3

Surface Samples

In this chapter we introduce some of the properties of surfaces and their samples in three dimensions. The results developed in this chapter are used in later chapters to design algorithms for surface reconstruction and prove their guarantees. Before we talk about these results, let us explain what we mean by smooth surfaces.

Consider a map $\pi : U \to V$ where U and V are the open sets in $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively. The map π has three components, namely $\pi(x) = (\pi_1(x), \pi_2(x), \pi_3(x))$ where $x = (x_1, x_2)$ is a point in $\mathbb{R}^2$. The three by two matrix of first-order partial derivatives $(\frac{\partial \pi_i(x)}{\partial x_j})_{i,j}$ is called the *Jacobian* of π at x. We say π is *regular* if its Jacobian at each point of U has rank 2. The map π is C^i-continuous if the ith order ($i > 0$) partial derivatives of π are continuous.

For $i > 0$, a subset $\Sigma \subset \mathbb{R}^3$ is a C^i*-smooth* surface if each point $x \in \Sigma$ satisfies the following condition. There is a neighborhood $W \subset \mathbb{R}^3$ of x and a map $\pi : U \to W \cap \Sigma$ of an open set $U \subset \mathbb{R}^2$ onto $W \cap \Sigma$ so that

(i) π is C^i-continuous,
(ii) π is a homeomorphism, and
(iii) π is regular.

The first condition says that π is continuously differentiable at least up to ith order. The second condition imposes one-to-one property which eliminates self-intersections of Σ. The third condition together with the first actually enforce the smoothness. It makes sure that the tangent plane at each point in Σ is well defined. All of these three conditions together imply that the functions like π defined in the neighborhood of each point of Σ overlap smoothly. There are two extremes of smoothness. If the partial derivatives of π of all orders are continuous, we say Σ is C^∞-smooth. On the other hand,

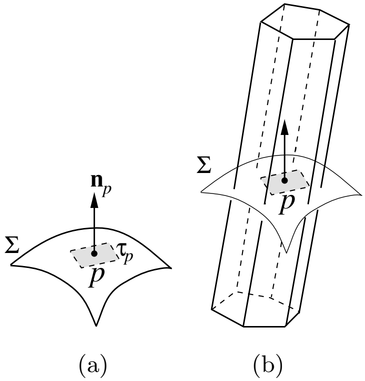

Figure 3.1. (a) Tangent plane and the normal at a point on a smooth surface and (b) a long thin Voronoi cell elongated along the normal direction.

if Σ is not C^1-smooth but is at least a 2-manifold, we say it is C^0-smooth or *nonsmooth*.

In this chapter and the chapters to follow, we assume that Σ is a C^2-smooth surface. Notice that, by the definition of smoothness (condition (ii)) Σ is a 2-manifold without boundary. We also assume that Σ is compact since we are interested in approximating Σ with a finite simplicial complex. We need one more assumption. Just like the curves, for a finite point set to be a ε-sample for some $\varepsilon > 0$, we need that $f(x) > 0$ for any point x in Σ. It is known that C^2-smooth surfaces necessarily have positive feature size everywhere. The example in Chapter 2 for curves can be extended to surfaces to claim that a C^1-smooth surface may not have nonzero local feature sizes everywhere.

As a C^2-smooth surface Σ has a tangent plane τ_x and a normal $\mathbf{n}_x$ defined at each point $x \in \Sigma$. We assume that the normals are oriented outward. More precisely, $\mathbf{n}_x$ points locally to the unbounded component of $\mathbb{R}^3 \setminus \Sigma$. If Σ is not connected, $\mathbf{n}_x$ points locally to the unbounded component of $\mathbb{R}^3 \setminus \Sigma'$ where x is in Σ', a connected component of Σ.

An important fact used in surface reconstruction is that, disregarding the orientation, the direction of the surface normals can be approximated from the sample. An illustration in $\mathbb{R}^2$ is helpful here. See Figure 2.4 in Chapter 2 which shows the Voronoi diagram of a dense sample on a smooth curve. This Voronoi diagram has a specific structure. Each Voronoi cell is elongated along the normal direction at the sample points. Fortunately, the same holds in three dimensions. The three-dimensional Voronoi cells are long, thin, and the direction of the elongation matches with the normal direction at the sample points when the sample is dense (see Figure 3.1).

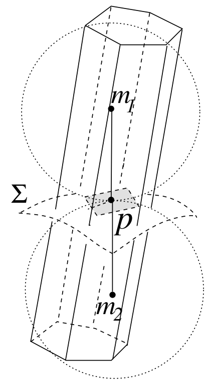

Figure 3.2. Medial axis points m_1 and m_2 are in the Voronoi cell V_p.

3.1 Normals

Let $P \subset \mathbb{R}^3$ be a ε-sample of Σ. If P is all we know about Σ, it is impossible to know the line of direction of $\mathbf{n}_p$ exactly at a point $p \in P$. However, it is conceivable that as P gets denser, we should have more accurate idea about the direction of $\mathbf{n}_p$ by looking at the adjacent points. This is what is done using the Voronoi cells in Vor P.

For further developments we will often need to talk about how one vector approximates another one in terms of the angles between them. We denote the angle between two vectors $\mathbf{u}$ and $\mathbf{v}$ as $\angle(\mathbf{u}, \mathbf{v})$. For vector approximations that disregard the orientation, we use a slightly different notation. This approximation measures the acute angle between the lines containing the vectors. We use $\angle_a(\mathbf{u}, \mathbf{v})$ to denote this acute angle between two vectors $\mathbf{u}$ and $\mathbf{v}$. Since any such angle is acute, we have the triangular inequality $\angle_a(\mathbf{u}, \mathbf{v}) \leq \angle_a(\mathbf{u}, \mathbf{w}) + \angle_a(\mathbf{v}, \mathbf{w})$ for any three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

3.1.1 Approximation of Normals

It turns out that the structure of the Voronoi cells contains information about normals. Indeed, if the sample is sufficiently dense, the Voronoi cells become long and thin along the direction of the normals at the sample points. One reason for this structural property is that a Voronoi cell V_p must contain the medial axis points that are the centers of the medial balls tangent to Σ at p (see Figure 3.2).

Lemma 3.1 (Medial). *Let m_1 and m_2 be the centers of the two medial balls tangent to Σ at p. The Voronoi cell V_p contains m_1 and m_2.*

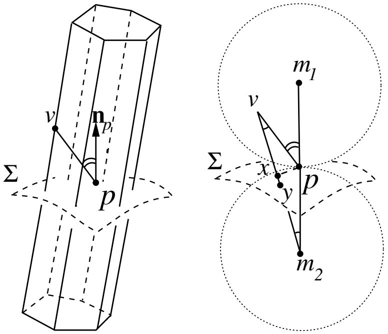

Figure 3.3. Illustration for the Normal Lemma 3.2.

Proof. Denote the medial ball with center m_1 as B. The ball B meets the surface Σ only tangentially at points, one of which is p. Thus, B is empty of any point from Σ and P in particular. Therefore, the center m_1 has p as the nearest point in P. By definition of Voronoi cells, m_1 is in V_p. A similar argument applies to the other medial axis point m_2. ■

We have already mentioned that the Voronoi cells are long and thin and they are elongated along the direction of the normals. The next lemma formalizes this statement by asserting that as we go further from p within V_p, the direction to p becomes closer to the normal direction.

Lemma 3.2 (Normal). *For* $\mu > 0$ *let* $v \notin \Sigma$ *be a point in* V_p *with* $\|v - p\| > \mu f(p)$. *For* $\varepsilon < 1$, $\angle_a(\overrightarrow{vp}, \mathbf{n}_p) \leq \arcsin \frac{\varepsilon}{\mu(1-\varepsilon)} + \arcsin \frac{\varepsilon}{1-\varepsilon}$.

Proof. Let m_1 and m_2 be the two centers of the medial balls tangent to Σ at p where m_1 is on the same side of Σ as v is. Both m_1 and m_2 are in V_p by the Medial Lemma 3.1. The line joining m_1 and p is normal to Σ at p by the definition of medial balls. Similarly, the line joining m_2 and p is also normal to Σ at p. Therefore, m_1, m_2, and p are colinear (see Figure 3.3). Consider the triangle pvm_2. We are interested in the angle $\angle m_1 pv$ which is equal to $\angle_a(\overrightarrow{pv}, \mathbf{n}_p)$. From the triangle pvm_2 we have

$$\angle m_1 pv = \angle pvm_2 + \angle vm_2 p.$$

To measure the two angles on the right-hand side, drop the perpendicular px from p onto the segment vm_2. The line segment vm_2 intersects Σ, say at y, since m_1 and m_2 and hence v and m_2 lie on opposite sides of Σ. Furthermore, y must lie inside V_p since any point on the segment joining two points v and m_2 in a convex set V_p must lie within the same convex set. This means y has p

as the nearest sample point and thus

$$\|x - p\| \leq \|y - p\| \leq \varepsilon f(y) \text{ by the } \varepsilon\text{-sampling condition.}$$

Using the Feature Translation Lemma 1.3 we get

$$\|x - p\| \leq \frac{\varepsilon}{1-\varepsilon} f(p)$$

when $\varepsilon < 1$. We have

$$\angle pvm_2 = \arcsin \frac{\|x-p\|}{\|v-p\|} \leq \arcsin \frac{\varepsilon}{\mu(1-\varepsilon)} \quad \text{as } \|v-p\| \geq \mu f(p).$$

Similarly,

$$\angle vm_2 p = \arcsin \frac{\|x-p\|}{\|m_2-p\|} \leq \arcsin \frac{\varepsilon}{1-\varepsilon} \quad \text{as } \|m_2-p\| \geq f(p).$$

The assertion of the lemma follows immediately. ∎

3.1.2 Normal Variation

The directions of the normals at nearby points on Σ cannot vary too abruptly. In other words, the surface looks flat locally. This fact is used later in many proofs.

Lemma 3.3 (Normal Variation). *If $x, y \in \Sigma$ are any two points with $\|x - y\| \leq \rho f(x)$ for $\rho < \frac{1}{3}$, $\angle(\mathbf{n}_x, \mathbf{n}_y) \leq \frac{\rho}{1-3\rho}$.*

Proof. Let $\ell(t)$ denote any point on the segment xy parameterized by its distance t from x. Let $x(t)$ be the nearest point on Σ from $\ell(t)$. The rate of change of normal $\mathbf{n}_{x(t)}$ at $x(t)$ is $\mathbf{n}'_t = \frac{dx(t)}{dt}$ as t changes. The total variation in normals between x and y is

$$\angle(\mathbf{n}_x, \mathbf{n}_y) \leq \int_{xy} |\mathbf{n}'_t| dt \leq \|x - y\| \max_t |\mathbf{n}'_t|.$$

The surface Σ is squeezed locally in-between two medial ball that are tangent to Σ at $x(t)$. The radius of the smaller medial ball cannot be larger than the radius of curvature of Σ at $x(t)$. This means Σ cannot turn faster than the smaller of the two medial balls at $x(t)$. Referring to Figure 3.4, we have

$$\begin{aligned} dt &= (\|m_2 - x(t)\| - \|x(t) - \ell(t)\|) \tan d\theta \\ &\geq (f(x(t)) - \|x(t) - \ell(t)\|) \tan d\theta. \end{aligned}$$

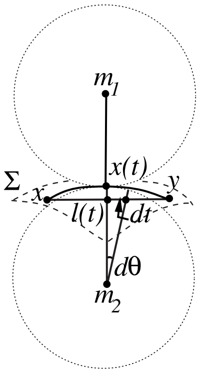

Figure 3.4. Illustration for the Normal Variation Lemma 3.3.

As

$$\lim_{d\theta \to 0} \frac{\tan d\theta}{d\theta} = 1$$

and

$$\|x(t) - \ell(t)\| \leq \|x - \ell(t)\| \leq \|x - y\| \leq \rho f(x)$$

we get

$$|\mathbf{n}'_t| = \lim_{d\theta \to 0} \left| \frac{d\theta}{dt} \right| \leq \frac{1}{(f(x(t)) - \|x(t) - \ell(t)\|)} \leq \frac{1}{(f(x(t)) - \rho f(x))}$$

provided $f(x(t)) - \rho f(x) > 0$. Also,

$$\|x(t) - x\| \leq \|x(t) - \ell(t)\| + \|x - \ell(t)\| \leq 2\rho f(x).$$

By the Lipschitz Continuity Lemma 1.2, $f(x(t)) \geq (1 - 2\rho) f(x)$. Therefore,

$$|\mathbf{n}'_t| \leq \frac{1}{(1 - 3\rho) f(x)} \quad \text{and} \quad \angle(\mathbf{n}_x, \mathbf{n}_y) \leq \frac{\rho}{1 - 3\rho}$$

provided

$$\begin{aligned} f(x(t)) - \rho f(x) &> 0 \\ \text{or, } (1 - 3\rho) f(x) &> 0 \\ \text{or, } \rho &< \frac{1}{3}. \end{aligned}$$

■

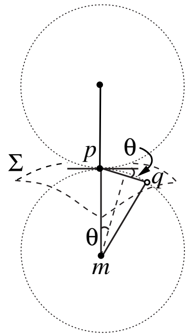

Figure 3.5. Illustration for the Edge Normal Lemma 3.4.

3.1.3 Edge and Triangle Normals

In Section 2.1, we saw that edges joining nearby points on a curve are almost parallel to the tangents at the endpoints of the edge. Similar results also hold for triangles connecting points on surfaces. But, the size is measured by circumradius. In fact, a triangle connecting three nearby points on a surface but with a large circumradius may lie almost perpendicular to the surface. However, if its circumradius is small compared to the local feature sizes at its vertices, it has to lie almost parallel to the surface. For an edge, half of its length is the same as its circumradius. Therefore, a small edge lies almost parallel to the surface. In essence if an edge or a triangle has a small circumradius, it must lie flat to the surface. We quantify these claims in the next two lemmas.

Lemma 3.4 (Edge Normal). *For an edge pq with $\|p-q\| \leq 2f(p)$, the angle $\angle_a(\overrightarrow{pq}, \mathbf{n}_p)$ is at least $\frac{\pi}{2} - \arcsin\frac{\|p-q\|}{2f(p)}$.*

Proof. Consider the two medial balls sandwiching the surface Σ at p. The point q cannot lie inside any of these two balls as they are empty of points from Σ. So, the smallest angle pq makes with $\mathbf{n}_p$ cannot be smaller than the angle pq makes with $\mathbf{n}_p$ when q is on the boundary of any of these two balls. In this case let θ be the angle between pq and the tangent plane at p. Clearly, (see Figure 3.5)

$$\sin\theta = \frac{\|p-q\|}{2\|m-p\|}$$

$$\leq \frac{\|p-q\|}{2f(p)}.$$

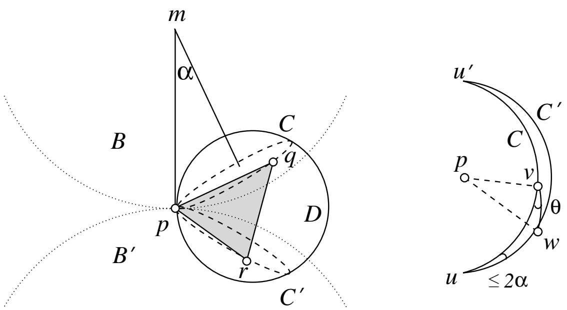

Figure 3.6. Illustration for the Triangle Normal Lemma 3.5. The two great arcs on the right picture are the intersections of the unit sphere with the planes containing C and C'.

Therefore,

$$\begin{aligned}\angle_a(\overrightarrow{pq}, \mathbf{n}_p) &= \frac{\pi}{2} - \theta \\ &\geq \frac{\pi}{2} - \arcsin \frac{\|p-q\|}{2f(p)}.\end{aligned}$$

■

It follows immediately from the Edge Normal Lemma 3.4 that small edges make a large angle with the surface normals at the vertices. For example, if pq has a length less than $\rho f(p)$ for $\rho < 2$, the angle $\angle_a(\overrightarrow{pq}, \mathbf{n}_p)$ is more than $\frac{\pi}{2} - \arcsin \frac{\rho}{2}$.

Next consider a triangle $t = pqr$ where p is the vertex subtending a maximal angle in pqr. Let R_{pqr} denote the circumradius of pqr.

Lemma 3.5 (Triangle Normal). *If* $R_{pqr} \leq \frac{f(p)}{\sqrt{2}}$,

$$\angle_a(\mathbf{n}_{pqr}, \mathbf{n}_p) \leq \arcsin \frac{R_{pqr}}{f(p)} + \arcsin\left(\frac{2}{\sqrt{3}} \sin\left(2 \arcsin \frac{R_{pqr}}{f(p)}\right)\right)$$

where $\mathbf{n}_{pqr}$ *is the normal of* pqr.

Proof. Consider the medial balls $B = B_{m,\ell}$ and $B' = B_{m',\ell'}$ that are tangent to Σ at p. Let D be the diametric ball of t (smallest circumscribing ball); refer to Figure 3.6. The radius of D is R_{pqr}. Let C and C' be the circles in which the boundary of D intersects the boundaries of B and B' respectively. The line normal to Σ at p passes through m, the center of B. Let α be the larger of the two angles this normal line makes with the normals to the planes containing C and C'. Since the radii of C and C' are at most R_{pqr} we have

$$\alpha \leq \arcsin \frac{R_{pqr}}{\|p-m\|} \leq \arcsin \frac{R_{pqr}}{f(p)}.$$

It follows from the definition of α that the planes containing C and C' make a wedge, say W, with an acute dihedral angle no more than 2α.

The other two vertices q, r of t cannot lie inside B or B'. This implies that t lies completely in the wedge W. Let π_t, π, and π' denote the planes containing t, C, and C' respectively. Consider a unit sphere centered at p. This sphere intersects the line $\pi \cap \pi'$ at two points, say u and u'. Within W let the lines $\pi_t \cap \pi$ and $\pi_t \cap \pi'$ intersect the unit sphere at v and w respectively. See the picture on the right in Figure 3.6. Without loss of generality, assume that the angle $\angle uvw \leq \angle uwv$. Consider the spherical triangle uvw. We are interested in the spherical angle $\theta = \angle uvw$ which is also the acute dihedral angle between the planes containing t and C. We have the following facts. The arc length of wv, denoted $|wv|$, is at least $\pi/3$ since p subtends the largest angle in t and t is in the wedge W. The spherical angle $\angle vuw$ is less than or equal to 2α. By standard sine laws in spherical geometry, we have

$$\sin\theta = \sin|uw|\frac{\sin\angle vuw}{\sin|wv|} \leq \sin|uw|\frac{\sin 2\alpha}{\sin|wv|}.$$

If $\pi/3 \leq |wv| \leq 2\pi/3$, we have

$$\sin|wv| \geq \sqrt{3}/2$$

and hence

$$\theta \leq \arcsin\left(\frac{2}{\sqrt{3}}\sin 2\alpha\right).$$

For the range $2\pi/3 < |wv| < \pi$, we use the fact that $|uw| + |wv| \leq \pi$. The arc length $|wv|$ cannot be longer than both $|wu'|$ and $|vu'|$ since $\angle vu'w \leq 2\alpha < \pi/2$ for $R_{pqr} \leq \frac{f(p)}{\sqrt{2}}$. If $|wv| \leq |wu'|$, we have

$$|uw| + |wv| \leq |uu'| = \pi.$$

Otherwise, $|wv| \leq |vu'|$. Then, we use the fact that $|uw| \leq |uv|$ as $\angle uvw \leq \angle uwv$. So, again

$$|uw| + |wv| \leq |uu'| = \pi.$$

Therefore, when $|wv| > \frac{2\pi}{3}$, we get

$$\frac{\sin|uw|}{\sin|wv|} < 1.$$

Thus, $\theta \leq \arcsin\left(\frac{2}{\sqrt{3}}\sin 2\alpha\right)$.

The normals to t and Σ at p make an acute angle at most $\alpha + \theta$ proving the lemma. ■

3.2 Topology

The sample P as a set of discrete points does not have the topology of Σ. A connection between the topology of Σ and P can be established through the restricted Voronoi and Delaunay diagrams. In particular, one can show that the underlying space of the restricted Delaunay triangulation Del $P|_\Sigma$ is homeomorphic to Σ if the sample P is sufficiently dense. Although we will not be able to compute Del $P|_\Sigma$, the fact that it is homeomorphic to Σ will be useful in the surface reconstruction later.

3.2.1 Topological Ball Property

The underlying space of Del $P|_\Sigma$ becomes homeomorphic to Σ when the Voronoi diagram Vor P intersects Σ nicely. This condition is formalized by the topological ball property which says that the restricted Voronoi cells in each dimension is a ball.

Definition 3.1. *Let F denote any Voronoi face of dimension k, $0 \leq k \leq 3$, in* Vor$P$ *which intersects Σ and $F|_\Sigma = F \cap \Sigma$ be the corresponding restricted Voronoi face. The face F satisfies the topological ball property if $F|_\Sigma$ is a (i) $(k-1)$-ball and (ii)* Int $F \cap \Sigma =$ Int $F|_\Sigma$. *The pair (P, Σ) satisfies the topological ball property if all Voronoi faces $F \in$ Vor P satisfy the topological ball property.*

Condition (i) means that Σ intersects a Voronoi cell in a single topological disk, a Voronoi facet in a single curve segment, a Voronoi edge in a single point, and does not intersect any Voronoi vertex (see Figure 3.7). Condition (ii) avoids any tangential intersection between a Voronoi face and Σ.

The following theorem is an important result relating the topology of a surface to a point sample.

Theorem 3.1. *The underlying space of* Del$P|_\Sigma$ *is homeomorphic to Σ if the pair (P, Σ) satisfies the topological ball property.*

Our aim is to show that, when P is a dense sample, the topology of Σ can be captured from P. Specifically, we prove that the pair (P, Σ) satisfies the topological ball property when ε is sufficiently small. The proof frequently uses the next two lemmas to reach a contradiction. The first one says that the points in a restricted Voronoi cell, that is, the points of Σ in a Voronoi cell, cannot be far apart. The second one says that any line almost normal to the surface cannot intersect it twice within a small distance.

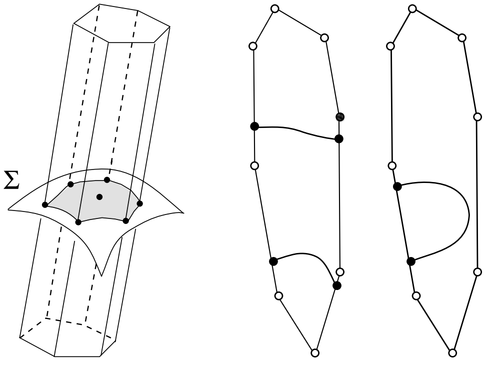

Figure 3.7. (a) A surface Σ intersects a Voronoi cell and its faces with the topological ball property, (b) a surface not intersecting a Voronoi facet in a 1-ball, and (c) a surface not intersecting a Voronoi edge in a 0-ball.

Lemma 3.6 (Short Distance). *Let x and y be any two points in a restricted Voronoi cell $V_p|_\Sigma$. For $\varepsilon < 1$, we have*

(i) $\|x - p\| \le \frac{\varepsilon}{1-\varepsilon} f(p)$ *and*
(ii) $\|x - y\| \le \frac{2\varepsilon}{1-\varepsilon} f(x)$.

Proof. Since x has p as the nearest sample point, $\|x - p\| \le \varepsilon f(x)$ for $\varepsilon < 1$. Apply the Feature Translation Lemma 1.3 to claim (i). For (ii), observe that

$$\begin{aligned}\|x - y\| &\le \|x - p\| + \|y - p\| \\ &\le \varepsilon(f(x) + f(y))\end{aligned}$$

By the Lipschitz Continuity Lemma 1.2

$$\begin{aligned}f(y) &\le f(x) + \|x - y\| \\ &\le f(x) + \varepsilon(f(x) + f(y)), \text{ or} \\ (1-\varepsilon)f(y) &\le (1+\varepsilon)f(x).\end{aligned}$$

Therefore, for $\varepsilon < 1$,

$$\|x - y\| \le \varepsilon\left(1 + \frac{1+\varepsilon}{1-\varepsilon}\right) f(x) \le \frac{2\varepsilon}{1-\varepsilon} f(x).$$

■

A restricted Delaunay edge pq is dual to a Voronoi facet that intersects Σ. Any such intersection point, say x, is within $\frac{\varepsilon}{1-\varepsilon} f(p)$ distance from p by the Short

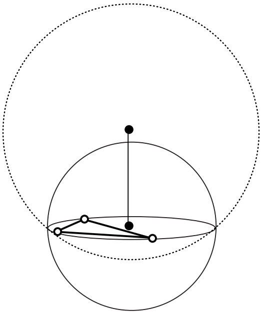

Figure 3.8. The circumradius of a triangle which is also the radius of its diametric ball (shown with solid circle) is no more than the radius of a circumscribing ball (shown with dotted circle).

Distance Lemma 3.6. The length of pq cannot be more than twice the distance between x and p. Hence, $\|p - q\| \leq \frac{2\varepsilon}{1-\varepsilon} f(p)$. We can extend this argument to the restricted Delaunay triangles too. A restricted Delaunay triangle t is dual to a Voronoi edge e that intersects Σ. The intersection point, say x, belongs to the Voronoi cells adjacent to e. Let V_p be any such cell. The point x is the center of a circumscribing ball of the triangle dual to e. By the Short Distance Lemma 3.6, x is within $\frac{\varepsilon}{1-\varepsilon} f(p)$ distance from p. The ball $B_{x,\|x-p\|}$ circumscribes t. The circumradius of t is no more than $\|x - p\|$ as the circumradius of a triangle cannot be more than any of its circumscribing ball (see Figure 3.8). Thus, the following corollary is immediate from the Short Distance Lemma 3.6.

Corollary 3.1. *For $\varepsilon < 1$, we have*

(i) the length of a restricted Delaunay edge e is at most $\frac{2\varepsilon}{1-\varepsilon} f(p)$ where p is any vertex of e and
(ii) the circumradius of any restricted Delaunay triangle t is at most $\frac{\varepsilon}{1-\varepsilon} f(p)$ where p is any vertex of t.

Lemma 3.7 (Long Distance). *Suppose a line intersects Σ in two points x and y and makes an angle no more than ξ with $\mathbf{n}_x$. One has $\|x - y\| \geq 2f(x)\cos\xi$.*

Proof. Consider the two medial balls at x as in Figure 3.9. The line meets the boundaries of these two balls at x and at points that must be at least $2r\cos\xi$ distance away from x where r is the radius of the smaller of the two balls. Since $r \geq f(x)$, the result follows as y cannot lie inside any of the two medial balls. ■

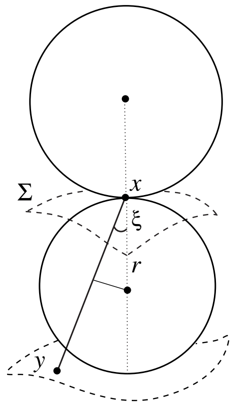

Figure 3.9. Illustration for the Long Distance Lemma 3.7.

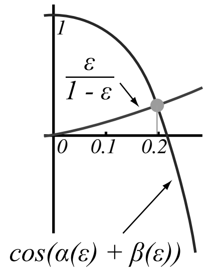

Figure 3.10. The graphs of the two functions on the left and right hand sides of the inequality in Condition A.

3.2.2 Voronoi Faces

Next we consider in turn the Voronoi edges, Voronoi facets, and Voronoi cells and show that they indeed satisfy the topological ball property if ε satisfies Condition A as stated below. For $\varepsilon < \frac{1}{3}$, let

$$\alpha(\varepsilon) = \frac{\varepsilon}{1 - 3\varepsilon}$$

$$\beta(\varepsilon) = \arcsin \frac{\varepsilon}{1-\varepsilon} + \arcsin\left(\frac{2}{\sqrt{3}} \sin\left(2 \arcsin \frac{\varepsilon}{1-\varepsilon}\right)\right).$$

$$\text{Condition A} \qquad \varepsilon < \frac{1}{3} \quad \text{and} \quad \cos\left(\alpha(\varepsilon) + \beta(\varepsilon)\right) > \frac{\varepsilon}{1-\varepsilon}.$$

Figure 3.10 shows that in the range $0 < \varepsilon \leq \frac{1}{3}$, Condition A holds for ε a little less than 0.2. So, for example, $\varepsilon \leq 0.18$ is a safe choice. Since Condition A stipulates $\varepsilon < \frac{1}{3}$, lemmas such as Normal Variation, Long Distance, Short Distance, and Corollary 3.1 can be applied under Condition A.

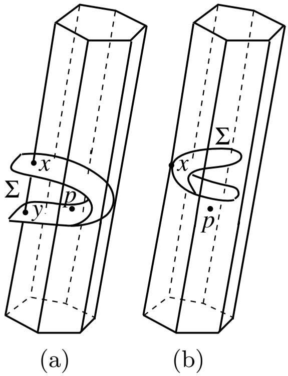

Figure 3.11. Illustration for the Voronoi Edge Lemma 3.8. A Voronoi edge intersecting the surface (a) at two points and (b) tangentially in a single point.

Lemma 3.8 (Voronoi Edge). *A Voronoi edge intersects* Σ *transversally in a single point if Condition* A *holds.*

Proof. Suppose for the sake of contradiction there is a Voronoi edge e in a Voronoi cell V_p intersecting Σ at two points x and y, or at a single point tangentially (see Figure 3.11). The dual Delaunay triangle, say pqr, is a restricted Delaunay triangle. By Corollary 3.1, its circumradius is no more than $\frac{\varepsilon}{1-\varepsilon} f(p)$. By the Triangle Normal Lemma 3.5, $\angle_a(\mathbf{n}_{pqr}, \mathbf{n}_p) \leq \beta(\varepsilon)$ if

$$\frac{1}{\sqrt{2}} > \frac{\varepsilon}{1-\varepsilon}$$

a restriction satisfied by Condition A.

The Normal Variation Lemma 3.3 puts an upper bound of $\alpha(\varepsilon)$ on the angle between the normals at p and x as $\|x - p\| \leq \varepsilon f(x)$. Let ξ denote the angle between $\mathbf{n}_x$ and the Voronoi edge e. We have

$$\begin{aligned} \xi = \angle_a(\mathbf{n}_x, \mathbf{n}_{pqr}) &\leq \angle_a(\mathbf{n}_x, \mathbf{n}_p) + \angle_a(\mathbf{n}_p, \mathbf{n}_{pqr}) \\ &\leq \alpha(\varepsilon) + \beta(\varepsilon). \end{aligned} \tag{3.1}$$

If e intersects Σ tangentially at x, we have $\xi = \frac{\pi}{2}$ requiring $\alpha(\varepsilon) + \beta(\varepsilon) \geq \frac{\pi}{2}$. Condition A requires $\varepsilon < 0.2$ which gives $\alpha(\varepsilon) + \beta(\varepsilon) < \frac{\pi}{2}$. Therefore, when Condition A is satisfied, e cannot intersect Σ tangentially. So, assume that e intersects Σ at two points x and y.

By the Short Distance Lemma 3.6, $\|x - y\| \leq \frac{2\varepsilon}{1-\varepsilon} f(x)$ and by the Long Distance Lemma 3.7, $\|x - y\| \geq 2 f(x) \cos \xi$. A contradiction is reached when

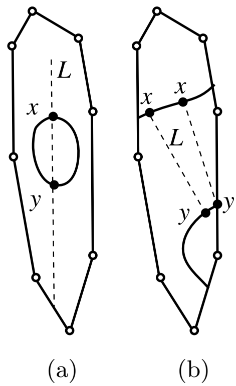

Figure 3.12. A Voronoi facet intersecting Σ (a) in a cycle and (b) in two segments.

$2\cos\xi > \frac{2\varepsilon}{1-\varepsilon}$, or

$$\cos(\alpha(\varepsilon) + \beta(\varepsilon)) > \frac{\varepsilon}{1-\varepsilon}. \tag{3.2}$$

Condition A satisfies Inequality 3.2 giving the required contradiction. ■

Lemma 3.9 (Voronoi Facet). *A Voronoi facet F intersects Σ transversally in a 1-ball if Condition* A *is satisfied.*

Proof. The intersection of F with Σ may contradict the assertion of the lemma if (i) Σ touches F tangentially at a point, (ii) Σ intersects F in a 1-sphere, that is, a cycle, or (iii) Σ intersects F in more than one component.

The dual Delaunay edge, say pq, of F is in the restricted Delaunay triangulation. Let $\mathbf{n}_F$ denote the normal to F. Its direction is the same as that of pq up to orientation. We have $\|p-q\| \le \frac{2\varepsilon}{1-\varepsilon} f(p)$ by Corollary 3.1. Therefore, the Edge Normal Lemma 3.4 gives

$$\angle_a(\mathbf{n}_p, \mathbf{n}_F) \ge \frac{\pi}{2} - \arcsin\frac{\varepsilon}{1-\varepsilon}$$

as long as $\varepsilon < 1$.

If Σ meets F tangentially at a point x, we have $\angle_a(\mathbf{n}_x, \mathbf{n}_F) = 0$ and by the Normal Variation Lemma 3.3 $\angle \mathbf{n}_p, \mathbf{n}_x \le \frac{\varepsilon}{1-3\varepsilon}$ when $\varepsilon < \frac{1}{3}$. This means, for $\varepsilon < \frac{1}{3}$, we have

$$\frac{\pi}{2} - \arcsin\frac{\varepsilon}{1-\varepsilon} \le \angle_a(\mathbf{n}_p, \mathbf{n}_F) \le \frac{\varepsilon}{1-3\varepsilon} = \alpha(\varepsilon).$$

The above inequality contradicts the upper bound for ε given by Condition A.

If Σ meets F in a cycle, let x be any point on it and L be the line on F intersecting the cycle at x orthogonally (see Figure 3.12(a)). This line must meet the cycle in another point, say y. The angle between L and $\mathbf{n}_x$ satisfies $\angle_a(L, \mathbf{n}_x) \le \angle_a(L', \mathbf{n}_x)$ for any other line L' on F passing through x. Choose L' that minimizes the angle with $\mathbf{n}_p$. The line L' being on the Voronoi facet F makes exactly $\frac{\pi}{2}$ angle with the dual restricted Delaunay edge, say pq. We know by the Edge Normal Lemma 3.4

$$\angle_a(\overrightarrow{pq}, \mathbf{n}_p) \ge \frac{\pi}{2} - \arcsin \frac{\varepsilon}{1-\varepsilon}.$$

Therefore, for $\varepsilon < 1$,

$$\angle_a(L', \mathbf{n}_p) = \frac{\pi}{2} - \angle_a(\overrightarrow{pq}, \mathbf{n}_p) \le \arcsin \frac{\varepsilon}{1-\varepsilon}.$$

These facts with the Normal Variation Lemma 3.3 give

$$\angle_a(L', \mathbf{n}_x) \le \angle_a(L', \mathbf{n}_p) + \angle(\mathbf{n}_p, \mathbf{n}_x) \le \arcsin \frac{\varepsilon}{1-\varepsilon} + \alpha(\varepsilon) \qquad (3.3)$$

for $\varepsilon < \frac{1}{3}$.

The right-hand side of Inequality 3.3 is less than the upper bound for ξ in the proof of the Voronoi Edge Lemma 3.8. Thus, we reach a contradiction between distances implied by the Short Distance Lemma 3.6 and the Long Distance Lemma 3.7 when Condition A holds.

In the case Σ meets F in two or more components as in Figure 3.12(b), consider any point x in one of the components. Let y be the closest point to x on any other component, say C. If the line L joining x and y meets C orthogonally at y we have the situation as in the previous case with only x and y interchanged. In the other case, y lies on the boundary of C on a Voronoi edge. The angle between L and $\mathbf{n}_y$ is less than the angle between the Voronoi edge and $\mathbf{n}_y$ which is no more than $\alpha(\varepsilon) + \beta(\varepsilon)$ as proved in the Voronoi Edge Lemma 3.8 (Inequality 3.1). We reach a contradiction again between two distances using the same argument. ■

Lemma 3.10 (Voronoi Cell). *A Voronoi cell V_p intersects Σ in a 2-ball if Condition* A *holds.*

Proof. We have $W = V_p \cap \Sigma$ contained in a ball B of radius $\frac{\varepsilon}{1-\varepsilon} f(p)$ by the Short Distance Lemma 3.6. If W is a manifold without boundary, B contains a medial axis point m by the Feature Ball Lemma 1.1. Then the radius of B is at least

$$\frac{\|m-p\|}{2} \ge \frac{f(p)}{2}.$$

We reach a contradiction if $\varepsilon < \frac{1}{3}$ which is satisfied by Condition A. So, assume that W is a manifold with boundary. It may not be a 2-ball only if it is nonorientable, has a handle, or has more than one boundary cycle. If W were nonorientable, so would be Σ, which is impossible. In case W has a handle, $B \cap \Sigma$ is not a 2-ball. By the Feature Ball Lemma 1.1, it contains a medial axis point reaching a contradiction again for $\varepsilon < \frac{1}{3}$ which is satisfied by Condition A.

The only possibility left is that W has more than one boundary cycles. Let L be the line of the normal at p. Consider a plane that contains L and intersects at least two boundary cycles. Such a plane exists since otherwise L must intersect W at a point other than p and we reach a contradiction between two distance lemmas. The plane intersects V_p in a convex polygon and W in at least two curves. We can argue as in the proof of the Voronoi Facet Lemma 3.9 to reach a contradiction between two distance lemmas. ■

Condition A holds for $\varepsilon \leq 0.18$. Therefore, the Voronoi Edge Lemma, Facet Lemma, and Cell Lemma hold for $\varepsilon \leq 0.18$. Then, Theorem 3.1 leads to the following result.

Theorem 3.2 (Topological Ball.). *Let P be an ε-sample of a smooth surface Σ. For $\varepsilon \leq 0.18$, (P, Σ) satisfies the topological ball property and hence the underlying space of* $\mathrm{Del}\, P|_\Sigma$ *is homeomorphic to* Σ.

3.3 Notes and exercises

The remarkable connection between ε-samples of a smooth surface and the Voronoi diagram of the sample points was first discovered by Amenta and Bern [4]. The Normal Lemma 3.2 and the Normal Variation Lemma 3.3 are two key observations made in this paper. The topological ball property that ensures the homeomorphism between the restricted Delaunay triangulation and the surface was discovered by Edelsbrunner and Shah [48]. Amenta and Bern observed that the Voronoi diagram of a sufficiently dense sample satisfies the topological ball property though the proof was not as rigorous as presented here. The proof presented here is adapted from Cheng, Dey, Edelsbrunner, and Sullivan [23].

Exercises

1. Let the restricted Voronoi cell $V_p|\Sigma$ be adjacent to the restricted Voronoi cell $V_q|\Sigma$ in the restricted Voronoi diagram $\mathrm{Vor}\, P|\Sigma$. Show that the distance between any two points x and y from the union of $Vp|\Sigma$ and $Vq|\Sigma$ is $\tilde{O}(\varepsilon) f(x)$ when ε is sufficiently small.

2. A version of the Edge Normal Lemma 3.4 can be derived from the Triangle Normal Lemma 3.5, albeit with a slightly worse angle bound. Derive this angle bound and carry out the proof of the topological ball property with this bound. Find out an upper bound on ε for the proof.
3. The topological ball property is a sufficient but not a necessary condition for the homeomorphism between a sampled surface and a restricted Delaunay triangulation of it. Establish this fact by an example.
4. Show an example where
   (i) all Voronoi edges satisfy the topological ball property, but the Voronoi cell does not,
   (ii) all Voronoi facets satisfy the topological ball property, but the Voronoi cell does not.
5. Show that for any $n > 0$, there exists a C^2-smooth surface for which a sample with n points has the Voronoi diagram where no Voronoi edge intersects the surface.

6^h. Let F be a Voronoi facet in the Voronoi diagram Vor P where P is an ε-sample of a C^2-smooth surface Σ. Let Σ intersect F in a single interval and the intersection points with the Voronoi edges lie within $\varepsilon f(p)$ away from p where $F \subset V_p$. Show that all points of $F \cap \Sigma$ lie within $\varepsilon f(p)$ distance when ε is sufficiently small.

7. Let F and Σ be as described in Exercise 6, but $F \cap \Sigma$ contains two or more topological intervals. Show that there exists a Voronoi edge $e \in F$ so that $e \cap \Sigma$ is at least $\lambda f(p)$ away from p where $\lambda > 0$ is an appropriate constant.

8^o. Let the pair (P, Σ) satisfy the topological ball property. We know that the underlying space of Del $P|_\Sigma$ and Σ are homeomorphic. Prove or disprove that they are isotopic.