $m \in V_p$. This means that there is a Voronoi vertex $v \in V_p$ with $\|v - p\| \geq \|m - p\|$ and $(v-p)^T\mathbf{v}_p < 0$. This immediately implies that such a Voronoi vertex p^-, which is furthest from p, is at least $f(p)$ away from p. Therefore, $h_p \geq f(p) \geq \frac{r_p}{1.3\varepsilon}$. Thus, the ratio condition is fulfilled for $\rho = 1.3\varepsilon$ where $\varepsilon \leq 0.01$. ■

Although the ratio condition holds for all interior sample points, the normal condition may not hold for all of them. Nevertheless, interior sample points with well-sampled neighborhoods satisfy the normal condition. To be precise we introduce the following definition.

Definition 5.5. *An interior sample point p is* deep *if there is no boundary sample point with p as its cocone neighbor.*

Theorem 5.1 (Interior Sample). *All deep interior sample points are (ρ, α)-flat for $\rho = 1.3\varepsilon$, $\alpha = 0.14$, and $\varepsilon \leq 0.01$.*

Proof. It follows from the Ratio Lemma 5.2 that for $\varepsilon \leq 0.01$, deep interior sample points satisfy the ratio condition. We show that they satisfy the normal condition as well. Let q be any Voronoi neighbor of p so that $p \in N_q$. The sample point q is interior by definition. Therefore, we can apply the Interior Normal Lemma 5.1 to assert that $\angle_a(\mathbf{v}_q, \mathbf{n}_q) \leq 2 \arcsin \frac{\varepsilon}{1-\varepsilon}$. Also, by the Ratio Lemma 5.2 any point $x \in C_q$ satisfies $\|x - q\| \leq 1.3\varepsilon f(q)$. In particular, there is such a point $x \in V_p \cap V_q$ since $p \in N_q$. With $||x - q|| \leq 1.3\varepsilon f(q)$ and $x \in V_p \cap V_q$ we have $||p - q|| \leq 2.6\varepsilon f(q)$. For $\varepsilon \leq 0.01$ we can apply the Normal Variation Lemma 3.3 to deduce $\angle(\mathbf{n}_p, \mathbf{n}_q) \leq 0.03$. Thus, we have

$$\begin{aligned}\angle_a(\mathbf{v}_q, \mathbf{v}_p) &\leq \angle_a(\mathbf{v}_q, \mathbf{n}_q) + \angle(\mathbf{n}_q, \mathbf{n}_p) + \angle_a(\mathbf{n}_p, \mathbf{v}_p)\\ &\leq 0.14\end{aligned}$$

which satisfies the normal condition for $\alpha = 0.14$. ■

Next, we aim to prove the converse of the above theorem, that is, a (ρ, α)-flat sample point is an interior sample when ρ and α are sufficiently small. In other

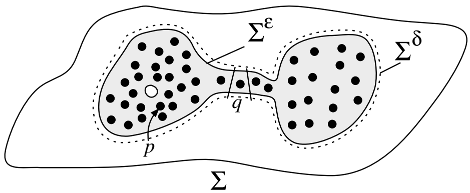

Figure 5.2. Σ^ε shown with the solid curves whereas Σ^δ is shown with the dotted curve. The point p is a boundary sample point in Σ^ε because of the small hole. This point may turn into an interior sample point for Σ^δ in which the hole may disappear. This is prohibited by the Boundary Assumption 5.1(i). The point q is a boundary sample point whose restricted Voronoi cell does not intersect that of any interior sample point violating the Boundary Assumption 5.1(ii).

words, these sample points cannot be boundary samples. This is the statement of the Boundary Sample Theorem 5.2.

For further development we will need to relate h_p with the local feature size $f(p)$. Since the Voronoi cell V_p contains the centers of the two medial balls at p, h_p is an upper bound on $f(p)$. Actually, for surfaces without boundary, it can be shown that h_p approximates the radius of the smaller of the two medial balls at p within a small factor of $\tilde{O}(\varepsilon^{\frac{2}{3}})$ (Exercise 3 in Chapter 6). We need a similar property for surfaces with boundary. However, for such surfaces h_p may not approximate $f(p)$ within a small factor dependent on ε. Nevertheless, we can bound the error with a surface-dependent constant which we use in the proof. Let $\Delta_p = \frac{h_p}{f(p)}$. We have an upper bound on $h_p = ||p^- - p||$ assuming that not all data points lie on a plane. By our assumption that Σ is compact and has a positive local feature size everywhere, $f(p)$ is greater than a surface-dependent constant. Thus, we have a surface-dependent constant, say Δ, so that $\Delta_p \leq \Delta$ for all $p \in \Sigma$.

The proof that the boundary sample points cannot be flat needs some assumptions. The first assumption (i) says that boundary sample points remain as boundary even if Σ^ε is expanded with a small collar around its boundary (see Figure 5.2). Assumption (ii) stipulates that the boundaries are "well separated" disallowing situations as shown in Figure 5.2.

Assumption 5.1 (Boundary Assumption).

(i) *We assume that both surfaces Σ^ε and Σ^δ define the same set of boundary sample points when $\delta = 1.3\Delta\varepsilon$ and $\Delta = \max \Delta_p$.*

(ii) *The restricted Voronoi cell of each boundary sample point in* Vor $P|_{\Sigma^\varepsilon}$ *intersects the restricted Voronoi cell of at least one interior sample point.*

The Boundary Assumption 5.1(i) is used to show the next lemma which leads to the Boundary Sample Theorem 5.2. This lemma says that a sample point satisfying the ratio condition that has a pole vector approximating the normal is an interior sample. Suppose p is such a sample point. The surface Σ has to lie within the cocone in V_p because of the ratio condition and the pole vector approximating the normal. This means that the subset of Σ^ε in V_p is small and, in particular, when expanded with a collar intersects V_p completely. This violates the Boundary Assumption 5.1(i) if p is a boundary sample. We formalize this argument now.

Lemma 5.3 (Interior). *Let p be a sample point which satisfies the ratio condition for $\rho = 1.3\varepsilon$. If $\angle_a(\mathbf{v}_p, \mathbf{n}_p) \leq 0.2$, p is an interior sample point when $\varepsilon \leq \frac{0.01}{1.3\Delta}$.*

Proof. Suppose, on the contrary, p is a boundary sample point. Since the ratio condition holds, we have $||x - p|| \leq \rho h_p = \rho \Delta_p f(p)$ for any point $x \in C_p$. With $\rho = 1.3\varepsilon$, we have $||x - p|| \leq \delta f(p)$ where $\delta = \Delta\rho = 1.3\Delta\varepsilon$. Therefore, for any $x \in C_p$ we have

$$\|x - p\| \leq \delta f(p). \tag{5.2}$$

Let y be any point on Σ with $||y - p|| \leq \delta f(p)$. The condition $\varepsilon \leq \frac{0.01}{1.3\Delta}$ gives $\delta < 2$. We can apply the proof of the Edge Normal Lemma 3.4 to claim $\angle_a(\overrightarrow{py}, \mathbf{n}_p) \geq \frac{\pi}{2} - \arcsin\frac{\delta}{2}$. Since $\angle_a(\mathbf{v}_p, \mathbf{n}_p) \leq 0.2$ by condition of the lemma, we have

$$\begin{aligned}\angle_a(\mathbf{v}_p, \overrightarrow{py}) &\geq \frac{\pi}{2} - \arcsin\frac{\delta}{2} - 0.2\\ &> \frac{3\pi}{8}.\end{aligned}$$

It implies that any point $y \in \Sigma$ with $||y - p|| \leq \delta f(p)$ cannot lie on the boundary of the double cone defining C_p. In other words, $\Sigma \cap V_p \in C_p$. Therefore, by Inequality 5.2 any point $y \in \Sigma \cap V_p$ satisfies $||y - p|| \leq \delta f(p)$. According to the Boundary Assumption 5.1 the surface $\Sigma^\delta \supseteq \Sigma^\varepsilon$ must define p as a boundary sample point. But, that would require a boundary point of Σ^δ to be in the interior of V_p. This would in turn require a point $y \in \Sigma$ with $||y - p|| > \delta f(p)$ to be in V_p. We reach a contradiction as each point $y \in \Sigma \cap V_p$ is at most $\delta f(p)$ distance away from p. ■

Theorem 5.2 (Boundary Sample). *Boundary sample points cannot be (ρ, α)-flat for $\rho = 1.3\varepsilon$, $\alpha = 0.14$, and $\varepsilon \leq \frac{0.01}{1.3\Delta}$.*

Proof. Let p be a boundary sample point. Suppose that, on the contrary, p is $(1.3\varepsilon, 0.14)$-flat. Consider an interior sample point q so that $V_q|_\Sigma \cap V_p|_\Sigma \neq \emptyset$ (Boundary Assumption 5.1(ii)). The sample point p is a cocone neighbor of q since $C_q \cap \Sigma = V_q \cap \Sigma$. The normal condition requires that $\angle_a(\mathbf{v}_p, \mathbf{v}_q) \leq 0.14$. Also, $||q - p|| \leq 2.6\varepsilon f(q)$ due to the ratio condition. It implies that $\angle(\mathbf{n}_p, \mathbf{n}_q) \leq 0.03$ (Normal Variation Lemma 3.3). Thus,

$$\begin{aligned}\angle_a(\mathbf{v}_p, \mathbf{n}_p) &\leq \angle_a(\mathbf{v}_p, \mathbf{v}_q) + \angle_a(\mathbf{v}_q, \mathbf{n}_q) + \angle(\mathbf{n}_q, \mathbf{n}_p)\\ &\leq 0.14 + 0.021 + 0.03 = 0.191.\end{aligned}$$

Thus, p satisfies the conditions of the Interior Lemma 5.3 and hence is an interior sample point reaching a contradiction. ■

5.3 Boundary Detection

The algorithm for boundary detection first computes the set of interior sample points, R, that are (ρ, α)-flat where ρ and α are two user-supplied parameters to check the ratio and normal conditions. If ρ and α are small enough, the Interior Sample Theorem 5.1 guarantees that R is not empty. In a subsequent phase R is expanded to include all interior sample points in an iterative procedure. A generic iteration proceeds as follows. Let p be any cocone neighbor of a sample point $q \in R$ so that $p \notin R$ and p satisfies the ratio condition. If $\mathbf{v}_p$ and $\mathbf{v}_q$ make small angle up to orientation, that is, if $\angle_a(\mathbf{v}_p, \mathbf{v}_q) \leq \alpha$, we include p in R. If no such sample point can be found, the iteration stops.

There is a subtle difference between the initial phase and the expansion phase of the boundary detection. The initial phase checks the normal condition for all cocone neighbors (step 3 in isFlat) whereas the expansion phase checks this condition only for cocone neighbors that have already been detected as interior (step 6 in Boundary). We argue that R includes all and only interior sample points at the end. The rest of the sample points are detected as boundary ones.

The following routine isFlat checks the ratio and normal conditions to detect flat sample points. The input is a sample point $p \in P$ with two parameters ρ and α. The return value is true if p is (ρ, α)-flat and false otherwise. The routine Boundary uses isFlat to detect the boundary sample points.

isFlat($p \in P, \alpha, \rho$)
1 compute the radius r_p and the height h_p;
2 `if` $r_p \leq \rho h_p$
3   `if` $\angle_a(\mathbf{v}_p, \mathbf{v}_q) \leq \alpha\ \forall q$ with $p \in N_q$
4   `return` TRUE;
5 `return` FALSE.

BOUNDARY(P, α, ρ)
1 $R := \emptyset$;
2 `for all` $p \in P$ `do`
3 `if` ISFLAT(p, α, ρ)
4 $R := R \cup p$;
5 `endfor`
6 `while` $(\exists p \notin R)$ and $(\exists q \in R$ with $p \in N_q)$ `do`
and $(r_p \leq \rho h_p)$ and $(\angle_a(\mathbf{v}_p, \mathbf{v}_q) \leq \alpha)$
7 $R := R \cup p$;
8 `endwhile`
9 `return` $P \setminus R$.

5.3.1 Justification

Now we argue that BOUNDARY outputs all and only boundary sample points. We need an interior assumption that says that all interior sample points have well-sampled neighborhoods.

Assumption 5.2 (Interior Assumption). *Each interior sample point is path connected to a deep interior sample point where the path lies only inside the restricted Voronoi cells of the interior sample points.*

Theorem 5.3. BOUNDARY *outputs all and only boundary sample points when* $\rho = 1.3\varepsilon$, $\alpha = 0.14$, *and* $\varepsilon \leq \frac{0.01}{1.3\Delta}$.

Proof. Inductively assume that the set R computed by BOUNDARY contains only interior sample points. Initially, the assumption is valid since steps 2 and 3 compute the set of flat sample points, R, which must be interior due to the Boundary Sample Theorem 5.2. The Boundary Assumption 5.1(ii) and the Interior Assumption 5.2 imply that each component of Σ^ε must have a deep interior sample point. Thus, R cannot be empty initially. In the *while* loop if a sample point p is included in the set R, it must satisfy the ratio condition. Also, there exists $q \in R$ so that $\angle_a(\mathbf{v}_p, \mathbf{v}_q) \leq 0.14$ since $\alpha = 0.14$ radians. Since q is an interior sample point by inductive assumption $\angle_a(\mathbf{v}_q, \mathbf{n}_q) \leq 2 \arcsin \frac{\varepsilon}{1-\varepsilon}$ (Interior Normal Lemma 5.1). It follows that $\angle_a(\mathbf{v}_p, \mathbf{n}_q) \leq 0.161$. Since p is a cocone neighbor of q, we have $\|q - p\| \leq 2.6\varepsilon f(q)$. Applying the Normal Variation Lemma 3.3 we get $\angle(\mathbf{n}_q, \mathbf{n}_p) \leq 0.03$ for $\varepsilon \leq 0.01$. Therefore,

$$\angle_a(\mathbf{v}_p, \mathbf{n}_p) \leq \angle_a(\mathbf{v}_p, \mathbf{n}_q) + \angle(\mathbf{n}_q, \mathbf{n}_p) \leq 0.2.$$

It follows from the Interior Lemma 5.3 that q is an interior sample point proving the inductive hypothesis.

Now we argue that each interior sample point p is included in R at the end of the *while* loop. The Interior Assumption 5.2 implies that one can reach p walking through adjacent cocones from a deep interior sample point. The proof of the Interior Sample Theorem 5.1 can be applied to show that any interior sample point that is a cocone neighbor of a sample point in R satisfies the condition of the *while* loop. It follows that p is encountered in the *while* loop during some iteration and is included in R. ■

5.3.2 Reconstruction

The COCONE algorithm described in Chapter 4 can be used to complete the surface reconstruction after the boundary sample points are detected. In the COCONE algorithm a sample point p chooses all triangles incident to it whose dual Voronoi edges are intersected by the cocone C_p. But, this causes the boundary sample points to choose undesirable triangles since the estimated normals at these sample points are not correct. So, in the modified algorithm BOUNDCOCONE, the boundary sample points are not allowed to choose any triangles. The desired triangles incident to boundary sample points are chosen by some interior sample points. As a result "garbage" triangles are eliminated and clean holes appear at the undersampled regions. Also, the manifold extraction step needs to be slightly modified so that it does not prune any boundary triangle incident to a boundary sample point.

```
BoundCocone(P, α, ρ)
compute Vor P;
B :=Boundary(P, α, ρ);
for each p ∈ P \ B do
   mark the triangle dual e where e ∩ C_p ≠ ∅;
endfor
T := ∅;
for each σ ∈ Del P do
   if σ is marked by all its vertices not in B
      T:=T ∪ σ;
   endif
endfor
extract a manifold from T using pruning and walking.
```

Figure 5.3 shows some examples of the boundary detection using BOUNDCOCONE. Obviously, in practice, sometimes the assumptions made for BOUNDCOCONE do not hold and the output may produce some artifacts.

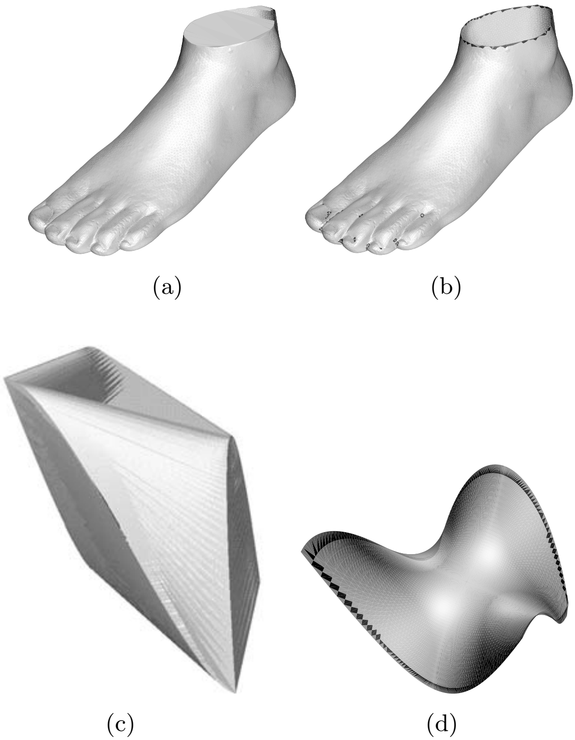

Figure 5.3. Reconstruction of the dataset FOOT: (a) without boundary detection; the big hole above the ankle is covered with triangles and (b) with boundary detection using the algorithm BOUNDCOCONE; the hole above the ankle is well detected. MONKEY saddle: (c) without boundary detection and (d) with boundary detection.

5.4 Notes and Exercises

The material for this chapter is taken from Dey and Giesen [30]. Undersampling is one of the major problems for surface reconstruction in practice. Systematic treatment of undersampling is scarce in the literature. Dey and Giesen gave the first provable algorithm for undersampling detection under some reasonable assumptions. The questions of relaxing these conditions and proving homeomorphisms between the reconstructed and original surfaces remain open. In practice, when BOUNDCOCONE is applied to reconstruct surfaces with boundaries sometimes it detects small holes in undersampled regions along with the intended boundaries. Theoretically, these small holes are correctly detected. However, often applications require that only the intended boundaries and not these small holes be detected. It would be interesting to find a solution which can recognize only distinct boundaries.

Recall that the presented theory is based on the assumption that the sampled surface Σ is C^2-smooth. However, it is observed that the boundary detection algorithm also detects undersampling in nonsmooth surfaces. The ability to handle nonsmooth surfaces stems from the fact that nonsmooth surfaces may be approximated with a smooth one that interpolates the sample points. Such a surface exists by a well-known result in mathematics that the class of C^2-smooth surfaces is dense in the class of C^0-smooth surfaces. For example, one can resort to the implicit surface that is C^2-smooth and interpolates the sample points using natural coordinates as explained in Boissonnat and Cazals [16] (see Section 9.7). These smooth surfaces have high curvatures near the sharp features of the original nonsmooth surface. The theory can be applied to the approximating smooth surface to ascertain that the sample points in the vicinity of sharp features act as boundary sample points in the vicinity of high curvatures for the smooth surface. Reconstructing nonsmooth surfaces with topological guarantees under the ε-sampling theory becomes difficult since the local feature size becomes zero at nonsmooth regions. Recently, Chazal, Cohen-Steiner, and Lieutier [19] proposed a sampling theory that alleviates this problem.

We assumed that the input point set samples a subset of a surface without boundary. It is not true that all surfaces with boundaries can be viewed as a subset of a surface without boundary. For example, nonorientable surfaces in $\mathbb{R}^3$ such as Möbius strip cannot be a subset of any surface without boundary. It remains open to develop a general reconstruction algorithm for any C^2-smooth, compact surface with boundaries (Exercise 6). Also, we did not prove any topological equivalence between the output and input surfaces. It remains open to develop an algorithm with such guarantees (Exercise 2).

Exercises

1. Let P oversample a C^2-smooth surface Σ, that is, P is unnecessarily dense. Devise an algorithm to eliminate points from P so that Σ can still be reconstructed from the decimated P.

2^o. Let Σ be a C^2-smooth surface with a boundary C. Suppose P is an ε-sample of Σ where the local feature size function is defined with respect to the medial axis of Σ taking C into account. Also, assume that the points $P' \subset P$ that sample C are known. Design an algorithm to reconstruct Σ from P with a proof of homeomorphism.

3^o. Devise an algorithm to detect the boundary sample points whose proof does not depend upon a global constant like Δ.

4^o. Prove or disprove that only the ratio condition as presented in this chapter is sufficient for detection of the boundary sample points.

5. Prove the Voronoi Cell Lemma 3.10 for Σ^ε when ε is sufficiently small.

6^o. BOUNDCOCONE and its proof depends on the fact that the surface Σ^ε is orientable. Devise an algorithm for reconstructing nonorientable surfaces.