0.1 Basics

Topological space: defined as a set of points, along with a set of neighborhoods.
Satisfy a set of axioms relating the points and the neighborhood.
Motivation: most general notion of a mathematical space that allows for

- continuity,
- connectedness,
- convergence

Extension: manifolds, metric spaces, etc
Commonly used: defined in terms of open sets.
More intuitive: neighborhoods.

X a set, together with $\mathbf{N} : X \mapsto 2^{2^X}$, $N \in \mathbf{N}(x)$ is a neighborhood of x, satisfying

1. Point in its own neighborhood;
2. Superset of neighborhood is a neighborhood, $N \subseteq M \implies M \in \mathbf{N}(x)$;
3. Intersection of neighborhoods is a neighborhood, $\forall N, M \in \mathbf{N}(x), N \cap M \in \mathbf{N}(x)$
4. $\forall N \in \mathbf{N}(x) \exists M, s.t. y \in M \implies N \in \mathbf{N}(y)$

Open sets: $x \in U \implies U \in \mathbf{N}(x)$

Equivalent definition with respect to open sets:
$(X, \tau), \tau \subseteq 2^{2^X}$, satisfying:

1. $\emptyset, X \subseteq \tau$
2. $N_i \in \tau, \bigcup_i N_i \in \tau$
3. $N_i \in \tau, \bigcap_i N_i \in \tau, i$ finite

These are called open sets.

Continuous: the inverse image of every open set is an open set.

Homotopy: Intuitively from the idea of continuous deformation; is strictly weaker than homeomorphism.

0.2 Homology

Intuitive view:

Path: continuous map $[0, 1] \mapsto X$; $x \sim y$ if $\gamma(0) = x, \gamma(1) = y$.
This is clearly an equivalence relation. Therefore, the path connected components of X are equivalence classes under $\sim$.

0.2.1 Simplicial Complexes

General idea is that simplicial complexes extend the notion of graph to include higher dimensional components in it.
Turn simplicial complexes to topological spaces: "embed" it into Euclidean space.
Embedding: map points to coordinates and all points affinely independent.
Induced map: $\hat{f} : K \to 2^{\mathbb{R}^d}, \{v_0, v_1, ..., v_r\} \mapsto Conv(f(v_0), ..., f(v_r))$
All embeddings $f : K \to \mathbb{R}^d, g : K \to \mathbb{R}^{d'}$, $\hat{f}$ and $\hat{g}$ are homeomorphic.