Correctness of the preflow push algorithm

Valid distance labels

We say that a distance function $d: N \rightarrow Z^{+} \cup \{0\}$ with respect to the residual capacities r_{ij} is valid with respect to a flow **x** if it satisfies the following two conditions: $d(t)$=0; and

$d(i) \leq d(j) + 1$ for every arc (*i,j*) in the residual network G(x).

Property 1. If the distance labels are valid, the distance label $d(i)$ is a lower bound on the length of the shortest (directed) path from node *i* to node t in the residual network.

Proof. Let $i = i_1 - i_2 - \cdots - i_k - i_{k+1} = t$ be any path of length k from node i to node t in the residual network. The validity conditions imply that

$$d(i_k) \leq d(i_{k+1}) + 1 = d(t) + 1 = 1,$$
$$d(i_{k-1}) \leq d(i_k) + 1 \leq 2,$$
$$d(i_{k-2}) \leq d(i_{k-1}) + 1 \leq 3,$$
$$\ldots$$
$$d(i) = d(i_1) \leq d(i_2) + 1 \leq k.$$

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Property 2. If $d(s) \geq n$, the residual network contains no directed path from the source node to the sink node.
Proof. No directed path can contain more than (n-1) nodes.

Proposition. *The preflow push algorithm will terminate if and only if x is the maximum flow.*
Proof.
First, when the algorithm terminates, there is no active node, so the current preflow is a flow.
Second, from Property 2, since d(s)=n, the residual network contains no path from the source to the sink.
Recall the termination criterion of the augmenting path algorithm, we can thus conclude that the preflow push algorithm will terminate at the solution of the max flow problem.