Hypergraph Product Codes

$$\boxed{\begin{aligned} H_x &= \begin{bmatrix} A \otimes I_{n_2} & I_{r_1} \otimes B^T \end{bmatrix} \\ H_z &= \begin{bmatrix} I_{n_1} \otimes B & A^T \otimes I_{r_2} \end{bmatrix} \end{aligned}}$$

$$A \in \mathbb{F}_2^{r_1 \times n_1}, \quad B \in \mathbb{F}_2^{r_2 \times n_2}.$$

$$H_x H_z^T = A \otimes B^T + A \otimes B^T = 0_{r_1 \times r_2}.$$

Example

$r_1 = r_2 = 2, \quad n_1 = n_2 = 3$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$H_x = \left[\begin{array}{ccc|ccc|ccc|cc|cc}
1 & & & 1 & & & & & & 0 & 1 & & \\
 & 1 & & & 1 & & & & & 1 & 0 & & \\
 & & 1 & & & 1 & & & & 1 & 1 & & \\
\hline
 & & & 1 & & & 1 & & & & & 0 & 1 \\
 & & & & 1 & & & 1 & & & & 1 & 0 \\
 & & & & & 1 & & & 1 & & & 1 & 1
\end{array}\right]$$

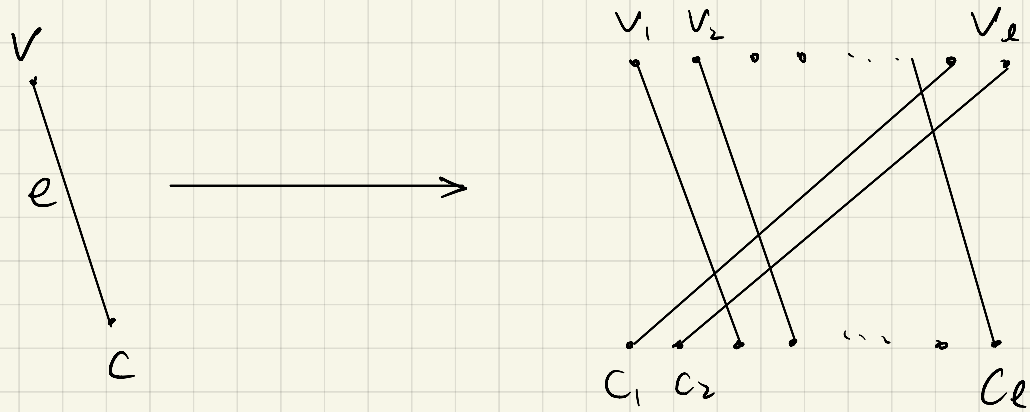
$$H_z = \left[\begin{array}{ccc|ccc|ccc|cc|cc}
0 & 1 & 1 & & & & & & & 1 & & & \\
1 & 0 & 1 & & & & & & & & 1 & & \\
\hline
 & & & 0 & 1 & 1 & & & & 1 & & 1 & \\
 & & & 1 & 0 & 1 & & & & & 1 & & 1 \\
\hline
 & & & & & & 0 & 1 & 1 & & & 1 & \\
 & & & & & & 1 & 0 & 1 & & & & 1
\end{array}\right]$$

Lifted Product Codes

$0 \le \alpha < 1$, $[[N, K, D]]$ - code

$K \sim \Omega(N^{\alpha} \log N)$, $D \sim \Omega(N^{1-\frac{\alpha}{2}} \log N)$.

"Lifted" versions of HGP codes.

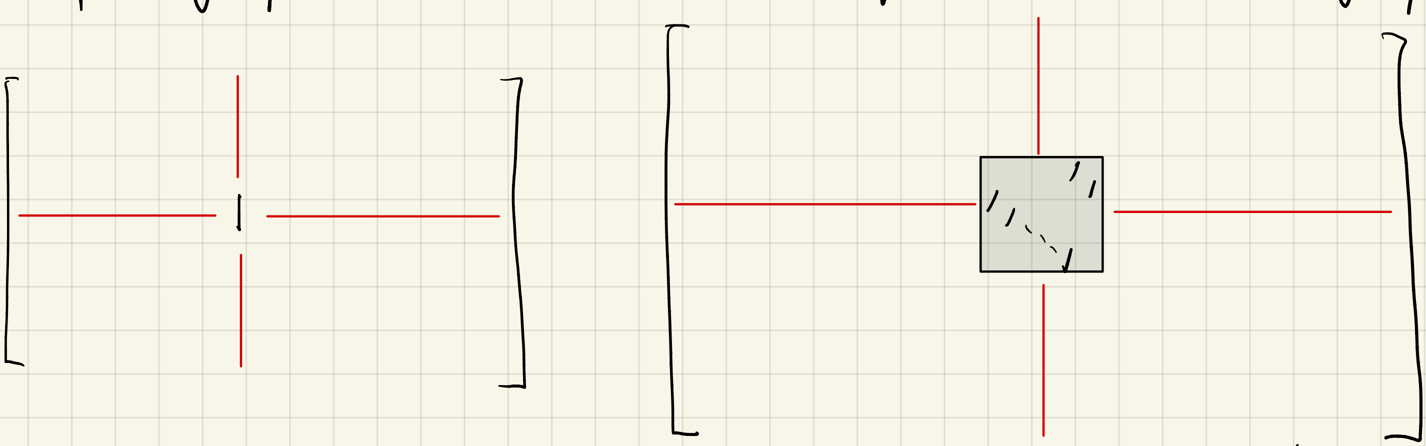


cyclic-shift.

Base graph / protograph.

Quasi-cyclic Codes

ℓ - lifts / ℓ-fold cover graphs

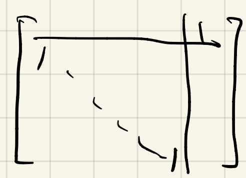


$1 \longmapsto x^2$

$\uparrow$

$\mathbb{F}_2[x]/(x^\ell - 1)$

$x =$



$$R = \mathbb{F}_2 G \longleftarrow \text{finite group.} \qquad |G| = l .$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \shortparallel$$

$$\downarrow$$

$$\left\{ \sum_{g \in G} \alpha_g \, g \;\middle|\; \alpha_g \in \mathbb{F}_2 \right\} \qquad \{g_1, \cdots, g_l\} .$$

$$\uparrow$$

$$(\alpha_{g_1}, \alpha_{g_2}, \cdots, \alpha_{g_l}) \in \mathbb{F}_2^{\,l}$$

$$a = \sum_{g \in G} \alpha_g \, g \quad , \quad b = \sum_{g \in G} \beta_g \, g$$

$$a + b = \sum_{g \in G} (\alpha_g + \beta_g) \, g . \qquad ab = \sum_{g \in G} \left(\sum_{hr = g} \alpha_h \beta_r \right) g$$

$$\begin{cases} \mathbf{b}(a) = (\alpha_{g_1}, \cdots, \alpha_{g_l}) \\ \mathbb{B}(a) = \sum_{g \in G} \alpha_g \, \mathbb{B}(g) . \end{cases} \qquad \begin{matrix} \mathbf{b}(a+b) = \mathbf{b}(a) + \mathbf{b}(b) \\ \mathbb{B}(ab) = \mathbb{B}(a)\,\mathbb{B}(b) \end{matrix}$$

$$\mathbb{B}(g)_{i,j} = \begin{cases} 1 & g_i = g g_j \\ 0 & \text{otherwise} \end{cases}$$

<u>Example:</u> $\quad G = \langle 1, g, g^2, \cdots, g^{l-1} \rangle . \qquad g_i = g^{i-1}$

$$\mathbb{B}(g) = \begin{array}{cc} & \begin{matrix} 1 & \cdots & \cdots & g^{l-1} \end{matrix} \\ \begin{matrix} 1 \\ \vdots \\ \vdots \\ \vdots \\ g^{l-1} \end{matrix} & \left[\begin{array}{cccc|c} & & & & 1 \\ \hline 1 & & & & \\ & \ddots & & & \\ & & \ddots & & \\ & & & 1 & \end{array} \right] \end{array}$$

$$A \in M_{m \times n}(R) \Longrightarrow \mathbb{B}(A) = [\mathbb{B}(a_{ij})]_{m \times n} \in M_{lm \times ln}(\mathbb{F}_2)$$

$$\mathbb{B}(AB) = \mathbb{B}(A)\,\mathbb{B}(B)$$

$$\mathbb{B}(Av) = \mathbb{B}(A)\,\mathbf{b}(v)$$

If $G = C_\ell = \langle 1, g, g^2, \cdots, g^{\ell-1} \rangle \rightarrow$ quasi-cyclic

G abelian $\rightarrow$ quasi-abelian.

<u>Example</u> : $G = C_3$, $\mathbb{F}_2 G = R_3$. $A \in M_{2\times 3}(R_3)$

$$A = \begin{bmatrix} 1 & 0 & 1+x^2 \\ 1+x & 1+x+x^2 & x^2 \end{bmatrix}$$

$$\mathbb{B}(A) = \left[\begin{array}{ccc|ccc|ccc} 1 & & & & & & 1 & 1 & \\ & 1 & & & & & & 1 & 1 \\ & & 1 & & & & 1 & & 1 \\ \hline 1 & & 1 & 1 & 1 & 1 & & 1 & \\ 1 & 1 & & 1 & 1 & 1 & & & 1 \\ & 1 & 1 & 1 & 1 & 1 & 1 & & \end{array}\right]$$

<u>LP(A, B)</u>

Replace elements in A, B of HGP codes with elements of $R \subseteq M_\ell(\mathbb{F}_2)$

$$\Downarrow$$

$$A \in M_{r_1 \times n_1}(R), \quad B \in M_{r_2 \times n_2}(R)$$

$$M = (m_{ij})_{m\times n} \longrightarrow M^* = (m_{ji}^T)_{n\times m}$$

conjugate transpose.

$$\mathbb{B}(M^*) = \mathbb{B}(M)^T.$$

$$H_X = [A \otimes I \quad I \otimes B]$$

$$H_Z = [I \otimes B^* \quad A^* \otimes I]$$

$$\mathbb{B}(H_X)\mathbb{B}(H_Z)^T = 0 \iff H_X H_Z^* = 0.$$

<u>Example</u>

$$A = \begin{bmatrix} 1 & x & x^3 \\ 1+x & x^2 & x^2+x+1 \end{bmatrix}$$

$$A^* = \begin{bmatrix} 1 & 1+x^{-1} \\ x^{-1} & x^{-2} \\ x^{-3} & 1+x^{-1}+x^{-2} \end{bmatrix}$$

$$B = [1+x] \qquad B^* = [1+x^{-1}]$$

$$H_X = \left[\begin{array}{ccc|cc} 1 & x & x^3 & 1+x & \\ 1+x & x^2 & x^2+x+1 & & 1+x \end{array}\right]$$

$$H_Z = \left[\begin{array}{ccc|cc} 1+x^{-1} & & & 1 & 1+x^{-1} \\ & 1+x^{-1} & & x^{-1} & x^{-2} \\ & & 1+x^{-1} & x^{-3} & 1+x^{-1}+x^{-2} \end{array}\right]$$

$$H_X H_Z^* = \left[\begin{array}{ccc|cc} 1 & x & x^3 & 1+x & \\ 1+x & x^2 & x^2+x+1 & & 1+x \end{array}\right] \left[\begin{array}{ccc} 1+x & & \\ & 1+x & \\ & & 1+x \\ \hline 1 & x & x^3 \\ 1+x & x^2 & 1+x+x^2 \end{array}\right]$$

$$= (1+x)\begin{bmatrix} 1 & x & x^3 \\ 1+x & x^2 & x^2+x+1 \end{bmatrix} + (1+x)\begin{bmatrix} 1 & x & x^3 \\ 1+x & x^2 & x^2+x+1 \end{bmatrix}$$

$$= 0_{2\times 3}.$$

Distances

both H, H^T have good expansion.

① A good base matrix H (protograph) constructed by expander codes $\leftarrow$ $T(G, C) \rightarrow \lambda_2(G) < 2\sqrt{d-1}+1$

② With positive probability, a random l-lift of H has $\lambda_2(\hat{G}) < c(2\sqrt{d-1}+1)$, for $|G| \geq 2\gamma \log l$, γ large enough. $\longrightarrow A, A^*$

③ $LP(A, 1+x)$ has $K \sim \Omega(\log N)$, $D \sim \Omega(N/\log N)$ for $l = \Omega(N/\log N) \longrightarrow$ $C_2 \longrightarrow C_1 \longrightarrow C_0$

④ $C_2 \xrightarrow{x} C_1 \xrightarrow{z} C_0$ $\qquad D_1 \longrightarrow D_0$.

Quantum. $[N, K, d_x, d_z]$ $\qquad$ Classical $[n, k, d]$

$$
\begin{tikzcd}
C_2 \times D_1 \arrow[r] \arrow[d] & C_1 \times D_1 \arrow[r] \arrow[d] & C_0 \times D_1 \\
C_2 \times D_0 \arrow[r] & C_1 \times D_0 &
\end{tikzcd}
$$

C_2' $\quad$ C_1' $\quad$ C_0'

$C_2' \longrightarrow C_1' \longrightarrow C_0'$ $\quad [N', K', d_x', d_z']$.

$N' \leq 2nN$. $\quad K' = kK$

$d_x' \geq d \cdot d_x$, $\quad d_z' \geq d_z$

$C'' \longrightarrow C_1'' \longrightarrow C_0''$ $\quad [N'', K'', d_x'', d_z'']$

$N'' = \Omega(n^2 N)$ $\quad K'' = k^2 K$.

$d_x'' \geq d \cdot d_x$, $\quad d_z' \geq d \cdot d_z$.

Let. $n = O(N^{\frac{\alpha}{2(1-\alpha)}})$, $\quad k = O(n)$, $\quad d = O(n)$.

$K = O(\log N)$, $\quad D = O(N / \log N)$

$N'' = O(N^{\frac{1}{1-\alpha}})$. $\quad N = \Omega(N''^{(1-\alpha)})$

$K'' = O(N^{\frac{\alpha}{1-\alpha}} \log N) = \Omega(N''^{\alpha} \log N'')$

$D'' = O(N^{\frac{\alpha}{2(1-\alpha)}+1} / \log N)$

$= O(N''^{(\frac{\alpha}{2}+1-\alpha)} / \log N'')$

$= O(N''^{(1-\frac{\alpha}{2})} / \log N'')$