Test

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目录

1 Expansion

1.1 from time to scale factor

We have two kinds of "time" in cosmology since our Universe are expanding.

If we focus on the coordinate, the metric is written as $(c = 1)$

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) \tag{1}$$

better to write in spherical coordinate system

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\mathrm{d}\chi^2 + \chi^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right)\right] \tag{2}$$

By scaling the time axis, we can get the conformal time, and have an "appearantly" Minkovski spacetime

$$\mathrm{d}s^2 = a^2(\tau)(-\mathrm{d}\tau^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) \tag{3}$$

The scale factor can be solved from the **Friedmann equation**, [1]

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho \tag{4}$$

[1] also deriveing the Friedmann equations from Einstein field equation is not that trival, see another block in preparation..

with **a model of energy-momentum** contained in the Universe like [2]

$$\frac{\rho(t)}{\rho_{\mathrm{cr}}} = \sum_{s=\gamma,\mathrm{m},\nu,\mathrm{DE}} \Omega_s a(t)^{-3(1+w_s)} \tag{5}$$

This model assumes a constant of equation of state.

In our ΛCDM model (reserve curvature)

$$\frac{\rho(t)}{\rho_{\mathrm{cr}}} = \Omega_{\mathrm{R}}\left(\frac{a}{a_0}\right)^{-4} + \Omega_{\mathrm{M}}\left(\frac{a}{a_0}\right)^{-3} + \Omega_{\Lambda} + \Omega_{\mathrm{K}}\left(\frac{a}{a_0}\right)^{-2} \tag{6}$$

just solve the differencial equation

$$\dot{a} = H_0 a_0 \sqrt{\Omega_R\left(\frac{a}{a_0}\right)^{-2} + \Omega_M\left(\frac{a}{a_0}\right)^{-1} + \Omega_\Lambda\left(\frac{a}{a_0}\right)^{2} + \Omega_K} \tag{7}$$

$$t(a_1) = \int_0^{a_1} \frac{\mathrm{d}a}{H_0 a_0 \sqrt{\Omega_R\left(\frac{a}{a_0}\right)^{-2} + \Omega_M\left(\frac{a}{a_0}\right)^{-1} + \Omega_\Lambda\left(\frac{a}{a_0}\right)^{2} + \Omega_K}} \tag{8}$$

It's not linear, so need numerical solution. See code/expansion.ipynb

But it's OK to see some exceptions.

Block 1. *matter donimated:*

$$\dot{a} = H_0(a_0\Omega_M)^{\frac{1}{2}} a^{-\frac{1}{2}} \tag{9}$$

then

$$t = \frac{2}{3H_0(a_0\Omega_M)^{\frac{1}{2}}} a^{\frac{3}{2}} \tag{10}$$

assumed $a(t=0)=0$.

1.2 to more notations..

It will be easy to use redshift and Hubble parameter $H(z)$ in the calculations.

1.2.1 Cosmological Redshift

Define redshift as

$$\frac{a}{a_0} = \frac{1}{1+z} \tag{11}$$

Redshift z is linked to a and t simply. How about the comoving spacial coordinate χ?

$$\mathrm{d}a = -\frac{a_0 \mathrm{d}z}{(1+z)^2} \tag{12}$$

The comoving distance from z_1 to now ($z_0 = 0$)

$$\chi(z_1) = \int_{a_1}^{a_0} \frac{\mathrm{d}a}{a^2 H(a)} = \int_{z_0}^{z_1} \frac{c\,\mathrm{d}z}{a_0 H(z)} \tag{13}$$

where $z_0 = 0$ by defination, c is filled to balance the unit.

[2] Friedmann equation itself has assumptions about the energy-momentum, it's ideal fluid which can be parameterized by only ρ and P..

1.2.2 Hubble parameter

And turn Eq. (4) to

$$H(a) \equiv \frac{\dot{a}}{a} = H_0 \sqrt{\frac{\rho(a)}{\rho_{\mathrm{cr}}}} \equiv H_0 E(a) \tag{14}$$

Eq. (6) turns out

$$E(a) = \sqrt{\Omega_{\mathrm{R}} \left(\frac{a}{a_0}\right)^{-4} + \Omega_{\mathrm{M}} \left(\frac{a}{a_0}\right)^{-3} + \Omega_\Lambda + \Omega_{\mathrm{K}} \left(\frac{a}{a_0}\right)^{-2}} \tag{15}$$

$$E(z) = \sqrt{\Omega_{\mathrm{R}}(1+z)^4 + \Omega_{\mathrm{M}}(1+z)^3 + \Omega_\Lambda + \Omega_{\mathrm{K}}(1+z)^2} \tag{16}$$

1.2.3 Conformal Time

Conformal time identic to the relation between χ and t in light cone.

$$\mathrm{d}\tau^2 = \frac{\mathrm{d}t^2}{a^2(t)} \tag{17}$$

$$\mathrm{d}\tau = \frac{\mathrm{d}t}{a(t)} = \mathrm{d}\chi \tag{18}$$

$$\tau(t_1) = \int_{t_i}^{t_1} \frac{\mathrm{d}t}{a(t)} \tag{19}$$

The last $=$ in Eq. (18) is for the convinence of coding.

1.3 Hubble sphere

Hubble sphere is where the Hubble flow velocity[3] equals to the speed of light.

$$v_{\mathrm{H}} = \dot{a}\chi \tag{20}$$

Hubble sphere as a function of time should be

$$\chi_{\mathrm{H}} = \frac{c}{aH(a)} \tag{21}$$

1.4 Light Cone

Light cone is defined as $\mathrm{d}s^2 = 0$, the photon starts from a_1 and ends at the observer now $(a_0, \chi = 0)$. so

$$\mathrm{d}t^2 = a^2(t)\mathrm{d}\chi^2 \tag{22}$$

$$\mathrm{d}t = a(t)\mathrm{d}\chi \tag{23}$$

$$\chi_{\mathrm{lc}}(a_1) = \int_{t_1}^{t_0} \frac{\mathrm{d}t}{a(t)} = \int_{a_1}^{a_0} \frac{\mathrm{d}a}{a\dot{a}} = \int_{a_1}^{a_0} \frac{\mathrm{d}a}{a^2 H(a)} \tag{24}$$

1.5 Event Horizon

the photon starts from a_1 and ends at the observer in the infinity future $(a \to \infty, \chi = 0)$.

$$\chi_{\mathrm{EH}}(a_1) = \int_{t_1}^{\infty} \frac{\mathrm{d}t}{a(t)} = \int_{a_1}^{\infty} \frac{\mathrm{d}a}{a^2 H(a)} \tag{25}$$

[3] in the proper length and is not real 'velovity'.

1.6 Particle Horizon

Particle horizon is the trajectory that a photon start from $\chi = 0$ at t_i (in 'our model', $t_i = a_i = 0$).

$$\chi_{\text{PH}}(a_1) = \int_{t_i}^{t_1} \frac{\mathrm{d}t}{a(t)} = \int_{a_i}^{a_1} \frac{\mathrm{d}a}{a\dot{a}} = \int_{a_i}^{a_1} \frac{\mathrm{d}a}{a^2 H(a)} \tag{26}$$

1.6.1 optical horizon

The optical horizon due to recombination $z(t_r) \approx 1100$.

$$d_{\text{opt}}(t) = a(\tau)(\tau - \tau_r) = a(t) \int_{t_r}^{t} \frac{\mathrm{d}t'}{a(t')} \tag{27}$$

LaTeX

$$\mathrm{H}\,. \tag{28}$$

$$\mathrm{H}\,|\psi\rangle \tag{29}$$

2 Boltzmann Equation

citation example[Lyth and Liddle(2009)]

参考文献

[Lyth and Liddle(2009)] D. H. Lyth and A. R. Liddle, *The Primordial Density Perturbation: Cosmology, Inflation and the Origin of Structure* (Cambridge University Press, Cambridge, UK, 2009).