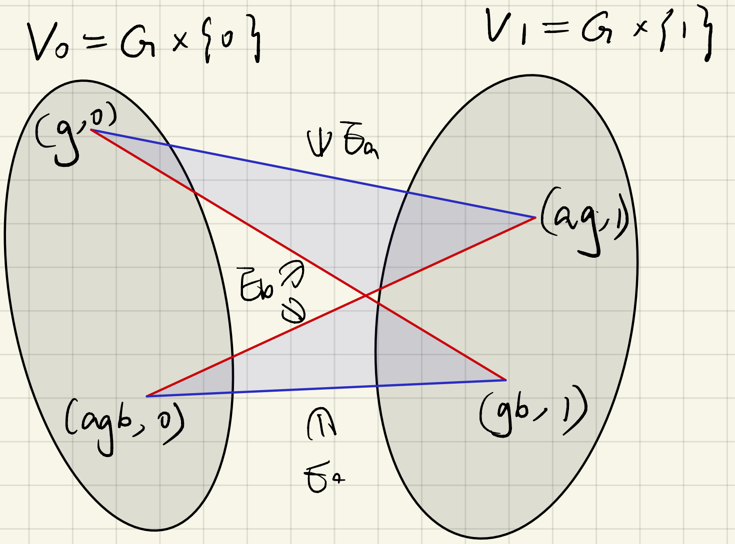
Quantum Tanner Codes

Left-right Caylay Complex



G: group

A, B: $A = A^{-1}$, $B = B^{-1}$

$A, B \subseteq G$.

$X(V, E, F)$

① $V = V_0 \cup V_1$

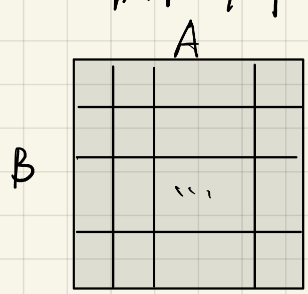
② $E = E_a \cup E_b$

$E_a = \{(g,0) \sim (ag,1)\}$ $\quad E_b = \{(g,0) \sim (gb,1)\}$

③ $F = \left\{ \begin{matrix} (g,0) \sim (ag,1) \\ \wr \qquad \wr \\ (gb,1) \sim (agb,0) \end{matrix} \right\}$

Properties:

① Each nodes has $|A| \times |B|$ neighboring faces.



$= F(v)$

② $F((g,0)) \cap F((g',1))$

$$= \begin{cases} \{(g,0) \sim (ag,1) \sim (agb,0) \sim (gb,1) \mid b \in B\} \cong B \\ \{(g,0) \sim (ag,1) \sim (agb,0) \sim (gb,1) \mid a \in A\} \cong A. \\ \phi, \quad \text{otherwise} \end{cases}$$

Construction:

$\boxed{ag \neq gb, \ \forall a \in A, b \in B, g \in G}$

$$\begin{cases} G, A, B, X(V, E, F) \\ \text{Codes } C_A \leq \mathbb{F}_2^{|A|}, \quad C_B \leq \mathbb{F}_2^{|B|} \end{cases}$$

$C_1 = C_A \boxtimes C_B$

$C_2 = C_A^\perp \boxtimes C_B^\perp$

① Put an X generator on $F((g,0))$, $\forall g \in G$.
with supports $\in C_A \otimes C_B$

$\mathcal{C}_0 = T(\mathcal{G}_0^\square, C_0^\perp)$

② Put a Z generator on $F((g,1))$, $\forall g \in G$.
with supports $\in C_A^\perp \otimes C_B^\perp$.

$\mathcal{C}_1 = T(\mathcal{G}_1^\square, C_1^\perp)$

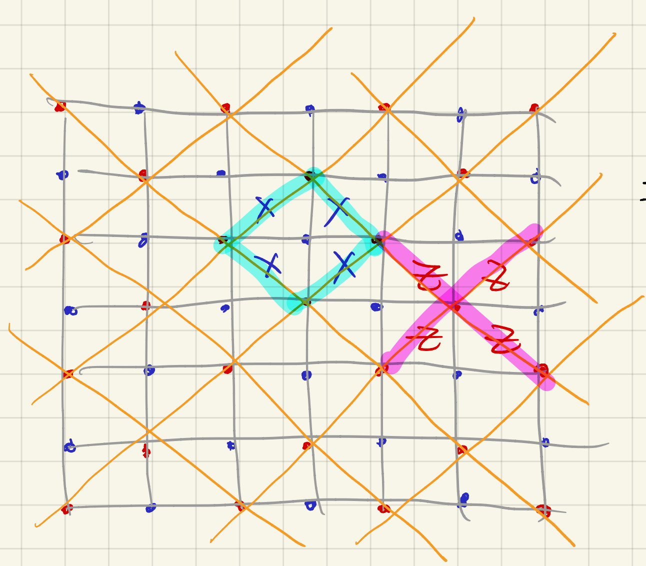
$\Longrightarrow$ All X, Z generators commute

Example:

$G = \mathbb{Z}_d \times \mathbb{Z}_d$, $\quad A = \{(\pm 1, 0)\}$, $B = \{(0, \pm 1)\}$.

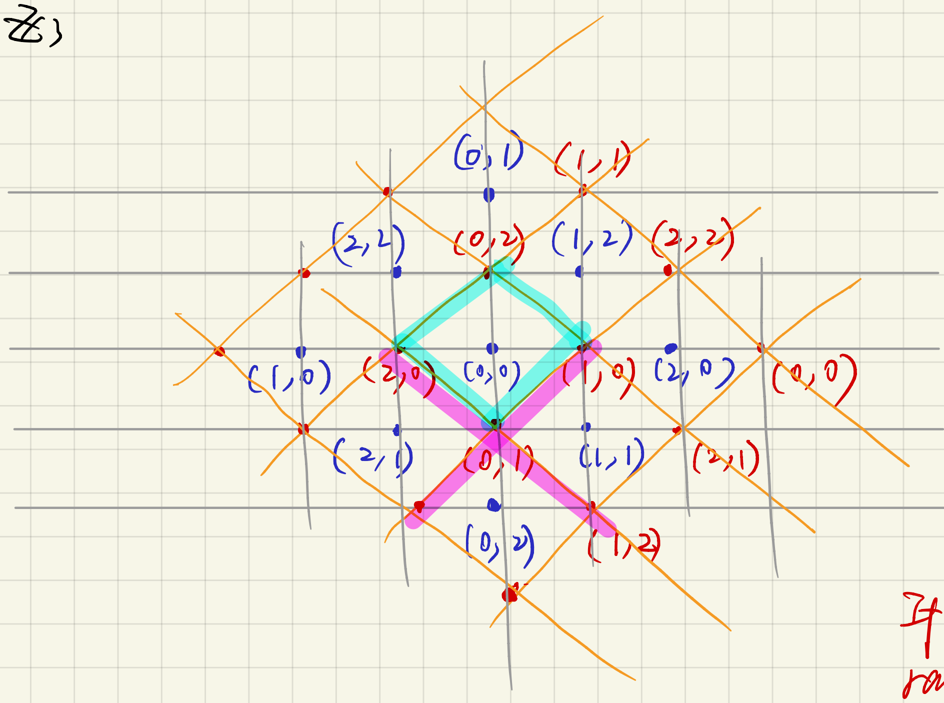
$d = 2k+1$, $\quad C_A = C_B = \{00, 11\}$.

$$\begin{tikzcd}
(i-1, j-1, 0) \arrow[r, dash] \arrow[d, dash] & (i, j-1, 1) \arrow[r, dash] \arrow[d, dash] & (i+1, j-1, 0) \arrow[d, dash] \\
(i-1, j, 1) \arrow[r, dash] \arrow[d, dash] & (i, j, 0) \arrow[r, dash] \arrow[d, dash] & (i+1, j, 1) \arrow[d, dash] \\
(i+1, j+1, 0) \arrow[r, dash] & (i, j+1, 1) \arrow[r, dash] & (i+1, j+1, 0)
\end{tikzcd}$$



$\Longrightarrow$ Toric Code.

$\mathbb{Z}_3 \times \mathbb{Z}_3$



If $\rho > H(\delta)$ random codes C_A, C_B has nonzero probability to satisfy all conditions (with $r = \lfloor \rho\Delta \rfloor$)

Asymptotically Good Quantum Tanner Code

$\rho, \varepsilon \in (0, \frac{1}{2})$, $\quad \gamma \in (\frac{1}{2}+\varepsilon, 1)$, $\delta > 0$

① $\quad 0 < \dim C_A \leq \rho\Delta$, $\quad \dim C_B = \Delta - \dim C_A$

② $\quad d_{min}\ C_A, C_B, C_A^{\perp}, C_B^{\perp} \geq \delta\Delta$

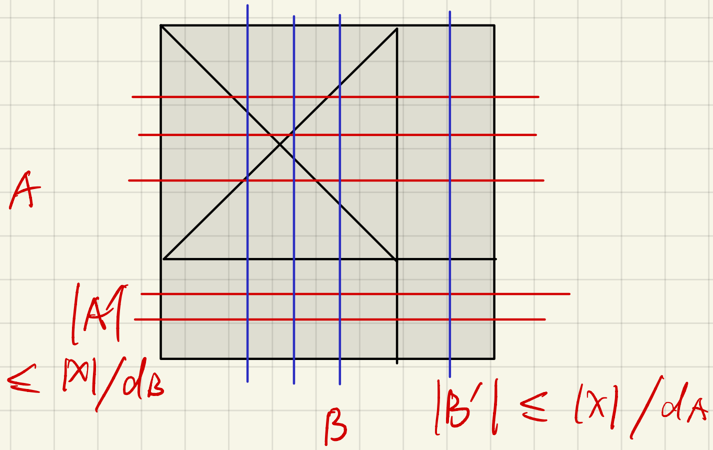
③ $\quad$ Both $(C_A \otimes C_B)^{\perp}$ and $(C_A^{\perp} \otimes C_B^{\perp})^{\perp}$ are $\Delta^{\frac{3}{2}-\frac{\varepsilon}{2}}$-robust with Δ^{γ}-resistence to puncturing

Then the resultant Quantum Tanner Code has parameters:

$$[[n,\ K \geq (1-2\rho)^2 n,\ d \geq \frac{\delta}{4\Delta^{\frac{3}{2}+\varepsilon}} n]]$$

Robustness

$0 \leq w \leq \Delta^2$, $C = C_A \otimes \mathbb{F}_2^B + \mathbb{F}_2^A \otimes C_B = (C_A^{\perp} \otimes C_B^{\perp})^{\perp}$ is w-robust if any codword of weight $|x| \leq w$ spans $\leq \frac{|x|}{d_A}, \frac{|x|}{d_B}$ rows and columns (while representing codewords by matrix of $|A| \times |B|$)



If C is w-robust with $w \leq d_A d_B$, then any codword x with $|x| \leq w$ can be decomposed to $x = c + r$, $c \in C_A \otimes \mathbb{F}_2^{|B'|}$, $r \in \mathbb{F}_2^{|A'|} \otimes C_B$, for some A', B' s.t. $|A'| \leq |x|/d_B$, $|B'| \leq |x|/d_A$

If C is w-robust with $w \leq d_A d_B/2$, then if any word x: $d(x, C_A \otimes \mathbb{F}_2^B) + d(x, \mathbb{F}_2^A \otimes C_B) \leq w$,

$$\Rightarrow d(x, C_A \otimes C_B) \leq \frac{3}{2}\left(d(x, C_A \otimes \mathbb{F}_2^B) + d(x, \mathbb{F}_2^A \otimes C_B)\right)$$

$(C_A^\perp \otimes C_B^\perp)^\perp$ is w-robust with p-resistence to puncturing if

If for any $|A'| - |B'| = \Delta - w'$ with $w' \leq p$.

$(C_{A'}^\perp \otimes C_{B'}^\perp)^\perp$ is w robust

For large Δ, with probability $\to 1$ such code exists (by random code)

Proving that $d \geqslant \frac{\delta}{4\Delta^{\frac{3}{2}+\varepsilon}} n \iff \forall\, x$ s.t. $|x| < \frac{\delta}{4\Delta^{\frac{3}{2}+\varepsilon}} n$,

$$\min\Big(\underbrace{d_{\min}(\mathcal{C}_1/\mathcal{C}_0^{\perp})}_{d_x},\ \underbrace{d_{\min}(\mathcal{C}_0/\mathcal{C}_1^{\perp})}_{d_y}\Big)$$

$x \in \mathcal{C}_1 \Rightarrow x \in \mathcal{C}_0^{\perp}$

$x \in \mathcal{C}_0 \Rightarrow x \in \mathcal{C}_1^{\perp}$.

①

Main Theorem:

$$\boxed{\forall\, x \in \mathcal{C}_1 \text{ s.t. } |x| < \frac{\delta}{4^{\frac{3}{2}+\varepsilon}} n,\ \exists\, y \in \mathcal{C}_1 \text{ s.t. } \exists\, v \in V_0 \text{ s.t. } y \leq \Phi(v),\ y \in C_A \otimes C_B,\ \text{and } |x+y| < |x|}$$

$\Rightarrow y \in \mathcal{C}_0^{\perp}$

Keep using the theorem: $x = y_1 + y_2 + \cdots + y_k$, $y_i \in \mathcal{C}_0^{\perp}$

$\Rightarrow x \in \mathcal{C}_0^{\perp}$. $\Rightarrow$ ① proved.

Reading: Quantum Tanner Code.
Claim 10–14.

Major Proof Steps.

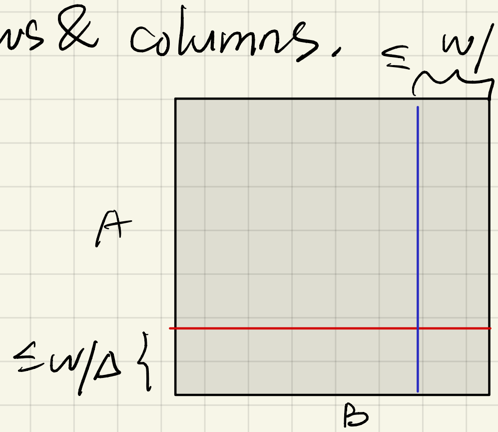
$x \in \mathcal{C}_1$, $|x| < \frac{\delta}{4^{\frac{3}{2}+\varepsilon}} n \Rightarrow$ vertex set $S \subset V_1$

$\forall\, v \in S$, $F(v) \in C_1^{\perp} = C_A \otimes \mathbb{F}_2^B + \mathbb{F}_2^A \otimes C_B$

$\downarrow$ w-robustness of C_1, $w = \Delta^{3/2-\varepsilon}$

if $|F(v)| \leq w = \Delta^{3/2-\varepsilon}$, $F(v)$ is supported on $\leq$ $w/\delta\Delta = \Delta^{1/2-\varepsilon}/\delta$ rows & columns.

Normal Vertices.



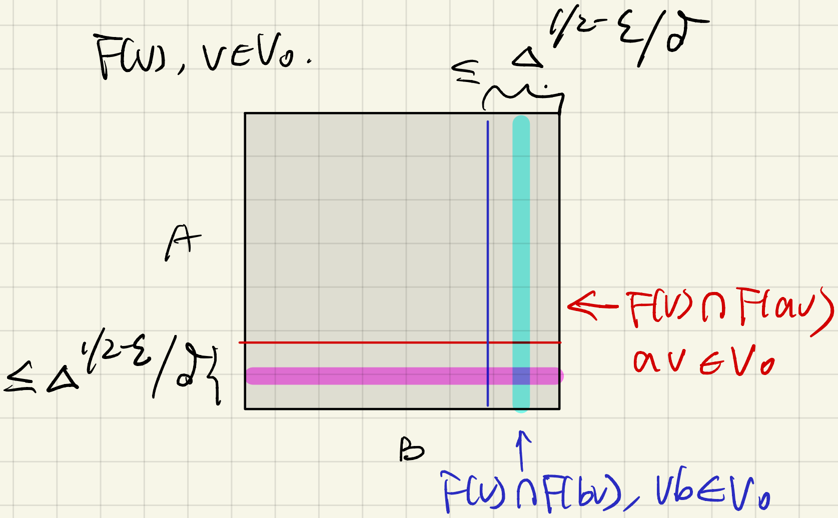
ves.

Normal Vertices: $d(v) \le \Delta^{3/2-\varepsilon}$ in subgraph induced by x.

Exceptional Vertices S_e: $d(v) > \Delta^{3/2-\varepsilon}$.

Expansion $\Longrightarrow$ $\boxed{|S_e| \le \frac{64}{\Delta^{1-2\varepsilon}} |S|.}$

$F(v)$, $v \in V_0$.



$T \subset V_0$: $u \in V_0$ s.t. $F(v) \cap F(u)$ intersects at a row/column with weight $\ge \delta\Delta - \Delta^{1/2-\varepsilon/\delta}$

For large enough Δ, $\boxed{|T| \le \frac{64}{\delta^2 \Delta} |S|}$.

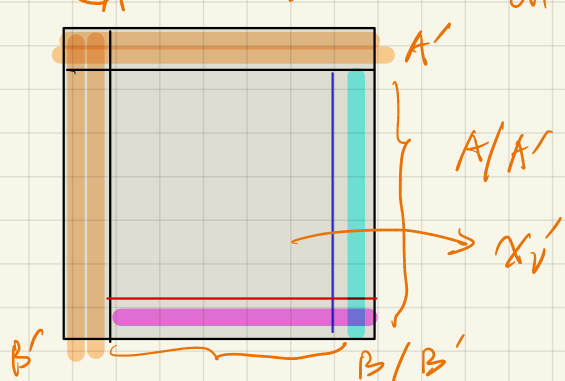
$\Downarrow$

$$\boxed{\begin{array}{l}\text{At least a fraction } \alpha/4 \text{ of vertices in } T \text{ satisfy:} \\ \quad (1)\ \text{ has } \ge \alpha\Delta \text{ rows/columns with weight} \ge \delta\Delta - \Delta^{1/2-\varepsilon}/\delta \\ \quad (2)\ \text{ are adjacent to} \le d_1 = \Delta^{\gamma} \text{ vertices of } S_e.\end{array}}$$

(for large enough Δ)

close to C_A or C_B.

d_1 punctured rows/colums.



$\Rightarrow \exists\, c' \in C_{A'} \otimes C_{B'}$ s.t.

$d(x_v', c') \le \frac{3}{2}\left(d(x_v', C_{A'} \otimes \mathbb{F}_2^{B'}) + d(x_v', \mathbb{F}_2^{A'} \otimes C_{B'})\right) \le 3\Delta \frac{\Delta^{1/2-\varepsilon}}{\delta}.$

$d_1 < \delta\Delta \Rightarrow \exists\, c \in C_A \otimes C_B$, s.t. $c' = c|_{A' \otimes B'}$.

$$\Rightarrow \quad \underbrace{d(x_v, c)}_{\sim O(\Delta^2)} \le \underbrace{d(x_v', c')}_{\sim O(\Delta^{\frac{3}{2}+\varepsilon})} + 2 d_1 \Delta$$

①

$$\Rightarrow \quad d(x_v, c) = |\underbrace{x_v + c}_{y}| < |x_v| \text{ for large enough } \Delta.$$