8

Noise and Reconstruction

The algorithms for surface reconstruction in previous chapters assume that the input is noise-free. Although in practice all of them can handle some amount of displacements of the points away from the surface, they are not designed in principle to handle such data sets. As a result when the points are scattered around the sampled surface, these algorithms are likely to fail. In this chapter we describe an algorithm that is designed to tolerate noise in data.

The algorithm works with the Delaunay/Voronoi diagrams of the input points and draws upon some of the principles of the power crust algorithm. The power crust algorithm exploits the fact that the union of the polar balls approximates the solid bounded by the sampled surface. Obviously, this property does not hold in the presence of noise. Nevertheless, we have observed in Chapter 7 that, under some reasonable noise model, some of the Delaunay balls remain relatively big and can play the role of the polar balls. These balls are identified and partitioned into inner and outer balls. We show that the boundary of the union of the outer (or inner) big Delaunay balls is homeomorphic to the sampled surface. This immediately gives a homeomorphic surface reconstruction though the reconstructed surface may not interpolate the sample points. The algorithm can be extended to compute a homeomorphic surface interpolating a subset of the input sample points. These points reside on the outer (or inner) big Delaunay balls. The rest of the points are deleted. The Delaunay triangulation of the chosen sample points restricted to the boundary of the chosen big Delaunay balls is output as an approximation to the sampled surface. Figure 8.1 illustrates this algorithm in two dimensions.

8.1 Preliminaries

As before we will assume that the sampled surface Σ is smooth, compact, and has no boundary. Also, we will assume that Σ is connected. The requirement

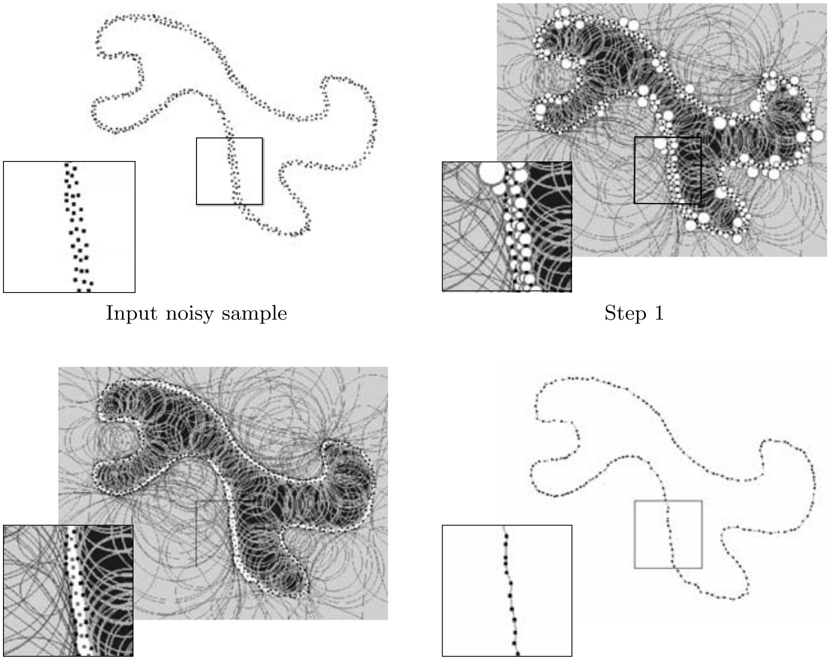


Figure 8.1. Step 1: big Delaunay balls (shaded) are separated from small ones (unshaded), Step 2: outer and inner big Delaunay balls are separated, Step 3: only the points on the outer balls are retained and the curve (surface) is reconstructed from them.

of connectedness is no more for mere simplicity but is indeed needed for the algorithm. Inner and outer big Delaunay balls are separated by a labeling step similar to that of PowerCrust, which we already know requires Σ to be connected. As in Chapter 7, we use the notations Ω_O to denote the unbounded component of $\mathbb{R}^3 \setminus \Sigma$ and $\Omega_I = \mathbb{R}^3 \setminus \Omega_O$. The normals of Σ are oriented to point outside, that is, toward Ω_O.

We will follow the noise model presented in Chapter 7. This noise model allows two separate parameters for the horizontal and the normal scatters. For simplicity we will make this general noise model a little more specific by assuming P to be a $(\varepsilon, \varepsilon^2, \kappa)$-sample of Σ. First, this removes one parameter from the general model. Second, the quadratic dependence of the normal scatter on ε makes the presentation simpler. Notice that the analysis we are going to present can be extended to the general model by carrying around an extra parameter δ in all calculations.

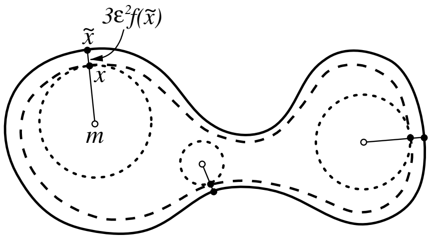


Figure 8.2. Feature balls for three different positions for x are shown with the dotted boundary. The points x, $\tilde{x}$, and the center m are collinear.

Recall that, for a point $x \in \mathbb{R}^3$ that is not on the medial axis, $\tilde{x}$ denotes its closest point in Σ. Under the assumed noise model, the following claims follow from the Close Sample Lemma 7.1 and the κ-Neighbor Lemma 7.4.

Lemma 8.1 (Sampling).

(i) Any point $x \in \Sigma$ has a sample point within $\varepsilon_1 f(x)$ distance where $\varepsilon_1 = \varepsilon(1+\varepsilon+\varepsilon^2)$.

(ii) Any sample point $p \in P$ has its κth closest sample point within $\varepsilon_2 f(\tilde{p})$ distance where $\varepsilon_2 = \left(\varepsilon + \frac{4\kappa+\varepsilon}{1-4\kappa\varepsilon}\right)\varepsilon = \tilde{O}(\varepsilon)$.

From this point onward we consider Ω_I to state all definitions and results unless specified otherwise. It should be clear that they also hold for Ω_O. We have already seen in the previous chapter (Empty Ball Lemma 7.2) that there are empty balls with radius almost as large as local feature sizes and with a boundary point close to Σ. These balls, which we call *feature balls* (Figure 8.2) will play an important role in the proofs. Because of their importance in the proofs, we give a formal definition of them.

Definition 8.1. *Let $B_{m,r}$ be the ball with the following conditions:*

(i) $m \in \Omega_I$; the boundary of $B_{m,r}$ has a point x where $\|x - \tilde{x}\| \leq 3\varepsilon^2 f(\tilde{x})$,

(ii) $r = (1 - 3\varepsilon^2) f(\tilde{x})$, and

(iii) the center m lies on the line of $\mathbf{n}_{\tilde{x}}$. In other words, $\tilde{m} = \tilde{x}$.

Call $B_{m,r}$ a feature ball.

The particular choice of the term $3\varepsilon^2$ in the definition of the feature balls is motivated by the Empty Ball Lemma 7.2. One can substitute δ with ε^2 in this lemma to claim that the feature balls are empty.

Also, the following observation will be helpful for our proofs. It says that if a ball with two points x and y on its boundary is big relative to the feature size of $\tilde{x}$, it remains big relative to the feature size of $\tilde{y}$ if x and y are close to Σ. The parameters λ and ε' will be close to 1 and ε respectively when we use this lemma later.

Lemma 8.2. *Let $B = B_{c,r}$ be a ball with two points x and y on its boundary where $\|x - \tilde{x}\| \leq \varepsilon' f(\tilde{x})$, $\|y - \tilde{y}\| \leq \varepsilon' f(\tilde{y})$. Then, $r \geq \frac{\lambda(1-\varepsilon')}{1+2\lambda+\varepsilon'} f(\tilde{y})$ given that $r \geq \lambda f(\tilde{x})$ for $\lambda > 0$.*

Proof. We get

$$\begin{aligned} r &\geq \lambda f(\tilde{x}) \\ &\geq \lambda(f(\tilde{y}) - \|\tilde{x} - \tilde{y}\|) \\ &\geq \lambda(f(\tilde{y}) - \|x - \tilde{x}\| - \|x - y\| - \|y - \tilde{y}\|) \\ &\geq \lambda(f(\tilde{y}) - \varepsilon' f(\tilde{x}) - 2r - \varepsilon' f(\tilde{y})) \end{aligned}$$

from which it follows that

$$\begin{aligned} (1 + 2\lambda + \varepsilon')r &\geq \lambda(1 - \varepsilon') f(\tilde{y}) \\ \text{or, } r &\geq \frac{\lambda(1 - \varepsilon')}{1 + 2\lambda + \varepsilon'} f(\tilde{y}). \end{aligned}$$

■

8.2 Union of Balls

As we indicated before, our goal is to filter out a subset of points from P that lie on big Delaunay balls. We do this by choosing Delaunay balls that are big compared to the distances between sample points and their κth nearest neighbors. Let d_p denote the distance to the κth nearest neighbor of a sample point $p \in P$. For an appropriate constant $K > 0$, we define

$\mathcal{B}(K) =$ set of Delaunay balls $B_{c,r}$ where $r > K d_p$ for all points $p \in P$ incident on the boundary of $B_{c,r}$.

Since we know that $d_p \geq \varepsilon f(\tilde{p})$ by the sampling condition, we have

Observation 8.1. *Let $B_{c,r} \in \mathcal{B}(K)$ be a Delaunay ball with $p \in P$ on its boundary. Then, $r > K\varepsilon f(\tilde{p})$.*

By definition $\mathbb{R}^3 = \Omega_I \cup \Omega_O$. So, we can write $\mathcal{B}(K) = \mathcal{B}_I \cup \mathcal{B}_O$ where $\mathcal{B}_I$ is the set of balls having their centers in Ω_I and $\mathcal{B}_O$ is the set of balls with their centers in Ω_O. We call the balls in $\mathcal{B}_I$ the *inner* big Delaunay balls and the ones in $\mathcal{B}_O$ the *outer* big Delaunay balls.

We will filter out those points from P that lie on the balls in $\mathcal{B}(K)$. A decomposition of $\mathcal{B}(K)$ induces a decomposition on these points, namely

$$P_I = \{p \in P \cap B \mid B \in \mathcal{B}_I\} \quad \text{and} \quad P_O = \{p \in P \cap B \mid B \in \mathcal{B}_O\}.$$

Notice that P_I and P_O may not be disjoint and they decompose only the set of points incident to the balls in $\mathcal{B}(K)$ and not necessarily the set P.

In the analysis to follow we will assume that ε is a sufficiently small positive value no more than 0.01. With this assumption we have

$$\varepsilon_1 = \varepsilon(1 + \varepsilon + \varepsilon^2) \leq 1.1\varepsilon.$$

We will use the General Normal Theorem 7.1 in the analysis. Substituting $\delta = 3\varepsilon^2$ and $\varepsilon_1 \leq 1.1\varepsilon$ we get the following corollary.

Corollary 8.1. *Let $B_{c,r}$ be a Delaunay ball whose boundary contains a sample point $p \in P$. Let c lie in Ω_I. If $r = \lambda f(\tilde{p})$ then the sin of the angle the vector $\overrightarrow{cp}$ makes with n_p is at most*

$$\left(2.2 + 4\sqrt{3} + \frac{3\sqrt{3}}{\sqrt{\lambda}} + \frac{3.3}{\lambda}\right)\varepsilon$$

when $\varepsilon \leq 0.01$ is sufficiently small.

In the rest of the chapter we use

$$\varepsilon_3 = \varepsilon_1 + 3\varepsilon^2 \quad \text{and} \quad \varepsilon_4 = \left(\frac{7\varepsilon_3}{(1 - 3\varepsilon^2) + 4\varepsilon_3}\right)^{\frac{1}{2}} (1 - 3\varepsilon^2).$$

Notice that $\varepsilon_3 = \tilde{O}(\varepsilon)$ and $\varepsilon_4 = \tilde{O}(\sqrt{\varepsilon})$.

The next lemma shows that not only do we have large Delaunay balls in Del P but also many of them covering almost the entire Ω_I.

Lemma 8.3 (Delaunay Ball). *For each point $x \in \Omega_I$ with $\|x - \tilde{x}\| = 3\varepsilon^2 f(\tilde{x})$, there is a Delaunay ball that enjoys the following properties when ε is sufficiently small.*

(i) The radius of the Delaunay ball is at least $\frac{3}{4}(1 - 3\varepsilon^2) f(\tilde{x})$.
(ii) The boundary of the Delaunay ball contains a sample point $p \in P_I$ within a distance $\varepsilon_4 f(\tilde{x}) = \tilde{O}(\sqrt{\varepsilon}) f(\tilde{x})$ from x.

Proof. Consider the feature ball $B = B_{m,r}$ whose boundary meets x. By definition,

$$r = (1 - 3\varepsilon^2) f(\tilde{x}).$$

We construct the Delaunay ball as claimed by deforming B as follows.

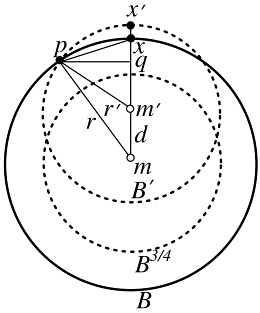


Figure 8.3. Deformation of B to B'.

Shrinking: Let $B^{3/4} = B_{m,3r/4}$ be a shrunk copy of B. The ball B and hence $B^{3/4}$ are empty.

Translation: Translate $B^{3/4}$ rigidly by moving the center m along the direction $\overrightarrow{mx}$ until its boundary hits a sample point $p \in P$. Let this new ball be $B' = B'_{m',r'}$, refer to Figure 8.3.

Delaunay deformation: Deform B' further to a larger Delaunay ball $B'' = B_{m'',r''}$ which we show has the claimed properties. The center m' of B' belongs to the Voronoi cell V_p since B' is empty of points from P. Move the center m' of B' continuously in V_p always increasing the distance $\|m' - p\|$ till m' meets a Voronoi vertex, say m'', in V_p. This motion is possible as the distance function from p reaches its maxima only at the Voronoi vertices.

Let x' be the closest point to x on the boundary of B'. The Sampling Lemma 8.1(i) implies that the point x has a sample point within $(\varepsilon_1 + 3\varepsilon^2)f(\tilde{x})$ distance. We have

$$\|x' - x\| \leq (\varepsilon_1 + 3\varepsilon^2)f(\tilde{x}) = \varepsilon_3 f(\tilde{x}) \tag{8.1}$$

since otherwise there is an empty ball centering x with radius $\varepsilon_3 f(\tilde{x})$.

Claim 8.1. $\|x - p\| \leq \varepsilon_4 f(\tilde{x})$.

First, we observe that both B and B' contain their centers in their intersection. Since B' has a radius smaller than B, it is sufficient to show that B' contains m inside. During the rigid translation when the ball $B^{3/4}$ touches B at x, its center moves by $\frac{1}{4}r$ distance. After that, we move $B^{3/4}$ by the distance $\|x' - x\| \leq$

$\varepsilon_3 f(\tilde{x})$ (Inequality 8.1). Thus,

$$\|m - m'\| \le \frac{1}{4}r + \varepsilon_3 f(\tilde{x}). \tag{8.2}$$

Therefore, the distance between m and m' is less than $\frac{3}{4}r$ for sufficiently small ε implying that m is in B'.

Now we prove the claimed bound for $\|x - p\|$. The point p can only be on that part of the boundary of B' which is outside the empty ball B. This with the fact that the centers of B and B' are in their intersection imply that the largest distance from x to p is realized when p is on the circle where the boundaries of B and B' intersect. Consider this situation as in Figure 8.3.

Let $d = \|m' - m\|$. First, observe that

$$\frac{1}{4}r \le d \le \frac{1}{4}r + \varepsilon_3 f(\tilde{x}). \tag{8.3}$$

The first half of the inequality holds since B is empty of samples and hence $B^{\frac{3}{4}}$ has to move out of it to hit a sample point. The second half of the inequality follows from Inequality 8.2. Since

$$\begin{aligned}\|p - q\|^2 &= \|m - p\|^2 - \|m - q\|^2,\\ &= r^2 - (\|m' - q\| + d)^2\end{aligned}$$

and also

$$\begin{aligned}\|p - q\|^2 &= \|m' - p\|^2 - \|m' - q\|^2,\\ &= (r')^2 - \|m' - q\|^2\end{aligned}$$

we have

$$\|m' - q\| = \frac{r^2 - (r')^2 - d^2}{2d}.$$

Hence,

$$\begin{aligned}\|x - p\|^2 &= \|p - q\|^2 + \|q - x\|^2\\ &= r^2 - (d + \|m' - q\|)^2\\ &\quad +(r - (d + \|m' - q\|))^2\\ &= 2r^2 - rd - \frac{r}{d}(r^2 - r'^2)\\ &\overset{\text{Ineq. 8.3}}{\le} \frac{\varepsilon_3(1 + \frac{3}{4})}{\frac{1}{4}(1 - 3\varepsilon^2) + \varepsilon_3} r^2\\ &\le \varepsilon_4 f(\tilde{x}).\end{aligned}$$

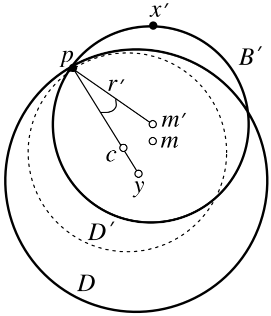


Figure 8.4. The balls B' and D incident to p. The reduced ball D' shown with dotted circle contains m.

Claim 8.2. $m'' \in \Omega_I$.

To prove this claim we first show that the radius r' of B', which is $\frac{3}{4}r = \frac{3}{4}(1-3\varepsilon^2)f(\tilde{x})$, is also large compared to $f(\tilde{p})$. Observe that $\|x' - \tilde{x}\|$ is at most $\varepsilon_3 f(\tilde{x})$ if $\tilde{x}$ lies between x and x' (Inequality 8.1). If $\tilde{x}$ does not lie between x and x', the distance $\|x' - \tilde{x}\|$ is no more than $\|x - \tilde{x}\|$ which and is at most $3\varepsilon^2 f(\tilde{x})$ since B is a feature ball (refer to Figure 8.2). Hence, $\|x' - \tilde{x}\| \leq \max\{\varepsilon_3, 3\varepsilon^2\} f(\tilde{x})$. We have $\varepsilon_3 > 3\varepsilon^2$. Therefore, we can say $\|x' - \tilde{x}\| \leq \varepsilon_3 f(\tilde{x})$. We know $\|p - \tilde{p}\| \leq \varepsilon^2 f(\tilde{p})$. So, we can apply Lemma 8.2 with $\varepsilon' = \varepsilon_3$ and $\lambda = \frac{3}{4}(1-3\varepsilon^2)$ to deduce that $r' = \|p - m'\| \geq \beta f(\tilde{p})$ where

$$\beta = \frac{3}{10}\frac{(1-3\varepsilon^2)(1-\varepsilon_3)}{1+\tilde{O}(\varepsilon)}.$$

This means

$$r' \geq \left(\frac{3}{10} - \tilde{O}(\varepsilon)\right) f(\tilde{p}). \tag{8.4}$$

Now we show that the center of B' cannot reach a point in Σ during its deformation to B'' establishing $m'' \in \Omega_I$. Suppose not, that is, the center of B' reaches a point $y \in \Sigma$ during the deformation. Then, we reach a contradiction.

First, observe that m' is in Ω_I as it is only within $\frac{1}{4}r + \varepsilon_3 f(\tilde{x})$ distance away from m. Next, consider the two balls B' and $D = B_{y,\|y-p\|}$ meeting at p (Figure 8.4). Both have radii larger than $(\frac{3}{10} - \tilde{O}(\varepsilon))f(\tilde{p})$ (Inequality 8.4) which is at least $\frac{f(\tilde{p})}{4}$ for sufficiently small ε. Both vectors $\overrightarrow{yp}$ and $\overrightarrow{m'p}$ make at most 35ε angle with $\mathbf{n}_{\tilde{p}}$ (Corollary 8.1) and hence make an angle of at most 70ε among themselves. Consider a smaller version of D by moving its center towards p till its radius becomes same as that of B'. Let this new ball be

$D' = B_{c,\|p-c\|}$ (Figure 8.4). We show that this D' and hence D contain m. We have

$$\begin{aligned}\|m-c\| &\le \|m-m'\| + \|m'-c\| \\ &\le \frac{1}{4}(1-3\varepsilon^2)f(\tilde{x}) + \varepsilon_3 f(\tilde{x}) + 70\varepsilon r' \\ &= \left(\frac{1}{4} + \tilde{O}(\varepsilon)\right)(1-3\varepsilon^2)f(\tilde{x}).\end{aligned}$$

On the other hand, the radius $\|p-c\|$ of D' is $r' = \frac{3}{4}(1-3\varepsilon^2)f(\tilde{x})$. Therefore, $\|m-c\|$ is smaller than this radius for a sufficiently small ε. Hence, m is in D' and therefore in D. Now we claim that y and m are far away and thus y cannot have any sample point nearby contradicting the Sampling Lemma 8.1. Let z be the point on the medial axis so that $\|\tilde{x}-m\| = f(\tilde{x}) = \|\tilde{x}-z\|$. Then, $\|y-m\| + 2\|\tilde{x}-m\| \ge \|y-z\| \ge f(y)$ giving $3\|y-m\| \ge f(y)$ or $\|y-m\| \ge f(y)/3$. Since D contains m, the ball centered at y with radius $\|y-m\|$ lies completely inside D and thus cannot contain any sample point. This means y cannot have a sample point within $f(y)/3$ distance, a contradiction to the Sampling Lemma 8.1 when ε is sufficiently small. This completes the claim that the center of B' always remains in Ω_I while deforming B' to B''.

Claim 8.3. $B'' \in \mathcal{B}_I$.

The ball B'' contains four sample points including p on its boundary. For any of these sample points u, we have $\|u-\tilde{u}\| \le \varepsilon^2 f(\tilde{u})$ by the sampling condition. Therefore, applying Lemma 8.2 to B'' with points p, $u \ne p$, and $\lambda = (\frac{3}{10} - \tilde{O}(\varepsilon))$ we get

$$r'' \ge \frac{\lambda(1-\varepsilon^2)}{1+2\lambda+\varepsilon^2} f(\tilde{u}) \ge \left(\frac{3}{16} - \tilde{O}(\varepsilon)\right) f(\tilde{u}).$$

Also, we have $d_u \le \varepsilon_2 f(\tilde{u})$ from the Sampling Lemma 8.1. Thus, B'' is in $\mathcal{B}(K)$ if

$$\frac{(\frac{3}{16} - \tilde{O}(\varepsilon))}{2} > K\varepsilon_2, \quad \text{or} \quad 1 > \tilde{O}(\varepsilon) + 11K\varepsilon_2,$$

a condition which is satisfied for a sufficiently small ε. Since $m'' \in \Omega_I$ by Claim 8.2, we have $B'' \in \mathcal{B}_I$.

Lemma claims: Clearly,

$$r'' \ge r' \ge \frac{3}{4}(1-3\varepsilon^2)f(\tilde{x}).$$

This proves (i). Claim 8.3 proves $p \in P_I$ which together with the Claim 8.1 gives (ii). ■

8.3 Proximity

We aim to prove that the boundary of $\bigcup \mathcal{B}_I$ is homeomorphic and close to Σ. The proof can be adapted in a straightforward manner for a similar result between the boundary of $\bigcup \mathcal{B}_O$ and Σ. We define

$$S_I = \text{bd}\left(\bigcup \mathcal{B}_I\right),$$
$$S_O = \text{bd}\left(\bigcup \mathcal{B}_O\right).$$

In the next two lemmas we establish that each point in S_I has a nearby point on Σ.

Lemma 8.4. *Let x be a point lying in Ω_O where $x \in S_I$. Then, $\|x - \tilde{x}\| \leq \frac{\varepsilon_1}{1-2\varepsilon_1} f(\tilde{x})$.*

Proof. Let $x \in B_{c,r}$ where $B_{c,r} \in \mathcal{B}_I$. The line segment joining x and c must intersect Σ since c lies in Ω_I while x lies in Ω_O. Let this intersection point be z. We claim that $\|x - z\| \leq \varepsilon_1 f(z)$. Otherwise, there is a ball inside $B_{c,r}$ centering z and radius at least $\varepsilon_1 f(z)$. This ball is empty since $B_{c,r}$ is empty. This violates the Sampling Lemma 8.1 for z. This means that the closest point $\tilde{x} \in \Sigma$ to x has a distance $\|x - \tilde{x}\| \leq \|x - z\| \leq \varepsilon_1 f(z)$. We also have $\|z - \tilde{x}\| \leq 2\|x - z\|$. Applying the Lipschitz property of f, we get the desired bound for $\|x - \tilde{x}\|$. ■

Lemma 8.5. *Let x be a point lying in Ω_I where $x \in S_I$. Then, for a sufficiently small ε, $\|x - \tilde{x}\| \leq 36\varepsilon f(\tilde{x})$.*

Proof. Let $y \in \Omega_I$ be a point where $\tilde{y} = \tilde{x}$ and $\|y - \tilde{x}\| = 3\varepsilon^2 f(\tilde{x})$. Observe that x, y, and $\tilde{x}$ are collinear. If x lies between $\tilde{x}$ and y, then $\|x - \tilde{x}\| \leq 3\varepsilon^2 f(\tilde{x})$ which is no more than $36\varepsilon f(\tilde{x})$.

So, assume that x is further away from $\tilde{x}$ than y is. Consider a Delaunay ball $B = B_{c,r} \in \mathcal{B}_I$ for y guaranteed by the Delaunay Ball Lemma 8.3. This ball has a sample point $p \in P$ on the boundary so that $\|y - p\| \leq \varepsilon_4 f(\tilde{x})$. Moreover, $r \geq \frac{3}{4}(1 - 3\varepsilon^2) f(\tilde{x})$. This ball was obtained by deforming a ball $B' = B_{m',r'}$ whose boundary passes through p and a point x' where $\|y - x'\| \leq \varepsilon_3 f(\tilde{x})$. Also, $r' = \frac{3}{4}(1 - 3\varepsilon^2) f(\tilde{x})$. Focus on the two balls B and B' incident to p. Since y and p and hence $\tilde{x}$ and p are close, both B and B' have radii larger

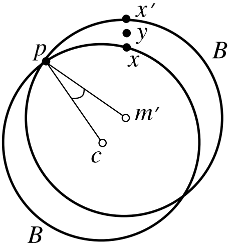


Figure 8.5. The balls B and B' incident to p. The point x' is furthest from x when B is the smallest possible.

than $\frac{3}{5}f(\tilde{p})$ when ε is sufficiently small. By Corollary 8.1, we obtain that the vectors $\overrightarrow{pc}$ and $\overrightarrow{pm'}$ make 23ε angle with $-\mathbf{n}_{\tilde{p}}$ and hence make an angle of 46ε among them.

We know that B has a radius at least as large as B' (proof of the Delaunay Ball Lemma 8.3). The points $x, x', \tilde{x}$, and y are collinear and y separates x and x'. Further, x cannot lie inside a Delaunay ball. With these constraints, the distance between x and x' is the most when x lies on the boundary of B and B is the smallest possible (see Figure 8.5). This means we can assume that both B and B' have the same radius to estimate the worst upper bound on $\|x - x'\|$. In that configuration, $\|x - x'\| \leq \|c - m'\| \leq 46r'\varepsilon$ which is at most $34\varepsilon(1 - 3\varepsilon^2)f(\tilde{x})$. Therefore,

$$\begin{aligned}\|x - \tilde{x}\| &\leq \|x - y\| + \|y - \tilde{x}\|\\ &\leq \|x - x'\| + \|x' - \tilde{x}\|\\ &\leq (34\varepsilon(1 - 3\varepsilon^2) + \varepsilon_3)f(\tilde{x})\\ &\leq 36\varepsilon f(\tilde{x}).\end{aligned}$$

■

From Lemma 8.4 and Lemma 8.5 we get the following theorem.

Theorem 8.1 (Small Hausdorff). *For a sufficiently small ε, each point x on S_I has a point in Σ within $36\varepsilon f(\tilde{x})$ distance.*

Lemma 8.6. *Let x be any point on the boundary of a ball $B_{c,r} \in \mathcal{B}_I$, we have $r \geq (K/2)\varepsilon f(\tilde{x})$ for a sufficiently small ε.*

Proof. Suppose the claim is not true. Then, consider a vertex $p \in P_I$ on the Delaunay ball $B_{c,r}$. Since this ball is in $\mathcal{B}_I$, we have $r \geq K\varepsilon f(\tilde{p})$. Since

$\|x - p\| \le 2r$, we have $\|x - p\| \le K\varepsilon f(\tilde{x})$ by our assumption. This means $\|x - \tilde{p}\| \le K\varepsilon f(\tilde{x}) + \varepsilon^2 f(\tilde{p})$. Since $\tilde{x}$ is closer to x than $\tilde{p}$, we have

$$\begin{aligned}\|\tilde{x} - \tilde{p}\| &\le \|\tilde{x} - x\| + \|x - \tilde{p}\| \\ &\le 2\left(K\varepsilon f(\tilde{x}) + \varepsilon^2 f(\tilde{p})\right).\end{aligned}$$

Using the Lipschitz property of f we get

$$f(\tilde{x}) \le \left(\frac{1 + 2\varepsilon^2}{1 - 2K\varepsilon}\right) f(\tilde{p}).$$

Therefore by our assumption,

$$r < \left(\frac{K}{2}\right)\left(\frac{1 + 2\varepsilon^2}{1 - 2K\varepsilon}\right) \varepsilon f(\tilde{p}).$$

We reach a contradiction if $\frac{K(1+2\varepsilon^2)}{2(1-2K\varepsilon)} \le K$, a condition which is satisfied for a sufficiently small ε. ■

Theorem 8.2 (Normal Approximation). *Let x be a point in S_I where $B_{c,r} \in \mathcal{B}_I$ contains x. For a sufficiently small ε, $\angle(\mathbf{n}_{\tilde{x}}, \overrightarrow{cx}) = 25\sqrt{\varepsilon} + \frac{26}{\sqrt{K}} + \frac{8}{K}$.*

Proof. We apply the General Normal Theorem 7.1 to x. Lemma 8.6 gives $\lambda = \frac{K\varepsilon}{2}$ and the Small Hausdorff Theorem 8.1 gives $\delta = 36\varepsilon$. With these substitutions we get the required angle bound. ■

8.4 Topological Equivalence

We have all ingredients to establish a homeomorphism between Σ and S_I using the map ν. Recall that ν maps all points of $\mathbb{R}^3$ except the medial axis points of Σ to their closest point in Σ.

Although the next theorem is stated for a large K, it is not as large in practice. The large value of K is due to the slacks introduced at various places of the analysis.

Theorem 8.3. *For any $K > 400$ there exists a $\varepsilon > 0$ and $\kappa \ge 1$ so that if P is a $(\varepsilon, \varepsilon^2, \kappa)$-sample of a surface Σ, the restriction ν' of ν to S_I defines a homeomorphism between S_I and Σ.*

Proof. For any fixed $K > 0$, Lemmas 8.3 to 8.6 hold for a sufficiently small ε. In particular, the Small Hausdorff Theorem 8.1 asserts that each point x in S_I is within $\tilde{O}(\varepsilon) f(\tilde{x})$ distance from $\tilde{x}$. Therefore, all points of S_I are far away from the medial axis when ε is sufficiently small. Thus ν' is well defined. Since S_I

and Σ are both compact we only need to show that ν' is continuous, one-to-one, and onto. The continuity of ν' follows from the continuity of ν.

To prove that ν' is one-to-one, assume on the contrary that there are points x and x' in S_I so that $\tilde{x} = \nu'(x) = \nu'(x')$. Without loss of generality assume x' is further away from $\tilde{x}$ than x is. Let $x \in B_{c,r}$ where $B_{c,r} \in \mathcal{B}_I$. The line ℓ_x passing through x and x' is normal to Σ at $\tilde{x}$ and according to the Normal Approximation Theorem 8.2, ℓ_x makes an angle of at most $\alpha = 25\sqrt{\varepsilon} + \frac{26}{\sqrt{K}} + \frac{8}{K}$ with the vector $\overrightarrow{cx}$. This angle is less than $\frac{\pi}{2}$ for $K > 400$. Thus, while walking on the line ℓ_x toward the inner medial axis starting from $\tilde{x}$, we encounter a segment of length at least $2r \cos\alpha$ inside $B_{c,r}$. By the Small Hausdorff Theorem 8.1 both x and x' are within $36\varepsilon f(\tilde{x})$ distance from $\tilde{x}$. We reach a contradiction if $2r \cos\alpha$ is more than $72\varepsilon f(\tilde{x})$. Since $r > (K/2)\varepsilon f(\tilde{x})$ this contradiction can be reached for a sufficiently small ε. Then, x and x' are the same.

The map ν' is also onto. Since S_I is a closed, compact surface without boundary and ν' maps S_I continuously to Σ, $\nu'(S_I)$ must consist of closed connected components of Σ. By our assumption Σ is connected. This means $\nu'(S_I) = \Sigma$ and hence ν' is onto. ■

We can also show an isotopy between S_I and Σ using the proof technique of the PC-Isotopy Theorem 6.4 in Section 6.1. To carry out the proof we need (i) S_I lives in a small tubular neighborhood of Σ which is ensured by the Small Hausdorff Theorem 8.1 and (ii) the normals to Σ intersects S_I in exactly one point within this neighborhood which is shown in the proof of the above theorem.

8.4.1 Labeling

To apply the previous results, we need to label the balls in $\mathcal{B}_I$ and the ones in $\mathcal{B}_O$. As in PowerCrust we achieve this by looking at how deeply the balls intersect. A ball in $\mathcal{B}_I$ can have only a shallow intersection with a ball in $\mathcal{B}_O$. However, adjacent balls in $\mathcal{B}_I$ or in $\mathcal{B}_O$ intersect deeply. In the case of PowerCrust we took two balls adjacent if they contribute a facet in the power diagram. Here we will define the adjacency slightly differently without referring to the power diagram. We call two balls in $\mathcal{B}_I$ ($\mathcal{B}_O$) *adjacent* if their boundaries intersect at a point lying in S_I (S_O respectively). The adjacent balls in $\mathcal{B}_I$ or in $\mathcal{B}_O$ intersect deeply. We measure the depth of intersection as before, that is, by the angle at which two balls intersect. We say a ball B_1 intersects another ball B_2 at an angle α if there is a point x in the intersection of their boundaries and $\angle(\overrightarrow{c_1x}, \overrightarrow{c_2x}) = \alpha$ where c_1 and c_2 are the centers of B_1 and B_2 respectively.

Lemma 8.7. *Any two adjacent balls B_1 and B_2 in $\mathcal{B}_I$ intersect at an angle of at most $50\sqrt{\varepsilon} + \frac{52}{\sqrt{K}} + \frac{16}{K}$ when ε is sufficiently small.*

Proof. Let $x \in B_1 \cap B_2$ be a point in S_I. The angle at which B_1 and B_2 intersect at x is equal to the angle between the vectors $\overrightarrow{c_1x}$ and $\overrightarrow{c_2x}$ where c_1 and c_2 are the centers of B_1 and B_2 respectively. By the Normal Approximation Theorem 8.2 both $\angle(\mathbf{n}_{\tilde{x}}, \overrightarrow{c_1x})$ and $\angle(\mathbf{n}_{\tilde{x}}, \overrightarrow{c_2x})$ are at most $25\sqrt{\varepsilon} + \frac{26}{\sqrt{K}} + \frac{8}{K}$. This implies $\angle(\overrightarrow{c_1x}, \overrightarrow{c_2x})$ is no more than the claimed bound. ■

Lemma 8.8. *For a sufficiently small ε, any ball $B_1 \in \mathcal{B}_I$ intersects any other ball $B_2 \in \mathcal{B}_O$ at an angle more than $\pi/2 - \arcsin((2/K)(1 + \tilde{O}(\varepsilon)))$.*

Proof. The line segment joining the center c_1 of B_1 and the center c_2 of B_2 intersects Σ as c_1 lies in Ω_I where c_2 lies in Ω_O. Let this intersection point be x. Without loss of generality, assume that x lies inside B_1. Let C be the circle of intersection of the boundaries of B_1 and B_2 and d be its radius. Clearly, d is smaller than the distance of x to the closest sample point as B_1 is empty. This fact and the Sampling Lemma 8.1 imply

$$d \le \varepsilon_1 f(x). \tag{8.5}$$

Next, we obtain a lower bound on the radius of B_1 in terms of $f(x)$. Let the segment c_1c_2 intersect the boundary of B_1 at y. The Sampling Lemma 8.1 implies $\|x - y\| \le \varepsilon_1 f(x)$. This also means $\|x - \tilde{y}\| \le 2\varepsilon_1 f(x)$. By Lipschitz property of f, we have

$$f(\tilde{y}) \ge (1 - 2\varepsilon_1) f(x).$$

The radius r of B_1 satisfies (Lemma 8.6)

$$\begin{aligned} r &\ge (K/2)\varepsilon f(\tilde{y}) \\ &\ge (K/2)\varepsilon(1 - 2\varepsilon_1) f(x). \end{aligned} \tag{8.6}$$

Combining Inequalities 8.5 and 8.6 we obtain that, for a point z on the circle C, $\overrightarrow{zc_1}$ makes an angle at least $\pi/2 - \arcsin((2/K)(1 + \tilde{O}(\varepsilon)))$ with the plane of C. The angle at which B_1 and B_2 intersect is greater than this angle. ■

Lemmas 8.7 and 8.8 say that, for a sufficiently large K and a small ε, one can find an angle $\theta > 0$ so that the adjacent balls in $\mathcal{B}_I$ and $\mathcal{B}_O$ intersect at an angle less than θ whereas a ball from $\mathcal{B}_I$ intersects a ball from $\mathcal{B}_O$ at an angle larger than θ. This becomes the basis of separating the inner balls from the outer ones. The boundary of the union of the outer balls, or the inner big balls can be output as the approximated surface. Alternatively, one can apply a technique to smooth this boundary. In fact, it is known how to produce a surface from the union of a set of balls with C^2-smoothness. These surfaces are called *skin surfaces*. However, these surfaces may not interpolate the input points. We take the help of the restricted Delaunay triangulation to compute a surface interpolating through the points on the outer (or inner) big Delaunay balls. The restricted Delaunay surfaces $\mathrm{Del}\, P_I|_{S_I}$ and $\mathrm{Del}\, P_O|_{S_O}$ can be shown to be homeomorphic to S_I and S_O respectively by showing that (S_I, P_I) and (S_O, P_O) satisfy the topological ball property when ε is sufficiently small.

Theorem 8.4. *For sufficiently small* $\varepsilon > 0$, $\mathrm{Del}\, P_I|_{S_I}$ *is homeomorphic to* Σ. *Further, each point* x *in* $\mathrm{Del}\, P_I|_{S_I}$ *has a point in* Σ *within* $\tilde{O}(\sqrt{\varepsilon})f(\tilde{x})$ *distance and conversely, each point* x *in* Σ *has a point in* $\mathrm{Del}\, P_I|_{S_I}$ *within* $\tilde{O}(\sqrt{\varepsilon})f(x)$ *distance.*

8.4.2 Algorithm

Now we have all ingredients to design an algorithm that computes a surface homeomorphic to Σ. We will describe the algorithm to compute $\mathrm{Del}\, P_O|_{S_O}$. Clearly, it can be adapted to compute $\mathrm{Del}\, P_I|_{S_I}$ as well. The algorithm uses three user-supplied parameters, κ, K, and θ. It first chooses each Delaunay ball whose radius is bigger than K times the distance between any sample point p on its boundary and the κth nearest sample point of p. Then, it starts walking from an infinite Delaunay ball circumscribing an infinite tetrahedron formed by a convex hull triangle and a point at infinity. This Delaunay ball is outer. The angle of intersection between an infinite Delaunay ball and other Delaunay balls intersecting it needs to be properly interpreted taking infinity into account. The algorithm continues to collect all big balls that intersect a ball already marked *outer* at an angle more than a threshold angle θ. Once all outer big Delaunay balls are identified, the set P_O is constructed.

To compute $\mathrm{Del}\, P_I|_{S_I}$ we first compute $\mathrm{Del}\, P_I$ and then determine the Voronoi edges of $\mathrm{Vor}\, P_I$ that intersect S_I. The dual Delaunay triangles of these Voronoi edges along with their vertices and edges form $\mathrm{Del}\, P_I|_{S_I}$.

RobustCocone(P,κ,K,θ)

compute $\mathrm{Del}\, P$;
mark all infinite Dealuany balls;
`for` each tetrahedron $pqrs \in \mathrm{Del}\, P$ `do`
  let $B_{c,r}$ be the Delaunay ball of $pqrs$;
  let the smallest κth neighbor distance for p, q, r, and s be d;

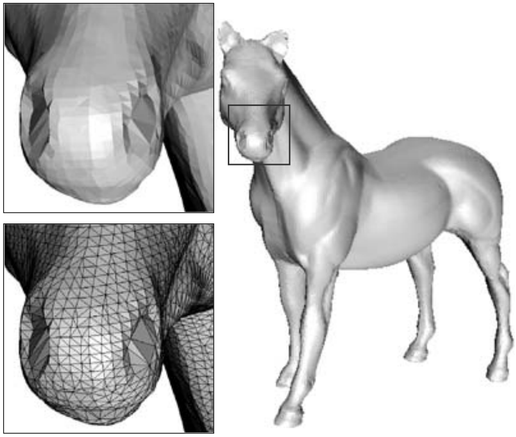
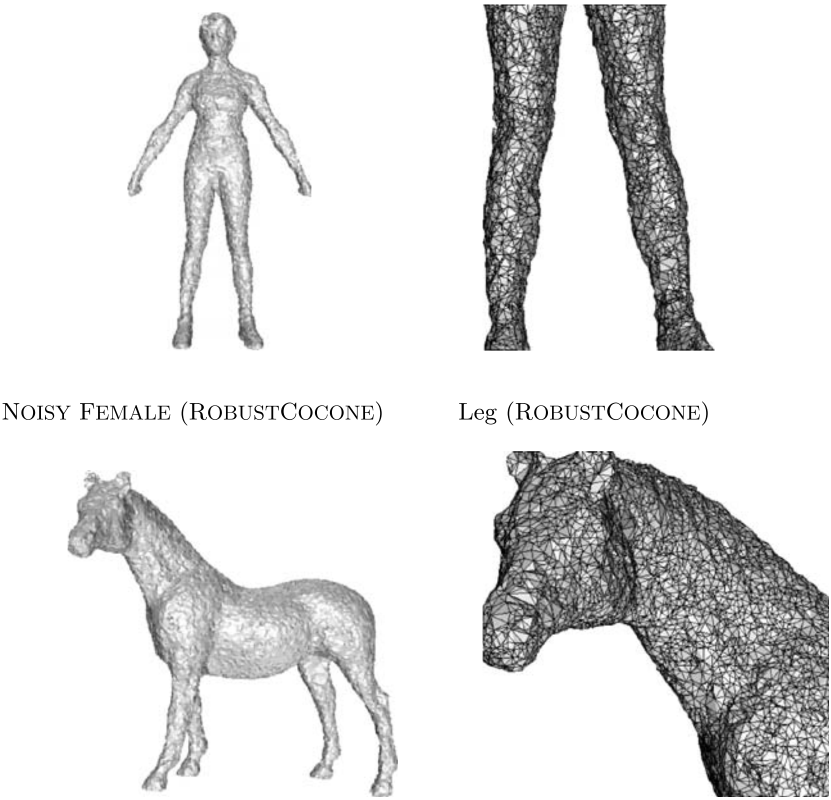


Figure 8.6. Surface reconstruction by RobustCocone from a noise-free sample.

    if $r \geq Kd$ then mark $B_{c,r}$;
endfor
initialize a stack S and a set U with all infinite Delaunay balls;
while $S \neq \emptyset$ do
    B:= pop S;
    for each marked ball $B' \notin U$ do
        if $B \neq B'$, and B and B' intersect at an angle less than θ
            U:=$U \cup B'$;
            push B' into S;
        endif
    endfor
endwhile
let P_O be the vertex set of tetrahedra circumscribed by balls in U;
compute Vor P_O;
E:= $\emptyset$;
for each Voronoi edge $e \in$ Vor P_O do
    if one vertex of e is in a ball in U and
        the other is in none of them
        E:=$E \cup$ dual e;
    endif
endfor
output E.

In Figures 8.6 and 8.7, we show the results of a slightly modified RobustCocone. It first filters the points as described using the parameters $K = 0.5$



Noisy Female (RobustCocone) Leg (RobustCocone)

Noisy Horse (RobustCocone) Neck (RobustCocone)

Figure 8.7. Reconstruction by RobustCocone on noisy samples.

and $\kappa = 3$. Then, instead of computing the restricted Delaunay triangulation Del $P_O|_{S_O}$, it applies TightCocone on the filtered point set. RobustCocone performs much better than TightCocone alone on noisy data where noise is reasonably high. One aspect of the algorithm is that it tends to produce much less nonmanifold vertices and edges. It should be clear that the RobustCocone is able to handle noise-free data sets as well. Figure 8.6 shows an example.

8.5 Notes and Exercises

The RobustCocone algorithm presented in this chapter is taken from Dey and Goswami [34]. This paper showed that the idea of power crust can be applied to noisy point cloud data.

The noise model is reasonable though variations in the sampling conditions are certainly possible. The sampling condition (ii) requires a quadratic dependence (ε^2) on the sampling parameter. One can relax this condition to be linearly dependent on ε by trading off the normal approximation guarantee. Corollary 8.1 will give an $\tilde{O}(\sqrt{\varepsilon})$ approximation to normals at the sample points. This will in turn give an $\tilde{O}(\sqrt{\varepsilon})f(\tilde{x})$ bound on the distances between any point x in S_I and $\tilde{x}$ in Σ in Lemma 8.5. As a consequence the Normal Approximation Theorem 8.2 will provide an $\tilde{O}(\varepsilon^{\frac{1}{4}} + \frac{1}{\sqrt{K\sqrt{\varepsilon}}} + \frac{1}{K\sqrt{\varepsilon}})$ approximation for the normals which will mean that $K\sqrt{\varepsilon}$ has to be large, or K has to be large, say $\Omega(\frac{1}{\varepsilon})$, to have a good normal approximations. This observation suggests that larger the noise amplitude, the bigger the parameter K should be for choosing big Delaunay balls. It would be interesting to see what kind of other tradeoffs can be achieved between the guarantees and the noise models.

We have assumed Σ to be connected. All definitions and proofs can be easily extended to the case when Σ has multiple components. However, it is not clear how to extend the labeling algorithm to separate the balls on two sides of a component of Σ when it has multiple components. It is important that all the big Delaunay balls on one side remain connected through the adjacency relation as defined in Section 8.4.1. When Σ has multiple components, we cannot appeal to Theorem 8.3 to claim the connectedness among the big Delaunay balls since the surface S_I may not be connected as Σ is not. This is also a bottleneck for the PowerCrust algorithm [7]. It would be interesting to devise a labeling algorithm which can handle multiple components with guarantee.

The RobustCocone algorithm requires that the sampled surface have no boundary. It is not clear how the algorithm should be adapted for surfaces with boundary. A reconstruction of surfaces with boundaries from noiseless point samples can be done by the BoundCocone algorithm described in Chapter 5. However, noise together with boundaries pose a difficult challenge. The spectral crust of Kolluri, O'Brien, and Shewchuk [64] is shown to work well for such data sets though no proofs are given.

Exercises

1^h. Consider the set of inner and outer big Delaunay balls $\mathcal{B}_I$ and $\mathcal{B}_O$ respectively. Consider the following algorithm for reconstructing Σ. Compute the power diagram of the centers of the balls in $\mathcal{B}_I \cup \mathcal{B}_O$ with their radii as the weights. Then output the facets that separate a power cell of an inner ball center from that of an outer ball center. Show that the surface output by this algorithm is homeomorphic to Σ if the sample is sufficiently dense.

2. Let $\phi(x) = (x - \tilde{x})^T \mathbf{n}_{\tilde{x}}$ for a point $x \in \mathbb{R}^3$. Consider the offset surface $\Sigma_{-\varepsilon}$ defined as:

$$\Sigma_{-\varepsilon} = \{x \mid |\phi(x)| = \varepsilon f(x) \text{ and } \phi(x) \text{ is negative}\}.$$

   (i) Is $\phi(x)$ continuous if Σ is C^1-smooth? What if Σ is C^2-smooth?
   (ii) Give an example that shows $\Sigma_{-\varepsilon}$ is not necessarily C^1-smooth even if Σ is.
   (iii) Prove that $\Sigma_{-\varepsilon}$ is homeomorphic to Σ for a sufficiently small ε.
3. Suppose one adopts the following intersection depth check to collect all outer big Delaunay balls. Let B be any big Delaunay ball that has been already collected. Let t be the tetrahedron circumscribed by B. For depth intersection with B check all the balls circumscribing the tetrahedra sharing a triangle with t. Does this algorithm work?

4^h. Carry out the entire analysis of topological and geometric guarantees of S_I assuming that P is a $(\varepsilon, \varepsilon, \kappa)$-sample for suitable ε and κ.

5^o. Prove that the COCONE algorithm applied to the points on the union of inner big Delaunay balls produces a surface homeomorphic to Σ for a sufficiently small ε in the noise model.

6. Instead of choosing big Delaunay balls with a threshold in ROBUSTCOCONE one can choose the largest polar balls among k-nearest neighbors for some $k \geq 1$ as in the feature approximation algorithm in Chapter 7. Show that, for a $(\varepsilon, \varepsilon^2, \kappa)$-sample, the surface S_I defined with these balls is isotopic to the sampled surface if k is close to κ and ε is sufficiently small.