Test

Siyi Zhao (赵思逸)

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目录

1 Expansion

1.1 from time to scale factor

We have two kinds of "time" in cosmology since our Universe are expanding.

If we focus on the coordinate, the metric is written as ($c = 1$)

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) \tag{1}$$

better to write in spherical coordinate system

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\mathrm{d}\chi^2 + \chi^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right)\right] \tag{2}$$

By scaling the time axis, we can get the conformal time, and have an "appearantly" Minkovski spacetime

$$\mathrm{d}s^2 = a^2(\tau)(-\mathrm{d}\tau^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) \tag{3}$$

The scale factor can be solved from the **Friedmann equation**, [1]

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho \tag{4}$$

with **a model of energy-momentum** contained in the Universe like [2]

$$\frac{\rho(t)}{\rho_{\mathrm{cr}}} = \sum_{s=\gamma,\mathrm{m},\nu,\mathrm{DE}} \Omega_s a(t)^{-3(1+w_s)} \tag{5}$$

This model assumes a constant of equation of state.

In our ΛCDM model (reserve curvature)

$$\frac{\rho(t)}{\rho_{\mathrm{cr}}} = \Omega_{\mathrm{R}}\left(\frac{a}{a_0}\right)^{-4} + \Omega_{\mathrm{M}}\left(\frac{a}{a_0}\right)^{-3} + \Omega_\Lambda + \Omega_{\mathrm{K}}\left(\frac{a}{a_0}\right)^{-2} \tag{6}$$

just solve the differencial equation

$$\dot{a} = H_0 a_0 \sqrt{\Omega_R\left(\frac{a}{a_0}\right)^{-2} + \Omega_M\left(\frac{a}{a_0}\right)^{-1} + \Omega_\Lambda\left(\frac{a}{a_0}\right)^{2} + \Omega_K} \tag{7}$$

$$t(a_1) = \int_0^{a_1} \frac{\mathrm{d}a}{H_0 a_0 \sqrt{\Omega_R\left(\frac{a}{a_0}\right)^{-2} + \Omega_M\left(\frac{a}{a_0}\right)^{-1} + \Omega_\Lambda\left(\frac{a}{a_0}\right)^{2} + \Omega_K}} \tag{8}$$

It's not linear, so need numerical solution. See code/expansion.ipynb

But it's OK to see some exceptions.

Block 1. *matter donimated:*

$$\dot{a} = H_0(a_0\Omega_M)^{\frac{1}{2}} a^{-\frac{1}{2}} \tag{9}$$

then

$$t = \frac{2}{3H_0(a_0\Omega_M)^{\frac{1}{2}}} a^{\frac{3}{2}} \tag{10}$$

assumed $a(t=0)=0$.

1.2 to more notations..

It will be easy to use redshift and Hubble parameter $H(z)$ in the calculations.

1.2.1 Cosmological Redshift

Define redshift as

$$\frac{a}{a_0} = \frac{1}{1+z} \tag{11}$$

Redshift z is linked to a and t simply. How about the comoving spacial coordinate χ?

$$\mathrm{d}a = -\frac{a_0 \mathrm{d}z}{(1+z)^2} \tag{12}$$

The comoving distance from z_1 to now ($z_0 = 0$)

$$\chi(z_1) = \int_{a_1}^{a_0} \frac{\mathrm{d}a}{a^2 H(a)} = \int_{z_0}^{z_1} \frac{c\,\mathrm{d}z}{a_0 H(z)} \tag{13}$$

where $z_0 = 0$ by defination, c is filled to balance the unit.

[1] also deriveing the Friedmann equations from Einstein field equation is not that trival, see another block in preparation..

[2] Friedmann equation itself has assumptions about the energy-momentum, it's ideal fluid which can be parameterized by only ρ and P..

1.2.2 Hubble parameter

And turn Eq. (4) to

$$H(a) \equiv \frac{\dot{a}}{a} = H_0 \sqrt{\frac{\rho(a)}{\rho_{\rm cr}}} \equiv H_0 E(a) \tag{14}$$

Eq. (6) turns out

$$E(a) = \sqrt{\Omega_{\rm R}\left(\frac{a}{a_0}\right)^{-4} + \Omega_{\rm M}\left(\frac{a}{a_0}\right)^{-3} + \Omega_\Lambda + \Omega_{\rm K}\left(\frac{a}{a_0}\right)^{-2}} \tag{15}$$

$$E(z) = \sqrt{\Omega_{\rm R}(1+z)^4 + \Omega_{\rm M}(1+z)^3 + \Omega_\Lambda + \Omega_{\rm K}(1+z)^2} \tag{16}$$

1.2.3 Conformal Time

Conformal time identic to the relation between χ and t in light cone.

$$\mathrm{d}\tau^2 = \frac{\mathrm{d}t^2}{a^2(t)} \tag{17}$$

$$\mathrm{d}\tau = \frac{\mathrm{d}t}{a(t)} = \mathrm{d}\chi \tag{18}$$

$$\tau(t_1) = \int_{t_i}^{t_1} \frac{\mathrm{d}t}{a(t)} \tag{19}$$

The last $=$ in Eq. (18) is for the convinence of coding.

1.3 Hubble sphere

Hubble sphere is where the Hubble flow velocity[3] equals to the speed of light.

$$v_{\rm H} = \dot{a}\chi \tag{20}$$

Hubble sphere as a function of time should be

$$\chi_{\rm H} = \frac{c}{aH(a)} \tag{21}$$

1.4 Light Cone

Light cone is defined as $\mathrm{d}s^2 = 0$, the photon starts from a_1 and ends at the observer now $(a_0, \chi = 0)$. so

$$\mathrm{d}t^2 = a^2(t)\mathrm{d}\chi^2 \tag{22}$$

$$\mathrm{d}t = a(t)\mathrm{d}\chi \tag{23}$$

$$\chi_{\rm lc}(a_1) = \int_{t_1}^{t_0} \frac{\mathrm{d}t}{a(t)} = \int_{a_1}^{a_0} \frac{\mathrm{d}a}{a\dot{a}} = \int_{a_1}^{a_0} \frac{\mathrm{d}a}{a^2 H(a)} \tag{24}$$

1.5 Event Horizon

the photon starts from a_1 and ends at the observer in the infinity future $(a \to \infty, \chi = 0)$.

$$\chi_{\rm EH}(a_1) = \int_{t_1}^{\infty} \frac{\mathrm{d}t}{a(t)} = \int_{a_1}^{\infty} \frac{\mathrm{d}a}{a^2 H(a)} \tag{25}$$

[3] in the proper length and is not real 'velovity'.

1.6 Particle Horizon

Particle horizon is the trajectory that a photon start from $\chi = 0$ at t_i (in 'our model', $t_i = a_i = 0$).

$$\chi_{\mathrm{PH}}(a_1) = \int_{t_i}^{t_1} \frac{\mathrm{d}t}{a(t)} = \int_{a_i}^{a_1} \frac{\mathrm{d}a}{a\dot{a}} = \int_{a_i}^{a_1} \frac{\mathrm{d}a}{a^2 H(a)} \tag{26}$$

1.6.1 optical horizon

The optical horizon due to recombination $z(t_r) \approx 1100$.

$$d_{\mathrm{opt}}(t) = a(\tau)(\tau - \tau_r) = a(t) \int_{t_r}^{t} \frac{\mathrm{d}t'}{a(t')} \tag{27}$$

LaTeX

$$\mathrm{H}\,. \tag{28}$$

$$\mathrm{H}\,|\psi\rangle \tag{29}$$

2 Boltzmann Equation

我们讨论 inflation 结束后到 recombination 之前，光子如何产生各向异性/光子的各向异性如何演化。我们将看到光子和物质之间通过 Thomson scattering 相互作用，将为光子贡献一部分各向异性。(interaction between matter and photon contributes to the **anisotropies of CMB**)

在微观层面，满足 Bosen-Einstein 分布

$$f(\eta, \mathbf{x}, E, \widehat{\mathbf{p}}) = \frac{1}{e^{E/T} - 1} \tag{30}$$

说明：

- 注意相空间是四维时空加上四动量。
- $\widehat{\mathbf{p}}$ 是因为光子动量由能量决定。
- thermal dynamic in local coordinate,
- 讨论问题：**thermal dynamic 如何推广到 Minkowski space with perturbation ？这样简单的推广成立吗？——似乎是个还在研究中的问题。**
- f is scalar since 相空间的体积元不变，且粒子数不变，$f =$ 粒子数/相空间的体积元。
- 相体积不变，熵不变，T 随 E 变。
- 讨论问题：在考虑温度 T 是否改变时聊到了：黑体谱在多普勒红移后还是黑体谱吗？所以聊的结论是啥？

EoM: Boltzmann equation

$$\frac{\mathrm{d}f}{\mathrm{d}\tau} = C[f_a] \tag{31}$$

$C[f_a]$ is the interaction with other particles (eg. baryon)

2.1 0th order

推出 $\bar{T} \propto \frac{1}{a}$

2.2 1st order

Step1: 对 f 做 Taylor expansion

$$f(\eta, \mathbf{x}, E, \widehat{\mathbf{p}}) = \frac{1}{e^{E/T} - 1} \tag{32}$$

- 算 Boltzmann equation - RHS - e-γ scattering in e- coordinate （Thomson scattering） - $\frac{d\sigma}{d\Omega}$ - 1 给出 monopole - cos theta 给出 quadrupole - 通过这些电子影响光子的分布，给光子 anisotropy ? - 洛伦兹变换

- fluid description - 和 f 比较，得到 CMB temperature multipoles 和 photon fluid variables energy-momentum tensor perturbation multipoles 的关系

- 以上两件事放在一起，得到 CMB temperature multipoles - 和 geodesic 联立，把 epsilon 换成 Bardeen's potential - $\mu = \dot{p}k$

- 得到 Θ 的演化方程 - 和 matter 的 energy-momentum tensor 比较，得到 baryon perturbation , dark matter perturbation 和 baryon perturbation 差一个散射项 - Einstein field equation 得到 Bardeen's potential 的演化。

LaTeXusage

citation example[Lyth and Liddle(2009)]

参考文献

[Lyth and Liddle(2009)] D. H. Lyth and A. R. Liddle, *The Primordial Density Perturbation: Cosmology, Inflation and the Origin of Structure* (Cambridge University Press, Cambridge, UK, 2009).