


Test

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目录

1	Expansion	1
1.1	from time to scale factor	1
1.2	to more notations.. . . .	2
1.2.1	Cosmological Redshift	2
1.2.2	Hubble parameter	3
1.2.3	Conformal Time	3
1.3	Hubble sphere	3
1.4	Light Cone	3
1.5	Event Horizon	3
1.6	Particle Horizon	4
1.6.1	optical horizon	4
2	Boltzmann Equation	4
2.1	0th order	5
2.2	1st order	5

1 Expansion

1.1 from time to scale factor

We have two kinds of “time” in cosmology since our Universe are expanding.

If we focus on the **coordinate**, the metric is written as ($c = 1$)

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) \quad (1)$$

better to write in spherical coordinate system

$$ds^2 = -dt^2 + a^2(t)[d\chi^2 + \chi^2(d\theta^2 + \sin^2\theta d\varphi^2)] \quad (2)$$

By scaling the time axis, we can get the **conformal time**, and have an “appearantly” Minkovski spacetime

$$ds^2 = a^2(\tau)(-d\tau^2 + dx^2 + dy^2 + dz^2) \quad (3)$$

The **scale factor** can be solved from the **Friedmann equation**,¹

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho \quad (4)$$

with **a model of energy-momentum** contained in the Universe like²

$$\frac{\rho(t)}{\rho_{\text{cr}}} = \sum_{s=\gamma, \text{m}, \nu, \text{DE}} \Omega_s a(t)^{-3(1+w_s)} \quad (5)$$

This model assumes a constant of equation of state.

In our Λ CDM model (reserve curvature)

$$\frac{\rho(t)}{\rho_{\text{cr}}} = \Omega_R \left(\frac{a}{a_0}\right)^{-4} + \Omega_M \left(\frac{a}{a_0}\right)^{-3} + \Omega_\Lambda + \Omega_K \left(\frac{a}{a_0}\right)^{-2} \quad (6)$$

just solve the differential equation

$$\dot{a} = H_0 a_0 \sqrt{\Omega_R \left(\frac{a}{a_0}\right)^{-2} + \Omega_M \left(\frac{a}{a_0}\right)^{-1} + \Omega_\Lambda \left(\frac{a}{a_0}\right)^2 + \Omega_K} \quad (7)$$

$$t(a_1) = \int_0^{a_1} \frac{da}{H_0 a_0 \sqrt{\Omega_R \left(\frac{a}{a_0}\right)^{-2} + \Omega_M \left(\frac{a}{a_0}\right)^{-1} + \Omega_\Lambda \left(\frac{a}{a_0}\right)^2 + \Omega_K}} \quad (8)$$

It's not linear, so need numerical solution. See code/expansion.ipynb

But it's OK to see some exceptions.

Block 1. matter dominated:

$$\dot{a} = H_0 (a_0 \Omega_M)^{\frac{1}{2}} a^{-\frac{1}{2}} \quad (9)$$

then

$$t = \frac{2}{3H_0 (a_0 \Omega_M)^{\frac{1}{2}}} a^{\frac{3}{2}} \quad (10)$$

assumed $a(t=0) = 0$.

1.2 to more notations..

It will be easy to use **redshift** and **Hubble parameter $H(z)$** in the calculations.

1.2.1 Cosmological Redshift

Define redshift as

$$\frac{a}{a_0} = \frac{1}{1+z} \quad (11)$$

Redshift z is linked to a and t simply. How about the comoving spacial coordinate χ ?

$$da = -\frac{a_0 dz}{(1+z)^2} \quad (12)$$

The comoving distance from z_1 to now ($z_0 = 0$)

$$\chi(z_1) = \int_{a_1}^{a_0} \frac{da}{a^2 H(a)} = \int_{z_0}^{z_1} \frac{c dz}{a_0 H(z)} \quad (13)$$

where $z_0 = 0$ by defination, c is filled to balance the unit.

¹also deriving the Friedmann equations from Einstein field equation is not that trivial, see another block in preparation..

²Friedmann equation itself has assumptions about the energy-momentum, it's **ideal fluid** which can be parameterized by only ρ and P ..

1.2.2 Hubble parameter

And turn Eq. (4) to

$$H(a) \equiv \frac{\dot{a}}{a} = H_0 \sqrt{\frac{\rho(a)}{\rho_{\text{cr}}}} \equiv H_0 E(a) \quad (14)$$

Eq. (6) turns out

$$E(a) = \sqrt{\Omega_{\text{R}} \left(\frac{a}{a_0}\right)^{-4} + \Omega_{\text{M}} \left(\frac{a}{a_0}\right)^{-3} + \Omega_{\Lambda} + \Omega_{\text{K}} \left(\frac{a}{a_0}\right)^{-2}} \quad (15)$$

$$E(z) = \sqrt{\Omega_{\text{R}}(1+z)^4 + \Omega_{\text{M}}(1+z)^3 + \Omega_{\Lambda} + \Omega_{\text{K}}(1+z)^2} \quad (16)$$

1.2.3 Conformal Time

Conformal time identic to the relation between χ and t in light cone.

$$d\tau^2 = \frac{dt^2}{a^2(t)} \quad (17)$$

$$d\tau = \frac{dt}{a(t)} = d\chi \quad (18)$$

$$\tau(t_1) = \int_{t_i}^{t_1} \frac{dt}{a(t)} \quad (19)$$

The last = in Eq. (18) is for the convinence of coding.

1.3 Hubble sphere

Hubble sphere is where the Hubble flow velocity³ equals to the speed of light.

$$v_{\text{H}} = \dot{a}\chi \quad (20)$$

Hubble sphere as a function of time should be

$$\chi_{\text{H}} = \frac{c}{aH(a)} \quad (21)$$

1.4 Light Cone

Light cone is defined as $ds^2 = 0$, the photon starts from a_1 and ends at the observer now ($a_0, \chi = 0$). so

$$dt^2 = a^2(t)d\chi^2 \quad (22)$$

$$dt = a(t)d\chi \quad (23)$$

$$\chi_{\text{lc}}(a_1) = \int_{t_1}^{t_0} \frac{dt}{a(t)} = \int_{a_1}^{a_0} \frac{da}{a\dot{a}} = \int_{a_1}^{a_0} \frac{da}{a^2 H(a)} \quad (24)$$

1.5 Event Horizon

the photon starts from a_1 and ends at the observer in the infinity future ($a \rightarrow \infty, \chi = 0$).

$$\chi_{\text{EH}}(a_1) = \int_{t_1}^{\infty} \frac{dt}{a(t)} = \int_{a_1}^{\infty} \frac{da}{a^2 H(a)} \quad (25)$$

³in the proper length and is not real ‘velocity’.

1.6 Particle Horizon

Particle horizon is the trajectory that a photon start from $\chi = 0$ at t_i (in ‘our model’, $t_i = a_i = 0$).

$$\chi_{\text{PH}}(a_1) = \int_{t_i}^{t_1} \frac{dt}{a(t)} = \int_{a_i}^{a_1} \frac{da}{a\dot{a}} = \int_{a_i}^{a_1} \frac{da}{a^2 H(a)} \quad (26)$$

1.6.1 optical horizon

The optical horizon due to recombination $z(t_r) \approx 1100$.

$$d_{\text{opt}}(t) = a(\tau)(\tau - \tau_r) = a(t) \int_{t_r}^t \frac{dt'}{a(t')} \quad (27)$$

L^AT_EX

$$H. \quad (28)$$

$$H|\psi\rangle \quad (29)$$

2 Boltzmann Equation

我们讨论 inflation 结束后到 recombination 之前，光子如何产生各向异性/光子的各向异性如何演化。我们将看到光子和物质之间通过 Thomson scattering 相互作用，将为光子贡献一部分各向异性。(interaction between matter and photon contributes to the **anisotropies of CMB**)

在微观层面，满足 Bosen-Einstein 分布

$$f(\eta, \mathbf{x}, E, \hat{\mathbf{p}}) = \frac{1}{e^{E/T} - 1} \quad (30)$$

说明：

- 注意相空间是四维时空加上四动量。
- $\hat{\mathbf{p}}$ 是因为光子动量由能量决定。
- thermal dynamic in local coordinate,
- 讨论问题: thermal dynamic 如何推广到 Minkowski space with perturbation ? 这样简单的推广成立吗? ——似乎是个还在研究中的问题。
- f is scalar since 相空间的体积元不变，且粒子数不变， $f =$ 粒子数/相空间的体积元。
- 相体积不变，熵不变， T 随 E 变。
- 讨论问题: 在考虑温度 T 是否改变时聊到了: 黑体谱在多普勒红移后还是黑体谱吗? 所以聊的结论是啥?

EoM: Boltzmann equation

$$\frac{df}{d\tau} = C[f_a] \quad (31)$$

$C[f_a]$ is the interaction with other particles (eg. baryon)

2.1 0th order

推出 $\bar{T} \propto \frac{1}{a}$

2.2 1st order

Step1: 对 f 做 Taylor expansion

$$f(\eta, \mathbf{x}, E, \hat{\mathbf{p}}) = \frac{1}{e^{E/T} - 1} \quad (32)$$

- 算 Boltzmann equation - RHS - e- γ scattering in e- coordinate (Thomson scattering)
- $\frac{d\sigma}{d\Omega} - 1$ 给出 monopole - $\cos \theta$ 给出 quadrupole - 通过这些电子影响光子的分布, 给光子 anisotropy ? - 洛伦兹变换
- fluid description - 和 f 比较, 得到 CMB temperature multipoles 和 photon fluid variables energy-momentum tensor perturbation multipoles 的关系
- 以上两件事放在一起, 得到 CMB temperature multipoles - 和 geodesic 联立, 把 epsilon 换成 Bardeen's potential - $\mu = \dot{\bar{\mu}}$
- 得到 Θ 的演化方程 - 和 matter 的 energy-momentum tensor 比较, 得到 baryon perturbation, dark matter perturbation 和 baryon perturbation 差一个散射项 - Einstein field equation 得到 Bardeen's potential 的演化。

L^AT_EX usage

citation example[[Lyth and Liddle\(2009\)](#)]

参考文献

- [Lyth and Liddle(2009)] D. H. Lyth and A. R. Liddle, *The Primordial Density Perturbation: Cosmology, Inflation and the Origin of Structure* (Cambridge University Press, Cambridge, UK, 2009).