0.1 Basics

Topological space: defined as a set of points, along with a set of neighborhoods.
Satisfy a set of axioms relating the points and the neighborhood.
Motivation: most general notion of a mathematical space that allows for

- continuity,
- connectedness,
- convergence

Extension: manifolds, metric spaces, etc
Commonly used: defined in terms of open sets.
More intuitive: neighborhoods.

X a set, together with $\mathbf{N} : X \mapsto 2^{2^X}$, $N \in \mathbf{N}(x)$ is a neighborhood of x, satisfying

1. Point in its own neighborhood;
2. Superset of neighborhood is a neighborhood, $N \subseteq M \implies M \in \mathbf{N}(x)$;
3. Intersection of neighborhoods is a neighborhood, $\forall N, M \in \mathbf{N}(x), N \cap M \in \mathbf{N}(x)$
4. $\forall N \in \mathbf{N}(x) \exists M, s.t. y \in M \implies N \in \mathbf{N}(y)$

Open sets: $x \in U \implies U \in \mathbf{N}(x)$

Equivalent definition with respect to open sets:
$(X, \tau), \tau \subseteq 2^{2^X}$, satisfying:

1. $\emptyset, X \subseteq \tau$
2. $N_i \in \tau, \bigcup_i N_i \in \tau$
3. $N_i \in \tau, \bigcap_i N_i \in \tau, i$ finite

These are called open sets.

Continuous: the inverse image of every open set is an open set.

Homotopy: Intuitively from the idea of continuous deformation; is strictly weaker than homeomorphism.

0.2 Homology

Intuitive view:

Path: continuous map $[0,1] \mapsto X$; $x \sim y$ if $\gamma(0) = x, \gamma(1) = y$.
This is clearly an equivalence relation. Therefore, the path connected components of X are equivalence classes under $\sim$.

0.2.1 Simplicial Complexes

General idea is that simplicial complexes extend the notion of graph to include higher dimensional components in it.
Turn simplicial complexes to topological spaces: "embed" it into Euclidean space.
Embedding: map points to coordinates and all points affinely independent.
Induced map: $\hat{f} : K \to 2^{\mathbb{R}^d}, \{v_0, v_1, ..., v_r\} \mapsto Conv(f(v_0), ..., f(v_r))$
All embeddings $f : K \to \mathbb{R}^d, g : K \to \mathbb{R}^{d'}$, $\hat{f}$ and $\hat{g}$ are homeomorphic.
Underlying space, $|K|$, the image of K through embedding, unique up to homeomorphism.
Triangulability: X is triangulable if $\exists K, h : X \to |K|$, h is homeomorphism.
Simplicial map: combinatorial equivalence of continuous map.
Topological realization of a map: simplicial $f : K \to L$ induces continuous $|f| : |K| \to |L|$
Reverse is not necessarily true.
Simplicial Approximation: continuous $f : |K| \to |L|$ is homotopic to $|f'| : |K| \to |L|$, where $f' : K' \to L$, for some simplicial subdivision K' of K.
Intuition: by dividing the simplicial complex sufficiently many times, we can approximate the topological space.

0.2.2 Simplicial Homology

Orientation: Ordering of vertices, unique up to even permutation, negative by odd permutation.

Chains

K finite simplicial complex and k a fixed field. Given $r \in \mathbb{N}$, we are interested in the $k-$linear combinations (formal sums) of $r-$simplices in K.
The linear space of all the formal sums for each r is a chain space of $k-$chains.
Note: Seems that the field k can be relaxed to a ring. Can study later.

Boundary Operator

Remove one vertex from a simplex, we get a face of the simplex. The linear combination together with interchanging orientation is the boundary.
$\partial_r : C_r(K, k) \to C_{r-1}(K, k)$
$[v_0, ..., v_r] \mapsto \sum_{j=0}^{r} (-1)^j [v_0, ...\hat{v_j} ..., v_r]$
Clearly ∂_r is distributive.
Note:$\partial^2 = 0$, and therefore is nilpotent. (Actually it's $\partial_r \circ \partial_{r+1} = 0$.

Homology Groups

Important motivation: want to find "cycles modulo the boundaries".
Note: need to further think about this.

r-cycles: $Z_r(K, k) := \ker\partial$
r-boundaries: $Z_r(K, k) := \mathrm{im}\partial_{r+1}$
Homology group: $Z_r(K, k)/Z_r(K, k)$

Algorithm for Homology

Since it's a vector space, it's isomorphic to k^{β_r},
Note: when relaxed to ring, it's a module, and therefore we still have the fundamental theorem for modules to decompose to a torsion part and a torsion-free part. Still have some kind of β_r.
Note: everything is linear space / linear transformation.
Matrix form M_r of ∂_r for each r:
$\#K_r$ columns and $\#K_{r-1}$ rows, $\beta_r = \#K_r - \mathrm{rank}M_r - \mathrm{rank}M_{r+1}$, just compute ranks to get the dimension, which I have learned and don't want to go into details.

Morphisms

Operator on spaces: $H_r : K \mapsto H_r(K, k)$, which we want to extend to maps as well.
Idea of a functor? Some category stuff?
Chain level: simplicial map induces a chain map. Details omitted.
The chain map commutes with boundary operator.
Functoriality:

0.2.3 From simplicial complexes to topological spaces

Theorem: X triangulable, then, $\forall K, L, H_r(K, k) \simeq H_r(L, k)$.
Conclusion: homology groups of triangulable spaces, and the morphisms between them, are uniquely defined, for different ways of triangulation. Moreover, morphisms are invariant under homotopy.
Corollary: $X \sim Y \implies H_r(X, k) \simeq H_r(Y, k)$.
However, homology does not completely characterize the topology of a space. Thus still much weaker than homeomorphism.

0.3 Exercise

Compute the homology groups, and in particular, the Betti numbers.
Often we only want the Betti numbers, so choose the simplest field, $\mathbb{Z}_2$, whereby we can ignore the orientation.
The key is

1. triangulation
2. identify the structures
3. computationally, matrix and linear algebra

Questions:

1. Circle, $\mathbb{S}^1 : \beta_0 = 1, \beta_1 = 1$
2. Disk, $\mathbb{B}^2 : \beta_0 = 1, \beta_1 = 1, \beta_2 = 0, etc$, same homology groups as a single point. In fact, homotopy equivalent to a point.
3. Cylinder, $\mathbb{S}^1 \times [0, 1] : \beta_0 = 1, \beta_1 = 1, \beta_2 = 0$, homotopy equivalent to a circle.
4. Sphere, $\mathbb{S}^2 : \beta_0 = 1, \beta_1 = 0, \beta_2 = 1$
5. Ball, $\mathbb{B}^3 : \beta_0 = 1, \beta_1 = 0, \beta_2 = 0, etc$, homotopy equivalent to a single point.
6. Torus, see online for triangulation, $\mathbb{T} : \beta_0 = 1, \beta_1 = 2, \beta_2 = 1, \beta_3 = 0$
7. Homology for sphere $\mathbb{S}^d$, $\beta_0 = 1, \beta_i = 0, \beta_d = 1$

Brouwer's fixed point theorem:
$\forall$continuous $f : \mathbb{B}^2 \to \mathbb{B}^2, \exists p \in \mathbb{B}^2, f(p) = p$.
Hairy ball theorem:
For d even, $\forall$continuous tangent vector field $V : \mathbb{S}^d \to \mathbb{R}^d, \exists p \in \mathbb{S}^d$, s.t. $V(p) = 0$.