Distributed dynamic event-triggered communication and control for multi-agent consensus: A co-design approach

***Abstract*—This paper investigates the output-based event-triggered communication and control for linear multi-agent consensus under directed graph with external disturbances. In order to further reduce the use of the system resources, we consider the scenario in which both the communication among agents and the controller update are determined by the event-triggering mechanisms. To simultaneously guarantee the significant properties including the asymptotic consensus, and strong Zeno-freeness (strictly positive inter-event times), a novel distributed dynamic event-triggered protocol is proposed. Unlike most-existing works in which the control gain is previously decided, a co-design method is proposed to design the control gain, the observer gain and the event-triggering mechanisms altogether in terms of solving a linear matrix inequality optimization problem. Based on the hybrid system framework, a novel decoupled model is constructed for the distributed closed-loop system, and the consensus is achieved asymptotically. Finally, a numerical example is illustrated to verify the effectiveness of the proposed methods.**

I. INTRODUCTION

Over the past decades, researchers in the field of system and control have put great efforts in designing the consensus protocol for networked multi-agent systems (MAS) due to the increasing applications including coordination of unmanned air vehicles, distributed sensor networks and attitude alignment for satellites, etc; see [1], [2] and their references. In many practical scenarios, constrained by the limited communication resources, the agents can not be assumed to have continuous access to others' states. The event-triggered idea for MAS has been proven to be an efficient way to reduce the usage of communication network as the event generator only triggers when it is needed, see [3]–[5].

The event-triggering mechanism decides when a control system should respond based on an event function, which characterizes the condition to trigger an event. One significant consideration in the design of the event triggering mechanism is the exclusion of infinite times of triggers in a finite time, called the Zeno-behavior. Some works have addressed this issue by implementing periodically sampled (or sampled-data) implementation to naturally avoid the Zeno-behavior, but it requires perfect synchronization among the agents which is neither practical nor scalable [6], [7]. Researchers have also put great efforts in continuous-time event-triggered consensus protocol [7]–[11]. To mention a few, in [7], a state-dependent event-triggering mechanism was proposed in which additional broadcast of each agent was triggered when it receives broadcasting information from neighbors, assuring that the event conditions always hold. [9] proposed a time-dependent event-triggering mechanism with a positive minimum inter-event time (MIET), but sacrificed asymptotic convergence in exchange for convergence to a neighborhood. In [10], a dynamic event-triggering mechanism which involves an auxiliary dynamic variable was designed to further reduce the communication burden among the agents. It's worth mentioning that all of them can guarantee the avoidance of Zeno-behavior but can not provide the strong Zeno-freeness property, i.e., a positive MIET, unless sacrificing the asymptotic convergence to the consensus. Moreover, the authors in [12] accurately point out that for some event-triggering mechanisms, even arbitrary small disturbances can disrupt the Zeno-freeness property. Therefore, it is not a trivial task to assure both the strong Zeno-freeness and the asymptotic convergence to the consensus. Recently, as shown in [13]–[15], the hybrid system approach seems to be an efficient tool to fulfill this task. However, this is still a major challenge for MAS and deserves more attention.

When handling the event-triggered control problem with network-induced communication, the method to design the control and the event-triggering mechanisms are classified into two categories: the emulation-based approach [15]–[18] and the co-design approach [19], [20]. Most works utilized the emulation-based approach, i.e., the control gain is pre-designed such that the closed-loop system can converge to the origin or consensus with ideal continuous communication. It is relatively simple as it avoids the complexity induced by the coupling of control gain and the event-triggering mechanisms. Specifically, for the design of the control gain in the field of the linear multi-agent consensus, the typical method is by solving an algebraic Riccati equation (ARE) of which some parameters have to be chosen deliberately, and the control gain takes a specific form related to the solution of the ARE [15], [17], [18]. The emulation-based approach in MAS has two main drawbacks compared with the co-design approach. One is the low generality and systematicness in the sense that each time the system varies, parameters in ARE have to be chosen again. The other is that the initial choice of the control gain limits the performance of the event-triggering scheme. It's worth mentioning that [21] starts the first and only attempt to co-design the control gain and the event-triggering mechanisms, with all the agents' event-triggering mechanisms the same. However, assigning agents individual communication schemes according to their own communication topology and communication resources seem to make more sense. Obviously, devising a method to co-design the control gain and the event-triggering mechanism for each agent respectively is of significance and full of challenges.

As noted by [22], wireless communication among the devices is extensively widespread in the field of Internet of Things (IoT) and other large-scale networks, and the communication resource seems extremely precious in this case. This leads to a resurgence of a kind of topic that the event-triggered idea is implemented not only to decide when the agents communicate with neighbors, but to determine when the agents update the control signals themselves. The above two events are usually described as 'event-triggered communication' and 'event-triggered control' respectively. Notably, such two kinds of the event-induced errors have a possible mutual effect on each other, leading to a more complex closed-loop system. There are some relative works on the event-triggered communication and control, for example, [18], [23], however, with the external disturbances ignored.

Although a considerable amount of works has considered the distributed event-triggered schemes for MAS, a complete solution is still inadequate for some technical difficulties. There are few results that possess all the following properties simultaneously:

(i) asymptotic consensus, i.e., the system achieves the consensus asymptotically;
(ii) communication resources are saved for both the communication among agents and the control update themselves;
(iii) strong Zeno-freeness, i.e., a positive MIET for all the event-triggering mechanisms even with the existence of disturbances;
(iv) each agent has its independent event-triggering mechanisms, and the trigger state is not interrupted by neighbors' abrupt broadcast.

Obviously, these properties are vital for the digital implementation in real life. In particular, property (iv) is also of high relevance to practical implementation which is easily ignored. Remember that we aim to make sure the event conditions hold for all the time. Since the event function of the event-triggered communication usually includes the latest broadcasting information of neighbors, one agent's event-triggered state may jump at the moment when its neighbors broadcast. Once it jumps from non-trigger to trigger, the agent's non-trigger neighbors have to recheck the trigger state, and it may even further induce another round of rechecking. The above process is assumed to be finished immediately in the relevant literatures, which is definitely undesirable. To this end, employing the event-triggering mechanism in which the trigger state will not jump is of interest.

In this paper, we consider the general linear MAS with external disturbances. Only partial state (output measurement) of each agent is available, and thus an observer is designed for each agent. Under the assumptions of stabilizability and detectability of each agent, we provide a systematic approach to achieve the consensus which is usually not a constant. To summarize, our work leads to the following contributions: (i) a novel decoupled hybrid model is constructed, and the closed-loop system can achieve both asymptotic consensus and strong Zeno-freeness properties; (ii) a distributed dynamic event-triggered scheme of each agent is proposed for both the communication and the control update; (iii) this paper presents one of the first attempts to co-design the control gain, the observer gain and the event-triggering mechanisms for MAS.

The remainder of this article is organized as follows. The preliminary and problem formulation are presented in section II. The main result is given in Section III. Simulation results are provided in Section IV and the conclusion is drawn in Section V.

Notations: The set of real numbers is denoted by $\mathbb{R}$ and the set of non-negative real integers is denoted by $\mathbb{R}_{\geq 0}$. For vectors $v_1,\ v_2,\ ...,\ v_n \in \mathbb{R}^n$, we denote the vector $[v_1^T, v_2^T, ..., v_n^T]^T$ by $(v_1, v_2, ...v_n)$. We write I_n to state the identity matrix of dimension n. For a vector x, we denote by $|x| := \sqrt{x^T x}$ its Euclidean norm and, for a matrix A, $|A| := \sqrt{\lambda_{\max}(A^T A)}$. A function α: $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$. It is said to be class $\mathcal{K}_\infty$ if in addition it is unbounded. A continuous function β: $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{KL}$ if, for each $t \in \mathbb{R}_{\geq 0}$, $\beta(\cdot, t)$ is of class $\mathcal{K}$, and for each $s \in \mathbb{R}_{\geq 0}$, $\beta(s, \cdot)$ is decreasing to 0. By $\langle \cdot, \cdot \rangle$, we denote the usual inner product of real vectors. We denote $C^\dagger$ as the Pseudo inverse of matrix C.

II. Preliminary and Problem Formulation

A. Graph Theory

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a directed graph of order n among N agents with a set of nodes $\mathcal{V} = \{1, 2, \cdots, N\}$, an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and an adjacency matrix $\mathcal{A} = [a_{ij}]_{N \times N}$. For any $i, j \in \mathcal{V}$, $i \neq j$, $a_{ij} = 1 \Leftrightarrow (i, j) \in \mathcal{E}$, otherwise, $a_{ij} = 0$, and the diagonal elements $a_{ii} = 0$. A directed path from node i_1 to node i_n is a sequence of ordered edges of the form (i_k, i_{k+1}), $k = 1, 2, \cdots, N-1$. A diagraph contains a directed spanning tree if there exists a node called the root such that there exist directed paths from this node to every other node. The in- and out- neighbor of node i is defined as $N_i^{in} = \{j \in \mathcal{V} : \mathcal{E}_{ij} \in \mathcal{E}\}$ and $N_i^{out} = \{j \in \mathcal{V} : \mathcal{E}_{ji} \in \mathcal{E}\}$. Let $\mathcal{D} = \mathrm{diag}\{d_1, d_2, \cdots, d_N\}$ represent the degree matrix with entries $d_i = \sum_{j=1}^{N} a_{ij}$. The Laplacian matrix $L = (l_{ij})_{N \times N} = \mathcal{D} - \mathcal{A}$.

Assumption 1: The diagraph $\mathcal{G}$ contains a directed spanning tree.

Define

$$\begin{aligned}\tilde{L} &= (\tilde{l}_{ij})_{(N-1)\times(N-1)} \\ &= \begin{pmatrix} l_{22} - l_{12} & \cdots & l_{2N} - l_{1N} \\ \dots & \ddots & \dots \\ l_{N2} - l_{12} & \cdots & l_{NN} - l_{1N} \end{pmatrix}.\end{aligned} \tag{1}$$

Lemma 1: (see [24]) Denote the eigenvalues of Laplacian mtrix L and the matrix $\tilde{L}$, respectively, by $\lambda_1, \lambda_2, \cdots, \lambda_N$ and $\mu_1, \mu_2, \cdots, \mu_{N-1}$, where $0 = |\lambda_1| \leq |\lambda_2| \leq \cdots \leq |\lambda_N|$ and $\mu_1| \leq |\mu_2| \leq \cdots \leq |\mu_{N-1}|$. Then we have $\mu_1 = \lambda_2$, $\mu_2 = \lambda_3$, $\cdots$, $\mu_{N-1} = \lambda_N$.

B. Hybrid System

We consider hybrid systems of the following form [25]:

$$\begin{cases} \dot{\xi} \in F(\xi, w), & \xi \in \mathcal{F} \\ \xi^+ \in J(\xi), & \xi \in \mathcal{J} \end{cases} \tag{2}$$

where $\xi \in \mathbb{R}^{n_\xi}$ is the state, $w \in \mathbb{R}^{n_w}$ is the external disturbance, $\mathcal{F}$ is the flow set, $\mathcal{J}$ is the jump set, F is the flow map, and J is the jump map. The performance output z is defined as:

$$z(t) = C_z \xi(t) + D_z w(t) \tag{3}$$

where C_z and D_z are constant matrices with appropriate dimensions. For the hybrid signal, we adopt the following definitions as in [20].

Definition 1: For a hybrid signal z defined on the hybrid time domain $\operatorname{dom} z = \bigcup_{j=0}^{J-1} [t_j, t_{j+1}] \times \{j\}$ with J possibly ∞ and/or $t_J = \infty$, the $\mathcal{L}_2$-norm of z is defined as $||z||_2 = \left(\sum_{j=0}^{J-1} \int_{t_j}^{t_{j+1}} |z(t,j)|^2 dt\right)^{\frac{1}{2}}$, provided that the right-hand side exists and is finite, in which case we write $z \in \mathcal{L}_2$.

With the definition of the hybrid signal, $\mathcal{L}_2$ stability for the hybrid system is given as follows.

Definition 2: The hybrid system (2) is $\mathcal{L}_2$ stable from input $w \in \mathcal{L}_2$ to the output z with gain less than or equal to γ if there exists $\beta \in \mathcal{K}_\infty$ such that any solution pair to (2) satisfies $||z||_2 \le \beta(|\xi(0,0)|) + \gamma ||w||_2$.

C. System Setup Description

We consider the multi-agent consensus problem for a network of N agents. Without loss of generality, we use the convention that an agent i can receive information from neighbors in N_i^{in} and send information to neighbors in N_i^{out}. The dynamics of each agent i, $i \in [1, N]$, is described by

$$\begin{cases} \dot{x}_i(t) = Ax_i(t) + Bu_i(t) + B_w w_i(t) \\ y_i(t) = Cx_i(t) \end{cases} \tag{4}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $B_w \in \mathbb{R}^{n\times l}$, $C \in \mathbb{R}^{p\times n}$ are constant matrices; $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$ and $y_i \in \mathbb{R}^p$ are agent i's state, control input, and measurement output respectively. $w_i \in \mathbb{R}^l$ is unknown external disturbances and satisfies $w_i(t) \in \mathcal{L}_2[0, \infty)$.

Assumption 2: The matrix pair (A, B) is stabilizable.

Assumption 3: The matrix pair (A, C) is detectable.

If we ignore the impact of limited resources caused by network communication, the consensus of the closed-loop system can be achieved with the following observer-based consensus protocol:

$$\dot{\hat{x}}_i(t) = A\hat{x}_i(t) + Bu_i(t) + F(y_i(t) - C\hat{x}_i(t)) \tag{5}$$

$$u_i(t) = K \sum_{j\in N_i^{in}} (\hat{x}_j(t) - \hat{x}_i(t)) \tag{6}$$

where $\hat{x}_i \in \mathbb{R}^n$ is the observer state, $F \in \mathbb{R}^{n\times p}$ is the observer matrix to be designed, and $K \in \mathbb{R}^{m\times n}$ is the feedback gain matrix to be determined. Revelant works which cope with linear MAS usually set the feedback gain matrix $K = \alpha R^{-1} B^T P$, where α is a positive constant to be designed and P is the unique solution to the following algebraic Riccati equation (ARE):

$$A^T P + PA - PBR^{-1}B^T P + Q = 0 \tag{7}$$

for appropriately chosen R and Q. It was proven in [18] that the consensus can be achieved if the parameter α satisfies some constraint. Obviously, the method to decide the feedback gain is kind of awkward and inflexible in the sense that the man-made adjustment of the chosen variables is required for a specific MAS. Furthermore, such aimless adjustment is likely to deteriorate the system performance and may even disrupt the consensus property when some uncertainty such as external disturbances arises.

Besides, the consensus protocol in (6) requires each controller to access the observer states of all the neighbors continuously and to update its control signal continuously, which is nearly impossible in the practical implementation. Thus, the event-triggering mechanism is utilized in this work to save the resources implementing on both the communication side among agents and the controller side of each agent.

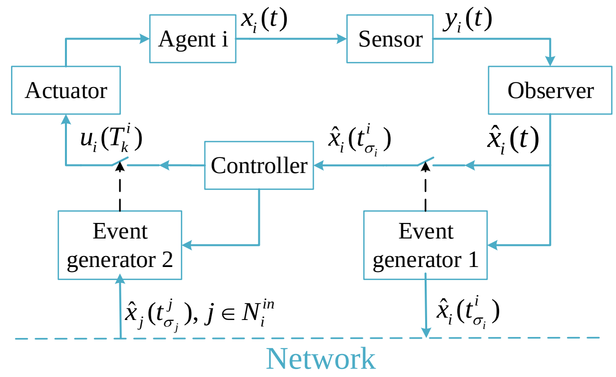

Fig. 1. The block diagram of the control system.

The structure of the whole control system is illustrated in Fig. 1. We assume that agent i has an estimator in its controller side, aiming to estimate its and its in-neighbors' states, denoted by $\zeta_i(t)$ and $\zeta_j(t)$ for $j \in \mathcal{N}_i^{in}$. Specifically, the dynamics of $\zeta_i(t)$ is described as

$$\begin{cases} \dot{\zeta}_i(t) = A\zeta_i(t), & t \in (t_{\sigma_i}^i, t_{\sigma_i+1}^i] \\ \zeta_i^+(t) = \hat{x}_i(t), & t = t_{\sigma_i}^i \end{cases} \tag{8}$$

where $t_{\sigma_i}^i$, $\sigma_i = 0, 1, 2, \cdots$, $i \in [1, N]$ is the communication time instants of agent i, and the notation $\zeta_i^+(t)$ denotes the state of $\zeta_i(t)$ right after jump, i.e., $\zeta_i^+(t) = \lim_{\epsilon\to 0^+} \zeta_i(t+\epsilon)$. With such an estimator, agent i is capable of estimating its in-neighbors' states in real time. For agent i's out-neighbors, their estimators also estimate agent i's state in real time as $\zeta_i(t)$, and $\zeta_i(t)$ is updated to be $\hat{x}_i(t)$ when agent i broadcasts at $t_{\sigma_i}^i$. In other words, $\zeta_i(t)$ is available for agent i and its out-neighbors. We then use an event-triggered implementation of controller (6) given by

$$u_i(t) = K \sum_{j\in N_i^{in}} \left(\zeta_j(T_k^i) - \zeta_i(T_k^i)\right), \ t \in (T_k^i, T_{k+1}^i] \tag{9}$$

where T_k^i, $k = 0, 1, 2, \cdots$, $i \in [1, N]$ is the control update time instants of agent i.

D. Dynamic Event-Triggering Mechanism

Constrained by limited communication resources, the event-triggering mechanism has been widely used as an efficient way to avoid unnecessary transmission since it answers the question when exactly should the control system respond or communicate.

As shown in Fig. 1, the event generators are implemented both on the communication side and the controller side for each agent. To this end, we respectively introduce the auxiliary dynamic variables η_{c_i} and η_{u_i}, $i = [1, N]$. The event-triggered communication strategy for agent i, which determines the communication time instants $t^i_{\sigma_i+1}$, is designed as

$$t^i_{\sigma_i+1} = \inf\{t > t^i_{\sigma_i} + \tau^{c_i}_{\text{miet}} | \eta_{c_i} = 0\} \tag{10}$$

where $\tau^{c_i}_{\text{miet}} \in \mathbb{R}_{\geq 0}$ is the minimum inter-event time to be designed at the communication side of agent i, and the dynamic variable η_{c_i} flows according to

$$\eta_{c_i} \in \psi_{c_i}(o_{c_i}) \tag{11}$$

where $\psi_{c_i}(\cdot)$ is the dynamic function to be designed and o_{c_i} represents locally available information at the event-triggered communication strategy of agent i. Similarly, the event-triggered control update strategy is designed as

$$T^i_{k+1} = \inf\{t > T^i_{k_i} + \tau^{u_i}_{\text{miet}} | \eta_{u_i} = 0\}, \tag{12}$$

where $\tau^{u_i}_{\text{miet}} \in \mathbb{R}_{\geq 0}$ is the minimum inter-event time to be designed at the controller side of agent i, and the dynamic variable η_{u_i} flows according to

$$\eta_{u_i} \in \psi_{u_i}(o_{u_i}) \tag{13}$$

where $\psi_{u_i}(\cdot)$ is the dynamic function to be designed and o_{u_i} represents locally available information at the i-th controller side.

It is worth pointing out that many literatures on event-triggered control for MAS start from the stability analysis point of view, then accordingly design the event-triggering mechanism to derive the consensus property. One subsequent result is that, although Zeno-behavior is excluded with a strictly proved positive scalar, the scalar usually infinitely approaches 0 as the agents converge to the consensus. This paper starts from the design of the event-triggering mechanism with strong Zeno-freeness, and then looks for mathematical tools to guarantee the consensus.

E. Problem Statement

Under the multi-agent model described above, our objective is to design a distributed event-triggered control protocol that allows each agent to asymptotically converge to the consensus and has a positive minimum inter-event time at the same time. Specifically, we aim to co-design the system matrices, including the control gain K in (9) and the observer matrix F in (5), and the event-triggering mechanisms on both the communication side in (10) and the controller side in (12), including the functions ψ_{c_i}, ψ_{u_i} and the parameters $\tau^{c_i}_{miet}$, $\tau^{u_i}_{miet}$.

III. Main Result

In this section, a distributed event-triggered communication and control strategy is proposed for linear MAS. Specifically, we construct a systematic methodology to co-design the observer, the controller and the event-triggering mechanisms. The parameters can be obtained by solving an LMI optimization problem, and the asymptotic consensus can be achieved with strong Zeno-freeness based on the stability theory for hybrid systems.

A. Hybrid Model Construction

Our whole analysis builds upon the hybrid model, which is given in this subsection. Since the Laplacian matrix L surely has an eigenvalue 0, there exists coupling in the whole distributed system. Adding that only partial state is available and mutual effects of the double event-triggering mechanisms on each other exist, constructing an appropriate hybrid model for the stability analysis of MAS is not a trivial task. Skillful transformations should be taken to decouple the whole system.

Before proceeding, we define the observation error $\tilde{x}_i(t)$, the communication measurement error $e_i(t)$, the controller measurement error $r_i(t)$, for agent i, respectively as:

$$\tilde{x}_i(t) = x_i(t) - \hat{x}_i(t), \tag{14}$$

$$e_i(t) = \zeta_i(t) - \hat{x}_i(t), \tag{15}$$

$$r_i(t) = \varphi_i(T^i_k) - \varphi_i(t), \tag{16}$$

where

$$\varphi_i(t) = \sum_{j \in N_i^{in}} \big(\zeta_j(t) - \zeta_i(t)\big). \tag{17}$$

According to the control structure of MAS as in Fig. 1, when agent i broadcasts its own state to its out-neighbors at $t^i_{\sigma_i}$, $\sigma_i = 0, 1, 2, \cdots$, the communication measurement error $e_i(t)$ is reset to 0. Equipped with the broadcasting information, the out-neighbors update their estimates of agent i's state and make adjustments of the event generators. When agent i sends the control signal to its actuator at T^i_k, $k = 0, 1, 2, \cdots$, the control measurement error $r_i(t)$ is reset to 0. The value of u_i is kept constant between two consecutive control update events in a zero-order-hold (ZOH) fashion. For simplicity of writing, we also define

$$\begin{aligned}
\delta_{x_i}(t) &= \sum_{j \in N_i^{in}} (x_j(t) - x_i(t)), \delta_x = (\delta_{x_1}, \delta_{x_2}, \cdots, \delta_{x_N}), \\
\delta_{\tilde{x}_i}(t) &= \sum_{j \in N_i^{in}} (\tilde{x}_j(t) - \tilde{x}_i(t)), \delta_{\tilde{x}} = (\delta_{\tilde{x}_1}, \delta_{\tilde{x}_2}, \cdots, \delta_{\tilde{x}_N}), \\
\delta_{e_i}(t) &= \sum_{j \in N_i^{in}} (e_j(t) - e_i(t)),\ \delta_e = (\delta_{e_1}, \delta_{e_2}, \cdots, \delta_{e_N}).
\end{aligned} \tag{18}$$

From the above definitions, one has $\delta_{x_i}(t) = -(L^i \otimes I_n)x$, where L^i denotes the i-th row of matrix L. Similarly, we have $\delta_{\tilde{x}_i}(t) = -(L^i \otimes I_n)\tilde{x}$, and $\delta_{e_i}(t) = -(L^i \otimes I_n)e$.

From the definition of $\varphi_i(t)$ in (17), it follows that

$$\begin{aligned}\varphi_i(t) =& \sum_{j\in N_i^{in}} \big(\zeta_j(t) - \zeta_i(t)\big)\\ =& \sum_{j\in N_i^{in}} \big(x_j(t) - x_i(t) - \hat{x}_j(t) + \hat{x}_i(t) + e_j(t) - e_i(t)\big)\\ =& \delta_{x_i}(t) - \delta_{\tilde{x}_i}(t) + \delta_{e_i}(t).\end{aligned} \tag{19}$$

In view of the control strategy in (9) and definitions of errors in (14–16), we obtain $u_i(t) = K\varphi_i(t) + Kr_i(t)$, and it gives that

$$\begin{aligned}\dot{x}_i(t) =& Ax_i(t) + BK\varphi_i(t) + BKr_i(t) + B_w w_i(t)\\ =& Ax_i(t) + BK\delta_{x_i}(t) - BK\delta_{\tilde{x}_i}(t) + BK\delta_{e_i}(t)\\ &+ BKr_i(t) + B_w w_i(t),\\ \dot{\tilde{x}}_i(t) =& (A - FC)\tilde{x}_i(t) + B_w w_i(t),\\ \dot{e}_i(t) =& A\zeta_i(t) - A\hat{x}_i(t) - Bu_i(t) - FC\tilde{x}_i(t)\\ =& A\zeta_i(t) - BK\delta_{x_i}(t) + BK\delta_{\tilde{x}_i}(t) - BK\delta_{e_i}(t)\\ &- BKr_i(t) - FC\tilde{x}_i(t),\\ \dot{r}_i(t) =& -A\delta_{x_i}(t) + A\delta_{\tilde{x}_i}(t) - A\delta_{e_i}(t).\end{aligned} \tag{20}$$

Let x, $\tilde{x}$, e, r be the concatenated vectors of x_i, $\tilde{x}_i$, e_i, r_i, respectively. Then, the dynamics of the distributed system can be written as

$$\begin{aligned}\dot{x}(t) =& (I_N \otimes A)x(t) + (I_N \otimes BK)\delta_x(t)\\ &- (I_N \otimes BK)\delta_{\tilde{x}}(t) + (I_N \otimes BK)\delta_e(t)\\ &+ (I_N \otimes BK)r(t) + (I_N \otimes B_w)w\\ =& (I_N \otimes A - L \otimes BK)x(t) + (L \otimes BK)\tilde{x}(t)\\ &- (L \otimes BK)e(t) - (L \otimes BK)r(t) + (I_N \otimes B_w)w(t),\\ \dot{\tilde{x}}(t) =& (I_N \otimes (A - FC))\tilde{x}(t) + (I_N \otimes B_w)w(t),\\ \dot{e}(t) =& (I_N \otimes A)e(t) - (I_N \otimes BK)\delta_e(t)\\ &- (I_N \otimes BK)\delta_x(t) + (I_N \otimes BK)\delta_{\tilde{x}}(t)\\ &- (I_N \otimes BK)r(t) - (I_N \otimes FC)\tilde{x}(t)\\ =& (L \otimes BK)x(t) - (L \otimes BK + I_N \otimes FC)\tilde{x}(t)\\ &+ (I_N \otimes A + L \otimes BK)e(t) - (I_N \otimes BK)r(t),\\ \dot{e}(t) =& -(I_N \otimes A)\delta_x(t) + (I_N \otimes A)\delta_{\tilde{x}}(t) + (I_N \otimes A)\delta_{\tilde{x}}(t)\\ =& (L \otimes A)x(t) - (L \otimes A)\tilde{x}(t) + (L \otimes A)e(t).\end{aligned} \tag{21}$$

Let $\varrho_i(t) = x_i(t) - x_1(t)$, $i = [1, N]$, and the disagreement vector $\varrho(t) = (\varrho_1(t), \varrho_{2-N}(t))$, where $\varrho_{2-N}(t) = (\varrho_2(t), \cdots, \varrho_N(t))$. Apparently we have $\varrho_1(t) \equiv 0$, and the consensus is achieved if and only if $\varrho_{2-N}(t) \to 0$. Derived from the dynamics of $x(t)$ in (21), the vector $\varrho_{2-N}(t)$ satisfies

$$\begin{aligned}\dot{\varrho}_{2-N} =& (I_{N-1} \otimes A - \tilde{L} \otimes BK)\varrho_{2-N}(t) + (W \otimes B_w)w(t)\\ &+ (\tilde{L}W \otimes BK)(\tilde{x}(t) - e(t) - r(t))\end{aligned} \tag{22}$$

where $\tilde{L}$ is defined in (1), and $W = [-\mathbf{1}_{N-1}, I_{N-1}] \in \mathbb{R}^{(N-1)\times N}$. According to Lemma 1, the matrix $\tilde{L}$ is full-rank and all of its eigenvalues have positive real parts.

A key factor for the subsequent analysis is to ensure that the matrix $(I_{N-1} \otimes A - \tilde{L} \otimes BK)$ is Hurwitz. In fact, the property of Hurwitz of the matrix $(I_{N-1} \otimes A - \tilde{L} \otimes BK)$ is shown in [24] if the controller gain takes a specific form $K = B^T P$. In this paper, we look for a more general method to design the controller gain, which will be specified in Section III.

Now we focus on the remaining work to decouple the whole distributed system. According to the definition of $\delta_{x_i}(t)$, one has

$$\begin{aligned}\delta_{x_i}(t) =& \sum_{j\in N_i^{in}} (x_j(t) - x_i(t))\\ =& \sum_{j\in N_i^{in}} (x_j(t) - x_1(t) - x_i(t) + x_1(t))\\ =& \sum_{j\in N_i^{in}} (\varrho_j(t) - \varrho_i(t))\\ =& -(L^i \otimes I_n)\varrho(t)\end{aligned}$$

Then we have $(L \otimes BK)x(t) = -(I_N \otimes BK)\delta_x(t) = (I_N \otimes BK)(L \otimes I_n)\varrho(t) = (L \otimes BK)\varrho(t) = (L_0 \otimes BK)\varrho_{2-N}$, where $L_0 \in \mathbb{R}^{N\times(N-1)}$ is defined as

$$L_0 = \begin{pmatrix} l_{12} & \cdots & l_{1N}\\ l_{22} & \cdots & l_{2N}\\ \cdots & \ddots & \cdots\\ l_{N2} & \cdots & l_{NN}\end{pmatrix} \tag{23}$$

Define the augmented variable $\delta(t) = (\varrho_{2-N}(t), \tilde{x}(t))$, then the dynamics for MAS can be described as:

$$\begin{aligned}\dot{\delta}(t) &= \mathcal{A}_1\delta(t) + \mathcal{B}_1 e(t) + \mathcal{M}_1 r(t) + \mathcal{E}_1 w(t)\\ \dot{e}(t) &= \mathcal{A}_2\delta(t) + \mathcal{B}_2 e(t) + \mathcal{M}_2 r(t)\\ \dot{r}(t) &= \mathcal{A}_3\delta(t) + \mathcal{B}_3 e(t)\end{aligned} \tag{24}$$

where

$$\begin{aligned}\mathcal{A}_1 &= \begin{bmatrix} I_{N-1} \otimes A - \tilde{L} \otimes BK & \tilde{L}W \otimes BK\\ 0 & I_N \otimes (A - FC)\end{bmatrix},\\ \mathcal{B}_1 &= \begin{bmatrix} -\tilde{L}W \otimes BK\\ 0\end{bmatrix}, \quad \mathcal{M}_1 = \begin{bmatrix} -\tilde{L}W \otimes BK\\ 0\end{bmatrix},\\ \mathcal{E}_1 &= \begin{bmatrix} W \otimes BK\\ I_N \otimes BK\end{bmatrix},\\ \mathcal{A}_2 &= \begin{bmatrix} L_0 \otimes BK & -L \otimes BK - I_N \otimes FC\end{bmatrix},\\ \mathcal{B}_2 &= \begin{bmatrix} I_N \otimes A + L \otimes BK\end{bmatrix}, \ \mathcal{M}_2 = \begin{bmatrix} -I_N \otimes BK\end{bmatrix},\\ \mathcal{A}_3 &= \begin{bmatrix} L_0 \otimes A & -L \otimes A\end{bmatrix}, \quad \mathcal{B}_3 = \begin{bmatrix} L \otimes A\end{bmatrix}.\end{aligned}$$

Before presenting the hybrid system, we introduce the auxiliary variables τ_{c_i}, τ_{u_i}, $i = [1, N]$. The variable τ_{c_i} denotes the time elapsed since the last broadcast of agent i, and τ_{u_i} represents the time elapsed since the last controller update of agent i. Thus, they have the following dynamics:

$$\begin{aligned}\dot{\tau}_{c_i}(t) &= 1, \quad t \in (t^i_{\sigma_i}, t^i_{\sigma_i+1}]\\ \tau_{c_i}(t^+) &= 0, \quad t = t^i_{\sigma_i}\end{aligned} \tag{25}$$

$$\begin{aligned}\dot{\tau}_{u_i}(t) &= 1, \quad t \in (T^i_k, T^i_{k+1}]\\ \tau_{u_i}(t^+) &= 0, \quad t = T^i_k\end{aligned} \tag{26}$$

Let τ_c and τ_u be the concatenated vectors of τ_{c_i} and τ_{u_i} respectively. Now the hybrid system state for MAS can be defined as $\xi = (\delta, e, r, \tau_c, \tau_u, \eta_c, \eta_u) \in \mathbb{X}$ with $\mathbb{X} = \mathbb{R}^{n(2N-1)} \times \mathbb{R}^{nN} \times \mathbb{R}^{nN} \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$, where η_c and η_u are the concatenated vectors of η_{c_i} and η_{u_i} respectively. In view of the event-triggering mechanisms in (10) and (12), the flow set $\mathcal{F}$ and the jump set $\mathcal{J}$ are given by

$$\begin{aligned}\mathcal{F} &= \mathcal{F}_c \bigcap \mathcal{F}_u,\\ \mathcal{J} &= \mathcal{J}_c \bigcup \mathcal{J}_u,\end{aligned} \tag{27}$$

where

$$\begin{aligned}&\mathcal{F}_c = \bigcap_{i=1}^{N} \mathcal{F}_{c_i},\ \mathcal{F}_u = \bigcap_{i=1}^{N} \mathcal{F}_{u_i},\\ &\mathcal{F}_{c_i} = \{\xi \in \mathbb{X} | \tau_{c_i} \le \tau_{miet}^{c_i} \vee \eta_{c_i} \ge 0\},\\ &\mathcal{F}_{u_i} = \{\xi \in \mathbb{X} | \tau_{u_i} \le \tau_{miet}^{u_i} \vee \eta_{u_i} \ge 0\},\end{aligned} \tag{28}$$

and

$$\begin{aligned}&\mathcal{J}_c = \bigcup_{i=1}^{N} \mathcal{J}_{c_i},\ \mathcal{J}_u = \bigcup_{i=1}^{N} \mathcal{J}_{u_i},\\ &\mathcal{J}_{c_i} = \{\xi \in \mathbb{X} | \tau_{c_i} > \tau_{miet}^{c_i} \wedge \eta_{c_i} \le 0\},\\ &\mathcal{J}_{u_i} = \{\xi \in \mathbb{X} | \tau_{u_i} > \tau_{miet}^{u_i} \wedge \eta_{u_i} \le 0\}.\end{aligned} \tag{29}$$

For $\xi \in \mathcal{F}$, the flow dynamics can be represented as

$$\dot{\xi} \in F(\xi, w) := \begin{pmatrix} \mathcal{A}_1\delta(t) + \mathcal{B}_1 e(t) + \mathcal{M}_1 r(t) + \mathcal{E}_1 w(t) \\ \mathcal{A}_2\delta(t) + \mathcal{B}_2 e(t) + \mathcal{M}_2 r(t) \\ \mathcal{A}_3\delta(t) + \mathcal{B}_3 e(t) \\ \mathbf{1}_N \\ \mathbf{1}_N \\ \psi_c \\ \psi_u \end{pmatrix} \tag{30}$$

where ψ_c and ψ_u are concatenated vectors of ψ_{c_i} and ψ_{u_i} respectively. For $\xi \in \mathcal{J}$, the jump dynamics is described by

$$J = J_c \bigcup J_u, \tag{31}$$

where

$$J_c = \bigcup_{i=1}^{N} J_{c_i}, \quad J_u = \bigcup_{i=1}^{N} J_{u_i}$$

with

$$\begin{aligned} J_{c_i} &= \begin{pmatrix} \delta \\ (I_{nN} - \Gamma_i \otimes I_n)e \\ r \\ (I_N - \Gamma_i)\tau_c \\ \tau_u \\ \eta_c \\ \eta_u \end{pmatrix}, \\ J_{u_i} &= \begin{pmatrix} \delta \\ e \\ (I_{nN} - \Gamma_i \otimes I_n)r \\ \tau_c \\ (I_N - \Gamma_i)\tau_u \\ \eta_c \\ \eta_u \end{pmatrix}, \end{aligned} \tag{32}$$

and $\Gamma_i \in \mathbb{R}^{N\times N}$ represents a diagonal matrix with the ii-th entry equaling to one and all the other entries are equal to zero.

With the hybrid model constructed, we then give a relevant definition of consensus for MAS.

Definition 3: For the MAS described by the hybrid model (30) and (31) with $w(t) = 0$, the set given by $\{\xi \in \mathbb{X} | \delta = 0_{n(2N-1)}, e = r = 0_{nN}\}$ is said to be uniformly globally asymptotically stable (UGAS) if there exist a function $\beta \in \mathcal{KL}$ such that, for any given initial condition $\xi(0) \in \mathbb{X}$, the following condition holds

$$\|\delta(t), e(t), r(t)\| \le \beta(\|(\delta(0), e(0), r(0))\|, t). \tag{33}$$

We further explore the dynamics of the communication measurement error e and control measurement r error, which is characterized in the following proposition.

Proposition 1: Consider the flow dynamics for MAS in (30). For almost all $e_i(t) \in \mathbb{R}^n$, $r_i(t) \in \mathbb{R}^n$, $i = [1, N]$, and all $(\delta, w) \in \mathbb{R}^{n(2N-1)+Nl}$, it holds that

$$|\dot{e}_i(t)| \le L_{c_i}|e_i(t)| + M_{c_i}|r_i(t)| + H_{c_i}(\delta(t), \bar{e}_i(t), \bar{r}_i(t)) \tag{34}$$

$$|\dot{r}_i(t)| \le M_{u_i}|e_i(t)| + H_{u_i}(\delta(t), \bar{e}_i(t)) \tag{35}$$

where

$$\begin{aligned}&L_{c_i} = |(R_i \otimes I_n)\mathcal{B}_2|, \quad M_{c_i} = |(R_i \otimes I_n)\mathcal{M}_2|,\\ &H_{c_i}(\delta(t), \bar{e}_i(t), \bar{r}_i(t)) = |A_{\delta_{ci}}\delta(t) + A_{e_{ci}}e(t) + A_{r_{ci}}r(t)|,\\ &M_{u_i} = |(R_i \otimes I_n)\mathcal{B}_3|,\\ &H_{u_i}(\delta(t), \bar{e}_i(t)) = |A_{\delta_{ui}}\delta(t) + A_{e_{ui}}e(t)|,\end{aligned} \tag{36}$$

where $R_i = [0 \cdots 1 \cdots 0] \in \mathbb{R}^{1\times N}$ with the i-th element being 1, and

$$\begin{aligned}&A_{\delta_{ci}} = (R_i \otimes I_n)\mathcal{A}_2, \quad A_{e_{ci}} = (R_i \otimes I_n)\mathcal{B}_2(I_{nN} - \Gamma_i \otimes I_n),\\ &A_{r_{ci}} = (R_i \otimes I_n)\mathcal{M}_2(I_{nN} - \Gamma_i \otimes I_n),\ A_{\delta_{ui}} = (R_i \otimes I_n)\mathcal{A}_3,\\ &A_{e_{ui}} = (R_i \otimes I_n)\mathcal{B}_3(I_{nN} - \Gamma_i \otimes I_n).\end{aligned}$$

Sketch of the Proof: Combining the dynamics in (21) with the facts $e_i(t) = (R_i \otimes I_n)e(t)$ and $r_i(t) = (R_i \otimes I_n)r(t)$, the result can be easily verified. Detailed proof is omitted due to space limitation.

Besides, we define the following compact variables

$$\begin{aligned}A_{\delta_c} &= (A_{\delta_{c_1}}, A_{\delta_{c_2}}, \cdots, A_{\delta_{c_N}}),\\ A_{e_c} &= (A_{e_{c_1}}, A_{e_{c_2}}, \cdots, A_{e_{c_N}}),\\ A_{r_c} &= (A_{r_{c_1}}, A_{r_{c_2}}, \cdots, A_{r_{c_N}}),\\ A_{\delta_u} &= (A_{\delta_{u_1}}, A_{\delta_{u_2}}, \cdots, A_{\delta_{u_N}}),\\ A_{e_u} &= (A_{e_{u_1}}, A_{e_{u_2}}, \cdots, A_{e_{uN}}).\end{aligned}$$

In view of the dynamics of the hybrid system, it can be verified that

$$\begin{aligned}&A_{\delta_c} = \mathcal{A}_2,\ A_{e_c} = [L \otimes BK - D_0 \otimes BK],\ A_{r_c} = 0,\\ &A_{\delta_u} = \mathcal{A}_3,\ A_{e_u} = [L \otimes A - D_0 \otimes A].\end{aligned} \tag{37}$$

Remark 1: Proposition 1 illustrates the growth of the communication measurement error $e_i(t)$ and the control measurement error $r_i(t)$. As we can see from (34) and (35), these two errors have an impact on each other, which will definitely increase the complexity of the distributed system. Such a mutual effect of these two errors on each other requires a more delicate design of the event-triggering mechanism, as we will specify in the following section.

B. Dynamic Event-Triggering Mechanism

We first define two useful functions $\phi_{c_i} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $\phi_{u_i} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $i = [1, N]$, with the initial values $\phi_{c_i}(0) = \lambda_{c_i}^{-1} \in (1, \infty)$, $\phi_{u_i}(0) = \lambda_{u_i}^{-1} \in (1, \infty)$, which respectively evolve according to

$$\dot{\phi}_{c_i}(\tau_{c_i}) = \varpi_{c_i}(\tau_{c_i})\Big(-2L_{c_i}\phi_{c_i}(\tau_{c_i}) - (1+\kappa_{c_i})\gamma_{c_i}\big(\phi_{c_i}^2(\tau_{c_i})+1\big)\Big) \tag{38}$$

$$\varpi_{c_i}(\tau_{c_i}) = \begin{cases} 1, & \tau_{c_i} \leq \tau_{miet}^{c_i} \\ 0, & \tau_{c_i} > \tau_{miet}^{c_i} \end{cases} \tag{39}$$

and

$$\dot{\phi}_{u_i}(\tau_{u_i}) = \varpi_{c_i}(\tau_{c_i})\Big(- (1 + \kappa_{u_i})\gamma_{u_i}(\phi_{u_i}^2(\tau_{u_i}) + 1)\Big) \tag{40}$$

$$\varpi_{u_i}(\tau_{u_i}) = \begin{cases} 1, & \tau_{u_i} \leq \tau_{miet}^{u_i} \\ 0, & \tau_{u_i} > \tau_{miet}^{u_i} \end{cases} \tag{41}$$

where L_{c_i} as in (36), and γ_{c_i}, γ_{u_i} are parameters in the event-triggering mechanisms to be determined. After fixing the parameters γ_{c_i}, γ_{u_i}, the parameters κ_{c_i}, κ_{u_i} are selected according to

$$\kappa_{c_i}\kappa_{u_i} \geq \max\left\{\frac{M_{u_i}}{\gamma_{c_i}}, \frac{M_{c_i}}{\gamma_{u_i}}\right\}. \tag{42}$$

with M_{c_i} and M_{u_i} as in (36).

Then the parameter $\tau_{miet}^{c_i}$ is chosen such that $\tau_{miet}^{c_i} \leq \mathcal{T}_c^i$ where

$$\mathcal{T}_c^i = \begin{cases} \frac{1}{L_{c_i}\upsilon_{c_i}}\arctan(\vartheta_{c_i}), & \gamma_{c_i}(1+\kappa_{c_i}) > L_{c_i} \\ \frac{1}{L_{c_i}}\frac{1-\lambda_{c_i}}{1+\lambda_{c_i}}, & \gamma_{c_i}(1+\kappa_{c_i}) = L_{c_i} \\ \frac{1}{L_{c_i}\upsilon_{c_i}}\operatorname{arctanh}(\vartheta_{c_i}), & \gamma_{c_i}(1+\kappa_{c_i}) < L_{c_i} \end{cases} \tag{43}$$

where

$$\vartheta_{c_i} = \frac{\upsilon_{c_i}(1-\lambda_{c_i})}{2\frac{\lambda_{c_i}}{1+\lambda_{c_i}}\left(\frac{\gamma_{c_i}(1+\kappa_{c_i})}{L_{c_i}} - 1\right) + 1 + \lambda_{c_i}},$$

$$\upsilon_{c_i} = \sqrt{\left|\left(\frac{\gamma_{c_i}(1+\kappa_{c_i})}{L_{c_i}}\right)^2 - 1\right|}.$$

In fact, $\mathcal{T}_c^i$ is equivalent to the time it takes for the function ϕ_{c_i} in (38) to decrease from $\phi_{c_i}(0) = \lambda_{c_i}^{-1}$ to $\phi_{c_i}(\mathcal{T}_c^i) = \lambda_{c_i}$.

Now we are ready to present the details about the event-triggering mechanisms. The dynamics of the functions η_{c_i} in (11) is given by

$$\begin{aligned} \psi_{c_i} =& \sigma_{c_i}|\ \varphi_i(t)|^2 - (1 - \varpi_{c_i}(\tau_{c_i}))\,\tilde{\gamma}_{c_i}|e_i(t)|^2 \\ &+ \left(\kappa_{c_i}\gamma_{c_i}^2 - \frac{M_{u_i}^2}{\kappa_{u_i}}\right)|e_i(t)|^2 \end{aligned} \tag{44}$$

where σ_{c_i} is the parameter to be determined, and $\tilde{\gamma}_{c_i} = 2L_{c_i}\gamma_{c_i}\phi_{c_i}(\tau_{miet}^{c_i}) + (1+\kappa_{c_i})\gamma_{c_i}^2(\phi_{c_i}^2(\tau_{miet}^{c_i}) + 1)$.

Similarly, $\tau_{miet}^{u_i}$ is chosen such that $\tau_{miet}^{u_i} \leq \mathcal{T}_u^i$ where

$$\mathcal{T}_u^i = \frac{1}{\gamma_{u_i}(1+\kappa_{u_i})}\arctan\frac{\lambda_{u_i}^{-1} - \lambda_{u_i}}{2}. \tag{45}$$

Then the dynamics of η_{u_i} is given by

$$\begin{aligned} \psi_{u_i} =& \sigma_{u_i}|\ \varphi_i(t)|^2 - (1 - \varpi_{u_i}(\tau_{u_i}))\,\tilde{\gamma}_{u_i}|r_i(t)|^2 \\ &+ \left(\kappa_{u_i}\gamma_{u_i}^2 - \frac{M_{c_i}^2}{\kappa_{c_i}}\right)|r_i(t)|^2 \end{aligned} \tag{46}$$

where σ_{u_i} is the parameter to be determined, and $\tilde{\gamma}_{u_i} = (1 + \kappa_{u_i})\gamma_{u_i}^2(\phi_{u_i}^2(\tau_{miet}^{u_i}) + 1)$.

Remark 2: Note that the parameters γ_{c_i} and γ_{u_i} are closely related to $\mathcal{T}_{c_i}$ and $\mathcal{T}_{u_i}$. Decreasing γ_{c_i} and γ_{u_i} will increase the lower bounds of the minimum inter-event time $\mathcal{T}_{c_i}$ and $\mathcal{T}_{u_i}$. Thus we aim to minimize γ_{c_i} and γ_{u_i} in the design procedure as we will see in the optimization problem.

Besides, the resulting event-triggering mechanisms have several features, which will be described in the following remarks. The event-triggering mechanism of the control update is similar to that of the communication and thus we take the latter for example.

Remark 3: In view of the dynamics of ψ_{c_i} in (38), the term $\left(\kappa_{c_i}\gamma_{c_i}^2 - \frac{M_{u_i}^2}{\kappa_{u_i}}\right)|e_i|^2$ is introduced to eliminate the mutual effect of two errors, which will be shown explicitly in the proof of the main result. The condition in (42) guarantees the positiveness of this term, and therefore, when $\tau_{c_i} \leq \tau_{miet}^{c_i}$, we have $\psi_{c_i} \geq 0$, which means η_{c_i} will not decrease to 0 before $\tau_{miet}^{c_i}$. The strong Zeno-freeness of the system is thus guaranteed regardless of external disturbances. Besides, since $\tilde{\gamma}_{c_i} > \left(\kappa_{c_i}\gamma_{c_i}^2 - \frac{M_{u_i}^2}{\kappa_{u_i}}\right)$, there is no need to worry η_{c_i} will always increase.

Remark 4: The event-triggering conditions of the communication strategy among agents in most literatures explicitly contain the latest broadcast information of neighbors. In this case, when one of its neighbors broadcasts information, its event-triggering condition will jump suddenly. It may result in a phenomenon that, some agent which does not intend to broadcast at some instant, has to trigger the event instead, due to the neighbor's newly arrived information. What's worse is that its trigger may further induce other agents' additional trigger. Such kind of 'propagation effect' is obviously unexpected. However, in our event-triggering mechanisms, the latest broadcasting information of neighbors do not explicitly appear in the event-triggering condition (10) but in the derivative of η_{c_i}. Even if the neighbors broadcast their states, the event-triggering condition of each agent will not jump, and thus this issue does not bother our method.

C. Convergence Analysis

Before proceeding the convergence result, the following lemma is proposed which will be used later.

Lemma 2: Consider the dynamics of MAS in (21), and the definition of $\varphi_i(t)$ in (17), it satisfies that

$$\sum_{i=1}^{N} |\varphi_i(t)|^2 \leq \ell_0(|e|^2 + |\delta|^2) \tag{47}$$

where $\hat{\ell} = \max_i\{d_i\}$.

Proof: According to the definitions (14)–(16), one has

$$\begin{aligned}
&\|\varphi_i(t)\| \\
&= \left\| \sum_{j\in N_i^{in}} (\zeta_j(t)-\zeta_i(t)) \right\| \\
&= \left\| \sum_{j\in N_i^{in}} (e_j(t)-e_i(t)+\hat{x}_j(t)-\hat{x}_i(t)) \right\| \\
&= \left\| \sum_{j\in N_i^{in}} (e_j(t)-e_i(t)+x_j(t)-x_i(t)-\tilde{x}_j(t)+\tilde{x}_i(t)) \right\| \\
&= \left\| \sum_{j\in N_i^{in}} (e_j(t)-e_i(t)+\xi_j(t)-\xi_i(t)-\tilde{x}_j(t)+\tilde{x}_i(t)) \right\| \\
&= \sum_{j\in N_i^{in}} \Big(\|e_j(t)\|+\|e_i(t)\|+\|\xi_j(t)\|+\|\xi_i(t)\| \\
&\quad + \|\tilde{x}_j(t)\|+\|\tilde{x}_i(t)\| \Big)
\end{aligned} \tag{48}$$

According to the mean value inequality, it follows that

$$\begin{aligned}
&\sum_{i=1}^{N} |\varphi_i(t)|^2 \\
&\le \sum_{i=1}^{N} 6\hat{\ell} \sum_{j\in N_i^{in}} \Big(\|e_j(t)\|^2+\|e_i(t)\|^2+\|\xi_j(t)\|^2+\|\xi_i(t)\|^2 \\
&\quad + \|\tilde{x}_j(t)\|^2+\|\tilde{x}_i(t)\|^2 \Big) \\
&\le 6\hat{\ell} \sum_{i=1}^{N} \Big(\|e(t)\|^2+\hat{\ell}\,\|e_i(t)\|^2+\|\xi(t)\|^2+\hat{\ell}\,\|\xi_i(t)\|^2 \\
&\quad + \|\tilde{x}(t)\|^2+\hat{\ell}\,\|\tilde{x}_i(t)\|^2 \Big) \\
&\le 6\hat{\ell}(N+\hat{\ell})\Big(|e|^2+|\xi|^2+|\tilde{x}|^2\Big) \\
&= \ell_0\Big(|\delta|^2+|e|^2\Big)
\end{aligned}$$

which completes the proof.

We are now ready to present the main result.

Theorem 1: If there exist a positive definite matrices $\boldsymbol{X} \in \mathbb{R}^{n\times n}$, $\hat{\boldsymbol{\Gamma}}_{\boldsymbol{c}} \in \mathbb{R}^{nN\times nN}$, $\hat{\boldsymbol{\Gamma}}_{\boldsymbol{u}} \in \mathbb{R}^{nN\times nN}$, matrices $\boldsymbol{Y} \in \mathbb{R}^{m\times n}$, $\boldsymbol{Z} \in \mathbb{R}^{n\times n}$, constants $\boldsymbol{\varepsilon_\delta}$, $\boldsymbol{\varepsilon_e}$, $\boldsymbol{\varepsilon_r}$, $\boldsymbol{\varepsilon_w}$, $\boldsymbol{\sigma_c}$, $\boldsymbol{\sigma_u}$, $\boldsymbol{\theta_0}$, $\boldsymbol{\theta_1}$, $\boldsymbol{\theta_2} > 0$, with $\overline{\boldsymbol{X}}_{\boldsymbol{N}} = I_N \otimes \boldsymbol{X}$, $\overline{\boldsymbol{X}}_{\boldsymbol{N-1}} = I_{N-1} \otimes \boldsymbol{X}$, $\overline{\boldsymbol{X}} = \mathrm{diag}(\overline{\boldsymbol{X}}_{\boldsymbol{N-1}}, \overline{\boldsymbol{X}}_{\boldsymbol{N}})$, $\overline{\boldsymbol{Y}}_{\boldsymbol{N}} = I_N \otimes \boldsymbol{Y}$, $\overline{\boldsymbol{Y}}_{\boldsymbol{N-1}} = I_{N-1} \otimes \boldsymbol{Y}$, $\overline{\boldsymbol{Z}}_{\boldsymbol{N}} = I_N \otimes \boldsymbol{Z}$, satisfying the following optimization

$$\min \quad \mathscr{F} = \boldsymbol{\theta_1} + \boldsymbol{\theta_2} - \boldsymbol{\theta_0} + \boldsymbol{\sigma_c^{-1}} + \boldsymbol{\sigma_u^{-1}} \tag{49}$$

subject to:

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ * & \Lambda_{22} \end{bmatrix} < 0, \tag{50}$$

$$\hat{\boldsymbol{\Gamma}}_{\boldsymbol{c}} \le \boldsymbol{\theta_1} I_{nN},\ \hat{\boldsymbol{\Gamma}}_{\boldsymbol{u}} \le \boldsymbol{\theta_2} I_{nN},\ \overline{\boldsymbol{X}}_{\boldsymbol{N}} > \boldsymbol{\theta_0} I_{n(2N-1)} \tag{51}$$

where the matrices are shown in (52).
Then, the controller gain is given by $K = \boldsymbol{Y}\boldsymbol{X}^{\boldsymbol{-1}}$, and the observer gain is given by $F = \boldsymbol{Y}\boldsymbol{X}^{\boldsymbol{-1}}C^{\dagger}$. The parameters in the event-triggering mechanisms are selected such that $\sigma_{c_i} = 1/\boldsymbol{\sigma_c^{-1}}$, $\sigma_{u_i} = 1/\boldsymbol{\sigma_u^{-1}}$, $i = [1, N]$, and γ_{c_i} and γ_{u_i} can be decided by solving another simple optimization problem

$$\min \quad \mathrm{Trace}(\boldsymbol{\Gamma_c} + \boldsymbol{\Gamma_u}) \tag{53}$$

subject to:

$$\boldsymbol{\Gamma_c} > \overline{\boldsymbol{X}}_{\boldsymbol{N}}^{\boldsymbol{-1}} \hat{\boldsymbol{\Gamma}}_{\boldsymbol{c}} \overline{\boldsymbol{X}}_{\boldsymbol{N}}^{\boldsymbol{-1}}, \quad \boldsymbol{\Gamma_u} > \overline{\boldsymbol{X}}_{\boldsymbol{N}}^{\boldsymbol{-1}} \hat{\boldsymbol{\Gamma}}_{\boldsymbol{u}} \overline{\boldsymbol{X}}_{\boldsymbol{N}}^{\boldsymbol{-1}} \tag{54}$$

where $\boldsymbol{\Gamma_c} = \mathrm{diag}(\gamma_{c_1} I_n, \gamma_{c_2} I_n, \cdots, \gamma_{c_N} I_n)$, $\boldsymbol{\Gamma_u} = \mathrm{diag}(\gamma_{u_1} I_n, \gamma_{u_2} I_n, \cdots, \gamma_{u_N} I_n)$. Suppose that the event-triggering mechanisms are implemented according to (10), (12), (42)–(45). Then, the (δ, e, r) dynamics is UGAS in case of $w = 0$, which implies the asymptotic consensus of the MAS. When $w \neq 0$, the system is finite gain $\mathcal{L}_2$ stable from input w to output z with $\mathcal{L}_2$ gain less than or equal to ε_w. Moreover, strong Zeno-freeness is guaranteed in every event-triggering mechanism.

The proof can be found in the Appendix.

Algorithm 1 Co-design Algorithm.

1: acquire system matrices
2: minimize $\boldsymbol{\theta_1} + \boldsymbol{\theta_2} - \boldsymbol{\theta_0} + \boldsymbol{\sigma_c^{-1}} + \boldsymbol{\sigma_u^{-1}}$ subject to (50) and (51), compute the controller gain $K = \boldsymbol{Y}\boldsymbol{X}^{\boldsymbol{-1}}$, the observer gain $F = \boldsymbol{Y}\boldsymbol{X}^{\boldsymbol{-1}}C^{\dagger}$, and the parameters $\hat{\boldsymbol{\Gamma}}_{\boldsymbol{c}}$, $\hat{\boldsymbol{\Gamma}}_{\boldsymbol{u}}$, $\boldsymbol{\sigma_c^{-1}}$, $\boldsymbol{\sigma_u^{-1}}$
3: minimize Trace $(\boldsymbol{\Gamma_c} + \boldsymbol{\Gamma_u})$ subject to (54), compute the parameters γ_{c_i}, and γ_{u_i}
4: select κ_{c_i} and κ_{u_i} such that (42) is satisfied
5: choose $\lambda_{c_i} \in (0, 1)$, $\lambda_{u_i} \in (0, 1)$, compute $\mathcal{T}_{c_i}$ and $\mathcal{T}_{u_i}$ according to (43) and (45) respectively, and select $\tau_{miet}^{c_i}$ and $\tau_{miet}^{u_i}$.

Remark 5: Since the matrices Γ_c and Γ_u have a specific structure, it can not be computed directly from the optimization problem. Motivated by [21], a linear scalarization mehod is utilized to address this problem. In fact, minimizing $\theta_1 + \theta_2 - \theta_0$ corresponds to minimizing Trace $(\Gamma_c + \Gamma_u)$, and the optimization problem in (49) actually aims to minimize $\sigma_c^{-1} + \sigma_u^{-1} + \sum_{i=1}^{N} (\gamma_{c_i} + \gamma_{u_i})$.

Remark 6: For the sake of reducing complexity, we set $\sigma_{c_i} = \sigma_c$ and $\sigma_{u_i} = \sigma_u$. However, agents' event-triggering mechanisms are not the same. One obvious feature is that the parameters τ_{c_i} and $\tilde{\gamma}_{c_i}$ are related to L_{c_i} and κ_{c_i}. Difference of L_{c_i} among agents manifests that the communication topology of each agent is usually not symmetric, and the parameter κ_{c_i} can be regulated to realize a mild resource allocation among agents. Decreasing κ_{c_i} will increase the value of $\tau_{miet}^{c_i}$ and decrease the value of $\tilde{\gamma}_{c_i}$, which indicates less frequent usage of communication resources for agent i. Compared with the co-design related works in [21], where the event-triggering parameters of all the agents are identical, our co-design method is naturally more flexible and sensible.

Remark 7: Most literatures on the linear MAS utilize the emulation-based approaches. It should be noted that separating these two design process, which brings brevity for the consensus analysis, can not save the communication resources efficiently. Instead, our co-design method extensively explores the feasible solution of the pair of the control law and the

$$
\begin{aligned}
&\Lambda_{11}=\begin{bmatrix} \Pi_{1,1} & \Pi_{1,2} & \Pi_{1,3} & \mathcal{E}_1 \\ \star & -\hat{\mathbf{\Gamma}}_{\mathbf{1}} & 0 & 0 \\ \star & \star & -\hat{\mathbf{\Gamma}}_{\mathbf{2}} & 0 \\ \star & \star & \star & -\boldsymbol{\varepsilon_w} I_{Nl} \end{bmatrix},\\
&\Lambda_{12}=\begin{bmatrix} \overline{\boldsymbol{X}} & 0 & \overline{\boldsymbol{X}} & 0 & \overline{\boldsymbol{X}} & 0 & 0 & \Pi_{1,12} & \Pi_{1,13} & \overline{\boldsymbol{X}} C_z^T \\ 0 & \overline{\boldsymbol{X}}_{\boldsymbol{N}} & 0 & \overline{\boldsymbol{X}}_{\boldsymbol{N}} & 0 & \overline{\boldsymbol{X}}_{\boldsymbol{N}} & 0 & \Pi_{2,12} & \Pi_{2,13} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \overline{\boldsymbol{X}}_{\boldsymbol{N}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_z^T \end{bmatrix},\\
&\Lambda_{22}=\mathrm{diag}(-\boldsymbol{\sigma_c^{-1}}\ell_0^{-1}I_{n(2N-1)},-\boldsymbol{\sigma_c^{-1}}\ell_0^{-1}I_{nN},-\boldsymbol{\sigma_u^{-1}}\ell_0^{-1}I_{n(2N-1)},-\boldsymbol{\sigma_u^{-1}}\ell_0^{-1}I_{nN},\\
&\qquad -\boldsymbol{\varepsilon_\delta^{-1}}I_{n(2N-1)},-\boldsymbol{\varepsilon_e^{-1}}I_{nN},-\boldsymbol{\varepsilon_r^{-1}}I_{nN},-I_{nN},-I_{nN},-I_{nN}),\\
&\Pi_{1,1}=\begin{bmatrix} \overline{A}_{N-1}\boldsymbol{X_{N-1}}+\boldsymbol{X_{N-1}}\overline{A}_{N-1}^T-\tilde{B}_L\boldsymbol{Y_{N-1}}-\boldsymbol{Y_{N-1}^T}\tilde{B}_L^T & \tilde{B}_{LW}\overline{\boldsymbol{Y}}_{\boldsymbol{N}} \\ \star & \overline{A}_N\overline{\boldsymbol{X}}_{\boldsymbol{N}}+\overline{\boldsymbol{X}}_{\boldsymbol{N}}\overline{A}_N^T-\overline{\boldsymbol{Z}}_{\boldsymbol{N}}-\overline{\boldsymbol{Z}}_{\boldsymbol{N}}^{\boldsymbol{T}} \end{bmatrix},\\
&\Pi_{1,2}=\begin{bmatrix} -\tilde{B}_{LW}\overline{\boldsymbol{Y}}_{\boldsymbol{N}} \\ 0 \end{bmatrix},\quad \Pi_{1,3}=\begin{bmatrix} -\tilde{B}_{LW}\overline{\boldsymbol{Y}}_{\boldsymbol{N}} \\ 0 \end{bmatrix},\quad \Pi_{1,12}=\begin{bmatrix} \overline{\boldsymbol{Y}}_{\boldsymbol{N-1}}^{\boldsymbol{T}}\overline{B}_{L_0}^T \\ -\overline{\boldsymbol{Y}}_{\boldsymbol{N}}^{\boldsymbol{T}}\overline{B}_L^T-\overline{\boldsymbol{Z}}_{\boldsymbol{N}}^{\boldsymbol{T}} \end{bmatrix},\\
&\Pi_{1,13}=\begin{bmatrix} \overline{\boldsymbol{X}}_{\boldsymbol{N-1}}\overline{A}_{L_0}^T \\ -\overline{\boldsymbol{X}}_{\boldsymbol{N}}\overline{A}_L^T \end{bmatrix},\quad \Pi_{2,12}=\begin{bmatrix} \overline{\boldsymbol{Y}}_{\boldsymbol{N}}^{\boldsymbol{T}}\overline{B}_L^T-\overline{\boldsymbol{Y}}_{\boldsymbol{N}}^{\boldsymbol{T}}\overline{B}_{D_0}^T \end{bmatrix},\quad \Pi_{2,13}=\begin{bmatrix} \overline{\boldsymbol{X}}_{\boldsymbol{N}}\overline{A}_L^T-\overline{\boldsymbol{X}}_{\boldsymbol{N}}^{\boldsymbol{T}}\overline{A}_{D_0}^T \end{bmatrix}\\
&\overline{A}_{N-1}=I_{N-1}\otimes A,\quad ,\overline{A}_N=I_N\otimes A,\quad \overline{A}_{L_0}=L_0\otimes A,\quad \overline{A}_L=L\otimes A,\quad \overline{A}_{D_0}=D_0\otimes A,\quad \overline{B}_{L_0}=L_0\otimes B,\\
&\quad \overline{B}_L=L\otimes B,\quad \overline{B}_{D_0}=D_0\otimes B,\quad \tilde{B}_L=\tilde{L}\otimes B,\quad \tilde{B}_{LW}=\tilde{L}W\otimes B,\quad D_0=\mathrm{diag}(d_1,d_2,\cdots,d_N)
\end{aligned} \tag{52}
$$

event-triggering condition, and finds an optimal solution based on the objective functions to mitigate the communication burden.

IV. Illustrative example

To verify the theoretical results, we consider a spacecraft formation flying problem which can be transformed into a consensus problem as mentioned in [15]. Each agent represents a spacecraft flying in the low Earth orbit with the state $x_i=(\overline{x}_i,\overline{y}_i,\overline{z}_i,v_i^x,v_i^y,v_i^z)$, where $(\overline{x}_i,\overline{y}_i,\overline{z}_i)$ is the distance from the desired position in the $X-Y-Z$ directions and (v_i^x,v_i^y,v_i^z) is the velocity in the three directions. The control vector u_i is defined as $u_i=(u_{x_i},u_{y_i},u_{z_i})$. The system matrices are given by

$$
\begin{aligned}
&A=\begin{bmatrix} 0 & I_3 \\ A_1 & A_2 \end{bmatrix},\quad A_1=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3\omega_0^2 & 0 \\ 0 & 0 & -\omega_0^2 \end{bmatrix},\\
&A_2=\begin{bmatrix} 0 & 2\omega_0 & 0 \\ -2\omega_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, C=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},\\
&B=\begin{bmatrix} 0 \\ I_3 \end{bmatrix},\quad B_w=[0;0;0;0.22;0.22;0.22],
\end{aligned}
$$

where ω_0 is angular rate of the satellite. C_z and D_z are respectively given as $C_z=[0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ \mathrm{zeros}(1,54)]$, $D_z=[0\ 1\ 0\ 1\ 0\ 1]$. The assumptions 2 and 3 can be verified with these matrices. Suppose that there are six agents which communicate with each other via a directed communication graph given in Fig. 2.

By following the co-design algorithm, the parameters can be obtained using MATLAB tools, which results in

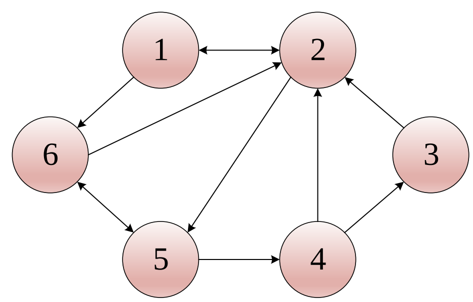

Fig. 2. Communication Topology.

$$
K=\begin{bmatrix} 5.5022 & 0.3558 & 0.3566 & 7.8528 & 0.8019 & 0.9114 \\ 0.3585 & 5.5061 & 0.3539 & 0.8029 & 7.8299 & 0.8035 \\ 0.3574 & 0.3604 & 5.5023 & 0.8114 & 0.8025 & 7.8270 \end{bmatrix},
$$

$$
F=\begin{bmatrix} 603 & 17 & 17 \\ 17 & 603 & 17 \\ 17 & 17 & 603 \\ 291 & 21 & 22 \\ 22 & 291 & 21 \\ 22 & 22 & 291 \end{bmatrix}, \sigma_c=0.4503, \sigma_u=0.4503,
$$

$$
\gamma_{c_i}=75.9460,\ \gamma_{u_i}=75.7784, i=[1,N]
$$

Next we select $\kappa_{c_i}=0.16$, $\kappa_{u_i}=0.15$, $i=[1,N]$, to satisfy (42). By choosing $\lambda_{c_i}=0.1$, $\lambda_{u_i}=0.1$, $i=[1,N]$, it derives that $\mathcal{T}_{c_1}=0.0139$, $\mathcal{T}_{c_2}=0.0112$, $\mathcal{T}_{c_3}=0.0139$, $\mathcal{T}_{c_4}=0.0139$, $\mathcal{T}_{c_5}=0.0129$, $\mathcal{T}_{c_6}=0.0129$, $\mathcal{T}_{u_i}=0.0157$, $i=[1,N]$. Finally we select $\tau_{miet}^{c_1}=\tau_{miet}^{c_3}=\tau_{miet}^{c_4}=T_1=0.012$, $\tau_{miet}^{c_2}=\tau_{miet}^{c_5}=\tau_{miet}^{c_6}=T_2=0.01$, $\tau_{miet}^{u_i}=T_3=0.01$, $i=[1,N]$.

The initial conditions are chosen arbitrarily, and the random disturbance w_i satisfies $|w_i(t)| \leq 0.4$ with $t \in [0, 5]$ and $w_i(t) \leq 0.5$ with $t \in (5, 10]$. The simulation results are illustrated in Figs. 3–9. Fig. 3 shows the evolution of the velocity in X-direction, the remaining two directions are similar thus omitted. Fig. 4 plots the state trajectories of six agents. In Fig. 5, the triggering instants for the communication among six agents are presented and the evolution of the dynamic variable η_{c_5} of agent 5 is also shown. Besides, the trigger instants for the control update are presented in Fig. 6, together with the evolution of the dynamic variable η_{u_1}. The inter-event interval for the communication and the control update is illustrated in Fig. 7 and 8 respectively. As we can see, all the event-triggering mechanisms have a positive MIET even when the system is influenced by external disturbances. Finally, the control input of six agents in the X-direction is shown in Fig. 9.

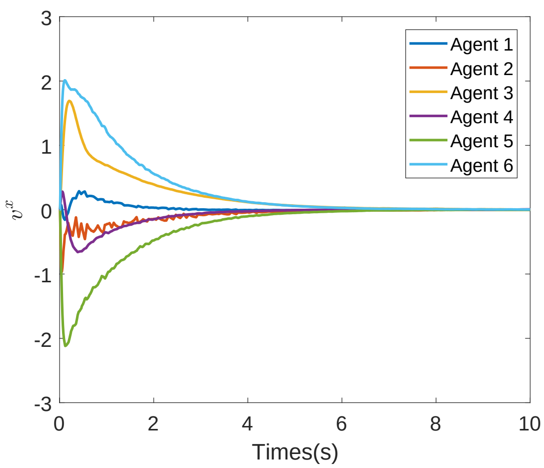

Fig. 3. Evolution of v^x.

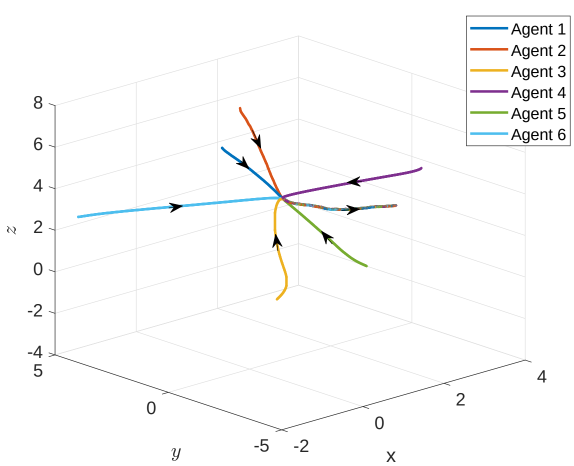

Fig. 4. State trajectories of six agents.

V. Conclusion

This paper proposes a co-design approach of the control law and the event-triggering mechanisms for the consensus of MAS. The event-triggered idea is implmented on both the communication side and the controller side. A distributed event-triggered consensus protocol is proposed that guarantee the strong Zeno-freeness and asymptotic consensus simultaneously even with external disturbances. The parameters of the control gain, the observer gain and the event-triggering mechanisms can be obtained by solving an LMI optimization problem. Based on a novel hybrid model, the convergence analysis is presented using the hybrid system approach. Finally, the spacecraft formation flying problem is utilized to verify our theoretical results.

The framework constructed in this paper can be extended in different directions, such as the consideration of quantization [26], and inclusion of delays [13]. The extension of the approach to the heterogeneous or nonlinear multi-agent systems is a potential future topic. Besides, designing a fully distributed event-triggered consensus protocol is desperate and necessary for large-scale MAS and also constitutes a challenging future work.

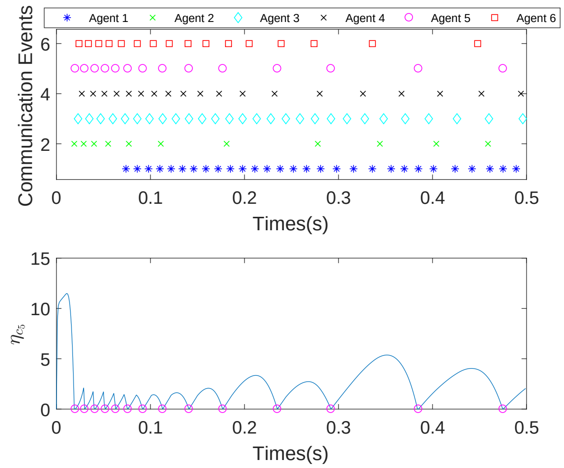

Fig. 5. Triggering instants of the communication for the first 0.5 second.

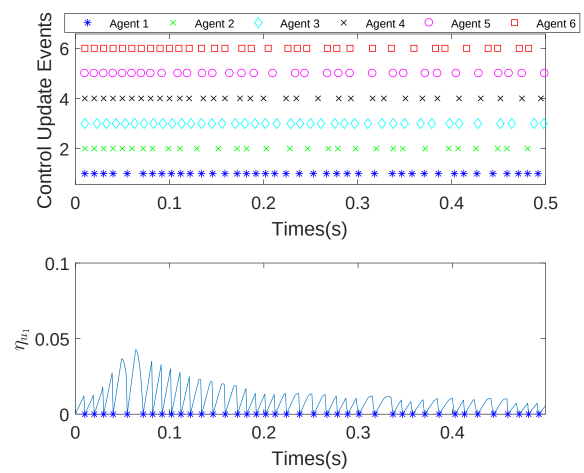

Fig. 6. Triggering instants of the control update for the first 0.5 second.

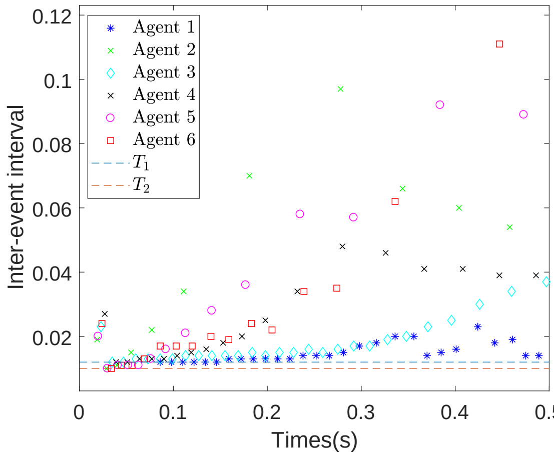

Fig. 7. Inter-event Interval for the communication and the dynamic variable η_{c_5} for the first 0.5 second.

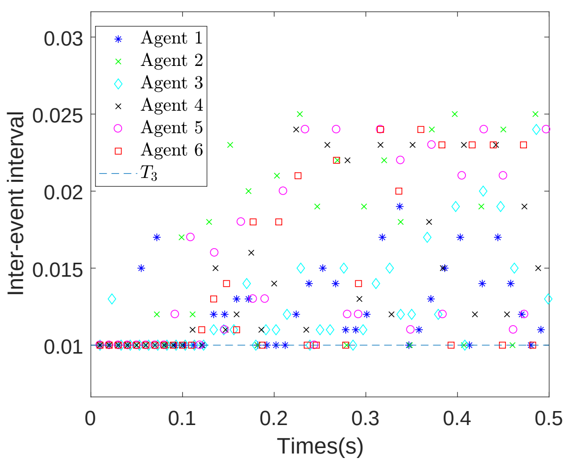

Fig. 8. Inter-event Interval for the control update and the dynamic variable η_{u_1} for the first 0.5 second.

Appendix

Proof of Theorem 1: Our design builds on the analysis of the evolution of the hybrid system characterized by the following Lyapunov function

$$V(\delta) = \delta^T \overline{X} \delta \tag{55}$$

Let $KX = Y$, and $FCX = Z$. After some direct calculations, in view of (37), we obtain that

$$\overline{X}^{-1}\Pi_{1,12} = A_{\delta_c}^T, \quad \overline{X}_N^{-1}\Pi_{2,12} = A_{e_c}^T,$$
$$\overline{X}^{-1}\Pi_{1,13} = A_{\delta_c}^T, \quad \overline{X}_N^{-1}\Pi_{2,13} = A_{e_u}^T.$$

We pre- and post- multiply inequality (50) by $\mathrm{diag}(\overline{X}^{-1}, \overline{X}_N^{-1}, \overline{X}_N^{-1}, 1,1,1,1,1,1,1,1,1,1,1,1)$, then combing (54) and applying Schur Complement Lemma, and we can deduce that the following inequality holds

$$\begin{bmatrix} \Omega_{11} & \Omega_{12} & \overline{X}\mathcal{M}_1 & \overline{X}\mathcal{E}_1 + C_z^T D_z \\ \star & \Omega_{22} & 0 & 0 \\ \star & \star & \varepsilon_r I_{nN} - \Gamma_u & 0 \\ \star & \star & \star & D_z^T D_Z - \varepsilon_w I_{Nl} \end{bmatrix} < 0 \tag{56}$$

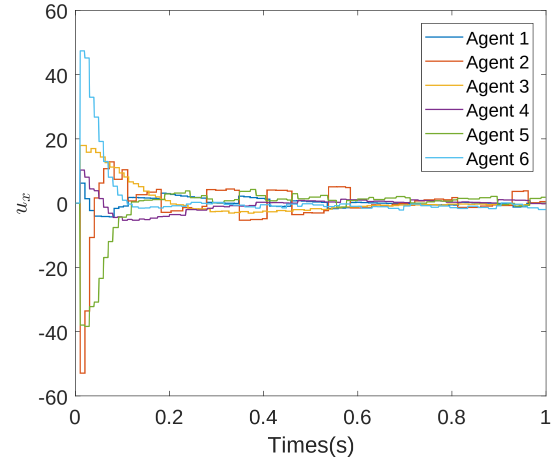

Fig. 9. Control signals of six agents in the X-direction for the first second.

where

$$\begin{aligned} \Omega_{11} &= \overline{X}\mathcal{A}_1 + \mathcal{A}_1^T\overline{X} + A_{\delta_c}^T A_{\delta_c} + A_{\delta_u}^T A_{\delta_u} + C_z^T C_z \\ &\quad + (\sigma_c + \sigma_u)\ell_0 I_{n(2N-1)} + \varepsilon_\delta I_{n(2N-1)}, \\ \Omega_{12} &= \overline{X}\mathcal{B}_1 + A_{\delta_c}^T A_{e_c} + A_{\delta_u}^T A_{e_u}, \\ \Omega_{22} &= A_{e_c}^T A_{e_c} + A_{e_u}^T A_{e_u} + (\sigma_c + \sigma_u)\ell_0 I_{nN} + \varepsilon_e I_{nN} - \Gamma_c. \end{aligned}$$

In view of Lemma 2, it follows that

$$\begin{aligned} &\langle \nabla V(\delta), \mathcal{A}_1\delta + \mathcal{B}_1 e + \mathcal{M}_1 r + \mathcal{E}_1 w \rangle \\ &\leq -\varepsilon_\delta|\delta|^2 - |z|^2 + \varepsilon_w|w|^2 - \varepsilon_e|e|^2 - \varepsilon_r|r|^2 \\ &\quad + \sum_{i=1}^N \left(-H_{c_i}^2 + \gamma_{c_i}^2|e_i|^2 - \sigma_{c_i}|\varphi_i|^2\right) \\ &\quad + \sum_{i=1}^N \left(-H_{u_i}^2 + \gamma_{u_i}^2|r_i|^2 - \sigma_{u_i}|\varphi_i|^2\right) \end{aligned} \tag{57}$$

Now consider the following hybrid Lyapunov function:

$$\begin{aligned} U(\xi) = V(\delta) &+ \sum_{j=1}^N \left(\gamma_{c_i}\phi_{c_i}|e_i|^2 + \eta_{c_i}\right) \\ &+ \sum_{j=1}^N \left(\gamma_{u_i}\phi_{u_i}|r_i|^2 + \eta_{u_i}\right) \end{aligned} \tag{58}$$

According to (38) and (40), it gives $\phi_{c_i} > 0$, ϕ_{u_i}. The event-triggering mechanisms in (10) and (12) indicate that $\eta_{c_i} > 0$, $\eta_{u_i} > 0$. Combining these with the facts that $V(\delta)$ is radially unbounded, we can conclude that $U(\xi)$ constitutes a suitable Lyapunov function. Next we will explore the dynamics of the Lyapunov function during flows and during jumps.

Dynamics during flows:

For $\xi \in \mathcal{F}$, we obtain that

$$\begin{aligned}
&\langle \nabla U(\xi), F(\xi, w)\rangle \\
&\leq \langle \nabla V(\delta), \mathcal{A}_1\delta + \mathcal{B}_1 e + \mathcal{M}_1 r + \mathcal{E}_1 w\rangle \\
&\quad + \sum_{i=1}^{N}\Bigg(\gamma_{c_i}|e_i|^2\varpi_{c_i}\big(-2L_{c_i}\phi_{c_i} - (1+\kappa_{c_i})\gamma_{c_i}(\phi_{c_i}^2+1)\big) \\
&\quad + 2\gamma_{c_i}\phi_{c_i}|e_i|(L_{c_i}|e_i| + M_{c_i}|r_i| + H_{c_i}) + \psi_{c_i}\Bigg) \\
&\quad + \sum_{i=1}^{N}\Bigg(\gamma_{u_i}|r_i|^2\varpi_{u_i}\big(-(1+\kappa_{u_i})\gamma_{u_i}(\phi_{u_i}^2+1)\big) \\
&\quad + 2\gamma_{u_i}\phi_{u_i}|r_i|(M_{u_i}|r_i| + H_{u_i}) + \psi_{u_i}\Bigg) \\
&\leq -\varepsilon_\delta|\delta|^2 - |z|^2 + \varepsilon_w|w|^2 - \varepsilon_e|e|^2 - \varepsilon_r|r|^2 \\
&\quad + \sum_{i=1}^{N} S_{c_i} + \sum_{i=1}^{N} S_{u_i}
\end{aligned}$$

where

$$\begin{aligned}
S_{c_i} =& \gamma_{c_i}|e_i|^2\varpi_{c_i}\big(-2L_{c_i}\phi_{c_i} - (1+\kappa_{c_i})\gamma_{c_i}(\phi_{c_i}^2+1)\big) \\
&+ 2\gamma_{c_i}\phi_{c_i}|e_i|(L_{c_i}|e_i| + M_{c_i}|r_i| + H_{c_i}) \\
&+ \psi_{c_i} - H_{c_i}^2 + \gamma_{c_i}^2|e_i|^2 - \sigma_{c_i}|\varphi_i|^2 \\
S_{u_i} =& \gamma_{u_i}|r_i|^2\varpi_{u_i}\big(-(1+\kappa_{u_i})\gamma_{u_i}(\phi_{u_i}^2+1)\big) \\
&+ 2\gamma_{u_i}\phi_{u_i}|r_i|(M_{u_i}|r_i| + H_{u_i}) + \psi_{u_i} \\
&- H_{u_i}^2 + \gamma_{u_i}^2|r_i|^2 - \sigma_{u_i}|\varphi_i|^2.
\end{aligned}$$

We now focus on the S_{c_i} term.

$$\begin{aligned}
&S_{c_i} \\
=& -\gamma_{c_i}|e_i|^2\big(2L_{c_i}\phi_{c_i} + (1+\kappa_{c_i})\gamma_{c_i}(\phi_{c_i}^2+1)\big) \\
&+ 2\gamma_{c_i}\phi_{c_i}|e_i|(L_{c_i}|e_i| + M_{c_i}|r_i| + H_{c_i}) \\
&+ \psi_{c_i} - H_{c_i}^2 + \gamma_{c_i}^2|e_i|^2 - \sigma_{c_i}|\varphi_i|^2 \\
&+ (1-\varpi_{c_i})\gamma_{c_i}|e_i|^2\big(2L_{c_i}\phi_{c_i} + (1+\kappa_{c_i})\gamma_{c_i}(\phi_{c_i}^2+1)\big) \\
=& -\Big(H_{c_i} - \gamma_{c_i}\phi_{c_i}e_i\Big)^2 - \Big(\sqrt{\kappa_{c_i}}\gamma_{c_i}\phi_{c_i}e_i - \frac{M_{c_i}}{\sqrt{\kappa_{c_i}}}r_i\Big)^2 \\
&+ \frac{M_{c_i}^2}{\kappa_{c_i}}|r_i|^2 - \sigma_{c_i}|\varphi_i|^2 - \kappa_{c_i}\gamma_{c_i}|e_i|^2 + \psi_{u_i} \\
&+ (1-\varpi_{c_i})\gamma_{c_i}|e_i|^2\big(2L_{c_i}\phi_{c_i} + (1+\kappa_{c_i})\gamma_{c_i}(\phi_{c_i}^2+1)\big).
\end{aligned}$$

Recalling the definition of $\tilde{\gamma}_{c_i}$, it follows that

$$\begin{aligned}
S_{c_i} \leq \frac{M_{c_i}^2}{\kappa_{c_i}}|r_i|^2 - \sigma_{c_i}|\varphi_i|^2 - \kappa_{c_i}\gamma_{c_i}|e_i|^2 + \psi_{c_i} \\
+(1-\varpi_{c_i})\tilde{\gamma}_{c_i}|e_i|^2.
\end{aligned}$$

Applying similar techniques, we can obtain that S_{u_i} satisfies

$$\begin{aligned}
S_{u_i} \leq \frac{M_{u_i}^2}{\kappa_{u_i}}|e_i|^2 - \sigma_{u_i}|\varphi_i|^2 - \kappa_{u_i}\gamma_{u_i}|r_i|^2 + \psi_{u_i} \\
+(1-\varpi_{u_i})\tilde{\gamma}_{u_i}|r_i|^2.
\end{aligned}$$

Recalling the defined event-triggering mechanisms, we obtain that for all $\xi \in \mathcal{F}$

$$\begin{aligned}
&\langle \nabla U(\xi), F(\xi, w)\rangle \\
\leq& -\varepsilon_\delta|\delta|^2 - |z|^2 + \varepsilon_w|w|^2 - \varepsilon_e|e|^2 - \varepsilon_r|r|^2 \\
&+ \sum_{i=1}^{N}\Big(-\sigma_{c_i}|\varphi_i|^2 - \kappa_{c_i}\gamma_{c_i}|e_i|^2 + (1-\varpi_{c_i})\tilde{\gamma}_{c_i}|e_i|^2 \\
&+ \frac{M_{u_i}^2}{\kappa_{u_i}}|e_i|^2 + \psi_{c_i}\Big) + \sum_{i=1}^{N}\Big(-\sigma_{u_i}|\varphi_i|^2 - \kappa_{u_i}\gamma_{u_i}|r_i|^2 \\
&+ (1-\varpi_{u_i})\tilde{\gamma}_{u_i}|r_i|^2 + \frac{M_{c_i}^2}{\kappa_{c_i}}|r_i|^2 + \psi_{u_i}\Big) \\
\leq& -\varepsilon_\delta|\delta|^2 - |z|^2 + \varepsilon_w|w|^2 - \varepsilon_e|e|^2 - \varepsilon_r|r|^2.
\end{aligned} \tag{59}$$

Dynamics during jumps:
When $\xi \in \mathcal{J}_{c_i}$, in view of the jump map (32), we obtain that

$$\begin{aligned}
&U(\xi^+) - U(\xi) \\
\leq& \gamma_{c_i}\phi_{c_i}(0)|e_i^+|^2 + \eta_{c_i} - \gamma_{c_i}\phi_{c_i}(\tau_{miet}^{c_i})|e_i|^2 - \eta_{c_i} \leq 0
\end{aligned}$$

Similarly, when $\xi \in \mathcal{J}_{u_i}$, we have

$$\begin{aligned}
&U(\xi^+) - U(\xi) \\
\leq& \gamma_{u_i}\phi_{u_i}(0)|r_i^+|^2 + \eta_{u_i} - \gamma_{u_i}\phi_{u_i}(\tau_{miet}^{u_i})|r_i|^2 - \eta_{u_i} \leq 0
\end{aligned}$$

Since $\mathcal{J} = \Big(\bigcup_{i=1}^{N}\mathcal{J}_{c_i}\Big) \bigcup \Big(\bigcup_{i=1}^{N}\mathcal{J}_{u_i}\Big)$, for all $\xi \in \mathcal{J}$, we can obtain that

$$U(\xi^+) - U(\xi) \leq 0 \tag{60}$$

Using similar techniques as in [15], we can conclude that the system is $\mathcal{L}_2$ stable from input w to output z with $\mathcal{L}_2$ gain less than or equal to ε_w. Particularly, when $w(t) = 0$, the dynamics during flows (59) turns into the following condition

$$\langle \nabla U(\xi), F(\xi, w)\rangle \leq -\varepsilon_\delta|\delta|^2 - |z|^2 - \varepsilon_e|e|^2 - \varepsilon_r|r|^2 \tag{61}$$

Combining (61) and (60), the UGAS property can be derived based on stability theory for hybrid systems, which indicates the asymptotic consensus for MAS.

References

[1] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1520–1533, 2004.

[2] W. Ren, R. W. Beard, and E. M. Atkins, "Information consensus in multivehicle cooperative control," *IEEE Control Systems Magazine*, vol. 27, no. 2, pp. 71–82, 2007.

[3] D. V. Dimarogonas and E. Frazzoli, "Distributed event-triggered control strategies for multi-agent systems," in *47th Annual Allerton Conference*, (Monticello, IL), pp. 906–910, 2009.

[4] D. V. Dimarogonas and K. H. Johansson, "Event-triggered control for multi-agent systems," in *48th IEEE Conference on Decision and Control*, (Shanghai, P.R., China), pp. 7131–7136, 2009.

[5] D. V. Dimarogonas, E. Frazzoli, and K. H. Johansson, "Distributed event-triggered control for multi-agent systems," *IEEE Transactions on Automatic Control*, vol. 57, no. 5, pp. 1291–1297, 2012.

[6] X. Meng and T. Chen, "Event based agreement protocols for multi-agent networks," *Automatica*, no. 49, p. 2125–2132, 2013.

[7] C. Nowzari and J. Cortes, "Distributed event-triggered coordination for average consensus on weight-balanced digraphs," *Automatica*, no. 68, p. 237 244, 2016.

[8] E. Garcia, Y. Cao, H. Yu, P. Antsaklis, D. Casbeer," Decentralised event-triggered cooperative control with limited communication," *International Journal of Control*, 86(9), 1479–1488, 2013.

[9] G. S. Seyboth, D. V. Dimarogonas, and K. H. Johansson, "Event-based broadcasting for multi-agent average consensus," *Automatica*, no. 49, p. 245–252, 2013.

[10] X. Yi, K. Liu, D. V. Dimarogonas and K. H. Johansson, "Dynamic Event-Triggered and Self-Triggered Control for Multi-agent Systems," *IEEE Transactions on Automatic Control*, vol. 64, no. 8, pp. 3300–3307, 2019.

[11] W. Hu, C. Yang, T. Huang and W. Gui, "A Distributed Dynamic Event-Triggered Control Approach to Consensus of Linear Multiagent Systems With Directed Networks," *IEEE Transactions on Cybernetics*, vol. 50, no. 2, pp. 869–874, 2020.

[12] D. N. Borgers and W. M. Heemels, "Event-separation properties of event-triggered control systems," *IEEE Transactions on Automatic Control*, vol. 59, no. 10, pp. 2644–2656, 2014.

[13] V. S. Dolk, R. Postoyan and W. P. M. H. Heemels, "Event-Triggered Consensus for Multi-Agent Systems with Guaranteed Robust Positive Minimum Inter-Event Times," in *IEEE 58th Conference on Decision and Control*, (Nice, France), pp. 2604-2609, 2019.

[14] J. Berneburg and C. Nowzari, "Robust Dynamic Event-Triggered Coordination With a Designable Minimum Inter-Event Time," *IEEE Transactions on Automatic Control*, doi: 10.1109/TAC.2020.3020809.

[15] G. Zhao and C. Hua, "A Hybrid Dynamic Event-Triggered Approach to Consensus of Multi-Agent systems with External Disturbances," *IEEE Transactions on Automatic Control*, 2020.

[16] M. Abdelrahim, R. Postoyan, J. Daafouz, D. Nesic, "Robust event-triggered output feedback controllers for nonlinear systems," *Automatica*, vol. 75, pp. 96–108, 2017.

[17] B. Cheng and Z. Li, "Fully distributed event-triggered protocols for linear multiagent networks," *IEEE Transactions on Automatic Control*, vol. 64, no. 4, pp. 1655–1662, 2018.

[18] P. Yu, C. Fischione and D. V. Dimarogonas, "Distributed Event-Triggered Communication and Control of Linear Multiagent Systems Under Tactile Communication," *IEEE Transactions on Automatic Control*, vol. 63, no. 11, pp. 3979–3985, 2018.

[19] C. Peng and Q. L. Han, "A novel event-triggered transmission scheme and L2 control co-design for sampled-data control systems," *IEEE Transactions on Automatic Control*, vol. 58, no. 10, pp. 2620–2626, 2013.

[20] M. Abdelrahim, R. Postoyan, J. Daafouz, D. Nesic and W. Heemels, "Co-design of output feedback laws and event-triggering conditions for the $\mathcal{L}_2$ -stabilization linear systems," *Automatica*, vol. 87, pp. 337–344, 2018.

[21] A. Amini, A. Asif and A. Mohammadi, "Dynamic Event-triggered Formation Control for Multi-agent Systems: A Co-design Optimization Approach," in *American Control Conference*, (Denver, CO, USA), pp. 707–712, 2020.

[22] C. Nowzari, E. Garcia and J. Corts, "Event-triggered communication and control of networked systems for multi-agent consensus", Automatica, 2019.

[23] Y. Wu, H. Zhang, Z. Wang and C. Huang, "Distributed Event-Triggered Consensus of General Linear Multiagent Systems Under Directed Graphs," *IEEE Transactions on Cybernetics*, 2020.

[24] Y. Zhang and Y. Tian, "Consentability and protocol design of multi-agent systems with stochastic switching topology," *Automatica*, vol. 45, no. 5, pp. 1195–1201, 2009.

[25] R. Goebel, R. G. Sanfelice, and A. R. Teel, *Hybrid Dynamical Systems: Modeling, Stability, Robustness.* Princeton, NJ, USA: Princeton Univ. Press, 2012.

[26] M. Abdelrahim, V. S. Dolk and W. P. M. H. Heemels, "Event-Triggered Quantized Control for Input-to-State Stabilization of Linear Systems With Distributed Output Sensors," *IEEE Transactions on Automatic Control*, vol. 64, no. 12, pp. 4952–4967, 2019.