


Test

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1 Expansion

1.1 from time to scale factor

We have two kinds of “time” in cosmology since our Universe are expanding.

If we focus on the **coordinate**, the metric is written as ($c = 1$)

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) \quad (1)$$

better to write in spherical coordinate system

$$ds^2 = -dt^2 + a^2(t)[d\chi^2 + \chi^2(d\theta^2 + \sin^2\theta d\varphi^2)] \quad (2)$$

By scaling the time axis, we can get the **conformal time**, and have an “apparently” Minkovski spacetime

$$ds^2 = a^2(\tau)(-d\tau^2 + dx^2 + dy^2 + dz^2) \quad (3)$$

The **scale factor** can be solved from the **Friedmann equation**,¹

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho \quad (4)$$

¹also deriving the Friedmann equations from Einstein field equation is not that trivial, see another block in preparation..

with a model of energy-momentum contained in the Universe like ²

$$\frac{\rho(t)}{\rho_{\text{cr}}} = \sum_{s=\gamma, \text{m}, \nu, \text{DE}} \Omega_s a(t)^{-3(1+w_s)} \quad (5)$$

This model assumes a constant of equation of state.

In our Λ CDM model (reserve curvature)

$$\frac{\rho(t)}{\rho_{\text{cr}}} = \Omega_R \left(\frac{a}{a_0} \right)^{-4} + \Omega_M \left(\frac{a}{a_0} \right)^{-3} + \Omega_\Lambda + \Omega_K \left(\frac{a}{a_0} \right)^{-2} \quad (6)$$

just solve the differencial equation

$$\dot{a} = H_0 a_0 \sqrt{\Omega_R \left(\frac{a}{a_0} \right)^{-2} + \Omega_M \left(\frac{a}{a_0} \right)^{-1} + \Omega_\Lambda \left(\frac{a}{a_0} \right)^2 + \Omega_K} \quad (7)$$

$$t(a_1) = \int_0^{a_1} \frac{da}{H_0 a_0 \sqrt{\Omega_R \left(\frac{a}{a_0} \right)^{-2} + \Omega_M \left(\frac{a}{a_0} \right)^{-1} + \Omega_\Lambda \left(\frac{a}{a_0} \right)^2 + \Omega_K}} \quad (8)$$

It's not linear, so need numerical solution. See code/expansion.ipynb

But it's OK to see some exceptions.

Block 1. *matter donimated:*

$$\dot{a} = H_0 (a_0 \Omega_M)^{\frac{1}{2}} a^{-\frac{1}{2}} \quad (9)$$

then

$$t = \frac{2}{3H_0 (a_0 \Omega_M)^{\frac{1}{2}}} a^{\frac{3}{2}} \quad (10)$$

assumed $a(t=0) = 0$.

1.2 to more notations..

It will be easy to use **redshift** and **Hubble parameter $H(z)$** in the calculations.

1.2.1 Cosmological Redshift

Define redshift as

$$\frac{a}{a_0} = \frac{1}{1+z} \quad (11)$$

Redshift z is linked to a and t simply. How about the comoving spacial coordinate χ ?

$$da = -\frac{a_0 dz}{(1+z)^2} \quad (12)$$

The comoving distance from z_1 to now ($z_0 = 0$)

$$\chi(z_1) = \int_{a_1}^{a_0} \frac{da}{a^2 H(a)} = \int_{z_0}^{z_1} \frac{c dz}{a_0 H(z)} \quad (13)$$

where $z_0 = 0$ by defination, c is filled to balance the unit.

²Friedmann equation itself has assumptions about the energy-momentum, it's **ideal fluid** which can be parameterized by only ρ and P ..

1.2.2 Hubble parameter

And turn Eq. (4) to

$$H(a) \equiv \frac{\dot{a}}{a} = H_0 \sqrt{\frac{\rho(a)}{\rho_{\text{cr}}}} \equiv H_0 E(a) \quad (14)$$

Eq. (6) turns out

$$E(a) = \sqrt{\Omega_{\text{R}} \left(\frac{a}{a_0}\right)^{-4} + \Omega_{\text{M}} \left(\frac{a}{a_0}\right)^{-3} + \Omega_{\Lambda} + \Omega_{\text{K}} \left(\frac{a}{a_0}\right)^{-2}} \quad (15)$$

$$E(z) = \sqrt{\Omega_{\text{R}}(1+z)^4 + \Omega_{\text{M}}(1+z)^3 + \Omega_{\Lambda} + \Omega_{\text{K}}(1+z)^2} \quad (16)$$

1.2.3 Conformal Time

Conformal time identic to the relation between χ and t in light cone.

$$d\tau^2 = \frac{dt^2}{a^2(t)} \quad (17)$$

$$d\tau = \frac{dt}{a(t)} = d\chi \quad (18)$$

$$\tau(t_1) = \int_{t_i}^{t_1} \frac{dt}{a(t)} \quad (19)$$

The last = in Eq. (18) is for the convinence of coding.

1.3 Hubble sphere

Hubble sphere is where the Hubble flow velocity³ equals to the speed of light.

$$v_{\text{H}} = \dot{a}\chi \quad (20)$$

Hubble sphere as a function of time should be

$$\chi_{\text{H}} = \frac{c}{aH(a)} \quad (21)$$

1.4 Light Cone

Light cone is defined as $ds^2 = 0$, the photon starts from a_1 and ends at the observer now ($a_0, \chi = 0$). so

$$dt^2 = a^2(t)d\chi^2 \quad (22)$$

$$dt = a(t)d\chi \quad (23)$$

$$\chi_{\text{lc}}(a_1) = \int_{t_1}^{t_0} \frac{dt}{a(t)} = \int_{a_1}^{a_0} \frac{da}{a\dot{a}} = \int_{a_1}^{a_0} \frac{da}{a^2 H(a)} \quad (24)$$

1.5 Event Horizon

the photon starts from a_1 and ends at the observer in the infinity future ($a \rightarrow \infty, \chi = 0$).

$$\chi_{\text{EH}}(a_1) = \int_{t_1}^{\infty} \frac{dt}{a(t)} = \int_{a_1}^{\infty} \frac{da}{a^2 H(a)} \quad (25)$$

³in the proper length and is not real ‘velocity’.

1.6 Particle Horizon

Particle horizon is the trajectory that a photon start from $\chi = 0$ at t_i (in ‘our model’, $t_i = a_i = 0$).

$$\chi_{\text{PH}}(a_1) = \int_{t_i}^{t_1} \frac{dt}{a(t)} = \int_{a_i}^{a_1} \frac{da}{a\dot{a}} = \int_{a_i}^{a_1} \frac{da}{a^2 H(a)} \quad (26)$$

1.6.1 optical horizon

The optical horizon due to recombination $z(t_r) \approx 1100$.

$$d_{\text{opt}}(t) = a(\tau)(\tau - \tau_r) = a(t) \int_{t_r}^t \frac{dt'}{a(t')} \quad (27)$$

L^AT_EX

$$\mathbf{H}. \quad (28)$$

$$\mathbf{H}|\psi\rangle \quad (29)$$

citation example[[Lyth and Liddle\(2009\)](#)]

参考文献

[Lyth and Liddle(2009)] D. H. Lyth and A. R. Liddle, *The Primordial Density Perturbation: Cosmology, Inflation and the Origin of Structure* (Cambridge University Press, Cambridge, UK, 2009).