A convex programming approach for ridesharing user equilibrium under fixed driver/rider demand

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Abstract

With the proliferation of smartphone-based ridesharing apps around the world, traffic assignment with ridesharing is drawing increasing attentions in recent years. A number of ridesharing user equilibrium (RUE) models have been proposed, but most of them are formulated as mixed complementary problems based on presumed ridesharing price and inconvenience functions, thus are inconvenient to implement in reality. In this study, we propose an alternative approach to modeling the RUE when the driver- and rider-demand for each OD pair are fixed and given. By redefining the set of feasible driver trajectories, and introducing market clearing conditions to characterize drivers' net income at market equilibrium, we show that the resulting RUE conditions can be equivalently transformed into a convex programming problem, and establish the existence and uniqueness of RUE link flows under mild conditions. A subgradient algorithm with averaging is proposed to solve the problem. The dual subproblem has a similar structure as Beckmann's formulation, therefore enables the application of classical traffic assignment algorithms such as Frank-Wolfe. Numerical examples are provided to demonstrate the effectiveness of the model and algorithm.

Keywords: Traffic assignment, ridesharing, user equilibrium, network

1. Introduction

Traffic congestion and emissions have long been headaches in metropolitan areas around

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the world. To ease congestion and reduce vehicle emissions, ridesharing, defined as two or more travelers with similar schedules and routessharing one car, has been widely viewed as an encouraging mode of green transport. In the recent decade, the increased popularity of smartphones and the fast development of GPS positioning and wireless communication technologies have led to the proliferation of digitalized dynamic ridesharing apps all over the world (e.g., Didi Hitch and Dida Pinche in China, Blablacar in over 20 countries, Flinc in Germany, and Carma in the U.S. and Ireland).

However, the technological progress is not sufficient to guarantee the success of dynamic ridesharing. As witnessed around the world, despite the lower price of ridesharing, the number of daily orders on ridesharing platforms is much smaller than the number of orders for ride-sourcing services (i.e., taxi-like service provided by private car drivers). According to the Corporate Citizenship Report published by Didi Chuxing (2018), to the end of 2017, the leading on-demand mobility service platform in China, managed more than 20 million orders every day, but less than 10% of these orders are contributed by ridesharing users. So to turn the potential of ridesharing in congestion mitigation into a reality, administrative stimulus like the implementation of High-Occupancy Vehicle (HOV) lanes, High-Occupancy-Toll (HOT) lanes are still necessary.

To design appropriate network-based policies to stimulate ridesharing, the first and fundamental step is to solve the traffic assignment problem with ridesharing. In 1952, Wardrop (1952) formalized the notion of "user equilibrium (UE)" as the first traffic assignment principle. Beckmann et al. (1956) then formulated the UE conditions with monotone link performance functions as a nonlinear optimization problem (known as Beckmann's transformation), and the well-known Frank-Wolfe algorithm (Frank and Wolfe, 1956) was used to solve traffic assignment problems (Leblanc et al., 1975). Since then, solving traffic assignment problems under diverse scenarios has developed into a substantial stream of research in the transportation literature (Yang and Huang, 2005; Patriksson, 2015). However, all of these UE models have been built for traditional transportation systems. They are not directly applicable to the new urban road systems, where ridesharing serves as an important alternative. *First*, a traditional traffic assignment problem only deals with the route choice problem of solo drivers, provided that all drivers choose the paths that minimize their travel costs. With ridesharing (RS), driving alone or serving riders between different OD pairs leads to different trajectories of drivers. Thus, in order to predict the traffic flow distribution in the presence of ridesharing, drivers' mode, rider and route choices have to be simultaneously determined. *Second*, while solo drivers only experience travel time and monetary costs en-route, RS drivers have to make additional effort (e.g., waiting) to become matched and receive payments from riders. The different net incomes that drivers can obtain from serving different riders affect the drivers' mode and rider choices, and vice versa. Therefore, to predict travelers' mode, rider and route choices at ridesharing user equilibrium (RUE), one needs to appropriately model the driver-rider matching cost and payment for each ridesharing trip. *Third*, the fundamental UE conditions and the development of efficient solution algorithms are built on the concept of "path" (i.e., a walk without any repetition of nodes (Ahuja et al.,1988)). However, due to the detour trips for pick-up and/or delivery, the trajectory of an RS driver may no longer be a conventional "path" in the road network. Thus, to solve the RUE problem, it is necessary to redefine the set of feasible driver trajectories. These different features brought about by the emerging ridesharing mode motivates new traffic assignment models being proposed to describe, prescribe and predict the different UE flow pattern in the presence of ridesharing.

1.1 Related literature

Before the emergence of dynamic ridesharing services, traffic assignment models that incorporate carpooling into travelers' mode choice have been proposed to optimize the HOV/HOT lane allocation over road networks. Daganzo (1981) developed the first user equilibrium model that describes the different route choice behavior of solo drivers and carpoolers, with the number of drivers in each type being fixed and given. Song et al. (2015) relaxed this assumption, and formulated drivers' mode and route choices in general networks with carpooling as a variational inequality (VI) problem. However, as carpooling usually occurs between family members and colleagues, these early models do not take drivers' rider choice into consideration, and the inconvenience cost of carpooling are often ignored.

For digitalized dynamic ridesharing service which facilitates matching between strangers, a number of traffic assignment models have been proposed in recent years. Xu et al. (2015a) firstly incorporated dynamic ridesharing into a static user equilibrium model with elastic travel demands, considering the mutual interactions between ridesharing price, traffic conditions, and driver/rider demand. As an initial attempt, they assumed that ridesharing drivers could only pick up riders between the same OD pairs, leading to a significant underestimate of potential matches. To relax this assumption, Xu et al. (2015b) proposed an alternative ridesharing user equilibrium (RUE) model based on an extended graph as in Figure 1. Assuming that each traveler is a car owner therefore can freely choose to drive alone, be an RS rider or be an RS driver, they proposed an extended network with each link in the original network being extended to three links for solo drivers, RS drivers, and RS riders, respectively. The number of riders in each link was restricted to be no more (no less) than the total number of RS seats (RS vehicles) in the link. The ridesharing price and inconvenience cost for RS riders and RS drivers were assumed to be explicit link-based functions of driver and rider flows in the link. The resulting RUE was formulated as a path-based mixed complementarity problem (MiCP). As this path-based MiCP requires path enumeration to solve, Di et al. (2018) reformulated it as a link-node-based complementarity problem, which avoids path enumeration and lowers the number of variables. They also pointed out that the RUE models built on the extended network proposed by Xu et al. (2015b) cannot guarantee each passenger to be transported to her/his destination by only one driver. Riders may have to take multiple vehicles to complete their trips, but such vehicle transfer costs (e.g., waiting time of each transfer) of riders cannot be easily incorporated into the models.

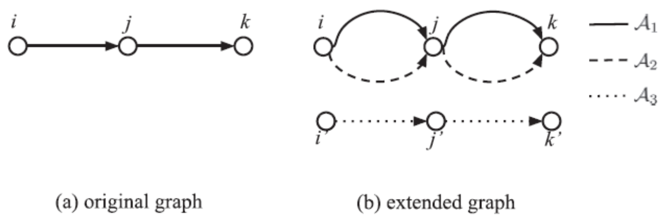

Figure 1. Graph extension in Xu et al. (2015b) and Di et al. (2018)

To make the model more realistic, Di et al. (2017) proposed a path-based RUE model, which restricts the number of RS riders on each path to be no more (no less) than the number of available RS seats (RS drivers) on the path. Under this assumption, each rider is guaranteed to be carried by one driver for the entire trip. The inconvenience costs and ridesharing prices are given by path-based explicit functions of RS path flows. As most ridesharing apps in reality adopt an OD-based surge pricing strategy, Ma et al. (2020) introduces OD-based price and inconvenience functions, and further differentiated RS drivers' role as 1-rider and 2-rider service providers. To facilitate modelling, in both works, RS drivers are restricted to serve riders between the same OD pairs only. Li et al. (2019a) and Li et al. (2019b) extended the RUE model in Di et al. (2017) by allowing RS drivers to serve riders between different OD pairs. Different path-based price and inconvenience functions are adopted in the two papers. But they both assume that a rider could be taken by a driver only if the rider-path was a sub-path of the driver-path. This assumption rules out potential matches that could be attractive in reality[1], and asks for an explicit path-subpath matrix in calculation, which is usually very difficult (if not infeasible) to derive for large networks.

Table 1. Differences between existing RUE models and our model

<table>
<tr><th>RUE models</th><th>Travelers' mode choice</th><th>Drivers serve riders between</th><th>Vehicle transfer required for riders?</th><th>Inconvenience costs and RS prices are given by</th><th>Math. model</th></tr>
<tr><td>Xu et al. (2015a)</td><td rowspan="7">solo driver, RS driver, RS rider</td><td>the same OD pairs</td><td>No</td><td>OD-based functions of RS rider demands</td><td>convex programming[2]</td></tr>
<tr><td>Xu et al. (2015b)</td><td rowspan="2">the same or different OD pairs</td><td rowspan="2">Yes</td><td rowspan="2">link-based functions of RS driver- and riders-link flows</td><td>path-based MiCP</td></tr>
<tr><td>Di et al. (2018)</td><td>link-based MiCP</td></tr>
<tr><td>Di et al. (2017)</td><td>the same OD pairs</td><td rowspan="5">No</td><td rowspan="2">path-based functions of RS driver- and rider-path flows</td><td rowspan="2">path-based MiCP</td></tr>
<tr><td>Li et al. (2019a)</td><td rowspan="2">the same or different OD pairs, but riders' path must be a sub-path of drivers' path</td></tr>
<tr><td>Li et al. (2019b)</td><td>path-based functions of travel time</td><td>path-based MiCP</td></tr>
<tr><td>Ma et al. (2020)</td><td>the same OD pairs</td><td>OD-based functions of RS supply and demand</td><td>path-based variational inequality problem</td></tr>
<tr><td>This study</td><td>solo driver, RS driver</td><td>the same or different OD pairs</td><td>multipliers of market clearing conditions for each rider OD pair</td><td>convex programming</td></tr>
</table>

[1] As mentioned in the previous subsection, the trajectory of a RS driver is not necessarily a 'path' of the network any more.

[2] As an initial trial on this subject, Xu et al. (2015a) is subject to several conceptual errors. Its definition of user equilibrium and proof of the equivalence between UE conditions and the convex programming program are problematic.

Table 1 provides a comparison of existing RUE models. In summary, under the assumption that all travelers are car owners therefore can freely choose to be solo drivers, RS riders and RS drivers, different RUE models have been proposed in recent years. Yet until now, these proposed models are still subject to unrealistic assumptions and/or difficult to implement. Some models (i.e., in Xu et al. (2015b) and Di et al. (2018)) cannot guarantee each rider to be served by only one driver, and some models (i.e., in Xu et al. (2015a), Di et al. (2017), Ma et al. (2020)) restrict RS drivers to serve RS riders between the same OD pairs only. The inconvenience costs are all presumed as explicit functions of RS driver and RS rider (path, link or OD) flows, so the predicted flow patterns are highly dependent on the forms and parameters of the functions, which are difficult, if not impossible, to estimate in practice. Path enumeration is extremely time-consuming even for middle-scale networks, but most existing RUE models are formulated as path-based MiCPs, which require path enumeration to implement.

1.2 Contributions of this work

In view of the abovementioned issues of existing RUE models, this paper takes one step backward to model drivers' mode, rider and route choices when the driver- and rider-demands are fixed. Assuming that all drivers travel with at most one RS rider, and each RS rider is served by only one driver, we redefine the set of driver trajectories with no restriction on feasible driver-rider matchings, and introduce market clearing conditions to characterize drivers' net income from ridesharing services at market equilibrium. The resulting RUE conditions can be equivalently transformed into a convex programming problem, based on which the existence of RUE solutions and the uniqueness of aggregate link flows can be established under mild conditions. A dual subgradient algorithm with averaging is proposed to solve the problem. And the dual subproblem has a similar structure as Beckmann's formulation, therefore enables the application of classical traffic assignment algorithms such as Frank-Wolfe.

In comparison with existing RUE models, our model is more restrictive on travelers' role flexibility. However, the different modelling approach adopted in this paper leads to a convex programming problem, which is more convenient to implement in reality. No estimation of inconvenience functions is needed before modelling, and no time-consuming path enumeration is required in calculation. The nice mathematical property of our model may make it an appealing alternative to practitioners and researchers in traffic planning.

The rest of this work is organized as follows. Section 2 introduces the problem and establishes RUE conditions. Section 3 transforms the RUE conditions into an equivalent convex programming problem, and establishes the existence and uniqueness of equilibrium flow patterns. In Section 4, a dual subgradient method with averaging is proposed to solve the convex programming problem. Numerical examples are presented in Section 5 to demonstrate the effectiveness of the proposed model and algorithm. Section 6 concludes the paper. A list of notations is provided in Appendix A.

2. Ridesharing user equilibrium

Consider an urban transportation system with a ridesharing platform in service. There are three types of users in the system: solo drivers, RS drivers and RS riders. RS riders announce their OD information and bid price on the platform; and drivers, after submitting their OD information on the platform, would receive a list of riding requests with different prices. The platform facilitates matching by filtering out impossible matches for drivers. Drivers can choose any rider they want to serve (considering the different paybacks and costs associated with the services), or simply drive alone if there is no satisfactory choice. Day by day, riders adjust their bid prices based on their own hailing experiences, and drivers adjust their choices of riders and driving 'paths' based on riders' bids and network traffic conditions respectively. An equilibrium will be reached when the demand and supply of ridesharing seats of each OD pair reach market equilibrium, and no driver has incentive to change his/her rider and 'path' choices.

To predict the ridesharing price, driver-rider matches and network flow pattern at equilibrium in the presence of such a ridesharing platform, let $G(N,A)$ be the general network of the urban transportation system, with N being the set of nodes and A being the set of links. The sets of OD pairs for drivers and riders are indicated by W and M respectively. Let d_w and q_m respectively be the driver and rider demand rates between driver OD pair $w \in W$ and rider OD pair $m \in M$ .In this study, we assume the driver and rider demand rates for each OD pair are fixed and given, and each RS driver (RS rider) serves (is served by) only one rider (driver) along her/his trip. For each link $a \in A$, the link travel time function, $t_a(x_a)$, is assumed to be nonnegative, differentiable and monotonically increasing with respect to its link flow x_a.

As mentioned in Section 1, the trajectories of RS drivers may not be "paths" in the network anymore. So before proceeding to establish the RUE conditions, we first propose to define the set of feasible driver trajectories[3] in the presence of ridesharing. For each driver OD pair $w \in W$, let R_{wm} be the set of trajectories of RS drivers serving riders between OD pair $m \in M$, and R_w be the set of solo-driver trajectories. As all solo drivers seek the minimum-cost paths between their OD pairs, the set R_w is identical with the set of paths between OD pair $w \in W$ in the network. For RS drivers, since we assume each rider is carried from her/his origin to destination by one driver, every trajectory of RS drivers between OD pair $w \in W$, serving riders between OD pair $m \in M$, can be divided into three segments as in **Figure 2**: 1) from the driver origin o_w to the rider origin o_m (before picking up a rider); 2) from the rider origin o_m to the rider destination e_m (travelling with the rider); and 3) from the rider destination e_m to the driver destination e_w (after dropping off the rider). And each part of the trajectory corresponds to a path between (o_w, o_m), (o_m, e_m) and (e_m, e_w). Thus R_{wm} can be defined as the set of trajectories that possess the following features: 1) connecting the origin and destination of driver OD pair $w \in W$; 2) sequentially traversing the origin and destination of rider OD pair $m \in M$; and 3) the driver and rider origins, the rider origin and destination, and the rider and driver destinations are all connected by paths. The set of all RS-driver trajectories between OD pair $w \in W$ then can be given by $\bigcup_{m \in M} R_{wm}$.

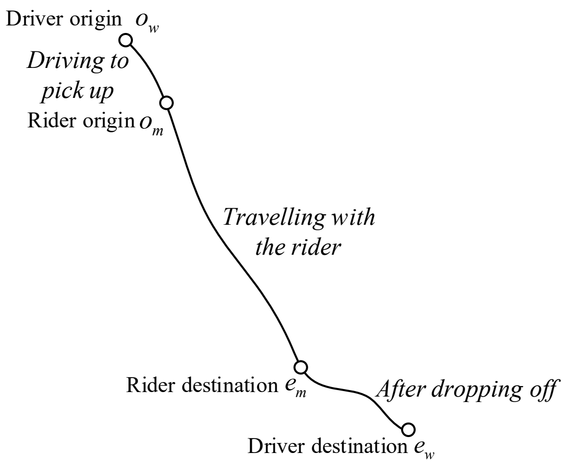

Figure 2. A typical trajectory of RS drivers between OD pair (o_w, e_w) serving riders between OD pair (o_m, e_m)

[3] We would like to clarify that these sets of driver trajectories are only defined to establish the RUE conditions. In the calculation, our model does not require trajectory enumeration.

Based on the driver trajectory sets, we then proceed to establish the RUE conditions under fixed rider and driver demands for each OD pair. Let $f_{r,wm}$ and $y_{r,w}$ respectively be the driver flows on trajectory $r \in R_{wm}, m \in M, w \in W$ and $r \in R_w, w \in W$. Then, for each OD pair $w \in W$ and $m \in M$, we have

$$\sum_{m \in M} \sum_{r \in R_{wm}} f_{r,wm} + \sum_{r \in R_w} y_{r,w} = d_w, \; w \in W \tag{1}$$

$$\sum_{w \in W} \sum_{r \in R_{wm}} f_{r,wm} \geq q_m, \; m \in M \tag{2}$$

Eq. (1) states that the summation of driver flows on all feasible solo-driver and RS-driver trajectories between OD pair $w \in W$ equals to the total driver demand between this OD pair. And Eq. (2) ensures that the driver supply for each rider OD pair is larger than its rider demand, such that every rider is guaranteed to be carried by one driver from her/his origin to destination. Meanwhile, let v_a and h_a respectively indicate the RS-driver flow and solo-driver flow on link $a \in A$. Then, from the relationships between link flows and trajectory flows, we have

$$v_a = \sum_{w \in W} \sum_{m \in M} \sum_{r \in R_{wm}} f_{r,wm} \delta_{a,r}, a \in A \tag{3}$$

$$h_a = \sum_{w \in W} \sum_{r \in R_w} y_{r,w} \delta_{a,r}, a \in A \tag{4}$$

where $\delta_{a,r}$ is the link-trajectory incident parameter, indicating the times that trajectory r traverses link $a \in A$. The total traffic flow on link $a \in A$, indicated by x_a, can be given by

$$x_a = v_a + h_a, \; a \in A \tag{5}$$

We next give the drivers' generalized cost on different trajectories $r \in \left(\bigcup_{m \in M} R_{wm} \right) \cup R_w$, $w \in W$. For each solo driver who chooses trajectory $r \in R_w$, $w \in W$, s/he experiences time and monetary costs along the trajectory, so the associated cost of the trajectory is given by

$$c_{r,w} = \sum_{a \in A} \delta_{a,r} \left[t_a \left(x_a \right) + c_a \right]. \tag{6}$$

However, for RS drivers along a trajectory $r \in R_{wm}$, in addition to the driving costs of their trips, they experience additional time costs for riders getting in and out, spend effort on being matched with riders between $m \in M$, and receive payments from riders. As in Ma et al. (2020), we assume riders pay an OD-based price to drivers, and the route choices are determined by drivers[4]. Let p_m indicate the net income for a driver serving a rider between OD pair $m \in M$, defined as the payment received from a rider between $m \in M$ minus the cost-equivalent effort for matching with the rider. Then the generalized cost for RS drivers along trajectory $r \in R_{wm}$ can be given by

$$c_{r,wm} = \sum_{a \in A} \delta_{a,r} \left[t_a \left(x_a \right) + c_a \right] + t_0 + \Delta - p_m \tag{7}$$

where the first term is the en-trajectory travel cost, t_0 is additional time cost for riders getting in and out, and Δ is the safety cost for travelling with a stranger. Both t_0 and Δ are assumed to be constant in this study.

Assuming that each driver always seeks to minimize his or her own travel cost, then the user equilibrium conditions under given $p_m, m \in M$ can be given by

$$f_{r,wm} \left(c_{r,wm} - \mu_w \right) = 0, \ \ f_{r,wm} \geq 0, \ \ c_{r,wm} - \mu_w \geq 0, \ \ r \in R_{wm}, w \in W, m \in M \tag{8}$$

$$y_{r,w} \left(c_{r,w} - \mu_w \right) = 0, \ \ y_{r,w} \geq 0, \ \ c_{r,w} - \mu_w \geq 0, \ \ r \in R_w, w \in W \tag{9}$$

$$\sum_{m \in M} \sum_{r \in R_{wm}} f_{r,wm} + \sum_{r \in R_w} y_{r,w} = d_w, w \in W \tag{10}$$

where $c_{r,w}$ and $c_{r,wm}$ are defined in Eqs (6) and (7), and μ_w is the minimal generalized travel cost of drivers between OD pair $w \in W$. Eqs (8) and (9) state that at user equilibrium, the travel cost of every used trajectory is equal to or less than that of any unused trajectory, or in other words, no one can reduce her/his travel cost by unilaterally changing her/his mode, rider, and/or route choices.

The net income p_m of drivers from serving riders between OD pair $m \in M$ is endogenously determined by the ridesharing market. At market equilibrium, drivers' net income from serving riders between OD pair $m \in M$ is positive only if all demand is served. Thus the following market clearing condition holds:

[4] Drivers and riders incur different link costs, so they may have different path preferences. In our study, as each rider bids and pays an OD-based price, rather than a path-based price, we assume the service paths are chosen by RS drivers. This is the reason that riders' path choices are modelled in the following. Furthermore, we note that riders' costs on different service paths (chosen by their drivers) are not necessarily equal at RUE, and they don't have to be equal in this case. If the ridesharing apps ask riders to announce their preferred service paths and bid path-based prices, then riders' generalized costs on different utilized paths could be equalized, as in previous studies (e.g., Di et al. (2017), Li et al. (2019)). However, to our knowledge, most ridesharing apps (at least for Didi Hitch, Dida Pingche) do not ask riders to announce their preferred paths, but only publish their OD information to drivers.

$$\left(\sum_{w\in W}\sum_{r\in R_{wm}} f_{r,wm} - q_m\right)p_m = 0, \sum_{w\in M}\sum_{r\in R_{wm}} f_{r,wm} - q_m \geq 0,\ p_m \geq 0,\ \ m\in M \tag{11}$$

At RUE under given driver and rider demands, no driver can further reduce her/his cost by unilaterally changing her/his own mode, rider and/or route choices, and the ridesharing demand and supply of each rider-OD pair satisfy the market clearing condition. So Eqs (8) -(11) hold simultaneously. Inserting Eqs (6) and (7) into Eqs (8) and (9), the RUE conditions can be written as the following mixed complementarity problem:

$$0 \leq f_{r,wm} \perp \left(\sum_{a\in A}\delta_{a,r}\left[t_a\left(x_a\right)+c_a\right]+t_0+\Delta - p_m - \mu_w\right) \geq 0, r\in R_{w,m}, w\in W, m\in M \tag{12}$$

$$0 \leq y_{r,w} \perp \left(\sum_{a\in A}\delta_{a,r}\left[t_a\left(x_a\right)+c_a\right]-\mu_w\right) \geq 0, r\in R_w, w\in W \tag{13}$$

$$0 \leq p_m \perp \left(\sum_{w\in W}\sum_{r\in R_{wm}} f_{r,wm} - q_m\right) \geq 0, m\in M \tag{14}$$

$$\sum_{m\in M}\sum_{r\in R_{wm}} f_{r,wm} + \sum_{r\in R_w} y_{r,w} = d_w, w\in W \tag{15}$$

where $a \perp b$ means that $a\cdot b = 0$, and $\mathbf{x} = \left(x_a, a\in A\right)$ in Eqs (12) and (13) satisfies Eqs (3), (4) and (5).

3. Equivalent optimization problem for RUE

Let $\mathbf{x} = \left(x_a, a\in A\right)$, $\mathbf{y} = \left(y_{r,w}, r\in R_w, w\in W\right)$ and $\mathbf{f} = \left(f_{r,wm}, r\in R_{wm}, m\in M, w\in W\right)$ respectively indicate the vectors of link flows, solo-driver trajectory flows and RS-driver trajectory flows. In this section, we show that the above RUE conditions (12)-(15) can be equivalently reformulated as a convex programming problem, based on which we can easily validate the existence and uniqueness of RUE flow patterns and design efficient solution algorithms.

Proposition 1. *Suppose that the link travel time function* $t_a\left(x_a\right)$ *is continuous and increasing with respect to* x_a *for all* $a\in A$. *Then the aggregate link flow pattern* $\mathbf{x}$ *at RUE can be obtained by solving the following convex programming problem:*

P0:

$$\min_{\mathbf{f},\mathbf{x},\mathbf{y}} Z(\mathbf{f},\mathbf{x}) = \sum_{a\in A}\int_0^{x_a}\left[t_a(w)+c_a\right]dw + (t_0+\Delta)\sum_{w\in W}\sum_{m\in M}\sum_{r\in R_{wm}} f_{r,wm} \tag{16}$$

s.t.

$$\sum_{m\in M}\sum_{r\in R_{wm}} f_{r,wm} + \sum_{r\in R_w} y_{r,w} = d_w,\ \ w\in W \tag{17}$$

$$\sum_{w\in W}\sum_{r\in R_{wm}} f_{r,wm} \ge q_m,\ \ m\in M \tag{18}$$

$$x_a = \sum_{w\in W}\sum_{m\in M}\sum_{r\in R_{wm}} f_{r,wm}\delta_{a,r} + \sum_{w\in W}\sum_{r\in R_w} y_{r,w}\delta_{a,r}, a\in A \tag{19}$$

$$(\mathbf{f},\mathbf{y})\ge 0 \tag{20}$$

Proof. We first show that P0 is a convex programming problem. Since all constraint functions of P0 are linear, the feasible region of P0 is convex. From Eq. (16), we have

$$\begin{array}{l} \dfrac{\partial^2 Z}{\partial x_a \partial x_b} = \begin{cases} t_a'(x_a), \forall a=b\in A \\ 0, \qquad \forall a\ne b\in A \end{cases}, \qquad \dfrac{\partial^2 Z}{\partial x_a \partial f_{r,wm}} = 0,\ \forall a\in A, r\in R_{wm}, m\in M, w\in W \\ \dfrac{\partial^2 Z}{\partial f_{r,wm}\partial f_{r',w'm'}} = 0,\ \ \forall r\in R_{wm}, r'\in R_{w'm'}, m,m'\in M, w,w'\in W \end{array} . \tag{21}$$

As we assume $t_a'(x_a)\ge 0$ for all $a\in A$, Eq. (21) implies that the Hessian matrix of the objective function $Z(\mathbf{f},\mathbf{x})$ is positive semi-definite, and thus the objective function of P0 is convex. Therefore, P0 is a convex programming problem with linear constraints, and the KKT points of P0 are thus identical to its optimal solutions. Let $(\tilde{\mu}_w, w\in W)$ and $(\tilde{p}_m, m\in M)$ respectively denote the Lagrange multipliers of Eqs (17) and (18). Based on the derivatives of function $Z(\mathbf{f},\mathbf{x})$ in Appendix B, the KKT conditions of P0 can be given by

$$0\le f_{r,wm}\perp\left(\sum_{a\in A}\delta_{a,r}\left[t_a(x_a)+c_a\right]+t_0+\Delta-\tilde{p}_m-\tilde{\mu}_w\right)\ge 0, r\in R_{wm}, w\in W, m\in M \tag{22}$$

$$0\le y_{r,w}\perp\left(\sum_{a\in A}\delta_{a,r}\left[t_a(x_a)+c_a\right]-\tilde{\mu}_w\right)\ge 0, r\in R_w, w\in W \tag{23}$$

$$0\le \tilde{p}_m\perp\left(\sum_{w\in W}\sum_{r\in R_{wm}} f_{r,wm}-q_m\right)\ge 0, m\in M \tag{24}$$

$$\sum_{m\in M}\sum_{r\in R_{wm}} f_{r,wm} + \sum_{r\in R_w} y_{r,w} = d_w, w\in W \tag{25}$$

where $\mathbf{x}$ satisfies Eq (19). Comparing Eqs (22)-(25) with Eqs (12)-(15), we can see that the

KKT conditions of P0 are identical with the RUE conditions, with the Lagrange multipliers $\tilde{\mu}_w$ corresponding to the minimal driver cost between OD pair $w \in W$ and the Lagrange multipliers $\tilde{p}_m$ corresponding to the drivers' net income from serving riders between OD pair $m \in M$. This completes the proof. ■

Let Ω be the set of feasible flow patterns defined by

$$\Omega=\left\{(\mathbf{f},\mathbf{y},\mathbf{x})\middle|\text{Eqs }(17)\text{-}(20)\right\} \tag{26}$$

The following proposition guarentees the existence of solutions to problem P0 based on the non-emptiness and compactness of Ω when $\sum_{w\in W} d_w \ge \sum_{m\in M} q_m$.

Proposition 2. *Suppose that total driver demand is no less than total rider demand, i.e.,* $\sum_{w\in W} d_w \ge \sum_{m\in M} q_m$, *and that the link travel time function* $t_a(x_a)$ *is continuous for all* $a \in A$. *Then there is at least one RUE solution.*

Proof. Please refer to Appendix C.

Based on Propositions 1 and 2, we can further validate the uniqueness of the aggregate link flow pattern $\mathbf{x}$ at RUE when the link travel time functions are strictly monotone.

Proposition 3. *Suppose that the total driver demand is no less than the total rider demand, i.e.,* $\sum_{w\in W} d_w \ge \sum_{m\in M} q_m$, *and the link travel time function* $t_a(x_a)$ *is continuous and strictly increasing with respect to* x_a *for all* $a \in A$. *Then the aggregate link flow* $\mathbf{x}$ *at RUE is unique.*

Proof. Please refer to Appendix D.

Provided the uniqueness of aggregate link flow $\mathbf{x}$ at RUE, the travel time cost for each link becomes uniquely determined. For any OD pair $w \in W$, if there are drivers who choose to drive alone at UE, e.g., $\sum_{r\in R_w} y_{r,w} > 0$, the uniqueness of the link travel time would lead to a unique minimal travel cost μ_w from Eq.(13). In this case, we can further infer from Eq.(12) that drivers' net income from riders between all of their served OD pairs $m \in M$ is uniquely determined.

Corollary 1. *Suppose that the link travel time function* $t_a\left(x_a\right)$ *is continuous and strictly increasing with respect to* x_a *for* $a \in A$*, then for OD pair* $w \in W$ *with solo drivers at user equilibrium, i.e.,* $\sum_{r \in R_w} y_{r,w} > 0$*, the drivers' net income from riders between all of their served OD pairs is uniquely determined at RUE.*

Proof. Please refer to Appendix E.

At the end of this section, it is worth pointing out that similar to a traditional UE without ridesharing mode, the trajectory flow patterns at RUE are generally not unique. Consequently, the driver-rider partnership is not uniquely determined at RUE.

4. Solution algorithm

In this section, we investigate how to efficiently solve problem P0 by exploring the network structure. In comparison with the classic Beckmann's formulation, P0 has an additional side constraint (18). This motivates us to penalize this side constraint to the objective and use the dual subgradient method with averaging (Larsson et al., 1999; Gustavsson et al., 2015) to solve it. The algorithm can ensure not only the convergence of dual solution sequences but also that of primal solution sequences. Moreover, the sub-problem has the same structure as the traditional Beckmann's formulation, therefore enables the application of well-established Frank-Wolfe algorithm to solve it. No path or trajectory enumeration is required.

4.1 Dual subgradient algorithm with averaging

Let $\mathbf{p} = \left(p_m, m \in M\right)$ be the vector of Lagrange multipliers associated with the constraints (18). Define the Lagrangian function with respect to the relaxation of constraints (18) by

$$L\left(\mathbf{p},\mathbf{f},\mathbf{x}\right) = Z\left(\mathbf{f},\mathbf{x}\right) + \sum_{m \in M} p_m \left(q_m - \sum_{w \in W} \sum_{r \in R_{wm}} f_{r,wm} \right) \tag{27}$$

For the primal problem P0, its dual objective function $\theta : \mathbb{R}_+^{|M|} \to \mathbb{R}$ is defined by the dual sub-problem:

$$\theta(\mathbf{p}) = \min_{(\mathbf{f},\mathbf{y},\mathbf{x})} L(\mathbf{p},\mathbf{f},\mathbf{x}) \quad \textbf{s.t.} \quad (\mathbf{f},\mathbf{y},\mathbf{x}) \in \tilde{\Omega} \tag{28}$$

where

$$\tilde{\Omega} = \left\{ (\mathbf{f},\mathbf{y},\mathbf{x}) \geq \mathbf{0} \left| \begin{array}{l} \sum_{m\in M}\sum_{r\in R_{wm}} f_{r,wm} + \sum_{r\in R_w} y_{r,w} = d_w, \ w \in W \\ x_a = \sum_{w\in W}\sum_{m\in M}\sum_{r\in R_{wm}} f_{r,wm}\delta_{a,r} + \sum_{w\in W}\sum_{r\in R_w} y_{r,w}\delta_{a,r}, a \in A \end{array} \right. \right\} \tag{29}$$

By the convexity of $Z(\mathbf{f},\mathbf{x})$ established in Proposition 1, $L(\mathbf{p},\cdot)$ is convex. Furthermore, the set $\tilde{\Omega}$ is non-empty, convex and compact. Thus, the dual subproblem (28) has at least one optimal solution.

Let θ^* be the optimal value of Lagrange dual to the primal problem P0, i.e.,

$$\theta^* = \max \theta(\mathbf{p}), \quad \text{s.t.} \quad \mathbf{p} \geq 0 \tag{30}$$

and Z^* be the optimal value of the primal problem P0. Since P0 is a convex programming problem with linear constraints, it follows that $Z^* = \theta^*$ by strong duality theorem (see Proposition 6.4.2 in Bertsekas et al., 2013). So the optimal value of P0 can be achieved by solving its dual problem (30).

Let $\mathbf{p}^*$ be an optimal solution of Eq. (30). Following a proof procedure similar to Proposition 3, the optimal link flow $\mathbf{x}(\mathbf{p}^*)$ generated by Eq. (28) is unique. However, the optimal trajectory flows, i.e., $(\mathbf{f}(\mathbf{p}),\mathbf{y}(\mathbf{p}))$ to Eq. (28), are generally not uniquely determined. Let

$$\Psi(\mathbf{p}) = \left\{ (\mathbf{f},\mathbf{y},\mathbf{x}) \in \tilde{\Omega} \middle| L(\mathbf{p},\mathbf{f},\mathbf{x}) \leq \theta(\mathbf{p}) \right\}$$

be the non-empty, convex and compact solution set to the dual sub-problem (28) with $\mathbf{p} \in \mathbb{R}_+^{|M|}$. Because $\Psi(\mathbf{p})$ is usually not a singleton, the resulting flow pattern $(\mathbf{f},\mathbf{y},\mathbf{x}) \in \Psi(\mathbf{p}^*)$ at the optimal dual $\mathbf{p}^*$ is not guaranteed to be feasible for the primal problem[5]. Although in many cases, only the aggregate UE link flow pattern $\mathbf{x}$ is important to transportation authorities[6], it is sometimes meaningful in both practice and theory to

[5] A flow pattern $(\mathbf{f},\mathbf{y},\mathbf{x}) \in \Psi(\mathbf{p}^*)$ at the optimal dual $\mathbf{p}^*$ may violate the relaxed constraints (18), thus is infeasible to P0.

[6] If this is the case, then a traditional dual subgradient method applies, i.e., in the following algorithm, step 3 can be skipped, and the algorithm terminates when the improvement to the dual objective becomes sufficiently small, i.e., $\left| Z(\hat{\mathbf{f}}^n,\hat{\mathbf{x}}^n) - \theta(\mathbf{p}^{n+1}) \right| / Z(\hat{\mathbf{f}}^n,\hat{\mathbf{x}}^n) \leq \varepsilon_2$.

observe drivers' mode and rider choices at RUE. To produce primal feasible solutions with dual subgradient techniques, we introduce the following dual subgradient method with averaging (Larsson et al., 1999; Gustavsson et al., 2015).

Algorithm 1. Dual subgradient method with averaging:

Step 0 (Initialization): Choose a starting point $\mathbf{p}^0 = \left(p_m^0, m \in M\right) \geq 0$ and set $n = 0$.

Step 1 (Solve the dual sub-problem): Solve the dual sub-problem $\theta\left(\mathbf{p}^{\mathbf{n}}\right)$ in Eq. (28) and obtain the optimal link and trajectory flow vectors $\mathbf{x}\left(\mathbf{p}^n\right)$, $\mathbf{f}\left(\mathbf{p}^n\right)$ and $\mathbf{y}\left(\mathbf{p}^n\right)$. Define the RS driver supply $\mathbf{s}\left(\mathbf{p}^{\mathbf{n}}\right) = \left(s_m\left(\mathbf{p}^{\mathbf{n}}\right), m \in M\right)$, where $s_m\left(\mathbf{p}^{\mathbf{n}}\right)$ is the RS driver supply for each rider OD pair $m \in M$, i.e.,

$$s_m\left(\mathbf{p}^{\mathbf{n}}\right) = \sum_{w \in W} \sum_{r \in R_{wm}} f_{r,wm}\left(\mathbf{p}^{\mathbf{n}}\right) \tag{31}$$

Step 2 (Dual update). Update $\mathbf{p}^{\mathbf{n+1}} = \max\left\{\mathbf{p}^{\mathbf{n}} + \alpha^n(\mathbf{q} - \mathbf{s}(\mathbf{p}^n)), 0\right\}$, where $\alpha^n = \hat{a}/(1+n)$ for some positive constant $\hat{a}$.

Step 3 (Generate ergodic sequence). Set

$$\hat{\mathbf{s}}^n = \sum_{t=0}^{n-1} \gamma_t^n \mathbf{s}\left(\mathbf{p}^t\right),\quad \hat{\mathbf{x}}^n = \sum_{t=0}^{n-1} \gamma_t^n \mathbf{x}\left(\mathbf{p}^t\right),\ \hat{\mathbf{f}}^n = \sum_{t=0}^{n-1} \gamma_t^n \mathbf{f}\left(\mathbf{p}^t\right),\ \hat{\mathbf{y}}^n = \sum_{t=0}^{n-1} \gamma_t^n \mathbf{y}\left(\mathbf{p}^t\right), \tag{32}$$

where $\gamma_t^n = \frac{(t+1)^k}{\sum_{l=0}^{n-1}(l+1)^k}$ and $k \geq 0$ is a constant;

Step 4 (check convergence). If $\frac{\left\|\min\left\{\hat{\mathbf{s}}^n - \mathbf{q}, 0\right\}\right\|_1}{\|\mathbf{q}\|_1} \leq \varepsilon_1$ and $\frac{\left|Z\left(\hat{\mathbf{f}}^n, \hat{\mathbf{x}}^n\right) - \theta\left(\mathbf{p}^{n+1}\right)\right|}{Z\left(\hat{\mathbf{x}}^n, \hat{\mathbf{v}}^n\right)} \leq \varepsilon_2$, where $\varepsilon_1, \varepsilon_2 > 0$ are tolerance parameters, stop. Otherwise, set $n = n+1$ and go to Step 1.

Compared with an ordinary dual subgradient method, Algorithm 1 constructs ergodic sequences $\left\{\hat{\mathbf{s}}^n\right\}$, $\left\{\hat{\mathbf{x}}^n\right\}$, $\left\{\hat{\mathbf{f}}^n\right\}$, and $\left\{\hat{\mathbf{y}}^n\right\}$ in Step 3. The vector $\left(\hat{\mathbf{x}}^n, \hat{\mathbf{f}}^n, \hat{\mathbf{y}}^n, \hat{\mathbf{s}}^n\right)$ is a convex combination of the sub-problem solutions found up to iteration $n-1$. When $\left\|\min\left\{\hat{\mathbf{s}}^n - \mathbf{q}, 0\right\}\right\|_1 / \|\mathbf{q}\|_1 \leq \varepsilon_1$ holds for sufficiently small $\varepsilon_1 > 0$, $\left(\hat{\mathbf{f}}^n, \hat{\mathbf{y}}^n\right)$ is approximately feasible for the primal problem. Therefore, $\left|Z\left(\hat{\mathbf{f}}^n, \hat{\mathbf{x}}^n\right) - \theta\left(\mathbf{p}^{n+1}\right)\right|$ provides a suitable upper estimate of the duality gap, and $\left(\hat{\mathbf{x}}^n, \hat{\mathbf{f}}^n, \hat{\mathbf{y}}^n\right)$ is optimal for problem P0 if $\frac{\left|Z\left(\hat{\mathbf{f}}^n, \hat{\mathbf{x}}^n\right) - \theta\left(\mathbf{p}^{n+1}\right)\right|}{Z\left(\hat{\mathbf{f}}^n, \hat{\mathbf{x}}^n\right)} \leq \varepsilon_2$

and $\frac{\left\|\min\left\{\hat{\mathbf{s}}^n-\mathbf{q},0\right\}\right\|_1}{\left\|\mathbf{q}\right\|_1}\le\varepsilon_1$. In the following Proposition, we show that the ergodic sequence $\left\{\hat{\mathbf{x}}^n,\hat{\mathbf{f}}^n,\hat{\mathbf{y}}^n\right\}$ generated in Step 3 is guaranteed to converge to the optimal solution of the primal problem P0 under mild conditions.

Proposition 5. *Suppose that* $\sum_{w\in W}d_w>\sum_{m\in M}q_m$ *. Then the sequences* $\left\{\mathbf{p}^{\mathbf{n}}\right\}$ *and* $\left\{\left(\hat{\mathbf{x}}^n,\hat{\mathbf{f}}^n,\hat{\mathbf{y}}^n\right)\right\}$ *generated by Alrogithm 1 respectively converge to optimal solutions of the dual problem (30) and the primal problem P0.*

Proof. The result follows directly from Theorem 2 in Gustavsson et al. (2015). According to the theorem in Gustavsson et al. (2015), the above statement holds if problem P0 satisfies Slater's constraint qualification. Following a proof procedure similar to Proposition 1, it is not difficult to see that the following Slater's constraint qualification, i.e.,

$$\left\{(\mathbf{f},\mathbf{y},\mathbf{x})\in\tilde{\Omega}\,\middle|\,q_m-\sum_{w\in W}\sum_{r\in R_{wm}}f_{r,wm}<0,m\in M\right\}\ne\varnothing$$

holds if $\sum_{w\in W}d_w>\sum_{m\in M}q_m$. ■

Remarks. We note that storing the trajectory flow vector $(\mathbf{f},\mathbf{y})$ is space-consuming. In the above algorithm, the objective $Z(\mathbf{f},\mathbf{x})$ can be obtained simply based on $\mathbf{x}$ and $\mathbf{s}$, so it is unnecessary to store the trajectory flows if one has no need on this information.

In Algorithm 1, the key step is to solve the dual sub-problem $\theta\left(\mathbf{p}^{\mathbf{n}}\right)$ in Eq. (28) to generate $\mathbf{x}\left(\mathbf{p}^n\right)$ and $\mathbf{s}(\mathbf{p}^n)$. In terms of structure, this dual sub-problem is similar to a traditional traffic assignment problem. Although there have been many efficient algorithms developed to solve the traditional traffic assignment problem in a much faster way (e.g., Bar-Gera, 2002; Dial, 2006; Nie, 2010, 2012; Xie et al., 2013; Xie and Xie, 2015, 2016), in the following we use the classic Frank-Wolfe algorithm, because it is the most well-known.

4.2 Solving the dual sub-problem with the Frank-Wolfe algorithm

The Frank-Wolfe algorithm is an iterative first-order optimization algorithm for compact-constrained convex programming. Starting from an initial feasible solution, the Frank-Wolfe algorithm considers a linear approximation of the objective function in each iteration, and then moves toward a minimizer of this linear function. In this study, the dual sub-problem (28) is solved as follows:

Algorithm 2: Frank-Wolfe algorithm.

Step 0 (Initialization): Let $\left(\mathbf{f}^0, \mathbf{y}^0, \mathbf{x}^0\right) \in \tilde{\Omega}$ be an initial feasible flow pattern for the dual sub-problem (28) and set $k = 0$.

Step 1(Searching direction generation): Let $\mathbf{z} = \left(\mathbf{f}, \mathbf{y}\right)$ and solve the following linear programming to obtain $\overline{\mathbf{z}}^k = \left(\overline{\mathbf{f}}^k, \overline{\mathbf{y}}^k\right)$ and $\overline{\mathbf{x}}^k$:

$$\left(\overline{\mathbf{z}}^k, \overline{\mathbf{x}}^k\right) \in \arg\min_{\mathbf{z},\mathbf{x}} \nabla_{\mathbf{z}} L\left(\mathbf{p}, \mathbf{f}^k, \mathbf{x}^k\right)^T \left(\mathbf{z} - \mathbf{z}^k\right), \text{ s.t. } \left(\mathbf{z}, \mathbf{x}\right) \in \tilde{\Omega} \tag{33}$$

Step 2 (Update step-size): Choose a step $\beta^k = \frac{2}{k+2}$.

Step 3 (Update variables): Let $\mathbf{x}^{k+1} = \mathbf{x}^k + \beta^k \left(\overline{\mathbf{x}}^k - \mathbf{x}^k\right)$, $\mathbf{y}^{k+1} = \mathbf{y}^k + \beta^k \left(\overline{\mathbf{y}}^k - \mathbf{y}^k\right)$ and $\mathbf{f}^{k+1} = \mathbf{f}^k + \beta^k \left(\overline{\mathbf{f}}^k - \mathbf{f}^k\right)$.

Step 4 (Check convergence): Let

$$\Delta\left(\mathbf{x}, \mathbf{f}, \mathbf{y}, \mathbf{p}\right) = \left| 1 - \frac{\sum_{w \in W} d_w \mu_w \left(\mathbf{x}, \mathbf{f}, \mathbf{y}, \mathbf{p}\right)}{\sum_{a \in A} x_a \left[t_a\left(x_a\right) + c_a\right] + \left(t_0 + \Delta\right) \sum_{m \in M} s_m\left(\mathbf{f}\right) - \sum_{m \in M} p_m s_m\left(\mathbf{f}\right)} \right| \tag{34}$$

where $\mu_w\left(\mathbf{x}, \mathbf{f}, \mathbf{y}, \mathbf{p}\right)$ is the drivers' minimal generalized travel cost between OD pair $w \in W$ under $\left(\mathbf{x}, \mathbf{f}, \mathbf{y}, \mathbf{z}\right)$ and $\mathbf{p}$, and $s_m\left(\mathbf{f}\right)$ is calculatd according to Eq. (31). If $\Delta\left(\mathbf{x}^{k+1}, \mathbf{f}^{k+1}, \mathbf{y}^{k+1}, \mathbf{p}\right) \le \varepsilon_3$, with $\varepsilon_3 > 0$ being a sufficiently small constant, stop and set $\mathbf{x}\left(\mathbf{p}\right) = \mathbf{x}^{k+1}$, $\mathbf{f}\left(\mathbf{p}\right) = \mathbf{f}^{k+1}$, $\mathbf{y}\left(\mathbf{p}\right) = \mathbf{y}^{k+1}$ and $\theta\left(\mathbf{p}\right) = L\left(\mathbf{p}, \mathbf{x}\left(\mathbf{p}\right), \mathbf{f}\left(\mathbf{p}\right)\right)$. Otherwise, set $k = k + 1$ and go to Step 1.

We next show how to solve the linear programming problem (33) by a shortest-path algorithm in Step 1. The linear programming problem (33) is equivalent to the following explicit form:

$$\min \sum_{m\in M}\sum_{w\in W}\sum_{r\in R_{wm}} f_{r,wm}\left(g_{r,wm}\left(\mathbf{x}^k\right)-p_m\right)+\sum_{w\in W}\sum_{r\in R_w} y_{r,w}g_r\left(\mathbf{x}^k\right) \tag{35}$$

$$\text{s.t.} \quad \left(\mathbf{z},\mathbf{x}\left(\mathbf{z}\right)\right)\in\tilde{\Omega}$$

where $g_{r,wm}\left(\mathbf{x}^k\right), r\in R_{wm}, m\in M, w\in W$ and $g_{r,w}\left(\mathbf{x}^k\right), r\in R_w, w\in W$ are respectively defined by

$$g_{r,wm}\left(\mathbf{x}^k\right)=\frac{\partial Z}{\partial f_{r,wm}}=\sum_{a\in A}\delta_{a,r}\left[t_a\left(x_a^k\right)+c_a\right]+t_0+\Delta \tag{36}$$

and

$$g_{r,w}\left(\mathbf{x}^k\right)=\frac{\partial Z}{\partial y_{r,w}}=\sum_{a\in A}\delta_{a,r}\left[t_a\left(x_a^k\right)+c_a\right] \tag{37}$$

Apparently, when $\mathbf{x}^k$ is given, both $g_{r,wm}\left(\mathbf{x}^k\right)$ and $g_{r,w}\left(\mathbf{x}^k\right)$ are constants, representing drivers' driving costs on trajectory $r\in R_{wm}, m\in M, w\in W$ and $r\in R_w, w\in W$ respectively under given link flow $\mathbf{x}^k$. So the optimal flow pattern $\overline{\mathbf{z}}=\left(\overline{\mathbf{f}},\overline{\mathbf{y}}\right)$ to Eq. (35) is to impose the driver demand d_w between each OD pair $w\in W$ onto one of the trajectory $r(w)$ with the minimal generalized cost, i.e.,

$$c_{r(w)}\left(\mathbf{x}^k,\mathbf{p}\right)=\min\left\{g_{r,wm}\left(\mathbf{x}^k\right)-p_m, r\in R_{wm}, m\in M; g_{r,w}\left(\mathbf{x}^k\right), r\in R_w\right\}.$$

To determine one of such trajectory $r(w)$ for each $w\in W$, we note that

$$\begin{aligned} c_{r(w)}\left(\mathbf{x}^k,\mathbf{p}\right)&=\min\left\{g_{r,wm}\left(\mathbf{x}^k\right)-p_m, r\in R_{wm}, m\in M; g_{r,w}\left(\mathbf{x}^k\right), r\in R_w\right\}\\ &=\min\left\{\min\left\{g_{r,wm}\left(\mathbf{x}^k\right), r\in R_{wm}\right\}-p_m, m\in M; \min\left\{g_{r,w}\left(\mathbf{x}^k\right), r\in R_w\right\}\right\}\end{aligned} \tag{38}$$

In Eq. (38), for any $w\in W$ and $m\in M$, the trajectory $r_{wm}\in R_{wm}$ with $g_{r_{wm}}\left(\mathbf{x}^k\right)=\min\left\{g_{r,wm}\left(\mathbf{x}^k\right), r\in R_{wm}\right\}$ can be easily obtained by respectively searching the shortest paths between $\left(o_w,o_m\right)$, $\left(o_m,e_m\right)$ and $\left(e_m,e_w\right)$ in the network and linking them together, where $o_w,o_m\in N$ and $e_w,e_m\in N$ are respectively the origins and destinations of driver OD pair $w\in W$ and rider OD pair $m\in M$. And for each driver OD pair $w\in W$, the trajectory $r_w\in R_w$ with $g_{r_w}\left(\mathbf{x}^k\right)=\min\left\{g_{r,w}\left(\mathbf{x}^k\right), r\in R_w\right\}$ is simply the shortest path between $\left(o_w,e_w\right)$. By comparing the costs $\left(g_{r_{wm}}\left(\mathbf{x}^k\right)-p_m, m\in M\right)$ and g_{r_w}, we can easily obtain the minimum-cost trajectory $r(w)$ and its corresponding cost $c_{r(w)}(\mathbf{x}^k,\mathbf{p})$ for drivers between OD pair $w \in W$:

$$c_{r(w)}(\mathbf{x}^k,\mathbf{p}) = \min\left\{g_{r_{wm}}(\mathbf{x}^k) - p_m, m \in M; g_{r_w}(\mathbf{x}^k)\right\}, w \in W . \tag{39}$$

Note that the value of $\mu_w(\mathbf{x},\mathbf{f},\mathbf{y},\mathbf{p})$ in step 4 is also derived during the process, and $\mu_w(\mathbf{x}^k,\mathbf{f}^k,\mathbf{y}^k,\mathbf{p}) = c_{r(w)}(\mathbf{x}^k,\mathbf{p}), w \in W$.

The termination criterion is chosen based on the RUE conditions introduced in Section 2. In Appendix F, we show that $\mathbf{x}(\mathbf{p})$, $\mathbf{y}(\mathbf{p})$ and $\mathbf{f}(\mathbf{p})$ solve the dual sub-problem $\theta(\mathbf{p})$ in Eq. (28) if and only if the termination criterion is met under sufficiently small $\varepsilon_3 > 0$.

5. Numerical examples

We now present several examples to illustrate the proposed model and algorithm intended to solve the traffic assignment problem in general networks with ridesharing.

Example 1 (a small network). Consider a simple network as in **Figure 3**. There are travelers commuting every day between three OD pairs: $W^0 = \{(i,j),(k,j),(k,i)\}$. For each OD pair, there are riders and drivers. The driver and rider demands for each OD pair are listed in **Table 2**. The travel time for each link follows the BPR function:

$$t_a(x_a) = t_a^0\left(1 + 0.15\left(\frac{x_a}{C_a}\right)^4\right), a \in A$$

with t_a^0 and C_a being defined as in **Table 2**. The non-time cost for each link c_a is proportional to the link travel time, given by $c_a(x_a) = 3t_a(x_a), a \in A$. The perceived safety cost of ridesharing is $\Delta = 5$, and the rider getting in and out cost is $t_0 = 4$.

For this small example, we enumerate all driver trajectories in **Table 3**, and explicitly establish the link-trajectory relationships for calculating the optimization problem P0 with an NLP solver. As we can see from **Table 3**, there are ten feasible driver trajectories in this example. Some trajectories traverse exactly the same links, but they belong to different trajectory sets, and are associated with different mode, rider and route choices. For example, both Trajectories 2 and 3 sequentially traverse links b, c and a, but drivers in Trajectory 2 take riders between (k,i), while drivers in Trajectory 3 take riders between (k,j).

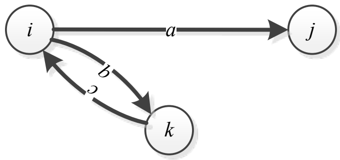

Figure 3. The network for Example 1

Table 2. Demand and link information for Example 1

OD index	OD pair	Driver demand	Rider demand	Link $a \in A$	Free flow travel time t_a^0	Link capacity C_a
1	***(i,j)***	15	5	a	10	20
2	***(k,j)***	12	8	b	3	20
3	***(k,i)***	20	30	c	3	20

Table 3. Trajectory information for Example 1

Trajectory No.	Drivers' OD pair	Riders' OD pair	Traversed links	Trajectory No.	Drivers' OD pair	Riders' OD pair	Traversed links
1	1	1	a	6	2	3	c, a
2	1	2	b, c, a	7	2	2	c, a
3	1	3	b, c, a	8	2	NA	c, a
4	1	NA	a	9	3	3	c
5	2	1	c,a	10	3	NA	c

Under this setup, we solve the optimization problem P0 with both the NLP solver KNITRO and the dual subgradient method with averaging introduced in Section 4. The convexity weights, step length, and error tolerances for the dual subgradient method with averaging are set as follows:

$$\gamma_t^n = (t+1)^2 \Big/ \sum_{l=0}^{n-1} (l+1)^2, \quad t = 0,\ldots,n-1$$

$$\alpha^n = \hat{a} \big/ (1+n), \quad \hat{a} = 1 \tag{40}$$

$$\varepsilon_1 = \varepsilon_2 = 0.001 \quad \text{and} \quad \varepsilon_3 = 0.0005$$

Both methods generate the same link flows, drivers' generalized cost, and their net income from serving different ODs at RUE as listed in **Table 4**. For this small network, we save the trajectory flow and trajectory cost information, as provided in **Table 5**. From **Table 5**, it is not difficult to see that the flow patterns generated by the proposed model and algorithm meet all of the RUE conditions. At RUE, drivers between (i, j) would serve all the three OD pairs, drivers between (k,j) would serve riders between (k,i) and (k,j), and drivers between (k,i) would only serve riders between the same OD pair. Further, because there are solo drivers between OD pair (i,j), and the RS drivers between (i,j) serve riders between all OD pairs in this example, the drivers' net income from serving all three OD pairs is uniquely determined and provided in **Table 4**.

Table 4. Link flows, drivers' costs and drivers' net income from serving different ODs at UE

OD pair	Driver's net income from serving the OD	Drivers' generalized cost	Link	Link flow
(i,j)	9.000	59.929	a	27.00
(*k,j*)	56.472	47.914	b	6.00
(*k,i*)	56.472	-12.014	c	38.00

Table 5. Trajectory flow and cost (without rider payment) at UE

Trajectory No.	Trajectory flow	Trajectory cost (without rider payment)	Trajectory cost (with rider payment)
1	5	68.929	59.929
2	3.265	116.401	59.929
3	2.735	116.401	59.929
4	4	59.929	59.929
5	0	104.387	95.378
6	7.265	104.387	47.906
7	4.735	104.387	47.906
8	0	95.387	95.378
9	20	44.458	-12.014
10	0	35.458	35.458

Example 2 (The Sioux Falls network). To further examine the efficiency of the proposed algorithm, we implement the algorithm in the Sioux Falls network as in **Figure 4**. The dataset of the Sioux Falls network was downloaded from the website called "Transportation Network Test Problems" (http://www.bgu.ac.il/bargera/tntp/.). The original data contains a network with 24 nodes and 76 links. The travel time for each link follows the BPR function, with the free-flow travel time and capacity of each link being the same as given on the website. The monetary cost of each link $b \in A$ is assumed to be three times of the time cost, i.e., $c_b = 3t_b(x_b)$. We randomly generate 20 node pairs as drivers' OD pairs. For riders, 10 of their OD pairs are chosen from drivers' OD pairs, whereas another 10 OD pairs are randomly generated. The detailed OD pairs and demands for both riders and drivers are provided in

Appendix G. The perceived safety cost for ridesharing is set at $\Delta = 5$, and the rider getting in and out cost is $t_0 = 4$.

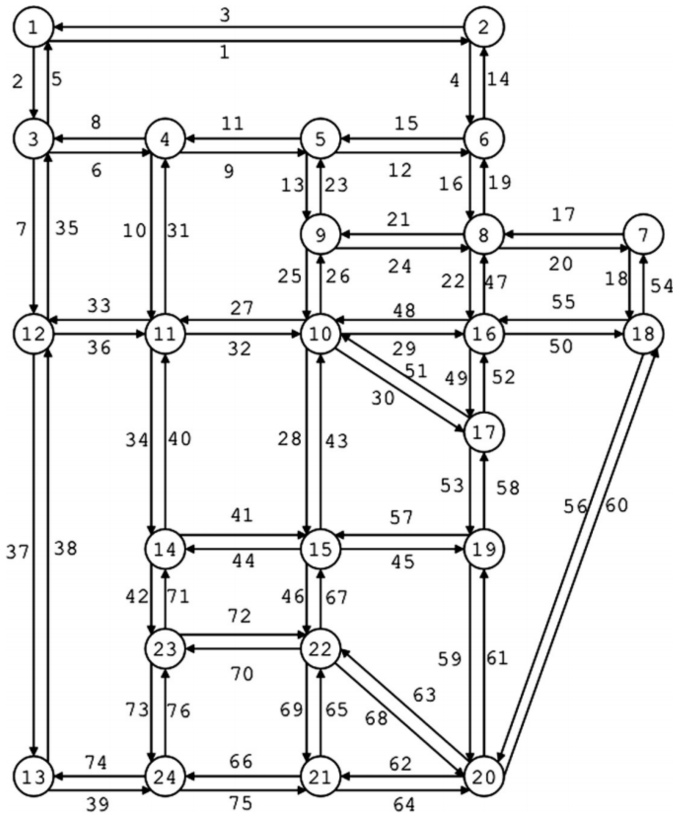

Figure 4. The Sioux Falls network

Following the same setup of convexity weights, step length, and error tolerances as in Eq (40), and starting from $p_m^0 = 55, m \in M$, the program terminates after 32,183 iterations. The calculation results of the drivers' generalized costs are respectively provided in **Table 6**. The drivers' net income from serving different ODs and the observed service relationship between drivers and riders are provided in **Table 7**. The UE link flows and costs are provided in Appendix G. The convergence performance of the dual subgradient algorithm with averaging is depicted in **Figure 5**. As we can see from **Figure 5**, both the feasibility gap and duality gap converge to tolerated errors with time. However, the speed of convergence is not fast. This is unsurprising, because the dual subgradient method is based on first-order derivative information only, and the Frank-Wolfe algorithm for the dual sub-problem is subject to a well-known drawback of low convergence rate when approaching the solution (Nie, 2010). It is in our interest to propose more efficient solving algorithms in the future.

Table 6. Drivers' generalized costs (with ridesharing payment) at UE

Driver OD index	Drivers' generalized cost	Driver OD index	Drivers' generalized cost
1	-22	11	-1.875
2	26.161	12	6.022
3	2.0502	13	-12.98
4	6.0502	14	22.046
5	44	15	26.58
6	2.5868	16	36.025
7	14.05	17	32.633
8	20.003	18	-0.392
9	-4.25	19	43.62
10	2.5869	20	40

Table 7. Drivers' net income for serving different ODs and driver-rider matchings at UE

Rider OD index	Drivers' net income from serving the OD (yuan/trip)	OD index of drivers in service
1	34.632	1
2	77.272	5
3	47.096	2, 3
4	51.449	2
5	82.597	2, 16
6	56.851	4, 6
7	46.955	2, 3, 6, 10
8	69.482	7
9	65.047	17
10	56.986	5
11	48.83	9, 18

12	35.518	6, 10, 18
13	42.13	13
14	40.969	12
15	52.677	11
16	96.558	14
17	87.198	10
18	79.198	15
19	121.88	15, 20
20	121.64	8

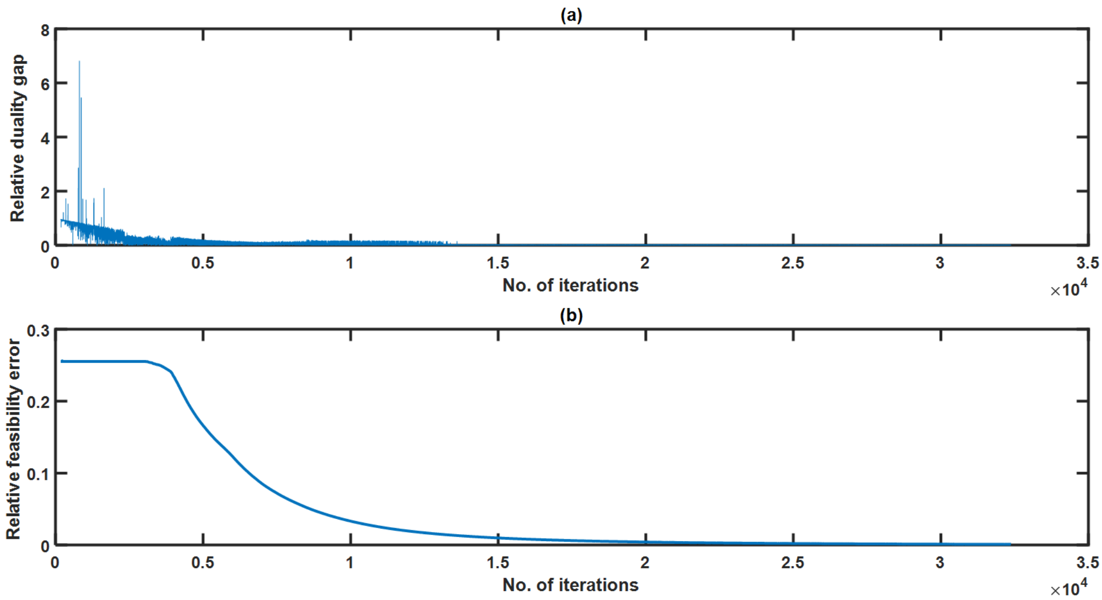

Figure 5. Convergence performance

6. Conclusions and discussions

With the growing popularity of smart-phone based ridesharing platforms, how to incorporate ridesharing into traffic assignment has received increasing attention in recent years. Most existing modes are inconvenient to implement in reality, for its presumed ridesharing price and inconvenience functions, and path-based mixed complementary formulations. In this study, assuming that drivers can serve riders between any accessible OD pair, and that each RS user travels with one RS partner along her/his trip, we redefine the set of feasible driver trajectories in the presence of ridesharing, and introduce complementary conditions to characterize the market clearing conditions for each rider OD pair. With the driver- and rider-demand of each OD pair being fixed and given, the coupled user equilibrium and market equilibrium can be formulated as a convex programming problem, and the existence of UE solutions and the uniqueness of aggregate link flows are then established under mild conditions. For this convex programming problem, we propose a dual subgradient algorithm with averaging to solve it. And the dual sub-problem has a similar structure as the basic model of traffic assignment, therefore enables the application of classical traffic assignment algorithms such as the Frank-Wolfe algorithm.

In comparison with existing RUE models, our model is more restrictive on rider and driver demand elasticity. But the different modeling approach adopted in this paper leads to an alternative RUE model, which is more convenient to implement in practice. The model is built on no presumed price or inconvenience functions, and no path numeration is required in calculation. The driver- and rider-demand information can be obtained by applying the third-step (mode choice) in the classic four-step method for transport planning. As the popularity of a traffic assignment model largely relies on its simplicity and implementability, the nice mathematical properties of our model may make it appealing to practitioners and researchers in traffic planning.

Nevertheless, to generate more accurate predictions of traffic flows at RUE, the model could be improved in many aspects. For example, provided the link travel time and drivers' net income generated by our RUE model under a given driver- and rider- demand pattern, one can infer riders' time and monetary costs on each utilized path. But the obtained costs for drivers and riders between the same OD pairs are generally not the same. This implies disequilibrium when travelers are free to choose between being drivers and riders. So to enhance the prediction accuracy, more general RUE models, with the number of drivers and riders for each OD pair being endogenously determined (as assumed in our predecessors), are highly desired. In this case, our model can be perceived as a subproblem, which provides an easier way to generate the driver- and rider-costs of each OD pair under a given driver- and rider-demand pattern. Furthermore, in this paper, we assume each driver serves at most one rider during her/his trip. Allowing drivers to serve multiple riders along their trips may further enhance the social benefits of ridesharing, so how to extend the model into the multiple-rider case could be an important extension. And users with different values of time would apparently have different mode, rider and route choices, so user heterogeneity may also be incorporated into the model. How to enhance the prediction accuracies of the model while sacrificing as little tractability as possible is an important and challenging topic for our future investigation.

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Appendix A. List of notations

Table A.1 Notations of parameters

G	The network
N	Set of nodes
A	Set of links
W	Set of driver OD pairs
M	Set of rider OD pairs
R_w	Set of solo-driver trajectories between OD pair $w \in W$
R_{wm}	Set of trajectories of RS drivers between OD pair $w \in W$ that serve riders between OD pair $m \in M$
d_w	Driver demand between OD $w \in W$
q_m	Rider demand between OD $m \in M$

x_b	Aggregate link flow on link $b \in A$
$t_a\left(x_a\right)$	Travel time function of link $a \in A$
t_0	A constant, indicating the rider get-in and get-out time costs
Δ	A constant, indicating the safety cost of traveling with a stranger

Table A.2 Notations of variables

$f_{r,wm}$	RS driver flow on trajectory $r \in R_{wm}, w \in W, m \in M$
$y_{r,w}$	Solo driver flow on trajectory $r \in R_w, w \in W$
v_a	RS driver flow on link $a \in A$
h_a	Solo driver flow on link $a \in A$
x_a	Aggregate link flow on link $a \in A$
p_m	Drivers' net income of serving riders between OD $m \in M$

Appendix B. Derivatives of the objective function of P0

First, from Eq. (16), we have

$$\begin{aligned}\frac{\partial Z}{\partial f_{r,wm}} &= \frac{\partial\left\{\sum_{a\in A}\int_0^{x_a}\left[t_a\left(w\right)+c_a\right]dw+\left(t_0+\Delta\right)\sum_{w\in W}\sum_{m\in M}\sum_{r\in R_{wm}} f_{r,wm}\right\}}{\partial f_{r,wm}} \\ &= \sum_{a\in A}\frac{\partial x_a}{\partial f_{r,wm}}\left[t_a\left(x_a\right)+c_a\right]+t_0+\Delta \\ &= \sum_{a\in A}\frac{\partial\left(v_a+h_a\right)}{\partial f_{r,wm}}\left[t_a\left(x_a\right)+c_a\right]+t_0+\Delta \\ &= \sum_{a\in A}\delta_{a,r}\left[t_a\left(x_a\right)+c_a\right]+t_0+\Delta\end{aligned} \tag{41}$$

where the last equity holds because for any $r \in R_{wm}, w \in W, m \in M$, we have $\partial h_a / f_{r,wm} = 0$ and $\partial v_a / \partial f_{r,wm} = \delta_{a,r}, a \in A$ from Eqs (4) and (3) respectively. On the other hand, from Eq. (16), we have

$$\frac{\partial Z}{\partial y_{r,w}} = \frac{\partial\left\{\sum_{a\in A}\int_0^{x_a}\left[t_a\left(w\right)+c_a\right]dw+\left(t_0+\Delta\right)\sum_{w\in W}\sum_{m\in M}\sum_{r\in R_{wm}} f_{r,wm}\right\}}{\partial y_{r,w}}$$

$$= \sum_{a\in A}\frac{\partial x_a}{\partial y_{r,w}}\left[t_a\left(x_a\right)+c_a\right]$$

$$= \sum_{a\in A}\frac{\partial\left(v_a+h_a\right)}{\partial y_{r,w}}\left[t_a\left(x_a\right)+c_a\right]$$

$$= \sum_{a\in A}\delta_{a,r}\left[t_a\left(x_a\right)+c_a\right]$$

where the last equity holds because for any $r\in R_w, w\in W$, we have $\partial h_a/y_{r,w}=\delta_{a,r}$ and $\partial v_a/\partial y_{r,w}=0, a\in A$ from Eqs (4) and (3) respectively.

Appendix C. Proof of Proposition 2

Proof. We first show that Ω is non-empty when $\sum_{w\in W} d_w \ge \sum_{m\in M} q_m$. From Eqs (17) -(20), the non-emptiness of Ω holds if and only if $\exists\left(\mathbf{f},\mathbf{y}\right)\in\mathbb{R}_+^{\sum_{m,w}|R_{wm}|+|R_w|}$ such that

$$\begin{aligned}&\sum_{m\in M}\sum_{r\in R_{wm}} f_{r,wm}+\sum_{r\in R_w} y_{r,w}=d_w,\ \ w\in W\\&\sum_{w\in W}\sum_{r\in R_{wm}} f_{r,wm}\ge q_m,\ \ m\in M\end{aligned} \tag{42}$$

Because $\sum_{w\in W} d_w \ge \sum_{m\in M} q_m$, we can always find a vector $\mathbf{d}'=\left(d'_w, w\in W\right)\in\mathbb{R}_+^{|W|}$ such that $\sum_{w\in W} d'_w=\sum_{m\in M} q_m=Q$ and $d'_w\le d_w$ for all $w\in W$. It then suffices to show that there exists $\left(\mathbf{f},\mathbf{y}\right)\in\mathbb{R}_+^{\sum_{m,w}|R_{wm}|+|R_w|}$ such that

$$\sum_{m\in M}\sum_{r\in R_{wm}} f_{r,wm}=d'_w,\ \ w\in W \tag{43}$$

$$\sum_{w\in W}\sum_{r\in R_{wm}} f_{r,wm}=q_m,\ \ m\in M \tag{44}$$

$$\sum_{r\in R_w} y_{r,w}=d_w-d'_w,\ \ w\in W. \tag{45}$$

Let

$$\sum_{r\in R_{wm}} f_{r,wm}=\frac{d'_w q_m}{Q},\ \ m\in M, w\in W. \tag{46}$$

We then have

$$\sum_{m\in M}\sum_{r\in R_{wm}} f_{r,wm} = \sum_{m\in M}\frac{d'_w q_m}{Q} = d'_w \frac{\sum_{m\in M} q_m}{Q} = d'_w,\quad w\in W$$

$$\sum_{w\in W}\sum_{r\in R_{wm}} f_{r,wm} = q_m \sum_{w\in W}\frac{d'_w}{Q} = q_m \frac{\sum_{w\in W} d'_w}{Q} = q_m,\quad m\in M$$

So, $\mathbf{f}\in\mathbb{R}_+^{\sum_{w,m}|R_{wm}|}$ satisfying Eq. (46) must be a solution to Eqs (43) and (44). Note that the existence of $\mathbf{f}\in\mathbb{R}_+^{\sum_{w,m}|R_{wm}|}$ ($\mathbf{y}\in\mathbb{R}_+^{\sum_w|R_w|}$) to Eq. (46) (Eq.(45)) is always guaranteed, because for each $m\in M, w\in W$ ($w\in W$), we can always construct a solution by letting $f_{r',wm} = \frac{d'_w q_m}{Q}$ ($y_{r',w} = d_w - d'_w$) for one trajectory $r'\in R_{wm}$ ($r'\in R_w$) and $f_{r,wm}=0$ ($y_{r,w}=0$) for the rest of the trajectory $r\in R_{wm}/r'$ ($r\in R_w/r'$). The existence of $(\mathbf{f},\mathbf{y})\in\mathbb{R}_+^{\sum_{m,w}|R_{wm}|+|R_w|}$ to Eqs (43)-(45) is therefore always guaranteed, and the set Ω is thus non-empty.

Meanwhile, because Eqs (17)-(20) are all linear constraints, and $0\le f_{r,wm}, y_{r,w}\le d_w$ holds for all $r\in R_w, r\in R_{wm}, w\in W, m\in M$, the compactness of Ω is apparent. From the well-known Weierstrass's theorem (see, e.g., Proposition 2.1.1 in Bertsekas et al., 2003), a continuous function attains a global minimum over a compact set. Provided the continuity of $t_a(x_a), a\in A$, we can readily conclude the existence of optimal solutions to the optimization problem P0, and the existence of user equilibrium with ridesharing is thus guaranteed by Proposition 1. ■

Appendix D. Proof of Proposition 3

Proof. In the proof for Propositions 1 and 2, we have shown that P0 is a convex programming problem with guaranteed optimal solutions. For the sake of simplicity, we rewrite the objective function (16) of P0 as $Z(\mathbf{f},\mathbf{y},\mathbf{x}) = G(\mathbf{x}) + H(\mathbf{f})$, where

$$G(\mathbf{x}) = \sum_{a\in A}\int_0^{x_a}\left[t_a(w)+c_a\right]dw \quad\text{and}\quad H(\mathbf{f}) = (t_0+\Delta)\sum_{w\in W}\sum_{m\in M}\sum_{r\in R_{wm}} f_{r,wm}.$$

From Eq. (21), it follows that the Hessian Matrix of $G(\mathbf{x})$ is positive definite if $t_a(x_a)$ is continuous and strictly increasing with respect to x_a for all $a\in A$, so $G(\mathbf{x})$ is strictly convex. Meanwhile, $H(\mathbf{f})$ is linear.

Now suppose that $(\mathbf{f}_1,\mathbf{y}_1,\mathbf{x}_1)\in\Omega$ and $(\mathbf{f}_2,\mathbf{y}_2,\mathbf{x}_2)\in\Omega$ are both optimal solutions to P0, and $\mathbf{x}_1\neq\mathbf{x}_2$. By the convexity of Ω, for any $0<\lambda<1$, we have

$$\lambda(\mathbf{f}_1,\mathbf{y}_1,\mathbf{x}_1)+(1-\lambda)(\mathbf{f}_2,\mathbf{y}_2,\mathbf{x}_2)=(\mathbf{f}_3,\mathbf{y}_3,\mathbf{x}_3)\in\Omega.$$

So $(\mathbf{f}_3,\mathbf{y}_3,\mathbf{x}_3)$ is also a feasible flow pattern to P0. However, since $G(\mathbf{x})$ is strictly convex and $H(\mathbf{f})$ is linear, we have

$$\begin{aligned} Z(\mathbf{f}_3,\mathbf{y}_3,\mathbf{x}_3)=G(\mathbf{x}_3)+H(\mathbf{f}_3)&=G(\lambda\mathbf{x}_1+(1-\lambda)\mathbf{x}_2)+H(\lambda\mathbf{f}_1+(1-\lambda)\mathbf{f}_2)\\ &<\lambda G(\mathbf{x}_1)+(1-\lambda)G(\mathbf{x}_2)+\lambda H(\mathbf{f}_1)+(1-\lambda)H(\mathbf{f}_2)\\ &=\lambda Z(\mathbf{f}_1,\mathbf{y}_1,\mathbf{x}_1)+(1-\lambda)Z(\mathbf{f}_2,\mathbf{y}_2,\mathbf{x}_2)\\ &=Z^* \end{aligned} \tag{47}$$

where $Z^*=Z(\mathbf{f}_1,\mathbf{y}_1,\mathbf{x}_1)=Z(\mathbf{f}_2,\mathbf{y}_2,\mathbf{x}_2)$ is the optimal value of problem P0. Eq (53) apparently contradicts the optimality of $(\mathbf{f}_1,\mathbf{y}_1,\mathbf{x}_1)$ and $(\mathbf{f}_2,\mathbf{y}_2,\mathbf{x}_2)$, and we therefore conclude that the aggregate link flow pattern $\mathbf{x}$ at optimum must be unique. ■

Appendix E. Proof of Corollary 1

Proof. Suppose that for an OD pair $w'\in W$, we have $\sum_{r\in R_w} y^*_{r,w'}>0$ at user equilibrium. Let $r'\in R_{w'}$ be a solo-driver trajectory between OD pair w' with $y^*_{r',w'}>0$, then from Eq. (13), we have

$$\mu^*_{w'}=\sum_{a\in A}\delta_{a,r'}\left[t_a\left(x^*_a\right)+c_a\right]$$

From Proposition 3, the aggregate link flow pattern $\mathbf{x}^*$ is unique at UE, so $\mu^*_{w'}$ is uniquely determined at RUE. Let $m'\in M$ be any rider OD pair served by drivers between OD pair w', i.e., $\sum_{r\in R_{w'm'}} f_{r,w'm'}>0$. Then, from Eq. (12), for at least one trajectory $r\in R_{w'm'}$, we have $f_{r,w'm'}>0$ and

$$p^*_{m'}=\sum_{a\in A}\delta_{a,r}\left[t_a\left(x^*_a\right)+c_a\right]+t_0+\Delta-\mu^*_{w'} \tag{48}$$

Given that both $\mu_{w'}^*$ and $\mathbf{x}^*$ are uniquely determined, Eq. (48) implies that $p_{m'}^*$ is uniquely determined. This completes the proof. ■

Appendix F. Validation of the termination criteria for the Frank-Wolfe algorithm

First, from Eq. (28), it is not difficult to see that the dual sub-problem $\theta(\mathbf{p})$ is a convex programming problem, and its KKT conditions can be given by

$$0 \le f_{r,wm} \perp \left(\sum_{a\in A} \delta_{a,r} \left[t_a\left(x_a\right) + c_a \right] + t_0 + \Delta - p_m - \mu_w \right) \ge 0, r \in R_{wm}, w \in W, m \in M \tag{49}$$

$$0 \le y_{r,w} \perp \left(\sum_{a\in A} \delta_{a,r} \left[t_a\left(x_a\right) + c_a \right] - \mu_w \right) \ge 0, r \in R_w, w \in W \tag{50}$$

$$\left(\mathbf{f}, \mathbf{y}, \mathbf{x}\right) \in \tilde{\Omega} \tag{51}$$

So to validate the termination criteria for the Frank-Wolfe algorithm is equivalent to show that Eqs (49)-(51) hold if and only if $\Delta\left(\mathbf{x}, \mathbf{f}, \mathbf{y}, \mathbf{p}\right) = 0$.

From the definition of $\Delta\left(\mathbf{x}, \mathbf{f}, \mathbf{y}, \mathbf{p}\right)$ in Eq. (34), $\Delta\left(\mathbf{x}, \mathbf{f}, \mathbf{y}, \mathbf{p}\right) = 0$ if and only if

$$\Lambda = \sum_{a\in A} x_a \left[t_a\left(x_a\right) + c_a \right] + \left(t_0 + \Delta\right) \sum_{m\in M} s_m - \sum_{m\in M} p_m s_m - \sum_{w\in W} \mu_w\left(\mathbf{x}, \mathbf{f}, \mathbf{y}, \mathbf{p}\right) d_w = 0$$

where $s_m = \sum_{w\in W} \sum_{r\in R_{wm}} f_{r,wm}$, and $\mu_w\left(\mathbf{x}, \mathbf{f}, \mathbf{y}, \mathbf{p}\right)$ is the minimal generalized travel cost for OD pair $w \in W$, i.e.,

$$\mu_w\left(\mathbf{x}, \mathbf{f}, \mathbf{y}, \mathbf{p}\right) = \min \left\{ \begin{array}{l} \sum_{a\in A} \delta_{a,r} \left[t_a\left(x_a\right) + c_a \right] + t_0 + \Delta - p_m, r \in R_{wm}, m \in M; \\ \sum_{a\in A} \delta_{a,r} \left[t_a\left(x_a\right) + c_a \right], r \in R_w \end{array} \right\} \tag{52}$$

During each iteration of the Frank-Wolfe algorithm, the feasibility of flow patterns is always maintained, i.e., $\left(\mathbf{f}^k, \mathbf{y}^k, \mathbf{x}^k\right) \in \tilde{\Omega}$ holds for all k. So, Λ can be further written into

$$\begin{aligned} \Lambda &= \sum_{a\in A} x_a \left[t_a\left(x_a\right) + c_a \right] + \left(t_0 + \Delta\right) \sum_{m\in M} s_m - \sum_{m\in M} p_m s_m - \sum_{w\in W} \mu_w\left(\mathbf{x}, \mathbf{f}, \mathbf{y}, \mathbf{p}\right) d_w \\ &= \sum_{a\in A} \left(v_a + h_a\right) \left[t_a\left(x_a\right) + c_a \right] + \left(t_0 + \Delta\right) \sum_{m\in M} \sum_{w\in W} \sum_{r\in R_{wm}} f_{r,wm} - \sum_{m\in M} p_m \sum_{w\in W} \sum_{r\in R_w} f_{r,wm} - \sum_{w\in W} \mu_w d_w \end{aligned} \tag{53}$$

From Eqs (3) and (4), we have

$$
\begin{aligned}
\sum_{a\in A}(v_a+h_a)\left[t_a(x_a)+c_a\right] &= \sum_{a\in A}\left(\sum_{m\in M}\sum_{w\in W}\sum_{r\in R_{wm}} f_{r,wm}\delta_{a,r}+\sum_{w\in W}\sum_{r\in R_w} y_{r,w}\delta_{a,r}\right)\left[t_a(x_a)+c_a\right] \\
&= \sum_{a\in A}\left\{\sum_{m\in M}\sum_{w\in W}\sum_{r\in R_{wm}} f_{r,wm}\left[t_a(x_a)+c_a\right]\delta_{a,r}+\sum_{w\in W}\sum_{r\in R_w} y_{r,w}\left[t_a(x_a)+c_a\right]\delta_{a,r}\right\} \\
&= \sum_{m\in M}\sum_{w\in W}\sum_{r\in R_{wm}} f_{r,wm}\sum_{a\in A}\left[t_a(x_a)+c_a\right]\delta_{a,r}+\sum_{w\in W}\sum_{r\in R_w} y_{r,w}\sum_{a\in A}\left[t_a(x_a)+c_a\right]\delta_{a,r}
\end{aligned}
\tag{54}
$$

Inserting Eq. (54) into (53) yields

$$
\begin{aligned}
\Lambda &= \sum_{m\in M}\sum_{w\in W}\sum_{r\in R_{wm}} f_{r,wm}\sum_{a\in A}\left[t_a(x_a)+c_a\right]\delta_{a,r}+\sum_{w\in W}\sum_{r\in R_w} y_{r,w}\sum_{a\in A}\left[t_a(x_a)+c_a\right]\delta_{a,r} \\
&\qquad +(t_0+\Delta)\sum_{m\in M}\sum_{w\in W}\sum_{r\in R_{wm}} f_{r,wm}-\sum_{m\in M} p_m\sum_{w\in W}\sum_{r\in R_w} f_{r,wm}-\sum_{w\in W}\mu_w d_w \\
&= \sum_{m\in M}\sum_{w\in W}\sum_{r\in R_{wm}} f_{r,wm}\left\{\sum_{a\in A}\left[t_a(x_a)+c_a\right]\delta_{a,r}+(t_0+\Delta)-p_m-\mu_w\right\} \\
&\qquad +\sum_{w\in W}\sum_{r\in R_w} y_{r,w}\left\{\sum_{a\in A}\left[t_a(x_a)+c_a\right]\delta_{a,r}-\mu_w\right\}
\end{aligned}
\tag{55}
$$

On the other hand, from Eq. (52) we have

$$
\sum_{a\in A}\left[t_a(x_a)+c_a\right]\delta_{a,r}+(t_0+\Delta)-p_m-\mu_w\ge 0,\quad \text{for any } r\in R_{wm}, m\in M, w\in W,
$$

and

$$
\sum_{a\in A}\delta_{a,r}\left[t_a(x_a)+c_a\right]-\mu_w\ge 0,\quad \text{for any } r\in R_w, w\in W.
$$

So, Eq. (55) implies that $\Lambda=0$ if and only if

$$
f_{r,wm}\left\{\sum_{a\in A}\left[t_a(x_a)+c_a\right]\delta_{a,r}+(t_0+\Delta)-p_m-\mu_w\right\}=0, r\in R_{wm}, w\in W, m\in M \tag{56}
$$

$$
y_{r,w}\left(\sum_{a\in A}\delta_{a,r}\left[t_a(x_a)+c_a\right]-\mu_w\right)=0, r\in R_w, w\in W \tag{57}
$$

We can therefore conclude that the optimal solution to the dual sub-problem (28) under any given $\mathbf{p}\in\mathbb{R}_+^{|M|}$ is obtained when $\Delta(\mathbf{x},\mathbf{f},\mathbf{y},\mathbf{p})<\varepsilon_3$ holds for sufficiently small $\varepsilon_3>0$.

Appendix G. UE link flows and costs for Example 2

Table G.1. Driver/rider OD and demand input

Driver OD and demand				Rider OD and demand			
OD Index	**Origin node**	**Destination node**	**Demand (trips/hr)**	**OD Index**	**Origin node**	**Destination node**	**Demand (trips/hr)**
1	1	2	100	1	1	2	100
2	1	16	500	2	2	13	300
3	3	8	200	3	3	8	200
4	4	18	100	4	3	10	300
5	4	13	600	5	3	15	100
6	5	16	500	6	4	18	100
7	5	20	100	7	4	8	700
8	6	2	400	8	5	20	100
9	8	10	1600	9	6	15	200
10	9	8	800	10	6	12	200
11	10	12	2000	11	8	10	1600
12	10	23	1800	12	9	10	2800
13	10	19	1800	13	10	19	1800
14	11	2	200	14	10	23	1800
15	13	16	600	15	10	12	2000
16	13	22	1300	16	11	2	200
17	16	22	1200	17	11	7	500
18	16	10	4400	18	13	16	600
19	17	24	300	19	13	2	300
20	21	12	300	20	15	2	100

Table G.2. Link flow and cost at UE

Link index	Beginning node	Tail node	Link flow	Link time and monetary cost
1	1	2	400.04	24.00
2	1	3	1099.94	16.00
3	2	1	599.97	24.00
4	2	6	0.27	20.00
5	3	1	300.02	16.00
6	3	4	682.21	16.00
7	3	12	920.14	16.00
8	4	3	300.02	16.00
9	4	5	2000.14	8.00
10	4	11	80.57	24.00
11	5	4	798.51	8.00
12	5	6	1800.08	16.04
13	5	9	404.69	20.00
14	6	2	900.20	20.00
15	6	5	200.03	16.00
16	6	8	1400.14	8.01
17	7	8	499.55	12.00
18	7	18	200.55	8.00
19	8	6	300.02	8.00
20	8	7	200.05	12.00
21	8	9	0.93	40.00
22	8	16	2397.03	20.15
23	9	5	203.10	20.00
24	9	8	0.00	40.00
25	9	10	3600.05	12.01

26	10	9	2597.54	12.00
27	10	11	3045.95	20.03
28	10	15	1254.39	24.00
29	10	16	3402.10	16.58
30	10	17	0.00	32.00
31	11	4	200.00	24.00
32	11	10	1100.11	20.00
33	11	12	2000.03	24.10
34	11	14	626.49	16.00
35	12	3	302.41	16.00
36	12	11	600.12	24.00
37	12	13	620.19	12.00
38	13	12	902.54	12.00
39	13	24	1317.46	16.01
40	14	11	0.01	16.00
41	14	15	80.13	20.00
42	14	23	546.37	16.00
43	15	10	0.10	24.00
44	15	14	0.00	20.00
45	15	19	197.76	12.00
46	15	22	2756.08	12.01
47	16	8	398.35	20.00
48	16	10	5999.71	21.60
49	16	17	3099.21	8.15
50	16	18	500.05	12.00
51	17	10	0.01	32.00
52	17	16	197.69	8.00
53	17	19	3399.19	8.30

54	18	7	500.05	8.00
55	18	16	0.50	12.00
56	18	20	100.06	16.00
57	19	15	1599.28	12.00
58	19	17	197.67	8.00
59	19	20	0.00	16.00
60	20	18	0.00	16.00
61	20	19	0.00	16.00
62	20	21	0.00	24.00
63	20	22	0.06	20.00
64	21	20	0.00	24.00
65	21	22	1317.65	8.00
66	21	24	599.77	12.00
67	22	15	20.14	12.00
68	22	20	0.00	20.00
69	22	21	299.99	8.00
70	22	23	1253.69	16.01
71	23	14	0.01	16.00
72	23	22	0.03	16.00
73	23	24	0.04	8.00
74	24	13	299.81	16.00
75	24	21	1317.43	12.01
76	24	23	0.02	8.00