Challenge1

Defining the following two proposition.

$$
\begin{aligned}
&a: \ A \ is \ knight\\
&b: \ B, \ C \ are \ knaves\\
&a \Rightarrow b: \ if \ A \ is \ a \ knight \ then \ his \ two \ friends \ B \ and \ C \ here \ are \ knaves
\end{aligned}
$$

We then generating the truth table as follows:

a	$a \Rightarrow b$	b	$Follow\ logic\ rule?$
0	0	0	No
0	0	1	No
1	1	0	No
1	1	1	Yes

From the table we have the following situation:
if a is false, that means A is not a knight. So he has to tells lies. This means the statement $a \Rightarrow b$ has to be false. However if a is false then $a \Rightarrow b$ cannot be false in any case. Therefore, a cannot be false, so A has to be a knight.

Now Since A is a knight, $a \Rightarrow b$ has to be true, because knight always tells the truth. Since a and $a \Rightarrow b$ are both true, statement b has to be true.
Therefore, B and C are Knaves. A is a knight.

Challenge 2

1. The CNF of φ is:

$$
\begin{aligned}
&\neg(((P \Rightarrow S) \wedge (Q \Rightarrow R) \wedge (R \Rightarrow P)) \Rightarrow S))\\
&\neg(\neg((P \Rightarrow S) \wedge (Q \Rightarrow R) \wedge (R \Rightarrow P)) \vee S)\\
&((\neg P \vee S) \wedge (\neg Q \vee R) \wedge (\neg R \vee P)) \wedge \neg S)\\
&\neg P \wedge \neg S \wedge (\neg Q \vee R) \wedge (\neg R \vee P)\\
&\neg P \wedge \neg S \wedge (\neg Q \vee R) \wedge \neg R\\
&\neg P \wedge \neg S \wedge \neg Q \wedge \neg R
\end{aligned}
$$

2. φ is non-valid by ssigning P, S, R, Q to False.

   So the formula would become:

$$
\begin{aligned}
&((F \Rightarrow F) \wedge (F \Rightarrow F) \wedge (F \Rightarrow F)) \Rightarrow F\\
&(T \wedge T \wedge T \Rightarrow F)\\
&False
\end{aligned}
$$

   Therefore, φ is non-valid as it can be made to False.

3. $\neg\psi$ in CNF form is:

$$
\begin{aligned}
&\neg((((P \vee Q) \Rightarrow S) \wedge (\neg P \Rightarrow (R \Rightarrow Q)) \wedge (R \vee S)) \Rightarrow S)\\
&\neg(\neg(((P \vee Q) \Rightarrow S) \wedge (\neg P \Rightarrow (R \Rightarrow Q)) \wedge (R \vee S)) \vee S)\\
&((P \vee Q) \Rightarrow S) \wedge (\neg P \Rightarrow (R \Rightarrow Q)) \wedge (R \vee S)) \wedge \neg S\\
&(\neg(P \vee Q) \vee S) \wedge (P \vee (\neg R \vee Q)) \wedge (R \vee S) \wedge \neg S\\
&(\neg P \wedge \neg Q \wedge \neg S) \wedge (P \vee (\neg R \vee Q)) \wedge (R \vee S)\\
&\neg P \wedge \neg Q \wedge \neg S \wedge (P \vee \neg R \vee Q) \wedge R
\end{aligned}
$$

4. We prove ψ is valid by deriving $\perp$ using refutation on $\neg\psi$:

   The refutation tree is as followed:

$$\begin{tikzcd}
P \vee Q \vee \neg R \arrow[dr, dash] & & \neg P \arrow[dl, dash] & & \\
& \neg R \vee Q \arrow[dr, dash] & & \neg R \arrow[dl, dash] & \\
& & Q \arrow[dr, dash] & & \neg Q \arrow[dl, dash] \\
& & & \perp &
\end{tikzcd}$$

   Therefore, from refutation tree, we know ψ is valid.

Challenge3

The non-valid case is when we set following interpretation to the formula:

$$\begin{aligned}&P(x,y):\ x\ does\ not\ equal\ to\ y\\&h(x):\ return\ x\ itself\end{aligned}$$

with the domain of x and y to be:

$$\begin{aligned}&x's\ domain\ is\ all\ even\ number\ in\ R\\&y's\ domain\ is\ all\ odd\ number\ in\ R\end{aligned}$$

So the equation would become:

$$\begin{aligned}&[(False \Rightarrow False) \Rightarrow (False \wedge False)]\\&True \Rightarrow Flase \equiv False\end{aligned}$$

Now the Satisfiable case is we set $P(x,y) = True$, whatever x and y is.So the formula would become

$$(True \Rightarrow True) \Rightarrow (True \wedge True)$$

which is True. This means the formula is satisfiable.
Therefore the formula is non-valid but satisfiable.

Challenge4

1. $S_1: \ \forall x \ (S(x) \wedge \forall y(\neg P(y,x))) \Rightarrow H(x)$
2. $S_2: \ \forall x \ (S(x) \wedge (\forall y(P(y,x) \Rightarrow R(y)))) \Rightarrow H(x)$
3. S_2 in CNF form is:

$$\begin{array}{l}
\forall x \ \neg(S(x) \wedge (\forall y(P(y,x) \Rightarrow R(y)))) \vee H(x) \\
\forall x \ \neg S(x) \vee \neg(\forall y(\neg P(y,x) \vee R(y)))) \vee H(x) \\
\forall x \ \neg S(x) \vee \exists y(P(y,x) \wedge \neg R(y)) \vee H(x) \\
\forall x \ \neg S(x) \vee (P(f(x),x) \wedge \neg R(f(x))) \vee H(x) \\
\forall x \ (\neg S(x) \vee H(x) \vee P(f(x),x)) \wedge (\neg S(x) \vee H(x) \vee \neg R(x))
\end{array}$$

   Therefore, the clausal form is:

$$\{\{\neg S(x) \vee H(x) \vee P(f(x),x\}, \{\neg S(x) \vee H(x) \vee \neg R(x)\}\}$$

4. $\neg S_1$ in CNF form is:

$$\begin{array}{l}
\neg(\forall x \ (S(x) \wedge \forall y(\neg P(y,x))) \Rightarrow H(x)) \\
\neg(\forall x \ \neg(S(x) \wedge \forall y(\neg P(y,x))) \vee H(x)) \\
\exists x \ S(x) \wedge \forall y \neg P(y,x) \wedge \neg H(x) \\
S(a) \wedge \neg P(y,a) \wedge \neg H(a)
\end{array}$$

   Therefore, the clausal form is :

$$\{\{S(a)\}, \{\neg P(y,a)\}, \{\neg H(a)\}\}$$

5. We now refutate $S_2 \wedge \neg S_1$ to get $\perp$, thus prove S_1 is a logical Consequence of S_2: ‘ The refutation tree is as follows:

$$\begin{tikzcd}
\neg S(x) \vee H(x) \vee P(f(x),x) \arrow[dr, dash, "x \mapsto a"'] & & S(a) \arrow[dl, dash] & & \\
& H(a) \vee P(f(a),a) \arrow[dr, dash] & & \neg H(a) \arrow[dl, dash] & \\
& & P(f(a),a) \arrow[dr, dash, "y \mapsto f(a)"] & & \neg P(y,a) \arrow[dl, dash] \\
& \neg S(X) \vee H(x) \vee \neg R(x) \arrow[dr, dash] & & \perp \arrow[dl, dash] & \\
& & \perp & &
\end{tikzcd}$$