

## Challenge1

Defining the following two proposition.

$a$  : *A is knight*

$b$  : *B, C are knaves*

$a \Rightarrow b$  : *if A is a knight then his two friends B and C here are knaves*

We then generating the truth table as follows:

$a$	$a \Rightarrow b$	$b$	Follow logic rule?
0	0	0	No
0	0	1	No
1	1	0	No
1	1	1	Yes

From the table we have the following situation:

if  $a$  is false, that means A is not a knight. So he has to tells lies. This means the statement  $a \Rightarrow b$  has to be false. However if  $a$  is false then  $a \Rightarrow b$  cannot be false in any case. Therefore,  $a$  cannot be false, so A has to be a knight.

Now Since A is a knight,  $a \Rightarrow b$  has to be true, because knight always tells the truth. Since  $a$  and  $a \Rightarrow b$  are both true, statement  $b$  has to be true.

Therefore, B and C are Knaves. A is a knight.

## Challenge 2

1. The CNF of  $\varphi$  is:

$$\begin{aligned}
 & \neg(((P \Rightarrow S) \wedge (Q \Rightarrow R) \wedge (R \Rightarrow P)) \Rightarrow S)) \\
 & \neg(\neg((P \Rightarrow S) \wedge (Q \Rightarrow R) \wedge (R \Rightarrow P)) \vee S) \\
 & ((\neg P \vee S) \wedge (\neg Q \vee R) \wedge (\neg R \vee P)) \wedge \neg S \\
 & \neg P \wedge \neg S \wedge (\neg Q \vee R) \wedge (\neg R \vee P) \\
 & \neg P \wedge \neg S \wedge (\neg Q \vee R) \wedge \neg R \\
 & \neg P \wedge \neg S \wedge \neg Q \wedge \neg R
 \end{aligned}$$

2.  $\varphi$  is non-valid by signing  $P, S, R, Q$  to False.

So the formula would become:

$$\begin{aligned}
 & ((F \Rightarrow F) \wedge (F \Rightarrow F) \wedge (F \Rightarrow F)) \Rightarrow F \\
 & (T \wedge T \wedge T \Rightarrow F) \\
 & \text{False}
 \end{aligned}$$

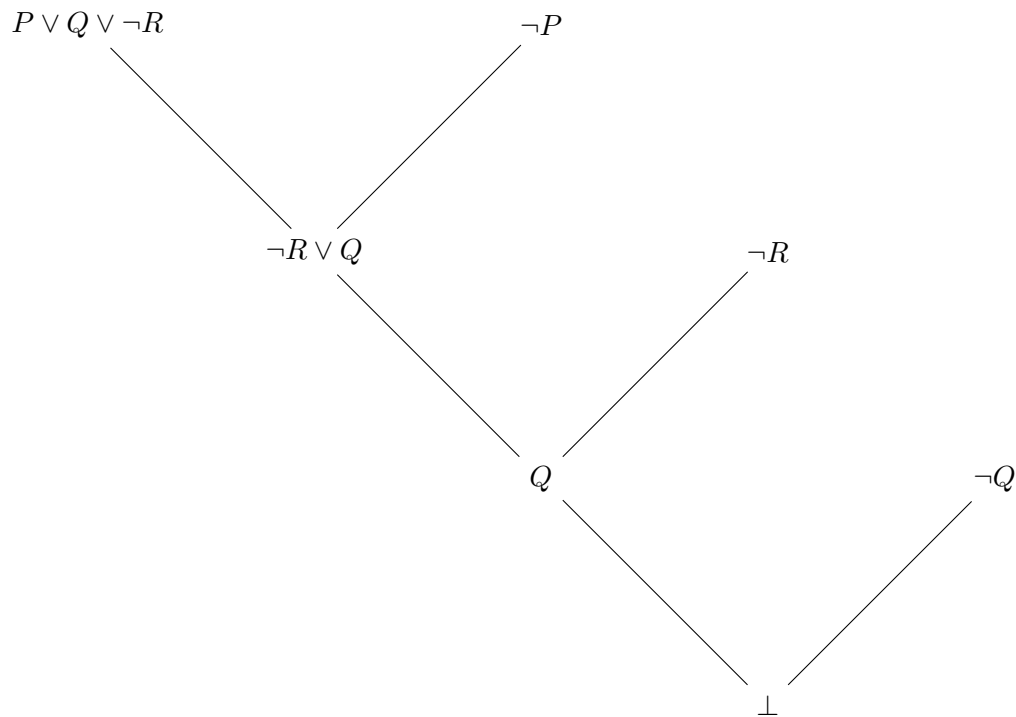
Therefore,  $\varphi$  is non-valid as it can be made to False.

3.  $\neg\psi$  in CNF form is:

$$\begin{aligned}
 & \neg((((P \vee Q) \Rightarrow S) \wedge (\neg P \Rightarrow (R \Rightarrow Q)) \wedge (R \vee S)) \Rightarrow S) \\
 & \neg(\neg(((P \vee Q) \Rightarrow S) \wedge (\neg P \Rightarrow (R \Rightarrow Q)) \wedge (R \vee S)) \vee S) \\
 & ((P \vee Q) \Rightarrow S) \wedge (\neg P \Rightarrow (R \Rightarrow Q)) \wedge (R \vee S) \wedge \neg S \\
 & (\neg(P \vee Q) \vee S) \wedge (P \vee (\neg R \vee Q)) \wedge (R \vee S) \wedge \neg S \\
 & (\neg P \wedge \neg Q \wedge \neg S) \wedge (P \vee (\neg R \vee Q)) \wedge (R \vee S) \\
 & \neg P \wedge \neg Q \wedge \neg S \wedge (P \vee \neg R \vee Q) \wedge R
 \end{aligned}$$

4. We prove  $\psi$  is valid by deriving  $\perp$  using refutation on  $\neg\psi$ :

The refutation tree is as followed:



Therefore, from refutation tree, we know  $\psi$  is valid.

### Challenge3

The non-valid case is when we set following interpretation to the formula:

$P(x, y) : x \text{ does not equal to } y$   
 $h(x) : \text{return } x \text{ itself}$

with the domain of  $x$  and  $y$  to be:

$x$ 's domain is all even number in  $R$   
 $y$ 's domain is all odd number in  $R$

So the equation would become:

$[(False \Rightarrow False) \Rightarrow (False \wedge False)]$   
 $True \Rightarrow False \equiv False$

Now the Satisfiable case is we set  $P(x, y) = True$ , whatever  $x$  and  $y$  is. So the formula would become

$(True \Rightarrow True) \Rightarrow (True \wedge True)$

which is True. This means the formula is satisfiable.

Therefore the formula is non-valid but satisfiable.

### Challenge4

1.  $S_1 : \forall x (S(x) \wedge \forall y (\neg P(y, x))) \Rightarrow H(x)$
2.  $S_2 : \forall x (S(x) \wedge (\forall y (P(y, x) \Rightarrow R(y)))) \Rightarrow H(x)$
3.  $S_2$  in CNF form is:

$$\begin{aligned}
 & \forall x \neg(S(x) \wedge (\forall y (P(y, x) \Rightarrow R(y)))) \vee H(x) \\
 & \forall x \neg S(x) \vee \neg(\forall y (\neg P(y, x) \vee R(y))) \vee H(x) \\
 & \forall x \neg S(x) \vee \exists y (P(y, x) \wedge \neg R(y)) \vee H(x) \\
 & \forall x \neg S(x) \vee (P(f(x), x) \wedge \neg R(f(x))) \vee H(x) \\
 & \forall x (\neg S(x) \vee H(x) \vee P(f(x), x)) \wedge (\neg S(x) \vee H(x) \vee \neg R(x))
 \end{aligned}$$

Therefore, the clausal form is:

$$\{\{\neg S(x) \vee H(x) \vee P(f(x), x)\}, \{\neg S(x) \vee H(x) \vee \neg R(x)\}\}$$

4.  $\neg S_1$  in CNF form is:

$$\begin{aligned}
 & \neg(\forall x (S(x) \wedge \forall y (\neg P(y, x))) \Rightarrow H(x)) \\
 & \neg(\forall x \neg(S(x) \wedge \forall y (\neg P(y, x))) \vee H(x)) \\
 & \exists x S(x) \wedge \forall y \neg P(y, x) \wedge \neg H(x) \\
 & S(a) \wedge \neg P(y, a) \wedge \neg H(a)
 \end{aligned}$$

Therefore, the clausal form is :

$$\{\{S(a)\}, \{\neg P(y, a)\}, \{\neg H(a)\}\}$$

5. We now refute  $S_2 \wedge \neg S_1$  to get  $\perp$ , thus prove  $S_1$  is a logical Consequence of  $S_2$ : ‘ The refutation tree is as follows:

