1

Basics

Simply stated, the problem we study in this book is: how to approximate a shape from the coordinates of a given set of points from the shape. The set of points is called a point sample, or simply a *sample* of the shape. The specific shape that we will deal with are curves in two dimensions and surfaces in three dimensions. The problem is motivated by the availability of modern scanning devices that can generate a point sample from the surface of a geometric object. For example, a range scanner can provide the depth values of the sampled points on a surface from which the three-dimensional coordinates can be extracted. Advanced hand held laser scanners can scan a machine or a body part to provide a dense sample of the surfaces. A number of applications in computer-aided design, medical imaging, geographic data processing, and drug designs, to name a few, can take advantage of the scanning technology to produce samples and then compute a digital model of a geometric shape with reconstruction algorithms. Figure 1.1 shows such an example for a sample on a surface which is approximated by a triangulated surface interpolating the input points.

The reconstruction algorithms described in this book produce a piecewise linear approximation of the sampled curves and surfaces. By approximation we mean that the output captures the topology and geometry of the sampled shape. This requires some concepts from topology which are covered in Section 1.1.

Clearly, a curve or a surface cannot be approximated from a sample unless it is dense enough to capture the features of the shape. The notions of features and dense sampling are formalized in Section 1.2.

All reconstruction algorithms described in this book use the data structures called *Voronoi diagrams* and their duals called *Delaunay triangulations*. The key properties of these data structures are described in Section 1.3.

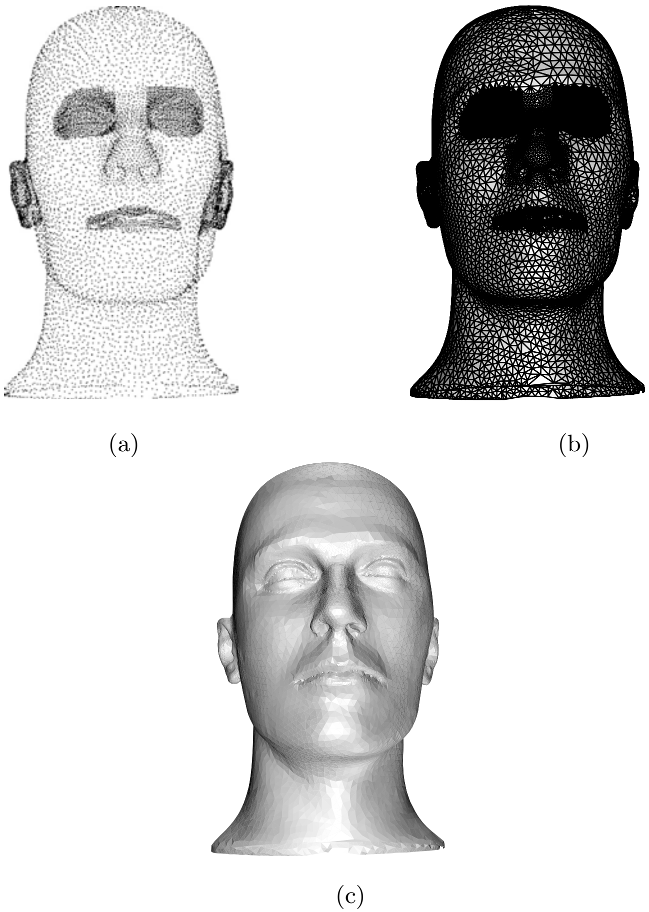


Figure 1.1. (a) A sample of MANNEQUIN, (b) a reconstruction, and (c) rendered MANNEQUIN model.

1.1 Shapes

The term *shape* can circumscribe a wide variety of meaning depending on the context. We define a shape to be a subset of an Euclidean space. Even this class is too broad for our purpose. So, we focus on a specific type of shapes called *smooth manifolds* and limit ourselves only up to three dimensions.

A global yardstick measuring similarities and differences in shapes is provided by *topology*. It deals with the connectivity of spaces. Various shapes are compared with respect to their connectivities by comparing functions over them called *maps*.

1.1.1 Spaces and Maps

In point set topology a *topological space* is defined to a be a point set $\mathbb{T}$ with a system of subsets $\mathcal{T}$ so that the following conditions hold.

1. $\emptyset, \mathbb{T} \in \mathcal{T}$,
2. $U \subseteq \mathcal{T}$ implies that the union of U is in $\mathcal{T}$,
3. $U \subseteq \mathcal{T}$ and U finite implies that the intersection of U is in $\mathcal{T}$.

The system $\mathcal{T}$ is the topology on the set $\mathbb{T}$ and its sets are *open* in $\mathbb{T}$. The *closed* sets of $\mathbb{T}$ are the subsets whose complements are open in $\mathbb{T}$. Consider the k-dimensional Euclidean space $\mathbb{R}^k$ and let us examine a topology on it. An *open ball* is the set of all points closer than a certain Euclidean distance to a given point. Define $\mathcal{T}$ as the set of open sets that are a union of a set of open balls. This system defines a topology on $\mathbb{R}^k$.

A subset $\mathbb{T}' \subseteq \mathbb{T}$ with a *subspace topology* $\mathcal{T}'$ defines a *topological subspace* where $\mathcal{T}'$ consists of all intersections between $\mathbb{T}'$ and the open sets in the topology $\mathcal{T}$ of $\mathbb{T}$. Topological spaces that we will consider are subsets of $\mathbb{R}^k$ which inherits their topology as a subspace topology. Let x denote a point in $\mathbb{R}^k$, that is, $x = \{x_1, x_2, \ldots, x_k\}$ and $\|x\| = (x_1^2 + x_2^2 + \cdots + x_k^2)^{\frac{1}{2}}$ denote its distance from the origin. Example of subspace topology are the k-ball $\mathbb{B}^k$, k-sphere $\mathbb{S}^k$, the halfspace $\mathbb{H}^k$, and the open k-ball $\mathbb{B}_o^k$ where

$$
\begin{aligned}
\mathbb{B}^k &= \{x \in \mathbb{R}^k \mid \|x\| \leq 1\} \\
\mathbb{S}^k &= \{x \in \mathbb{R}^{k+1} \mid \|x\| = 1\} \\
\mathbb{H}^k &= \{x \in \mathbb{R}^k \mid x_k \geq 0\} \\
\mathbb{B}_o^k &= \mathbb{B}^k \setminus \mathbb{S}^k.
\end{aligned}
$$

It is often important to distinguish topological spaces that can be covered with finitely many open balls. A *covering* of a topological space $\mathbb{T}$ is a collection of open sets whose union is $\mathbb{T}$. The topological space $\mathbb{T}$ is called *compact* if every covering of $\mathbb{T}$ can be covered with finitely many open sets included in the covering. An example of a compact topological space is the k-ball $\mathbb{B}^k$. However, the open k-ball is not compact. The *closure* of a topological space $\mathbb{X} \subseteq \mathbb{T}$ is the smallest closed set $\mathrm{Cl}\mathbb{X}$ containing $\mathbb{X}$.

Continuous functions between topological spaces play a significant role to define their similarities. A function $g\colon \mathbb{T}_1 \to \mathbb{T}_2$ from a topological space $\mathbb{T}_1$ to a topological space $\mathbb{T}_2$ is *continuous* if for every open set $U \subseteq \mathbb{T}_2$, the set $g^{-1}(U)$ is open in $\mathbb{T}_1$. Continuous functions are called *maps*.

Homeomorphism

Broadly speaking, two topological spaces are considered the same if one has a correspondence to the other which keeps the connectivity same. For example, the surface of a cube can be deformed into a sphere without any incision or attachment during the process. They have the same topology. A precise definition for this topological equality is given by a map called *homeomorphism*. A homeomorphism between two topological spaces is a map $h : \mathbb{T}_1 \to \mathbb{T}_2$ which is bijective and has a continuous inverse. The explicit requirement of continuous inverse can be dropped if both $\mathbb{T}_1$ and $\mathbb{T}_2$ are compact. This is because any bijective map between two compact spaces must have a continuous inverse. This fact helps us proving homeomorphisms for spaces considered in this book which are mostly compact.

Two topological spaces are *homeomorphic* if there exists a homeomorphism between them. Homeomorphism defines an equivalence relation among topological spaces. That is why two homeomorphic topological spaces are also called *topologically equivalent*. For example, the open k-ball is topologically equivalent to $\mathbb{R}^k$. Figure 1.2 shows some more topological spaces some of which are homeomorphic.

Homotopy

There is another notion of similarity among topological spaces which is weaker than homeomorphism. Intuitively, it relates spaces that can be continuously deformed to one another but may not be homeomorphic. A map $g : \mathbb{T}_1 \to \mathbb{T}_2$ is *homotopic* to another map $h : \mathbb{T}_1 \to \mathbb{T}_2$ if there is a map $H : \mathbb{T}_1 \times [0, 1] \to \mathbb{T}_2$ so that $H(x, 0) = g(x)$ and $H(x, 1) = h(x)$. The map H is called a *homotopy* between g and h.

Two spaces $\mathbb{T}_1$ and $\mathbb{T}_2$ are *homotopy equivalent* if there exist maps $g : \mathbb{T}_1 \to \mathbb{T}_2$ and $h : \mathbb{T}_2 \to \mathbb{T}_1$ so that $h \circ g$ is homotopic to the identity map $\iota_1 : \mathbb{T}_1 \to \mathbb{T}_1$ and $g \circ h$ is homotopic to the identity map $\iota_2 : \mathbb{T}_2 \to \mathbb{T}_2$. If $\mathbb{T}_2 \subset \mathbb{T}_1$, then $\mathbb{T}_2$ is a *deformation retract* of $\mathbb{T}_1$ if there is a map $r : \mathbb{T}_1 \to \mathbb{T}_2$ which is homotopic to ι_1 by a homotopy that fixes points of $\mathbb{T}_2$. In this case $\mathbb{T}_1$ and $\mathbb{T}_2$ are homotopy equivalent. Notice that homotopy relates two maps while homotopy equivalence relates two spaces. A curve and a point are not homotopy equivalent. However, one can define maps from a 1-sphere $\mathbb{S}^1$ to a curve and a point in the plane which have a homotopy.

One difference between homeomorphism and homotopy is that homeomorphic spaces have same dimension while homotopy equivalent spaces need not have same dimension. For example, the 2-ball shown in Figure 1.2(e) is homotopy equivalent to a single point though they are not homeomorphic. Any of

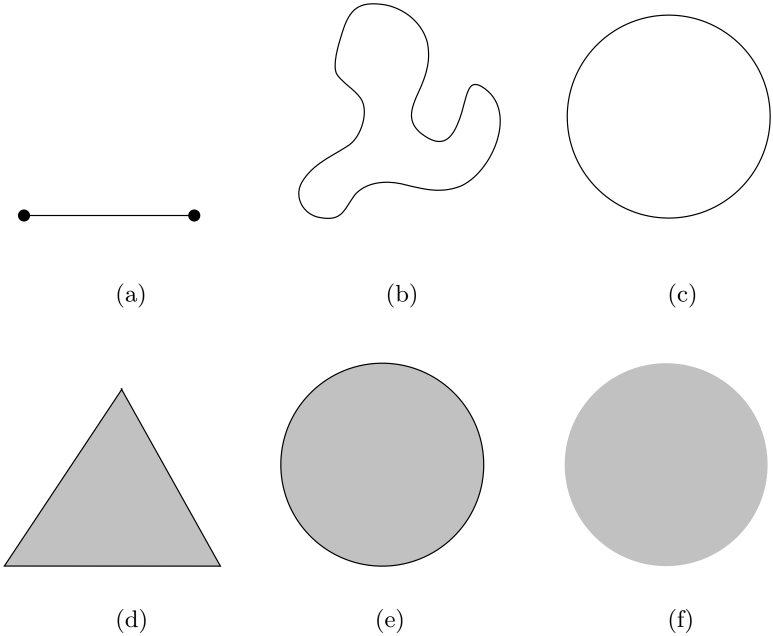


Figure 1.2. (a) 1-ball, (b) and (c) spaces homeomorphic to the 1-sphere, (d) and (e) spaces homeomorphic to the 2-ball, and (f) an open 2-ball which is not homeomorphic to the 2-ball in (e).

the end vertices of the segment in Figure 1.2(a) is a deformation retract of the segment.

Isotopy

Homeomorphism and homotopy together bring a notion of similarity in spaces which, in some sense, is stronger than each one of them individually. For example, consider a standard torus embedded in $\mathbb{R}^3$. One can knot the torus (like a knotted rope) which still embeds in $\mathbb{R}^3$. The standard torus and the knotted one are both homeomorphic. However, there is no continuous deformation of one to the other while maintaining the property of embedding. The reason is that the complement spaces of the two tori are not homotopy equivalent. This requires the notion of *isotopy*.

An *isotopy* between two spaces $\mathbb{T}_1 \subseteq \mathbb{R}^k$ and $\mathbb{T}_2 \subseteq \mathbb{R}^k$ is a map $\xi : \mathbb{T}_1 \times [0, 1] \to \mathbb{R}^k$ such that $\xi(\mathbb{T}_1, 0)$ is the identity of $\mathbb{T}_1$, $\xi(\mathbb{T}_1, 1) = \mathbb{T}_2$ and for each $t \in [0, 1]$, $\xi(\cdot, t)$ is a homeomorphism onto its image. An *ambient isotopy* between $\mathbb{T}_1$ and $\mathbb{T}_2$ is a map $\xi : \mathbb{R}^k \times [0, 1] \to \mathbb{R}^k$ such that $\xi(\cdot, 0)$ is the identity of $\mathbb{R}^k$, $\xi(\mathbb{T}_1, 1) = \mathbb{T}_2$ and for each $t \in [0, 1]$, $\xi(\cdot, t)$ is a homeomorphism of $\mathbb{R}^k$.

Observe that ambient isotopy also implies isotopy. It is also known that two spaces that have an isotopy between them also have an ambient isotopy between them. So, these two notions are equivalent. We will call $\mathbb{T}_1$ and $\mathbb{T}_2$ *isotopic* if they have an isotopy between them.

When we talk about reconstructing surfaces from sample points, we would like to claim that the reconstructed surface is not only homeomorphic to the sampled one but is also isotopic to it.

1.1.2 Manifolds

Curves and surfaces are a particular type of topological space called *manifolds*. A *neighborhood* of a point $x \in \mathbb{T}$ is an open set that contains x. A topological space is a k*-manifold* if each of its points has a neighborhood homeomorphic to the open k-ball which in turn is homeomorphic to $\mathbb{R}^k$. We will consider only k-manifolds that are subspaces of some Euclidean space.

The plane is a 2-manifold though not compact. Another example of a 2-manifold is the sphere $\mathbb{S}^2$ which is compact. Other compact 2-manifolds include *torus* with one through-hole and *double torus* with two through-holes. One can glue g tori together, called *summing* g tori, to form a 2-manifold with g through-holes. The number of through-holes in a 2-manifold is called its *genus*. A remarkable result in topology is that all compact 2-manifolds in $\mathbb{R}^3$ must be either a sphere or a sum of g tori for some $g \geq 1$.

Boundary

Surfaces in $\mathbb{R}^3$ as we know them often have boundaries. These surfaces have the property that each point has a neighborhood homeomorphic to $\mathbb{R}^2$ except the ones on the boundary. These surfaces are 2-manifolds with boundary. In general, a k*-manifold with boundary* has points with neighborhood homeomorphic to either $\mathbb{R}^k$, called the *interior points*, or the halfspace $\mathbb{H}^k$, called the *boundary points*. The boundary of a manifold M, bd M, consists of all boundary points. By this definition a manifold as defined earlier has a boundary, namely an empty one. The interior of M consists of all interior points and is denoted Int M.

It is a nice property of manifolds that if M is a k-manifold with boundary, $\text{bd}\, M$ is a $(k-1)$-manifold unless it is empty. The k-ball $\mathbb{B}^k$ is an example of a k-manifold with boundary where $\text{bd}\, \mathbb{B}^k = \mathbb{S}^{k-1}$ is the $(k-1)$-sphere and its interior $\text{Int}\, \mathbb{B}^k$ is the the open k-ball. On the other hand, $\text{bd}\, \mathbb{S}^k = \emptyset$ and $\text{Int}\, \mathbb{S}^k = \mathbb{S}^k$. In Figure 1.2(a), the segment is a 1-ball where the boundary is a 0-sphere consisting of the two endpoints. In Figure 1.2(e), the 2-ball is a manifold with boundary and its boundary is the circle, a 1-sphere.

Orientability

A 2-manifold with or without boundary can be either *orientable* or *nonorientable*. We will only give an intuitive explanation of this notion. If one travels along any curve on a 2-manifold starting at a point, say p, and considers the oriented normals at each point along the curve, then one gets the same oriented normal at p when he returns to p. All 2-manifolds in $\mathbb{R}^3$ are orientable. However, 2-manifolds in $\mathbb{R}^3$ that have boundaries may not be orientable. For example, the Möbius strip, obtained by gluing the opposite edges of a rectangle with a twist, is nonorientable. The 2-manifolds embedded in four and higher dimensions may not be orientable no matter whether they have boundaries or not.

1.1.3 Complexes

Because of finite storage within a computer, a shape is often approximated with finitely many simple pieces such as vertices, edges, triangles, and tetrahedra. It is convenient and sometimes necessary to borrow the definitions and concepts from combinatorial topology for this representation.

An *affine combination* of a set of points $P = \{p_0, p_1, \ldots, p_n\} \subset \mathbb{R}^k$ is a point $p \in \mathbb{R}^k$ where $p = \Sigma_{i=0}^{n} \alpha_i p_i$, $\Sigma_i \alpha_i = 1$ and each α_i is a real number. In addition, if each α_i is nonnegative, the point p is a *convex combination*. The *affine hull* of P is the set of points that are an affine combination of P. The *convex hull* Conv P is the set of points that are a convex combination of P. For example, three noncollinear points in the plane have the entire $\mathbb{R}^2$ as the affine hull and the triangle with the three points as vertices as the convex hull.

A set of points is affinely independent if none of them is an affine combination of the others. A *k-polytope* is the convex hull of a set of points which has at least $k+1$ affinely independent points. The affine hull aff μ of a polytope μ is the affine hull of its vertices.

A *k-simplex* σ is the convex hull of exactly $k+1$ affinely independent points P. Thus, a vertex is a 0-simplex, an edge is a 1-simplex, a triangle is a 2-simplex, and a tetrahedron is a 3-simplex. A simplex $\sigma' = \text{Conv}\, T$ for a nonempty subset $T \subseteq P$ is called a *face* of σ. Conversely, σ is called a *coface* of σ'. A face $\sigma' \subset \sigma$ is *proper* if the vertices of σ' are a proper subset of σ. In this case σ is a *proper* coface of σ'.

A collection $\mathcal{K}$ of simplices is called a *simplicial complex* if the following conditions hold.

(i) $\sigma' \in \mathcal{K}$ if σ' is a face of any simplex $\sigma \in \mathcal{K}$.
(ii) For any two simplices $\sigma, \sigma' \in \mathcal{K}$, $\sigma \cap \sigma'$ is a face of both unless it is empty.

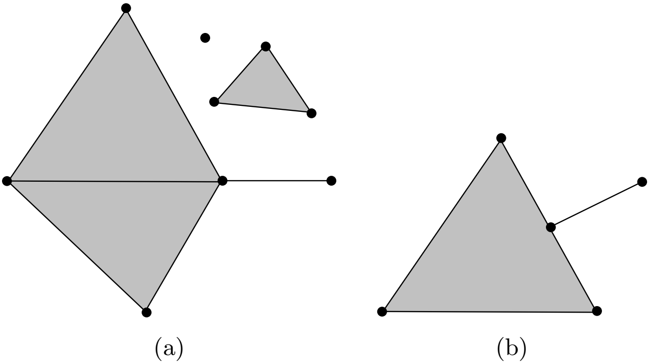


Figure 1.3. (a) A simplicial complex and (b) not a simplicial complex.

The above two conditions imply that the simplices meet nicely. The simplices in Figure 1.3(a) form a simplicial complex whereas the ones in Figure 1.3(b) do not.

Triangulation

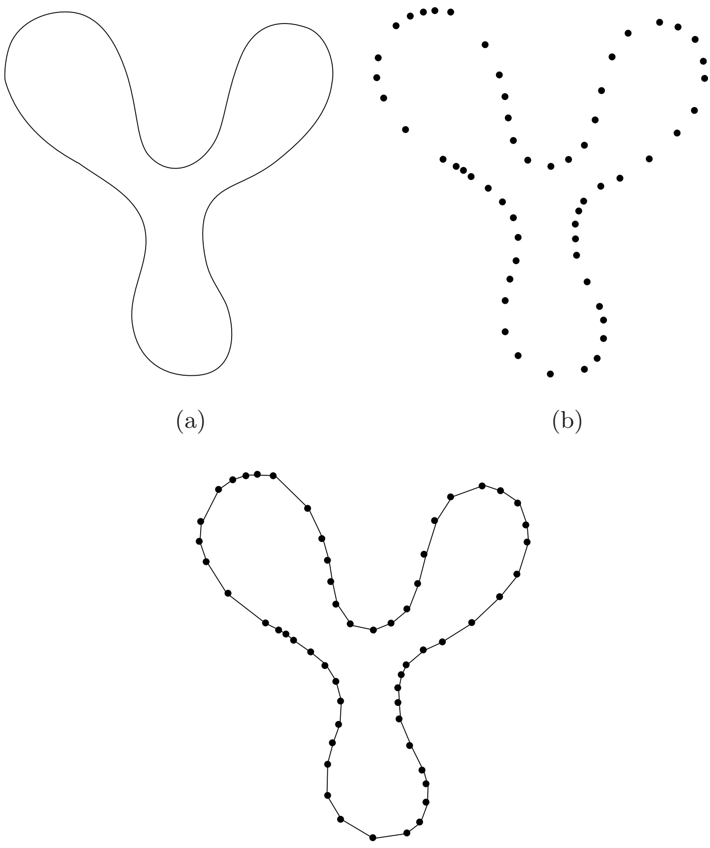
A triangulation of a topological space $\mathbb{T}$ is a simplicial complex $\mathcal{K}$ whose underlying point set is $\mathbb{T}$. Figure 1.1(b) shows a triangulation of a 2-manifold with boundary.

Cell Complex

We also use a generalized version of simplicial complexes in this book. The definition of a cell complex is exactly same as that of the simplicial complex with simplices replaced by polytopes. A cell complex is a collection of polytopes and their faces where any two intersecting polytopes meet in a face which is also in the collection. A cell complex is a k-complex if the largest dimension of any polytope in the complex is k. We also say that two elements in a cell complex are *incident* if they intersect.

1.2 Feature Size and Sampling

We will mainly concentrate on smooth curves in $\mathbb{R}^2$ and smooth surfaces in $\mathbb{R}^3$ as the sampled spaces. The notation Σ will be used to denote this generic sampled space throughout this book. We will defer the definition of smoothness until Chapter 2 for curves and Chapter 3 for surfaces. It is sufficient to assume



(a) (b) (c)

Figure 1.4. (a) A curve in the plane, (b) a sample of it, and (c) the reconstructed curve.

that Σ is a 1-manifold in $\mathbb{R}^2$ and a 2-manifold in $\mathbb{R}^3$ for the definitions and results described in this chapter.

Obviously, it is not possible to extract any meaningful information about Σ if it is not sufficiently sampled. This means features of Σ should be represented with sufficiently many sample points. Figure 1.4 shows a curve in the plane which is reconstructed from a sufficiently dense sample. But, this brings up the question of defining features. We aim for a measure that can tell us how complicated Σ is around each point $x \in \Sigma$. A geometric structure called the *medial axis* turns out to be useful to define such a measure.

Before we define the medial axis, let us fix some notations about distances and balls that will be used throughout the rest of this book. The Euclidean distance between two points $p = (p_1, p_2, \ldots, p_k)$ and $x = (x_1, x_2, \ldots, x_k)$ in $\mathbb{R}^k$ is the length $\|p - x\|$ of the vector $\overrightarrow{xp} = (p - x)$ where

$$\|p - x\| = \left\{(p_1 - x_1)^2 + (p_2 - x_2)^2 + \cdots + (p_k - x_k)^2\right\}^{\frac{1}{2}}.$$

Also, we have

$$\begin{aligned}\|p - x\| &= \{(p - x)^T (p - x)\}^{\frac{1}{2}} \\ &= \{p^T p - 2p^T x + x^T x\}^{\frac{1}{2}} \\ &= \{\|p\|^2 - 2p^T x + \|x\|^2\}^{\frac{1}{2}}.\end{aligned}$$

For a set $P \subseteq \mathbb{R}^k$ and a point $x \in \mathbb{R}^k$, let $d(x, P)$ denote the Euclidean distance of x from P; that is,

$$d(x, P) = \inf_{p \in P} \{\|p - x\|\}.$$

We will also consider distances called *Hausdorff distances* between two sets. For $X, Y \subseteq \mathbb{R}^k$ this distance is given by

$$\max\{\sup_{x \in X} d(x, Y), \sup_{y \in Y} d(y, X)\}.$$

Roughly speaking, the Hausdorff distance tells how much one set needs to be moved to be identical with the other set.

The set $B_{x,r} = \{y \mid y \in \mathbb{R}^k, \|y - x\| \leq r\}$ is a *ball* with center x and radius r. By definition $B_{x,r}$ and its boundary are homeomorphic to $\mathbb{B}^k$ and $\mathbb{S}^{k-1}$ respectively.

1.2.1 Medial Axis

The medial axis of a curve or a surface Σ is meant to capture the middle of the shape bounded by Σ. There are slightly different definitions of the medial axis in the literature. We adopt one of them and mention the differences with the others.

Assume that Σ is embedded in $\mathbb{R}^k$. A ball $B \subset \mathbb{R}^k$ is empty if the interior of B is empty of points from Σ. A ball B is maximal if every empty ball that contains B equals B. The *skeleton* Sk_Σ of Σ is the set of centers of all maximal balls. Let M_Σ^o be the set of points in $\mathbb{R}^k$ whose distance to Σ is realized by at least two points in Σ. The closure of M_Σ^o is M_Σ, that is, $M_\Sigma = \mathrm{Cl}\, M_\Sigma^o$. The following inclusions hold:

$$M_\Sigma^o \subseteq Sk_\Sigma \subseteq M_\Sigma.$$

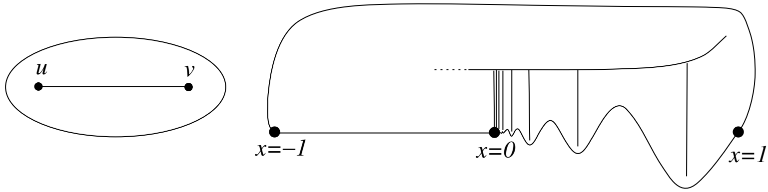


Figure 1.5. (a) The two endpoints on the middle segment are not in M_Σ^o, but are in Sk_Σ and M_Σ, and (b) right half of the bottom curve is $y = x^3 \sin \frac{1}{x}$. Sk_Σ does not include the segment in M_Σ at $x = 0$.

There are examples where the inclusions are strict. For example, consider the curve in Figure 1.5(a). The two endpoints u and v are not in M_Σ^o though they are in Sk_Σ. These are the centers of the curvature balls that meet the curve only at a single point. Consider the curve in Figure 1.5(b):

$$y = \begin{cases} 0 & \text{if } -1 \le x \le 0 \\ x^3 \sin \frac{1}{x} & \text{if } 0 < x \le 1. \end{cases}$$

The two endpoints $(-1, 0)$ and $(1, \sin 1)$ can be connected with a smooth curve so that the resulting curve Σ is closed, that is, without any boundary point, see Figure 1.5(b). The set M_Σ^o has infinitely many branches, namely one for each oscillation of the $y = x^3 \sin \frac{1}{x}$ curve. The closure of M_Σ^o has a vertical segment at $x = 0$, which is not part of Sk_Σ and thus Sk_Σ is a strict subset of M_Σ. However, this example is a bit pathological since it is known that a large class of curves and surfaces have $Sk_\Sigma = M_\Sigma$. All curves and surfaces that are at least C^2-smooth[1] have $Sk_\Sigma = M_\Sigma$. The example we considered in Figure 1.5(b) is a C^1-smooth curve which is tangent continuous but not curvature continuous.

In our case we will consider only the class of curves and surfaces where $Sk_\Sigma = M_\Sigma$ and thus define the *medial axis* of Σ as M_Σ. For simplicity we write M in place of M_Σ.

Definition 1.1. *The medial axis M of a curve (surface) $\Sigma \subset \mathbb{R}^k$ is the closure of the set of points in $\mathbb{R}^k$ that have at least two closest points in Σ.*

Each point of M is the center of a ball that meets Σ only tangentially. We call each ball $B_{x,r}$, $x \in M$, a *medial ball* where $r = d(x, \Sigma)$. If a medial ball $B_{x,r}$ is tangent to Σ at $p \in \Sigma$, we say $B_{x,r}$ is a medial ball *at* p.

[1] See the definition of C^i-smoothness for curves in Chapter 2 and for surfaces in Chapter 3.

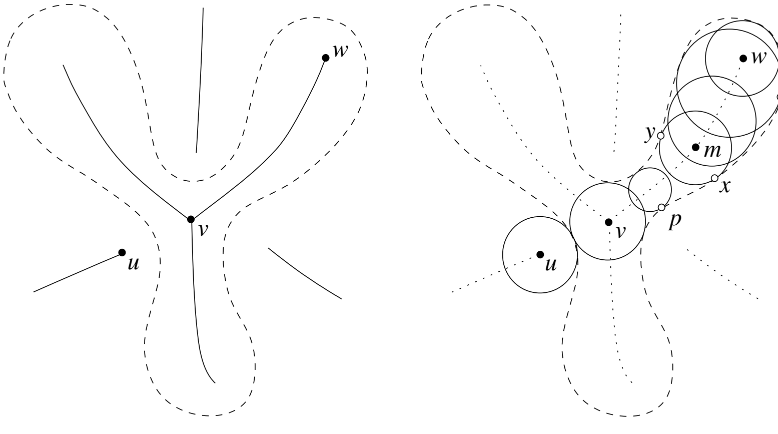


Figure 1.6. (a) A subset of the medial axis of the curve in Figure 1.4 and (b) medial ball centered at v touches the curve in three points, whereas the ones with centers u and w touch it in only one point and coincide with the curvature ball.

Figure 1.6(a) shows a subset of the medial axis of a curve. Notice that the medial axis may have a branching point such as v and boundary points such as u and w. Also, the medial axis need not be connected. For example, the part of the medial axis in the region bounded by the curve may be disjoint from the rest (see Figure 1.6(a)). In fact, if Σ is C^2-smooth, the two parts of the medial axis are indeed disjoint. The subset of the medial axis residing in the unbounded component of $\mathbb{R}^2 \setminus \Sigma$ is called the *outer* medial axis. The rest is called the *inner* medial axis.

It follows from the definition that if one grows a ball around a point on the medial axis, it will meet Σ for the first time tangentially in one or more points (see Figure 1.6(b)). Conversely, for a point $x \in \Sigma$ one can start growing a ball keeping it tangent to Σ at x until it hits another point $y \in \Sigma$ or becomes maximally empty. At this moment the ball is medial and the segments joining the center m to x and y are normal to Σ at x and y respectively (see Figure 1.6).

If we move along the medial axis and consider the medial balls as we move, the radius of the medial balls increases or decreases accordingly to maintain the tangency with Σ. At the boundaries it coincides with the radius of the *curvature ball* where all tangent points merge into a single one (see Figure 1.6(b)).

It will be useful for our proofs later to know the following property of balls intersecting the sampled space Σ. The proof of the lemma assumes that Σ

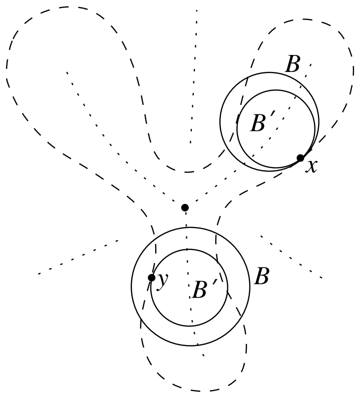


Figure 1.7. The ball B intersecting the upper right lobe is shrunk till it becomes tangent to another point other than x. The new ball B' intersects the medial axis. The ball B intersecting the lower lobe is shrunk radially to the ball B' that is tangent to the curve at y and also intersects the curve in other points. B' can further be shrunk till it meets the curve only tangentially.

is either a smooth curve or a smooth surface whose definitions are given in later chapters. Also, the proof uses some concepts from differential topology (critical point theory) some of which are exposed in Chapter 10. The readers may skip the proof at this point if they are not familiar with these concepts.

We say that a topological space is a k-ball or a k-sphere if it is homeomorphic to $\mathbb{B}^k$ or $\mathbb{S}^k$ respectively.

Lemma 1.1 (Feature Ball). *If a d-ball $B = B_{c,r}$ intersects a k-manifold $\Sigma \subset \mathbb{R}^d$ at more than one point where either (i) $B \cap \Sigma$ is not a k-ball or (ii)* bd $(B \cap \Sigma)$ *is not a $(k-1)$-sphere, then a medial axis point is in B.*

Proof. First we show that if B intersects Σ at more than one point and B is tangent to Σ at some point, B contains a medial axis point. Let x be the point of this tangency. Shrink B further keeping it tangent to Σ at x. This means the center of B moves toward x along a normal direction at x. We stop when B meets Σ only tangentially. Observe that, since $B \cap \Sigma \neq x$ to start with, this happens eventually when B is maximally empty. At this moment B becomes a medial ball and its center is a medial axis point which must lie in the original ball B, refer to Figure 1.7.

Now consider when condition (ii) holds. Define a function $h\colon B \cap \Sigma \to \mathbb{R}$ where $h(x)$ is the distance of x from the center c of B. The function h is a scalar function defined over a smooth manifold. At the critical points of h where its gradient vanishes the ball B becomes tangent to Σ when shrunk appropriately.

Let m be a point in Σ so that $h(m)$ is a global minimum. If there is more than one such global minimum, the ball B meets Σ only tangentially at more than one point when radially shrunk to a radius of $h(m)$. Then, B becomes a medial ball which implies that the original B contains a medial axis point, namely its center. So, assume that there is only global minimum m of h.

We claim that the function h has a critical point p in $\mathrm{Int}\,(B \cap \Sigma)$ other than m where B becomes tangent to Σ. If not, as we shrink B centrally the level set $\mathrm{bd}\,(B \cap \Sigma)$ does not change topology until it reaches the minimum m when it vanishes. This follows from the Morse theory of smooth functions over smooth manifolds.[2] Since m is a minimum, there is a small enough $\delta > 0$ so that $B_{c,h(m)+\delta} \cap \Sigma$ is a k-ball. The boundary of this k-ball given by $(\mathrm{bd}\, B_{c,h(m)+\delta}) \cap \Sigma$ should be a $(k-1)$-sphere. This contradicts the fact that $\mathrm{bd}\,(B \cap \Sigma)$ is not a $(k-1)$-sphere and remains that way till the end. Therefore, there is a critical point, say $y \neq m$ of h in $\mathrm{Int}\,(B \cap \Sigma)$. At this point y, the ball $B_{c,\|y-c\|}$ becomes tangent to Σ, see also Figure 1.7. Now we can apply our previous argument to claim that B contains a medial axis point.

Next, consider when condition (i) holds. If condition (ii) also holds, we have the previous argument. So, assume that $\mathrm{bd}\,(B \cap \Sigma)$ is a $(k-1)$-sphere and $B \cap \Sigma$ is not a k-ball. Again, we claim that the function h as defined earlier has a critical point other than m. If not, consider the subset of Σ swept by B while shrinking it till it meets Σ only at m. This subset is homeomorphic to a space which is formed by taking the product of $\mathbb{S}^{k-1}$ with the closed unit interval I in $\mathbb{R}$ and then collapsing one of its boundary to a single point, that is, the quotient space $(\mathbb{S}^{k-1} \times I)/(\mathbb{S}^{k-1} \times \{0\})$. This space is a k-ball which contradicts the fact that $B \cap \Sigma$ is not a k-ball to begin with. Therefore, as B is continually shrunk, it becomes tangent to Σ at a point $y \neq m$. Apply the previous argument to claim that B has a medial axis point. ■

Figure 1.8 illustrates the different cases of Feature Ball Lemma in $\mathbb{R}^2$.

1.2.2 Local Feature Size

The medial axis M with the distance to Σ at each point $m \in M$ captures the shape of Σ. In fact, Σ is the boundary of the union of all medial balls centering points of the inner (or outer) medial axis. So, as a first attempt to capture local feature size one may define the following two functions on Σ.

[2] See Milnor (1963) for an exposition on Morse theory.

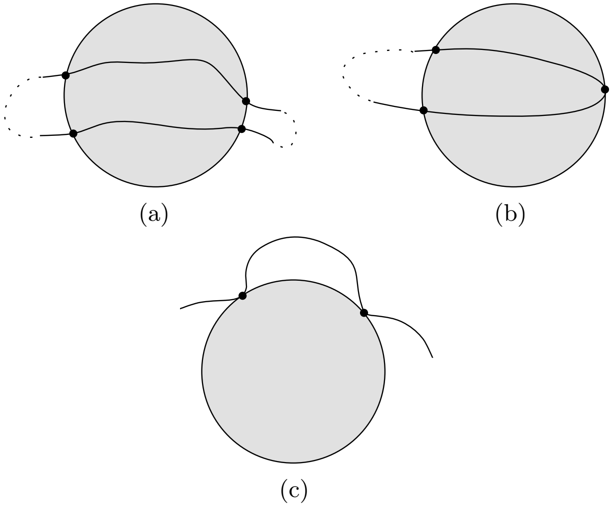


Figure 1.8. (a) $B \cap \Sigma$ is not a 1-ball, (b) $B \cap \Sigma$ is a 1-ball, but bd $B \cap \Sigma$ is not a 0-sphere, and (c) bd $B \cap \Sigma$ is a 0-sphere, but $B \cap \Sigma$ is not a 1-ball.

$\rho_i, \rho_o : \Sigma \to \mathbb{R}$ where $\rho_i(x)$, $\rho_o(x)$ are the radii of the inner and outer medial balls respectively both of which are tangent to Σ at x.

The functions ρ_i and ρ_o are continuous for a large class of curves and surfaces. However, we need a stronger form of continuity on the local feature size function to carry out the proofs. This property, called the *Lipschitz property*, stipulates that the difference in the function values at two points is bounded by a constant times the distance between the points. Keeping this in mind we define the following.

Definition 1.2. *The local feature size at a point $x \in \Sigma$ is the value of a function*

$$f : \Sigma \to \mathbb{R} \text{ where } f(x) = d(x, M).$$

In words, $f(x)$ is the distance of $x \in \Sigma$ to the medial axis M.

Figure 1.9 illustrates how the local feature size can vary over a shape. As one can observe, the local feature sizes at the leg and tail are much smaller than the local feature sizes at the middle in accordance with our intuitive notion of features. For example, $f(b)$ is much smaller than $f(a)$. Local feature size can be determined either by the inner or outer medial axis. For example, $f(c)$ is determined by the outer medial axis whereas $f(d)$ is determined by the inner one.

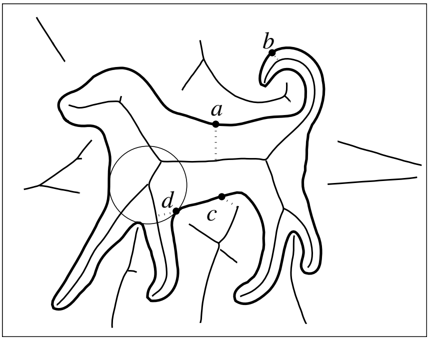


Figure 1.9. Local feature sizes $f(a)$, $f(b)$, $f(c)$, and $f(d)$ are the lengths of the corresponding dotted line segments.

It follows from the definitions that $f(x) \le \min\{\rho_i(x), \rho_o(x)\}$. In Figure 1.9, $f(d)$ is much smaller than the radius of the drawn medial ball at d. Lipschitz property of the local feature size function f follows easily from the definition.

Lemma 1.2 (Lipschitz Continuity). *$f(x) \le f(y) + \|x - y\|$ for any two points x and y in Σ.*

Proof. Let m be a point on the medial axis so that $f(y) = \|y - m\|$. By triangular inequality,

$$\|x - m\| \le \|y - m\| + \|x - y\|, \quad \text{and}$$
$$f(x) \le \|x - m\| \le f(y) + \|x - y\|.$$

■

1.2.3 Sampling

A *sample* P of Σ is a set of points from Σ. Once we have quantized the feature size, we would require the sample respect the features, that is, we require more sample points where the local feature size is small compared to the regions where it is not.

Definition 1.3. *A sample P of Σ is a ε-sample if each point $x \in \Sigma$ has a sample point $p \in P$ so that $\|x - p\| \le \varepsilon f(x)$.*

The value of ε has to be smaller than 1 to have a dense sample. In fact, practical experiments suggest that $\varepsilon < 0.4$ constitutes a dense sample for reconstructing

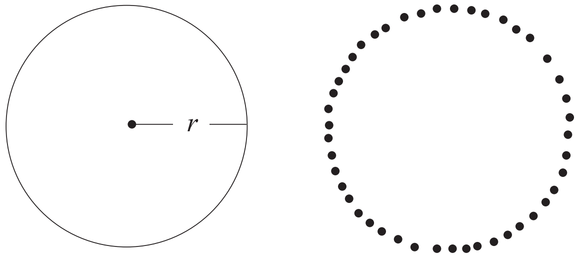


Figure 1.10. Local feature size at any point on the circle is equal to the radius r. Each point on the circle has a sample point within $0.2r$ distance.

Σ from P. A ε-sample is also a ε'-sample for any $\varepsilon' > \varepsilon$. The definition of ε-sample allows a sample to be arbitrarily dense anywhere on Σ. It only puts a lower bound on the density. Figure 1.10 illustrates a sample of a circle which is a 0.2-sample. By definition, it is also a 0.3-sample of the same.

A useful application of the Lipschitz Continuity Lemma 1.2 is that the distance between two points expressed in terms of the local feature size of one can be expressed in terms of that of the other.

Lemma 1.3 (Feature Translation). *For any two points x, y in Σ with $\|x - y\| \leq \varepsilon f(x)$ and $\varepsilon < 1$ we have*

(i) $f(x) \leq \frac{1}{1-\varepsilon} f(y)$ *and*
(ii) $\|x - y\| \leq \frac{\varepsilon}{1-\varepsilon} f(y)$.

Proof. We have

$$f(x) \leq f(y) + \|x - y\|$$
$$\text{or, } f(x) \leq f(y) + \varepsilon f(x).$$

For $\varepsilon < 1$ the above inequality gives

$$f(x) \leq \frac{1}{1-\varepsilon} f(y) \text{ proving (i).}$$

Plug the above inequality in $\|x - y\| \leq \varepsilon f(x)$ to obtain (ii). ∎

Uniform Sampling

The definition of ε-sample allows nonuniform sampling over Σ. A *globally uniform* sampling is more restrictive. It means that the sample is equally dense everywhere. Local feature size does not play a role in such sampling. There could be various definitions of globally uniform samples. We will say a sample

$P \subset \Sigma$ is *globally δ-uniform* if any point $x \in \Sigma$ has a point in P within $\delta > 0$ distance. In between globally uniform and nonuniform samplings, there is another one called the *locally uniform sampling*. This sampling respects feature sizes and is uniform only locally. We say $P \subset \Sigma$ is *locally* (ε, δ)-*uniform* for $\delta > 1 > \varepsilon > 0$ if each point $x \in \Sigma$ has a point in P within $\varepsilon f(x)$ distance and no point $p \in P$ has another point $q \in P$ within $\frac{\varepsilon}{\delta} f(p)$ distance. This definition does not allow two points to be arbitrarily close which may become a severe restriction for sampling in practice. So, there is an alternate definition of local uniformity. A sample P is *locally* (ε, κ)-*uniform* for some $\varepsilon > 0$ and $\kappa \geq 1$ if each point $x \in \Sigma$ has at least one and no more than κ points within $\varepsilon f(x)$ distance.

$\tilde{O}(\varepsilon)$ *notation*

Our analysis for different algorithms obviously involve the sampling parameter ε. To ease these analyses, sometimes we resort to $\tilde{O}$ notation which provides the asymptotic dependences on ε. A value is $\tilde{O}(\varepsilon)$ if there exist two constants $\varepsilon_0 > 0$ and $c > 0$ so that the value is less than $c\varepsilon$ for any positive $\varepsilon \leq \varepsilon_0$. Notice that $\tilde{O}$ notation is slightly different from the well-known big-O notation since the latter would require ε greater than or equal to ε_0.

1.3 Voronoi Diagram and Delaunay Triangulation

Voronoi diagrams and Delaunay triangulations are important geometric data structures that are built on the notion of "nearness." Many differential properties of curves and surfaces are defined on local neighborhoods. Voronoi diagrams and their duals, Delaunay triangulations, provide a tool to approximate these neighborhoods in the discrete domain. They are defined for a point set in any Euclidean space. We define them in two dimensions and mention the extensions to three dimensions since the curve and surface reconstruction algorithms as dealt in this book are concerned with these two Euclidean spaces. Before the definitions we state a nondegeneracy condition for the point set P defining the Voronoi and Delaunay diagrams. This nondegeneracy condition not only makes the definitions less complicated but also makes the algorithms avoid special cases.

Definition 1.4. *A point set* $P \subset \mathbb{R}^k$ *is nondegenerate if (i) the affine hull of any* ℓ *points from* P *with* $1 \leq \ell \leq k$ *is homeomorphic to* $\mathbb{R}^{\ell-1}$ *and (ii) no* $k+2$ *points are cospherical.*

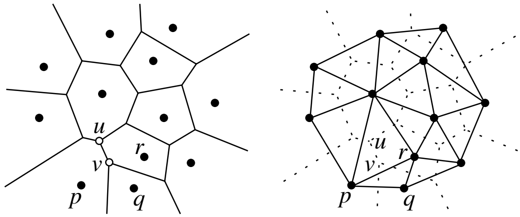


Figure 1.11. (a) The Voronoi diagram and (b) the Delaunay triangulation of a point set in the plane.

1.3.1 Two Dimensions

Let P be a set of nondegenerate points in the plane $\mathbb{R}^2$.

Voronoi Diagrams

The Voronoi cell V_p for each point $p \in P$ is given as

$$V_p = \{x \in \mathbb{R}^2 \mid d(x, P) = \|x - p\|\}.$$

In words, V_p is the set of all points in the plane that have no other point in P closer to it than p. For any two points p, q the set of points closer to p than q are demarked by the perpendicular bisector of the segment pq. This means the Voronoi cell V_p is the intersection of the closed half-planes determined by the perpendicular bisectors between p and each other point $q \in P$. An implication of this observation is that each Voronoi cell is a convex polygon since the intersection of convex sets remains convex.

Voronoi cells have *Voronoi faces* of different dimensions. A Voronoi face of dimension k is the intersection of $3 - k$ Voronoi cells. This means a k-dimensional Voronoi face for $k \leq 2$ is the set of all points that are equidistant from $3 - k$ points in P. A zero-dimensional Voronoi face, called *Voronoi vertex* is equidistant from three points in P, whereas a one-dimensional Voronoi face, called *Voronoi edge* contains points that are equidistant from two points in P. A Voronoi cell is a two-dimensional Voronoi face.

Definition 1.5. *The Voronoi diagram* Vor P *of* P *is the cell complex formed by Voronoi faces.*

Figure 1.11(a) shows a Voronoi diagram of a point set in the plane where u and v are two Voronoi vertices and uv is a Voronoi edge.

Some of the Voronoi cells may be unbounded with unbounded edges. It is a straightforward consequence of the definition that a Voronoi cell V_p is unbounded if and only if p is on the boundary of the convex hull of P. In Figure 1.11(a), V_p and V_q are unbounded and p and q are on the convex hull boundary.

Delaunay Triangulations

There is a *dual* structure to the Voronoi diagram Vor P, called the *Delaunay triangulation*.

Definition 1.6. *The Delaunay triangulation of P is a simplicial complex*

$$\mathrm{Del}\ P = \{\sigma = \mathrm{Conv}\, T \mid \bigcap_{p \in T \subseteq P} V_p \neq \emptyset\}.$$

In words, $k+1$ points in P form a Delaunay k-simplex in Del P if their Voronoi cells have nonempty intersection. We know that $k+1$ Voronoi cells meet in a $(2-k)$-dimensional Voronoi face. So, each k-simplex in Del P is dual to a $(2-k)$-dimensional Voronoi face. Thus, each Delaunay triangle pqr in Del P is dual to a Voronoi vertex where V_p, V_q, and V_r meet, each Delaunay edge pq is dual to a Voronoi edge shared by Voronoi cells V_p and V_q, and each vertex p is dual to its corresponding Voronoi cell V_p. In Figure 1.11(b), the Delaunay triangle pqr is dual to the Voronoi vertex v and the Delaunay edge pr is dual to the Voronoi edge uv. In general, when μ is a dual Voronoi face of a Delaunay simplex σ we say $\mu = \mathrm{dual}\,\sigma$ and conversely $\sigma = \mathrm{dual}\ \mu$.

A *circumscribing ball* of a simplex σ is a ball whose boundary contains the vertices of the simplex. The smallest circumscribing ball of σ is called its *diametric* ball. A triangle in the plane has only one circumscribing ball, namely the diametric one. However, an edge has infinitely many circumscribing balls among which the diametric one is unique, namely the one with the center on the edge.

A dual Voronoi vertex of a Delaunay triangle is equidistant from its three vertices. This means that the center of the circumscribing ball of a Delaunay triangle is the dual Voronoi vertex. It implies that no point from P can lie in the interior of the circumscribing ball of a Delaunay triangle. These balls are called *Delaunay*. A ball is *empty* if its interior does not contain any point from P. Clearly, the Delaunay balls are *empty*. The converse also holds.

Property 1.1 (Triangle Emptiness). *A triangle is in the Delaunay triangulation if and only if its circumscribing ball is empty.*

The triangle emptiness property of Delaunay triangles also implies a similar emptiness for Delaunay edges. Clearly, each Delaunay edge has an empty circumscribing ball passing through its endpoints. It turns out that the converse is also true, that is, any edge pq with an empty circumscribing ball must also be in the Delaunay triangulation. To see this, grow the empty ball of pq always keeping p, q on its boundary. If it never meets any other point from P, the edge pq is on the boundary of Conv P and is in the Delaunay triangulation since V_p and V_q has to share an edge extending to infinity. Otherwise, when it meets a third point, say r from P, we have an empty circumscribing ball passing through p, q, and r. By the triangle emptiness property pqr must be in the Delaunay triangulation and hence the edge pq.

Property 1.2 (Edge Emptiness). *An edge is in the Delaunay triangulation if and only if the edge has an empty circumscribing ball.*

The Delaunay triangulation form a planar graph since no two Delaunay edges intersect in their interiors. It follows from the property of planar graphs that the number of Delaunay edges is at most $3n - 6$ for a set of n points. The number of Delaunay triangles is at most $2n - 4$. This means that the dual Voronoi diagram also has at most $3n - 6$ Voronoi edges and $2n - 4$ Voronoi vertices. The Voronoi diagram and the Delaunay triangulation of a set of n points in the plane can be computed in $O(n \log n)$ time and $O(n)$ space.

Restricted Voronoi Diagrams

When the input point set P is a sample of a curve or a surface Σ, the Voronoi diagram Vor P imposes a structure on Σ. It turns out that this diagram plays an important role in reconstructing Σ from P. Formally, a restricted Voronoi cell $V_p|_\Sigma$ is defined as the intersection of the Voronoi cell V_p in Vor P with Σ, that is,

$$V_p|_\Sigma = V_p \cap \Sigma \quad \text{where } p \in P.$$

Similar to the Voronoi faces, we can define *restricted Voronoi faces* as the intersection of the restricted Voronoi cells. They can also be viewed as the intersection of Voronoi faces with Σ. In Figure 1.12(a), the white circles represent restricted Voronoi faces of dimension zero. The curve segments between them are restricted Voronoi faces of dimension one which are restricted Voronoi cells in this case. Notice that the restricted Voronoi cell $V_p|_\Sigma$ in the figure consists of two curve segments whereas $V_r|_\Sigma$ consists of a single curve segment.

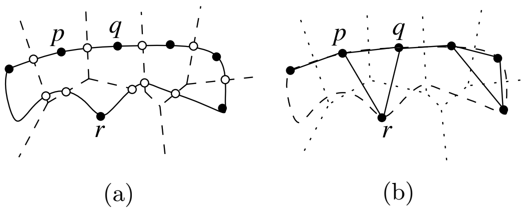


Figure 1.12. (a) Restricted Voronoi diagram for a point set on a curve and (b) the corresponding restricted Delaunay triangulation.

Definition 1.7. *The restricted Voronoi diagram* Vor $P|_\Sigma$ *of* P *with respect to* Σ *is the collection of all restricted Voronoi faces.*

Restricted Delaunay Triangulations.

As with Voronoi diagrams we can define a simplicial complex dual to a restricted Voronoi diagram Vor $P|_\Sigma$.

Definition 1.8. *The restricted Delaunay triangulation of* P *with respect to* Σ *is a simplicial complex* Del $P|_\Sigma$ *where a* k*-simplex with* $k+1$ *vertices,* $R \subseteq P$*, is in this complex if and only if*

$$\bigcap V_p|_\Sigma \neq \emptyset, \textit{ for } p \in R.$$

In words, a simplex in Del P is in Del $P|_\Sigma$ if and only if its dual Voronoi face intersects Σ. The simplicial complex Del $P|_\Sigma$ is called the *restricted Delaunay triangulation* of P with respect to Σ. Figure 1.12(b) shows the restricted Delaunay triangulation for the restricted Voronoi diagram in (a). The vertex p is connected to q and r in the restricted Delaunay triangulation since $V_p|_\Sigma$ meets both $V_q|_\Sigma$ and $V_r|_\Sigma$. However, the triangle pqr is not in the triangulation since $V_p|_\Sigma$, $V_q|_\Sigma$ and $V_r|_\Sigma$ do not meet at a point.

1.3.2 Three Dimensions

We chose the plane to explain the concepts of the Voronoi diagrams and the Delaunay triangulations in the previous subsection. However, these concepts extend to arbitrary dimensions. We will mention these extensions for three dimensions which will be important for later expositions.

Voronoi cells of a point set P in $\mathbb{R}^3$ are three-dimensional convex polytopes some of which are unbounded. There are four types of Voronoi faces: Voronoi vertices, Voronoi edges, Voronoi facets, and Voronoi cells in increasing order of dimension starting with zero and ending with three. Four Voronoi cells meet

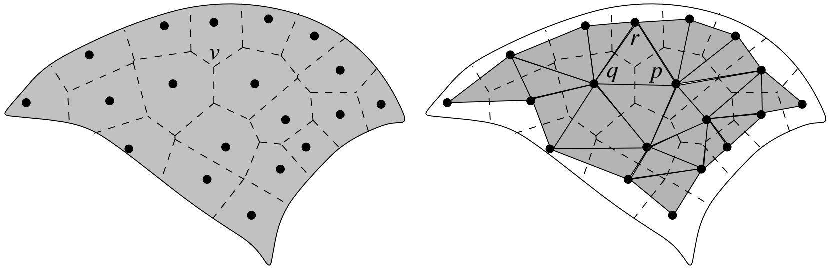


Figure 1.13. (a) The restricted Voronoi diagram and (b) the restricted Delaunay triangulation for a sample on a surface.

at a Voronoi vertex which is equidistant from four points in P. Three Voronoi cells meet at a Voronoi edge, and two Voronoi cells meet at a Voronoi facet.

The Delaunay triangulation of P contains four types of simplices dual to each of the four types of Voronoi faces. The vertices are dual to the Voronoi cells, the Delaunay edges are dual to the Voronoi facets, the Delaunay triangles are dual to the Voronoi edges, and the Delaunay tetrahedra are dual to the Voronoi vertices. The circumscribing ball of each tetrahedron is empty. Conversely, any tetrahedron with empty circumscribing ball is in the Delaunay triangulation. Further, each Delaunay triangle and edge has an empty circumscribing ball. Conversely, an edge or a triangle belongs to the Delaunay triangulation if there exists an empty ball circumscribing it.

The number of edges, triangles, and tetrahedra in the Delaunay triangulation of a set of n points in three dimensions can be $O(n^2)$ in the worst case. By duality the Voronoi diagram can also have $O(n^2)$ Voronoi faces. Both of the diagrams can be computed in $O(n^2)$ time and space.

We can define the restricted Voronoi diagram and its dual restricted Delaunay triangulation for a point sample on a surface in $\mathbb{R}^3$ in the same way as we did for a curve in $\mathbb{R}^2$. Figure 1.13 shows the restricted Voronoi diagram and its dual restricted Delaunay triangulation for a set of points on a surface. The triangle pqr is in the restricted Delaunay triangulation since $V_p|_\Sigma$, $V_q|_\Sigma$, and $V_r|_\Sigma$ meet at a common point v.

1.4 Notes and Exercises

The books by Munkres [71] and Weeks [81] are standard books on point set topology where the definitions of topological spaces and maps can be found in details. Munkres [72] and Stillwell [79] are good sources for algebraic and combinatorial topology where simplicial complexes and their use in triangulation of topological spaces are described. A number of useful definitions in topology are collected in the survey paper by Dey, Edelsbrunner, and Guha [29].

The concept of the medial axis was introduced by Blum [14] in the context of image analysis. Variants of this concept as discussed in the Medial axis section appeared later. Choi, Choi, and Moon [25] established that the medial axis of a piecewise real analytic curve is a finite graph. Chazal and Soufflet [21] extended this result to semianalytic curves. See Matheron [66], Wolter [82], and Chazal and Lieutier [20] for more expositions on the medial axis.

The concept of local feature size was first used by Ruppert [76] for meshing a polygonal domain with guaranteed qualities. His definition was somewhat different from the one described in this chapter. The local feature size as defined in this chapter and used throughout the book appeared in Amenta, Bern, and Eppstein [5].

The Voronoi diagrams and the Delaunay triangulations are well-known data structures named after Georges Voronoi [80] and Boris Delaunay [28] respectively. They are frequently used in various computational problems. A good source for the materials on the Delaunay triangulation is Edelsbrunner [43]. Voronoi diagrams are discussed in great detail in Okabe, Boots, and Sugihara [74]. Various references to the algorithms for computing Voronoi diagrams and Delaunay triangulations can be found in the *Handbook of Discrete and Computational Geometry* [50]. The concepts of the restricted Voronoi and Delaunay diagrams were used by Chew [24] for meshing surfaces. Edelsbrunner and Shah [48] formalized the notion.

Exercises

1. Construct an explicit deformation retraction of $\mathbb{R}^k \setminus \{0\}$ onto $\mathbb{S}^{k-1}$. Also, show $\mathbb{R}^k \cup \{\infty\}$ is homeomorphic to $\mathbb{S}^k$.
2. Deduce that homeomorphism is an equivalence relation. Show that the relation of homotopy among maps is an equivalence relation.
3. Construct a triangulation of $\mathbb{S}^2$ and verify that $v - e + f = 2$ where v is the number of vertices, e is the number of edges, and f is the number of triangles. Prove that the number $v - e + f$ (Euler characteristic) is always 2 for any triangulation of $\mathbb{S}^2$.
4. Let p be a vertex in Del P in three dimensions. Show that a point $x \in V_p$ if and only if $\|p - x\| \leq \|q - x\|$ for each vertex q where pq is a Delaunay edge.
5. Show that for any Delaunay simplex σ and its dual Voronoi face $\mu = \operatorname{dual} \sigma$, the affine hulls aff μ and aff σ intersect orthogonally.

6. An edge e in a triangulation $T(P)$ of a point set $P \subset \mathbb{R}^2$ is called *locally Delaunay* if e is a convex hull edge or the circumscribing ball of one triangle incident to e does not contain the other triangle incident to e completely inside. Show that $T(P) = \mathrm{Del}\, P$ if and only if each edge of $T(P)$ is locally Delaunay.
7. Given a point set $P \subset \mathbb{R}^2$, an edge connecting two points p, q in P is called a nearest neighbor edge if no point in P is closer to q than p is. Show that pq is a Delaunay edge.
8. Given a point set $P \subset \mathbb{R}^2$, an edge connecting two points in P is called *Gabriel* if its diametric ball is empty. The Gabriel graph for P is the graph induced by all Gabriel edges. Give an $O(n \log n)$ algorithm to compute the Gabriel graph for P where P has n points.
9. Let pq be a Delaunay edge in Del P for a point set $P \subset \mathbb{R}^3$. Show that if pq does not intersect its dual Voronoi facet $g = \mathrm{dual}\, pq$, the line of pq does not intersect g either.
10. For $\alpha > 0$, a function $f : \Sigma \to \mathbb{R}$ is called α-Lipschitz if $f(x) \le f(y) + \alpha \|x - y\|$ for any two points x, y in Σ. Given an arbitrary function $f : \Sigma \to \mathbb{R}$, consider the functions

$$f_m(x) = \min_{p \in \Sigma} \{ f(p) + \alpha \|x - p\| \},$$
$$f_M(x) = \max_{p \in \Sigma} \{ f(p) - \alpha \|x - p\| \}.$$

Show that both f_m and f_M are α-Lipschitz.
11. Consider the functions ρ_i and ρ_o as in Section 1.2.2. Show that these functions may be continuous but not 1-Lipschitz.