# CSCI-H 335: Homework 2: Number representation

Siyi Xian

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1. Answer to question 1:

   Assume the largest number for $N$ digits is $L(N)$.
   Base case ($N = 0$):
   For binary number, $L(0) = 0 = 1 - 1 = r^0 - 1 = r^N - 1$.
   Induction case :
   Assume $L(N) = r^N - 1$ for some $r, N \in \mathbb{N}$.

   $$\begin{aligned} L(N+1) &= r^{N+1-1} \cdot (r-1) + L(N) && \text{Add largest number in r-radix to highest digit.} \\ &= r^N - 1 + r^N \cdot (r-1)N && \text{By IH} \\ &= r^N - 1 + r^{N+1} - r^N \\ &= r^{N+1} - 1 \end{aligned}$$

2. Answer to question 2

   Because $B$ is magnitude less than $2^{N2}$, $-B$ can be represent in $N$ bits. We define the twos complement of a binary number by inverting its bits and adding 1. Thus $-(b_{N-1}..b_0)_{\bar{2}} = (\overline{b_{N-1}..b_0})_{\bar{2}} + 1$.

3. Answer to question 3

   Assume $I$ is $k$ digits binary number, $m > k$, and $a_1$ is 0 or 1.
   Positive case:

   $$I = \sum_{0}^{k} a_i \cdot 2^i = 0 + \sum_{0}^{k} a_i \cdot 2^i = \sum_{k+1}^{m} 0 \cdot 2^i + \sum_{0}^{k} a_i \cdot 2^i \tag{1}$$

   Thus, it's value prevente for positive value because the highest digit of positive number always be 0.
   Negative case:
   For all two's complement number, highest digit is always 1 if the given number is in range. Inverse 1 we get 0, and we proved in positive case. Thus extension will always keep value same.

4. Answer to question 4

   Left-shift:
   Assume $b_{-1} = 0$

   $$(b_{N-1}..b_0)_{\bar{2}} \cdot 2 = (-b_{N-1} \cdot 2^{N-1} + \sum_{i=0}^{N-2} b_i \cdot 2^i) \cdot 2 = -b_{N-1} \cdot 2^{(N+1)-1} + \sum_{i=-1}^{(N+1)-2} b_{i+1} \cdot 2^i = (b_{N-1}..b_0, b_{-1})_{\bar{2}} = (b_{N-1}..b_0, 0)_{\bar{2}} \tag{2}$$

Arithmetic right-shift:

$$(b_{N-1}..b_0)_{\bar{2}}/2 = (-b_{N-1}\cdot 2^{N-1}+\sum_{i=0}^{N-2} b_i\cdot 2^i)/2 = -b_{N-1}\cdot 2^{N-3}+\sum_{i=1}^{N-3} b_i\cdot 2^i = (b_{N-1}..b_1)_{\bar{2}} = (b_{N-1}, b_{N-1}..b_1)_{\bar{2}} \tag{3}$$

5. Answer to question 5

(a)

| Repeated Division (mod 2) | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Quotients | 273 | 136 | 68 | 34 | 17 | 8 | 4 | 2 | 1 |
| Remainders | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |

$(273)_{10} = (100010001)_2$

(b)

| Repeated Division (mod 2) | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Quotients | 1327 | 663 | 331 | 165 | 82 | 41 | 20 | 10 | 5 | 2 | 1 |
| Remainders | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 1 |

$(1327)_{10} = (10101001111)_2$

(c)

| Repeated Division (mod 2) | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Quotients | 31673 | 15836 | 7918 | 3959 | 1979 | 989 | 494 | 247 | 123 | 61 | 30 | 15 | 7 | 3 | 1 |
| Remainders | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 |

$(31673)_{10} = (111101110111001)_2$

(d)

| Repeated Division (mod 2) | | | | | | | |
|---|---|---|---|---|---|---|---|
| Quotients | 107 | 53 | 26 | 13 | 6 | 3 | 1 |
| Remainders | 1 | 1 | 0 | 1 | 0 | 1 | 1 |

$(107)_{10} = (1101011)_2$

6. Answer to question 6

(a)

| | | | |
|---|---|---|---|
| $\boxed{0}$ | $\boxed{0}$ | $\boxed{0}$ | |
| | 2 | 3 | 3 |
| - | | 2 | 1 |
| | 2 | 1 | 2 |

$233 - 21 = 212$

(b)

| | | | | |
|---|---|---|---|---|
| $\boxed{0}$ | $\boxed{1}$ | $\boxed{1}$ | $\boxed{1}$ | |
| | 1 | 0 | 0 | 1 |
| - | | 9 | 0 | 2 |
| | | | 9 | 9 |

$1001 - 902 = 99$

(c)

| | | | |
|---|---|---|---|
| $\boxed{0}$ | $\boxed{0}$ | $\boxed{0}$ | |
| | 1 | 7 | 3 |
| - | 1 | 2 | 3 |
| | | 5 | 0 |

$173 - 123 = 50$

(d)

$$\begin{array}{ccccc} \boxed{0} & \boxed{0} & \boxed{0} & \boxed{1} & \\ & 2 & 7 & 8 & 0 \\ - & & 1 & 7 & 7 \\ \hline & 1 & 6 & 0 & 3 \end{array}$$

$2780 - 177 = 1603$

7. Answer to question 6

(a)

| Repeated Division (mod 2) | | | | | | | |
|---|---|---|---|---|---|---|---|
| Quotients | 111 | 55 | 27 | 13 | 6 | 3 | 1 |
| Remainders | 1 | 1 | 1 | 1 | 0 | 1 | 1 |

$(111)_{10} = (01101111)_2$
$(-111)_{10} = (10010000 + 1)_2 = (10010001)_2$

(b)

| Repeated Division (mod 2) | | | | | | |
|---|---|---|---|---|---|---|
| Quotients | 59 | 29 | 14 | 7 | 3 | 1 |
| Remainders | 1 | 1 | 0 | 1 | 1 | 1 |

$(59)_{10} = (00111011)_2$
$(-59)_{10} = (11000100 + 1)_2 = (11000101)_2$

(c)

| Repeated Division (mod 2) | | | | | | | |
|---|---|---|---|---|---|---|---|
| Quotients | 105 | 52 | 26 | 13 | 6 | 3 | 1 |
| Remainders | 1 | 0 | 0 | 1 | 0 | 1 | 1 |

$(105)_{10} = (01101001)_2$
$(-105)_{10} = (10010110 + 1)_2 = (10010111)_2$

(d)

| Repeated Division (mod 2) | | | | | |
|---|---|---|---|---|---|
| Quotients | 31 | 15 | 7 | 3 | 1 |
| Remainders | 1 | 1 | 1 | 1 | 1 |

$(31)_{10} = (00011111)_2$

(e)

| Repeated Division (mod 2) | | | | | | | |
|---|---|---|---|---|---|---|---|
| Quotients | 127 | 63 | 31 | 15 | 7 | 3 | 1 |
| Remainders | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

$(127)_{10} = (01111111)_2$
$(-127)_{10} = (10000000 + 1)_2 = (10000001)_2$

8. Answer to question 8:

Base case ($i = 0$):

$V((c_{i+1}, s_i)_2) = V((c_1, s_0)_2) = V((a_0)_2) + V((b_0)_2) + V((c_0)_2) = V((a_i)_2) + V((b_i)_2) + V((c_i)_2)$

Induction case :

Assume $V((c_N, s_{N-1}, ..., s_0)_2) = V((a_{N-1}, ..., a_0)_2) + V((b_{N-1}, ..., b_0)_2) + V((c_0)_2)$.

$V((c_{N+2}, s_{N+1}, ..., s_0)_2 = V((a_N, 0, ..., 0)_2) + V((b_N, ..., b_0)_2) + V((c_N, 0, ..., 0)_2) + V((c_N, s_{N-1}, ..., s_0)_2$
$= V((a_N, 0, ..., 0)_2) + V((b_N, ..., b_0)_2) + V((c_N, 0, ..., 0)_2) + V((a_{N-1}, ..., a_0)_2) + V((b_{N-1}, ..., b_0)_2) + V((c_0)_2$ By IH

$= V((a_N, ..., a_0)_2) + V((b_N, ..., b_0)_2) + V((c_0)_2)$