Symmedian Point

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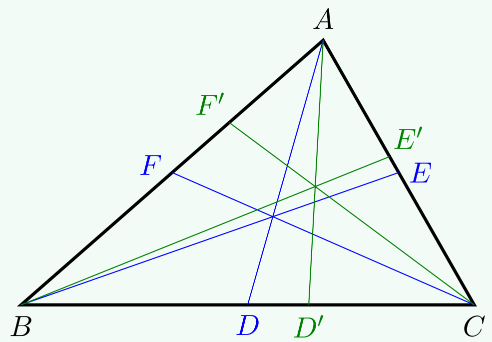
1 Introduction

Symmedian Point is the intersection of three *Symmedians* of a triangle. *Symmedians* are lines that are isogonal to the triangle's medians. *Émile Lemoine*, a french mathematician, discovered symmedian point in 1873. So sometimes it is also called *Lemoine Point*.

Definition 1. (Symmedians)

In a triangle, the Symmedians are the lines isogonal(Topic 7) to the medians. In the following $\triangle ABC$, let AD, BE, CF be the medians on the respective sides. The lines AD', BE', CF' are isogonal to AD, BE, CF respectively, which are $\angle D'AC = \angle BAD, \angle E'BA = \angle CBE, \angle F'CA = \angle FCB$, and therefore are called the triangle's symmedians. [a]

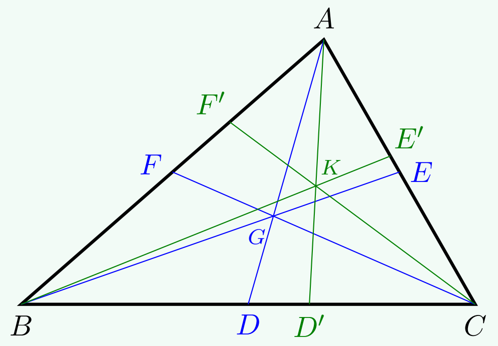


[a] X(6): Symmedian Point is the 6th point in the Encyclopedia of Triangle Centers while the first point is "incenter" and the second point is "centroid"

Definition 2. (Symmedian Point)

Symmedian Point is intersection of the symmedians. By Topic 7, it is isogonal conjugate point of the centroid. In the following picture, let G be the centroid of $\triangle ABC$, and let K be the symmedian point. Then G and K are a part of isogonal conjugate points.

[1] The author thanks Dr. Zhiqin Lu for his help.



Theorem 1

Three symmedians of a triangle are concurrent.

Proof: From theorem 1 in Topic 3 and Ceva's Theorem, three medians AD, BE, CF are concurrent at $\triangle ABC$'s centroid G and

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.$$

By definition of the isotomic conjugate lines, we have

$$\frac{BD'}{D'C} = \frac{DC}{BD} = \left(\frac{BD}{DC}\right)^{-1}.$$

Similarly, we have

$$\frac{CE'}{E'A} = \left(\frac{CE}{EA}\right)^{-1}, \qquad \frac{AF'}{F'B} = \left(\frac{AF}{FB}\right)^{-1}.$$

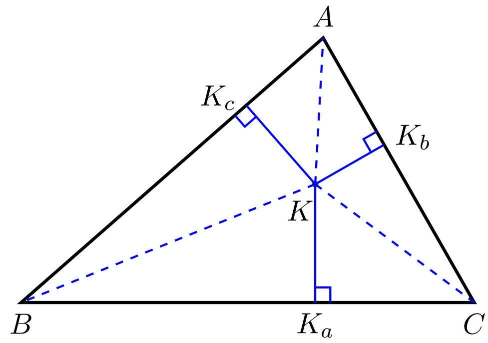
Thus we have

$$\frac{BD'}{D'C} \cdot \frac{CE'}{E'A} \cdot \frac{AF'}{F'B} = \left(\frac{BD}{DC}\right)^{-1} \cdot \left(\frac{CE}{EA}\right)^{-1} \cdot \left(\frac{AF}{FB}\right)^{-1} = 1,$$

and hence AD', BE', CF' are concurrent at symmedian point K. ∎

2 Properties of Symmedian Point

Here are some basic properties of the symmedian point. Let us make a line through K so that it is perpendicular to BC and name its intersection with BC K_a. Similarly, we make K_b, K_c, as shown in the figure below.



Let S be the area of $\triangle ABC$, then we have

$$KK_a = \frac{2 \cdot a \cdot S}{a^2 + b^2 + c^2}$$

$$KK_b = \frac{2 \cdot b \cdot S}{a^2 + b^2 + c^2}$$

$$KK_c = \frac{2 \cdot c \cdot S}{a^2 + b^2 + c^2}$$

From formulas above, we have

$$KK_a : KK_b : KK_c = a : b : c$$

Areas of $\triangle BKC, \triangle AKC, \triangle AKB$ will be:

$$S_{\triangle BKC} = \frac{a^2 \cdot S}{a^2 + b^2 + c^2}$$

$$S_{\triangle AKC} = \frac{b^2 \cdot S}{a^2 + b^2 + c^2}$$

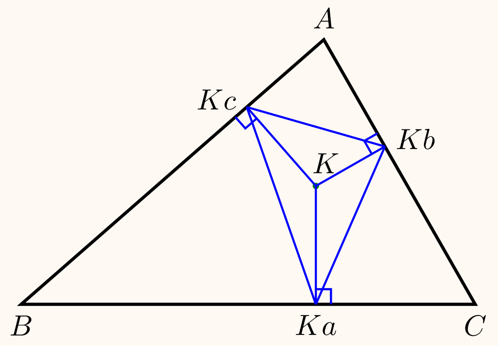
$$S_{\triangle AKB} = \frac{c^2 \cdot S}{a^2 + b^2 + c^2}$$

$$S_{\triangle BKC} : S_{\triangle AKC} : S_{\triangle AKB} = a^2 : b^2 : c^2$$

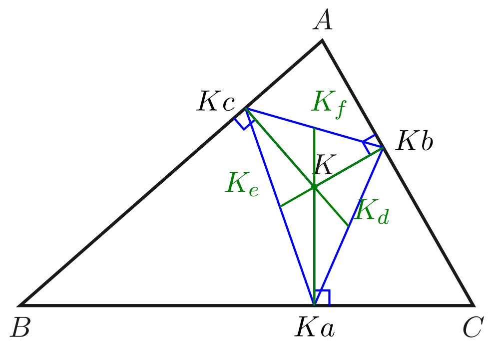
Therefore K has trilinear coordinates $a : b : c$ and barycentric coordinates $a^2 : b^2 : c^2$.

Lemma 1

$\triangle K_aK_bK_c$ is a pedal triangle of $\triangle ABC$ with orthocenter at K. K, in this case, is centroid of $\triangle K_aK_bK_c$.



Proof: Connect K_a and k and extend K_aK so that it intersects with K_cK_b at Kf. Similarly, we can make K_d, K_e.



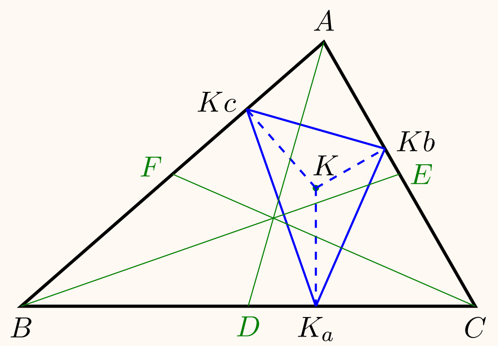
We first prove that K_fK_a is one median of $\triangle K_aK_bK_c$.

$$\frac{K_cK_f}{K_fK_b} = \frac{KK_c}{KK_b} \cdot \frac{\sin(\angle AKK_b)}{\sin(\angle AKK_c)} = \frac{AC}{AB} \cdot \frac{\sin(\angle C)}{\sin(\angle B)} = 1$$

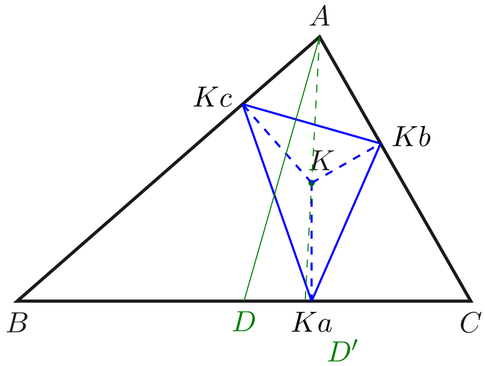
Therefore, K_fK_a is one median of $\triangle K_aK_bK_c$ and we can prove K_eK_b, K_cK_d are medians of $\triangle K_aK_bK_c$ in a similar way. ■

Lemma 2

Sides of $\triangle K_aK_bK_c$ are perpendicular to the three medians of $\triangle ABC$.



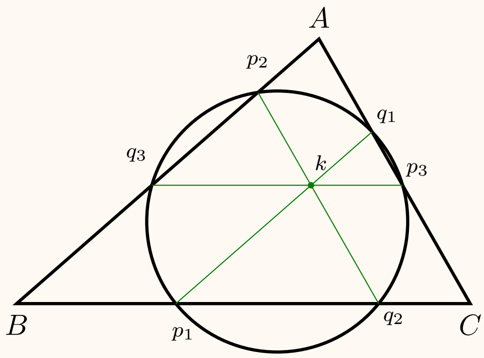
Proof:



Since $\angle AK_cK = \angle AK_bK$=90°, the four points A, Kb, K, Kc are con-cylic. Then $\angle AK_cKb = \angle AKK_b$. Since $\angle BAD = \angle CAD'$, $\angle AxK = \angle AK_cK = 90°$. Hence K_bK_c is perpendicular to AD. Similarly, K_aK_b is perpendicular to CF and K_aK_c is perpendicular to BE.

Theorem 2. (First Lemoine Circle)

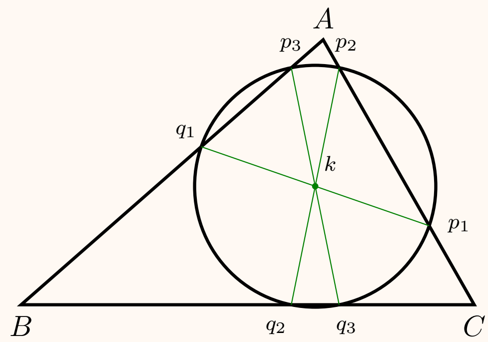
Let K be Symmedian Point of $\triangle ABC$, draw a line that is parallel to AB interacts BC, AC at point p_1, q_1. Similarly, draw a line that is parallel to AC interacts BC, AB at point p_2, q_2 and draw a line that is parallel to AC interacts AB, BC at point p_3, q_3. $p_1, p_2, p_3, q_1, q_2, q_3$ form a circle and is named the First Lemoine Circle.



For a proof of the theorem, see Topic 17. There is also the Second Lemoine Circle

Theorem 3. (Second Lemoine Circle)

Let K be Symmedian Point of $\triangle ABC$, draw a line that is antiparallel to AB interacts BC, AC at point p_1, q_1. Similarly, draw a line that is antiparallel to AC interacts BC, AB at point p_2, q_2 and draw a line that is antiparallel to AC interacts AB, BC at point p_3, q_3. $p1, p2, p3, q1, q2, q3$ form a circle and is named the Second Lemoine Circle. ALso known as cosine circle



For a proof of the theorem, see Topic 17.

Theorem 4. (Lemoine line)

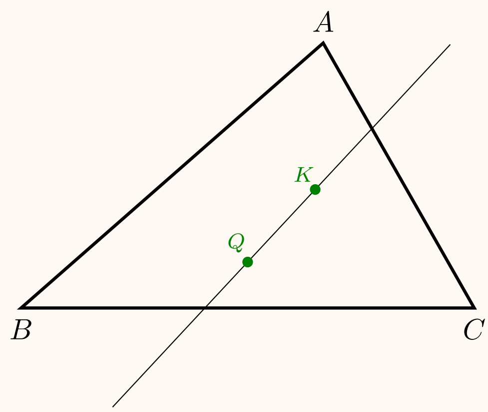
Lemonine Line, or the Lemonine Axis is the line passing through three intersections of three tangent lines to a triangle's incircle and three extension lines of a triangle's sides. The proof of why three points are collinear to form a Lemonine line is proven in Topic 18.

Lemma 3

The Brocard axis is perpendicular to the Lemoine line

Theorem 5. (Brocard Axis)

Brocard axis is the line passing through a triangle's symmedian point(K) and circumcenter(Q).(Circumcenter is the center of a triangle's circumcircle, which discussed in detail in Topic 3.) $P'Q$ is the Brocard diameter.



Warning: Brocard axis is not the same as Brocard line

Lemma 4

The Symmedian Point, Circumcenter, First an the Second Isodynamic Points, and Brocardmidpoint(Topic 25) all lie along the Brocard axis.
[a]

[a]The Brocard Midpoint has the trilinear coordinates

$$(a(b^2+c^2), b(c^2+a^2), c(a^2+b^2)).$$

The first isodynamic point has the trilinear coordinates

$$(\sin(\alpha+\pi/3), \sin(\beta+\pi/3), \sin(\gamma+\pi/3)),$$

and the second isodynamic point has the trilinear coordinates

$$(\sin(\alpha-\pi/3), \sin(\beta-\pi/3), \sin(\gamma-\pi/3)).$$

Here α, β, γ are the three angles of triangle ABC.

Proof: To prove that these 5 points are collinear, we have to take groups of three at a time and prove them collinear individually.

First, we show that the symmedian pointand the isodynamic points are collinear. To do that, we need to show the determinant of the following matrix is zero:

$$\det\begin{bmatrix} \sin(\alpha+\pi/3) & \sin(\beta+\pi/3) & \sin(\gamma+\pi/3) \\ \sin(\alpha-\pi/3) & \sin(\beta-\pi/3) & \sin(\gamma-\pi/3) \\ \sin\alpha & \sin\beta & \sin\gamma \end{bmatrix}.$$

Using the sum to product identity for sin, we add the second equation to the first to get

$$\det\begin{bmatrix} 2\sin\alpha\cdot\cos\frac{\pi}{3} & 2\sin\beta\cdot\cos\frac{\pi}{3} & 2\sin\gamma\cdot\cos\frac{\pi}{3} \\ \sin(\alpha-\pi/3) & \sin(\beta-\pi/3) & \sin(\gamma-\pi/3) \\ \sin\alpha & \sin\beta & \sin\gamma \end{bmatrix}.$$

Then, divide the first row by $2\cos\frac{\pi}{3}$ to get

$$\det\begin{bmatrix} \sin\alpha & \sin\beta & \sin\gamma \\ \sin(\alpha-\pi/3) & \sin(\beta-\pi/3) & \sin(\gamma-\pi/3) \\ \sin\alpha & \sin\beta & \sin\gamma \end{bmatrix}.$$

Since two rows are the same, the determinant of this matrix is 0. As such, the symmedian point is collinear with the first and second isodynamic points.

Next, we show that the two isodynamic points are collinear with the circumcenter.

$$\det\begin{bmatrix} \sin(\alpha+\pi/3) & \sin(\beta+\pi/3) & \sin(\gamma+\pi/3) \\ \sin(\alpha-\pi/3) & \sin(\beta-\pi/3) & \sin(\gamma-\pi/3) \\ \cos\alpha & \cos\beta & \cos\gamma \end{bmatrix}.$$

Now, we use the sum to product identity and subtract the second row from the first to get

$$\det\begin{bmatrix} 2\cos\alpha\cdot\sin\frac{\pi}{3} & 2\cos\beta\cdot\sin\frac{\pi}{3} & 2\cos\gamma\cdot\sin\frac{\pi}{3} \\ \sin(\alpha-\pi/3) & \sin(\beta-\pi/3) & \sin(\gamma-\pi/3) \\ \cos\alpha & \cos\beta & \cos\gamma \end{bmatrix}.$$

Dividing both sides of the first row by $2\sin\frac{\pi}{3}$ gives

$$\det\begin{bmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \sin(\alpha-\pi/3) & \sin(\beta-\pi/3) & \sin(\gamma-\pi/3) \\ \cos\alpha & \cos\beta & \cos\gamma \end{bmatrix}.$$

Again, since two rows of the matrix are the same, the determinant is 0, which shows that The two isodynamic points are collinear with the circumcenter.

Since the isodynamic points are collinear with the symmedian point (see Problem 7) and the isodynamic points are also collinear with the circumcenter, these 4 points are all collinear.

Finally, we need to prove that the Brocard Midpoint is collinear with the symmedian point and the circumcenter.

$$\det\begin{bmatrix} \sin\alpha & \sin\beta & \sin\gamma \\ \cos\alpha & \cos\beta & \cos\gamma \\ a(b^2+c^2) & b(c^2+a^2) & c(a^2+b^2) \end{bmatrix}.$$

Applying the law of cosines to the second row, applying the law of sines to the first row and multiplying $2abc$ gives

$$\det\begin{bmatrix} a & b & c \\ a(b^2+c^2-a^2) & b(c^2+a^2-b^2) & c(a^2+b^2-c^2) \\ a(b^2+c^2) & b(c^2+a^2) & c(a^2+b^2) \end{bmatrix}.$$

Now, we multiply the third row by 2 and subtract row 2 from it to get

$$\det\begin{bmatrix} a & b & c \\ a(b^2+c^2-a^2) & b(c^2+a^2-b^2) & c(a^2+b^2-c^2) \\ a(a^2+b^2+c^2) & b(a^2+b^2+c^2) & c(a^2+b^2+c^2) \end{bmatrix}.$$

Divide the third row by $a^2+b^2+c^2$ to get

$$\det\begin{bmatrix} a & b & c \\ a(b^2+c^2-a^2) & b(c^2+a^2-b^2) & c(a^2+b^2-c^2) \\ a & b & c \end{bmatrix}.$$

Since the first and the third row are the same, the determinant is 0, which shows that the symmedian point, the circumcenter, and the Brocard midpoint are collinear.

Thus, the symmedian point, the circumcenter, the Brocard midpoint, and the first and second isodynamic points are collinear.

■