A Survey of Shallow Water Equations Finite Volume Methods Neural Network Riemann Solvers Physics-Informed Machine Learning Conservation Law Modeling and Computational Fluid Dynamics

Abstract

The integration of mathematical modeling, numerical analysis, and machine learning has significantly advanced fluid dynamics, particularly through the synergy of shallow water equations, finite volume methods, neural network Riemann solvers, physics-informed machine learning, conservation law modeling, and computational fluid dynamics (CFD). This multidisciplinary approach enhances simulation accuracy and efficiency, addressing complex fluid dynamics scenarios by offering robust solutions that improve predictive capabilities and computational performance. Notable advancements include the development of high-order structure-preserving finite volume schemes, the integration of neural network Riemann solvers, and the application of physics-informed frameworks, which collectively enhance the fidelity of fluid dynamics simulations. Challenges remain in improving boundary layer resolution, handling wetting and drying processes, and optimizing computational efficiency. Future research should focus on refining these methodologies, exploring their applications in diverse environmental contexts, and integrating real-time data to enhance predictive accuracy. By continuing to explore the potential of combining mathematical modeling, numerical analysis, and machine learning, researchers can further advance the field of fluid dynamics, addressing complex challenges and expanding the applicability of these techniques to new and emerging areas.

1 Introduction

1.1 Multidisciplinary Approach in Fluid Dynamics

The integration of diverse disciplines in fluid dynamics has propelled significant advancements, particularly through the synergy of mathematical modeling, numerical methods, and machine learning. Notably, the combination of machine learning with traditional computational fluid dynamics (CFD) solvers enhances simulation efficiency, demonstrating the complementary nature of these methodologies [1]. This coupling not only improves simulation accuracy but also addresses complex fluid dynamics challenges, as evidenced by the integration of finite volume methods with neural networks [2].

Deep Neural Networks (DNNs) represent a substantial advancement in applying AI to scientific domains, effectively solving irregular Partial Differential Equations (PDEs) such as shock fronts [3]. Hybrid ML-PDE solvers that rectify errors from coarse-grid simulations significantly enhance the accuracy of high-resolution field estimations [4].

The optimization of data processing through machine learning underscores the interplay between computational efficiency and resource management, crucial for advancing fluid dynamics research.

Identifying areas where machine learning can enhance CFD—such as simulation speed and reduced-order models—further elucidates its transformative potential [1]. Additionally, the integration of neural operators with traditional numerical methods for solving PDEs exemplifies this multidisciplinary approach, enhancing both accuracy and efficiency in simulations.

The necessity for a multidisciplinary approach that incorporates deep learning with physical simulation data arises from the limitations of current machine learning models, which often struggle to predict complex physical systems accurately. By leveraging physics-informed machine learning methods that integrate noisy empirical data with established physical laws, challenges like high-dimensional parameterization and mesh generation complexities can be addressed. Specialized neural network architectures that respect physical invariants can improve training efficiency and generalization, leading to more reliable predictions across various applications in physics and engineering [5, 6]. These advancements illustrate the transformative potential of integrating mathematical modeling, numerical analysis, and machine learning in fluid dynamics.

1.2 Relevance and Applications

The integration of physics-informed machine learning (PIML) within fluid mechanics underscores the practical significance of this survey's core topics, particularly in complex turbulent flow scenarios. PIML enhances predictive capabilities vital for climate modeling and other domains requiring precise fluid dynamics simulations [7]. This survey also reviews multiscale modeling approaches in fluid dynamics and thermal hydraulics, emphasizing the synergy between physics-based methodologies and data-driven techniques [8].

The application of DNNs to PDEs provides both theoretical insights and practical applications in scientific research and engineering contexts [9]. The development of physics-informed neural operators (PINOs) further enhances predictive performance in solving PDEs, demonstrating their significance across various physical contexts. The intersection of machine learning (ML) and CFD plays a pivotal role in accelerating direct numerical simulations (DNS), improving turbulence closure models, and developing reduced-order models (ROMs) [10]. These advancements indicate ML's potential to revolutionize fluid dynamics by optimizing simulation speed and accuracy.

Moreover, addressing knowledge gaps in ML applications in CFD—including forward modeling, inverse design, and control methods—is crucial [1]. The optimization of machine learning algorithms extends beyond fluid dynamics, impacting sectors such as healthcare, finance, and autonomous systems. The proposed Dynamic Resource Allocation Framework (DRAF) exemplifies this by enhancing performance and scalability in large-scale machine learning applications [11].

The integration of deep learning techniques with physical simulation data, as seen in hybrid models, enhances predictive performance in fields such as protein structure prediction [12]. This illustrates the transformative potential of combining mathematical modeling, numerical analysis, and machine learning across diverse scientific and engineering disciplines. The proposed Distributed Physics Informed Neural Network (DPINN) aims to enhance existing methods' flexibility and capability by incorporating physics-based constraints, further underscoring these advancements' practical significance [4].

1.3 Structure of the Survey

This survey is systematically structured to explore the multidisciplinary approach in fluid dynamics, focusing on the integration of mathematical modeling, numerical analysis, and machine learning techniques. It begins with an introduction that establishes the synergy between these disciplines and their collective impact on advancing fluid dynamics research. Following the introduction, a detailed background section defines core concepts and terminologies, laying a foundation for subsequent discussions.

The survey then examines specific areas, starting with a thorough analysis of shallow water equations, including their formulation, numerical methods, applications, challenges, and future research directions. This section is crucial for understanding the role of shallow water equations in modeling fluid motion with a free surface.

Next, the focus shifts to finite volume methods, detailing their application in computational fluid dynamics. This section discusses high-order schemes, well-balanced properties, handling complex geometries, entropy stability, and recent innovations in numerical flux and reconstruction techniques, providing insights into their advantages over other numerical methods.

The survey continues with an exploration of neural network Riemann solvers, detailing their introduction, data-driven approaches, integration with traditional numerical methods, and innovative architectures. This section emphasizes the potential of neural networks in approximating complex solution behaviors and enhancing numerical solvers.

Following this, the paper discusses physics-informed machine learning, bridging physical laws with data-driven models to improve simulation accuracy and reliability. The section on physics-informed neural networks and their applications in turbulence modeling further exemplifies the integration of machine learning with physical constraints.

Conservation law modeling is then examined, highlighting its importance in adhering to fundamental conservation principles. This section explores numerical approaches, techniques for handling discontinuities and shocks, and applications in fluid dynamics.

The penultimate section provides an overview of computational fluid dynamics as a tool for simulating fluid interactions, discussing the integration of shallow water equations, finite volume methods, neural network Riemann solvers, and physics-informed machine learning within the CFD context. Challenges and future directions in CFD are also identified.

Finally, the survey concludes by summarizing key points and emphasizing the significance of the multidisciplinary approach in advancing fluid dynamics. The potential for future research and development is highlighted, focusing on the transformative impact of integrating mathematical modeling, numerical analysis, and machine learning in fluid dynamics. This integration could enhance CFD by improving simulation accuracy, accelerating computational processes, and enabling complex analyses of turbulent flows. Specifically, machine learning techniques, including physics-informed machine learning, hold the potential to refine turbulence models, develop reduced-order models, and facilitate real-time applications across various scientific fields such as aerodynamics and plasma physics. Addressing key challenges like multi-scale representation and encoding physical knowledge will enable future studies to leverage these advanced computational methods to overcome traditional numerical simulation limitations [1, 7, 13, 9, 10].The following sections are organized as shown in Figure 1.

2 Background and Definitions

2.1 Background and Definitions

Fluid dynamics is founded on several key concepts and terminologies crucial for its multidisciplinary exploration. The shallow water equations, a fundamental set of partial differential equations (PDEs), model fluid motion with a free surface, as seen in oceans and rivers, playing a significant role in environmental and engineering applications by effectively simulating wave propagation and related phenomena [9].

Finite volume methods are vital numerical techniques used in solving PDEs, ensuring the conservation of mass, momentum, and energy across discrete volumes. These methods are integral to computational fluid dynamics (CFD), particularly effective in simulating fluid interactions within complex geometries and unstructured meshes [14]. Their high-order schemes and well-balanced properties enhance the accuracy and efficiency of simulations, addressing computational challenges in traditional CFD approaches [10].

Neural network Riemann solvers exemplify the integration of machine learning with traditional numerical methods, enhancing the accuracy and efficiency of fluid dynamics simulations. This approach manages high-dimensional data and reduces computational costs, demonstrating the synergy between machine learning and numerical analysis [15].

Physics-informed machine learning (PIML) embeds physical laws into data-driven models, significantly enhancing predictive capabilities. This framework is particularly effective for simulating complex turbulent flows and addressing scaling gaps in fluid flow simulations, bridging theoretical physics with practical applications [8, 7].

Conservation law modeling is crucial for maintaining core conservation principles in fluid dynamics. It involves developing numerical methods that uphold the invariance of physical quantities—such as mass, momentum, and energy—across computational domains, employing advanced techniques to manage discontinuities and shocks, thereby ensuring stability and accuracy in simulations [16].

CFD provides a comprehensive framework for simulating fluid interactions across various engineering and scientific applications, integrating shallow water equations, finite volume methods, and neural network Riemann solvers to offer detailed insights into fluid behavior. The complexity of generating numerical solvers from differential equations highlights the importance of these concepts for both novice and experienced practitioners [17]. Additionally, the application of AI libraries in solving PDEs underscores the limitations of traditional methods and the need for more efficient, architecture-agnostic solutions [18].

These foundational concepts and terminologies support fluid dynamics research, enabling the simulation and analysis of complex fluid systems. Their integration within a multidisciplinary framework underscores the transformative potential of combining mathematical modeling, numerical analysis, and machine learning to advance the field [1].

3 Shallow Water Equations

The shallow water equations (SWEs) are pivotal in fluid dynamics, modeling water flows across diverse terrains. This section delves into the core aspects of SWEs, focusing on their formulation and numerical methods, highlighting their significance in practical modeling scenarios. Figure 2 illustrates the hierarchical structure of SWEs, encompassing their formulation and numerical methods, applications in environmental and engineering contexts, challenges and limitations, and recent advancements with future directions. The subsequent subsection provides a detailed analysis of SWEs, emphasizing recent advancements that enhance their applicability in various contexts.

3.1 Formulation and Numerical Methods for Shallow Water Equations

SWEs are essential for modeling free-surface fluid dynamics, particularly in coastal and riverine environments. Typically formulated as hyperbolic PDEs, they derive from conservation laws of mass and momentum, using depth-averaged quantities to capture fluid dynamics. Recent advancements, such as residual distribution schemes, have improved unsteady problem modeling and facilitated sophisticated numerical discretizations [19, 20]. Covariant forms often integrate along local normals to accurately depict flow dynamics over complex terrains.

Numerical methods for SWEs have evolved to enhance simulation precision and address computational challenges. The Well-Balanced Positive-Preserving Scheme (WB-PPS) maintains steady states and positive water depth, crucial for stability and realism [21]. The adNOC scheme balances flux gradients and source terms, enhancing solution robustness [22]. Finite volume methods, such as the Automatically Well-Balanced Pressure Forcing (AWBPF) method, offer a well-balanced formulation under variable topography [23]. The Newton multigrid method (NMGM) improves convergence by applying Newton's iteration to nonlinear algebraic systems from finite volume discretization [24].

High-order numerical techniques, inspired by penalization methods, enhance accuracy in scenarios with stiff friction [25]. The Structure-preserving split finite element method (SPSFEM) maintains physical structures while simplifying computations [26]. Quasi-two-layer models improve depth-averaged solutions of the Euler equations, offering nuanced fluid dynamics understanding [27]. Machine learning integration, such as hybrid quantum physics-informed neural networks, combines classical and quantum methods for efficient fluid dynamics solutions [28].

The hierarchical structure of SWEs formulation and numerical methods is illustrated in Figure 3, which categorizes key advancements into formulation techniques, numerical methods, and advanced techniques. This figure highlights recent innovations and their contributions to fluid dynamics modeling, providing a visual representation that complements the textual discussion.

Advancements in SWEs formulation and numerical methods reflect efforts to enhance simulation accuracy, stability, and applicability in environmental and engineering contexts. Libraries like SWASHES provide benchmark solutions for validating numerical methods, and innovative solvers,

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including second-order discontinuous Galerkin methods, offer superior predictive capabilities for shallow flows [29, 30, 31, 32].

3.2 Applications in Environmental and Engineering Contexts

SWEs are crucial in environmental and engineering applications, modeling fluid dynamics in free surface contexts like coastal and riverine systems. They are valuable for predicting flow dynamics around structures such as bridge piers, where traditional methods may be too resource-intensive. Surrogate models based on SWEs have demonstrated accuracy and efficiency in these scenarios [33].

In urban planning and disaster management, SWEs simulate urban flooding scenarios, enhancing flood risk assessments and mitigation strategies [34]. Modeling tsunami-induced flooding relies on SWEs to capture debris dynamics and evaluate potential risks [35]. SWEs accommodate diverse flow conditions, incorporating factors like rain and soil friction for realistic simulations, adaptable to various environmental contexts [36].

In engineering, SWEs infer underwater topography from wave dynamics, reconstructing hidden structures based on partial data [2]. By leveraging SWEs' predictive power, engineers enhance underwater survey accuracy and improve maritime infrastructure design and maintenance.

SWEs are essential for understanding and managing fluid dynamics in natural and built environments, critical for infrastructure and human safety. Recent advancements include a compilation of analytic solutions to SWEs, serving as a benchmark library for hydraulic and environmental studies. Innovative formulations adapt these equations for complex terrains, emphasizing bottom geometry's significance in flow behavior [37, 29].

3.3 Challenges and Limitations

SWEs in fluid dynamics face challenges impacting simulation accuracy and efficiency. Accurately representing discontinuities and shock waves is significant, as traditional methods like finite volume and discontinuous Galerkin often struggle with incompressibility constraints, necessitating complex stabilization techniques [38]. These methods frequently fail to incorporate manifold curvature, leading to inaccuracies in complex domains [39].

Unphysical Gibbs oscillations occur in numerical solutions with sharp gradients, exacerbated by SWEs' nonlinearity and non-divergent properties [40]. Balancing flux gradients and source terms can yield unphysical results, especially with mobile wet-dry boundaries [22]. Methods often produce spurious solutions due to static forcing terms from pressure representation [23].

SWEs' computational complexity, particularly in maintaining hyperbolicity, limits effectiveness in practical applications. The curse of dimensionality exacerbates this issue, as increased resolution leads to exponentially growing computational demands [41]. Convergence of numerical methods, especially for long-time simulations with explicit methods, is challenging [24]. Preserving potential vorticity in numerical schemes is believed unattainable due to conservation laws' complexities in discrete systems [42].

Despite advancements in hybrid quantum physics-informed neural networks (HQPINNs), challenges like gradient vanishing and scalability limitations persist [28]. Continued research is needed to develop advanced numerical techniques, improving SWEs models' reliability and applicability.

3.4 Advancements and Future Directions

Recent SWEs advancements have improved simulation precision and robustness. The Neural Particle Method (NPM) uses a neural network to approximate velocity and pressure fields, enabling incompressible fluid flow simulation without a fixed mesh, enhancing flexibility and reducing costs [38]. Reduced-order models (ROMs) preserve conserved quantities while lowering computational expenses [43].

High-order numerical methods have progressed, preserving asymptotic behavior and capturing steady states accurately, validated through extensive tests [25]. The adNOC scheme manages challenges with discontinuous topography, ensuring well-balanced conditions [22].

High-order entropy stable schemes, like the entropy-stable nodal discontinuous Galerkin method, integrate energy and entropy conservation, enhancing robustness against oscillations [40]. The Dyn-SGS model stabilizes high-order Galerkin methods, eliminating Gibbs oscillations [40].

Innovations in computational frameworks include the Automatically Well-Balanced Pressure Forcing method, improving shallow water dynamics modeling under variable topography [23]. Noether identities allow potential vorticity conservation in discrete schemes, converging to smooth conservation laws [42].

Future research will explore these advancements' real-world implications, particularly in developing numerical methods for complex fluid flows. Enhancing existing methods for two-dimensional problems and refining transversal velocity components are promising study avenues [27]. Group classification techniques offer a novel approach to achieving accurate and efficient simulations.

SWEs advancements, including a new friction model with an interactive viscous layer and complex terrain-adapted formulation, highlight ongoing research efforts to enhance fluid dynamics modeling accuracy and efficiency. These innovations pave the way for improved applications across environmental and engineering contexts, addressing critical factors like friction and topography interplay [37, 44].

4 Finite Volume Methods

4.1 High-Order Schemes and Well-Balanced Properties

Method Name	Numerical Techniques	Conservation Properties	Integration Methods
mPDeC-WENO[45]	Weno Reconstruction	Unconditional Positivity Preservation	Patankar Deferred Correction
FVGC[46]	Graph Convolutions	Mass, Momentum, Energy	Graph Convolutions
NPM[38]	Runge Kutta Methods	Measuring Conservation Properties	Runge Kutta Methods
Dyn-SGS[40]	High-order Approximations	Mass, Momentum, Energy	Three-stage Esdirk
FENI[42]	Noether's Theorem	Potential Vorticity	Variational Problems

Table 1: Comparison of various high-order numerical methods for finite volume simulations, detailing their numerical techniques, conservation properties, and integration methods. This table highlights the diversity of approaches in achieving accuracy and stability in fluid dynamics simulations through different conservation and integration strategies.

Finite volume methods (FVMs) are pivotal in simulating fluid dynamics governed by conservation laws, necessitating high-order schemes and well-balanced properties for accurate mass, momentum, and energy conservation. High-order Weighted Essentially Non-Oscillatory (WENO) schemes, like the mPDeC-WENO method, exemplify strategies enhancing precision and stability through high-order reconstruction and modified Patankar Deferred Correction time integration, reducing numerical dissipation while preserving sharp gradients [45]. Structure-preserving and entropy-stable schemes, such as the dual-pairing (DP) summation-by-parts (SBP) finite difference framework, integrate upwind features and entropy stability, ensuring robust solutions in complex flows [47]. Table 1 presents a comparative analysis of high-order numerical methods used in finite volume simulations, emphasizing their distinct numerical techniques, conservation properties, and integration methods.

Integrating geometric representations with finite volume methods enhances computational fluid dynamics (CFD) accuracy, illustrating the synergy between geometry and numerical techniques [46]. Mixed finite elements conserve energy and enstrophy, enhancing stability, while parameterization schemes maintain conservation laws and symmetries, improving model consistency [48, 49]. The MUSCL-Hancock approach advances fluid dynamics by evolving physical and metric variables within a hyperbolic framework [39].

High-order implicit Runge Kutta methods for temporal integration in the Neural Particle Method (NPM) combine machine learning with traditional numerical methods, offering accurate fluid dynamics simulations [38]. The Dyn-SGS model stabilizes high-order solutions of shallow water equations by dynamically adjusting viscosity based on solution residuals, mitigating Gibbs oscillations [40]. The Automatically Well-Balanced Pressure Forcing method ensures well-balanced conditions without further adjustments, while the Newton multigrid method (NMGM) efficiently addresses nonlinearities and achieves convergence [23, 24].

Frameworks like FENI leverage Lagrangian invariance to derive conservation laws in numerical schemes, preserving fundamental physical properties [42]. These advancements in high-order schemes and well-balanced properties within FVMs underscore their critical role in preserving physical properties and enhancing numerical accuracy, providing robust tools for scientific and engineering applications.

4.2 Handling Complex Geometries and Unstructured Meshes

Finite volume methods adeptly address challenges posed by complex geometries and unstructured meshes in fluid dynamics simulations, ensuring accurate flow representation across diverse topographies. Utilizing a two-dimensional discrete de Rham complex for moist shallow water equations effectively couples moisture and buoyancy variables, enhancing simulation of intricate physical processes [50]. Techniques combining centered and upwinded fluxes maintain energy conservation and control entropy dynamics, crucial for stability and accuracy in simulations with complex geometries [51]. The PCCU-AENO method, employing a central-upwind approach with AENO reconstruction, achieves second-order accuracy in complex geometries [52].

Multi-GPU strategies integrating CUDA and MPI enhance processing efficiency for larger meshes and complex simulations, significantly outperforming traditional CPU methods [13, 53]. This advancement benefits high-resolution simulations over extensive domains, such as tsunami modeling, where Lagrangian remapping techniques are combined with finite volume methods. The Reduced-Order Correction Method (ROCM) reduces simulation errors on coarse grids, maintaining efficiency essential for unstructured mesh simulations [54]. Machine learning enhances Reynolds-averaged Navier-Stokes (RANS) and large-eddy simulations (LES), improving closure models and accuracy for complex geometries [10].

Numerical tests across finite element spaces on triangular and quadrilateral meshes demonstrate robustness in resolving complex flow patterns, establishing a reliable framework for meteorological and environmental applications [55]. These techniques highlight the versatility and capability of finite volume methods in managing complexities associated with complex geometries and unstructured meshes, paving the way for accurate and efficient fluid dynamics simulations.

4.3 Entropy Stability and Conservation Properties

Entropy stability and conservation properties are fundamental to finite volume methods, ensuring numerical simulations accurately reflect the governing physical laws. High-order schemes, like the Entropy Stable Discontinuous Galerkin Spectral Element Method (ESDGSEM), integrate high-order accuracy with entropy stability, adeptly handling sharp gradients and discontinuities [56]. Achieving entropy stability and conservation poses challenges in maintaining numerical stability and convergence amid discontinuities, often leading to oscillations and inaccuracies. The hybrid FD-FV method offers a compelling solution, achieving higher-order accuracy with a compact formulation that reduces computational overhead while maintaining stability [57].

Additional dissipation mechanisms enhance stability and performance by ensuring positivity preservation, crucial for robust simulations with sharp gradients [58]. The geometrically intrinsic Lagrangian-Eulerian scheme eliminates the need for Riemann solvers, reducing numerical dissipation and maintaining accuracy amidst complex geometries [59]. This method accounts for junction geometric configurations, influencing flow characteristics and ensuring conservation laws are satisfied [60].

Advancements like broad-class boundary conditions enhance convergence properties and reduce drift-off errors, suitable for a wider range of applications [61]. The learned conservative semi-Lagrangian finite volume method reduces computational complexity while improving accuracy in capturing fine-scale features, maintaining mass conservation without extensive upstream cell tracking [62]. The mPDeC-WENO method stands out for its unconditional positivity preservation and high accuracy, circumventing stringent CFL restrictions [45]. Employing convolutional neural networks (CNNs) for shock detection minimizes false positives, underscoring the significance of entropy stability and conservation properties in finite volume methods [63].

Advancements in entropy stability and conservation properties within finite volume methods underscore their critical role in enhancing numerical simulation performance and reliability. These properties provide robust tools for scientific and engineering applications, ensuring simulations accurately reflect fluid dynamics systems' physical behavior. The development of methods that reduce numerical dissipation while maintaining accuracy, such as deepMTBVD, further illustrates the potential for innovation in this field [64].

4.4 Innovations in Numerical Flux and Reconstruction Techniques

Innovations in numerical flux and reconstruction techniques within finite volume methods have significantly enhanced the accuracy and stability of fluid dynamics simulations. The integration of convolutional neural networks (CNNs) as multigrid solvers exemplifies a novel approach to enforcing incompressibility constraints in fluid dynamics, enhancing computational efficiency while maintaining precision [18]. High-order methods, such as the WENO finite difference method, have been refined to eliminate order-dependent corrections and avoid computing integrals involving quadrature points, streamlining the numerical process [65]. The DG2 method contributes by producing more accurate and efficient velocity field predictions without additional calibration efforts associated with eddy viscosity terms [66].

The introduction of flux-conserving differential operators that maintain local conservation without relying on global mesh structures represents a significant innovation in mesh-free methods [67]. This approach ensures the preservation of stationary solutions, crucial for maintaining the physical fidelity of simulations. The ML-SL-FV method, which integrates machine learning to replace traditional complex upstream cell tracking with a learned discretization, achieves sharp shock transitions and high accuracy without necessitating a finer grid [62].

Innovations integrating tensor-train decomposition with high-order reconstruction schemes, such as Upwind and WENO, achieve significant computational speed-ups while maintaining traditional methods' accuracy [41]. This highlights the importance of developing methods that extend traditional schemes' accuracy without incurring significant computational costs. The second-order finite volume scheme effectively models shallow water equations over manifolds, allowing for the accurate evolution of both physical and geometric variables [39].

The dynamic viscosity approach, which adjusts based on the governing equations' residuals, differs from traditional fixed or parameterized diffusion methods, providing a more responsive and accurate simulation environment [40]. Additionally, the use of physics-informed neural networks and DeepONets to learn solutions to shallow-water equations without a specific inundation model simplifies modeling processes and enhances the applicability of these techniques [68].

These innovations in numerical flux and reconstruction techniques reflect ongoing efforts to refine finite volume methods, providing more accurate and efficient tools for simulating complex fluid dynamics. Recent advancements in machine learning techniques are significantly transforming the field of Computational Fluid Dynamics (CFD) by enhancing simulation accuracy, reducing computational time, and enabling complex analyses across various scientific and engineering applications, such as aerodynamics, combustion, and multiphysics simulations. These innovations include the development of data-driven and physics-informed models that improve turbulence predictions and streamline the solution of partial differential equations, thereby addressing longstanding challenges in the field [5, 1, 69, 9, 10].

5 Neural Network Riemann Solvers

5.1 Introduction to Neural Network Riemann Solvers

Neural network Riemann solvers are a groundbreaking development in computational fluid dynamics (CFD), leveraging machine learning to tackle the complexities of Riemann problems. These solvers approximate solutions to hyperbolic partial differential equations (PDEs) that govern conservation laws in fluid dynamics, thereby enhancing CFD's accuracy and efficiency. This approach supports high-fidelity simulations, turbulence modeling, and reduced-order models across various fields, including aerodynamics and climate modeling. By combining domain expertise with machine learning, researchers achieve stable and cost-effective predictions, transforming fluid mechanics and scientific computing [1, 7, 13, 70, 10].

The fundamental aim of neural network Riemann solvers is to ensure outputs comply with physical laws, particularly conservation laws in fluid dynamics [71]. Neural networks address limitations of

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ideal gas assumptions, providing accurate fluid behavior representations under diverse conditions. For instance, feed-forward neural networks can solve coupled nonlinear PDEs by minimizing residuals and boundary conditions, highlighting their potential in complex fluid dynamics [3].

Advanced architectures like DeepONet, Fourier Neural Operator (FNO), and U-Net have demonstrated efficacy in Riemann problems, managing complex mappings and delivering high-fidelity solutions [72]. Adaptive learning rate optimization further enhances training and solution accuracy through real-time feedback [73].

Neural networks as global ansatz functions, exemplified by the Neural Particle Method, showcase machine learning's potential to surpass traditional numerical methods, enabling incompressible fluid flow simulations without fixed meshes, thus enhancing flexibility and reducing costs.

Neural network Riemann solvers mark a significant advance in fluid dynamics, offering improved accuracy, efficiency, and flexibility. Integrating advanced neural architectures with traditional methods underscores machine learning's transformative potential in CFD, enhancing simulation accuracy, accelerating simulations, and developing superior turbulence models and reduced-order models. This convergence addresses complex turbulent flows using physics-informed machine learning, enhancing data efficiency and prediction stability, ultimately revolutionizing the field [1, 10, 7].

5.2 Data-Driven Approaches and Classifiers

Data-driven approaches in neural network Riemann solvers leverage machine learning to enhance complex fluid dynamics problem-solving accuracy and efficiency. Utilizing extensive datasets, these methods enable neural networks to learn intricate fluid flow patterns. CFDBench, for example, provides 302,000 frames of velocity and pressure fields from classic CFD problems, serving as a benchmark for evaluating neural operators' performance [72].

Classifiers are crucial in these methodologies, categorizing flow regimes and extracting features to improve model accuracy and reliability. Physics-informed machine learning techniques refine flow characteristic identification by integrating domain knowledge, yielding robust predictions that can augment or replace traditional simulations [10, 1, 69, 7]. Advanced classification algorithms enable neural network solvers to distinguish between shocks, rarefaction waves, and contact discontinuities, ensuring numerical solutions adhere to physical principles. This capability optimizes training and enhances generalization across diverse fluid dynamics scenarios.

The integration of data-driven approaches and classifiers in neural network Riemann solvers signifies a major advancement in CFD. By leveraging large datasets and sophisticated classification techniques, solvers using physics-informed machine learning significantly enhance simulation accuracy and efficiency for complex fluid systems. This fusion of domain knowledge with machine learning improves data efficiency and stabilizes predictions, facilitating realistic turbulent flow modeling. These advancements pave the way for replacing traditional high-fidelity simulations with reliable approaches adaptable to real-world scenarios, including noisy or incomplete data [1, 7, 74, 13, 3].

5.3 Integration with Traditional Numerical Methods

Integrating neural networks with traditional numerical methods marks a substantial advancement in CFD, enhancing the performance and accuracy of complex fluid dynamics solutions. This hybrid approach leverages the strengths of machine learning and classical techniques, addressing each method's limitations. Neural operators learn mappings between functional spaces, effectively approximating PDE solution operators and capturing complex flow dynamics [5].

Pathak et al. proposed a method using machine learning to model and correct coarse-grid simulation discrepancies, enabling high-resolution feature recovery and bridging low-resolution data with high-fidelity results [75]. Similarly, the MCL-NN approach combines machine learning with flux limiting to produce accurate subgrid fluxes, illustrating machine learning's potential to enhance traditional methods [76].

Frameworks like WaterLily.jl demonstrate the flexibility and efficiency of integrating machine learning with CFD solvers, utilizing the Julia programming language for seamless integration with machine learning frameworks [77]. Trained CNNs paired with high-order solvers improve performance by managing discontinuities effectively [63].

In data-driven ODE solvers, FINN uses spatial data to predict temporal dynamics, showcasing the synergy between data-driven and traditional methods to enhance predictive capabilities [2]. The Adaptive Learning Rate Optimization (ALRO) method incorporates a feedback loop for real-time learning rate adjustments, enhancing training and solution accuracy [73].

Recent advancements in integrating neural networks with traditional methods underscore their transformative potential in CFD, enabling accurate simulations and innovative turbulence models. This integration enhances data efficiency through physics-informed machine learning and addresses complex turbulent flows previously deemed computationally prohibitive. It paves the way for improved simulation fidelity, reduced costs, and tackling real-world fluid dynamics challenges across various disciplines [1, 10, 7]. The synergy between machine learning and classical approaches continues to drive progress, offering robust solutions for scientific and engineering applications.

5.4 Innovative Neural Network Architectures

Innovative neural network architectures have significantly advanced Riemann solvers in CFD, enhancing accuracy and robustness in solving complex problems. RiemannONets, for instance, use a two-step training approach to improve traditional single-stage training, enhancing neural network solutions' accuracy and robustness for intricate flow dynamics [78].

The Generalized Riemann Problem (GRP) solver simplifies rarefaction wave treatment and derives L(Q)-equations, effectively managing wave interactions and discontinuities, underscoring its importance in advancing numerical methods for fluid dynamics [79].

Further advancements include the Constraint-Resolving Layer Method (CRes) and the Constraint-Adapted Loss Method (CAL), which incorporate constraints directly into neural network learning, enhancing simulation reliability and accuracy by adhering to physical laws [71]. These methods maintain conservation laws' integrity, crucial for accurate fluid dynamics modeling.

The Adaptive Learning Rate Optimization (ALRO) technique improves convergence and accuracy by dynamically adjusting learning rates based on real-time feedback, optimizing training processes [73].

Innovative architectures like DeepONet and U-Net exemplify the transformative potential of integrating machine learning with traditional methods in Riemann solvers. They enhance simulation accuracy and efficiency for complex compressible flow problems with strong shocks and pressure discontinuities. These architectures yield data-driven solutions that respect physical constraints and facilitate real-time Riemann problem forecasting, bridging classical numerical approaches with modern machine learning methodologies [71, 78, 9]. By addressing fluid dynamics simulations' challenges, these architectures pave the way for more accurate and efficient computational models, advancing CFD.

6 Physics-Informed Machine Learning

6.1 Bridging Physical Laws and Machine Learning

Physics-informed machine learning (PIML) innovatively integrates physical laws into machine learning frameworks, enhancing model interpretability and robustness in the face of imperfect data. This approach merges traditional data-driven models with fundamental fluid dynamics principles, offering a comprehensive understanding of fluid interactions and transport processes. Central to PIML is the incorporation of established physical principles, such as those from Reynolds-Averaged Navier-Stokes (RANS) models, which enables machine learning frameworks to capture flow physics more effectively, thereby improving generalization [1].

The combination of deep neural networks (DNNs) with physics-based models exemplifies methods that explore parameter spaces while integrating experimental data into the solution process [28]. Physics-informed neural networks (PINNs) further this by utilizing automatic differentiation for computing derivatives, adeptly handling complex geometries and dynamic interfaces beyond traditional numerical methods [38]. Their capability to accurately track fluid interfaces while maintaining physical properties is well-documented [70].

The Distributed Physics Informed Neural Network (DPINN) exemplifies PIML's ability to tackle complex, nonlinear problems through piecewise approximations and physics integration in the optimization process [76]. Additionally, frameworks utilizing existing Direct Numerical Simulation (DNS) databases systematically rectify discrepancies in RANS-predicted Reynolds stresses, thereby enhancing simulation accuracy and reliability [80].

Future research should focus on improving DNN accuracy in practical applications, exploring hybrid models, and developing methodologies that merge prior knowledge from partial differential equations (PDEs) with data-driven approaches. Leveraging neural operators can bridge traditional numerical methods and data-driven techniques, enhancing simulation accuracy and efficiency [28]. Adaptive learning rate optimization (ALRO) can be pivotal in tailoring training processes to prevent overfitting and underfitting.

PIML establishes a robust framework for advancing fluid dynamics and related scientific fields by continuously integrating physical laws into machine learning models. This integration fosters the development of computational models that achieve superior accuracy, interpretability, and reliability, propelling significant advancements in computational modeling, particularly in Computational Fluid Dynamics (CFD). By incorporating machine learning techniques, these models can accelerate simulations, enhance turbulence modeling, and efficiently solve complex PDEs, transforming scientific and engineering applications [1, 10, 5].

6.2 Physics-Informed Neural Networks (PINNs)

Physics-Informed Neural Networks (PINNs) represent a significant advancement in merging machine learning with physical laws, providing a robust framework for modeling complex physical phenomena. By embedding physical principles into the learning process, PINNs reduce dependency on extensive datasets and enhance model interpretability [81]. Their capability to accurately simulate steady turbulent jet flows without traditional turbulence models underscores their potential in capturing complex flow dynamics [69].

PINNs augment traditional turbulence models, such as the Reynolds-Averaged Navier-Stokes (RANS) equations, through innovations like the Physics-Informed Machine Learning Approach for Augmenting Turbulence Models (PIML-ATM), which improves Reynolds stress and mean velocity predictions in turbulent flows [70]. This method effectively decomposes the Reynolds stress tensor into linear and nonlinear components, addressing ill-conditioning issues linked with RANS equations and enhancing turbulence model accuracy.

The versatility of PINNs extends to multi-layer models and complex fluid dynamics scenarios, as ongoing research indicates [23]. By embedding physical constraints within the neural network architecture, PINNs not only enhance prediction accuracy and generalization but also provide a comprehensive framework for advancing the study and simulation of complex fluid dynamics and other physical systems.

The concept and applications of PINNs illustrate their transformative potential in accurately modeling complex physical phenomena by integrating mathematical models with data, effectively addressing challenges such as noisy data incorporation, high-dimensional problems, and the computational demands of traditional numerical methods for PDEs. This innovative approach enhances predictive accuracy and generalization while facilitating the discovery of hidden physical relationships and offering efficient solutions for both forward and inverse problems across various scientific domains, including fluid dynamics and multiphysics scenarios [81, 5, 7, 6, 82]. By bridging the divide between machine learning and conventional physics-based methods, PINNs serve as a powerful tool for improving simulation accuracy and reliability across diverse scientific and engineering fields.

6.3 Innovations in Physics-Informed Frameworks

Recent innovations in physics-informed frameworks have markedly improved machine learning models' capacity to capture complex physical phenomena, particularly in fluid dynamics. These frameworks integrate physical laws into the training process, reducing reliance on large datasets and improving generalization across various geometries. A pivotal advancement is the dynamic weighting strategy for loss terms introduced by Raghu et al., which enhances training convergence compared to traditional static weight approaches [81]. This innovation addresses the optimization challenges in physics-informed neural networks (PINNs), facilitating more efficient training and accurate predictions.

The Coupled Integral Physics-Informed Neural Network (CI-PINN) represents another significant advancement, employing two neural networks to approximate both the solution and its integral. This method eliminates the need for spatial and temporal discretization, thereby improving accuracy in scenarios involving shocks and complex boundary conditions [83]. Similarly, Physics-Informed Neural Operators (PINOs) incorporate physical laws directly into the loss function, enabling faster learning and more accurate predictions with less training data, underscoring their potential in advancing physics-informed frameworks [82].

The adaptation of Compact Approximate Taylor (CAT) methods to systems of balance laws, as discussed by Carrillo et al., preserves stationary solutions, distinguishing these methods from prior approaches [84]. This adaptation is critical for ensuring stability and accuracy in complex fluid dynamics simulations. Furthermore, integrating quantum layers into neural network architectures, as explored by Sedykh et al., significantly enhances accuracy in solving the Navier-Stokes equations, presenting a promising direction for future research in hybrid quantum-classical approaches [28].

Innovations in parameterized smoothing kernels, introduced by Woodward et al., enhance the flexibility of Smoothed Particle Hydrodynamics (SPH) models, improving accuracy at resolved scales [80]. This refinement underscores the importance of advancing numerical techniques for achieving more precise simulations. Additionally, Wang et al. propose a novel method that formulates discrepancies as functions of mean flow features rather than physical coordinates, facilitating better extrapolation across diverse flow scenarios [85].

Collectively, these innovations in physics-informed frameworks reflect ongoing efforts to refine and expand machine learning applications in modeling complex physical systems. By integrating fundamental physical laws into the machine learning process, these frameworks enhance prediction efficiency and stability, making them invaluable for studying and simulating fluid dynamics and other scientific fields. This approach allows for augmenting or replacing traditional high-fidelity numerical simulations, particularly in intricate turbulent flows, by leveraging data-driven insights to address challenges such as noisy measurements and inverse problems. Consequently, these physics-informed machine learning techniques not only enhance simulation accuracy but also facilitate exploring high-dimensional problems that conventional methods typically struggle to tackle [6, 70, 7].

6.4 Applications in Turbulence Modeling

Physics-informed machine learning (PIML) has emerged as a critical tool in turbulence modeling, enhancing simulation accuracy and efficiency by integrating fundamental physical laws within machine learning frameworks. This approach effectively captures the complex and chaotic nature of turbulent flows, characterized by intricate interactions and rapid fluctuations. By embedding physical principles directly into the learning process, PIML methods offer robust solutions to ill-posed and inverse problems, maintaining accuracy even with noisy or incomplete data [1].

Innovative techniques, such as compatible finite element discretization for moist shallow water equations, have been developed to improve robustness under extreme atmospheric conditions, demonstrating PIML's applicability in real-world turbulence scenarios [46]. Future research aims to refine these techniques by exploring additional parameterizations and expanding their application to various atmospheric phenomena [39].

AdjointNet exemplifies a novel approach that constrains machine learning models through adjoint methods, optimizing high-dimensional parameter spaces—a common challenge in turbulence modeling. Integrating AdjointNet with other machine learning frameworks, such as physics-informed neural networks and tensor-basis neural networks, could significantly advance the accurate modeling of turbulent flows by ensuring physics-based constraints are incorporated throughout the modeling process, thereby enhancing prediction reliability in complex fluid dynamics scenarios [86, 1, 87, 70, 10].

The PICKLE method represents a significant advancement by improving covariance function estimation and managing non-smooth parameter fields, thus enhancing turbulence simulation reliability. Integrating PICKLE with existing PIML frameworks presents promising avenues for research, particularly in enhancing the accuracy of temporal coefficient estimations and investigating its utility in more intricate fluid dynamics scenarios. This includes addressing complex challenges such as incorporating thermal effects and adapting to varying flow conditions, which are critical for improving predictive capabilities in turbulent flow simulations and overcoming traditional numerical methods' limitations [88, 77, 87, 7].

Despite these advancements, challenges persist in applying PIML to turbulence modeling, particularly regarding the accuracy and convergence of physics-informed neural networks (PINNs) in complex flow scenarios. Future research should focus on extending existing solvers, such as the Generalized Riemann Problem (GRP) solver, to multidimensional applications and refining methods to address intricate flow configurations. Comprehensive validation studies, incorporating additional calibrations, and applying these methods to a broader range of flows and machine learning frameworks are crucial for advancing predictive turbulence modeling [1].

Integrating physics-informed machine learning (PIML) into turbulence modeling presents a transformative opportunity for the field, enabling the enhancement of Reynolds-averaged Navier–Stokes (RANS) equations by accurately predicting Reynolds stress discrepancies. This approach improves turbulence model predictive capabilities and addresses critical challenges in estimating mean flow velocities and related quantities, such as drag and lift, potentially replacing traditional high-fidelity numerical simulations that are often computationally expensive. Recent advancements demonstrate a comprehensive framework that systematically incorporates data from high-fidelity simulations, allowing for better mean velocity predictions by leveraging machine learning techniques to model linear and nonlinear components of the Reynolds stress tensor separately [88, 70, 69, 7]. By integrating physical laws into machine learning models, researchers can develop more accurate and reliable simulations, paving the way for improved understanding and prediction of turbulent flows, ultimately contributing to the advancement of computational fluid dynamics.

7 Conservation Law Modeling

7.1 Fundamental Principles and Importance

Conservation law modeling is pivotal in fluid dynamics and other scientific fields, ensuring that specific physical properties remain unchanged over time within a system [19]. These laws are mathematically expressed through equations that govern the conservation of mass, momentum, and energy, which are essential for accurately modeling complex systems. A notable feature of conservation law modeling is its ability to represent both smooth regions and discontinuities, crucial for depicting fluid flow dynamics accurately [89]. This capability is particularly significant in computational fluid dynamics (CFD), where precise predictions of shock waves and contact discontinuities are vital for simulation fidelity. The efficacy of two-dimensional conservation laws underscores the need for robust methods to capture such complex behaviors [89].

Finite volume methods (FVMs) are frequently used in conservation law modeling due to their robustness and ability to handle complex geometries. Achieving high accuracy requires careful attention to numerical fluxes and reconstruction techniques [90]. Balancing accuracy and computational efficiency is a recurring challenge, especially when converting time derivatives into spatial derivatives without intensive computations [84]. Parameterization schemes that preserve conservation laws are critical for closing the differential equations governing atmosphere-ocean dynamics, ensuring numerical models remain physically consistent and enhancing simulation reliability [49]. Additionally, identifying first-order conservation laws for shallow water equations in Lagrangian variables has revealed further conservation laws for specific bottom profiles, highlighting the complexity of conservation law modeling [91]. The ability to predict solutions for conservation laws with discontinuous profiles exemplifies the importance of this modeling approach in addressing the challenges posed by complex fluid systems [92].

Conservation law modeling establishes a robust framework for analyzing and simulating physical system behaviors, particularly through FVMs, which are essential in various fields such as fluid mechanics, meteorology, and biological processes. These methods approximate solutions to conservation law systems while ensuring numerical robustness, local conservation properties, and adaptability to complex geometries. By employing discrete maximum principles and high-order accurate schemes, conservation law modeling enhances understanding of scalar nonlinear hyperbolic conservation laws and facilitates the development of innovative computational techniques, including physics-informed neural networks, that tackle challenges such as shock wave behavior and discontinuities [90, 93, 83, 94, 95]. This approach not only improves numerical simulation accuracy but also advances scientific knowledge in fluid dynamics and related fields.

7.2 Numerical Approaches and Techniques

Numerical approaches in conservation law modeling are essential for accurately capturing fluid dynamics governed by hyperbolic conservation laws. Stability and conservation properties are crucial for ensuring simulation fidelity. The Entropy-Stable Conservative Flux Form Neural Network (CFN) exemplifies a novel method employing the Kurganov-Tadmor scheme to achieve stability and conservation in predictions, effectively addressing challenges associated with hyperbolic conservation laws [92]. Evaluating these numerical techniques involves deriving equivalence groups and applying classification methods to identify zero-order conservation laws, refining model accuracy and reliability by preserving fundamental conservation properties [19]. This systematic classification allows for the development of robust numerical methods capable of handling the complexities of fluid dynamics.

FVMs are widely utilized in modeling conservation laws due to their numerical robustness, local conservation properties, and adaptability to complex geometries and discontinuities. These methods are effective across various applications, including fluid mechanics, porous media flow, meteorology, and semiconductor device simulation, as they can be implemented on unstructured meshes and accurately capture non-oscillatory discontinuities. The design of FVMs often incorporates discrete maximum principles and advanced techniques such as monotone fluxes and TVD discretization to ensure high-order accuracy in approximating solutions to nonlinear hyperbolic conservation laws [93, 90, 96]. These methods are particularly adept at capturing shock waves and contact discontinuities, critical for simulating fluid flow dynamics. The interplay between accuracy and computational efficiency remains a key theme in conservation law modeling, particularly regarding the transformation of time derivatives into spatial derivatives without resorting to intensive computations.

Advancements in numerical approaches, particularly FVMs, are essential for enhancing the accuracy and reliability of fluid dynamics simulations, as these techniques effectively address a wide range of conservation law systems across various engineering fields. By ensuring numerical robustness through discrete maximum principles and facilitating the use of unstructured meshes, these methods significantly improve the modeling of complex phenomena such as incompressible flows and non-linear hyperbolic conservation laws, thereby contributing to more precise and stable fluid dynamics simulations [93, 90, 97]. By conserving key physical properties, these methods advance scientific knowledge and foster the development of more effective computational models for complex fluid systems.

7.3 Handling Discontinuities and Shocks

Addressing discontinuities and shocks in conservation law modeling poses significant challenges due to the complexities of capturing sharp gradients and nonlinear interactions. Effective methods must ensure stability and accuracy while preserving essential conservation properties. The Entropy-Stable Conservative Flux Form Neural Network (CFN) provides a promising approach, demonstrating robustness against noise and accurately predicting shock propagation, which is crucial for managing discontinuities and ensuring reliable simulations in the presence of abrupt flow dynamics [92]. A key aspect of managing discontinuities is identifying 'troubled cells' in numerical solutions of hyperbolic conservation laws. Convolutional neural networks (CNNs) have proven effective in this context, accurately detecting regions where traditional numerical methods may struggle to maintain accuracy. By identifying these troubled cells, CNN-based methods enable targeted corrections, enhancing simulation fidelity [63].

These advancements highlight the contributions of innovative numerical techniques, such as CNNs for discontinuity detection and physics-informed machine learning for automating artificial viscosity model discovery, as well as advanced methods like Coupled Integral Physics-Informed Neural Networks (PINNs) in effectively managing discontinuities and shock phenomena in conservation law modeling, thereby improving solution accuracy and robustness across various contexts [83, 94, 63]. By integrating these methods, researchers can develop more robust and accurate computational models, ultimately advancing fluid dynamics and expanding the applicability of conservation law modeling to complex real-world scenarios.

7.4 Applications in Fluid Dynamics

Conservation law modeling is pivotal in fluid dynamics, providing a robust framework for simulating complex fluid interactions governed by fundamental conservation principles. These models are essential for accurately capturing fluid system dynamics, particularly in scenarios involving shock waves, contact discontinuities, and other nonlinear phenomena. The application of conservation laws in fluid dynamics is exemplified by their use in simulating atmospheric and oceanic flows, where the conservation of mass, momentum, and energy is critical for understanding large-scale environmental processes [49].

In computational fluid dynamics (CFD), conservation law modeling is integral to developing numerical methods that ensure stability and accuracy across diverse flow regimes. For instance, FVMs are widely employed due to their ability to handle complex geometries and preserve conservation properties, making them suitable for simulating turbulent flows and other challenging fluid dynamics scenarios [90]. These methods effectively capture intricate interactions between fluid particles, which are essential for predicting flow behavior in engineering applications such as aerodynamics and hydrodynamics [89].

The capacity of conservation law models to manage discontinuities and shocks is critical for their application in fluid dynamics. By employing advanced numerical techniques, such as the Entropy-Stable Conservative Flux Form Neural Network (CFN), researchers can accurately simulate shock propagation and other discontinuous phenomena, ensuring that simulations remain stable and reliable even under extreme conditions [92]. This capability is vital for applications in fields such as meteorology, where accurate predictions of weather patterns and extreme events depend on precise modeling of atmospheric dynamics [19].

Furthermore, integrating machine learning approaches with conservation law modeling has opened new avenues for enhancing the accuracy and efficiency of fluid dynamics simulations. By leveraging data-driven techniques, researchers can develop models that not only adhere to conservation principles but also adapt to complex flow scenarios, improving predictive capabilities and computational performance [63]. This integration underscores the transformative potential of combining traditional conservation law models with modern computational tools, paving the way for more comprehensive and reliable simulations in fluid dynamics.

8 Computational Fluid Dynamics

8.1 Overview of Computational Fluid Dynamics

Computational Fluid Dynamics (CFD) is an essential analytical tool for simulating fluid flows via numerical methods, widely applied in engineering and environmental sciences. Its capability to model complex physical phenomena aids significantly in system design and optimization involving fluid interactions. Recent machine learning (ML) advancements have further enhanced CFD by speeding up simulations, developing advanced turbulence models, and creating superior reduced-order models. These innovations enable more sophisticated analyses in aerodynamics, combustion, and environmental sciences, highlighting ML's transformative impact on CFD research and applications [1, 10].

CFD employs various numerical methods, including finite volume, finite element, and finite difference techniques, to discretize governing equations like the Navier-Stokes equations. The finite volume method is particularly valued for its conservation properties and adaptability to complex geometries, demonstrated by the differentiable CFD solver WaterLily.jl [77]. Integrating modern computational techniques with traditional methods enhances simulation efficiency and flexibility.

The SL-Div method exemplifies CFD advancements by approximating solutions to linear second-order balance equations in divergence form, using a modified semi-Lagrangian framework to manage variable diffusivity [98]. Such innovations are crucial for addressing flow dynamics complexities, leading to more precise simulations.

Beyond computational capabilities, CFD is vital for advancing scientific research and engineering applications. It provides a robust framework for simulating fluid interactions, facilitating innovative design exploration and system optimization, thereby enhancing predictive capabilities. Incorporating machine learning has notably accelerated high-fidelity simulations and improved turbulence modeling, solidifying CFD's role in advancing technology and knowledge across various fields [46, 13, 1, 10].

8.2 Integration of Shallow Water Equations in CFD

Integrating shallow water equations (SWEs) into CFD frameworks is crucial for simulating free surface flows in coastal and riverine environments, enhancing predictive capabilities by modeling wave propagation and hydrodynamic processes. Recent developments include high-order structure-preserving finite volume schemes that maintain well-balanced and positivity-preserving properties in SWEs [21].

The mPDeC-WENO method achieves high-order accuracy while preserving positivity for water height in shallow water flow simulations [45]. Similarly, the TT-FV method demonstrates high-order accuracy and significant computational speed-ups in solving SWEs [41], ensuring stable and accurate numerical solutions amid complex flow dynamics.

Meshless methods, employing geometric conservation laws with the HLL Riemann solver, facilitate the simulation of shallow water flows over varying topographies [99]. The ML-SL-FV method, utilizing a convolutional neural network (CNN) to predict coefficients for the SL FV scheme, enables efficient transport simulations while ensuring mass conservation [62].

Addressing complex boundary conditions and geometrical discontinuities is also essential in integrating SWEs into CFD. Implementing broad-class boundary conditions has improved convergence properties, reducing drift-off errors and enhancing accuracy [61]. Additionally, incorporating angle-dependent coupling conditions into numerical solvers enhances simulation accuracy for shallow-water canals [60].

The mixed finite element method proposed by McRae conserves energy and enstrophy in numerical simulations of SWEs, demonstrating improved stability over traditional methods [48]. The quasi-two-layer finite volume scheme further enhances simulation accuracy for shallow water flows [27].

The adNOC scheme is adept at modeling processes such as pollutant or sediment transport, showcasing its versatility in environmental simulations [22]. Its effectiveness lies in maintaining stability amid stiff friction, allowing consistent discretization as the small parameter approaches zero [25]. The structure-preserving split approach, treating topological and metric properties separately, enhances computational efficiency in integrating shallow water equations within CFD [26].

The Newton multigrid method (NMGM) has been effectively applied to both 1D and 2D steady-state shallow water equations, assessing efficiency and robustness using various numerical fluxes [24]. This method is particularly relevant for modeling oceanic flows influenced by tides, storm surges, and tsunamis [23].

The integration of SWEs into CFD significantly advances the simulation of complex fluid interactions, improving precision and efficiency while facilitating high-fidelity turbulence models. This integration addresses challenges in resolving intricate spatiotemporal features, propelling fluid dynamics research and applications across diverse domains, including climate science, aerodynamics, and combustion [13, 75, 10, 100].

8.3 Finite Volume Methods in CFD

Finite volume methods (FVM) are foundational to CFD, conserving physical quantities locally and globally, making them effective for simulating fluid flows across complex geometries and unstructured meshes [93]. These methods discretize the computational domain into control volumes, applying conservation laws to ensure the integral form of governing equations is satisfied, which is critical for capturing fluid dynamics, especially in shock wave scenarios.

The MacLABSWE method exemplifies advancements in FVM by simplifying simulations through the direct application of physical variables as boundary conditions, reducing memory requirements and streamlining computations [101]. This method's versatility in handling complex boundary conditions highlights its effectiveness in CFD applications.

Integrating well-balanced reconstruction operators in implicit and semi-implicit schemes enhances FVM by preserving stationary states in numerical solutions, crucial for maintaining stability and accuracy in applications [102]. The use of piecewise linear interpolants in traditional FVM ensures accurate entropy solutions, even amid complex flow features [95].

Hybrid FD-FV methods combine finite difference and finite volume techniques for discretizing first-order hyperbolic conservation laws, achieving a balance between accuracy and computational efficiency [57]. The Flux Conserving Generalized Finite Difference Method (FC-GFDM) enhances PDE discretization robustness by incorporating local control cells to enforce approximate conservation of numerical fluxes [67].

The Interleaved Continuous Discontinuous Galerkin (ICDTG) method improves accuracy by interleaving gradient calculations with FVM iterations without incurring substantial computational overhead [96]. Moreover, numerical approaches for solving nonlinear differential equations achieve high polynomial accuracy, showcasing FVM's potential for precise solutions less sensitive to grid resolution [100].

FVM continues to play a crucial role in CFD, offering robust tools for simulating fluid dynamics across diverse applications. Its ability to conserve critical physical quantities while managing complex geometries is essential for enhancing simulation accuracy and reliability. The integration of machine learning, particularly physics-informed approaches, significantly improves turbulence modeling, addressing discrepancies in Reynolds stress predictions, and facilitating reliable mean flow calculations. This combination of advanced computational methods enables higher fidelity in turbulent flow simulations, leading to precise predictions of key performance metrics in engineering applications [10, 97, 69, 7].

8.4 Neural Network Riemann Solvers in CFD

Neural network Riemann solvers represent a significant advancement in CFD, utilizing machine learning to solve hyperbolic partial differential equations (PDEs) with enhanced efficiency and accuracy. The NNLCI method, for example, shows promise in hyperbolic equation applications, underscoring its relevance in CFD [89].

The Finite Volume Graph Convolution (FVGC) method integrates finite volume characteristics into graph convolutional networks, enhancing the predictive capabilities of graph neural networks (GNNs) in CFD contexts [46]. By combining finite volume methods with neural networks, FVGC offers a novel approach to improving CFD simulation accuracy and efficiency.

Neural network Riemann solvers also facilitate the development of adaptive and efficient numerical methods. By leveraging advanced machine learning techniques, these solvers can dynamically respond to fluctuating flow conditions, improving solution accuracy while reducing computational costs. This adaptability is achieved through physics-informed machine learning (PIML), which integrates domain knowledge with data-driven models, enhancing efficiency and stability in predictions. Consequently, these solvers produce high-fidelity simulations and develop effective turbulence models, enabling complex analyses across scientific and engineering disciplines [13, 1, 10, 7].

The incorporation of neural network Riemann solvers into CFD significantly enhances simulation accuracy and efficiency for complex fluid dynamics. Leveraging machine learning capabilities allows for modeling intricate flow behaviors, improving turbulence modeling, and accelerating high-fidelity simulations. Advanced neural architectures, such as DeepONet and U-Net, enable precise solutions for Riemann problems, particularly in extreme pressure variation scenarios. This approach also aids in interpreting flow features through data-driven basis representations, outperforming traditional methods in speed and robustness. Overall, integrating neural networks into CFD streamlines computational processes and opens avenues for addressing challenging fluid mechanics problems [103, 1, 78, 10].

8.5 Physics-Informed Machine Learning in CFD

The integration of physics-informed machine learning (PIML) within CFD represents a transformative advancement, enhancing the accuracy, reliability, and efficiency of fluid dynamics simulations. By embedding physical laws into machine learning frameworks, PIML bridges traditional numerical methods and data-driven models, providing a robust framework for simulating complex fluid interactions. Utilizing machine learning to predict Reynolds stress discrepancies with a physics-based implicit treatment exemplifies PIML's potential to improve the conditioning of Reynolds-Averaged Navier-Stokes (RANS) equations, thus enhancing simulation fidelity [70].

PIML applications extend to predicting near-equilibrium and non-equilibrium flow regimes, with the Neural Particle Method (NPM) achieving accurate results for incompressible fluid flows with free surfaces, even under large deformations and irregular discretization [38]. This method showcases PIML's capability to manage complex geometries and varying flow conditions with enhanced stability and accuracy.

The Hybrid Quantum Physics-Informed Neural Networks (HQPINNs) leverage the expressivity of quantum circuits alongside the flexibility of Physics-Informed Neural Networks (PINNs), enhancing the handling of complex fluid dynamics problems [28]. This innovative approach highlights the potential of integrating quantum computing elements into PIML frameworks to further boost CFD simulation capabilities.

Advancements in PIML underscore its potential to revolutionize CFD by integrating domain knowledge with machine learning algorithms, enhancing data efficiency and stability in predictions while providing a viable alternative to traditional high-fidelity numerical simulations of complex turbulent flows. By addressing challenges like solving inverse problems and simulating realistic conditions, PIML paves the way for more accurate modeling of intricate fluid systems, advancing the field toward comprehensive analytical capabilities across scientific and engineering disciplines [1, 7]. By harnessing the strengths of both physics-based and data-driven approaches, PIML enhances the predictive capabilities and computational efficiency of CFD simulations.

8.6 Challenges and Future Directions in CFD

CFD faces several pressing challenges that require ongoing research to improve accuracy and applicability in complex fluid dynamics scenarios. A primary challenge is enhancing boundary layer resolution, particularly in simulations involving intricate geometries and fluid behaviors. Future research should focus on extending existing methods to address these complexities, thereby improving CFD simulation fidelity. Incorporating wetting and drying techniques remains critical for accurately capturing fluid dynamics in natural environments. Improving shock capturing methods is another priority, contributing to more stable and reliable simulations [104].

Future investigations should optimize parameter selection for combined machine learning and CFD schemes, exploring hybrid methodologies that integrate physics-informed neural networks with traditional numerical approaches. This could enhance performance in accurately capturing complex flow phenomena, addressing key challenges in multi-scale representation and physical knowledge encoding. Additionally, examining the scalability of these hybrid approaches may yield significant improvements in simulation speed and accuracy, particularly for turbulence modeling and transient flow simulations [14, 1, 10, 4]. The velocity-based Discontinuous Galerkin (DG) scheme shows promise for simulating compressible flow problems, with potential enhancements in robustness and accuracy for wetting and drying treatments.

Energy and entropy conservation are critical for maintaining stability and accuracy in long-term simulations, especially in atmospheric models. Methods that effectively conserve mass, momentum, and energy are essential for advancing CFD applications in environmental and atmospheric sciences. Finite volume methods, which inherently maintain local conservation principles and resist numerical oscillations, can enhance simulation accuracy in these fields. Additionally, integrating machine learning techniques can optimize CFD processes, enabling faster and more reliable modeling of complex fluid behaviors, thereby improving our understanding of environmental phenomena [90, 97, 93, 10, 4]. Optimizing memory exchanges in multi-GPU implementations and exploring GPU-Direct RDMA capabilities could significantly enhance CFD simulation efficiency on larger meshes.

Enhancing the computational efficiency of numerical methods, such as the primal-dual mimetic finite element approach, is essential for solving complex problems. Future research should extend these methods to three-dimensional applications, improving scalability and accuracy in modeling fluid dynamics and atmospheric processes critical for weather and climate models [96, 90, 100, 93, 105]. The extension of GoRINNs to more complex PDE systems, incorporating additional physical constraints, and applying the method to real-world fluid dynamics and traffic flow problems are promising research directions. Refining physics-informed neural network (PINN) methodologies to enhance their applicability in complex fluid dynamics scenarios and integrating them with other computational techniques could further advance the field.

The ROCM provides a more precise representation of subgrid dynamics, enhancing predictive capabilities in geophysical fluid dynamics. Future research will focus on extending existing methodologies to more intricate systems and higher-dimensional spaces, while also addressing challenges posed by problems with both stiff and non-stiff components. This exploration will integrate data-driven approaches, such as deep learning and physics-informed machine learning, with traditional numerical methods for solving PDEs. By leveraging advanced computational techniques, the goal is to improve efficiency and accuracy in simulating complex physical phenomena across various scientific and engineering disciplines, including CFD and multiphysics simulations. Additionally, research will tackle limitations related to noisy data incorporation, mesh generation complexities, and high-dimensional parameterized PDEs, ultimately contributing to more robust solutions in real-world applications [1, 5, 2, 6, 9]. The proposed parallel implementation of the CSPH-TVD method significantly reduces computation time, indicating potential for further parallelization strategies. Future research could refine this method to enhance computational efficiency and explore its applicability in more complex fluid dynamics scenarios.

Incorporating conservation laws as hard constraints into the modeling framework and investigating the potential of Physics-Informed Neural Networks (PINNs) in intricate meteorological models that integrate physical parameterizations represent promising avenues for future research. Recent advancements, such as the Coupled Integral PINN methodology, enhance shock wave modeling and address challenges associated with nonlinear conservation laws, as well as the development of parameterization schemes that ensure conservation laws are preserved in numerical weather prediction models [103, 49, 83]. Future research may also focus on optimizing the ESDGSEM for different computational architectures and enhancing positivity preservation mechanisms for challenging scenarios. The extension of the framework to more complex systems and exploring adaptive techniques to improve performance are additional areas of interest.

Addressing potential smoothness loss in the vorticity field, which may affect accuracy in turbulent flows, remains a notable limitation. Future research will aim to improve the computational efficiency of the CNN method and explore its integration with other numerical techniques [63]. Collectively, these challenges and future directions highlight ongoing efforts to refine CFD methodologies, ensuring their continued relevance and utility in scientific and engineering applications. Future research should focus on improving modeling techniques and exploring their applications across different environmental contexts to enhance prediction accuracy.

9 Conclusion

The integration of mathematical modeling, numerical analysis, and machine learning has catalyzed significant advancements in fluid dynamics, enhancing both simulation accuracy and computational efficiency. By utilizing shallow water equations, finite volume methods, neural network Riemann solvers, and physics-informed machine learning, this multidisciplinary approach effectively addresses complex fluid dynamics scenarios. This integration not only improves predictive capabilities but also enhances computational performance.

The DG2 solver exemplifies this by accurately capturing detailed velocity fields without additional calibration, effectively handling small-scale transients. Similarly, the CFN method demonstrates significant improvements in stability and accuracy, even in the presence of noise and discontinuities, highlighting areas for future exploration. The NMGM has proven efficient and robust for steady-state shallow water equations, adeptly managing the challenges of varying topography and dry areas.

The study of elementary wave interactions via characteristic methods reveals the intricate behaviors in shallow water systems, emphasizing the need for further investigation. The well-balanced RKDG method successfully resolves complex flow structures while preventing nonphysical oscillations, highlighting the importance of achieving high accuracy and stability in multilayer shallow water flows. Additionally, the variational formulation of momentum equations through Hamilton's principle underscores the role of non-holonomic constraints in deriving equations for non-dissipative shear shallow water flows and ideal turbulence.

Future research should focus on expanding benchmarks like CFDBench to cover a broader spectrum of fluid dynamics scenarios and refining neural network architectures to enhance generalization. Applying DeepONets to complex tsunami scenarios and integrating these methods with real-time data could significantly improve tsunami forecasting. Further exploration of conservation laws and their implications in complex fluid dynamics scenarios presents a promising direction for continued advancement.

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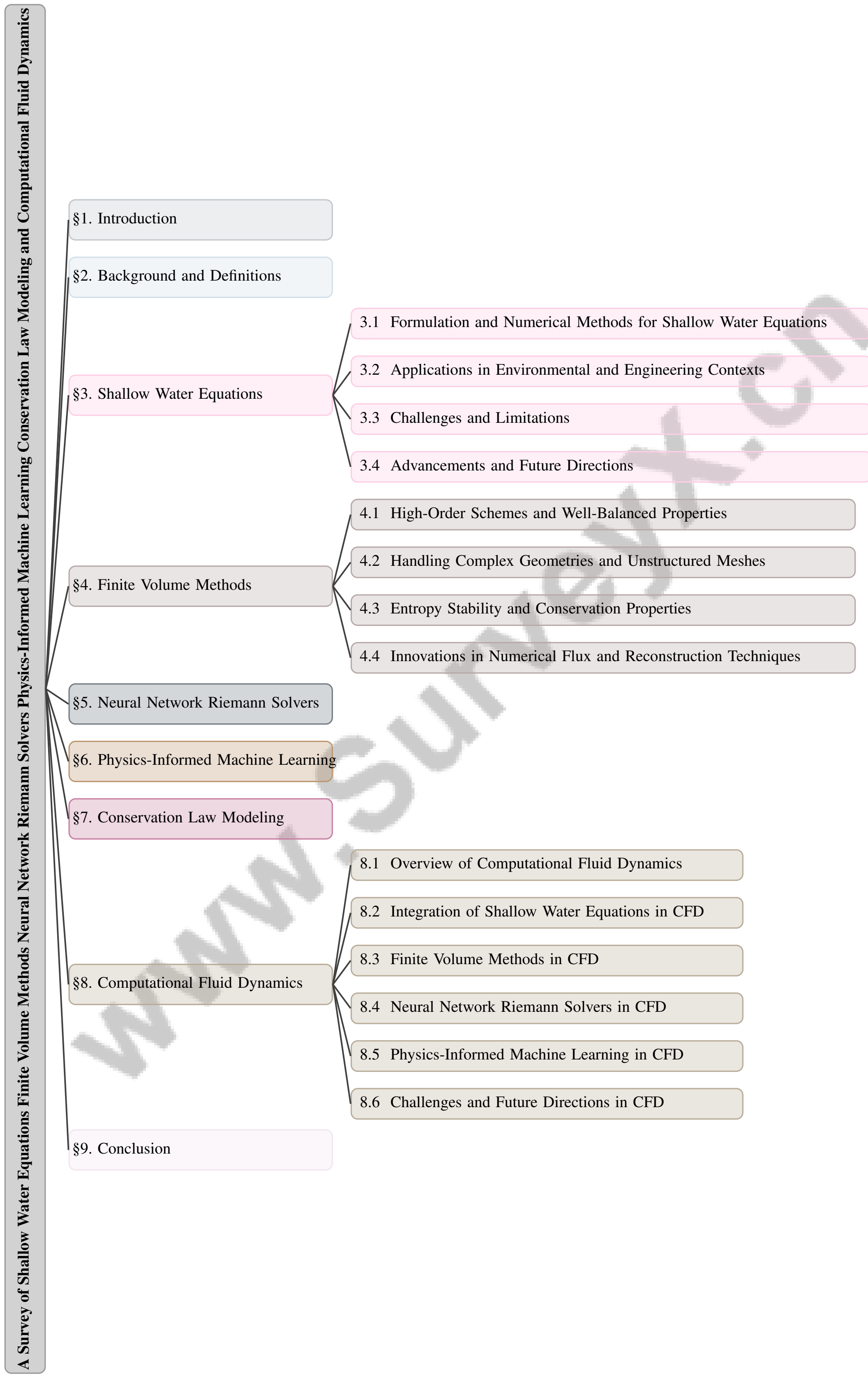

Figure 1: chapter structure

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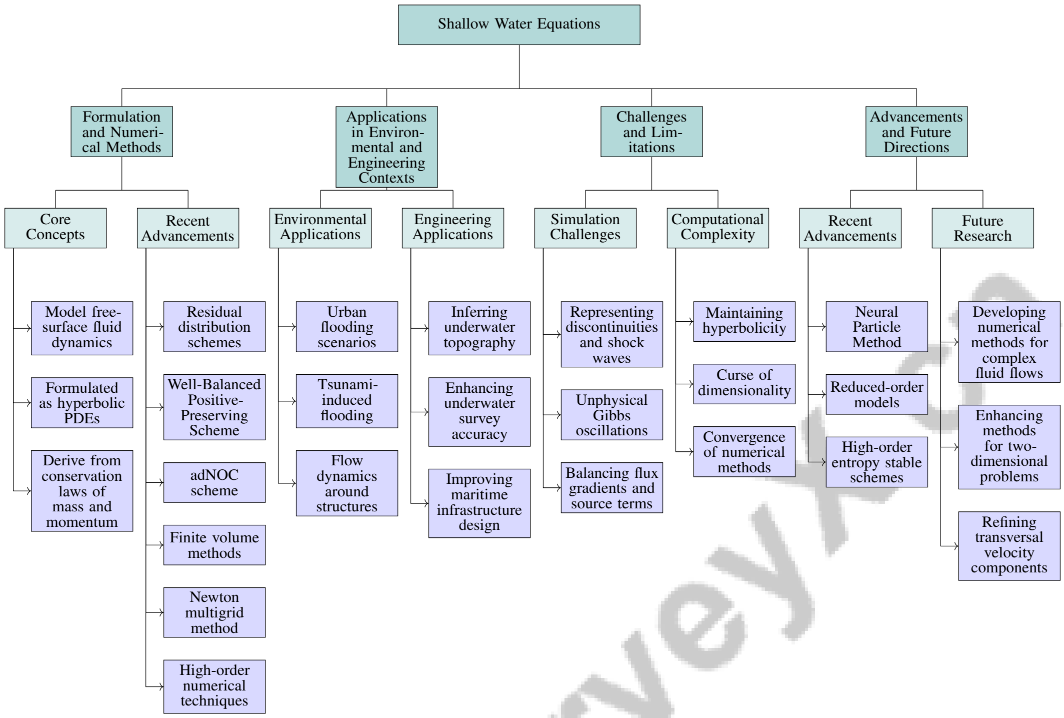

Figure 2: This figure illustrates the hierarchical structure of Shallow Water Equations (SWEs), encompassing their formulation and numerical methods, applications in environmental and engineering contexts, challenges and limitations, and recent advancements with future directions.

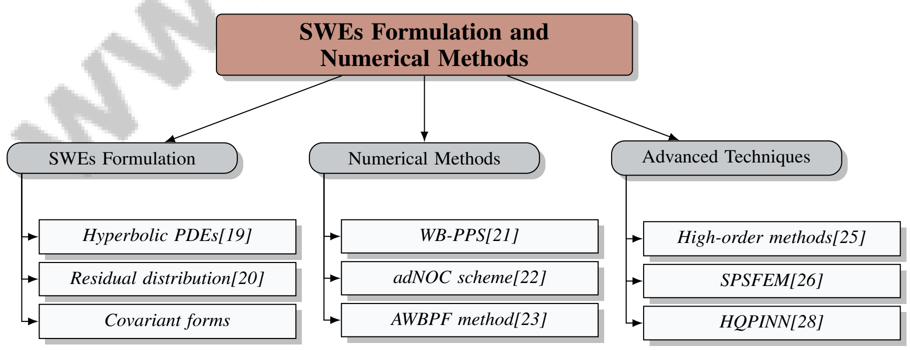

Figure 3: This figure illustrates the hierarchical structure of Shallow Water Equations (SWEs) formulation and numerical methods. It categorizes key advancements in SWEs into formulation techniques, numerical methods, and advanced techniques, highlighting recent innovations and their contributions to fluid dynamics modeling.

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