# Relativistic Perturbation Theory

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## 目录



# 1 Metric Perturbation

$$\mathrm{d}s^2 = a^2(\eta)\left[-(1+2A)\mathrm{d}\eta^2 + 2B_i\,\mathrm{d}x^i\,\mathrm{d}\eta + (\delta_{ij}+2E_{ij})\,\mathrm{d}x^i\,\mathrm{d}x^j\right] \tag{1}$$

## 1.1 SVT Decomposition

$$B_i = \underbrace{\partial_i B}_{\text{scalar}} + \underbrace{\hat{B}_i}_{\text{vector}} \tag{2}$$

$$E_{ij} = \underbrace{C\delta_{ij} + \partial_{\langle i}\partial_{j\rangle}E}_{\text{scalar}} + \underbrace{\partial_{(i}\hat{E}_{j)}}_{\text{vector}} + \underbrace{\hat{E}_{ij}}_{\text{tensor}}\,, \tag{3}$$

where

$$\begin{aligned}\partial_{\langle i}\partial_{j\rangle}E &\equiv \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)E \\ \partial_{(i}\hat{E}_{j)} &\equiv \frac{1}{2}\left(\partial_i\hat{E}_j + \partial_j\hat{E}_i\right).\end{aligned} \tag{4}$$

# 2 Coordinate Transformations

$$x^\mu(q) \mapsto \tilde{x}^\mu(q) \equiv x^\mu(q) + \xi^\mu(q), \quad \text{where} \quad \begin{aligned} &\xi^0 \equiv T, \\ &\xi^i \equiv L^i = \partial^i L + \hat{L}^i. \end{aligned} \tag{5}$$

# 3 Gauge Transformations of Metric Perturbations

Assume $A$, $B_i$ and $E_{ij}$ are in order $\xi$.

**Example (The transformation of $B_i$)**

$$\begin{aligned} a^2(\eta) B_i = g_{0i}(x) &= \frac{\partial \tilde{x}^\alpha}{\partial \eta} \frac{\partial \tilde{x}^\beta}{\partial x^i} \tilde{g}_{\alpha\beta}(\tilde{x}) \\ &= a^2(\eta + T) \left[ \underbrace{-\partial_i T}_{\text{00-term}} + \underbrace{\delta_i^j \tilde{B}_j}_{\text{0j-term}} + \underbrace{L^{j\prime} \delta_i^k \delta_{jk}}_{\text{jk-term}} \right] + \mathcal{O}(\xi^2) \end{aligned} \tag{6}$$

where j0-term is 0 since both $\frac{\partial \tilde{x}^j}{\partial \eta}$ and $\frac{\partial \tilde{\eta}}{\partial x^i}$ are perturbations.

$$B_i = (1 + 2\mathcal{H}T)\left[-\partial_i T + \tilde{B}_i + L^{i\prime}\right] + \mathcal{O}(\xi^2) \tag{7}$$

So

$$B_i \to \tilde{B}_i = B_i + \partial_i T - L^{i\prime} + \mathcal{O}(\xi^2) \tag{8}$$

**Example (The transformation of $E_{ij}$)**

$$g_{ij}(x) = \frac{\partial \tilde{x}^\alpha}{\partial x^i} \frac{\partial \tilde{x}^\beta}{\partial x^j} \tilde{g}_{\alpha\beta}(\tilde{x}) \tag{9}$$

**Exercise 6.1** SVT

# 4 Energy-momentum Tensor

$$\begin{aligned} &T_0^0 \equiv -(\bar{\rho} + \delta\rho) \\ &T_i^0 \equiv (\bar{\rho} + \bar{P}) v_i = -T_0^i \\ &T_j^i \equiv (\bar{P} + \delta P)\delta_j^i + \Pi_j^i, \quad \Pi_i^i \equiv 0 \end{aligned} \tag{10}$$

## 4.1 The Coordinate Transformations of Energy-momentum Tensor

$$T^\mu{}_\nu(x) = \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \frac{\partial \tilde{x}^\beta}{\partial x^\nu} \tilde{T}_\beta^\alpha(\tilde{x}) \tag{11}$$

The Jacobi

$$\frac{\partial \tilde{x}^\alpha}{\partial x^\mu} = \begin{pmatrix} \partial\tilde{\eta}/\partial\eta & \partial\tilde{\eta}/\partial x^i \\ \partial\tilde{x}^i/\partial\eta & \partial\tilde{x}^i/\partial x^j \end{pmatrix} = \begin{pmatrix} 1+T' & \partial_i T \\ L^{i\prime} & \delta^i_j + \partial_j L^i \end{pmatrix} \tag{12}$$

The inverse matrix of Jacobi should be delivered with perturbation theory

$$\frac{\partial x^\mu}{\partial \tilde{x}^\alpha} = \begin{pmatrix} \partial\eta/\partial\tilde{\eta} & \partial\eta/\partial\tilde{x}^i \\ \partial x^i/\partial\tilde{\eta} & \partial x^i/\partial\tilde{x}^j \end{pmatrix} = \begin{pmatrix} 1-T' & -\partial_i T \\ -L^{i\prime} & \delta^i_j - \partial_j L^i \end{pmatrix} \tag{13}$$

证明. $T$ 和 $L^i$ 是小量。用 $a \sim b_i \sim c_i \sim d_{ij} \sim \epsilon$ 代替。

$$\begin{pmatrix} 1+a & b_i \\ c_i & \delta^i_j + d_{ij} \end{pmatrix} \begin{pmatrix} 1-a & -b_i \\ -c_i & \delta^i_j - d_{ij} \end{pmatrix} = \begin{pmatrix} 1+\mathcal{O}(\epsilon^2) & \mathcal{O}(\epsilon^2) \\ \mathcal{O}(\epsilon^2) & \delta^i_j + \mathcal{O}(\epsilon^2) \end{pmatrix} \tag{14}$$

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