Boltzmann equation in CMB calculation

徐睿
清华大学 天文系

2023/11/15
DoA Cosmology Club

Outline

- Homogenous and isotropic cosmology

- CMB anisotropy

Outline

- Homogenous and isotropic cosmology
- CMB anisotropy

FLRW metric

Cosmological principle:

On the large scale, spacetime of the universe is

- homogeneous
- isotropy

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-Kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right)$$

Friedmann-Lemaitre-Robertson-Walker metric

FLRW metric

$$\frac{dt}{d\eta} = a$$

Cosmological principle:

On the large scale, spacetime of the universe is

- homogeneous
- isotropy

focusing on $K = 0$

$$\begin{aligned} ds^2 &= -dt^2 + a^2(t)\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right) \\ &= a^2\left(-d\eta^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right) \\ &= a^2\left(-d\eta^2 + dx^2 + dy^2 + dz^2\right) \end{aligned}$$

FLRW metric

$$
\begin{aligned}
ds^2 &= -dt^2 + a^2(t)\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right) \\
&= a^2\left(-d\eta^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right) \\
&= \boxed{a^2\left(-d\eta^2 + dx^2 + dy^2 + dz^2\right)}
\end{aligned}
$$

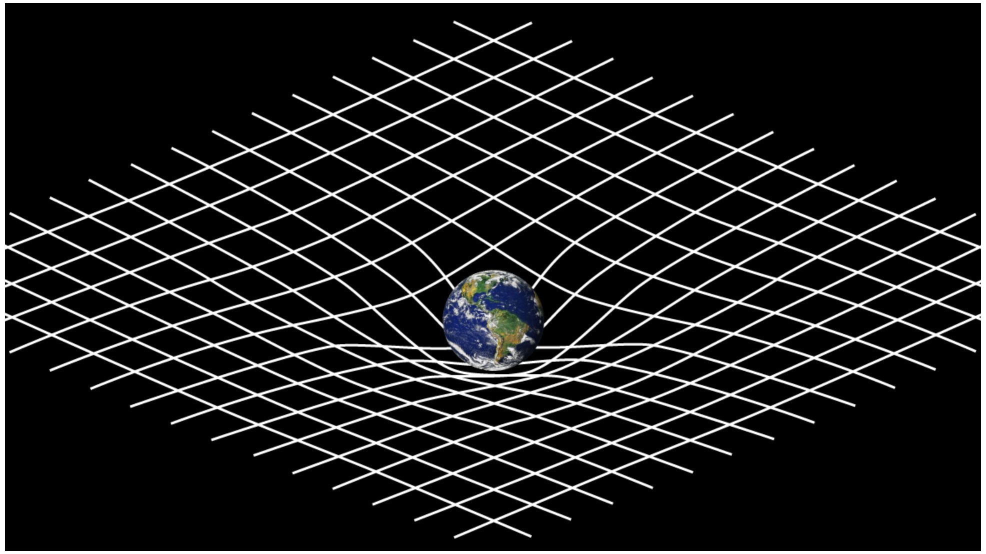
conformally flat

- t comoving time
- η conformal time

Friedmann equation in general relativity

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2 & 0 & 0 \\ 0 & 0 & a^2 & 0 \\ 0 & 0 & 0 & a^2 \end{pmatrix}$$



$$\Gamma^{\alpha}{}_{\mu\nu} = \tfrac{1}{2} g^{\alpha\lambda} \left(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu} \right),$$

$$R^{\alpha}{}_{\beta\gamma\delta} = \partial_\gamma \Gamma^{\alpha}{}_{\delta\beta} + \Gamma^{\alpha}{}_{\gamma\lambda} \Gamma^{\lambda}{}_{\delta\beta} - (\gamma \leftrightarrow \delta),$$

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} = \partial_\lambda \Gamma^{\lambda}{}_{\mu\nu} + \Gamma^{\lambda}{}_{\lambda\sigma} \Gamma^{\sigma}{}_{\mu\nu} - \partial_\nu \Gamma^{\lambda}{}_{\lambda\mu} - \Gamma^{\lambda}{}_{\nu\sigma} \Gamma^{\sigma}{}_{\lambda\mu},$$

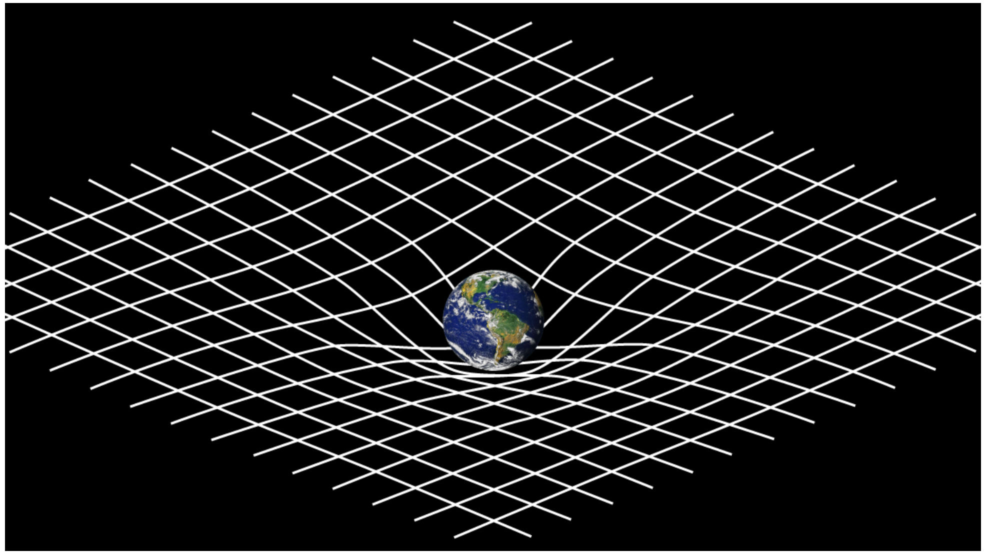
$$R = g^{\mu\nu} R_{\mu\nu},$$

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R.$$

Friedmann equation in general relativity

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$T_{\mu\nu} = (\epsilon + p)u_\mu u_\nu + p g_{\mu\nu}$$



Friedmann equation in general relativity

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$T_{\mu\nu} = (\epsilon + p) u_\mu u_\nu + p g_{\mu\nu}$$

isotropy $\quad u^\mu = (u^t, 0,0,0)$

Normalization
$g_{\mu\nu} u^\mu u^\nu = -1$ $\longrightarrow$ $u^t = 1$

The coarse-grained cosmic fluid's proper time is the comoving time

Friedmann equation in general relativity

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$T_{\mu\nu} = (\epsilon + p) u_\mu u_\nu + p g_{\mu\nu}$$

isotropy

$$u^\mu = (u^\eta, 0,0,0)$$

Normalization

$$g_{\mu\nu} u^\mu u^\nu = -1 \longrightarrow u^\eta = \frac{1}{a}$$

$$g_{\eta\eta} = -a^2$$

Friedmann equation in general relativity

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$$T_{\mu\nu} = (\epsilon + p) u_\mu u_\nu + p g_{\mu\nu}$$

$$u_\mu = (-1,0,0,0)$$

isotropy

$$u^\mu = (u^t, 0,0,0)$$

Normalization

$g_{\mu\nu} u^\mu u^\nu = -1$ $\longrightarrow$ $u^t = 1$

$$T_{\mu\nu} = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & a^2 p & 0 & 0 \\ 0 & 0 & a^2 p & 0 \\ 0 & 0 & 0 & a^2 p \end{pmatrix}$$

The coarse-grained cosmic fluid's proper time is the comoving time

Friedmann equation in general relativity

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

- matter: $p_m = 0$
- radiation: $p_r = \frac{1}{3}\epsilon_r$
- dark energy: $p_\Lambda = -\epsilon_\Lambda$

$$T_{\mu\nu} = (T_m)_{\mu\nu} + (T_r)_{\mu\nu} + (T_\Lambda)_{\mu\nu}$$

$$(T_m)_{\mu\nu} = \begin{pmatrix} \epsilon_m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(T_r)_{\mu\nu} = \begin{pmatrix} \epsilon_r & 0 & 0 & 0 \\ 0 & \frac{1}{3}a^2\epsilon_r & 0 & 0 \\ 0 & 0 & \frac{1}{3}a^2\epsilon_r & 0 \\ 0 & 0 & 0 & \frac{1}{3}a^2\epsilon_r \end{pmatrix}$$

$$(T_\Lambda)_{\mu\nu} = \begin{pmatrix} \frac{\Lambda}{\kappa} & 0 & 0 & 0 \\ 0 & -\frac{a^2\Lambda}{\kappa} & 0 & 0 \\ 0 & 0 & -\frac{a^2\Lambda}{\kappa} & 0 \\ 0 & 0 & 0 & -\frac{a^2\Lambda}{\kappa} \end{pmatrix}$$

Friedmann equation with dark energy

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\kappa(\epsilon_m + \epsilon_r + \epsilon_\Lambda) = \frac{{H_0}^2\Omega_{m0}}{a^3} + \frac{{H_0}^2\Omega_{r0}}{a^4} + {H_0}^2\Omega_{\Lambda 0}$$

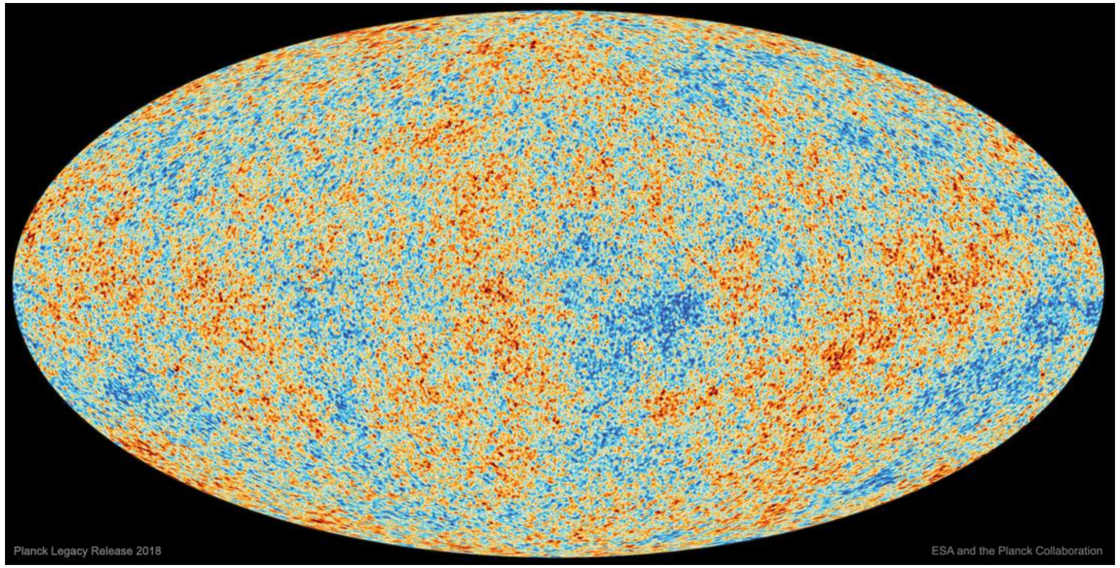
$$\frac{\ddot{a}}{a} = -\frac{1}{2}\frac{{H_0}^2\Omega_{m0}}{a^3} - \frac{{H_0}^2\Omega_{r0}}{a^4} + {H_0}^2\Omega_{\Lambda 0}$$

Outline

- Homogenous and isotropic cosmology
- CMB anisotropy

Inhomogeneity and anisotropy

Matter clustering: nonlinear simulation



CMB anisotropy: linear perturbation

CMB

- Radiation component of the universe

- Macroscopic description: fluid

$$G_{\mu\nu} = \frac{8\pi G}{c^4}\left[(T_m)_{\mu\nu} + (T_r)_{\mu\nu} + (T_\Lambda)_{\mu\nu}\right]$$

Energy-momentum conservation $D_\mu[(T_m)^{\mu\nu} + (T_r)^{\mu\nu}] = 0$

CMB

- Radiation component of the universe

- Macroscopic description: fluid

$$G_{\mu\nu} = \frac{8\pi G}{c^4}\left[(T_m)_{\mu\nu} + (T_r)_{\mu\nu} + (T_\Lambda)_{\mu\nu}\right]$$

Energy-momentum conservation $\quad D_\mu[(T_m)^{\mu\nu} + (T_r)^{\mu\nu}] = 0$

$$D_\mu(T_m)^{\mu\nu} = 0$$

$$D_\mu(T_r)^{\mu\nu} = 0$$

Only okay for isotropic universe

CMB

- Radiation component of the universe

- Macroscopic description: fluid

$$G_{\mu\nu} = \frac{8\pi G}{c^4}\left[(T_m)_{\mu\nu} + (T_r)_{\mu\nu} + (T_\Lambda)_{\mu\nu}\right]$$

Energy-momentum conservation $D_\mu[(T_m)^{\mu\nu} + (T_r)^{\mu\nu}] = 0$

$$D_\mu (T_m)^{\mu\nu} = \Gamma^\nu$$

$$D_\mu (T_r)^{\mu\nu} = -\Gamma^\nu$$

interaction between matter and photon contributes to anisotropies in CMB

CMB

- Microscopic description: Bose-Einstein system

$$f(\eta, \boldsymbol{x}, E, \hat{\boldsymbol{p}}) = \frac{1}{\exp[E/T] - 1}$$

- Equation of motion: Boltzmann equation

$$\frac{df}{d\eta} = C[f_a]$$

Interaction with other particles

CMB

- Statistic description: Bose-Einstein system

$$f(\eta, \boldsymbol{x}, E, \widehat{\boldsymbol{p}}) = \frac{1}{\exp[E/T] - 1}$$

- Equation of motion: Boltzmann equation

$$\frac{df}{d\eta} = C[f_a]$$

$$D_\mu (T_m)^{\mu\nu} = \Gamma^\nu$$

$$D_\mu (T_r)^{\mu\nu} = -\Gamma^\nu$$

Interaction with other particles

Boltzmann equation: 0th order

$$g_{\mu\nu} = \begin{pmatrix} -a^2(1+2\Psi) & 0 & 0 & 0 \\ 0 & a^2(1-2\Phi) & 0 & 0 \\ 0 & 0 & a^2(1-2\Phi) & 0 \\ 0 & 0 & 0 & a^2(1-2\Phi) \end{pmatrix}$$

Prerequisites:

✓ Photon 4-momentum in the perturbed FLRW spacetime

$$P^\mu = (E/a)[1-\Psi, (1+\Phi)\hat{\mathbf{p}}]\,,$$
$$P_\nu = (aE)[1+\Psi, -(1-\Phi)\hat{\mathbf{p}}]\,.$$

✓ Photon geodesic in the perturbed FLRW spacetime

$$P^\mu D_\mu P^\nu = 0 \quad \Longrightarrow \quad \frac{d\ln\epsilon}{d\eta} = -\frac{d\Psi}{d\eta} + \Phi' + \Psi'$$

$$\epsilon = aE \qquad \frac{d}{d\eta} = \frac{\partial}{\partial\eta} + \hat{\boldsymbol{p}}\cdot\boldsymbol{\nabla}$$

Boltzmann equation: 0th order

$$\frac{df}{d\eta} = 0 \qquad\qquad f(\eta, \boldsymbol{x}, E, \hat{\boldsymbol{p}}) = \frac{1}{\exp[E/T] - 1}$$

$$\frac{d\ln\epsilon}{d\eta} = 0$$

$$\epsilon = aE$$

$$\bar{T} \propto \frac{1}{a}$$

Boltzmann equation: 1st order

$$T = \bar{T}(1+\Theta)$$

$$f(\eta, \boldsymbol{x}, E, \widehat{\boldsymbol{p}}) = \frac{1}{\exp[E/T]-1} = \bar{f} + \bar{f}^2 \frac{E}{\bar{\bar{T}}} \exp\left[\frac{E}{\bar{\bar{T}}}\right] \Theta(\eta, \boldsymbol{x}, \widehat{\boldsymbol{p}}) + O(2)$$

$$\bar{f}(\eta, E) = \frac{1}{\exp[E/\bar{T}]-1}$$

$E = E(\eta)$ as the photon propagates in the expanding universe

$$\frac{df}{d\eta} = \frac{d\bar{f}}{d\eta} + \bar{f}^2 \frac{E}{\bar{\bar{T}}} \exp\left[\frac{E}{\bar{\bar{T}}}\right] \frac{d\Theta}{d\eta} + O(2) = -\bar{f}^2 \frac{E}{\bar{\bar{T}}} \exp\left[\frac{E}{\bar{\bar{T}}}\right] \left(\frac{d\ln E}{d\eta} - \frac{d\ln\bar{T}}{d\eta} - \frac{d\Theta}{d\eta}\right) + O(2)$$

Boltzmann equation: 1st order

$$T = \bar{T}(1+\Theta)$$

$$f(\eta, \boldsymbol{x}, E, \widehat{\boldsymbol{p}}) = \frac{1}{\exp[E/T]-1} = \bar{f} + \bar{f}^2 \frac{E}{\bar{T}} \exp\left[\frac{E}{\bar{T}}\right] \Theta(\eta, \boldsymbol{x}, \widehat{\boldsymbol{p}}) + O(2)$$

$$\bar{f}(\eta, E) = \frac{1}{\exp[E/\bar{T}]-1}$$

$E = E(\eta)$ as the photon propagates in the expanding universe

$\bar{T} \propto \frac{1}{a}$ used

$$\frac{df}{d\eta} = -\bar{f}^2 \frac{E}{\bar{T}} \exp\left[\frac{E}{\bar{T}}\right]\left(\frac{d\ln(aE)}{d\eta} - \frac{d\Theta}{d\eta}\right) + O(2) = -\bar{f}^2 \frac{E}{\bar{T}} \exp\left[\frac{E}{\bar{T}}\right]\left(\frac{d\ln\epsilon}{d\eta} - \frac{d\Theta}{d\eta}\right) + O(2)$$

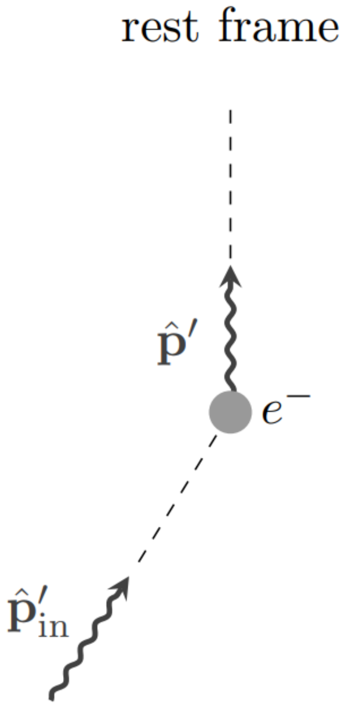
Boltzmann equation: 1st order

$$\frac{df}{d\eta} = C[f_a]$$

$$\text{LHS} = \frac{df}{d\eta} = -\bar{f}^2 \frac{E}{\bar{\bar{T}}} \exp\left[\frac{E}{\bar{\bar{T}}}\right]\left(\frac{d\ln\epsilon}{d\eta} - \frac{d\Theta}{d\eta}\right) + O(2)$$

$$\text{RHS} = C[f_a] = \text{electron photon scattering} = ?$$

Electron-photon scattering: electron rest frame

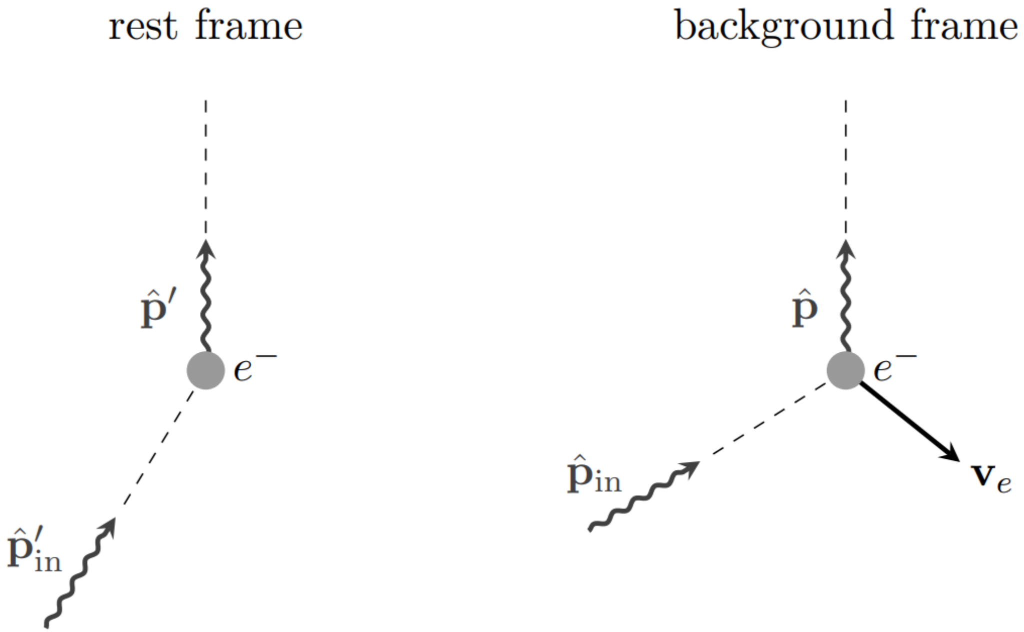


$$C'[f'(\epsilon', \hat{\mathbf{p}}')] \equiv \left.\frac{\mathrm{d}f'(\epsilon', \hat{\mathbf{p}}')}{\mathrm{d}\tau'}\right|_{\text{scatt.}} = n_e \int \mathrm{d}\hat{\mathbf{p}}'_{\text{in}} \, \frac{d\sigma}{d\Omega} \, [\underset{\substack{\uparrow \\ \text{in}}}{f'(\epsilon', \hat{\mathbf{p}}'_{\text{in}})} - \underset{\substack{\uparrow \\ \text{out}}}{f'(\epsilon', \hat{\mathbf{p}}')}]$$

$$\frac{d\sigma}{d\Omega} = \frac{3\sigma_T}{16\pi}\left[1 + \cos^2\theta\right] \qquad \sigma_T = \frac{8\pi}{3}\left(\frac{q_e^2}{4\pi\epsilon_0 m_e c^2}\right)^2 = 6.65 \times 10^{-29}\,\mathrm{m}^2$$

Daniel Baumann, *Cosmology,* Cambridge University Press (2022)

Electron-photon scattering: background frame



$$C[f(\epsilon, \hat{\mathbf{p}})] \equiv \left.\frac{\mathrm{d}f(\epsilon, \hat{\mathbf{p}})}{\mathrm{d}\eta}\right|_{\text{scatt.}} = a\left.\frac{\mathrm{d}f'(\epsilon', \hat{\mathbf{p}}')}{\mathrm{d}\tau}\right|_{\text{scatt.}} + O(2)$$

$$C'[f'(\epsilon', \hat{\mathbf{p}}')] \equiv \left.\frac{\mathrm{d}f'(\epsilon', \hat{\mathbf{p}}')}{\mathrm{d}\tau'}\right|_{\text{scatt.}}$$

Daniel Baumann, *Cosmology,* Cambridge University Press (2022)

Electron-photon scattering: background frame

$$
\begin{aligned}
C[f(E,\hat{\boldsymbol{p}})] &= an_e \int d\hat{\boldsymbol{p}}'_{\rm in} \tfrac{d\sigma}{d\Omega} \left[f'(E'_{\rm in},\hat{\boldsymbol{p}}'_{\rm in}) - f'(E',\hat{\boldsymbol{p}}')\right] \\
&= an_e \int d\hat{\boldsymbol{p}}'_{\rm in} \tfrac{d\sigma}{d\Omega} \left[f(E_{\rm in},\hat{\boldsymbol{p}}_{\rm in}) - f(E,\hat{\boldsymbol{p}})\right] \\
&= an_e \int d\hat{\boldsymbol{p}}'_{\rm in} \tfrac{d\sigma}{d\Omega} \left[\bar{f}(E_{\rm in}) + \bar{f}(E_{\rm in})^2 \tfrac{E_{\rm in}}{T} \exp\left(\tfrac{E_{\rm in}}{T}\right)\Theta(\eta,\boldsymbol{x},\hat{\boldsymbol{p}}_{\rm in}) - \bar{f}(E) - \bar{f}(E)^2 \tfrac{E}{T} \exp\left(\tfrac{E}{T}\right)\Theta(\eta,\boldsymbol{x},\hat{\boldsymbol{p}})\right] + O(2) \\
&= -\bar{f}(E)^2 \tfrac{E}{T} \exp\left(\tfrac{E}{T}\right) an_e \int d\hat{\boldsymbol{p}}'_{\rm in} \tfrac{d\sigma}{d\Omega} \left[\tfrac{E_{\rm in}-E}{E} - \Theta(\eta,\boldsymbol{x},\hat{\boldsymbol{p}}'_{\rm in}) + \Theta(\eta,\boldsymbol{x},\hat{\boldsymbol{p}})\right] + O(2) \\
&= -\bar{f}(E)^2 \tfrac{E}{T} \exp\left(\tfrac{E}{T}\right)\Gamma \left[-\boldsymbol{v}_e\cdot\hat{\boldsymbol{p}} + \Theta(\eta,\boldsymbol{x},\hat{\boldsymbol{p}}) - \tfrac{3}{16\pi}\int d\hat{\boldsymbol{p}}'_{\rm in} \left(1 + (\hat{\boldsymbol{p}}_{\rm in}\cdot\hat{\boldsymbol{p}})^2\right)\Theta(\eta,\boldsymbol{x},\hat{\boldsymbol{p}}'_{\rm in})\right] + O(2)
\end{aligned}
$$

use $E_{\rm in} = \gamma E'_{\rm in}\left(1 + \boldsymbol{v}_e\cdot\hat{\boldsymbol{p}}'_{\rm in}\right) = \gamma E'\left(1 + \boldsymbol{v}_e\cdot\hat{\boldsymbol{p}}'_{\rm in}\right) = \gamma^2 E\left(1 - \boldsymbol{v}_e\cdot\hat{\boldsymbol{p}}\right)\left(1 + \boldsymbol{v}_e\cdot\hat{\boldsymbol{p}}'_{\rm in}\right) = E\left[1 + \boldsymbol{v}_e\cdot(\hat{\boldsymbol{p}}'_{\rm in} - \hat{\boldsymbol{p}})\right] + O(2)$

Boltzmann equation: 1st order

$$\frac{df}{d\eta} = C[f_a]$$

$$\mathrm{LHS} = \frac{df}{d\eta} = -\bar{f}^2 \frac{E}{\bar{\bar{T}}} \exp\left[\frac{E}{\bar{\bar{T}}}\right]\left(\frac{d\ln\epsilon}{d\eta} - \frac{d\Theta}{d\eta}\right) + O(2)$$

$$\mathrm{RHS} = C[f_a] = -\bar{f}^2 \frac{E}{\bar{\bar{T}}} \exp\left[\frac{E}{\bar{\bar{T}}}\right] \times \Gamma\left[-\boldsymbol{v}_e \cdot \hat{\boldsymbol{p}} + \Theta(\eta, \boldsymbol{x}, \hat{\boldsymbol{p}}) - \frac{3}{16\pi}\int d\hat{\boldsymbol{p}}'_{\mathrm{in}}\left(1 + (\hat{\boldsymbol{p}}_{\mathrm{in}} \cdot \hat{\boldsymbol{p}})^2\right)\Theta(\eta, \boldsymbol{x}, \hat{\boldsymbol{p}}'_{\mathrm{in}})\right] + O(2)$$

$$\Downarrow$$

$$\boxed{\frac{d\Theta}{d\eta} = \frac{d\ln\varepsilon}{d\eta} - \Gamma\left[-\boldsymbol{v}_e \cdot \hat{\boldsymbol{p}} + \Theta(\eta, \boldsymbol{x}, \hat{\boldsymbol{p}}) - \frac{3}{16\pi}\int d\hat{\boldsymbol{p}}'_{\mathrm{in}}\left(1 + (\hat{\boldsymbol{p}}_{\mathrm{in}} \cdot \hat{\boldsymbol{p}})^2\right)\Theta(\eta, \boldsymbol{x}, \hat{\boldsymbol{p}}'_{\mathrm{in}})\right]}$$

Relation between photon fluid variables and CMB temperature multipoles

- Radiation component of the universe

$$g_{\mu\nu} = \begin{pmatrix} -a^2(1+2\Psi) & 0 & 0 & 0 \\ 0 & a^2(1-2\Phi) & 0 & 0 \\ 0 & 0 & a^2(1-2\Phi) & 0 \\ 0 & 0 & 0 & a^2(1-2\Phi) \end{pmatrix}$$

- Fluid description

$$G_{\mu\nu} = \frac{8\pi G}{c^4}\left[(T_m)_{\mu\nu} + (T_r)_{\mu\nu} + (T_\Lambda)_{\mu\nu}\right]$$

$$(T_m)_{\mu\nu} = \begin{pmatrix} \epsilon_m + \delta\epsilon_m + 2\Psi\epsilon_m & -\epsilon_m u_{mx} & -\epsilon_m u_{my} & -\epsilon_m u_{mz} \\ -\epsilon_m u_{mx} & 0 & 0 & 0 \\ -\epsilon_m u_{my} & 0 & 0 & 0 \\ -\epsilon_m u_{mz} & 0 & 0 & 0 \end{pmatrix}$$

$$(T_\gamma)_{\mu\nu} = \begin{pmatrix} \epsilon_\gamma + \delta\epsilon_\gamma + 2\Psi\epsilon_\gamma & -\frac{4}{3}\epsilon_\gamma u_{\gamma x} & -\frac{4}{3}\epsilon_\gamma u_{\gamma y} & -\frac{4}{3}\epsilon_\gamma u_{\gamma z} \\ -\frac{4}{3}\epsilon_\gamma u_{\gamma x} & \frac{a^2}{3}\left(\epsilon_\gamma + \delta\epsilon_\gamma - 2\Phi\epsilon_\gamma\right) + \Pi_{xx} & \Pi_{xy} & \Pi_{xz} \\ -\frac{4}{3}\epsilon_\gamma u_{\gamma y} & \Pi_{xy} & \frac{a^2}{3}\left(\epsilon_\gamma + \delta\epsilon_\gamma - 2\Phi\epsilon_\gamma\right) + \Pi_{yy} & \Pi_{yz} \\ -\frac{4}{3}\epsilon_\gamma u_{\gamma z} & \Pi_{xz} & \Pi_{yz} & \frac{a^2}{3}\left(\epsilon_\gamma + \delta\epsilon_\gamma - 2\Phi\epsilon_\gamma\right) + \Pi_{zz} \end{pmatrix}$$

Relation between photon fluid variables and CMB temperature multipoles

Bose-Einstein distribution:

$$f(\eta, \mathbf{x}, \epsilon, \hat{\mathbf{p}}) = \left[\exp\left(\frac{\epsilon}{a\bar{T}(\eta)[1+\Theta(\eta,\mathbf{x},\hat{\mathbf{p}})]}\right) - 1\right]^{-1}$$

$$\Downarrow$$

$$T^\mu{}_\nu = \int \frac{\mathrm{d}^3 p}{E(p)}\, f\, P^\mu P_\nu\,,$$

$$P^\mu = (E/a)[1-\Psi, (1+\Phi)\hat{\mathbf{p}}]\,,$$
$$P_\nu = (aE)[1+\Psi, -(1-\Phi)\hat{\mathbf{p}}]\,.$$

$$\delta\epsilon_\gamma = 4\epsilon_\gamma \Theta_0$$

$$u_{\gamma i} = -3ia\hat{k}^i\Theta_1$$

$$\Pi_{ij} = -4a^2\epsilon_\gamma(\hat{k}^i\hat{k}^j - \frac{1}{3}\delta_{ij})\Theta_2$$

Summary

At the first order of perturbation, energy-momentum tensors of photon and baryon are not conserved separately due to the scattering between photon and electron

$$\Downarrow$$

Conservation equations need to be replaced by Boltzmann equations

Summary

$$\frac{\mathrm{d}f}{\mathrm{d}\eta} = C[\{f, f_e\}]$$

$$f(\eta, \mathbf{x}, \epsilon, \hat{\mathbf{p}}) = \left[\exp\left(\frac{\epsilon}{a\bar{T}(\eta)[1+\Theta(\eta,\mathbf{x},\hat{\mathbf{p}})]}\right) - 1\right]^{-1}$$

$$\frac{d\Theta}{d\eta} = \frac{d\ln\varepsilon}{d\eta} - \Gamma\left[-\boldsymbol{v}_e\cdot\hat{\boldsymbol{p}} + \Theta(\eta,\boldsymbol{x},\hat{\boldsymbol{p}}) - \frac{3}{16\pi}\int d\hat{\boldsymbol{p}}'_{\rm in}\left(1+(\hat{\boldsymbol{p}}_{\rm in}\cdot\hat{\boldsymbol{p}})^2\right)\Theta(\eta,\boldsymbol{x},\hat{\boldsymbol{p}}'_{\rm in})\right]$$

Fourier transformation

$$\Theta \rightarrow \Theta e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

$$\frac{d\ln\epsilon}{d\eta} = -\frac{d\Psi}{d\eta} + \Phi' + \Psi'$$

$$\Theta' + ik\mu\Theta = \Phi' - ik\mu\Psi - \Gamma\left[\Theta - \Theta_0 - i\mu v_b + \frac{1}{2}\Theta_2 P_2(\mu)\right]$$

$$\boldsymbol{v}_e = \boldsymbol{v}_b = i\boldsymbol{k}v_b$$

$$\mu = \hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{p}}$$

Summary (continue)

$$\Theta' + ik\mu\Theta = \Phi' - ik\mu\Psi - \Gamma\left[\Theta - \Theta_0 - i\mu v_b + \frac{1}{2}\Theta_2 P_2(\mu)\right]$$

$$\Theta(\eta, \boldsymbol{k}, \hat{\boldsymbol{p}}) = \sum_{l=0}^{\infty}(-i)^l(2l+1)\Theta_l(\eta, \boldsymbol{k})P_l(\hat{\boldsymbol{p}} \cdot \hat{\boldsymbol{k}})$$

$$\Theta_l' + \tfrac{k}{2l+1}\left[(l+1)\Theta_{l+1} - l\Theta_{l-1}\right] = \delta_{l0}\Phi' + \delta_{l1}\tfrac{k}{3}\Psi - \Gamma\left[(1 - \delta_{l0} - \tfrac{1}{10}\delta_{l2})\Theta_l + \delta_{l1}\tfrac{v_b}{3}\right]$$

Summary of the perturbation equations

$$\Theta_0' = \Phi' - k\Theta_1,$$
$$\Theta_1' = \frac{k}{3}\Theta_0 + \frac{k}{3}\Psi - \frac{2k}{3}\Theta_2 - \Gamma\left(\Theta_1 + \frac{1}{3}v_b\right),$$
$$\Theta_l' + \frac{k}{2l+1}\left[(l+1)\Theta_{l+1} - l\Theta_{l-1}\right] = -\Gamma\left(1 - \frac{1}{10}\delta_{l2}\right)\Theta_l, \quad l \geq 2,$$
$$\delta_b' = 3\Phi' + kv_b,$$
$$v_b' = -k\Psi - \mathcal{H}v_b - \frac{4\epsilon_\gamma^{(0)}}{3\epsilon_b^{(0)}}\Gamma\left(3\Theta_1 + v_b\right),$$
$$\delta_c' = 3\Phi' + kv_c,$$
$$v_c' = -k\Psi - \mathcal{H}v_c,$$

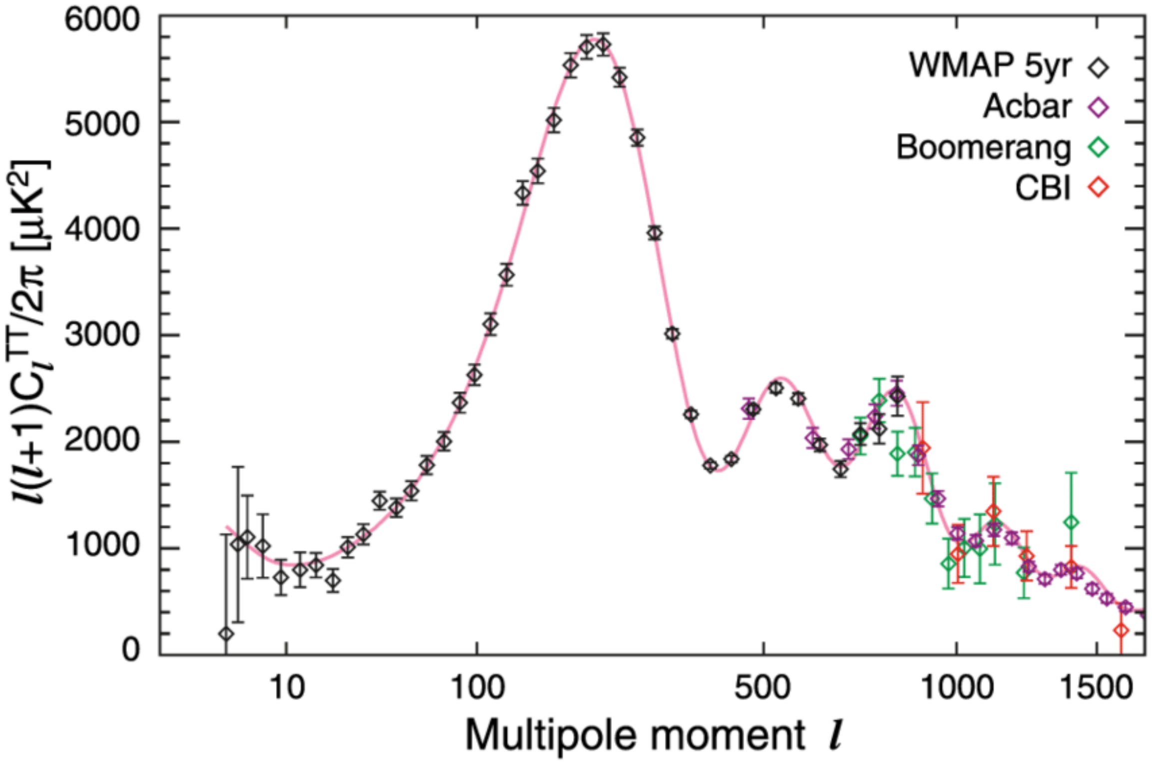
$$0 = 4\kappa a^4\epsilon_\gamma\Theta_1 - \kappa a^4\epsilon_c v_c - \kappa a^4\epsilon_b v_b - 2kaa'\Psi - 2ka^2\Phi',$$
$$0 = \frac{4\kappa a^2\epsilon_\gamma\Theta_2}{k^2} - \Phi + \Psi$$

Initial condition

When $a \to 0,\ k \ll \frac{a'}{a}$

$$\delta_m' = 3\Phi'$$
$$\delta_r' = 4\Phi'$$

CMB temperature spectrum



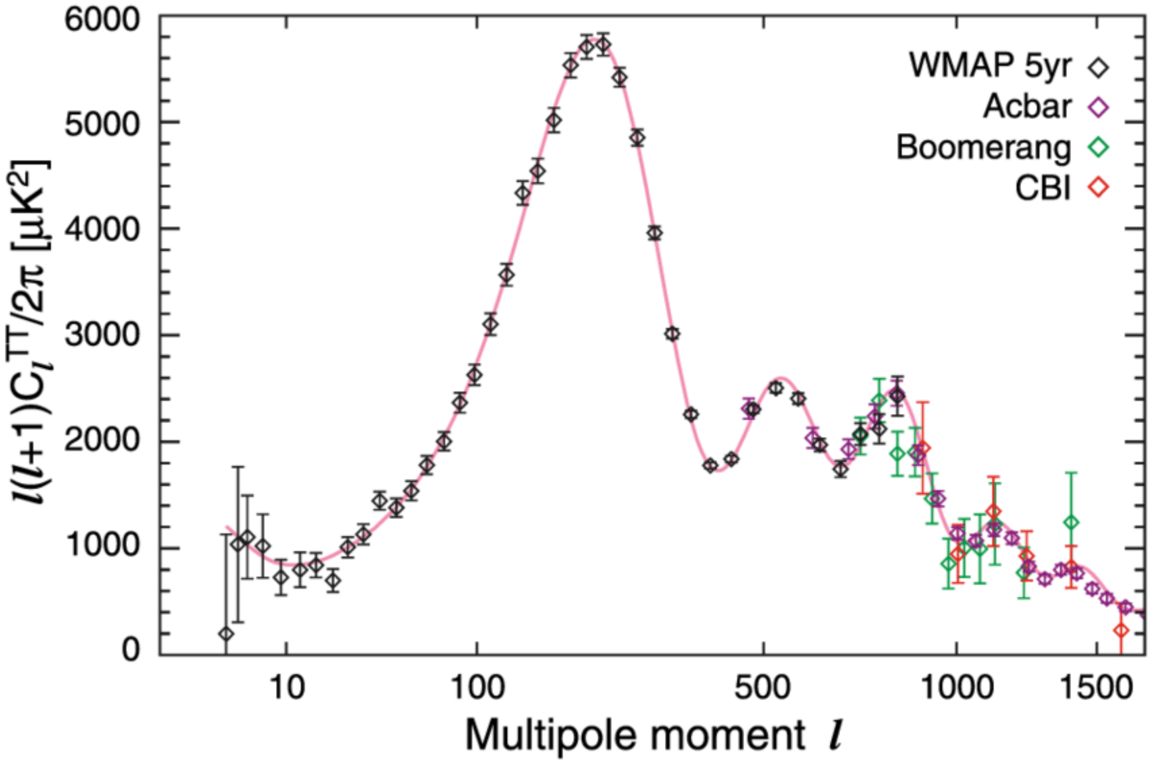
Observation:

$$T(\hat{\mathbf{n}}) \equiv \bar{T}_0 \left[1 + \Theta(\hat{\mathbf{n}})\right]$$

$$C(\theta) = \langle \Theta(\hat{\boldsymbol{n}})\Theta(\hat{\boldsymbol{n}}') \rangle = \sum_l \frac{2l+1}{4\pi} C_l P_l(\cos\theta).$$

Angular power spectrum

CMB temperature spectrum



Multipole moments of the temperature fluctuation

Theory:

$$\Theta(\hat{\boldsymbol{n}}) = \sum_l i^l(2l+1)\int \frac{d^3k}{(2\pi)^3}\mathcal{R}_i(\boldsymbol{k})P_l(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{n}})\Theta_l(k)$$

Statistically isotropic initial conditions

$$\langle \mathcal{R}_i(\boldsymbol{k})\mathcal{R}_i(\boldsymbol{k}')\rangle = \frac{2\pi^2}{k^3}\Delta^2_{\mathcal{R}}(k)(2\pi)^3\delta^3(\boldsymbol{k}+\boldsymbol{k}')$$

$$C_l = 4\pi\int \frac{dk}{k}\Theta_l^2(k)\Delta^2_{\mathcal{R}}(k)$$