7

Noisy Samples

In the previous chapters we have assumed that the input points lie exactly on the sampled surface. Unfortunately, in practice, the input sample often does not satisfy this constraint. Noise introduced by measurement errors scatters the sample points away from the surface. Consequently, all analysis as presented in the previous chapters becomes invalid for such input points. In this chapter we develop a noise model that accounts for the scatter of the inputs and then analyze noisy samples based on this model. We will see that, as in the noise-free case, some key properties of the sampled surface can be computed from the Delaunay triangulation of a noisy sample. Specifically, we show that normals of the sampled surface can still be estimated from the Delaunay/Voronoi diagrams. Furthermore, the medial axis and hence the local feature sizes of the sampled surface can also be estimated from these diagrams. These results will be used in Chapters 8 and 9 where we present algorithms to reconstruct surfaces from noisy samples.

7.1 Noise Model

In the noise-free case ε-sampling requires each point on the surface have a sample point within a distance of ε times the local feature size. When noise is allowed, the sample points need not lie exactly on the surface and may scatter around it. Therefore, the sampling model needs to specify both a *tangential scatter*, that is, the sparseness of the points along the tangential directions of the surface and also a *normal scatter*, that is, the sparseness of the points along the normal directions. We use two independent parameters ε and δ for these two scatters to reveal the dependence of the approximation errors on these two parameters separately.

We also need a third parameter to specify a local uniformity condition in the sampling. In the noise-free case we do not need any such condition. However, in the noisy case, the points can collaborate to form a dense sample of a spurious surface. For example, a set of points near the actual sampled surface can form a dense sample of a spurious toroidal handle. In that case, an ambiguity creeps in as the input becomes a dense sample of two topologically different surfaces. We prevent this ambiguity by a local uniformity condition.

As before we assume that the sampled surface $\Sigma \subset \mathbb{R}^3$ is a compact C^2-smooth surface without boundary. Recall that for any point $x \in \mathbb{R}^3 \setminus M$, $\tilde{x}$ denotes its closest point on Σ.

Definition 7.1. *A point set $P \subset \mathbb{R}^3$ is a $(\varepsilon, \delta, \kappa)$-sample of Σ if the following conditions hold.*

(i) $\tilde{P} = \{\tilde{p}\}_{p \in P}$ *is a ε-sample of Σ,*
(ii) $\|p - \tilde{p}\| \leq \delta f(\tilde{p})$,
(iii) $\|p - q\| \geq \varepsilon f(\tilde{p})$ *for any point $p \in P$ and its κth nearest sample point q.*

The first condition says that the projection of the input point set P on the surface makes a dense sample and the second one says that P is close to the surface. The third condition enforces the sample to be locally uniform. We will see that the third condition is not needed for the analyses of the normal and medial axis approximation results in Sections 7.3 and 7.4 respectively. However, it is needed for the algorithms that estimate the normals and features based on these analyses. When the third condition is ignored, we say P is a $(\varepsilon, \delta, -)$-sample.

The analysis that we are going to present holds for surfaces that may not be connected. However, for simplicity in presentation, we assume that Σ is connected. We have already observed that such a surface partitions $\mathbb{R}^3$ into two components, Ω_I and Ω_O where Ω_O is the unbounded component of $\mathbb{R}^3 \setminus \Sigma$ and $\Omega_I = \mathbb{R}^3 \setminus \Omega_O$. As before assume the normal $\mathbf{n}_x$ at any point $x \in \Sigma$ to be directed locally outward, that is, toward Ω_O.

In the analysis we concentrate only on the bounded component Ω_I together with the inner medial axis. The results also hold for the unbounded component and outer medial axis except for the points at infinity. To avoid these points, one can take a large enough bounded open set containing Σ and then extend the results to the outer medial axis defined with maximal empty balls within the bounded set. This does not make any change to the inner medial axis though. For a point $x \in \Sigma$, let m_x denote the center of the inner medial ball meeting Σ at x and ρ_x its radius. In what follows assume that P is a $(\varepsilon, \delta, \kappa)$-sample of Σ for $0 < \varepsilon < 1$, $0 < \delta < 1$, and $\kappa \geq 1$.

It follows almost immediately from the sampling conditions that all points of Σ and all points not far away from Σ have sample points nearby. The Close Sample Lemma 7.1 and Corollary 7.1 formalize this idea.

Lemma 7.1 (Close Sample). *Any point $x \in \Sigma$ has a sample point within $\varepsilon_1 f(x)$ distance where $\varepsilon_1 = (\delta + \varepsilon + \delta\varepsilon)$.*

Proof. From the sampling condition (i), we must have a sample point p so that $\|x - \tilde{p}\| \leq \varepsilon f(x)$. Also, $\|p - \tilde{p}\| \leq \delta f(\tilde{p}) \leq \delta(1 + \varepsilon) f(x)$. Thus,

$$\begin{aligned} \|x - p\| &\leq \|x - \tilde{p}\| + \|\tilde{p} - p\| \\ &\leq \varepsilon f(x) + \delta(1 + \varepsilon) f(x) \\ &= (\delta + \varepsilon + \delta\varepsilon) f(x). \end{aligned}$$

■

Since $f(x) \leq \rho_x$ for any point $x \in \Sigma$, the following corollary is immediate.

Corollary 7.1. *Any point $y \in \mathbb{R}^3$ with $\|y - \tilde{y}\| \leq \delta\rho_{\tilde{y}}$ has a sample point within $\varepsilon_2 \rho_{\tilde{y}}$ distance where $\varepsilon_2 = (2\delta + \varepsilon + \delta\varepsilon)$.*

7.2 Empty Balls

A main ingredient in our analysis will be the existence of large balls that remain empty of the points from P. They in turn lead to the existence of large Delaunay balls that circumscribe Delaunay tetrahedra in Del P. The centers of such Delaunay balls which are also Voronoi vertices in Vor P play crucial roles in the algorithms for normal and feature approximations. In this section, we present two lemmas that assure the existence of large empty balls with certain conditions.

The Empty Ball Lemma 7.2 below assures that for each point $x \in \Sigma$ there is a large empty ball of radius almost as large as (i) $f(x)$ and (ii) ρ_x. Notice the differences between the distances of these balls from x. Also, see Figure 7.1.

Lemma 7.2 (Empty Ball). *Let $B_{m,r}$ be a ball and $x \in \Sigma$ be a point so that either*

(i) $\tilde{m} = x$, $\|m - x\| = f(x)$, and $r = (1 - 3\delta) f(x)$, or
(ii) $m = m_x$ and $r = (1 - \delta)\rho_x$.

Then, $B_{m,r}$ is empty of points in P.

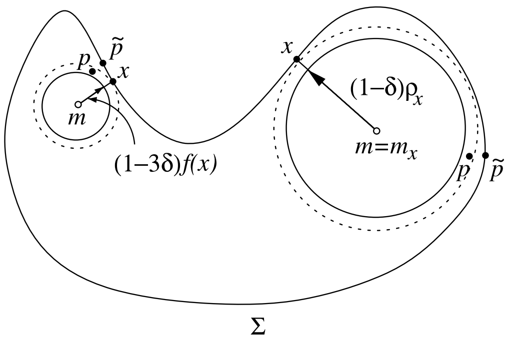

Figure 7.1. Illustration for the Empty Ball Lemma 7.2. The dotted big balls are not empty of sample points but their slightly shrunk copies (shown with solid boundaries) are.

Proof. Let p be any point in P (Figure 7.1). For (i) we have

$$\begin{aligned} f(\tilde{p}) &\leq f(x) + \|x - \tilde{p}\| \\ &\leq f(x) + \|x - m\| + \|m - \tilde{p}\| \\ &= 2f(x) + \|m - \tilde{p}\|. \end{aligned}$$

Therefore,

$$\begin{aligned} \|m - p\| &\geq \|m - \tilde{p}\| - \|p - \tilde{p}\| \\ &\geq \|m - \tilde{p}\| - \delta f(\tilde{p}) \\ &\geq \|m - \tilde{p}\| - \delta(2f(x) + \|m - \tilde{p}\|) \\ &= (1 - \delta)\|m - \tilde{p}\| - 2\delta f(x) \\ &\geq (1 - 3\delta) f(x) \end{aligned}$$

as $\|m - \tilde{p}\| \geq \|m - x\| = f(x)$. Hence, p cannot be in the interior of $B_{m,r}$. Now consider (ii). We get

$$\begin{aligned} \|m_x - p\| &\geq \|m_x - \tilde{p}\| - \|p - \tilde{p}\| \\ &\geq \|m_x - \tilde{p}\| - \delta f(\tilde{p}) \\ &\geq \|m_x - \tilde{p}\| - \delta\|m_x - \tilde{p}\| \\ &= (1 - \delta)\|m_x - \tilde{p}\| \\ &\geq (1 - \delta)\rho_x \end{aligned}$$

as $\|m_x - \tilde{p}\| \geq \|m_x - x\| = \rho_x$. Again, p cannot lie in the interior of $B_{m,r}$. ■

Next, we show that, for each point x of Σ, there is a nearby large ball which is not only empty but also has a boundary that passes through a sample point

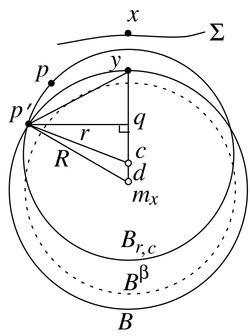

Figure 7.2. Illustration for the Deformed Ball Lemma 7.3.

close to x. Eventually this ball will be deformed to a Delaunay ball for medial axis point approximations.

Lemma 7.3 (Deformed Ball). *For each point $x \in \Sigma$ there is an empty ball $B_{c,r}$ with $c \in \Omega_I$ that enjoys the following properties when ε and δ are sufficiently small.*

(i) m_x is in $B_{c,r}$, $(1 - 2\sqrt{\varepsilon_2})\rho_x \leq r \leq \rho_x$, and $\|c - m_x\| \leq 2\sqrt{\varepsilon_2}\rho_x$ where $\varepsilon_2 = \tilde{O}(\varepsilon + \delta)$ is defined in Corollary 7.1,
(ii) The boundary of $B_{c,r}$ contains a sample point p within a distance $\varepsilon_3\rho_x$ from x where $\varepsilon_3 = 2\varepsilon_2^{\frac{1}{4}} + \delta$.

Proof. We describe a construction of $B_{c,r}$ which is also used later. Consider the empty ball $B = B_{m_x,R}$ whose boundary passes through a point y where $\tilde{y} = x$, $\|y - x\| = \delta\rho_x$, and $R = (1 - \delta)\rho_x$. Such a ball exists by the Empty Ball Lemma 7.2.

Shrinking: Let $B^\beta = B_{m_x,\beta R}$ for $\beta < 1$. The ball B^β is obtained by shrinking B by a factor of β. The ball B and hence B^β are empty.

Rigid motion: Translate B^β rigidly by moving the center along the direction $\overrightarrow{m_x x}$ until its boundary hits a sample point $p \in P$. Let this new ball be denoted $B_{c,r}$, refer to Figure 7.2.

Obviously, $r = \beta R$. Let $d = \|c - m_x\|$. First, we claim

$$(1 - \beta)R \leq d \leq (1 - \beta)R + \varepsilon_2\rho_x. \tag{7.1}$$

The first half of the inequality holds since B is empty of sample points and hence B^β has to move out of it to hit a sample point. The second half of the inequality holds since from Corollary 7.1, a ball centered at y with radius $\varepsilon_2\rho_x$ cannot be empty of sample points.

Next, we obtain an upper bound on $\|y - p\|$. Since

$$\|p' - q\|^2 = \|c - p'\|^2 - \|c - q\|^2 = \|m_x - p'\|^2 - \|m_x - q\|^2,$$

we have

$$\begin{aligned} r^2 - \|c - q\|^2 &= R^2 - \|m_x - q\|^2 \\ \text{or, } r^2 - \|c - q\|^2 &= R^2 - (d + \|c - q\|)^2 \\ \text{or, } r^2 &= R^2 - d^2 - 2d\|c - q\| \\ \text{or, } \|c - q\| &= \frac{R^2 - r^2 - d^2}{2d}. \end{aligned}$$

Hence,

$$\begin{aligned} \|y - p\|^2 \leq \|y - p'\|^2 &= \|p' - q\|^2 + \|q - y\|^2 \\ &= R^2 - (d + \|c - q\|)^2 + (R - (d + \|c - q\|))^2 \\ &= 2R^2 - Rd - \frac{R}{d}(R^2 - r^2) \end{aligned}$$

which, by Inequality 7.1, gives

$$\|y - p\|^2 \leq \frac{\varepsilon_2(1 + \beta)}{(1 - \delta)(1 - \beta) + \varepsilon_2} R^2. \tag{7.2}$$

Since we want both $\|c - m_x\|$ and $\|x - p\|$ to be small, we take $\beta = 1 - \sqrt{\varepsilon_2}$. Hence, with $R = (1 - \delta)\rho_x$,

$$r = \beta R = (1 - \delta)(1 - \sqrt{\varepsilon_2})\rho_x$$

which gives, for sufficiently small δ and ε,

$$(1 - 2\sqrt{\varepsilon_2})\rho_x \leq r \leq \rho_x.$$

Also,

$$\begin{aligned} \|c - m_x\| &= d \\ &\leq (1 - \beta)R + \varepsilon_2\rho_x \\ &\leq ((1 - \delta)\sqrt{\varepsilon_2} + \varepsilon_2)\rho_x \\ &\leq 2\sqrt{\varepsilon_2}\rho_x \end{aligned}$$

for $\varepsilon_2 < 1, \delta < 1$. Given that ε and δ are sufficiently small, $\sqrt{\varepsilon_2}\rho_x$ is small implying that m_x stays inside $B_{c,r}$. In addition, from Inequality 7.2 we have

$$\|y - p\| \leq \sqrt{\left(\frac{2 - \sqrt{\varepsilon_2}}{1 - \delta + \sqrt{\varepsilon_2}}\right)} \varepsilon_2^{\frac{1}{4}} R \leq 2\varepsilon_2^{\frac{1}{4}} \rho_x.$$

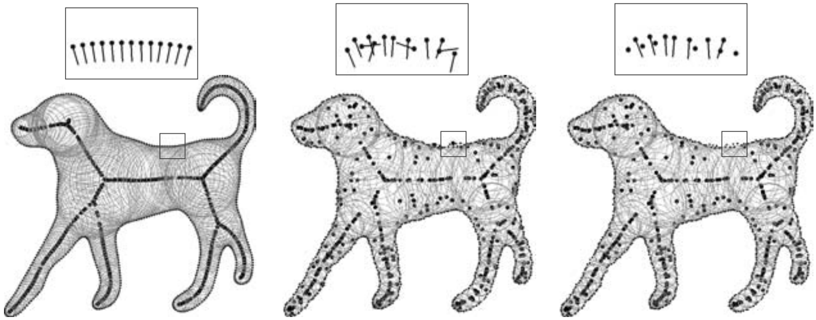

Figure 7.3. Black dots are the centers of Delaunay balls. Left: the normals are estimated correctly by pole vectors in the noise-free case. Left-middle: pole vectors do not estimate the normals correctly when noise is present. Right: vectors from the sample points to the center of the big Delaunay balls estimate the normals even when noise is present.

The bound on $\|p - x\|$ follows as $\|x - y\| = \delta\rho_x$ and $\|p - x\| \leq \|y - p\| + \|x - y\|$. ■

In the above proof, if we make the ball $B_{c,r}$ smaller, we will get a sample point closer to x. For example, if we choose β to be a constant, say $\frac{3}{4}$, the above proof gives $\varepsilon_3 = \tilde{O}(\sqrt{\varepsilon_2}) = \tilde{O}(\sqrt{\varepsilon + \delta})$. Also, the entire proof remains valid when we replace ρ_x with $f(x)$. We will use this version of the lemma in the next chapter.

7.3 Normal Approximation

In noise-free case we saw that poles help approximate the normals (Pole Lemma 4.1). When noise is present poles may come arbitrarily close to the surface and the pole vector may not approximate the normals. Figure 7.3 illustrates this point. Nevertheless, some Voronoi vertices that are the centers of some large Delaunay balls still help in estimating the normal directions. This idea is formalized in the analysis below.

7.3.1 Analysis

The normal approximation theorem says that if there is a large empty ball incident to a sample point p, then the vector from p to the center of the ball approximates the normal direction $\mathbf{n}_{\tilde{p}}$. The idea is that one cannot tilt a large ball too much and keep it empty if it is anchored at p and has its center in the direction of $\mathbf{n}_{\tilde{p}}$.

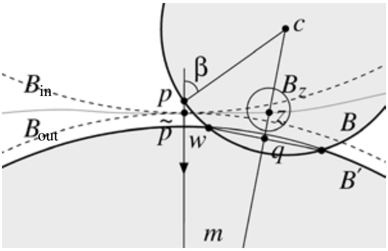

Figure 7.4. Illustration for the General Normal Theorem 7.1.

Theorem 7.1 (General Normal). *Let $p \in P$ be incident to a ball $B_{c,r}$ empty of sample points where $r = \lambda f(\tilde{p})$ and $c \in \Omega_I$. One has*

$$\sin \angle(\overrightarrow{cp}, \mathbf{n}_{\tilde{p}}) \leq \left(4 + \frac{3}{\sqrt{\lambda}}\right)\sqrt{\delta} + \left(2 + \frac{3}{\lambda}\right)\varepsilon_1$$

for a sufficiently small $\varepsilon > 0$ and $\delta > 0$.

Proof. Let $B = B_{c,r}$ and $\beta = \angle(\overrightarrow{cp}, \mathbf{n}_{\tilde{p}})$. Let B_{in} and B_{out} be two balls with radius $f(\tilde{p})$ that tangentially meet the surface at point $\tilde{p}$ as in Figure 7.4. Let m be the center of B_{out}. We know the surface Σ is outside these two balls. By the Empty Ball Lemma 7.2, the ball $B' = B_{m,(1-3\delta)f(\tilde{p})}$, a shrunk copy of B_{out}, is empty of sample points. Therefore, no sample point is inside the shaded area of Figure 7.4.

Observation A. Let D be the disk bounded by the circle in which the boundaries of B and B' intersect. Let cm intersect D at q. As β increases, the radius of D increases, that is, $\|w - q\|$ increases and vice versa.

Observation B. Suppose that $\sin\beta$ has the claimed bound when $\|w - q\| = \sqrt{2}\varepsilon_1 f(\tilde{p})$. Then, if we show $\|w - q\| < \sqrt{2}\varepsilon_1 f(\tilde{p})$, we are done following Observation A.

Assume $\|w - q\| = \sqrt{2}\varepsilon_1 f(\tilde{p})$. Let z be the intersection point between Σ and the segment mc.

Consider the triangle formed by p, m, and c. We have

$$(1 - \delta)f(\tilde{p}) \leq \|m - p\| \leq (1 + \delta)f(\tilde{p})$$

and

$$\|c - p\| = \|c - w\| = \lambda f(\tilde{p})$$

and also

$$\|c - m\| = \sqrt{\|c - w\|^2 - \|w - q\|^2} + \sqrt{\|m - w\|^2 - \|w - q\|^2}.$$

We obtain

$$\begin{aligned}\cos\beta &= \frac{\|c - m\|^2 - \|c - p\|^2 - \|m - p\|^2}{2\|c - p\|\|m - p\|} \\ &\geq \frac{\left(\sqrt{\lambda^2 - 2\varepsilon_1^2} + \sqrt{(1 - 3\delta)^2 - 2\varepsilon_1^2}\right)^2 - \lambda^2 - (1 + \delta)^2}{2\lambda(1 + \delta)}\end{aligned}$$

which after some calculations gives

$$1 - \cos\beta \leq \left(7 + \frac{4}{\lambda}\right)\delta + 2\left(1 + \frac{2}{\lambda^2}\right)\varepsilon_1^2.$$

Hence,

$$\begin{aligned}\sin\beta \leq 2\sin\frac{\beta}{2} &= \sqrt{2(1 - \cos\beta)} \\ &\leq \sqrt{\left(14 + \frac{8}{\lambda}\right)\delta + 4\left(1 + \frac{2}{\lambda^2}\right)\varepsilon_1^2} \\ &\leq \left(4 + \frac{3}{\sqrt{\lambda}}\right)\sqrt{\delta} + \left(2 + \frac{3}{\lambda}\right)\varepsilon_1. \qquad (7.3)\end{aligned}$$

Now we show that $\|w - q\| < \sqrt{2}\varepsilon_1 f(\tilde{p})$ as required by Observation B. Again, first assume that $\|w - q\| = \sqrt{2}\varepsilon_1 f(\tilde{p})$. One can show $\|\tilde{p} - z\| \leq 3\|\tilde{p} - m\| \tan\beta$. Therefore, from equation 7.3 $\|\tilde{p} - z\| = \tilde{O}(\varepsilon_1 + 2\sqrt{\delta}) f(\tilde{p})$ which by Lipschitz property gives $f(z) < \sqrt{2} f(\tilde{p})$ given a sufficiently small δ and ε. We know $B_z = B_{z,\varepsilon_1 f(z)}$ with radius $\varepsilon_1 f(z) < \sqrt{2}\varepsilon_1 f(\tilde{p})$ has to contain at least one sample point by the Close Sample Lemma 7.1. This is impossible since B_z has a radius at most $\sqrt{2}\varepsilon_1 f(\tilde{p}) = \|w - q\|$ which means it lies completely in the shaded area. Therefore, $\|w - q\| \neq \sqrt{2}\varepsilon_1 f(\tilde{p})$. Now consider increasing $\|w - q\|$ beyond this distance while keeping z fixed. Notice that now z is not the intersection point between Σ and the segment mc. It is obvious that B_z remains inside the shaded area. Therefore, again we reach a contradiction to the Close Sample Lemma 7.1. Hence, $\|w - q\|$ cannot be larger than $\sqrt{2}\varepsilon_1 f(\tilde{p})$. ∎

The General Normal Theorem 7.1 gives a general form of the normal approximation under a fairly general sampling assumption. One can derive different normal approximation bounds under different sampling assumptions from this general result. For example, if P is a $(\varepsilon, \varepsilon^2, -)$-sample we get an $\tilde{O}(\varepsilon)$ bound on the normal approximation error for a large Delaunay ball with $\lambda = \Omega(1)$.

In the case where P is a $(\varepsilon, \varepsilon, -)$-sample, this error bound becomes $\tilde{O}(\sqrt{\varepsilon})$. When there is no noise, that is, P is a $(\varepsilon, 0, -)$-sample, we obtain $\tilde{O}(\varepsilon)$ error bound that agrees with the Normal Lemma 3.2. Another important implication of the General Normal Theorem is that the Delaunay balls need not be too big to give good normal estimates. One can observe that if λ is only $\sqrt{\max\{\varepsilon, \delta\}}$, we get $\tilde{O}(\varepsilon^{\frac{1}{2}} + \delta^{\frac{1}{4}})$ error. The algorithmic implication of this fact is that a lot of Delaunay balls can qualify for normal approximation.

We also observe that the proof of the General Normal Theorem 7.1 remains valid even if the sample point p is replaced with any point $x \in \mathbb{R}^3$ meeting the conditions as stated in the corollary below. We use this fact later in feature approximation.

Corollary 7.2. *Let $x \in \mathbb{R}^3$ be any point with $\|x - \tilde{x}\| \leq \delta\rho_{\tilde{x}}$ and $B_{c,r}$ be any empty ball incident to x so that $r = \Omega(\rho_{\tilde{x}})$ and $c \in \Omega_I$. Then, $\angle(\overrightarrow{cx}, \mathbf{n}_{\tilde{x}}) = \tilde{O}(\varepsilon + \sqrt{\delta})$ for sufficiently small ε and δ.*

7.3.2 Algorithm

We know from the General Normal Theorem 7.1 that if there is a big Delaunay ball incident to a sample point p, the vector from the center of the ball to p estimates the normal direction at the point $\tilde{p}$. On the other hand, the Deformed Ball Lemma 7.3 assures that, for each point $x \in \Sigma$, there is a sample point p within $\tilde{O}(\varepsilon^{\frac{1}{4}} + \delta^{\frac{1}{4}})f(x)$ distance with an empty ball of radius $\Omega(f(x))$. This means there is a big Delaunay ball incident to p where the vector $\overrightarrow{cp}$ approximates $\mathbf{n}_{\tilde{p}}$ and hence $\mathbf{n}_x$. Algorithmically we can exploit this fact by picking up sample points that are incident to big Delaunay balls only if we have a scale to measure "big" Delaunay balls. For this we assume the third condition in the sampling which says that the sample is locally uniform.

Let d_p be the distance of p to its κth nearest neighbor. The locally uniform condition in the noise model gives $d_p \geq \varepsilon f(\tilde{p})$. Therefore, any Delaunay ball incident to p with radius more than τd_p will give a normal approximation with an error

$$\tilde{O}\left(\left(1 + \frac{1}{\sqrt{\tau\varepsilon}}\right)\sqrt{\delta} + \left(1 + \frac{1}{\tau\varepsilon}\right)\varepsilon\right)$$

according to the General Normal Theorem 7.1. If we assume that P is a $(\varepsilon, \varepsilon^2, \kappa)$-sample, the error bound is $\tilde{O}(\varepsilon + \frac{1}{\tau} + \sqrt{\frac{\varepsilon}{\tau}})$. Notice that the error decreases as τ increases. However, we cannot increase τ arbitrarily since then no Delaunay ball may meet the condition that its radius is at least as large as τd_p.

In fact, we also need an upper bound on d_p to assert that it is not arbitrarily large.

Lemma 7.4 (κ-Neighbor). *$d_p \leq \varepsilon' f(\tilde{p})$ where $\varepsilon' = \left(\varepsilon + \frac{4\kappa+\varepsilon}{1-4\kappa\varepsilon}\right)\varepsilon$.*

Proof. Consider the sample $\tilde{P}$ which is locally uniform. It is an easy consequence of the sampling condition (i) and the Lipschitz property of f that, for each $x \in \Sigma$ there exists a sample point p so that $\|\tilde{p} - x\| \leq \frac{\varepsilon}{1-\varepsilon} f(\tilde{p})$. This means that, for sufficiently small ε, balls of radius $2\varepsilon f(\tilde{p}) > \frac{\varepsilon}{1-\varepsilon} f(\tilde{p})$ around each point $\tilde{p} \in \tilde{P}$ cover Σ. Consider the graph where a point $\tilde{p} \in \tilde{P}$ is joined with $\tilde{q} \in \tilde{P}$ with an edge if the balls $B_{\tilde{p},r_1}$ and $B_{\tilde{q},r_2}$ intersect where $r_1 = 2\varepsilon f(\tilde{p})$ and $r_2 = 2\varepsilon f(\tilde{q})$. Take a simple path Π of κ edges in this graph with one endpoint at $\tilde{p}$. An edge between any two points $\tilde{q_i}$ and $\tilde{q_j}$ in the graph has a length at most $2\varepsilon(f(\tilde{q_i}) + f(\tilde{q_j}))$. The path Π thus has length at most

$$\ell = 2\varepsilon(f(\tilde{p}) + 2f(\tilde{q_1}) + \cdots + 2f(\tilde{q}_{\kappa-1}) + f(\tilde{q_\kappa}))$$

where $\tilde{q_i}, i = 1, \ldots, \kappa$ are the vertices ordered along the path. Denoting $f_{\max}$ as the maximum of the feature sizes of all vertices on the considered path we get

$$\begin{aligned}\ell &\leq 4\kappa\varepsilon f_{\max} \\ &\leq \frac{4\kappa\varepsilon}{1-4\kappa\varepsilon} f(\tilde{p}).\end{aligned}$$

The distance from p to the farthest point, say q, among the κ closest points to p cannot be more than the distance

$$\|p - \tilde{p}\| + \|\tilde{p} - \tilde{q}\| + \|\tilde{q} - q\|$$

which is no more than

$$\varepsilon^2 f(\tilde{p}) + \frac{4\kappa\varepsilon}{1-4\kappa\varepsilon} f(\tilde{p}) + \frac{\varepsilon^2}{1-4\kappa\varepsilon} f(\tilde{p}) \leq \varepsilon' f(\tilde{p}).$$

■

The previous lemma and the locally uniform sampling condition together confirm that a Delaunay ball with radius τd_p has at least $\tau\varepsilon f(\tilde{p})$ radius and at most $\tau\tilde{O}(\varepsilon) f(\tilde{p})$ radius. The quantity τd_p can be made as small as $\sqrt{\varepsilon} f(\tilde{p})$ to give an $\tilde{O}(\sqrt{\varepsilon})$ error. This means the Delaunay ball with radius as small as $\sqrt{\varepsilon} f(\tilde{p})$ provides a good approximation of the true normal. This explains why a large number of Delaunay balls give good normal approximations in practice. See Figure 7.3 for an illustration in two dimensions. Thus, we have the following algorithm:

ApproximateNormal(P,τ)
compute Del P;
`for` each $p \in P$ `do`
  compute d_p;
  `if` there is a Delaunay ball incident to p with radius larger than τd_p
    compute the largest Delaunay ball $B_{c,r}$ incident to p;
    store the normal direction at p as pc;
  `endif`
`endfor`.

Notice that, alternatively we could have eliminated the parameter τ in the algorithm by looking for the largest Delaunay ball incident to a set of k-nearest neighbors of p for some suitable k. Again, thanks to the Deformed Ball Lemma 7.3, we are assured that for a suitable k, one or more neighbors have Delaunay balls with radius almost equal to the medial balls. However, this approach limits the number of sample points where the normals are estimated. Because of our earlier observation, the normals can be estimated at more points where the Delaunay ball is big but not necessarily as big as the medial balls. However, as we see next, for feature approximation we need the Delaunay balls almost as big as the medial ones.

7.4 Feature Approximation

We approximate the local feature size at a sample point p by first approximating the medial axis with a set of discrete points and then measuring the distance of p from this set. In the noise-free case it is known that poles approximate the medial axis. Therefore, feature sizes can be estimated by computing distance to the poles. Unfortunately, as we have seen already, the poles do not necessarily lie near the medial axis when noise is present. We circumvented this difficulty by considering big Delaunay balls for normal approximations. The big Delaunay balls were chosen by an input threshold. The case for feature approximations is more difficult. This is because unlike normal approximations, not all centers of the big Delaunay balls approximate the medial axis. Only the centers of the Delaunay balls that approximate the medial balls lie near the medial axis. These Delaunay balls are difficult to choose with a size threshold. If the threshold is relatively small, a number of centers remain which do not approximate the medial axis. See the right picture in Figure 7.3. On the other hand, if the threshold is large, the medial axis for some parts of the models may not be approximated

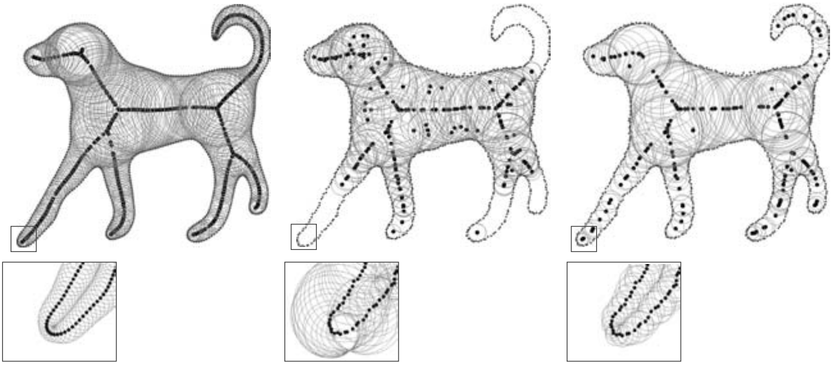

Figure 7.5. Left: poles approximate the medial axis when no noise is present. Middle: Delaunay balls of big size are selected to eliminate unwanted centers, some significant parts of the medial axis are not approximated. Right: centers of polar balls chosen with the nearest neighbors approach approximate the medial axis everywhere. Approximated feature sizes are indicated in the highlighted boxes.

at all; see the middle picture in Figure 7.5. As a result no threshold may exist for which large Delaunay balls' centers approximate the medial axis. The HORSE data in Figure 7.9 is another such example in three dimensions.

We design a different algorithm to choose the Delaunay balls for approximating the medial axis. We consider k-nearest neighbors for some k and take the largest polar ball's center among these neighbors to approximate the medial axis. Our analysis leads to this algorithm. It frees the user from the burden of choosing a size threshold. Experiments suggest that k can be chosen fairly easily, generally in the range of 5–10. The most important thing is that a k can be found for which the medial axis is well approximated where no such size threshold may exist.

We are guaranteed by the Deformed Ball Lemma 7.3 that there are lots of sample points which are incident to big Delaunay balls. The furthest Voronoi vertices from these sample points in Ω_I and Ω_O approximate the inner and outer medial axis respectively. For a point $p \in P$, we call the furthest Voronoi vertex from p in $V_p \cap \Omega_I$ as the inner pole p^+ of p. Similarly, one may define the outer pole p^- of p which resides in Ω_O.

It turns out that the entire medial axis cannot be approximated by poles. Certain parts of the medial axis needs to be excluded. This exclusion is also present in the medial axis approximations with poles in the noise-free case (Exercises 3 and 4 in Chapter 6). Of course, the excluded part is small. In fact, it vanishes to zero in the limit that ε and δ go to zero. Let x and x' be two points where the medial ball B centered at m meets Σ. Call $\angle xmx'$ the *medial angle*

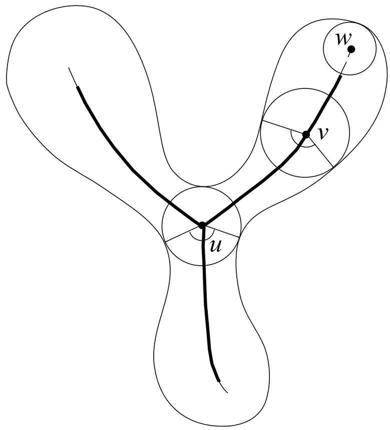

Figure 7.6. The medial angles at u, v are indicated. The medial angle at w is zero. M_α for a small α is shown with thicker curves.

at m if it is the largest angle less than π made by any two such points of $B \cap \Sigma$. Let $M_\alpha \subseteq M$ be the subset where each point $m \in M_\alpha$ has a medial angle at least α (see Figure 7.6).

7.4.1 Analysis

We show that each medial axis point m_x with a large enough medial angle is approximated by a pole. The idea is as follows. Consider the large ball incident to a sample point p guaranteed by the Deformed Ball Lemma 7.3. We deform it to a large Delaunay ball centering the pole p^+. First, during this deformation the ball cannot be tilted too much since the vector from the center to p has to approximate the normal $\mathbf{n}_{\tilde{p}}$ by the General Normal Theorem 7.1. Second, the center in the tilted direction cannot move too much due to Lemma 7.5 as stated below. The result of these constraints is that the center p^+ of the Delaunay ball remains close to the center of the original ball which in turn is close to m_x.

Lemma 7.5. *Let $B = B_{c,r}$ be an empty ball whose boundary passes through a sample point p. Let z be a point on Σ and the distance from z to the boundary of B be less than $\varepsilon' \rho_z$. Suppose B is expanded to an empty ball $B' = B_{c',r'}$ where c' is on the ray $\overrightarrow{pc}$ and* bd B' *passes through p (Figure 7.7). If $\beta \rho_z \leq r \leq \rho_z$, one has*

$$\|c - c'\| \leq \frac{(\varepsilon_1 + \varepsilon')(2 + \varepsilon')}{2\beta(1 - \cos \angle pcz) - 2\varepsilon_1 - 2\varepsilon' \cos \angle pcz} \rho_z.$$

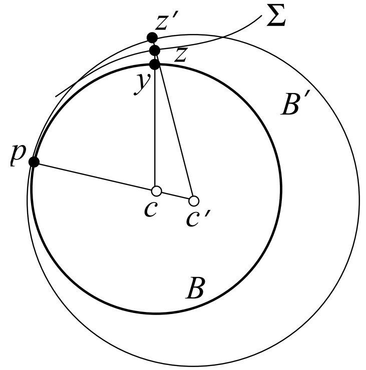

Figure 7.7. Illustration for Lemma 7.5.

Proof. Let y be the closest point to z on the boundary of B. Obviously, y, c, and z are collinear. Let z' be the point where the line of $c'z$ intersects the boundary of B' (see Figure 7.7). We have $\|y - z\| \leq \varepsilon' \rho_z$. Since a ball centered at z with radius $\varepsilon_1 f(z)$ cannot be empty of sample points by the Close Sample Lemma 7.1, we have $\|z' - z\| \leq \varepsilon_1 f(z) \leq \varepsilon_1 \rho_z$.

Consider the triangle made by c, c', and z. For convenience write $\angle pcz = \alpha$, $\|c - c'\| = \Delta c$, $\|z - z'\| = \Delta z$, and $\|y - z\| = \Delta y$.

$$\|c' - z\|^2 = (\Delta c)^2 + \|c - z\|^2 + 2\Delta c \|c - z\| \cos\alpha$$
$$\text{or, } (r + \Delta c - \Delta z)^2 = (\Delta c)^2 + (r + \Delta y)^2 + 2\Delta c(r + \Delta y)\cos\alpha$$

from which we get

$$\begin{aligned}\Delta c &= \frac{(r + \Delta y)^2 - (r - \Delta z)^2}{2(r - \Delta z) - 2(r + \Delta y)\cos\alpha} \\ &= \frac{(\Delta y + \Delta z)(2r + \Delta y - \Delta z)}{2r(1 - \cos\alpha) - 2\Delta z - 2\Delta y \cos\alpha} \\ &\leq \frac{(\varepsilon_1 + \varepsilon')(2 + \varepsilon')}{2\beta(1 - \cos\alpha) - 2\varepsilon_1 - 2\varepsilon' \cos\alpha}\rho_z.\end{aligned}$$

by plugging in $\Delta z \leq \varepsilon_1 \rho_z$, $\Delta y \leq \varepsilon' \rho_z$, and $\beta \rho_z \leq r \leq \rho_z$. ■

Theorem 7.2 (Medial Axis Approximation). *For each point $m_x \in M_\alpha$ in Ω_I where $\alpha = \varepsilon^{\frac{1}{4}} + \delta^{\frac{1}{4}}$ with ε and δ being sufficiently small, there is a sample point p within $\tilde{O}(\varepsilon^{\frac{1}{4}} + \delta^{\frac{1}{4}})\rho_x$ distance of x so that the pole p^+ lies within $\tilde{O}(\varepsilon^{\frac{1}{4}} + \delta^{\frac{1}{4}})\rho_x$ distance from m_x.*

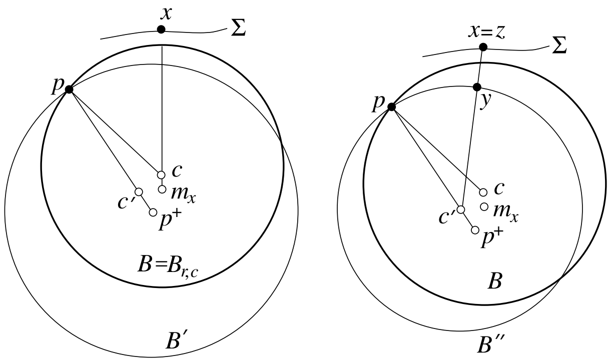

Figure 7.8. Illustration for the Medial Axis Approximation Theorem 7.2. The ball $B_{c,r}$ is deformed to the Delaunay ball $B' = B_{p^+,r'}$. The ball $B'' = B_{c',\|p-c'\|}$ on the right is a shrunk version of B'.

Proof. Consider the ball $B = B_{c,r}$ guaranteed by the Deformed Ball Lemma 7.3 whose boundary passes through a sample point p. We have

$$\begin{aligned} r &\geq (1 - 2\sqrt{\varepsilon_2})\rho_x, \\ \|p - x\| &\leq \varepsilon_3 \rho_x, \text{ and} \\ \|c - m_x\| &\leq 2\sqrt{\varepsilon_2}\rho_x. \end{aligned}$$

Let $B' = B_{p^+,r'}$ where p^+ is the inner pole of p and $r' = \|p - p^+\|$. The ball B' is Delaunay and has radius $r' \geq r \geq (1 - 2\sqrt{\varepsilon_2})\rho_x$.

Focus on the two balls B and B' passing through p (see Figure 7.8). The ball B has m_x inside it which means that its radius is at least $(1-\delta)f(\tilde{p})/2$. So, the radius of B' being bigger than that of B is also at least $(1-\delta)f(\tilde{p})/2$. Therefore, by plugging $\lambda = \Omega(1)$ in the General Normal Theorem 7.1, the vectors $\overrightarrow{pc}$ and $\overrightarrow{pp^+}$ make $\tilde{O}(\varepsilon + \sqrt{\delta})$ angle with $\mathbf{n}_{\tilde{p}}$ and at most double of this angle among them. Let c' be the point on the segment pp^+ so that pc' has the same length as pc. Clearly,

$$\|c - c'\| \leq \|p - c\|\angle cpc' \leq (1 - 2\sqrt{\varepsilon_2})\tilde{O}(\varepsilon + \sqrt{\delta})\rho_x. \tag{7.4}$$

Now we can bound the distance $\|c - p^+\|$ if we have a bound on $\|c' - p^+\|$. We will apply Lemma 7.5 to the ball $B'' = B_{c',\|p-c'\|}$ to bound $\|c' - p^+\|$. Since $m_x \in M_\alpha$ there are two points x and x' in Σ so that $\angle x m_x x' \geq \alpha$. Take z in Lemma 7.5 as the point x or x' which makes the angle $\angle z m_x p$ at least $\alpha/2$.

With this set up we show that β and ε' in Lemma 7.5 are $1 - \tilde{O}(\sqrt{\varepsilon} + \sqrt{\delta})$ and $\tilde{O}(\sqrt{\varepsilon} + \sqrt{\delta})$ respectively. Since the radius of B'' is $r \geq (1 - 2\sqrt{\varepsilon_2})\rho_x = (1 - 2\sqrt{\varepsilon_2})\rho_z$, the claim for β follows.

For ε', consider the point y where the ray $\overrightarrow{c'z}$ meets the boundary of B'', refer to Figure 7.8. We have $\|m_x - c'\| \leq \|c - m_x\| + \|c - c'\| = \tilde{O}(\sqrt{\varepsilon} + \sqrt{\delta})\rho_x$ and hence

$$\begin{aligned}\|y - z\| = \|c' - z\| - \|c' - y\| &\leq \|m_x - z\| + \|c' - m_x\| - \|c' - y\| \\ &\leq \rho_z + \tilde{O}(\sqrt{\varepsilon} + \sqrt{\delta})\rho_z - (1 - 2\sqrt{\varepsilon_2})\rho_z \\ &= \tilde{O}(\sqrt{\varepsilon} + \sqrt{\delta})\rho_z.\end{aligned}$$

So, we can apply Lemma 7.5 with $\varepsilon' = \tilde{O}(\sqrt{\varepsilon} + \sqrt{\delta})$ and $\beta = 1 - \tilde{O}(\sqrt{\varepsilon} + \sqrt{\delta})$. Observe that, since the points c' and m_x are nearby, the angle $\angle pc'y$ is almost equal to $\angle zm_x p$. So, we can take $\angle pc'y \geq \frac{\alpha}{4}$. With $\alpha = \varepsilon^{\frac{1}{4}} + \delta^{\frac{1}{4}}$, Lemma 7.5 gives

$$\|p^+ - c'\| = (\tilde{O}(\sqrt{\varepsilon} + \sqrt{\delta})/\,\Omega(\varepsilon^{\frac{1}{4}} + \delta^{\frac{1}{4}}))\rho_z = \tilde{O}(\varepsilon^{\frac{1}{4}} + \delta^{\frac{1}{4}})\rho_z.$$

The claim of the theorem follows as

$$\begin{aligned}\|p^+ - m_x\| &\leq \|p^+ - c'\| + \|c' - m_x\| \\ &= \tilde{O}(\varepsilon^{\frac{1}{4}} + \delta^{\frac{1}{4}})\rho_x + \tilde{O}(\varepsilon^{\frac{1}{2}} + \delta^{\frac{1}{2}})\rho_x \\ &= \tilde{O}(\varepsilon^{\frac{1}{4}} + \delta^{\frac{1}{4}})\rho_x.\end{aligned}$$

■

For each point $x \in \Sigma$ where $m_x \in M_\alpha$, the previous theorem guarantees the existence of a sample point p whose pole approximates m_x. Actually, the proof technique can be used to show that any Delaunay ball with radius almost as big as ρ_x and incident to a sample point close to x has its center close to m_x.

Theorem 7.3 (Feature). *Let $x \in \Sigma$ be a point so that $m_x \in M_\alpha$ for $\alpha = \varepsilon^{\frac{1}{4}} + \delta^{\frac{1}{4}}$ where ε and δ are sufficiently small. For any point $p \in P$ within $\varepsilon_3\rho_x$ distance of x and with an incident Delaunay ball of radius at least $(1 - \tilde{O}(\sqrt{\varepsilon} + \sqrt{\delta}))\rho_x$, the pole p^+ lies within $\tilde{O}(\varepsilon^{\frac{1}{8}} + \delta^{\frac{1}{8}})\rho_x$ distance from m_x.*

Proof. [sketch]. Notice that if $\mathbf{n}_{\tilde{p}}$ and $\mathbf{n}_x$ make small angle, we will be done. Then, we have two segments pp^+ and xm_x almost parallel where p and x are close. Also, these segments can be shown to be of almost same lengths by the given condition and a proof similar to that of the Medial Axis Approximation Theorem 7.2. This would imply m_x and p^+ are close.

Observe that we cannot assert that $\angle(\mathbf{n}_{\tilde{p}}, \mathbf{n}_x)$ is small directly from the Normal Variation Lemma 3.3. We could have applied this lemma had the distance between $\tilde{p}$ and x been $\tilde{O}(\varepsilon)f(x)$. The Deformed Ball Lemma 7.3 only gives that this distance is at most $\tilde{O}(\varepsilon)\rho_x$ and not $\tilde{O}(\varepsilon)f(x)$. Since p and x are at most $\varepsilon_3\rho_x$ apart and the distance of p^+ to p and hence to x is $\Omega(\rho_x)$, $\angle pp^+x = \tilde{O}(\varepsilon_3)$. By Corollary 7.2, it can be shown that p^+x makes $\tilde{O}(\sqrt{\varepsilon_3})$ angle with $\mathbf{n}_x$. Therefore, pp^+ makes $\tilde{O}(\sqrt{\varepsilon_3})$ angle with $\mathbf{n}_x$. It is easy to show that ρ_x is at least $\Omega(f(\tilde{p}))$. So, the angle between pp^+ and $\mathbf{n}_{\tilde{p}}$ is $\tilde{O}(\sqrt{\varepsilon_3})$ completing the claim that $\angle(\mathbf{n}_{\tilde{p}}, \mathbf{n}_x) = \tilde{O}(\sqrt{\varepsilon_3}) = \tilde{O}(\varepsilon^{\frac{1}{8}} + \delta^{\frac{1}{8}})$. ■

7.4.2 Algorithm

The Medial Axis Approximation Theorem 7.2 and the Feature Theorem 7.3 suggest the following algorithm for feature approximation at points $x \in \Sigma$ where $m_x \in M_{\varepsilon^{\frac{1}{4}}}$. The Medial Axis Approximation Theorem 7.2 says that x has a sample point p within a neighborhood of $\varepsilon_3\rho_x$ whose pole p^+ approximates m_x. Also, the Feature Theorem 7.3 says that *all* sample points within $\varepsilon_3\rho_x$ neighborhood of x with a large enough Delaunay ball have their poles approximate m_x. Therefore, if we take the pole of a sample point q whose distance to q is largest among all sample points within a small neighborhood of x, we will get an approximation of m_x.

We search the neighborhood of x by taking k nearest neighbors of a sample point s close to x. If we assume that P is a $(\varepsilon, \varepsilon, \kappa)$-sample for some $\kappa \geq 1$, k nearest neighbors cannot be arbitrarily close to x. Notice that if we do not prevent oversampling by the third condition of noisy sampling, we cannot make this assertion. In the algorithm, we simply allow a user supplied parameter k to search the k nearest neighbors. Since we want to cover all points of Σ, we simply take all points of P and carry out the following computations.

For each point $p \in P$ we select k-nearest neighbors for a suitable k. Let N_p be this set of neighbors. First, for each $q \in N_p$, we determine the Voronoi vertex v_q in V_q which is furthest from q. This is one of the poles of q. Let $\ell_1(q) = \|v_q - q\|$. Select the point $p_1 \in N_p$ so that $\ell_1(p_1)$ is maximum among all points in N_p. By Medial Axis Approximation Theorem 7.2 and the Feature Theorem 7.3, v_{p_1} approximates a medial axis point m_x if $x \in M_{\varepsilon^{\frac{1}{4}}}$. However, we do not know if m_x is an inner medial axis point or an outer one. Without loss of generality assume that m_x is an inner medial axis point. To approximate the outer medial axis point for x, we determine the Voronoi vertex u_q in V_q for each $q \in N_p$ so that $\overrightarrow{qu_q}$ makes more than $\frac{\pi}{2}$ angle with $\overrightarrow{p_1v_{p_1}}$. Let $\ell_2(q) = \|u_q - q\|$. Then, we select the point $p_2 \in N_p$ so that $\ell_2(p_2)$ is maximum among all points in N_p. Again, appealing to the Medial Axis Approximation Theorem 7.2 and the Feature Theorem 7.3 for outer medial axis, we can assert that u_{p_2} approximates a medial axis point for x.

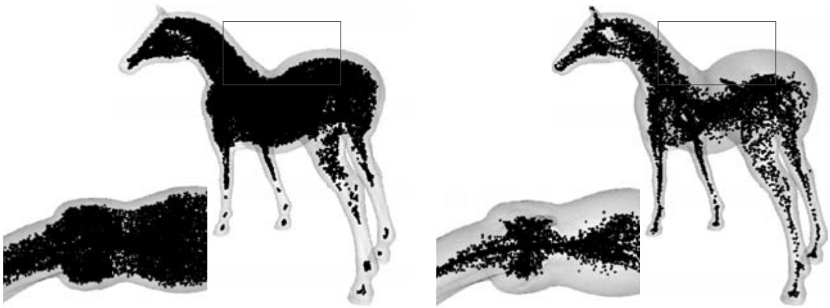

Figure 7.9. Left: medial axis approximated by centers of big Delaunay balls for a noisy HORSE. For a chosen threshold, some parts of the legs do not have medial axis approximated though still many centers lie near the surface. Right: medial axis well approximated by the poles as computed by APPROXIMATEFEATURE.

APPROXIMATEFEATURE(P,k)

compute Del P;
$L := \emptyset$;
`for` each $p \in P$ compute k nearest neighbors N_p;
  compute $p_1 \in N_p$ whose distance to one of its pole v_{p_1}
    is maximum among all points in N_p;
  compute $p_2 \in N_p$ with a pole v_{p_2} so that $\angle(\overrightarrow{p_2 v_{p_2}}, \overrightarrow{p_1 v_{p_1}}) \geq \frac{\pi}{2}$
    and $\|p_2 - v_{p_2}\|$ is maximum among all such points $p_2 \in N_p$;
  $L := L \cup \{v_{p_1}, v_{p_2}\}$;
`endfor`
`for` each $p \in P$ store the distance of p to L.

As we have observed already, a subset of the medial axis is not approximated by the poles. These are exactly the points on the medial axis which have a small medial angle. The implication of this exclusion is that features cannot be properly estimated for points whose closest point on the medial axis resides in the excluded part. However, if the sampling is sufficiently dense, the excluded part is indeed small in most cases. Figure 7.9 shows the result of feature approximations for a three-dimensional model.

7.5 Notes and Exercises

The material in this chapter is taken from Dey and Sun [39]. The noise model with a condition for each of tangential scatter, normal scatter, and local uniformity was first proposed by Dey and Goswami [34]. They used the same parameter ε for both the scatters. Later, Kolluri [63] proposed a slightly different model in the context of smoothing noisy point samples.

In the noise-free case normals can be approximated by poles as we have seen already. Amenta, Choi, and Kolluri [7] as well as Boissonnat and Cazals [16] showed independently that poles also approximate the medial axis. Later, Dey and Zhao [42] and Chazal and Lieutier [20] showed how to approximate the medial axis with a subset of Voronoi facets and not necessarily with a set of discrete points.

In case of noise, an analysis of normals with big Delaunay balls appeared in Dey and Sun [40] and also in Mederos et al. [67]. Normal approximation under the general noise model as adopted in this chapter was put forward by Dey and Sun [39]. They also provided the analysis and the algorithm for feature approximation under this noise model.

For noisy point samples, optimization-based techniques also work well for normal approximations in practice. See, for example, Mitra, Nguyen, and Guibas [69] and Pauly, Keiser, Kobbelt, and Gross [75]. A comparison between the optimization and the Delaunay-based approaches can be found in Dey, Li, and Sun [37].

Exercises

1. Show an example where a point set P is a $(\varepsilon, \delta, -)$-sample of two topologically different surfaces.
2. Call a point set P a (ε, κ)-sample of Σ if (i) each point $x \in \Sigma$ has a point in P within $\varepsilon f(x)$ distance and (ii) each point $p \in P$ has its κth nearest neighbor at least $\varepsilon f(\tilde{p})$ distance away. Show that P is also a $(\varepsilon', \varepsilon'', \kappa)$-sample of Σ for some ε' and ε'' dependent upon ε.
3. Formulate and prove a version of the Empty Ball Lemma 7.2 when P is a (ε, κ)-sample.
4. In the proof of the Deformed Ball Lemma 7.3 if we choose β to be a fraction, say $\frac{3}{4}$, what bound do we get for ε_3?
5. Derive from the General Normal Theorem 7.1 that the pole vectors in noise-free samples approximate the normals within an angle of $\arcsin 5\varepsilon$ when ε is sufficiently small.
6. Prove Feature Theorem 7.3 rigorously.