Liquidity trap revisited: when wages are sticky

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Abstract

I show that the puzzling implications of the standard New Keynesian model in a liquidity trap are resolved by incorporating sticky wages. In the standard New Keynesian model, price flexibility is harmful to the economy, forward guidance has a magical power to stimulate output, and the fiscal multiplier is paradoxically large during the liquidity trap. With more flexible prices, the economy suffers from a deeper recession and greater deflation, and the impacts of the forward guidance and the fiscal policy grow explosively to infinity. We find that these limit puzzles disappear when wages are sticky. The economy behaves normally, and policies have reasonable effects during the liquidity trap. Price flexibility is beneficial, while wage flexibility can be either beneficial or harmful depending on whether the zero lower bound (ZLB) constraint is binding

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1 Introduction

The Great Recession led central banks to set their policy rates at the zero lower bound. However, the solution of the standard New Keynesian model has some unusual predictions during the liquidity trap. Forward guidance, which lowers the nominal interest rate in the future after the liquidity trap, can dramatically boost consumption today. Government spending has huge output multipliers. Moreover, the cumulative current effects of these policies that persist for multiple periods in the future grow explosively with their expected duration. Price flexibility is harmful as greater price flexibility worsens the recession in a liquidity trap. The economic path shows discontinuity: the output gap increases with price flexibility to arbitrarily large but goes to zero when prices are fully flexible.

I find that these counterintuitive implications of the standard New Keynesian model disappear when I introduce the stickiness of wages. Price flexibility is no longer harmful to the economy, forward guidance has limited power, and fiscal multiplier lies within a reasonable range. The mechanism I emphasize in this paper is an additional inflation channel when wages are sticky. Suppose the economy suffers from a large negative TFP shock. The real interest rate is negative. We will see a deflationary depression in the liquidity trap for the standard model for the feedback loop of the IS equation and Phillips curves. When the nominal interest rate hits the zero lower bound at T, the negative real interest rate implies $\pi_T < 0$ and $x_T < 0$. Since x_{T-1} and π_{T-1} is positively related to π_T and x_T and ZLB is still binding at $T-1$, x_{T-1} and π_{T-1} take more negative values. As t goes backward, the deflation and output drop grow explosively. However, when wages are sticky, this feedback loop is much attenuated. Suppose the economy has great deflation today; it implies higher current real wages, which increases the labor supply and consequently leads to larger output and a greater inflation rate. Since the real wage cannot move too much at each period with both sticky wages and prices, it restricts the equilibrium path during the trap as the initial real wage w_{-1} is determined and cannot grow explosively. In other words, lower productivity implies lower real wages, and the inflation rate tends to increase to lower the real wage during the trap, which cuts off the feedback loop of deflation and depression.

The model predicts two distinct patterns from the standard New Keynesian model. Firstly, the economy suffers from output drop after the trap. The reason is the discrepancy between the real wage and labor productivity for nominal stickiness. Although the economy benefits from a lower output drop during the trap, it cannot jump back to an efficient one immediately at the end of the recession. Besides, as prices are more flexible, the duration of the liquidity trap gets shorter, and the economy finally escapes from the liquidity trap and converges to the price frictionless equilibrium.

The paper relates to the recent literature on forward guidance(Lacker, 2018; Andrade et al., 2019; Nakata et al., 2019; Bilbiie, 2019; Bielecki et al., 2019; Bundick and Smith, 2020; Gibbs and McClung, 2023; Fujiwara and Waki, 2022). Carlstrom et al. (2012) show that a promise by the central bank to peg the interest rate below the natural interest rate generates explosive dynamics for the inflation and output in the standard New Keynesian model. Negro et al. (2013) refer to this phenomenon as the forward guidance puzzle. They argue that it is unreasonable to assume that the central bank can engender substantial changes in the long-term interest rates, which is why the forward guidance puzzle arises. McKay et al. (2016) show that the magnitude of forward guidance is substantially reduced when the markets are incomplete rather than complete for the precautionary saving effects. They assume small redistribution effects. Redistributional effects of real interest rate changes can be significant in an incomplete market model, as emphasized by Auclert (2019). Varying the distributional assumption of McKay et al. (2016), Hagedorn et al. (2019) show that forward guidance can be more, equally, or less effective in the incomplete markets model than the complete markets model. Adding information frictions, as shown by Angeletos and Lian (2018) and Kiley (2016), can also substantially reduce the impact of forward guidance as it makes the IS equation or Phillips curve less forward-looking.

The paper also relates to the literature on the fiscal multiplier puzzle(Cook and Devereux, 2013; Carrillo and Poilly, 2013; Fujiwara and Ueda, 2013; Benhabib et al., 2014; Michau, 2019; Lemoine and Lindé, 2023) . Woodford (2011) and Christiano et al. (2011) find large fiscal multipliers in New Keynesian zero-bound models. Erceg and Lindé (2014) find fiscal multipliers decrease substantially at higher spending levels when the duration of the liquidity trap is endogenously determined. Cook and Devereux (2011) study the spillover effects of fiscal policy in open economy models of the liquidity trap.

Closely related is the work of Cochrane (2017). Cochrane (2017) resolves the limit puzzles by picking an equilibrium out of multiple equilibria, which either ensures zero inflation at time 0 or approaches the steady state as time goes backward. He names them "local-to-frictionless" equilibria. Given his choice of equilibria, the economy performs normally in the liquidity trap, and there are no counterintuitive implications of policies and flexibilities. However, in order to choose such equilibria, the government must at least have some partial commitments. I studied a different case where the government had no commitment, so our findings were complementary. The ability to make commitments matters a lot in our setting. As Werning (2011) shows, if the government has the full commitment, the economy circles around the steady state instead of having explosive dynamics even for the standard New Keynesian model. But there are also many similarities between the sticky wage equilibrium and "local-to-frictionless" equilibria. First, we both have positive inflation rates during the liquidity trap. Second, the economy still has lower real output after the trap. Last, we both require the time of 0 state to be close to the steady state.

Finally, Diba and Loisel (2021) resolve the New Keynesian puzzles by assuming that the central bank can only control the interest rate on bank reserve (IOR), and the nominal interest rate adjusts to satisfy the balance sheet of the central bank. Specifically, Diba and Loisel (2021) introduce a money-demand equation where the demand for money relates positively to the real output gap y_t and negatively to the difference of nominal interest rates and IOR $i_t - i_t^{IOR}$. The central bank sets the IOR rate and nominal stock of bank reserves exogenously during the liquidity trap. The nominal interest rate responds to the real output gap without zero lower bound constraints. Given the initial price level p_{-1}, they show policies have reasonable effects. Our approach is similar to Diba and Loisel (2021) in the sense that we both have one initial condition (in my case is w_{-1} and in their case is p_{-1}), which restricts the equilibrium to behave explosively. But the mechanisms are totally different: we emphasize the stickiness of wages. The equilibria are also different. In their model, the inflation rate is still negative, but in our model, it is positive. Besides, when prices become more flexible, the ZLB constraint is not necessarily binding for the whole recession period, so the nominal interest rate is not necessarily zero in our case.

The rest of the paper is organized as follows. In Section 2 and Section 3, I study the equilibrium path and policy implications during the liquidity trap theoretically for two cases: a non-forward-looking government that takes the future state as given and a forward-looking government that rationally realizes the impact of its own policy on the future path. I show for both cases, the limit puzzles disappear, and the economy behaves normally during the liquidity trap. In Section 4, I show quantitatively the behavior of the economy and the effects of policies within plausible ranges of price and wage stickiness. Additionally, I compute the welfare losses. The quantitative results are consistent with our theoretical findings. Besides, I find that price flexibility is beneficial, but wage flexibility can be either beneficial or harmful, depending on whether the ZLB constraint is binding or not. Section 5 concludes.

2 Non-forward looking government

I use the New Keynesian model with sticky wages introduced by Erceg et al. (2000):

Households: The economy is assumed to be inhabited by a large number of identical households. Each household is made up of a continuum of members, each specialized in a different labor service and indexed by $j \in [0,1]$. Income is pooled within each household. The period utility takes the form

$$U(C_t, \{N_t(j)\}) = log(C_t) - \int_0^1 \frac{N_t(j)^{1+\varphi}}{1+\varphi} dj$$

where C_t is the consumption index given

$$C_t = (\int_0^1 C_t(i)^{1-\frac{1}{\epsilon_p}} di)^{\frac{\epsilon_p}{\epsilon_p - 1}}$$

A typical household maximizes his expected life-long utility function $\sum_{t=0}^{\infty} \beta^t U(C_t, \{N_t(j)\})$ subject to the budget constraint:

$$\int_0^1 P_t(i)C_t(i) + Q_t B_t \leq B_{t-1} + \int_0^1 N_t(j)W_t(j)dj + D_t$$

where B_t represents purchases of one-period discount bonds at a price Q_t, and D_t denotes dividends from the ownership of firms. The household's optimality condition of C_t becomes

$$Q_t = \beta E_t \left\{ \left(\frac{C_t}{C_{t+1}} \right) \left(\frac{P_t}{P_{t+1}} \right) \right\}$$

Firms: For the production side, there is a continuum of firms indexed by $i \in [0,1]$. Each firm produces a differentiated good with a technology represented by the production function

$$Y_t(i) = A_t N_t(i)$$

where $Y_t(i)$ denotes the output of good i, A_t is an exogenous aggregate TFP (which goes down during the liquidity trap) and $N_t(i)$ is an index of labor input used by firm i and defined by

$$N_t(i) = (\int_0^1 N_t(i,j)^{1-\frac{1}{\epsilon_w}} dj)^{\frac{\epsilon_w}{\epsilon_w - 1}}$$

where $N_t(i,j)$ denotes the quantity of type-j labor employed by firm i in period t. The parameter ϵ_w represents the elasticity of substitution among labor varieties. Let $W_t(j)$ denote the nominal wage for type-j labor prevailing in period t, for all $j \in [0,1]$. The nominal wages $W_t(j)$ are taken as given by firms, so the aggregate wage index is

$$W_t = (\int_0^1 W_t(j)^{1-\epsilon_w} dj)^{\frac{1}{\epsilon_w - 1}}$$

Following Calvo (1983), the firm may reset its price only with probability 1–θ_p in any given period. The firm reoptimizing in period t will choose the price that maximizes the present value of profits while that price remains in effect:

$$\max \sum_{k=0}^{\infty} \beta^k \Lambda_{t,t+k} \left(P_{t+k|t} Y_{t+k|t} - \frac{W_{t+k}}{A_{t+k}} Y_{t+k|t}) \right)$$

subject to the sequence of demand constraints:

$$Y_{t+k|t} = \left(\frac{P_{t+k|t}}{P_t} \right)^{-\epsilon_p} Y_t$$

for $k = 0, 1, 2, \ldots$, where $\Lambda_{t,t+k} = \beta^k \frac{U_{c,t+k}}{U_{c,t}}$ is the stochastic discount factor, and $Y_{t+k|t}$ denotes the output in period $t+k$ for firms that last reset their price in period t. The optimality condition associated with the firm's pricing problem above takes the form

$$\sum_{k=0}^{\infty} \theta_p^k E_t \left\{ \Lambda_{t,t+k} Y_{t+k|t} \left(\frac{1}{P_{t+k}} \right) \left(P_t^* - \mathcal{M}_p \frac{W_{t+k}}{A_{t+k}} \right) \right\} = 0$$

where $\mathcal{M}_p = \frac{\theta_p}{\theta_p - 1}$.

Sticky Wages: There is the union of each labor type j, which presents its members' wages in period t. Wage stickiness is introduced following the Calvo staggered pricing, where the union can reset their nominal wage only with probability $1 - \theta_w$ each period, independently of the time elapsed since they last adjusted their wage. Let $W_t^*(j)$ denote the newly set wage. In a way consistent with the utility maximization of its members' households, the union seeks to maximize

$$\sum_{k=0}^{\infty}(\beta\theta_w)^k(C_{t+k}^{-\sigma}\frac{W_t^*(j)}{P_{t+k}}N_{t+k|t}-\frac{N_{t+k|t}^{1+\varphi}}{1+\varphi})$$

where $N_{t+k|t}$ is the labor demand and defined by

$$N_{t+k|t}=(\frac{W_t^*(j)}{W_{t+k}})^{-\epsilon_w}(\int_0^1 N_t(i)di),$$

The first-order condition associated with the labor unions is given by:

$$\sum_{k=0}^{\infty}(\beta\theta_w)^k E_t\left\{N_{t+k|t}\left(C_{t+k}^{-\sigma}\frac{W_t^*}{P_{t+k}}+\mathcal{M}_w N_{t+k|t}^{\phi}\right)\right\}=0$$

where $\mathcal{M}_w=\frac{\theta_w}{\theta_w-1}$.

New Keynesian Equations: For this sticky wage New Keynesian model, a first-order Taylor expansion of equilibrium conditions at the the zero-inflation steady state yields the IS equation and Phillips curves:

$$x_t = x_{t+1}-(i_t-{r_t}^n-{\pi^p}_{t+1}) \tag{1}$$

$${\pi^p}_t=\beta{\pi^p}_{t+1}+\kappa_p w_t, \tag{2}$$

$${\pi^w}_t=\beta{\pi^w}_{t+1}+\aleph_w x_t-\kappa_w w_t, \tag{3}$$

$$w_t=w_{t-1}+{\pi^w}_t-{\pi^p}_t-\Delta{w_t}^n, \tag{4}$$

where x_t, π_t^p, π_t^w, w_t denote the real output gap, price inflation rate, wage inflation rate, and the real wage gap, respectively. ${w_t}^n$ is the change of natural real wage. κ_w and κ_p are flexibilities of wages and prices. κ_p, κ_w, $\aleph_w$ are functions of the elasticity of substitutions ϵ_w and ϵ_p, the Fischer elasticity $\frac{1}{\varphi}$, and the nominal stickinesses θ_w and θ_p. The detailed proof of these equations is provided in the appendix. When wages become fully flexible ($\theta_w\to 0$), with $\aleph_w=\lambda_w(1+\varphi)$ and $\lambda_w\to\infty$, it follows that $w_t\to(1+\varphi)x_t$. Substituting this expression into equation (2), it gives us the IS equation and the

Phillips curve for the standard New Keynesian model featuring only price stickiness.

2.1 A motivating example: limit cases

Cochrane (2017) shows the jump in the inflation rate at the end of the recession:

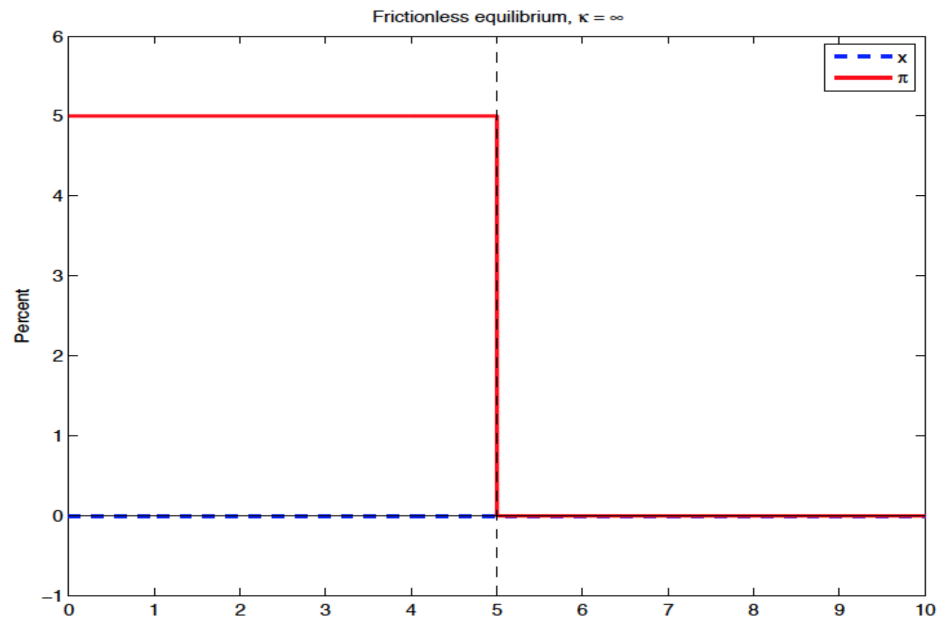

FIGURE 1: THE JUMP OF INFLATION IN THE STANDARD EQUILIBRIUM (COCHRANE, 2017)

The discontinuity of the economic path (explosive when prices are sticky; zero gap when prices are fully flexible) is related to this jump. Suppose the government has no commitment and uses discretionary policy. Let T be the first time when the natural interest rate returns to be positive. The standard model predicts that $\pi_T = 0$ when prices are sticky even though the stickiness is very small, but π_T can be any positive value when prices are fully flexible. The flexible price equilibrium allows for such a jump at the end of recession because the optimal path $\{\pi_t, x_t\}$ is not unique: we could allow for any inflation π_t when inflation has no cost. To uniquely pin down inflation, I introduce a new nominal rigidity: wage rigidity. First, I consider a simpler motivating case where prices are fully flexible, but wages are not. I show the following lemma:

Lemma 1. ***(Sticky Wage Only)*** *If wages are fully flexible but wages are not, and the government lacks the commitment, there exists a unique equilibrium where* $\pi_t{}^p = -g_t$ [1] *where* g_t *is*

[1] if $w_{-1} \neq 0$, $\pi_0{}^p = -g_0 + w_{-1}$.

the TFP growth rate at t and $i_t = \rho$.

Proof. When $\theta_p = 0$, it implies $\kappa_p = \infty$ and then $w_t = 0$ from (2). (1)-(4) become [2] [3]

$$x_t = x_{t+1} - (i_t - r_t{}^n - \pi^p{}_{t+1}), \tag{5}$$

$$\pi^w{}_t = \beta\pi^w{}_{t+1} + \aleph_w x_t, \tag{6}$$

$$\pi^w{}_t - \pi^p{}_t = \Delta w_t{}^n, \tag{7}$$

The change of the natural real wage (in log value) $\Delta w_t{}^n$ equals the change of the labor productivity, and I have $\Delta w_t{}^n = log(A_t/A_{t-1}) \equiv g_t$. The natural interest rate $r_t{}^n = \rho + g_{t+1}$ where ρ is the discount factor. Substituting (7) into (5),

$$x_t = x_{t+1} - (i_t - r_t{}^n - \pi^w{}_{t+1} + \Delta w^n_{t+1}) = x_{t+1} - (i_t - \rho - \pi^w{}_{t+1})$$

Like the standard New Keynesian model, the welfare loss at t is approximated by $x_t{}^2 + \mu_w(\pi_t{}^w)^2$. The government minimizes the current welfare loss function when it lacks the commitment. FOC implies

$$x_t + \mu_w N_w \pi_t{}^w = 0$$

Substituting this into (6), I have

$$\pi^w{}_t = \frac{\beta}{1 + \mu_w \aleph_w^2}\pi^w{}_{t+1},$$

Restricting the path that does not explode as t goes *forward*, I have a unique solution $x_t = \pi_t{}^w = 0$ at any t. We check that such policy satisfies all conditions and that ZLB is not binding. (6) is satisfied immediately and (7) implies $\pi_t{}^p = -\Delta w_t{}^n = -g_t$. Then we know from (5) $i_t = r_t{}^n + \pi_{t+1}{}^p = \rho > 0$. Thus, a zero lower bound (ZLB) is not binding. □

I have a constant and positive nominal interest rate $i_t = \rho$. When TFP goes down, the natural interest rate decreases, but at the same time, the inflation rate increases to the same level. The reason is that when there is only wage rigidity, the economy adjusts

[2]it can also be derived by using the model with only wage rigidities.

[3]if $w_{-1} \neq 0$, at $t = 0$ (7) becomes $\pi^w{}_0 - \pi^p{}_0 = \Delta w_0{}^n - w_{-1}$.

prices to match the change in the real wage. Therefore, when there are adverse productivity shocks, the economy faces both inflation pressure and a lower real interest rate. This inflation pressure attenuates the feedback loop of the deflation and the depression in the standard New Keynesian model, as shown in the proposition below.

Lemma 2. ***(Sticky Price Only)*** *If wages are fully flexible but prices are not, and the government lacks commitment, when the sequence of negative shocks g_t is large enough, the zero lower bound (ZLB) will bind, leading the economy into severe deflation and depression.*

Proof. When $\theta_w = 0$, it implies $\lambda_w = \infty$, and thus $w_t = (1+\varphi)x_t$ using (3) from the previous discussion. Equations (1)-(4) become:

$$x_t = x_{t+1} - (i_t - {r_t}^n - {\pi^p}_{t+1}),$$

$${\pi^p}_t = \beta {\pi^p}_{t+1} + \kappa_p (1+\varphi) x_t,$$

$${\pi^w}_t = (1+\varphi)(x_t - x_{t-1}) + {\pi^p}_t - \Delta {w_t}^n,$$

The welfare loss at time t is approximated by ${x_t}^2 + \mu_p (\pi_t^p)^2$. Similarly, the government aims to minimize the current welfare loss function in the absence of commitment. Assuming that the lower bound constraint for the nominal interest rate i_t is not binding, the resulting first-order condition indicates:

$$x_t + \mu_p \kappa_p (1+\varphi) \pi_t^p = 0$$

Substituting this into the Phillips curve, I derive:

$${\pi^p}_t = \frac{\beta}{1 + \mu_p \kappa_p^2 (1+\varphi)^2} \pi_{t+1}^p, \tag{8}$$

Thus, if the assumption that the zero lower bound constraint is not binding holds, then $x_t = {\pi_t}^p = 0$ for any t. However, the nominal rate $I_t = r_t^n = \rho + g_{t+1}$ can become negative if $g_{t+1} < -\rho$, which contradicts this assumption that the zero lower bound constraint is not binding. This scenario differs from the flexible-price, sticky-wage economy, where inflation fully offsets the decrease in the nominal interest rate when the growth rate declines. Consider that the negative shocks $g_t = -\bar{g}$ persist for $T+1$ periods (t from 1 to $T+1$) and then return to 0. The optimal interest rate is 0 for $t \leq T$ for a government that focuses solely on the current welfare. For $t > T$, the optimal nominal interest rate I_t is implied by (8). From the above discussion, we know that $x_t = {\pi_t}^p = 0$ for any $t > T$. Thus, for $t \leq T$, I have:

$$x_t = x_{t+1} + \hat{g} + {\pi^p}_{t+1},$$

$${\pi^p}_t = \beta {\pi^p}_{t+1} + \kappa_p (1 + \varphi) x_t,$$

where $\hat{g} \equiv -\bar{g} + \rho < 0$. Starting from $x_{T+1} = \pi^p_{T+1} = 0$, it is straightforward to check that x_t and π_t are negative for any $t \leq T$, becoming increasingly negative as t moves back from T to 0. Furthermore, if prices become more flexible and λ_p increases, the economy experiences greater deflation and depression. When prices are fully flexible (θ_p is also 0), the output gap is 0 ($x_t = 0$), the welfare loss is 0 independent of the inflation rate (as $\mu_p = 0$), and I only have the equation

$$i_t = r^n_t + \pi^p_{t+1}$$

Inflation can take any value as long as the implied nominal interest is non-negative. This demonstrates the discontinuity of the equilibrium path when prices are sticky while wages are flexible. □

As shown in lemma 1 and lemma 2, the sticky price and sticky wage equilibrium paths are different, as zero lower bound constraint is binding in one case but not in another case. Besides, there is a jump of the equilibrium path for large κ_p and for $\kappa_p = \infty$ when only prices are sticky. The rest of this section aims to show that the equilibrium path is continuous if wages are also sticky: with larger κ_p, the equilibrium path converges to the path of the fully flexible prices, as shown in my lemma 1.

Consider the case where wages and prices are both sticky. I characterize the discretionary policy of the government at each time t. First, I assume that the government is non-forward looking in the sense that he takes the future state, especially π^p_{t+1}, π^w_{t+1}, x_{t+1} as given. In the next section, I will discuss the case where the government is looking forward.

Using the quadratic approximation, the optimization problem becomes

$$\min\{{x_t}^2 + \mu_p ({\pi_t}^p)^2 + \mu_w ({\pi_t}^w)^2\}$$

subject to (1)-(4)

$$i_t \geq 0$$

$$\text{given } w_{t+1}, {\pi_{t+1}}^p, {\pi_{t+1}}^w, x_{t+1}$$

The government at each period t minimizes the current welfare loss subject to the IS equation and Phillips Curves and takes whatever happens in the future as given. From (4) at $t+1$, we know that w_t is given. And from (2), we know that π_t^p is also given. Thus the government at time t sets the interest rate i_t, which determines both π_t^w and x_t, to minimize the current welfare loss $\{{x_t}^2 + \mu_w({\pi_t}^w)^2\}$ (since the government treats ${\pi_t}^p$ as given).

The problem can be simplified as

$$\min\{{x_t}^2 + \mu_w({\pi_t}^w)^2\} \tag{9}$$

$$s.t. {\pi^w}_t = \beta {\pi^w}_{t+1} + \aleph_w x_t - \kappa_w w_t,$$

$$i_t \geq 0$$

$$\text{given } w_t$$

FOC implies (if ZLB is not binding)

$$x_t + \mu_w N_w {\pi_t}^w = 0 \tag{10}$$

I assume that ZLB is not binding for the rest of the discussion. In reality, ZLB can be binding for small values of κ_p (as we show in quantitative examples when prices are very sticky). But our purpose is to focus on the limited behavior of this economy when prices become very flexible. I will verify that the nominal interest rate i_t is indeed positive (it actually converges to ρ) later for extremely large κ_p.

Substituting (10) into (3) and after rearranging the terms, I get the backward evolution equation of $\{{\pi_t}^w, {\pi_t}^p, w_{t-1}\}$ (let $\phi = \mu_w N_w$):

$$\begin{bmatrix} {\pi_t}^w \\ {\pi_t}^p \\ w_{t-1} \end{bmatrix} = \begin{bmatrix} \frac{\beta}{1+\phi N_w} & 0 & \frac{-\kappa_w}{1+\phi N_w} \\ 0 & \beta & \kappa_p \\ \frac{-\beta}{1+\phi N_w} & \beta & 1 + \frac{\kappa_w}{1+\phi N_w} + \kappa_p \end{bmatrix} \begin{bmatrix} {\pi_{t+1}}^w \\ {\pi_{t+1}}^p \\ w_t \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ g_t \end{bmatrix} \tag{11}$$

2.2 After the recession

I assume the natural interest rate takes the following values: ${r_t}^n = -r_a < 0$ when $t = 1, 2, ..., T-1$, and ${r_t}^n = r_b > 0$ when $t = T, T+1, ...$ (there could be a liquidity trap before T). Since ${r_t}^n = \rho + g_{t+1}$, it implies $g_t = -r_a - \rho \equiv g_a$ when $t = 1, 2, ..., T$ and $g_t = r_b - \rho \equiv g_b$ when $t = T+1, T+2, ...$ Thus I focus on the time at and after $T+1$ so that (11) becomes:

$$\begin{bmatrix} {\pi_t}^w \\ {\pi_t}^p \\ w_{t-1} \end{bmatrix} = \begin{bmatrix} \frac{\beta}{1+\phi N_w} & 0 & \frac{-\kappa_w}{1+\phi N_w} \\ 0 & \beta & \kappa_p \\ \frac{-\beta}{1+\phi N_w} & \beta & 1 + \frac{\kappa_w}{1+\phi N_w} + \kappa_p \end{bmatrix} \begin{bmatrix} {\pi_{t+1}}^w \\ {\pi_{t+1}}^p \\ w_t \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ g_b \end{bmatrix}, \forall t \geq T+1 \quad (12)$$

(12) can be transformed into

$$\begin{bmatrix} \Delta_b {\pi_t}^w \\ \Delta_b {\pi_t}^p \\ \Delta_b w_{t-1} \end{bmatrix} = \begin{bmatrix} \frac{\beta}{1+\phi N_w} & 0 & \frac{-\kappa_w}{1+\phi N_w} \\ 0 & \beta & \kappa_p \\ \frac{-\beta}{1+\phi N_w} & \beta & 1 + \frac{\kappa_w}{1+\phi N_w} + \kappa_p \end{bmatrix} \begin{bmatrix} \Delta_b {\pi_{t+1}}^w \\ \Delta_b {\pi_{t+1}}^p \\ \Delta_b w_t \end{bmatrix} \equiv A \begin{bmatrix} \Delta_b {\pi_{t+1}}^w \\ \Delta_b {\pi_{t+1}}^p \\ \Delta_b w_t \end{bmatrix}, \forall t \geq T+1 \quad (13)$$

where

$$\begin{bmatrix} \Delta_b {\pi_t}^w \\ \Delta_b {\pi_t}^p \\ \Delta_b w_{t-1} \end{bmatrix} \equiv \begin{bmatrix} {\pi_t}^w \\ {\pi_t}^p \\ w_{t-1} \end{bmatrix} - \begin{bmatrix} {\pi_b}^w \\ {\pi_b}^p \\ w_b \end{bmatrix}, \begin{bmatrix} {\pi_b}^w \\ {\pi_b}^p \\ w_b \end{bmatrix} = - \begin{bmatrix} \frac{-\kappa_w}{1-\beta+\phi N_w} \\ \frac{\kappa_p}{1-\beta} \\ 1 \end{bmatrix} \frac{g_b}{\frac{\kappa_p}{1-\beta} + \frac{\kappa_w}{1-\beta+\phi N_w}} \quad (14)$$

$\begin{bmatrix} {\pi_b}^w \\ {\pi_b}^p \\ w_b \end{bmatrix}$ is the steady state value with respect to a constant growth rate g_b. Especially when $\kappa_p \to \infty$, $\begin{bmatrix} {\pi_b}^w \\ {\pi_b}^p \\ w_b \end{bmatrix}$ converges to $\begin{bmatrix} 0 \\ -g_b \\ 0 \end{bmatrix}$, the same as the fully flexible case. To solve the economic path after the trap, I need to find eigenvalues of matrix A in (13).

Lemma 3. *If prices and wages are not fully flexible, the transition matrix A has one eigenvalue strictly larger than $1 + \frac{\kappa_w}{1+\phi N_w} + \kappa_p$, and two eigenvalues in between zero and one.*

Proof. Please see the appendix. □

Since I have one eigenvalue strictly larger than one, $\Delta_b {\pi_t}^w$, $\Delta_b {\pi_t}^p$, $\Delta_b w_{t-1}$ are not necessarily zero after the trap as they are in the standard model without wage rigidities. Especially I have the following property:

Lemma 4. *The path of* $\{\Delta_b {\pi_t}^w, \Delta_b {\pi_t}^p, \Delta_b w_{t-1}\}$ *after the liquidity trap depends and only depends on* w_{T-1}.

Proof. Please see the appendix. □

Unlike the standard model where $\pi_t = x_t = 0$ for $t \geq T$, now the equilibrium path after the trap depends on w_T. This dependence gives the reason for the government's policy after the trap to be affected by the economic performance during the trap.

In the proof of lemma 4, I find $\begin{bmatrix} \Delta_b {\pi_t}^w \\ \Delta_b {\pi_t}^p \end{bmatrix} = \begin{bmatrix} x_w \\ x_p \end{bmatrix} \Delta_b w_{t-1}$ Next, I show some properties related to x_w and x_p. It is useful for our final proof.

Lemma 5. *(1)*$-\frac{\kappa_w}{1+\phi N_w - \beta} < x_w < 0, 0 < x_p < 1$ *(2) when* $\kappa_p \to \infty$, $x_p \to 1$ *and* $x_w \to 0$.

Proof. Please see the appendix. □

2.3 During the recession

When $t \leq T$, $g_t = g_a < 0$ by our setting. Using the same approach, I have the backward evolution equation

$$\begin{bmatrix} \Delta_a {\pi_t}^w \\ \Delta_a {\pi_t}^p \\ \Delta_a w_{t-1} \end{bmatrix} = \begin{bmatrix} \frac{\beta}{1+\phi N_w} & 0 & \frac{-\kappa_w}{1+\phi N_w} \\ 0 & \beta & \kappa_p \\ \frac{-\beta}{1+\phi N_w} & \beta & 1 + \frac{\kappa_w}{1+\phi N_w} + \kappa_p \end{bmatrix} \begin{bmatrix} \Delta_a {\pi_{t+1}}^w \\ \Delta_a {\pi_{t+1}}^p \\ \Delta_a w_t \end{bmatrix} \equiv A \begin{bmatrix} \Delta_a {\pi_{t+1}}^w \\ \Delta_a {\pi_{t+1}}^p \\ \Delta_a w_t \end{bmatrix}, \forall t \leq T \tag{15}$$

where

$$\begin{bmatrix} \Delta_a {\pi_t}^w \\ \Delta_a {\pi_t}^p \\ \Delta_a w_{t-1} \end{bmatrix} \equiv \begin{bmatrix} {\pi_t}^w \\ {\pi_t}^p \\ w_{t-1} \end{bmatrix} - \begin{bmatrix} {\pi_a}^w \\ {\pi_a}^p \\ w_a \end{bmatrix}, \begin{bmatrix} {\pi_a}^w \\ {\pi_a}^p \\ w_a \end{bmatrix} = - \begin{bmatrix} \frac{-\kappa_w}{1-\beta+\phi N_w} \\ \frac{\kappa_p}{1-\beta} \\ 1 \end{bmatrix} \frac{g_a}{\frac{\kappa_p}{1-\beta} + \frac{\kappa_w}{1-\beta+\phi N_w}} \tag{16}$$

and when $\kappa_p \to \infty$, $\begin{bmatrix} {\pi_a}^w \\ {\pi_a}^p \\ w_a \end{bmatrix} \to \begin{bmatrix} 0 \\ -g_a \\ 0 \end{bmatrix}$.

To find the economic path after T, I need to know w_T. I have the initial condition w_{-1}. So it's necessary to construct the bridge between w_{-1} and w_T from the above backward equation:

$$\begin{aligned} \begin{bmatrix} \Delta_a {\pi_{T+1}}^w \\ \Delta_a {\pi_{T+1}}^p \\ \Delta_a w_T \end{bmatrix} &= \begin{bmatrix} {\pi_{T+1}}^w \\ {\pi_{T+1}}^p \\ w_T \end{bmatrix} - \begin{bmatrix} {\pi_b}^w \\ {\pi_b}^p \\ w_b \end{bmatrix} + \begin{bmatrix} {\pi_b}^w \\ {\pi_b}^p \\ w_b \end{bmatrix} - \begin{bmatrix} {\pi_a}^w \\ {\pi_a}^p \\ w_a \end{bmatrix} \\ &= \begin{bmatrix} \Delta_b {\pi_{T+1}}^w \\ \Delta_b {\pi_{T+1}}^p \\ \Delta_b w_T \end{bmatrix} - \begin{bmatrix} {\pi_{a-b}}^w \\ {\pi_{a-b}}^p \\ w_{a-b} \end{bmatrix} = \begin{bmatrix} x_w \\ x_p \\ 1 \end{bmatrix} \Delta_b w_T - \begin{bmatrix} {\pi_{a-b}}^w \\ {\pi_{a-b}}^p \\ w_{a-b} \end{bmatrix} \end{aligned} \tag{17}$$

where

$$\begin{bmatrix} {\pi_{a-b}}^w \\ {\pi_{a-b}}^p \\ w_{a-b} \end{bmatrix} \equiv - \begin{bmatrix} \frac{-\kappa_w}{1-\beta+\phi N_w} \\ \frac{\kappa_p}{1-\beta} \\ 1 \end{bmatrix} \frac{g_a - g_b}{\frac{\kappa_p}{1-\beta} + \frac{\kappa_w}{1-\beta+\phi N_w}} \tag{18}$$

Thus

$$\begin{bmatrix} \Delta_a {\pi_0}^w \\ \Delta_a {\pi_0}^p \\ \Delta_a w_{-1} \end{bmatrix} = A^{T+1} (\begin{bmatrix} x_w \\ x_p \\ 1 \end{bmatrix} \Delta_b w_T - \begin{bmatrix} {\pi_{a-b}}^w \\ {\pi_{a-b}}^p \\ w_{a-b} \end{bmatrix}) \tag{19}$$

(19) gives the connection between $\Delta_a w_{-1}$ and $\Delta_b w_T$. Then I show the main proposition:

Proposition 1. *when $\kappa_p \to \infty$, the economic path converges to the path of fully flexible prices: $w_t \to 0$, ${\pi_t}^w \to 0$, $\forall t \geq 0$; ${\pi_t}^p \to -g_a$, $\forall t \leq T$ and ${\pi_t}^p \to -g_b$, $\forall t \geq T+1$; $i_t \to \rho$ $\forall t$.*[4] [5]

Proof. Please see the appendix. □

[4] if $w{-}1 \neq 0$, ${\pi_0}^p \to -g_a + w_{-1}$

[5] Our proof is mainly to show that the explosive economic path that appears in the standard model when prices become very flexible does not appear in the model with sticky wages.

2.4 Fiscal Multiplier

I follow Woodford (2011) in introducing fiscal policy. Suppose the government increases its consumption from zero to a small positive value G_t. Let $g_t^c \equiv G_t/Y_t^n$. The log linearized IS equation and Phillips curves are:

$$(x_t - g_t^c) = (x_{t+1} - g_{t+1}^c) - (i_t - {r_t}^n - {\pi^p}_{t+1}), \tag{20}$$

$${\pi^p}_t = \beta {\pi^p}_{t+1} + \kappa_p w_t, \tag{21}$$

$${\pi^w}_t = \beta {\pi^w}_{t+1} + \aleph_w(x_t - \Gamma g_t^c) - \kappa_w w_t, \tag{22}$$

$$w_t = w_{t-1} + {\pi^w}_t - {\pi^p}_t - \Delta {w_t}^n, \tag{23}$$

where $\Gamma = \frac{1}{1+\varphi}$. And the quadratic welfare loss function is

$$L = \{(x_t - \Gamma g_t^c)^2 + \mu_p ({\pi_t}^p)^2 + \mu_w ({\pi_t}^w)^2\} \tag{24}$$

Rewrite (24) as follows

$$(x_t - \Gamma g_t^c) = (x_{t+1} - \Gamma g_{t+1}^c) - (i_t - {r_t}^n - {\pi^p}_{t+1}) - (1-\Gamma)\Delta g_{t+1}^c, \tag{25}$$

where $\Delta g_{t+1}^c = g_{t+1}^c - g_t^c$. If we treat $(x_t - \Gamma g_t^c)$ as x_t, the only difference from the baseline sticky wage model is the additional term $(1-\Gamma)\Delta g_{t+1}^c$ in the IS equation. That's how fiscal policies come into the equation and affect the economy. Γ is also the frictionless fiscal multiplier. Let $\kappa_w = \infty$, (26)-(27) are the Phillips Curve and welfare loss function for the standard model with fiscal policies:

$${\pi^p}_t = \beta {\pi^p}_{t+1} + \aleph_p(x_t - \Gamma g_t^c), \tag{26}$$

$$L = \{(x_t - \Gamma g_t^c)^2 + \mu_p ({\pi_t}^p)^2\} \tag{27}$$

where $\aleph_p = (1+\varphi)\kappa_p$. (25)-(27) are the set of equations that characterize the standard model. They are the same as Woodford (2011) and Werning (2011).

Consider the following fiscal policy: $g_t^c = \bar{g}^c, \forall t < T$ and $g_t^c = 0, \forall t \geq T$. First I show why the fiscal multiplier of the standard model is large and converges to infinity with price flexibility.

From our setting, we know $\Delta g_t^c = -\bar{g^c}$, if $t = T$ and $\Delta g_t^c = 0$, if $t \neq T$. If we treat $(x_t - \Gamma g_t^c)$ as x_t, the only difference between the standard New Keynesian equations without fiscal policies and (25)-(27) is at $T-1$ the IS equation has $(1-\Gamma)\bar{g^c}$. Since it is positive, it actually helps to relieve the binding constraint of ZLB. Suppose $\bar{g^c}$ is small so that ZLB is still binding at $T-1$. Then ZLB will be always binding before T, and $i_t = 0$ $\forall t < T$. Given $i_t = 0$, I can compute the increase of $(x_t - \Gamma g_t^c)$ is $(\aleph_p + 1)^{T-1-t}(1-\Gamma)\bar{g^c}$ by using (25) and (26). Thus the fiscal multiplier is

$$\frac{dY}{dG}|t = (\aleph_p + 1)^{T-1-t}(1-\Gamma) + \Gamma \tag{28}$$

We know for any κ_p, the above term grows exponentially to infinity with $T-1-t$. The fiscal multiplier increases exponentially as time goes backward. And such fiscal multiplier is always larger than one and also larger than its frictionless counterpart Γ. Besides when $\kappa_p \to \infty$, the fiscal multiplier increases to infinity at any time $t < T-1$.

Then, I show for the sticky wage model, the fiscal multiplier converges to its frictionless level Γ. The approach is the same as I do in the baseline model without fiscal policy. If I treat $(x_t - \Gamma g_t^c)$ as x_t and I assume ZLB is not binding, the backward equation is the same as our baseline model (the change of IS equation does not affect anything once ZLB is not binding). So $(x_t - \Gamma g_t^c)$ must converge to 0 in the limit case. In other words, $x_t \to \Gamma g_t^c$. I get the fiscal multiplier Γ. And I only need to check that ZLB is not binding in the limit case. Using (25) I can show that $i_t \to \rho$ if $t \neq T-1$ and $i_t \to \rho + (1-\Gamma)\bar{g^c}$ if $t = T-1$. They are both positive numbers.

3 Forward looking government

Following Kydland and Prescott (1977), I consider the case where the government is forward-looking: they do not take future equilibrium and policies as given but instead consider the impact of their current policy on the future path. I focus on the policy rule, which is consistent and stationary after time T (after the liquidity trap, when ZLB is not binding, the policy rule is $i(w)$, which only depends on the state variable w). In other words, at any time $T+k$ after the liquidity trap, when the initial real wage gap is w_{T+k-1}, the government understands that his policy affects the future state and assumes that government in the future always employs the same policy rule $i(w_t)$ for state w_t, $\forall t \geq T+k$, and his own optimal policy is still $i(w_{T+k-1})$. I guess that the optimal policy is *linear*: $\Delta i(w_t) \equiv i(w_t) - \rho = i_w w_t$ and verify that it is true given the quadratic approximation of the welfare function and linear constraints [6]. Given linear policies, the key economic variables are linear functions of the state variable w_t:

$$\pi_t^p = C_p w_{t-1}, \pi_t^w = C_w w_{t-1}, x_t = C_x w_{t-1} \tag{29}$$

and

$$w_{t-1}/w_t = \gamma_w \tag{30}$$

for the reason I discussed in the non-forward looking case (see Lemma 4) [7]. Since the welfare loss function $Q(w_t)$ is the discounted sum of $x_t^2 + \mu_p(\pi_t^p)^2 + \mu_w(\pi_t^w)^2$, we know $Q(w_t)$ takes the quadratic form:

$$Q(w_t) = Q_w w_t^2 \tag{31}$$

The parameters $\{C_w, C_x, C_p, \gamma_w, Q_w\}$ are functions of i_w and i_w is determined by the FOC of the government. I solve for $\{C_w, C_x, C_p, \gamma_w, Q_w\}$ and i_w simultaneously. Combining (29)(30) with IS equations and Phillips curves after the trap, I know

$$C_x \gamma_w = (C_x + C_p) - i_w \gamma_w; \tag{32}$$

$$C_p \gamma_w = \beta C_p + \kappa_p; \tag{33}$$

$$C_w \gamma_w = \beta C_w - \kappa_w + N_w C_x \gamma_w; \tag{34}$$

$$1 = \gamma_w (1 + C_w - C_p); \tag{35}$$

For the welfare loss function, it can be written recursively as

[6] When I do the approximations of the welfare and evolution equations around the steady state, the corresponding optimal policy should be linear as it can be seen as a linear approximation of the policy function $i(w_t)$ around $w_t = 0$.

[7] I have linear evolution functions for $\{\pi_t^p, \pi_t^w, x_t, w_{t-1}\}$ given i_w, and the uniqueness implies that there are three eigenvalues within (-1,1) and one eigenvalue beyond (-1.1). Therefore I will find three equations $a_i \pi_t^p + b_i \pi_t^w + c_i x_t + d_i w_{t-1} = 0, i = 1, 2, 3$ which implies that $\{\pi_t^p, \pi_{t-1}^w, x_t\}$ are linear functions of w_{t-1}

$$Q(w_t) = x_t^2 + \mu_p(\pi_t^p)^2 + \mu_w(\pi_t^w)^2 + \beta Q(w_{t+1}) \tag{36}$$

Combining with (29) (30), I get

$$Q_w = C_x^2 + \mu_p C_p^2 + \mu_w C_w^2 + \beta \frac{Q_w}{\gamma_w^2}; \tag{37}$$

Next, I determine i_w by finding the optimal policy function. With some abuse of notation[8], the government at time $s(s > T)$ assumes that the policy rule is $i(w_t) = \rho + i_w w_{t-1}$ afterwards, and thus he can assume that at time $s+1$, the key economic variables are linear functions of the real wage gap w_s:

$$\pi_{s+1}^p = C_p w_s, \pi_{s+1}^w = C_w w_s, x_{s+1} = C_x w_s \tag{38}$$

He tries to minimize the following welfare loss function (by choosing i_s):

$$x_s^2 + \mu_p(\pi_s^p)^2 + \mu_w(\pi_s^w)^2 + \beta Q_w w_s^2 \tag{39}$$

while x_s, π_s^p and π_s^w satisfy the IS equations and Phillips curves at time s. Substituting (38) into (1)-(4), I have

$$w_s = \frac{w_{s-1} - N_w(i_s - \rho)}{1 + \kappa_p + \kappa_w + \beta(C_p - C_w) - N_w(C_x + C_p)} \tag{40}$$

Combining (1)-(4) with (33) (34) (35) (40),

$$x_s = \frac{(C_x + C_p)w_{s-1} - \gamma_w(1 + N_w C_x)(i_s - \rho)}{1 + \kappa_p + \kappa_w + \beta(C_p - C_w) - N_w(C_x + C_p)} \tag{41}$$

$$\pi_s^p = \frac{\gamma_w C_p(w_{s-1} - N_w(i_s - \rho))}{1 + \kappa_p + \kappa_w + \beta(C_p - C_w) - N_w(C_x + C_p)} \tag{42}$$

$$\pi_s^w = \frac{(\beta C_w - K_w + N_w(C_p + C_w))w_{s-1} - (1 + \gamma_w C_p)N_w(i_s - \rho)}{1 + \kappa_p + \kappa_w + \beta(C_p - C_w) - N_w(C_x + C_p)} \tag{43}$$

They are functions of the real wage gap w_{s-1} and policy i_s. When the government at s chooses his policy, he does not need to make i_s linearly related to w_{s-1}. But combining (40)-(43) and (39), we can see that the welfare loss function can be rewritten as a quadratic form of w_{s-1} and i_s, and the optimal policy i_s should be a linear function of

[8] I use i_w to denote the coefficient for the interest rate function, while I use i_s and i_t to denote the interest rate at time s and t.

w_{s-1}, and it verifies our guess. Since the policy rule and the functions of key economic variables are stationary, I have $i_s - \rho = i_w w_{s-1}$, $\pi_s^p = C_p w_{s-1}, \pi_s^w = C_w w_{s-1}, x_s = C_x w_{s-1}$. Substituting them into the FOC, I get

$$N_w \gamma_w \mu_p C_p^2 + N_w \mu_w (1 + C_p \gamma_w) C_w + \gamma_w (1 + N_w C_x) C_x + \beta N_w \frac{Q_w}{\gamma_w} = 0; \tag{44}$$

Now I have six equations (32)-(35), (37) and (44) to determine the six endogenous parameters $\{C_p, C_w, C_x, \gamma_w, Q_w, i_w\}$. The solution characterizes the equilibrium path after the liquidity trap when ZLB is not binding. To learn how the equilibrium path changes with price flexibility κ_p, I only need to know how the parameters $\{C_p, C_w, C_x, \gamma_w, Q_w, i_w\}$ (the solution to six equations) change with κ_p. And I have the following lemma:

Lemma 6. *For $\kappa_p > 0$, the solution to equations(32)-(35), (37) and (44) with $\gamma_w > 1$ exists. As $\kappa_p \to \infty$, it satisfies $C_w \to 0$, $C_x \to 0$, $C_p \to 1$, $i_w \to 0$, $Q_w \to 0$, $\gamma_w \to \infty$.*

Proof. Please see the appendix. □

Lemma 6 implies that after the trap ($t > T$) we will have a stationary policy rule, and this policy rule converges to the flexible price policy rule when price becomes sufficiently flexible. Besides, the equilibrium path after the trap converges to the path of fully flexible price equilibrium. Specifically, suppose initially we have a positive real wage gap. In that case, such a gap will be filled in by price inflation ($C_p \to 1$), and the equilibrium path will come back to the first best equilibrium immediately for the next period ($w_{t-1}/w_t \equiv \gamma_w \to \infty$). Wage inflation rate and output gap both converge to zero ($C_w \to 0$ and $C_x \to 0$).

Now I characterize the equilibrium path during the trap ($t \leq T$) by using the backward induction. Suppose the policy rule after time t is $i_s(w_{s-1})$ $\forall s > t$. At $t+1$ the price inflation rate is $\pi_{t+1}^p(w_t)$, the wage inflation rate is $\pi_{t+1}^w(w_t)$, the output gap is $x_{t+1}(w_t)$, the evolution function is $w_{t+1}(w_t)$, and the welfare loss function is $Q_{t+1}^w(w_t)$. The IS equations and Phillips Curves at time t become

$$x_t = x_{t+1}(w_t) - (i_t - {r_t}^n - \pi_{t+1}^p(w_t)), \tag{45}$$

$$\pi^p{}_t = \beta \pi^p{}_{t+1}(w_t) + \kappa_p w_t, \tag{46}$$

$$\pi^w{}_t = \beta \pi^w{}_{t+1}(w_t) + \aleph_w x_t - \kappa_w w_t, \tag{47}$$

$$w_t = w_{t-1} + \pi^w{}_t - \pi^p{}_t - \Delta w_t{}^n, \tag{48}$$

The welfare loss function is

$$Q_t^w = x_t^2 + \mu_p(\pi_t^p)^2 + \mu_w(\pi_t^w)^2 + \beta Q_{t+1}^w(w_t), \tag{49}$$

Combining with (45)-(49), I find the evolution of w_{t-1} follows

$$\begin{aligned} w_{t-1} = N_w(i_t - r_t^n) + w_t(1+\kappa_p+\kappa_w) + \beta(\pi_{t+1}^p(w_t) - \pi_{t+1}^w(w_t)) - \\ N_w(x_{t+1}(w_t) + \pi_{t+1}^p(w_t)) + \Delta w_t^n \end{aligned} \tag{50}$$

Given the initial state w_{t-1}, the policy rule i_t determines the real wage gap w_t for the next period from (50). By using the implicit function theorem [9], I find

$$\frac{dw_t}{di_t} = -\frac{N_w}{1+\kappa_p+\kappa_w+\beta(\frac{d\pi_{t+1}^p(w_t)}{dw_t} - \frac{d\pi_{t+1}^w(w_t)}{dw_t}) - N_w(\frac{dx_{t+1}(w_t)}{dw_t} + \frac{d\pi_{t+1}^p(w_s)}{dw_t})} \tag{51}$$

Combining with (45)-(49), we know $\frac{dx_t}{di_t}$, $\frac{d\pi_t^p}{di_t}$ and $\frac{d\pi_t^w}{di_t}$ using the chain rule. Then I can write out the FOC of the government and find the optimal policy i_t. The following lemma shows the limit properties of key economic variables during the trap.

Lemma 7. *If $\pi_{t+1}^p(w_t) \to w_t - \Delta w_{t+1}^n$, $\pi_{t+1}^w(w_t) \to 0$, $x_{t+1}(w_t) \to 0$ and $Q_{t+1}^w(w_t) \to 0$ as $\kappa_p \to \infty$, then $\pi_t^p(w_{t-1}) \to w_{t-1} - \Delta w_t^n$, $\pi_t^w(w_{t-1}) \to 0$, $x_t(w_{t-1}) \to 0$ and $Q_t^w(w_{t-1}) \to 0$ and $w_t(w_{t-1}) \to 0$.*

Proof. Please see the appendix. □

A direct result of lemma 6 and lemma 7 is the following proposition:

Proposition 2. *When wages are sticky, and the government is forward-looking, as $\kappa_p \to \infty$, the nominal interest rate converges to the discount rate, and the economic path converges to the path of a price-frictionless economy.*

[9] I need $\pi_{t+1}^w(w_t)$, $\pi_{t+1}^p(w_t)$, $x_{t+1}(w_t)$ and $Q_{t+1}^w(w_t)$ to be differentiable. We know that after time T, they are either linear or quadratic, so they are differentiable. And I can also show by backward induction that those functional forms remain the same during the trap when I *do not* consider the ZLB constraint. Lemma 6 shows that ZLB is not binding in the limit, so those functions are indeed differentiable when the wage flexibility is large and the initial real wage gap w_{-1} is small.

Instead of an explosive solution, I show that the economic path is continuous with discretionary monetary policy and converges to the efficient one when wages are sticky. Therefore, no matter whether the government takes whatever happens in the future as given or expects the impact of its current policy on the future states in a rational way, larger price flexibility will not lead to a worse outcome but will bring this economy back to the frictionless equilibrium. The connection between a non-forward-looking government and a forward-looking government is shown in the following proposition:

Proposition 3. *When $\kappa_p \to \infty$ or $\kappa_w \to 0$, the first-order conditions of the forward-looking and non-forward-looking government converge to be the same.*

Proof. Please see the appendix. □

When I have flexible prices or sticky wages, the equilibria of a non-forward-looking and a forward-looking government converge to be the same. The economic intuitions are as follows: (1) when prices are sufficiently flexible, the adjustment cost of prices is small. The level of the real wage gap for the next period (the state variable for the next period) does not matter much, as the government for the next period has enough room to mitigate the real wage gap by price inflation (since the cost is small). So it is like the current government (though he might be forward-looking) takes the future state as $w_{t+1} = 0$ (as if prices are fully flexible), and the future path is independent of his current policy. (2) when wages are sufficiently sticky, the wage inflation rate hardly depends on the output gap and the real wage gap (since κ_w is small). The Phillips curves imply that the government can hardly change the wage inflation and thus the real wage gap for the next period through its policy. Thus, it is like the current government (though he might be forward-looking) takes whatever happens in the future as given in spite of its own policy.

Finally, I show that the fiscal multiplier converges to its frictionless level when the government is forward-looking. Consider the same fiscal policy as I defined in Section 2. Since $g_t^c = 0, \forall t \geq T$, the optimal policy function $i(w)$ and the functions for critical economic variables should be the same as our previous expressions for $t > T$.

For $t \leq T$, when introducing fiscal policy, let $\pi^{*p}_{t+1}(w_t)$, $\pi^{*w}_{t+1}(w_t)$, $x^*_{t+1}(w_t)$ and $Q^{*w}_{t+1}(w_t)$ denote the corresponding functions of key economic variables and the welfare loss function at time $t+1$. The IS equations and Phillips curves are

$$x_t - \Gamma g_t^c = (x^*_{t+1}(w_t) - \Gamma g^c_{t+1}) - (i_t - {r_t}^n + (1-\Gamma)\Delta g^c_{t+1} - \pi^{*p}_{t+1}(w_t)), \tag{52}$$

$$\pi^p{}_t = \beta\pi^{*p}{}_{t+1}(w_t) + \kappa_p w_t, \tag{53}$$

$$\pi^w{}_t = \beta\pi^{*w}{}_{t+1}(w_t) + \aleph_w(x_t - \Gamma g^c_t) - \kappa_w w_t, \tag{54}$$

The welfare loss function is

$$Q^w_t = (x_t - \Gamma g^c)^2 + \mu_p(\pi^p_t)^2 + \mu_w(\pi^w_t)^2 + \beta Q^{*w}_{t+1}(w_t), \tag{55}$$

Treating $x_t - \Gamma g^c_t$ as x_t and $i_t - r_t{}^n + (1-\Gamma)\Delta g^c_{t+1}$ as $i_t - r_t{}^n$ (denoted by Δi_t) , I get the same expression as before as if there is no fiscal policy. If ZLB is not binding in this case, I must have $x_t - \Gamma g^c_t \rightarrow 0$ (or $x_t \rightarrow \Gamma g^c$) since initially I have $x_t \rightarrow 0$ when there is no government spending. In other words, the fiscal multiplier converges to the frictionless level Γ. To show that the ZLB is not binding, recall that we have already found $i_t - r_t{}^n \rightarrow -\Delta w^n_{t+1}$ in lemma 6. Thus with fiscal policies I must have $i_t - r_t{}^n + (1-\Gamma)\Delta g^c_{t+1} \rightarrow -\Delta w^n_{t+1}$ (or $i_t \rightarrow \rho - (1-\Gamma)\Delta g^c_{t+1}$). Δg^c_{t+1} is either -$\bar{g^c}$ (when $t = T-1$) or 0 (when $t \neq T-1$) so i_t converges to either ρ or $\rho + (1-\Gamma)\bar{g^c}$. Therefore ZLB is indeed not binding.

4 Quantitative results

4.1 Methods

The above analysis focuses on the limit case where $\kappa_p \rightarrow \infty$. It does not tell us what happens in this economy when prices are very sticky. To characterize the economic path for small κ_p, I use the quantitative methods. From now on, I cannot impose the assumption that ZLB is not binding (especially when κ_p is small). So I need the backward evolution equation for both cases where ZLB is binding and is not binding.

Non-forward looking government:

Consider the case when the government is non-forward looking. If ZLB is not binding, the backward evolution equation is:

$$\begin{bmatrix} \pi_t{}^w \\ \pi_t{}^p \\ w_{t-1} \\ x_t \end{bmatrix} = \begin{bmatrix} \frac{\beta}{1+\phi N_w} & 0 & \frac{-\kappa_w}{1+\phi N_w} & 0 \\ 0 & \beta & \kappa_p & 0 \\ \frac{-\beta}{1+\phi N_w} & \beta & 1 + \frac{\kappa_w}{1+\phi N_w} + \kappa_p & 0 \\ \frac{-\phi\beta}{1+\phi N_w} & 0 & \frac{\phi\kappa_w}{1+\phi N_w} & 0 \end{bmatrix} \begin{bmatrix} \pi_{t+1}{}^w \\ \pi_{t+1}{}^p \\ w_t \\ x_{t+1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ g_t \\ 0 \end{bmatrix}, \tag{56}$$

If ZLB is binding, it becomes:

$$
\begin{bmatrix} {\pi_t}^w \\ {\pi_t}^p \\ w_{t-1} \\ x_t \end{bmatrix} = \begin{bmatrix} \beta & N_w & -\kappa_w & N_w \\ 0 & \beta & \kappa_p & 0 \\ -\beta & \beta - N_w & 1+\kappa_w+\kappa_p & -N_w \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} {\pi_{t+1}}^w \\ {\pi_{t+1}}^p \\ w_t \\ x_{t+1} \end{bmatrix} + \begin{bmatrix} N_w r_t \\ 0 \\ g_t - N_w r_t \\ r_t \end{bmatrix}, \tag{57}
$$

To find whether ZLB is binding or not, I check the implied i_t from FOC:

$$
i_t = \begin{bmatrix} \frac{\phi\beta}{1+\phi N_w}, & 1, & \frac{-\phi\kappa_w}{1+\phi N_w}, & 1 \end{bmatrix} \begin{bmatrix} {\pi_{t+1}}^w \\ {\pi_{t+1}}^p \\ w_t \\ x_{t+1} \end{bmatrix} + r_t. \tag{58}
$$

ZLB is binding iff (58) is negative. From our analysis in Section 2, I also know that

$$
\begin{bmatrix} {\pi_{T+1}}^w \\ {\pi_{T+1}}^p \\ w_T \\ x_{T+1} \end{bmatrix} = \begin{bmatrix} x_w \\ x_p \\ 1 \\ -\phi x_w \end{bmatrix} w_T. \tag{59}
$$

Then, I use the following steps to solve the model quantitatively:

Step 1: Find the eigenvalue λ_3 (the one larger than 1) of A;

Step 2: Compute x_w and x_p;

Step 3: Guess w_{T-1} and compute w_{-1} backwardly by (56)(57)(58)(59)

Step 4: Solve w_{T-1} and the whole economic path using the bisection method.

Forward looking government:

Now consider the case when the government is forward-looking. The optimal policy function and functions of key economic variables can be solved backwardly as well. Specifically, I use the following steps to solve the model:

Step 1: solve equations (32)-(35), (37) and (44) to find the equilibrium after the recession and the optimal policy rule.

Step 2: solve the policy rule and key economic variables during the recession backwardly. Suppose I know $\pi_p^{t+1}(w_t)$, $\pi_w^{t+1}(w_t)$, $x_{t+1}(w_t)$, and $Q_{t+1}(w_t)$ and the initial real wage gap is w_{t-1}. Instead of choosing i_t, here I choose w_t, as w_t implies the current policy i_t by using the IS equation and Phillips Curves, and it is easier for us to calculate.

The interest rate i_t is very sensitive to our chosen w_t [10]. Thus I use linear interpolation for π_p^{t+1}, π_w^{t+1}, x_{t+1} and quadratic interpolation for Q^w_{t+1}.

Step 3: construct the path for given w_{-1}. As I use interpolation in Step 2, $w_t(w_{t-1})$ might not be a grid point. So I use linear interpolation again to find $\pi_p^{t+1}(w_t)$, $\pi_w^{t+1}(w_t)$, $x_{t+1}(w_t)$ and construct the path.

4.2 Quantitative Results

Our model period is one quarter. For the baseline model, I use Gali's calibration and set $w_{-1} = 0$ and the length of the recession to be 10 quarters. Our calibration is summarized in Table 1. The results for the non-forward-looking government are shown as follows (for the baseline model, the results are plotted by the solid lines):

Parameter	Description	Value
β	Discount factor	0.99
$1/\psi$	Frisch elasticity	0.2
θ_p	Price revision rate	3/4
θ_w	Wage revision rate	3/4
ϵ_p	Elasticity of substitution across intermediate goods	9
ϵ_w	Elasticity of substitution across labor types	4.5
g_a	Productivity growth rate during the trap	-0.02
g_b	Productivity growth rate after the trap	0

TABLE 1: CALIBRATION

[10]the coefficient $\sim 1/N_w$ where $1/N_w$ is 45 under our calibration.

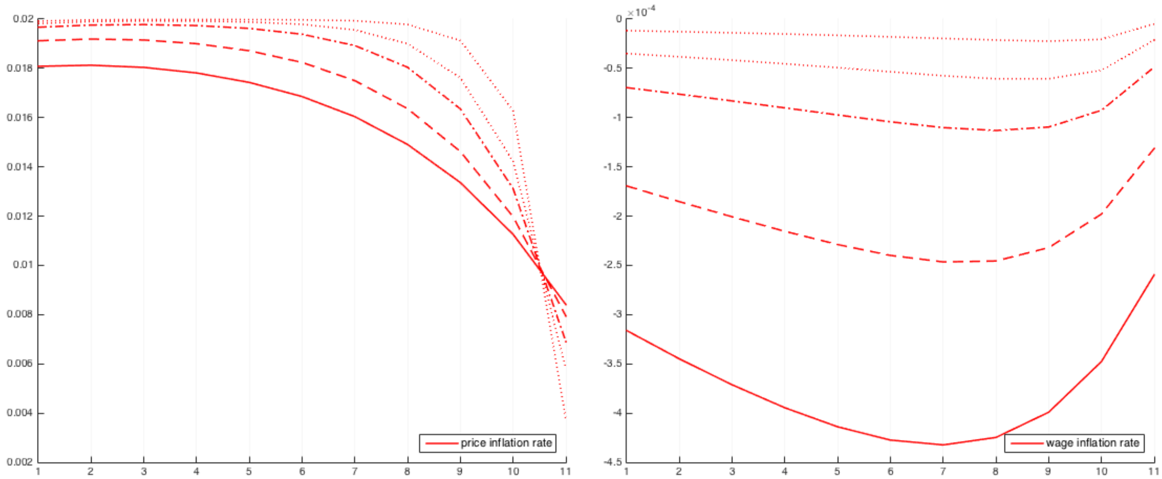

FIGURE 2: NON-FORWARD LOOKING GOVERNMENT: PRICE INFLATION AND WAGE INFLATION RATE

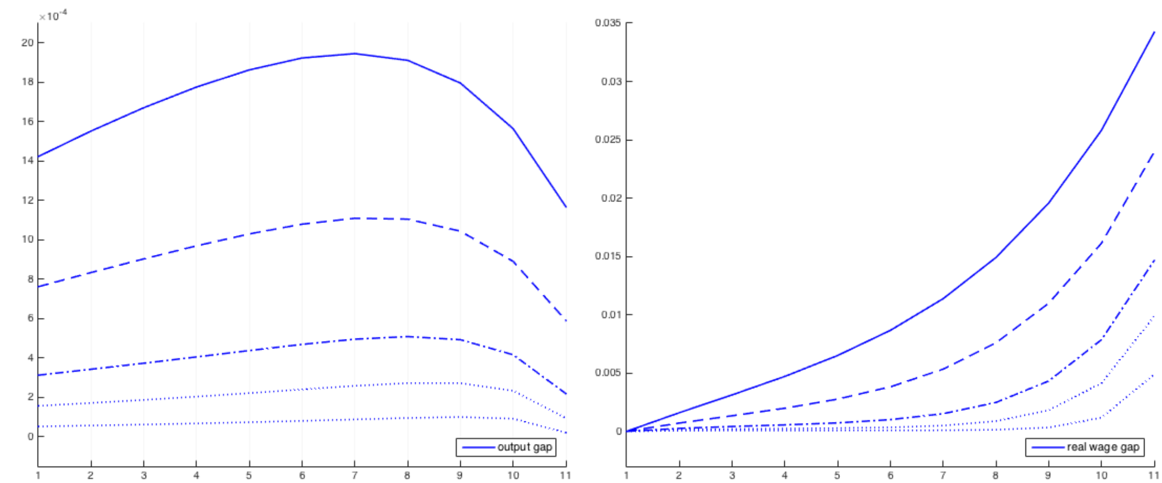

FIGURE 3: NON-FORWARD LOOKING GOVERNMENT: OUTPUT GAP AND REAL WAGE GAP

The results for the forward-looking government (for the solid line, I use Gali's calibration):

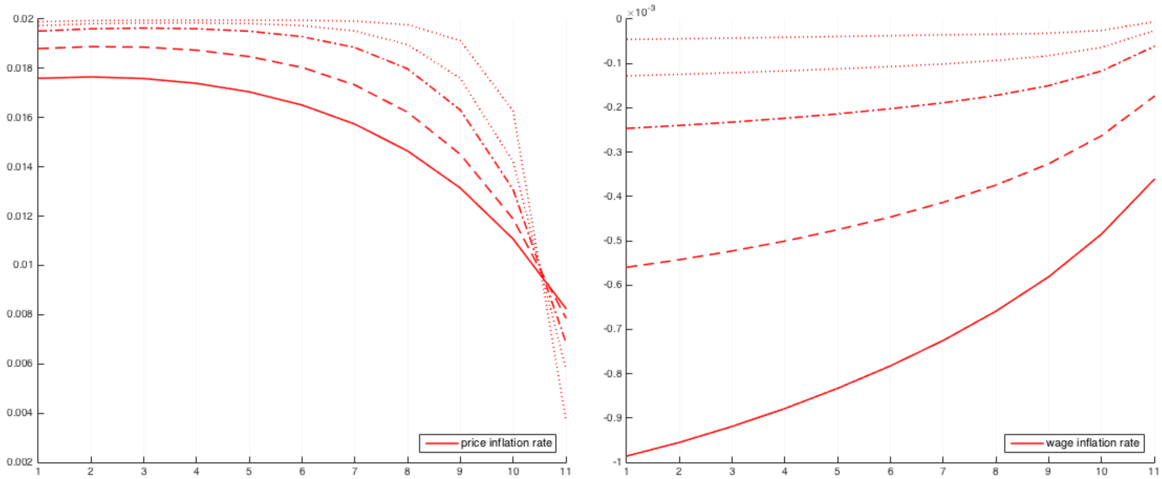

FIGURE 4: FORWARD-LOOKING GOVERNMENT: PRICE INFLATION AND WAGE INFLATION RATE

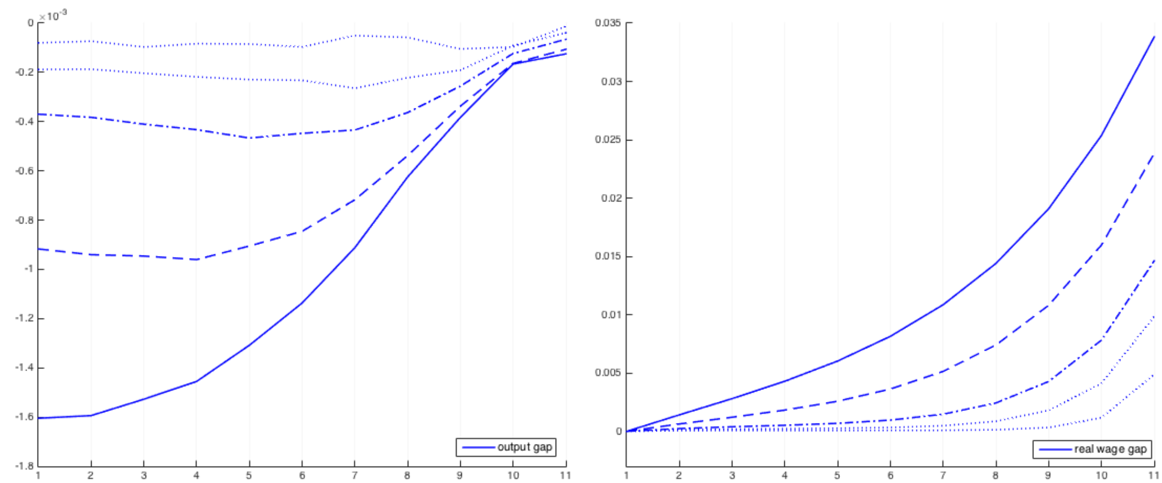

FIGURE 5: FORWARD-LOOKING GOVERNMENT: OUTPUT GAP AND REAL WAGE GAP

The quantitative results for the non-forward-looking government (Figure 2 and Figure 3) of our baseline calibration are very similar to the forward-looking counterparts (Figure 4 and Figure 5). Especially for the output gap and inflation rate, the values are nearly the same. As proposition 3 implies (the convergence of two equilibria), as I set $\theta_w = 3/4$ under rhe baseline calibration, the wages are sticky, and the results for two equilibria should be similar to each other.

The dotted lines from Figure 2 to Figure 5 imply larger κ_p or more flexible price. In particular, the dashed line represents relatively small price flexibility (but larger than our baseline calibration), the dash-dotted line represents the medial price flexibility, and the dotted line represents the largest price flexibility. As is shown in the graph, the wage inflation rate converges to zero while the price inflation rate converges to the inverse of the productivity growth rate (it's -2% when $t \leq 10$ and 0 when $t > 10$). For the output gap, when prices are sticky (κ_p is small), the economy suffers from great depression during the recession periods (x_t decreases exponentially for the solid line) when the government is forward-looking. When the government is non-forward looking, I have a small positive output gap because of the inflation of prices. But as prices become more flexible, both output gaps converge to zero. The real wage is above the frictionless level for both non-forward-looking and forward-looking cases. Besides, the real wage gap increases with time. The reason is that the decline of the real wage is smaller than the decline of productivity, for we have both sticky wages and sticky prices. Adjusting the real wage is costly, and the real wage goes down more slowly than productivity. Similarly, the real wage gap converges to zero as prices become more flexible.

Figure 6 plots the nominal interest rate, which shows an interesting pattern:

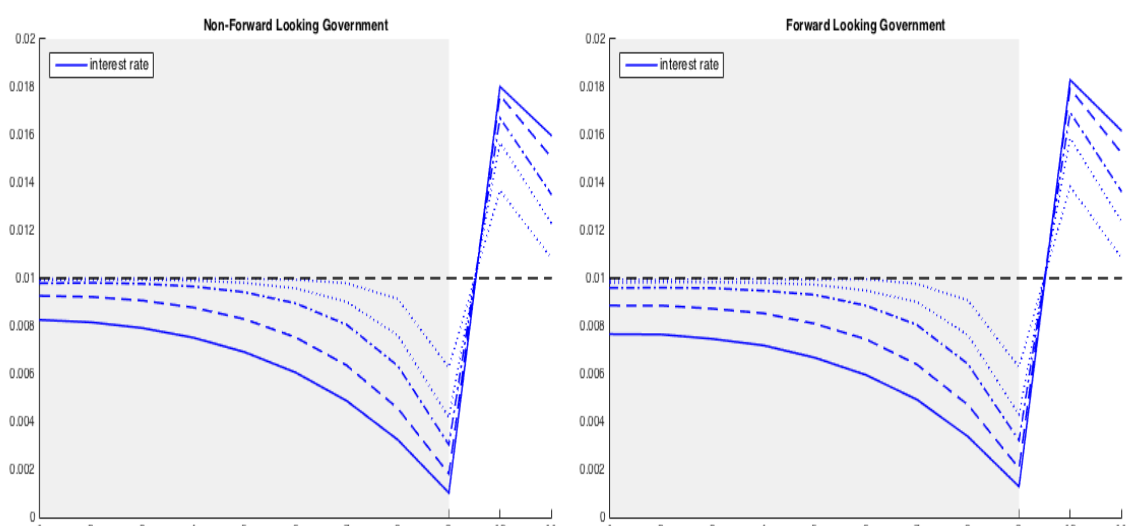

FIGURE 6: NOMINAL INTEREST RATE

When prices are sticky, ZLB turns out to be binding at the end of the recession rather than at the beginning (the solid line). It further confirms our statement that the feedback loop of the deflation and the depression disappears when wages are sticky. With more flexible prices, ZLB is not binding (the dotted lines), and the nominal interest rate converges to the discount rate ($\rho = 0.01$).

4.3 Welfare Loss

I answer one of the interesting questions in this section: Is price flexibility harmful?

I compute the total welfare loss (according to the second-order approximation) with respect to both price and wage flexibility. The results are as follows:

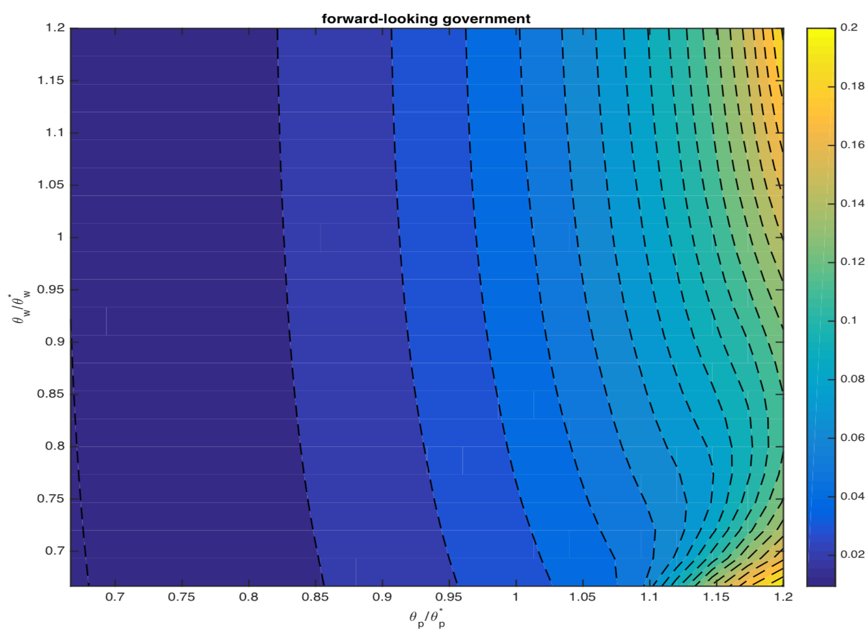

FIGURE 7: WELFARE LOSS FUNCTION: FORWARD-LOOKING GOVERNMENT

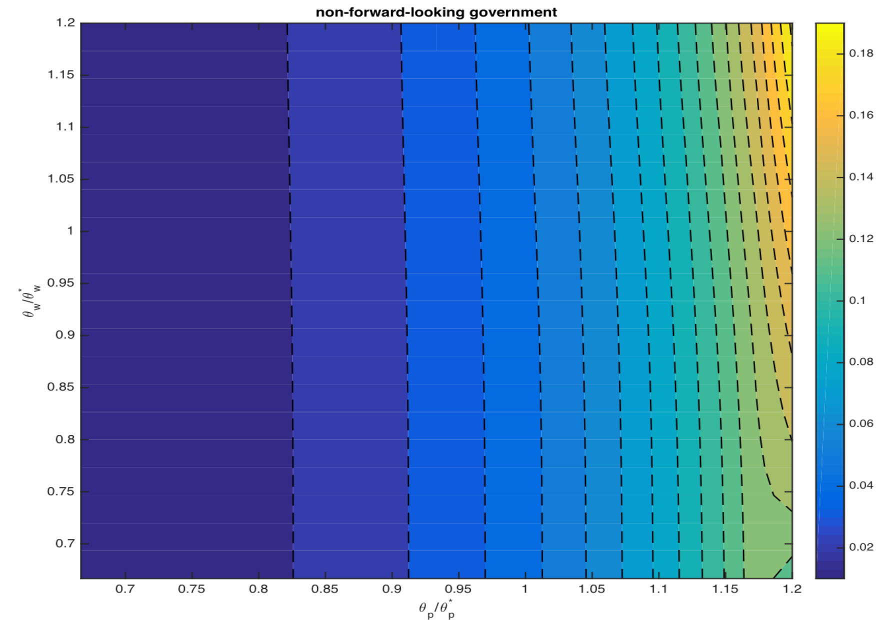

FIGURE 8: WELFARE LOSS FUNCTION: NON-FORWARD LOOKING GOVERNMENT

Figure 7 shows the welfare loss when the government is forward-looking. Figure 8 shows the welfare loss when the government is not forward-looking. The light colors (green $\rightarrow$ orange $\rightarrow$ yellow) imply larger welfare loss, while the dark colors (dark blue $\rightarrow$ blue) imply smaller welfare loss. From Figure 7 and Figure 8, we can see the following pattern *for our ranges of parameters*:

(1) Price flexibility is always beneficial, given the level of wage flexibility.

(2) Wage flexibility could either be beneficial or harmful.

The economic reasons are as follows. Consider the case where ZLB is not binding, but the economy suffers from some negative productivity shocks. The efficient outcome requires the real wage to decrease to the same level of productivity, while the nominal rigidities of prices and wages make such adjustments costly. When prices or wages are more flexible, the economy experiences lower costs for such adjustments. When wages are not fully flexible, there are two channels to reduce the real wage: by the inflation of prices or the deflation of wages. When prices become more flexible, the inflation channel is less costly, and the government has a stronger incentive for greater inflation. This incentive after and during the recession makes ZLB less binding.That's why the economy can ultimately escape from the vicious cycle of deflation and depression for sufficiently flexible prices. From both aspects, price flexibility is beneficial as it lowers the cost of the real wage adjustment and makes ZLB less binding. However, when wages are more flexible, the government prefers the deflation channel (the deflation of wages) instead of the inflation channel (the inflation of prices). So, it lowers the incentive for greater inflation and makes ZLB more binding. When ZLB is binding, monetary policy is ineffective in reducing the costly change of the real wage gap, and the economy suffers from a greater welfare loss. Such negative impact is especially larger when ZLB has already been binding, as we can see in the graph above (1) when θ_p is large (prices are very sticky and ZLB is close to being binding), increasing wage flexibility is at first beneficial (because it lowers the adjustment cost of the real wage) but then harmful (because it makes ZLB be binding); (2) when θ_p is small (prices are flexible), ZLB is always not binding, and thus increasing wage flexibility is always beneficial as it lowers the adjustment cost of the real wage gap.

4.4 Forward Guidance

In this section, I consider the impact of forward guidance. Forward guidance is employed in the following way: the government at time T is not allowed to optimize, and he must set $i_T = 0$. Here I show two distinct results about the effect of the forward guidance with wage rigidities:

(1) The effect of the forward guidance becomes much smaller with a sticky wage.

(2) Such effect converges to infinity with price flexibility for the standard New Keynesian model but converges to zero for the sticky wage model.

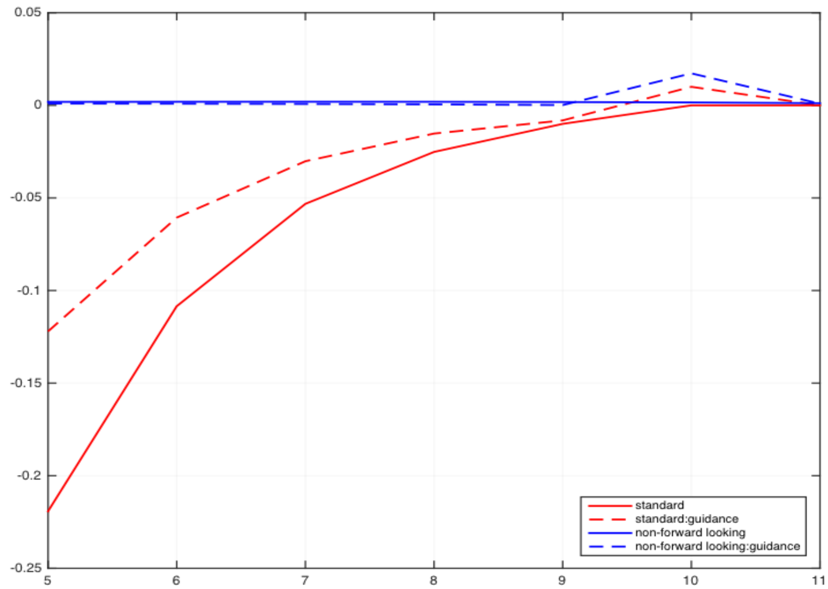

FIGURE 10: FORWARD GUIDANCE: OUTPUT GAP (NON-FORWARD LOOKING GOVERNMENT)

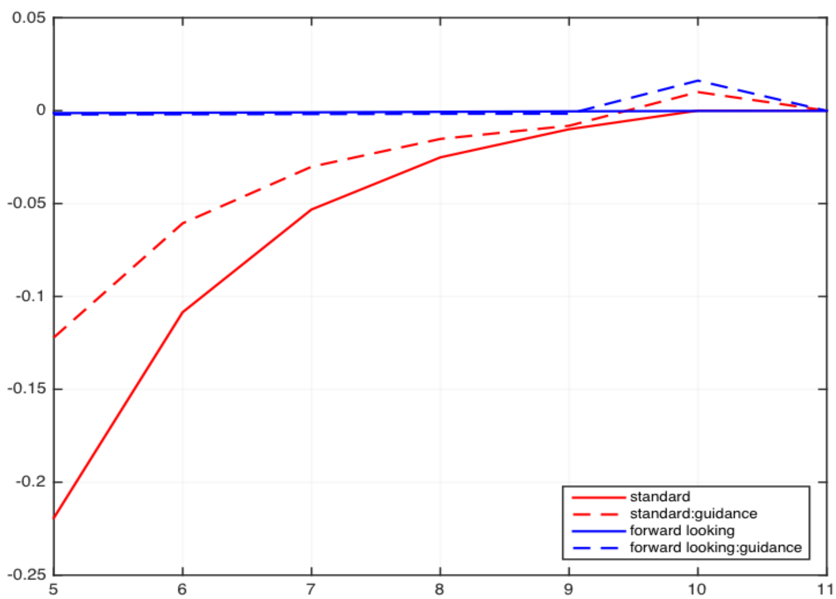

FIGURE 11: FORWARD GUIDANCE: OUTPUT GAP (FORWARD LOOKING GOVERNMENT)

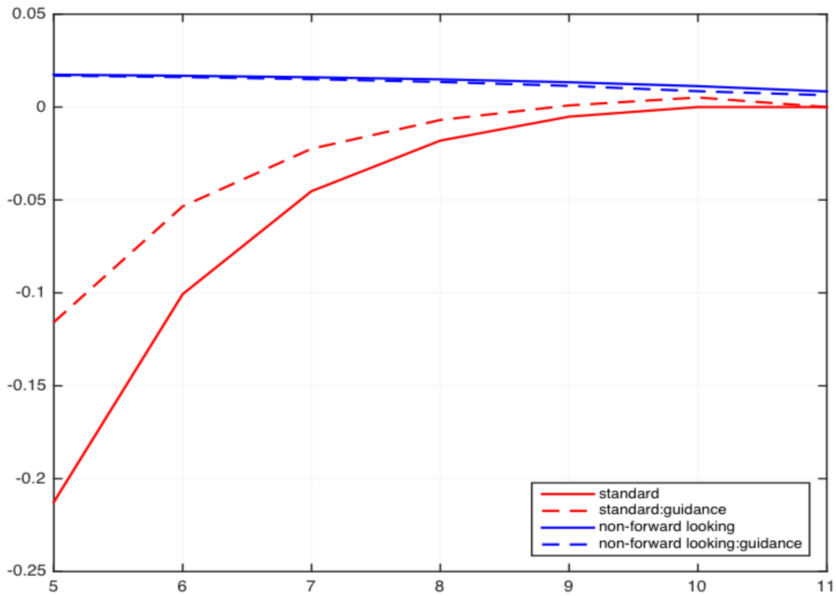

FIGURE 12: FORWARD GUIDANCE: INFLATION RATE (NON-FORWARD LOOKING GOVERNMENT)

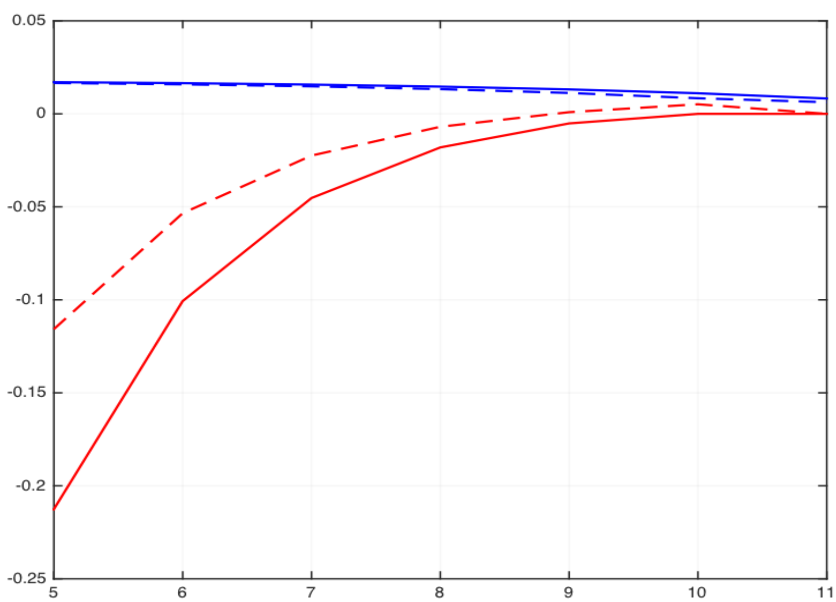

FIGURE 13: FORWARD GUIDANCE: INFLATION RATE (FORWARD LOOKING GOVERNMENT)

Figure 10 and Figure 11 show the effect of the forward guidance on the output gap.

Figure 12 and Figure 13 show the effect of the forward guidance on the inflation rate. The solid lines represent the sticky wage model and the dotted lines represent the standard model. I use the same calibration as the sticky wage model for the standard model, except that I assume wages are fully flexible. Before we look at the effect of the forward guidance, I first compare the equilibria of sticky wage (blue solid line) and flexible wage models (red solid line) when there is no forward guidance. We can see that without forward guidance, the economy with sticky wages behaves much more normally than the economy without sticky wages, whether the government is forward-looking or not, for the reasons I discussed previously.

When there is forward guidance, we observe that the effect of the forward guidance on the real output gap (the difference between the light blue and the dark blue lines) and on the inflation rate (the difference between the orange and the red lines) is tiny for the sticky wage model while their effects are much greater for the standard New Keynesian model. The reason is similar to what I discuss above. Forward guidance has magical power in the standard model as it raises the inflation rate at the end of the liquidity trap, which helps the economy escape from the vicious cycle of deflation and depression. However, when wages are sticky, there is an incentive for greater inflation to reduce the adjustment cost of the real wage during the recession, so the power of forward guidance is much muted.

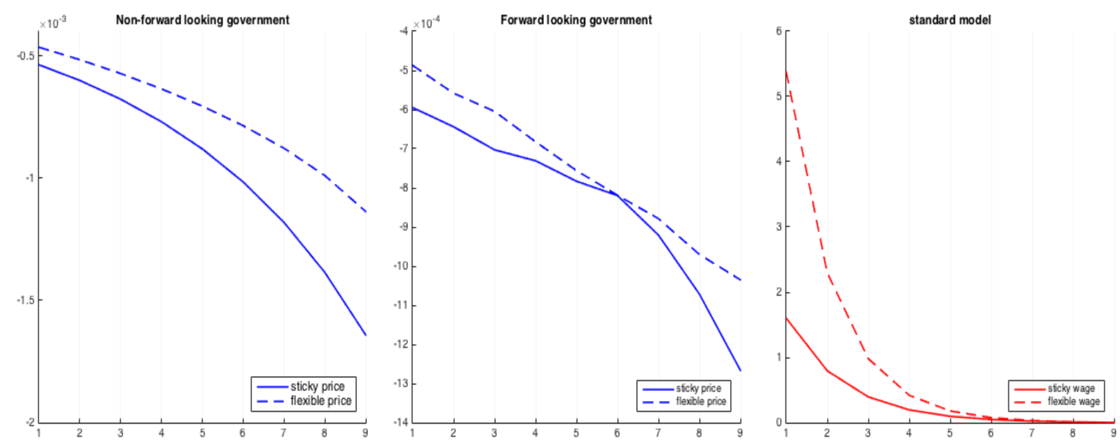

FIGURE 14: THE IMPACT OF FORWARD GUIDANCE ON THE OUTPUT GAP FOR DIFFERENT PRICE FLEXIBILITY

Figure 14 shows how the effect of the forward guidance varies with different price flexibilities. I measure such effect by the real output difference with and without the forward guidance, i.e. $x_t^* - x_t$. Solid lines are for stickier prices, and dotted lines are for more flexible prices. The blue color is for the sticky wage model, and the red color is for the standard model. The standard model predicts that when prices are more flexible, the power of the forward guidance is growing larger. This is counterintuitive: as the nominal rigidity goes down, monetary policy should have a smaller impact on the real economy by controlling the nominal interest rate. For the sticky wage model, the results are consistent with our intuition and observations. When prices are more flexible, the effect of the forward guidance becomes smaller. The reason is that when prices become more flexible, the price inflation channel becomes cheaper than the wage deflation channel in order to reduce the real wage. Thus, the government has a stronger incentive for higher inflation, and it further diminishes the effect of the forward guidance as it affects the equilibrium path during the recession by affecting the inflation rate.

4.5 Fiscal Multiplier

For the standard model, we know the fiscal multiplier is paradoxically large (much greater than one) during the liquidity trap. In this section, I show two distinct results for the fiscal multiplier for the sticky wage model:

(1) Fiscal multiplier becomes much smaller when wages are sticky.

(2) Fiscal multiplier converges to infinity with price flexibility for the standard model but converges to the frictionless level for the sticky wage model.

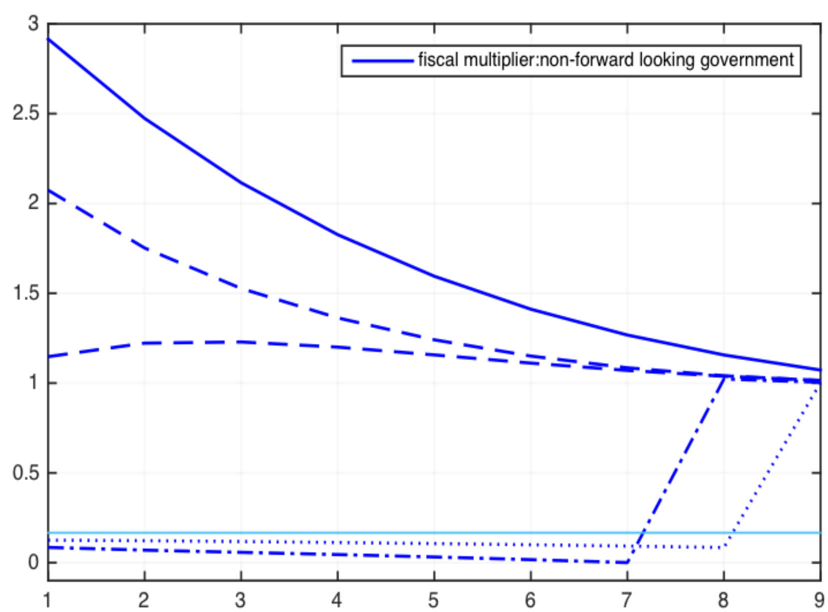

FIGURE 15: FISCAL MULTIPLIER: NON-FORWARD LOOKING GOVERNMENT

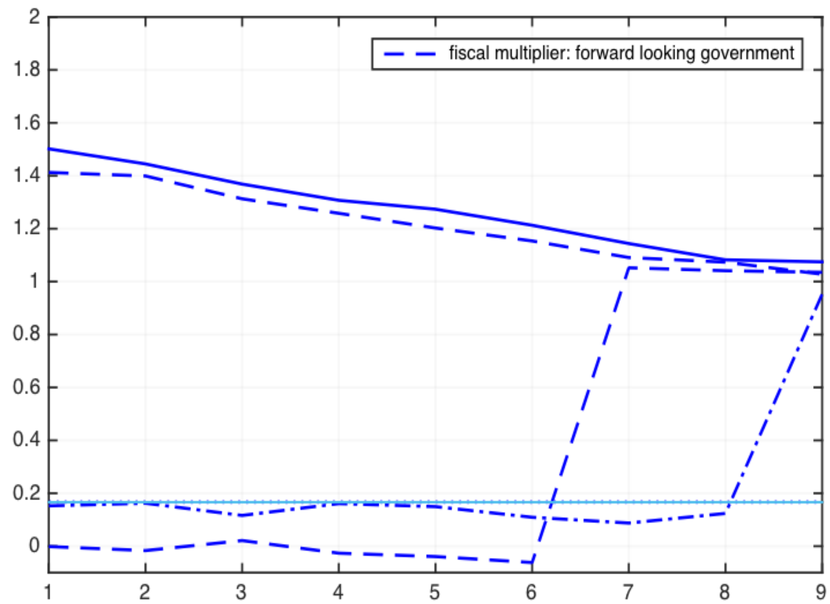

FIGURE 16: FISCAL MULTIPLIER: FORWARD-LOOKING GOVERNMENT

Figure 15 shows the fiscal multiplier of a non-forward-looking government. Figure 16 shows the fiscal multiplier of a forward-looking government. The solid line uses our calibration, while the dotted lines are for more flexible prices (the same parameters except θ_p). The light blue lines represent the frictionless fiscal multiplier. From (52), we know the fiscal multiplier of the standard model is $(\aleph_p + 1)^{T-1-t}(1 - \Gamma) + \Gamma$ during the liquidity trap when ZLB is binding. According to our baseline calibration, it is 144.1 at time $t = 1$ and 81.4 at time $t = 2$, and it is much larger than the level of the sticky wage model which is below 3. The reason why we have a smaller fiscal multiplier is again the incentive for inflation. For the standard model, fiscal policy has a great impact on the economy during the liquidity trap as it plays the role the monetary policy should play, but not because of the ZLB constraint. For the standard model, when the government spending increases by $\bar{g^c}$ at time $T - 1$, the output increases by $\bar{g^c}$ and the real output gap increases by $(1 - \Gamma)\bar{g^c}$ (since the output increases by $\Gamma\bar{g^c}$ for the economy without nominal rigidities) at time $T - 1$. A greater output gap stimulates a greater inflation rate and relieves the impact of the vicious feedback loop. However, when wages are sticky, the government has an incentive for inflation, and ZLB tends not to be binding. Therefore, there is less room for fiscal policy to substitute the role of the monetary policy during the liquidity trap, and we observe a much smaller fiscal multiplier. Besides, with more flexible prices, fiscal policy has less impact, as shown in Figure 15 and Figure 16.

5 Conclusion

I introduce wage stickiness into the standard New Keynesian model to resolve the New Keynesian puzzles and paradoxes. Sticky wages add an inflation channel of prices to lower the real wage gap, which attenuates the feedback loop of the IS equation and the Phillips Curve. I study a forward-looking and a non-forward-looking government and find that; in both cases, the equilibria converge with the price frictionless equilibria with increasing price flexibility. This convergence is guaranteed by the uniqueness of the solution when wages are sticky. Our quantitative results show that the economy performs much normally in the liquidity trap. Forward guidance and fiscal policy are not as effective as the standard New Keynesian model implies, and structural reforms that increase price flexibility are not contractionary and are beneficial for the economy. The fiscal multiplier decreases to the frictionless level instead of growing explosively with the flexibility of prices. In literature, sticky wage New Keynesian models are usually employed to study labor market performances, such as the unemployment rate and the labor income share over the business cycle. So, our study emphasizes the importance of wage stickiness in a zero lower-bound interest rate economy.

References

Andrade, Philippe, Gaetano Gaballo, Eric Mengus, and Benoît Mojon, "Forward Guidance and Heterogeneous Beliefs," *American Economic Journal: Macroeconomics*, 2019, *11* (3), 1–29.

Angeletos, George-Marios and Chen Lian, "Forward Guidance without Common Knowledge," *American Economic Review*, 2018, *108* (9), 2477–2512.

Auclert, Adrien, "Monetary Policy and the Redistribution Channel," *American Economic Review*, 2019, *109* (6), 2333–2367.

Benhabib, Jess, George W. Evans, and Seppo Honkapohja, "Liquidity Traps and Expectation Dynamics: Fiscal Stimulus or Fiscal Austerity?," *Journal of Economic Dynamics and Control*, 2014, *45*, 220–238.

Bielecki, Marcin, Michał Brzoza-Brzezina, and Marcin Kolasa, "Comment on "The Limits of Forward Guidance" by Jeffrey R. Campbell, Filippo Ferroni, Jonas D. M. Fisher and Leonardo Melosi," *Journal of Monetary Economics*, 2019, *108*, 135–139.

Bilbiie, Florin O., "Optimal Forward Guidance," *American Economic Journal: Macroeconomics*, 2019, *11* (4), 310–345.

Bundick, Brent and A. Lee Smith, "The Dynamic Effects of Forward Guidance Shocks," *The Review of Economics and Statistics*, 2020, *102* (5), 946–965.

Calvo, Guillermo A., "Staggered Prices in a Utility-Maximizing Framework," *Journal of Monetary Economics*, September 1983, *12* (3), 383–398.

Carlstrom, Charles T., Timothy S. Fuerst, and Matthias Paustian, "Inflation and Output in New Keynesian Models with a Transient Interest Rate Peg," Working Paper 1234, Federal Reserve Bank of Cleveland 2012.

Carrillo, Julio A. and Céline Poilly, "How do Financial Frictions Affect the Spending Multiplier During a Liquidity Trap?," *Review of Economic Dynamics*, 2013, *16* (2), 296–311.

Christiano, Lawrence, Martin Eichenbaum, and Sergio Rebelo, "When Is the Government Spending Multiplier Large?," *Journal of Political Economy*, 2011, *119* (1), 78–121.

Cochrane, John H., "The New-Keynesian Liquidity Trap," *Journal of Monetary Economics*, 2017, *92*, 47–63.

Cook, David and Michael B. Devereux, "Optimal fiscal policy in a world liquidity trap," *European Economic Review*, 2011, *55* (4), 443–462.

_ and _ , "Sharing the Burden: Monetary and Fiscal Responses to a World Liquidity Trap," *American Economic Journal: Macroeconomics*, 2013, *5* (3), 190–228.

Diba, Behzad and Olivier Loisel, "Pegging the interest rate on bank reserves: A resolution of New Keynesian puzzles and paradoxes," *Journal of Monetary Economics*, 2021, *118*, 230–244.

Erceg, Christopher J. and Jesper Lindé, "Is There a Fiscal Free Lunch in a Liquidity Trap?," *Journal of the European Economic Association*, 2014, *12* (1), 73–107.

_ , Dale W. Henderson, and Andrew T. Levin, "Optimal Monetary Policy with Staggered Wage and Price Contracts," *Journal of Monetary Economics*, October 2000, *46* (2), 281–313.

Fujiwara, Ippei and Kozo Ueda, "The Fiscal Multiplier and Spillover in a Global Liquidity Trap," *Journal of Economic Dynamics and Control*, 2013, *37* (7), 1264–1283.

_ and Yuichiro Waki, "The Delphic Forward Guidance Puzzle in New Keynesian Models," *Review of Economic Dynamics*, 2022, *46*, 280–301.

Galí, Jordi, *Monetary Policy, Inflation, and the Business Cycle: An Introduction to the New Keynesian Framework and Its Applications*, Princeton, NJ: Princeton University Press, 2015.

Gibbs, Christopher G. and Nigel McClung, "Does My Model Predict a Forward Guidance Puzzle?," *Review of Economic Dynamics*, 2023, *51*, 393–423.

Hagedorn, Marcus, Jinfeng Luo, Iourii Manovskii, and Kurt Mitman, "Forward guidance," *Journal of Monetary Economics*, 2019, *102*, 1–23.

Kiley, Michael T., "Policy Paradoxes in the New Keynesian Model," *Review of Economic Dynamics*, 2016, *21*, 1–15.

Kydland, Finn E. and Edward C. Prescott, "Rules Rather Than Discretion: The Inconsistency of Optimal Plans," *Journal of Political Economy*, 1977, *85* (3), 473–491.

Lacker, Jeffrey M., "Forward Guidance a Comment," *Journal of Monetary Economics*, 2018, *102*, 24–28.

Lemoine, Matthieu and Jesper Lindé, "Fiscal Stimulus in Liquidity Traps: Conventional or Unconventional Policies?," *European Economic Review*, 2023, *151*, 104324.

McKay, Alisdair, Emi Nakamura, and Jón Steinsson, "The Power of Forward Guidance Revisited," *American Economic Review*, 2016, *106* (10), 3133–3158.

Michau, Jean-Baptiste, "Monetary and Fiscal Policy in a Liquidity Trap with Inflation Persistence," *Journal of Economic Dynamics and Control*, 2019, *100*, 1–28.

Nakata, Taisuke, Ryota Ogaki, Sebastian Schmidt, and Paul Yoo, "Attenuating the Forward Guidance Puzzle: Implications for Optimal Monetary Policy," *Journal of Economic Dynamics and Control*, 2019, *105*, 90–106.

Negro, Marco Del, Marc Giannoni, and Christina Patterson, "The Forward Guidance Puzzle," Staff Report 574, Federal Reserve Bank of New York 2013.

Werning, Iván, "Managing a Liquidity Trap: Monetary and Fiscal Policy," Working Paper 17344, National Bureau of Economic Research 2011.

Woodford, Michael, *Interest and Prices: Foundations of a Theory of Monetary Policy*, Princeton, NJ: Princeton University Press, 2003.

_ , "Simple Analytics of the Government Expenditure Multiplier," *American Economic Journal: Macroeconomics*, 2011, *3* (1), 1–35.

Online Appendix of "Liquidity trap revisited: when wages are sticky"

The New Keynesian Equations:

I derive the IS equations and Phillips curves for prices and wages. The intertemporal Euler condition can be represented in a log-linearized form as

$$c_t = E_t\{c_{t+1}\} - \left(i_t - E_t\{\pi^p_{t+1}\} - \rho\right)$$

where $\rho = -\log \beta$. When the goods market is clearing, I have $y_t = c_t$. Rewriting the above equation in terms of the output gap, I have

$$x_t = E_t\left[x_{t+1}\right] - \left(i_t - E_t\left[\pi^p_{t+1}\right] - r^n_t\right)$$

Log-linearizing the first-order condition for the wage setting around that steady state yields

$$\omega^*_t = \mu^w + (1-\beta\theta_w)\sum_{k=0}^{\infty}(\beta\theta_w)^k E_t\left\{c_{t+k} + \varphi n_{t+k|t} + p_{t+k}\right\}$$

Up to a first-order approximation, one can write the log-linearized output as

$$y_t = n_t + a_t$$

The change in natural real wage, that is, the real wage that would prevail in the absence of nominal rigidities, is given by

$$w^n_t = a_t$$

Incorporating the expression for $N_{t+k|t}$, I have: [11]

$$n_{t+k|t} = -\epsilon_w(\omega^*_t - \omega_{t+k}) + n_{t+k}$$

[11] Throughout this paper, ω_t and ω^*_t denote the log of the nominal wage, w^n_t represents the natural real wage, and w_t signifies the real wage gap.

Recursively, w_t^* can be expressed as

$$\omega_t^* = (1-\beta\theta_w)\sum_{k=0}^{\infty}(\beta\theta_w)^k E_t\left\{\mu^w + c_{t+k} - \epsilon_w\varphi(\omega_t^* - \omega_{t+k}) + \varphi n_{t+k} + p_{t+k}\right\}$$

$$= \frac{1-\beta\theta_w}{1+\epsilon_w\varphi}\sum_{k=0}^{\infty}(\beta\theta_w)^k E_t\left\{(\epsilon_w\varphi+1)\omega_{t+k} + (y_{t+k} + \varphi n_{t+k} - (\omega_{t+k} - p_{t+k}) + \mu_w)\right\}$$

$$= \beta\theta_w E_t\{\omega_{t+1}^*\} + (1-\beta\theta_w)\left(\omega_t - (1+\epsilon_w\varphi)^{-1}\hat{\mu}_t^w\right)$$

where $\hat{\mu}_t^w \equiv \mu_w - (\omega_{t+k} - p_{t+k} - (y_{t+k} + \varphi n_{t+k}))$. $\hat{\mu}_t^w$should equal zero when prices and wages are flexible. Therefore,

$$\hat{\mu}_t^w = (1+\varphi)x_t - w_{t+k}$$

Combining with the wage evolution equation:

$$\omega_t = \theta_w\omega_{t-1} + (1-\theta_w)\omega_t^*$$

I have the Phillips curve for wages (3). The real wage gap follows (4) from its definition. The IS equation and Phillips curve for prices are derived similarly to the standard New Keynesian model.

Proof of Lemma 2:

For the characteristic function, I have

$$f(\lambda) \equiv |\lambda I - A| = (\lambda-\beta)(\lambda - (1 + \frac{\kappa_w}{1+\phi N_w} + \kappa_p))(\lambda - \frac{\beta}{1+\phi N_w})$$
$$-\beta\kappa_p(\lambda - \frac{\beta}{1+\phi N_w}) - \frac{\beta}{1+\phi N_w}\frac{\kappa_w}{1+\phi N_w}(\lambda-\beta)$$

Firstly, I show $f(0) < 0$:

$$f(0) = -\beta\frac{\beta}{1+\phi N_w}(1 + \frac{\kappa_w}{1+\phi N_w} + \kappa_p) + \beta\kappa_p\frac{\beta}{1+\phi N_w} + \beta\frac{\beta}{1+\phi N_w}\frac{\kappa_w}{1+\phi N_w}$$
$$= -\beta\frac{\beta}{1+\phi N_w} < 0$$

Secondly, I show $f(\frac{\beta}{1+\phi N_w}) > 0$:

$$f(\frac{\beta}{1+\phi N_w}) = -\frac{\beta}{1+\phi N_w}\frac{\kappa_w}{1+\phi N_w}(\frac{\beta}{1+\phi N_w} - \beta) > 0$$

Then, I show $f(1) < 0$:

$$f(1) = -(1-\beta)(\frac{\kappa_w}{1+\phi N_w} + \kappa_p)(1 - \frac{\beta}{1+\phi N_w}) - \beta\kappa_p(1 - \frac{\beta}{1+\phi N_w}) - \frac{\beta}{1+\phi N_w}\frac{\kappa_w}{1+\phi N_w}(1-\beta) < 0$$

where the inequality holds because each term is negative.

Finally, I show $f(1 + \frac{\kappa_w}{1+\phi N_w} + \kappa_p) < 0$:

$$f(1 + \frac{\kappa_w}{1+\phi N_w} + \kappa_p) = -\beta\kappa_p(1 + \frac{\kappa_w}{1+\phi N_w} + \kappa_p - \frac{\beta}{1+\phi N_w}) - \frac{\beta}{1+\phi N_w}\frac{\kappa_w}{1+\phi N_w}(1 + \frac{\kappa_w}{1+\phi N_w} + \kappa_p - \beta) < 0$$

where the inequality holds because each term is negative. Notice that when $\lambda \to \infty$, $f(\lambda) \to \infty$ (since the coefficient of λ^3 is 1). Thus we must find a number $K > 1 + \frac{\kappa_w}{1+\phi N_w} + \kappa_p$ satisfying $f(K) > 0$. By the continuity of $f(\lambda)$, we know three eigenvalues lie in ranges $(0, \frac{\beta}{1+\phi N_w})$, $(\frac{\beta}{1+\phi N_w}, 1)$ and $(1 + \frac{\kappa_w}{1+\phi N_w} + \kappa_p, K)$.

Proof of Lemma 3:

Let λ_1 denote the eigenvalue in $(0, \frac{\beta}{1+\phi N_w})$ and λ_2 denote the eigenvalue in $(\frac{\beta}{1+\phi N_w}, 1)$. Let λ_3 denote the eigenvalue larger than one. (13) can be written as:

$$\begin{bmatrix} \Delta_b \pi_t{}^w \\ \Delta_b \pi_t{}^p \\ \Delta_b w_{t-1} \end{bmatrix} = A \begin{bmatrix} \Delta_b \pi_{t+1}{}^w \\ \Delta_b \pi_{t+1}{}^p \\ \Delta_b w_t \end{bmatrix} = C^{-1} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} C \begin{bmatrix} \Delta_b \pi_{t+1}{}^w \\ \Delta_b \pi_{t+1}{}^p \\ \Delta_b w_t \end{bmatrix} \tag{60}$$

Thus

$$C \begin{bmatrix} \Delta_b \pi_t{}^w \\ \Delta_b \pi_t{}^p \\ \Delta_b w_{t-1} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} C \begin{bmatrix} \Delta_b \pi_{t+1}{}^w \\ \Delta_b \pi_{t+1}{}^p \\ \Delta_b w_t \end{bmatrix} \tag{61}$$

Let $C = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$. Since we focus on the bounded economic path and $\lambda_1 < 1$ $\lambda_2 < 1$, by iterating (61) we must have

$$\begin{bmatrix} a_1 & b_1 & c_1 \end{bmatrix} \begin{bmatrix} \Delta_b \pi_t{}^w \\ \Delta_b \pi_t{}^p \\ \Delta_b w_{t-1} \end{bmatrix} = \begin{bmatrix} a_2 & b_2 & c_2 \end{bmatrix} \begin{bmatrix} \Delta_b \pi_t{}^w \\ \Delta_b \pi_t{}^p \\ \Delta_b w_{t-1} \end{bmatrix} = 0 \tag{62}$$

Or

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} \Delta_b \pi_t{}^w \\ \Delta_b \pi_t{}^p \end{bmatrix} = - \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \Delta_b w_{t-1} \tag{63}$$

At this moment, suppose $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ is invertible. Then

$$\begin{bmatrix} \Delta_b \pi_t{}^w \\ \Delta_b \pi_t{}^p \end{bmatrix} = - \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}^{-1} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \Delta_b w_{t-1} \equiv \begin{bmatrix} x_w \\ x_p \end{bmatrix} \Delta_b w_{t-1} \tag{64}$$

$\Delta_b \pi_t{}^w$ and $\Delta_b \pi_t{}^p$ are linearly related to $\Delta_b w_{t-1}$. Substituting (61) into (13),

$$\Delta_b w_{t-1} = \lambda_3 \Delta_b w_t, \forall t \geq T+1 \tag{65}$$

From (64) and (65), we could characterize the whole economic path at and after T+1 once we know $\Delta_b w_T$ (or w_T). In other words, the economic path after the trap only depends on w_T.

To make it rigorous, I show that $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ is invertible. If not, we must find

$$m_1 a_1 + m_2 a_2 = m_1 b_1 + m_2 b_2 = 0, \{m_1, m_2\} \neq \{0, 0\} \tag{66}$$

Since $A = C^{-1} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} C$, it implies

$$\begin{bmatrix} a_1 & b_1 & c_1 \end{bmatrix} A = \lambda_1 \begin{bmatrix} a_1 & b_1 & c_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 & c_2 \end{bmatrix} A = \lambda_2 \begin{bmatrix} a_2 & b_2 & c_2 \end{bmatrix} \tag{67}$$

Then

$$(\frac{m_1}{\lambda_1}\begin{bmatrix} a_1 & b_1 & c_1 \end{bmatrix} + \frac{m_2}{\lambda_2}\begin{bmatrix} a_2 & b_2 & c_2 \end{bmatrix})A = \begin{bmatrix} 0 & 0 & m_1c_1 + m_2c_2 \end{bmatrix} \tag{68}$$

By using the specific values of A in (13), we get

$$\frac{m_1}{\lambda_1}a_1 + \frac{m_2}{\lambda_2}a_2 = \frac{m_1}{\lambda_1}b_1 + \frac{m_2}{\lambda_2}b_2; \tag{69}$$

Combining with (66),

$$(\frac{m_1}{\lambda_1} - \frac{m_2}{\lambda_2})a_1 = (\frac{m_1}{\lambda_1} - \frac{m_2}{\lambda_2})b_1, \tag{70}$$

We get $a_1 = b_1$ or $\frac{m_1}{\lambda_1} = \frac{m_2}{\lambda_2}$. If $\frac{m_1}{\lambda_1} = \frac{m_2}{\lambda_2}$, by using (66) and (67) again, we can also get $a_1 = b_1$. Substituting $a_1 = b_1$ into (67),

$$\frac{\beta}{1+\phi N_w} = \beta \tag{71}$$

It is a contradiction.

Proof of Lemma 4:

Substituting (64) (65) into (13) we get

$$\begin{bmatrix} x_w \\ x_p \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\beta}{1+\phi N_w} & 0 & \frac{-\kappa_w}{1+\phi N_w} \\ 0 & \beta & \kappa_p \\ \frac{-\beta}{1+\phi N_w} & \beta & 1 + \frac{\kappa_w}{1+\phi N_w} + \kappa_p \end{bmatrix} \begin{bmatrix} x_w \\ x_p \\ 1 \end{bmatrix} {\lambda_3}^{-1} \tag{72}$$

Then we have

$$x_w = \frac{\frac{{\lambda_3^{-1}}\kappa_w}{1+\phi N_w}}{\frac{\lambda_3^{-1}\beta}{1+\phi N_w} - 1}, x_p = \frac{{\lambda_3^{-1}}\kappa_p}{1 - {\lambda_3^{-1}}\beta}, 1 = x_p - x_w + \lambda_3^{-1} \tag{73}$$

Using $\lambda_3 > 1$, we have $-\frac{\kappa_w}{1+\phi N_w - \beta} < x_w < 0$ and $x_p > 0$. Since $1 = x_p - x_w + \lambda_3^{-1}$, we know $x_p < 1$.

Using $\lambda_3 > 1 + \frac{\kappa_w}{1+\phi N_w} + \kappa_p$, we know $\lambda_3^{-1} \to 0$ when $\kappa_p \to \infty$. Thus $x_w \to 0$. Since $1 = x_p - x_w + \lambda_3^{-1}$, we know $x_p \to 1$.

Proof of Proposition 1:

Step 1: The trick is to use the dominant part of matrix A, ignoring the higher order infinitesimal. Rewrite A as

$$A = \kappa_p \begin{bmatrix} \frac{\beta}{1+\phi N_w}\kappa_p^{-1} & 0 & \frac{-\kappa_w}{1+\phi N_w}\kappa_p^{-1} \\ 0 & \beta\kappa_p^{-1} & 1 \\ \frac{-\beta}{1+\phi N_w}\kappa_p^{-1} & \beta\kappa_p^{-1} & (1+\frac{\kappa_w}{1+\phi N_w})\kappa_p^{-1}+1 \end{bmatrix} \equiv \kappa_p B(\kappa_p) \tag{74}$$

when $\kappa_p \to \infty$, we know

$$B(\kappa_p) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} + O_(\kappa_p^{-1}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} + o_(1), \tag{75}$$

where $o(m)$ indicates higher order infinitesimal w.r.t m. Thus

$$A = \kappa_p \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} + o_(\kappa_p), \tag{76}$$

And

$$A^{T+1} = \kappa_p^{T+1} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{T+1} + o_(\kappa_p^{T+1}) = \kappa_p^{T+1} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} + o_(\kappa_p^{T+1}) \tag{77}$$

First I show $\Delta_b w_T \to 0$. If not, we know from lemma 4 that x_w and x_p are bounded when $\kappa_p \to \infty$. Thus (19) can be written as

$$\begin{aligned} \begin{bmatrix} \Delta_a \pi_0{}^w \\ \Delta_a \pi_0{}^p \\ \Delta_a w_{-1} \end{bmatrix} &= (\kappa_p^{T+1} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} + o_(\kappa_p^{T+1}))(\begin{bmatrix} x_w \\ x_p \\ 1 \end{bmatrix} \Delta_b w_T - \begin{bmatrix} \pi_{a-b}{}^w \\ \pi_{a-b}{}^p \\ w_{a-b} \end{bmatrix}) \\ &= \kappa_p^{T+1} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \Delta_b w_T - w_{a-b} \\ 0 & 0 & \Delta_b w_T - w_{a-b} \end{bmatrix} + o_(\kappa_p^{T+1}) \end{aligned} \tag{78}$$

Since $w_{a-b} \to 0$,

$$\begin{bmatrix} \Delta_a {\pi_0}^w \\ \Delta_a {\pi_0}^p \\ \Delta_a w_{-1} \end{bmatrix} = \kappa_p^{T+1} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \Delta_b w_T \\ 0 & 0 & \Delta_b w_T \end{bmatrix} + o_(\kappa_p^{T+1}) \tag{79}$$

Then $\Delta_a w_{-1}$ (also w_{-1}) grows exponentially to infinity with κ_p. It leads to a contradiction. Thus we show $\Delta_b w_T \to 0$.

Combining with (64) and (65), we know $\begin{bmatrix} \Delta_b {\pi_t}^w \\ \Delta_b {\pi_t}^p \\ \Delta_b w_{t-1} \end{bmatrix} \to 0, \forall t \geq T+1$. Then $\begin{bmatrix} {\pi_t}^w \\ {\pi_t}^p \\ w_{t-1} \end{bmatrix} \to \begin{bmatrix} {\pi_b}^w \\ {\pi_b}^p \\ w_b \end{bmatrix} \to \begin{bmatrix} 0 \\ -g_b \\ 0 \end{bmatrix}, \forall t \geq T+1.$

Step 2: we show eigenvalues $\lambda_1 \to 0$ and $\lambda_2 \to \frac{\beta}{1+\phi N_w}$:

We rewrite $f(\lambda)$ as a function of κ_p: $f(\lambda) = \kappa_p(\frac{\beta}{1+\phi N_w} - \lambda)\lambda + g(\lambda)$. For any λ slightly above 0, $(\frac{\beta}{(1+\phi N_w)} - \lambda)\lambda > 0$. For large κ_p, we must have $f(\lambda) > 0$. As we have already proved $f(0) < 0$ in lemma 2, by continuity, we know $\lambda_1 \to 0$. Using the same approach, for λ slightly above $\frac{\beta}{1+\phi N_w}$, we know $(\frac{\beta}{(1+\phi N_w)} - \lambda) < 0$ and then $f(\lambda) < 0$ for large enough κ_p. As $f(\frac{\beta}{1+\phi N_w}) > 0$, by continuity we know $\lambda_2 \to \frac{\beta}{1+\phi N_w}$.

Step 3: we show $\begin{bmatrix} a_1 & b_1 & c_1 \end{bmatrix} \to \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$, $\begin{bmatrix} a_2 & b_2 & c_2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$, where $\begin{bmatrix} a_1 & b_1 & c_1 \end{bmatrix}$ and $\begin{bmatrix} a_2 & b_2 & c_2 \end{bmatrix}$ are normalized eigenvectors for λ_1 and λ_2:

Since $\lambda_1 \to 0$ and $\begin{bmatrix} a_1 & b_1 & c_1 \end{bmatrix}$ is bounded, we have

$$\begin{bmatrix} a_1 & b_1 & c_1 \end{bmatrix} \begin{bmatrix} \frac{\beta}{1+\phi N_w} & 0 & \frac{-\kappa_w}{1+\phi N_w} \\ 0 & \beta & \kappa_p \\ \frac{-\beta}{1+\phi N_w} & \beta & 1 + \frac{\kappa_w}{1+\phi N_w} + \kappa_p \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$
$$\Rightarrow a_1 - c_1 \to 0, b_1 + c_1 \to 0$$

Then $\begin{bmatrix} a_1 & b_1 & c_1 \end{bmatrix} \to \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$. Since $\lambda_2 \to \frac{\beta}{1+\phi N_w}$:

$$\begin{bmatrix} a_2 & b_2 & c_2 \end{bmatrix} \begin{bmatrix} 0 & 0 & \frac{-\kappa_w}{1+\phi N_w} \\ 0 & \beta - \frac{\beta}{1+\phi N_w} & \kappa_p \\ \frac{-\beta}{1+\phi N_w} & \beta & 1 + \frac{\kappa_w - \beta}{1+\phi N_w} + \kappa_p \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow c_2 \to 0, (\beta - \frac{\beta}{1+\phi N_w}) b_2 + \beta c_2 \to 0 \Rightarrow c_2 \to 0, b_2 \to 0$$

Then $\begin{bmatrix} a_2 & b_2 & c_2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$.

Step 4: we show $\Delta_a w_t \to 0, \forall t \leq T$:

Replacing A by $C^{-1} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} C$ where $C = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ contains all eigenvectors,

$$C \begin{bmatrix} \Delta_a {\pi_t}^w \\ \Delta_a {\pi_t}^p \\ \Delta_a w_{t-1} \end{bmatrix} = \begin{bmatrix} {\lambda_1}^{T-t} & 0 & 0 \\ 0 & {\lambda_2}^{T-t} & 0 \\ 0 & 0 & {\lambda_3}^{T-t} \end{bmatrix} C(\begin{bmatrix} x_w \\ x_p \\ 1 \end{bmatrix} \Delta_b w_T - \begin{bmatrix} {\pi_{a-b}}^w \\ {\pi_{a-b}}^p \\ w_{a-b} \end{bmatrix}) \tag{80}$$

Using $\Delta_b w_T \to 0$ and combining steps 2-3, we find the above two elements of the RHS vector converge to zero, then

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} \begin{bmatrix} \Delta_a {\pi_t}^w \\ \Delta_a {\pi_t}^p \\ \Delta_a w_{t-1} \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{81}$$

If $\Delta_a w_{t-1}$ does not converge to 0, divide (81) by $\Delta_a w_{t-1}$. Let $x_{w,t} = \Delta_a {\pi_t}^w / \Delta_a w_{t-1}$ and $x_{p,t} = \Delta_a {\pi_t}^p / \Delta_a w_{t-1}$, we can get

$$\begin{bmatrix} x_{w,t} \\ x_{p,t} \end{bmatrix} \to \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{82}$$

$\begin{bmatrix} x_{w,t} \\ x_{p,t} \end{bmatrix}$ shares the same proprerty as $\begin{bmatrix} x_w \\ x_p \end{bmatrix}$ in lemma 4. Since $\begin{bmatrix} x_{w,t} \\ x_{p,t} \end{bmatrix}$ is bounded, we apply the same approach in step 1 to show that w_{-1} grows exponentially with κ_p which leads to the contradiction. Then we prove $\Delta_a w_{t-1} \to 0$. By using (81) again, we

know $\begin{bmatrix} \Delta_a {\pi_t}^w \\ \Delta_a {\pi_t}^p \\ \Delta_a w_{t-1} \end{bmatrix} \to 0, \forall t \leq T$. Thus $\begin{bmatrix} {\pi_t}^w \\ {\pi_t}^p \\ w_{t-1} \end{bmatrix} \to \begin{bmatrix} {\pi_a}^w \\ {\pi_a}^p \\ w_a \end{bmatrix} \to \begin{bmatrix} 0 \\ -g_a \\ 0 \end{bmatrix}, \forall t \leq T$.
Using the IS equation (1), we can check that $i_t \to \rho$. Thus the ZLB is indeed not binding as $\kappa_p \to \infty$.

Proof of Lemma 5:

First, we show the existence of such a solution for $\kappa_p > 0$:

Let $\gamma_w = 1$, we find $C_p = \frac{\kappa_p}{1-\beta}$, $C_w = \frac{\kappa_p}{1-\beta} + 1$, $C_x = \frac{\kappa_p + \kappa_w + 1 - \beta}{N_w}$, and $Q_w = \frac{C_x^2 + \mu_p C_p^2 + \mu_w C_w^2}{1-\beta}$. They are all positive, so the LHS of (44) is positive. Let $\gamma_w \to \infty$, we find $C_p \to 0$, $C_w \to -1$, $C_x \to -\frac{1}{N_w}$, $\frac{Q_w}{\gamma_w} \to 0$, $\gamma_w C_p^2 \to 0$, $(1 + N_w C_x)\gamma_w \to \kappa_p + \kappa_w + 1 + \beta$, $C_p \gamma_w \to \kappa_p$, and the LHS of (44) converges to minus infinity. So, there must be a solution with $\gamma_w > 1$.

Then we prove $\gamma_w \to \infty$ and C_p is bounded:

From (33) we know $C_p = \frac{\kappa_p}{\gamma_w - \beta}$. If γ_w does not converge to infinity, we have $C_p > 0$ as $\gamma_w > 1$ and $C_p \to +\infty$ when $\kappa_p \to +\infty$. From (34), we know $C_w \to +\infty$ and from (6) we know $C_x \to +\infty$. From (37) we find $Q_w \to \infty$. Using the fact that C_w, C_p, C_x and Q_w all converge to infinity and γ_w is bounded, the LHS of (44) should converge to infinity, which leads to the contradiction. Thus $\gamma_w \to \infty$, and for the same reason C_p is bounded.

Finally we prove the lemma:

As $\gamma_w \to \infty$, from (35) we know $C_w \to C_p - 1$. From (34) we know $N_w C_x \to C_w$. From (37) we find $Q_w \to C_x^2 + \mu_p C_p^2 + \mu_w C_w^2 \to C_x^2 + \mu_w C_w^2$ since $\mu_p \to 0$. So Q_w is also bounded. Combining with (44), we have

$$\gamma_w (N_w(\mu_p C_p^2 + \mu_w C_p C_w) + (1 + N_w C_x) C_x) + N_w \mu_w C_w \to 0; \tag{83}$$

which requires (necessary but not sufficient condition)

$$N_w(\mu_p C_p^2 + \mu_w C_p C_w) + (1 + N_w C_x) C_x \to 0; \tag{84}$$

Combining with $\mu_p \to 0$, $C_w + 1 \to C_p$ and $N_w C_x \to C_w$, we get

$$C_p C_w (\mu_w N_w + \frac{1}{N_w}) \to 0; \tag{85}$$

Thus $C_p \to 0$ or $C_w \to 0$:

(1) If $C_p \to 0$, we have $C_w \to -1$ and since $C_p = \frac{\kappa_p}{\gamma_w - \beta}$, we know $C_p \gamma_w \to \infty$. Besides, $\gamma_w \mu_p C_p^2 = \kappa_p \mu_p \frac{\gamma_w}{\gamma_w - \beta} C_p \to 0$. The LHS of (44) becomes

$$\gamma_w C_p C_w (\mu_w N_w + \frac{1}{N_w}) + N_w \mu_w C_w \to -\infty \tag{86}$$

which leads to a contradiction.

(2) We can check that the solution with $C_w \to 0$ satisfies all the conditions. Using the limit relationships, we can easily prove the rest limit properties: $C_p \to 1$, $i_w \to 0$, $Q_w \to 0$. For the previous discussion, we assume that ZLB is not binding (when we derive the FOC, we do not consider the ZLB constraint of i_s). We show here that it is true: as $i_w \to 0$, our policy rule $i(w) \to \rho$. Since ρ is positive, ZLB is indeed not binding.

Proof of Lemma 6:

Using the chain rule, we have

$$\frac{dx_t}{di_t} = \frac{dx_{t+1}(w_t)}{dw_t}\frac{dw_t}{di_t} + 1 + \frac{d\pi^p_{t+1}(w_t)}{dw_t}\frac{dw_t}{di_t} \tag{87}$$

$$\frac{d\pi^p_t}{di_t} = \beta \frac{d\pi^p_{t+1}(w_t)}{dw_t}\frac{dw_t}{di_t} + \kappa_p \frac{dw_t}{di_t} \tag{88}$$

$$\frac{d\pi^w_t}{di_t} = \beta \frac{d\pi^w_{t+1}(w_t)}{dw_t}\frac{dw_t}{di_t} + \aleph_w \frac{dx_t}{di_t} - \kappa_w \frac{dw_t}{di_t} \tag{89}$$

And the first order condition is

$$0 = x_t \frac{dx_t}{di_t} + \mu_p \pi^p_t \frac{d\pi^p_t}{di_t} + \mu_w \pi^w_t \frac{d\pi^w_t}{di_t} + \frac{1}{2}\beta \frac{dQ^w_t}{dw_t}\frac{dw_t}{di_t} \tag{90}$$

As $\pi^w_{t+1}(w_t)$, $x_{t+1}(w_t)$, $Q^w_{t+1}(w_t)$ converge to the constant zero, we have $\frac{d\pi^w_{t+1}(w_t)}{dw_t} \to 0$, $\frac{dx_{t+1}(w_t)}{dw_t} \to 0$, $\frac{dQ^w_{t+1}(w_t)}{dw_t} \to 0$. Since $\pi^p_{t+1}(w_t) \to w_t - \Delta w^n_{t+1}$, we know $\frac{d\pi^p_{t+1}(w_t)}{dw_t} \to 1$. Combining with (51)-(89), we have

$$\frac{dw_t}{di_t} \to 0, \kappa_p \frac{dw_t}{di_t} \to -\aleph_w, \frac{dx_t}{di_t} \to 1, \frac{d\pi^p_t}{di_t} \to -\aleph_w, \frac{d\pi^w_t}{di_t} \to \aleph_w \tag{91}$$

Combining with (91), FOC requires

$$x_t + \aleph_w \mu_w \pi^w_t \to 0 \tag{92}$$

When we replace the "limit" relation in (85) with equality, it becomes the FOC of the government when it is non-forward-looking. This implies that when the price is sufficiently flexible, the two equilibria with forward-looking and non-forward-looking governments forward-looking and non-forward-looking governments converge to be the same.

Substituting (92) into (47),

$$\pi_t^w \to -\frac{\kappa_w w_t}{1+\mu_w \aleph_w^2}, x_t \to \frac{\mu_w \aleph_w \kappa_w w_t}{1+\mu_w \aleph_w^2} \tag{93}$$

Combining (93) and (45),

$$i_t - r_t^n \to -\frac{\mu_w \aleph_w \kappa_w w_t}{1+\mu_w \aleph_w^2} + (w_t - \Delta w_{t+1}^n) \tag{94}$$

Combining (50), (94) and our assumption that $\pi_{t+1}^p(w_t) \to w_t - \Delta w_{t+1}^n$, $\pi_{t+1}^w(w_t) \to 0$, $x_{t+1}(w_t) \to 0$, we have

$$(1+\kappa_p + \frac{1}{1+\mu_w \aleph_w^2} k_w + \aleph_w k_w + \beta - \aleph_w) w_t \to w_{t-1} - \Delta w_t^n + \beta \Delta w_{t+1}^n \tag{95}$$

As $\kappa_p \to \infty$, $\kappa_p w_t$ in the LHS becomes dominant. It requires $\kappa_p w_t \to w_{t-1} - \Delta w_t^n + \beta \Delta w_{t+1}^n$ and $w_t \to 0$. Using (93) again, we know $\pi_t^w \to 0$ and $x_t \to 0$ regardless of w_{t-1}. Thus $\pi_t^w(w_{t-1}) \to 0$, $x_t(w_{t-1}) \to 0$, and $w_t(w_{t-1}) \to 0$. Similarly we find $\pi_s^p(w_{t-1}) \to -\beta \Delta w_{t+1}^n + (w_{t-1} - \Delta w_t^n + \beta \Delta w_{t+1}^n) = w_{t-1} - \Delta w_t^n$. From (49) we find $Q_t^w(w_{t-1}) \to 0$.

Previous discussions assume that ZLB is not binding (when we derive the FOC, we do not consider the ZLB constraint of i_t). We check it here. Using (94), we know $i_t \to r_t^n - \Delta w_{t+1}^n$ as $w_t \to 0$. Since $r_t^n - \Delta w_{t+1}^n = \rho$ and ρ is positive, ZLB is not binding.

Proof of Proposition 3:

We have proved the case where $\kappa_p \to \infty$ in lemma 6. So we only need to prove the case where $\kappa_w \to 0$.

Firstly we prove $\frac{d\pi_t^w(w_{t-1})}{dw_{t-1}} \to 0, \forall t \geq 0$ as $\kappa_w \to 0$:

(1) For $t > T$, we only need to prove $C_w \to 0$. From (34), we know as $\kappa_w \to 0$, $\aleph_w \to 0$. Thus we must have $C_w \to 0$.

(2) For $t \leq T$, we use backward induction. Assume $\frac{d\pi_{t+1}^w(w_t)}{dw_t} \to 0$. From (47), we know

$$\frac{d\pi_t^w(w_{t-1})}{dw_{t-1}} = \beta \frac{d\pi_{t+1}^w(w_t)}{dw_t}\frac{dw_t}{dw_{t-1}} + \aleph_w \frac{dx_t(w_{t-1})}{dw_{t-1}} - \kappa_w \frac{dw_t(w_{t-1})}{dw_{t-1}} \tag{96}$$

Since $\frac{d\pi_{t+1}^w(w_t)}{dw_t} \to 0$ and $\frac{dx_t(w_{t-1})}{dw_{t-1}}$, $\frac{dw_t(w_{t-1})}{dw_{t-1}}$ are bounded (when $\kappa_w \to 0$, there is no reason for the derivatives to be unbounded), we must have $\frac{d\pi_t^w(w_{t-1})}{dw_{t-1}} \to 0$.

Then we prove the lemma:

From (51) we know $\frac{dw_t}{di_t} \to 0$ and $\mu_w \frac{dw_t}{di_t}$ converges to a constant. From (87) and (88), we have $\frac{d\pi_t^p}{di_t} \to 0$ and $\frac{dx_t}{di_t} \to 1$. From (89):

$$\mu_w \frac{d\pi_t^w}{di_t} = \beta \frac{d\pi_{t+1}^w(w_t)}{dw_t}\mu_w \frac{dw_t}{di_t} + \mu_w \aleph_w \frac{dx_t}{di_t} - \mu_w \kappa_w \frac{dw_t}{di_t} \to 0 + \mu_w \aleph_w + 0 = \mu_w \aleph_w \tag{97}$$

by using $\frac{d\pi_{t+1}^w(w_t)}{dw_t} \to 0$. Thus, the FOC becomes:

$$x_t + \aleph_w \mu_w \pi_t^w \to 0 \tag{98}$$

which is the same as the FOC for the non-forward-looking government.