Rigidity of joinings for time-changes of unipotent flows on quotients of Lorentz groups

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Abstract. Let u_X^t be a unipotent flow on $X = SO(n,1)/\Gamma$, u_Y^t be a unipotent flow on $Y = G/\Gamma'$. Let $\tilde{u}_X^t$, $\tilde{u}_Y^t$ be time-changes of u_X^t, u_Y^t respectively. We show the disjointness (in the sense of Furstenberg) between u_X^t and $\tilde{u}_Y^t$ (or $\tilde{u}_X^t$ and u_Y^t) in certain situations.

Our method refines the works of Ratner's shearing argument. The method also extends a recent work of Dong, Kanigowski and Wei.

Contents

1. Introduction

1.1. **Main results.** In this paper, we study the rigidity of joinings of time-changes of unipotent flows. First, let

- $G_X = SO(n_X, 1)$, G_Y be a semisimple Lie group with finite center and no compact factors and $\Gamma_X \subset G_X$, $\Gamma_Y \subset G_Y$ be irreducible lattices,
- (X, m_X), (Y, m_Y) be the homogeneous spaces $X = G_X/\Gamma_X$, $Y = G_Y/\Gamma_Y$ equipped with the Lebesgue measures m_X, m_Y respectively,
- u_X^t, u_Y^t be unipotent flows on X and Y respectively,
- τ_X, τ_Y be positive functions with integral $m_X(\tau_X) = m_Y(\tau_Y) = 1$ under certain regularity on X and Y respectively,
- $\tilde{u}_X^t$, $\tilde{u}_Y^t$ be the *time-changes* of u_X^t, u_Y^t induced by τ_X, τ_Y, respectively,
- $d\mu = \tau_X dm_X$, $d\nu = \tau_Y dm_Y$ be the $\tilde{u}_X$-, $\tilde{u}_Y$-invariant measures respectively.

We shall verify the disjointness and so classify the joinings of u_X^t and $\tilde{u}_Y^t$ (or $\tilde{u}_X^t$ and u_Y^t) in certain situations.

Recall that a *joining* of $\tilde{u}_X^t$ and $\tilde{u}_Y^t$ is a $(\tilde{u}_X^t \times \tilde{u}_Y^t)$-invariant probability measure on $X \times Y$, whose marginals on X and Y are μ and ν respectively. It was first introduced by Furstenberg in [Fur81], and is a natural generalization of *measurable conjugacies*. The classical results on classifying joinings under this context were established by Ratner [Rat82], [Rat83], [Rat86], [Rat87], [Rat90]. First, the celebrated *Ratner's theorem* indicates that all joinings between u_X^t and u_Y^t have to be **algebraic**. Besides, for $G_X = SO(2, 1)$, Ratner studied the *H-property* (or *Ratner's property*) of horocycle flows u_X^t, as well as their time-changes $\tilde{u}_X^t$, and then showed that any nontrivial (i.e. not the product measure $\mu \times \nu$) ergodic joining of $\tilde{u}_X^t$ and $\tilde{u}_Y^t$ is a **finite extension** of ν. (In fact, this is even true for any measure-preserving system on (Y, ν).) Using this, Ratner was able to show that for $G_X = G_Y = SO(2, 1)$, the existence of a nontrivial ergodic joining of $\tilde{u}_X^t$ and $\tilde{u}_Y^t$ implies that τ_X and τ_Y are *algebraically cohomologous*. In other words, whether $\tilde{u}_X^t$ and $\tilde{u}_Y^t$ are disjoint is determined by cohomological equations.

It is natural to ask if it is possible to extend the results to $G_X = SO(n_X, 1)$ for $n_X \geq 3$. The difficulty is that the time-change $\tilde{u}_X^t$ needs not have the *H-property*. It is one of the main ingredient of unipotent flows. Roughly speaking, H-property states that the divergence of nearby unipotent orbits happens always along some direction from the centralizer $C_{G_X}(u_X)$ of the flow u_X^t. In particular, for $G_X = SO(2, 1)$, the direction can only be the flow direction u_X^t itself. Moreover, Ratner [Rat87] naturally extended this notion to the general measure-preserving systems and verified it for the time-changes $\tilde{u}_X^t$ of horocycle flows. However, for $n_X \geq 3$, it seems that there is no suitable way to describe the "centralizer" of the time-change $\tilde{u}_X^t$. Thus, classifying joinings of $\tilde{u}_X^t$ and $\tilde{u}_Y^t$ for $n_X \geq 3$ becomes a difficult problem.

Recently, Dong, Kanigowski and Wei [DKW20] considered the case when $G_X = SO(2, 1)$, G_Y is semisimple as above, Γ_X and Γ_Y are cocompact lattices. After comparing the H-property of $\tilde{u}_X^t$ and u_Y^t, they showed that $\tilde{u}_X^t$ and u_Y^t are disjoint once the Lie algebra $\mathfrak{g}_Y$ of G_Y contains at least one weight vector of weight at least 1 other than the $\mathfrak{sl}_2$-triples generated by u_Y^t.

In this paper, we try to generalize the results stated above for $n_X \geq 3$. First, we follow the idea of Ratner and study the H-property of u_X^t and deduce:

Theorem 1.1. *Let (Y, ν, S) be a measure-preserving system of some map $S : Y \to Y$, ρ be an ergodic joining of u_X^1 and S. Then either $\rho = \mu \times \nu$ or $(u_X^1 \times S, \rho)$ is a compact extension of (S, ν). More precisely, if $\rho \neq \mu \times \nu$, then there exists a compact subgroup $C^\rho \subset C_{G_X}(u_X)$, and $n > 0$ such that for ν-a.e. $y \in Y$, there exist $x_1^y, \dots, x_n^y$ in the support of ρ_y with*

$$\rho_y(C^\rho x_i^y) = \frac{1}{n}$$

for $i = 1, \dots, n$, where $\rho = \int_Y \rho_y d\nu(y)$ is the disintegration along Y.

By Theorem 1.1, for any nontrivial ergodic joining ρ of u_X^t and $\tilde{u}_Y^t$, there are measurable maps $\psi_1, \dots, \psi_n : Y \to X$ such that

$$\rho(f) = \int_Y \int_{C^\rho} \frac{1}{n} \sum_{p=1}^{n} f(k\psi_p(y), y) dm(k) d\nu(y) \tag{1.1}$$

for $f \in C(X \times Y)$ where m is the Lebesgue measure of the compact group C^ρ. Projecting ρ to $(C^\rho \backslash X) \times Y$, we get

$$\overline{\rho}(f) = \int_Y \frac{1}{n} \sum_{p=1}^{n} f(\overline{\psi}_p(y), y) d\nu(y)$$

for $f \in C((C^\rho \backslash X) \times Y)$. Then, we can study the rigidity of ρ by thinking about $\overline{\psi}_1, \dots, \overline{\psi}_n$. Also, $\overline{\rho}$ is a nontrivial ergodic joining of u_X^t and $\tilde{u}_Y^t$.

Then we can establish the rigidity of $\overline{\psi}_p$ by studying the shearing of u_X^t. The idea comes from [Rat86], [Tan20]. We require the time-changes having the effective mixing property. Thus, let $\mathbf{K}(Y)$ be the set of all positive integrable functions τ on Y such that τ, τ^{-1} are bounded and satisfies

$$\left| \int_Y \tau(y) \tau(u_Y^t y) d\nu(y) - \left(\int_Y \tau(y) \nu(y) \right)^2 \right| \leq D_\tau |t|^{-\kappa_\tau}$$

for some $D_\tau, \kappa_\tau > 0$. In other words, elements $\tau \in \mathbf{K}(Y)$ have polynomial decay of correlations. Let $\langle u_X, a_X, \overline{u}_X \rangle$, $\langle u_Y, a_Y, \overline{u}_Y \rangle$ be $\mathfrak{sl}_2$-triples of G_X and G_Y, respectively. Let $N_{G_Y}(u_Y)$ be the normalizer of u_Y. Then we obtain the following:

Theorem 1.2 (Extra central invariance of ρ)**.** *Let $\tau_Y \in \mathbf{K}(Y)$, $\tilde{u}_Y^t$ be the time-change of u_Y^t induced by τ_Y and ρ be a nontrivial ergodic joining of $u_X^t, \tilde{u}_Y^t$. Then there exist maps $\alpha : N_{G_Y}(u_Y) \times Y \to \mathbf{R}$, $\beta : N_{G_Y}(u_Y) \to C_{G_Y}(u_Y)$ such that*

(1) Restricted to the centralizer $C_{G_Y}(u_Y)$, $\alpha : C_{G_Y}(u_Y) \times Y \to \mathbf{R}$ is a cocycle, $\beta : C_{G_Y}(u_Y) \to C_{G_X}(u_X)$ is a homomorphism. Besides, $\tau_Y(cy)$ and $\tau_Y(y)$

are (measurably) cohomologous along u_Y^t via the transfer function $\alpha(c,y)$ for all $c \in C_{G_Y}(u_Y)$; in other words,

$$\int_0^T \tau_Y(cu_Y^t y) - \tau_Y(u_Y^t y)dt = \alpha(c, u_Y^T y) - \alpha(c,y).$$

(2) There is a map $S : N_{G_Y}(u_Y) \times X \times Y \to X \times Y$ that satisfies the following:

- *For $c \in C_{G_Y}(u_Y)$, the map $S_c : X \times Y \to X \times Y$ defined by*
$$S_c : (x,y) \mapsto (\beta(c)x, \tilde{u}_Y^{-\alpha(c,y)}(cy))$$
commutes with $u_X^t \times \tilde{u}_Y^t$, and is ρ-invariant. Besides, $S_{c_1c_2} = S_{c_1} \circ S_{c_2}$ for any $c_1, c_2 \in C_{G_Y}(u_Y)$, and $S_{u_Y^t} = \mathrm{id}$ for $t \in \mathbf{R}$.
- *For $r \in \mathbf{R}$, the map $S_{a_Y^r} : X \times Y \to X \times Y$ defined by*
$$S_{a_Y^r} : (x,y) \mapsto \left(\beta(a_Y^r)a_X^r x, \tilde{u}_Y^{-\alpha(a_Y^r,y)}(a_Y^r y)\right)$$
satisfies
$$S_{a_Y^r} \circ (u_X^t \times \tilde{u}_Y^t) = (u_X^{e^{-r}t} \times \tilde{u}_Y^{e^{-r}t}) \circ S_{a_Y^r}$$
and is ρ-invariant. Besides, $S_{a_Y^{r_1+r_2}} = S_{a_Y^{r_1}} S_{a_Y^{r_2}}$ for any $r_1, r_2 \in \mathbf{R}$, and
$$S_{a_Y} \circ S_c \circ S_{a_Y^{-1}} = S_{a_Y c a_Y^{-1}}$$
for any $c \in C_{G_Y}(u_Y)$.

For the opposite unipotent direction $\overline{u}_Y$, we cannot obtain the invariance for ρ directly. However, we can fix it by making the "a-adjustment". Here we further require τ_Y being smooth and $\alpha(c,\cdot)$ being integrable. The idea comes from [Rat87]. Then since $\overline{u}_Y$ and $C_{G_Y}(u_Y)$ generate the whole group G_Y, we are able to use *Ratner's theorem* to get the rigidity of $\overline{\psi}_1, \ldots, \overline{\psi}_n$.

Theorem 1.3 (Cohomological criterion). *Let $G_X = SO(n_X, 1)$, G_Y be a semisimple Lie group with finite center and no compact factors and $\Gamma_X \subset G_X$, $\Gamma_Y \subset G_Y$ be irreducible lattices. Let $U_Y \in \mathfrak{g}_Y$ be a nilpotent vector so that $C_{\mathfrak{g}_Y}(U_Y)$ only contains vectors of weight at most 2, and let $u_Y = \exp(U_Y)$. Let $\tau_Y \in \mathbf{K}(Y) \cap C^1(Y)$ so that $\tau_Y(cy)$ and $\tau_Y(y)$ are L^1-cohomologous along u_Y^t for any $c = \exp(v) \in C_{G_Y}(u_Y)$ with positive weight. If there is a nontrivial ergodic joining ρ of u_X^t and $\tilde{u}_Y^t$, then $\tau_X \equiv 1$ and τ_Y are joint cohomologous (see Definition 2.4 for the precise definition).*

Remark 1.4. When $\tau_X \equiv 1$ and τ_Y are joint cohomologous, one can deduce that 1 (on Y) and τ_Y are (measurably) cohomologous over the flow u_Y^t. See Proposition 2.14 for further discussion.

In [Tan20], we see that for $G_Y = SO(n_Y, 1)$, some cocompact lattice Γ_Y, there exists a function $\tau_Y \in \mathbf{K}(Y) \cap C^1(Y)$ such that

- τ_Y and 1 are not measurably cohomologous,
- for any $c \in C_{G_Y}(u_Y)$, $\tau_Y(cy)$ and $\tau_Y(y)$ are not measurably cohomologous if they are not L^2-cohomologous.

Applying Theorem 1.2 (1) and Theorem 1.3 to τ_Y, we get

Corollary 1.5 (Existence of nontrivial time-changes)**.** *For $G_Y = SO(n_Y, 1)$, there exists a cocompact lattice Γ_Y, and a function τ_Y on $Y = G_Y/\Gamma_Y$ such that u_X^t and $\tilde{u}_Y^t$ are disjoint (i.e. the only joining of u_X^t and $\tilde{u}_Y^t$ is the product measure $\mu \times \nu$).*

Besides, the homomorphism $\beta|_{C_{G_Y}(u_Y)}$ obtained by Theorem 1.2 also provide some information. Combining Ratner's theorem, we conclude that the existence of non-trivial joinings requires the algebraic structure G_Y to be similar to G_X.

Theorem 1.6 (Algebraic criterion)**.** *Let the notation and assumptions be as in Theorem 1.3. If there is a nontrivial ergodic joining ρ of u_X^t and $\tilde{u}_Y^t$, then ρ is a finite extension of ν (i.e. the C^ρ provided by Theorem 1.1 is trivial). Besides, consider the decomposition (see (2.7)):*

$$C_{\mathfrak{g}_Y}(U_Y) = \mathbf{R}U_Y \oplus V_{C_Y}^{\perp}, \quad C_{\mathfrak{g}_X}(U_X) = \mathbf{R}U_X \oplus V_{C_X}^{\perp}.$$

Then the derivative $d\beta|_{V_C^{\perp}} : V_{C_Y}^{\perp} \to V_{C_X}^{\perp}$ is an injective Lie algebra homomorphism.

Remark 1.7. Theorem 1.3 and 1.6 provide criteria for the disjointness of u_X^t and $\tilde{u}_Y^t$. However, they require that the functions $\tau_Y(cy)$ and $\tau_Y(y)$ are L^1-cohomologous for all $c \in C_{G_Y}(u_Y)$ with positive weight (Theorem 1.2 (1) indicates that they are always measurably cohomologous whenever u_X^t and $\tilde{u}_Y^t$ are not disjoint). This condition seems in general is not easy to verify.

On the other hand, when the time-changes happen on quotients X of Lorentz groups, we no longer have Theorem 1.1, because of the lack of H-property. However, if there exists a joining ρ as in (1.1), we can follow the same idea as in Theorem 1.2 and obtain the rigidity in certain situations:

Theorem 1.8. *Let $G_X = SO(n_X, 1)$, G_Y be a semisimple Lie group with finite center and no compact factors and $\Gamma_X \subset G_X$, $\Gamma_Y \subset G_Y$ be irreducible lattices. Let $U_Y \in \mathfrak{g}_Y$ be nilpotent. Let $\tau_Y \equiv 1$ and $\tau_X \in \mathbf{K}(X)$. Suppose that there exists an ergodic joining ρ of $\tilde{u}_X^t$ and u_Y^t that is a compact extension of ν, i.e. satisfies (1.1). Then there exist maps $\alpha : N_{G_Y}(u_Y) \times Y \to \mathbf{R}$, $\beta : N_{G_Y}(u_Y) \to C_{G_Y}(u_Y)$ such that*

1. *Restricted to the centralizer $C_{G_Y}(u_Y)$, $\alpha : C_{G_Y}(u_Y) \times Y \to \mathbf{R}$ is a cocycle, $\beta : C_{G_Y}(u_Y) \to C_{G_X}(u_X)$ is a homomorphism. Besides, $\tau_X(cx)$ and $\tau_X(x)$ are (measurably) cohomologous for all $c \in C_{G_X}(u_X)$.*
2. *There is a map $\widetilde{S} : N_{G_Y}(u_Y) \times X \times Y \to X \times Y$ that satisfies the following:*
   - *For $c \in C_{G_Y}(u_Y)$, the map $\widetilde{S}_c : X \times Y \to X \times Y$ defined by*

   $$\widetilde{S}_c : (x, y) \mapsto (u_X^{\alpha(c,y)}\beta(c)x, cy)$$

   commutes with $\tilde{u}_X^t \times u_Y^t$, and is ρ-invariant. Besides, $\widetilde{S}_{c_1c_2} = \widetilde{S}_{c_1} \circ \widetilde{S}_{c_2}$ for any $c_1, c_2 \in C_{G_Y}(u_Y)$, and $\widetilde{S}_{u_Y^t} = \tilde{u}_X^t$ for $t \in \mathbf{R}$.

- *The map $S_{a_Y} : X \times Y \to X \times Y$ defined by for $r \in \mathbf{R}$,*

$$\widetilde{S}_{a_Y^r} : (x, y) \mapsto \left(u_X^{\alpha(a_Y^r, y)} \beta(a_Y^r) a_X^r x, a_Y^r y\right)$$

is ρ-invariant. Besides, $\widetilde{S}_{a_Y^{r_1+r_2}} = \widetilde{S}_{a_Y^{r_1}} \widetilde{S}_{a_Y^{r_2}}$ for any $r_1, r_2 \in \mathbf{R}$, and

$$\widetilde{S}_{a_Y} \circ \widetilde{S}_c \circ \widetilde{S}_{a_Y^{-1}} = \widetilde{S}_{a_Y c a_Y^{-1}}$$

for any $c \in C_{G_Y}(u_Y)$.

Moreover, for any weight vector $v \in V_{C_Y}^{\perp}$ of positive weight, the derivative

$$d\beta|_{V_C^{\perp}}(v) \neq 0. \tag{1.2}$$

Remark 1.9. In other words, (1.2) asserts that $d\beta$ is injective on the nilpotent part of $V_{C_Y}^{\perp}$. One direct consequence of (1.2) is that $C_{\mathfrak{g}_Y}(U_Y)$ (under the assumptions of Theorem 1.8) does not contain any weight vector of weight $\neq 0, 2$ (see Lemma 6.5).

In particular, recall that [Rat87] showed that when $G_X = SO(2,1)$, any time-change $\tilde{u}_X^t$ with $\tau_X \in \mathbf{K}(X) \cap C^1(X)$ has H-property. It meets all the requirements of Theorem 1.8. Then combining [Rat87], we obtain a slight extension of [DKW20]:

Theorem 1.10. *Let $G_X = SO(2,1)$, G_Y be a semisimple Lie group with finite center and no compact factors and $\Gamma_X \subset G_X$, $\Gamma_Y \subset G_Y$ be irreducible lattices. Let $\tau_X \in \mathbf{K}(X) \cap C^1(X)$. If the Lie algebra $\mathfrak{g}_Y \not\cong \mathfrak{sl}_2$, then $\tilde{u}_X^t$ and u_Y^t are disjoint.*

1.2. **Structure of the paper.** In Section 2 we recall basic definitions, including some basic material on the Lie algebra $\mathfrak{so}(n,1)$ (in Section 2.1, Section 2.2), as well as time-changes (Section 2.3) and coboundaries (Section 2.4). In Section 3, we make use of the H-property of unipotent flows and deduce Theorem 1.1. This requires studying the shearing property of u_X^t for nearby points of the form (x, y) and (gx, y). In Section 4 we state and prove a number of results which will be used as tools to prove the extra invariance of joinings ρ (Theorem 1.2), in particular Proposition 4.19 which pulls the shearing phenomenon on the homogeneous space X back to the Lie group G_X. We also give a quantitative estimate of the difference between two nearby points in terms of the length of the shearing (Lemma 4.14). In Section 5, we present the proof of Theorem 1.2 (Section 5.1 Section 5.2) and a technical result for the opposite unipotent direction (Theorem 5.15). The latter result also requires studying the H-property of unipotent flows. Finally, using the results we got and Ratner's theorem, we present in Section 6 the proof of Theorem 1.3, 1.6 (in Section 6.1), 1.8 and 1.10 (in Section 6.2).

Acknowledgements. The original motivation of this paper came from the questions that Adam Kanigowski asked during the conversations. I am thankful to him for asking the questions. The paper was written under the guidance of my advisor David Fisher for my PhD thesis, and I am sincerely grateful for his help. I would also like to thank Livio Flaminio for helpful discussions. I also thank the anonymous referee for his effort and his detailed and helpful comments.

2. Preliminaries

2.1. **Definitions.** Let $G \coloneqq SO(n,1)$ be the set of $g \in SL_{n+1}(\mathbf{R})$ satisfying

$$\begin{bmatrix} I_n & \\ & -1 \end{bmatrix} g^T \begin{bmatrix} I_n & \\ & -1 \end{bmatrix} = g^{-1}$$

where I_n is the $n \times n$ identity matrix. The corresponding Lie algebra $\mathfrak{g}$ then consists of $v \in \mathfrak{sl}_{n+1}(\mathbf{R})$ satisfying

$$\begin{bmatrix} I_n & \\ & -1 \end{bmatrix} v^T \begin{bmatrix} I_n & \\ & -1 \end{bmatrix} = -v.$$

Then the *Cartan decomposition* can be given by

$$\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p} = \left\{ \begin{bmatrix} \mathbf{l} & \\ & 0 \end{bmatrix} : \mathbf{l} \in \mathfrak{so}(n) \right\} \oplus \left\{ \begin{bmatrix} 0 & \mathbf{p} \\ \mathbf{p}^T & 0 \end{bmatrix} : \mathbf{p} \in \mathbf{R}^n \right\}.$$

Let E_{ij} be the $(n \times n)$-matrix with 1 in the (i,j)-entry and 0 otherwise. Let $e_k \in \mathbf{R}^n$ be the k-th standard basis (vertical) vector. Set

$$Y_k \coloneqq \begin{bmatrix} 0 & e_k \\ e_k^T & 0 \end{bmatrix}, \quad \Theta_{ij} \coloneqq \begin{bmatrix} E_{ji} - E_{ij} & 0 \\ 0 & 0 \end{bmatrix}.$$

Then Y_i, Θ_{ij} form a basis of $\mathfrak{g} = \mathfrak{so}(n,1)$.

Let $\mathfrak{a} = \mathbf{R}Y_n \subset \mathfrak{p}$ be a maximal abelian subspace of $\mathfrak{p}$. Then the root space decomposition of $\mathfrak{g}$ is given by

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{g}_1. \tag{2.1}$$

Denote by $\mathfrak{n} \coloneqq \mathfrak{g}_1$ the sum of the positive root spaces. Let ρ be the half sum of positive roots. We also adopt the convention by identifying $\mathfrak{a}^*$ with $\mathbf{C}$ via $\lambda \mapsto \lambda(Y_n)$. Thus, $\rho = \rho(Y_n) = (n-1)/2$.

Let $\Gamma \subset G$ be a lattice, $X \coloneqq G/\Gamma$, μ be the Haar probability measure on X. Fix a nilpotent $U \in \mathfrak{g}_{-1}$. On G/Γ, denote by

- $\phi_t^{Y_n}(x) \coloneqq \exp(tY_n)x = a^t x$ a geodesic flow,
- $\phi_t^U(x) \coloneqq \exp(tU)x = u^t x$ a unipotent flow.

It is worth noting that

$$[Y_n, U] = -U.$$

Then there exists $\overline{U} \in \mathfrak{g}$ such that $\{U, Y_n, \overline{U}\}$ is an $\mathfrak{sl}_2$-*triple.* Denote

$$\overline{u}^t \coloneqq \exp(t\overline{U}).$$

For convenience, we choose

$$U \coloneqq \begin{bmatrix} 0 & e_{n-1} & e_{n-1} \\ -e_{n-1}^T & 0 & 0 \\ e_{n-1}^T & 0 & 0 \end{bmatrix}, \quad \overline{U} \coloneqq \begin{bmatrix} 0 & -e_{n-1} & e_{n-1} \\ e_{n-1}^T & 0 & 0 \\ e_{n-1}^T & 0 & 0 \end{bmatrix}. \tag{2.2}$$

Then $\langle u^t, a^t, \overline{u}^t \rangle$ generates $SO(2,1) \subset SO(n,1)$.

2.2. **$\mathfrak{sl}_2$-weight decomposition.** First, consider an arbitrary Lie algebra $\mathfrak{g}$ as a $\mathfrak{sl}_2$-representation via the adjoint map (after identifying an image of $\mathfrak{sl}_2$ by *Jacobson-Morozov theorem*), then by the complete reducibility of $\mathfrak{sl}_2$, there is a decomposition of $\mathfrak{sl}_2$-representations

$$\mathfrak{g} = \mathfrak{sl}_2 \oplus V^\perp \tag{2.3}$$

where $V^\perp \subset \mathfrak{g}$ is the sum of $\mathfrak{sl}_2$-irreducible representations other than $\mathfrak{sl}_2$. In particular, for $\mathfrak{g} = \mathfrak{so}(n,1)$, we have

$$V^\perp = \sum_i V_i^0 \oplus \sum_j V_j^2 \tag{2.4}$$

where V_i^0 and V_j^2 are $\mathfrak{sl}_2$-irreducible representations with highest weights 0 and 2. More precisely, we have

Lemma 2.1. *By the weight decomposition, an irreducible $\mathfrak{sl}_2$-representation V^ς is the direct sum of weight spaces, each of which is 1 dimensional. More precisely, there exists a basis $v_0, \ldots, v_\varsigma \in V^\varsigma$ such that*

$$U.v_i = (i+1)v_{i+1}, \quad Y_n.v_i = \frac{\varsigma - 2i}{2} v_i.$$

Thus, if V^ς is an irreducible representation of $\mathfrak{sl}_2$ with the highest weight $\varsigma \le 2$, then for any $v = b_0 v_0 + \cdots + b_\varsigma v_\varsigma \in V^\varsigma$, we have

$$\exp(tU).v = \sum_{j=0}^{\varsigma} \sum_{i=0}^{j} b_i \binom{j}{i} t^{j-i} v_j, \tag{2.5}$$

$$\exp(\omega Y_n).v = \sum_{j=0}^{\varsigma} b_j e^{(\varsigma - 2j)\omega/2} v_j. \tag{2.6}$$

For elements $g \in \exp \mathfrak{g}$ close to identity, we decompose

$$g = h \exp(v), \quad h \in SO_0(2,1), \quad v \in V^\perp$$

where $SO_0(2,1)$ is the connected component of $SO(2,1)$. Moreover, it is convenient to think about $h \in SO_0(2,1)$ as a (2×2)-matrix with determinant 1. Thus, consider the two-to-one isogeny $\iota : SL_2(\mathbf{R}) \to SO(2,1) \subset G$ induced by $\mathfrak{sl}_2(\mathbf{R}) \to \operatorname{Span}\{U, Y_n, \bar{U}\} \subset \mathfrak{g}$. In the following, for $h \in SO_0(2,1)$ and v in an irreducible representation, we write

$$h = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad v = b_0 v_0 + \cdots + b_\varsigma v_\varsigma$$

where v_i are weight vectors in $\mathfrak{g}$ of weight i. Notice that h should more appropriately be written as $\iota(h)$. Besides, for notational simplicity, we shall usually assume that $v \in V^\perp$ lies in a single irreducible representation, since the proofs will mostly focus on the $\operatorname{Ad} u^t$-action and so the general case will be identical but tedious to write down.

For the centralizer $C_{\mathfrak{g}}(U)$ (for an arbitrary Lie algebra $\mathfrak{g}$), we have the corresponding decomposition:

$$C_{\mathfrak{g}}(U) = \mathbf{R}U \oplus V_C^{\perp} \tag{2.7}$$

where $V_C^{\perp} \subset V^{\perp}$ consists of highest weight vectors other than U. In particular, for $\mathfrak{g} = \mathfrak{so}(n,1)$, under the setting (2.2), one may calculate

$$\begin{aligned} C_{\mathfrak{g}}(U) =& \mathbf{R}U \oplus V_C^{\perp} = \mathbf{R}U \oplus \mathfrak{k}_C^{\perp} \oplus \mathfrak{n}_C^{\perp} \\ =& \mathbf{R}U \oplus \begin{bmatrix} \mathfrak{so}(n-2) & \\ & 0 \end{bmatrix} \oplus \left\{ \begin{bmatrix} 0 & 0 & \mathbf{u} & \mathbf{u} \\ 0 & 0 & 0 & 0 \\ -\mathbf{u}^T & 0 & 0 & 0 \\ \mathbf{u}^T & 0 & 0 & 0 \end{bmatrix} : \mathbf{u} \in \mathbf{R}^{n-2} \right\}. \end{aligned} \tag{2.8}$$

Note that $\mathfrak{k}_C^{\perp}$ consists of semisimple elements, and $\mathfrak{n}_C^{\perp}$ consists of nilpotent elements, and they satisfy $[\mathfrak{k}_C^{\perp}, \mathfrak{n}_C^{\perp}] = \mathfrak{n}_C^{\perp}$.

2.3. **Time-changes.** Let Y be a homogeneous space and U be a nilpotent. Let $\phi_t^{U,\tau}$ be a *time change* for the unipotent flow ϕ_t^U, $t \in \mathbf{R}$. Thus, we assume that

- $\tau : Y \to \mathbf{R}^+$ is a integrable nonnegative function on Y satisfying
$$\int_Y \tau(y) dm_Y(y) = 1,$$
- $\xi : Y \times \mathbf{R} \to \mathbf{R}$ is the cocycle determined by
$$t = \int_0^{\xi(y,t)} \tau(u^s y) ds = \int_0^{\xi(y,t)} \tau(\phi_t^U y) ds.$$
- $\phi_t^{U,\tau} : Y \to Y$ is given by the relation
$$\phi_t^{U,\tau}(y) := u^{\xi(y,t)} y.$$

Remark 2.2. Note that $\phi_t^{U,1} = \phi_t^U$. Besides, one can check that $\phi_t^{U,\tau}$ preserves the probability measure on Y defined by $d\nu := \tau dm_Y$ where m_Y is the Lebesgue measure on Y. On the other hand, if τ is smooth, then the time-change $\phi_t^{U,\tau}$ is the flow on Y generated by the smooth vector field $U_\tau := U/\tau$.

In practice, we define $z : Y \times \mathbf{R} \to \mathbf{R}$ by

$$z(y,t) = \int_0^t \tau(u^s y) ds.$$

It follows that

$$t = z(y, \xi(y,t)), \quad \phi_{z(y,t)}^{U,\tau}(x) = \phi_t^U(y) = u^t y. \tag{2.9}$$

Let $\kappa > 0$ and $\mathbf{K}_\kappa(Y)$ be the collection of all positive integrable functions τ on Y such that τ, τ^{-1} are bounded and satisfies

$$\left| \int_Y \tau(y)\tau(u^t y) d\nu(y) - \Big(\int_Y \tau(y)\nu(y) \Big)^2 \right| \leq D_\tau |t|^{-\kappa} \tag{2.10}$$

for some $D_\tau > 0$. Let $\mathbf{K}(Y) = \bigcup_{\kappa>0} \mathbf{K}_\kappa(Y)$. This is the effective mixing property of the unipotent flow ϕ_t^U. Note that [KM99] (see also [Ven10]) has shown that there is $\kappa > 0$ such that

$$\left|\langle \phi_t^U(f), g\rangle - \left(\int_Y f(y)\nu(y)\right)\left(\int_Y g(y)\nu(y)\right)\right| \ll (1+|t|)^{-\kappa}\|f\|_{W^s}\|g\|_{W^s}$$

for $f, g \in C^\infty(X)$, where $s \geq \dim(K)$ and W^s denotes the Sobolev norm on $Y = G/\Gamma$. According to Lemma 3.1 [Rat86], when $\tau \in \mathbf{K}_\kappa(Y)$, we have the effective ergodicity: there is $K \subset Y$ with $\nu(K) > 1 - \sigma$ and $t_K > 0$ such that

$$|t - z(y,t)| = O(t^{1-\kappa}) \tag{2.11}$$

for all $t \geq t_K$ and $y \in K$. Later on, we shall make use of the effective mixing/ergodicity to study the shearing property of unipotent flows (see Section 4 (5.1)).

2.4. **Cohomology.** We first introduce the 1-coboundary of two functions.

Definition 2.3 (Cohomology)**.** We say that two functions τ_1, τ_2 on Y are *measurable (respectively L^2, smooth, etc.) cohomologous over the flow ϕ_t* if there exists a measurable (respectively L^2, smooth, etc.) function f on Y, called the *transfer function*, such that

$$\int_0^T \tau_1(\phi_t y) - \tau_2(\phi_t y)dt = f(\phi_T y) - f(y). \tag{2.12}$$

For $i \in \{1,2\}$, let $(Y_i, \mathcal{Y}_i, \nu_i, \phi_t^{(i)})$ be measure-preserving flows, and let $\tau_i : Y_i \to \mathbf{R}$ be measurable functions on Y_i. Besides, we extend τ_i to $Y_1 \times Y_2$ by setting

$$\tau_i : (y_1, y_2) \mapsto \tau_i(y_i), \quad i = 1, 2.$$

Definition 2.4 (Joint cohomology)**.** Let $\rho \in J(\phi_t^{(1)}, \phi_t^{(2)})$ be a joining of $\phi_t^{(1)}$ and $\phi_t^{(2)}$. We say that τ_1 and τ_2 are *jointly cohomologous via ρ* if τ_1 and τ_2 (considered as functions on $Y_1 \times Y_2$) are cohomologous over $\phi_t^{(1)} \times \phi_t^{(2)}$ on $(Y_1 \times Y_2, \rho)$. More specifically, if τ_1 and τ_2 are cohomologous over $\phi_t^{(1)} \times \phi_t^{(2)}$ with a transfer function $f : Y_1 \times Y_2 \to \mathbf{R}$, then we say that τ_1 and τ_2 are *jointly cohomologous via* (ρ, f), and we have

$$\int_0^T (\tau_1 - \tau_2)(\phi_t^{(1)} y_1, \phi_t^{(2)} y_2)dt = f(\phi_T^{(1)} y_1, \phi_T^{(2)} y_2) - f(y_1, y_2) \tag{2.13}$$

for ρ-a.e. $(y_1, y_2) \in Y_1 \times Y_2$ and all $T \in \mathbf{R}$.

Let $\mathcal{A}_1 \coloneqq \{A \times Y_2 : A \in \mathcal{Y}_1\}$, $\mathcal{A}_2 \coloneqq \{Y_1 \times A : A \in \mathcal{Y}_2\}$. Then there is a unique family $\{\rho_{y_1}^{\mathcal{A}_1} : y_1 \in Y_1\}$ of probability measure, called the *conditional measures*, on Y_2 such that

$$E^\rho(g|\mathcal{A}_1)(y_1) = \int_{Y_2} g(y_1, y_2)d\rho_{y_1}^{\mathcal{A}_1}(y_2), \quad \rho^{\mathcal{A}_1}_{\phi_t^{(1)} y_1} = (\phi_t^{(2)})_* \rho_{y_1}^{\mathcal{A}_1} \tag{2.14}$$

for every $g \in L^1(Y_1 \times Y_2, \rho)$, $t \in \mathbf{R}$, and ν_1-a.e. $y_1 \in Y_1$. Taking the integration over $\rho_{y_1}^{\mathcal{A}_1}$, expressions (2.13) and (2.14) show that if the transfer function $f(y_1, \cdot) \in$

$L^1(Y_2, \rho_{y_1}^{\mathcal{A}_1})$ for ν_1-a.e. $y_1 \in Y_1$, then τ_1 and $E^\rho(\tau_2|\mathcal{A}_1)$ are cohomologous along $\phi_t^{(1)}$ via $E^\rho(f|\mathcal{A}_1)$. We have just proved the following:

Proposition 2.5. *Let $\tau_i : Y_i \to \mathbf{R}$ be measurable functions on Y_i, $i = 1, 2$. Suppose that τ_1 and τ_2 are jointly cohomologous via (ρ, f) with $f(y_1, \cdot) \in L^1(Y_2, \rho_{y_1}^{\mathcal{A}_1})$ for μ_1-a.e. $y_1 \in Y_1$. Then τ_1 and $E^\rho(\tau_2|\mathcal{A}_1)$ are cohomologous over $\phi_t^{(1)}$ via $E^\rho(f|\mathcal{A}_1)$.*

3. Shearing property I, H-flow on one factor

3.1. **Joinings.** Let $G = SO(n, 1)$, Γ be a lattice of G, (X, μ) be the homogeneous space $X = G/\Gamma$ equipped with the Lebesgue measure μ, and let ϕ_t^U be a unipotent flow on X. Let (Y, ν, S) be a measure-preserving system. We want to study the joinings of (X, μ, ϕ_1^U) and (Y, ν, S). Thus, let ρ be an ergodic *joining* of ϕ_1^U and S, i.e. ρ is a probability measure on $X \times Y$, whose marginals on X and Y are μ and ν respectively, and which is $(\phi_1^U \times S)$-ergodic.

Let $C(\phi_1^U)$ be the *commutant* of ϕ_1^U, i.e. collection of all measure-preserving transformations on X that commute with ϕ_1^U. The following is a basic criterion for ρ in terms of the commutant of ϕ_1^U:

Lemma 3.1. *Let the notation and assumptions be as above. Assume further that $T \in C(\phi_1^U)$ is ergodic on (X, μ). Then*

$$\textit{either } (T \times \mathrm{id})_*\rho \perp \rho \quad \textit{or} \quad \rho = \mu \times \nu.$$

Proof. First, by the commutative property of T, we easily see that $(T \times \mathrm{id})_*\rho$ is again $(\phi_1^U \times S)$-ergodic on $X \times Y$. It implies that either $(T \times \mathrm{id})_*\rho \perp \rho$ or $(T \times \mathrm{id})_*\rho = \rho$. Now assume that $(T \times \mathrm{id})_*\rho = \rho$, i.e. ρ is $(T \times \mathrm{id})$-invariant. Then via disintegration, we know that ρ_y is T-invariant on X for ν-a.e. $y \in Y$, where

$$\rho = \int_Y \rho_y d\nu(y). \tag{3.1}$$

Now assume for contradiction that there exists $B \subset Y$ with $\nu(B) > 0$ such that $\rho_y \neq \mu$ for $y \in B$. It follows that for $y \in B$, there is $A_y \subset X$ with $\mu(A_y) > 0$ such that for $x \in A_y$, we have

$$(\rho_y)_x^{\mathcal{E}} \neq \mu \tag{3.2}$$

where $(\rho_y)_x^{\mathcal{E}}$ is given by the T-ergodic decomposition

$$\rho_y = \int_X (\rho_y)_x^{\mathcal{E}} d\mu(x).$$

Notice that by the ergodicity, there is a μ-conull set $\Omega \subset X$, namely the set of T-generic points of μ, such that $(\rho_y)_x^{\mathcal{E}}(\Omega) = 0$ for the measures $(\rho_y)_x^{\mathcal{E}}$ in (3.2). Then by the assumption of joining, we have

$$\begin{aligned}\mu(\Omega) = \rho(\pi_X^{-1}(\Omega)) &= \int_Y \rho_y(\Omega) d\nu(y) \\ &= \int_B \rho_y(\Omega) d\nu(y) + \int_{Y\setminus B} \rho_y(\Omega) d\nu(y) \\ &\leq \int_B \int_X (\rho_y)_x^{\mathcal{E}}(\Omega) d\mu(x) d\nu(y) + \nu(Y \setminus B) \\ &= \int_B \int_{X\setminus A_y} (\rho_y)_x^{\mathcal{E}}(\Omega) d\mu(x) d\nu(y) + \nu(Y \setminus B) \\ &\leq \int_B \mu(X \setminus A_y) d\nu(y) + \nu(Y \setminus B) \\ &< \nu(B) + \nu(Y \setminus B) = 1\end{aligned}$$

which is a contradiction. Thus, we conclude that $\rho_y = \mu$ for ν-a.e. $y \in Y$ and so $\rho = \mu \times \nu$. ☐

By *Moore's ergodicity theorem*, we deduce that

Corollary 3.2. *If $w \in C_{\mathfrak{g}}(U)$ so that $\langle \exp tw\rangle_{t\in\mathbf{R}}$ is not compact, then*

$$\textit{either } (\phi_1^w \times \mathrm{id})_* \rho \perp \rho \quad \textit{or} \quad \rho = \mu \times \nu.$$

3.2. **H-property.** In this section, we want to introduce the *H-property* (or *Ratner property*) in order to study the joining ρ in terms of the unipotent flow ϕ_t^U on X. The classic H-property can be formulated as follows:

Theorem 3.3 (H-property, [Wit85])**.** *Let u be a unipotent element of G. Given any neighborhood Q of e in $C_G(u)$, there is a compact subset ∂Q of $Q \setminus \{e\}$ such that for any $\epsilon > 0$ and $M > 0$, there are $\alpha = \alpha(u, Q, \epsilon) > 0$ and $\delta = \delta(u, Q, \epsilon, M) > 0$ such that if $x_1, x_2 \in X$ with $d_X(x_1, x_2) < \delta$ then one of the following holds:*

- *$x_2 = cx_1$ for some $c \in C_G(u)$ with $d_G(e, c) < \delta$,*
- *there are $L > M/\alpha$ and $q \in \partial Q$ such that*

$$d_X(u^n x_2, qu^n x_1) < \epsilon \tag{3.3}$$

whenever $n \in [L, (1 + \alpha)L]$.

Remark 3.4. In fact, for $x_2 = gx_1$ with $g = \exp(v) \in B_\delta^G$, the element $q \in C_{\mathfrak{g}}(U)$ in Theorem 3.3 is chosen by

$$q = \pi_{C_{\mathfrak{g}}(U)} \exp(LU).v \tag{3.4}$$

where $\pi_{C_{\mathfrak{g}}(U)} : \mathfrak{g} \to C_{\mathfrak{g}}(U)$ is the natural projection and $\exp(LU).v$ is the adjoint representation (see (2.5)). We often call q as the *fastest relative motion* between x_1, x_2; see [Mor05] for more discussion. In what follows, we choose $Q = B_\lambda^{C_G(u)}$ to be the ball of radius λ of e in $C_G(u)$ for sufficiently small λ (independent of ϵ), and then ∂Q is the sphere of radius λ. Now by (2.3) (2.4), we have the decomposition

$$v = v_0 + v_2$$

where $v_0 \in \sum_i V_i^0$ and $v_2 \in \mathfrak{sl}_2 + \sum_j V_j^2$. Thus, $\|v_0\|, \|v_2\| < \delta$ and

$$q = v_0 + \pi_{C_{\mathfrak{g}}(U)} \exp(LU).v_2.$$

Since $\|q\| = \lambda$, we see that v_0 is negligible. In other words, we can replace q by

$$q' := \pi_{C_{\mathfrak{g}}(U)} \exp(LU).v_2 \tag{3.5}$$

and then Theorem 3.3 still holds. On the other hand, note that $q' \in \mathfrak{n} = \mathbf{R}U + \mathfrak{n}_C^\perp$ (cf. (2.8)). Thus, the one-parameter group $\langle \exp(tq') \rangle_{t\in\mathbf{R}}$ generated by q' is not compact.

In the following, we shall generalize the idea in [Rat83] and prove Theorem 1.1.

Theorem 3.5. *Let the notation and assumptions be as above. Then either $\rho = \mu \times \nu$ or $(\phi_1^U \times S, \rho)$ is a compact extension of (S, ν). More precisely, if $\rho \neq \mu \times \nu$, then there exists a ν-conull set $\Theta \subset Y$, a compact subgroup $C^\rho \subset C_G(u)$, and $n > 0$ such that for any $y \in \Theta$, there exist $x_1^y, \dots, x_n^y$ in the support of ρ_y with*

$$\rho_y(C^\rho x_i^y) = \frac{1}{n}$$

for $i = 1, \dots, n$, where $\rho = \int_Y \rho_y d\nu(y)$ is the disintegration along Y (cf. (3.1)).

Assume that $\rho \neq \mu \times \nu$. Then by Corollary 3.2, there is a ρ-conull set $\Omega \subset X \times Y$, namely the set of $(\phi_1^U \times S)$-generic points, such that $(\phi_1^w \times \mathrm{id})(\Omega) \cap \Omega = \emptyset$ for all $w \in \mathfrak{n}$. Given a sufficiently small $\lambda > 0$, we define the sphere of radius λ of 0 by

$$B_\lambda^{\mathfrak{n}} := \{w \in \mathfrak{n} : \|w\| = \lambda\}.$$

Then, one can find a compact subset $K_1 \subset \Omega$ with $\mu(K_1) > 199/200$. Then

$$\bigcup_{w \in B_\lambda^{\mathfrak{n}}} (\phi_1^w \times \mathrm{id})(K_1)$$

is compact. Thus, there are $\epsilon > 0$ and $K_2 \subset K_1$ with $\mu(K_2) > 99/100$ such that

$$d_{X\times Y}\left(K_2, \bigcup_{w \in B_\lambda^{\mathfrak{n}}} (\phi_1^w \times \mathrm{id})(K_1)\right) > \epsilon.$$

It follows that if $(x_1, y), (x_2, y) \in K_2$ then

$$d_X(x_2, \phi_1^w x_1) \geq \epsilon \tag{3.6}$$

for all $w \in B_\lambda^{\mathfrak{n}}$. Let $\alpha = \alpha(\epsilon) > 0$ be as in Theorem 3.3. Comparing (3.6) with (3.3), we conclude

Lemma 3.6. *Assume that $\rho \neq \mu \times \nu$. There is a positive number $\delta = \delta(\epsilon) > 0$, a measurable set $K_4 \subset \Omega$ with $\rho(K_4) > 0$ such that if $(x_1, y), (x_2, y) \in K_4$ and $d_X(x_1, x_2) < \delta$, then $x_2 \in C_G(u)x_1$.*

Proof. Suppose that M, δ, K_4 are given, and $x_2 \notin C_G(u)x_1$ with $d_X(x_1, x_2) < \delta$. Then by the H-property of the unipotent flow (Theorem 3.3 and Remark 3.4), we know that there are $L > M/\alpha$ and $w \in B_\lambda^{\mathfrak{n}}$ such that

$$d_X(\phi_n^U x_1, \phi_1^w \phi_n^U x_2) < \epsilon \tag{3.7}$$

for $n \in [L, (1+\alpha)L]$. Next, we shall find some qualified $x_1, x_2 \in X$ such that the distance between $\phi_n^U x_1$ and $\phi_1^w \phi_n^U x_2$ is at least ϵ. This will lead to a contradiction.

First, applying the ergodic theorem, there is a measurable set $K_3 \subset \Omega$ with $\rho(K_3) > 1 - \alpha/2(100+\alpha)$, a number $M_1 > 0$ such that

$$\frac{1}{n}\left|\{k \in [0,n] : (\phi_1^U \times S)^k(x,y) \in K_2\}\right| > \frac{9}{10} \tag{3.8}$$

for $(x,y) \in K_3$ and $n > M_1$. Applying the ergodic theorem one more time, there is a measurable set $K_4 \subset \Omega$ with $\rho(K_4) > 0$, a number $M_2 > 0$ such that

$$\frac{1}{n}\left|\{k \in [0,n] : (\phi_1^U \times S)^k(x,y) \in K_3\}\right| > 1 - \frac{\alpha}{10+\alpha} \tag{3.9}$$

for $(x,y) \in K_4$ and $n > M_2$.

Choose $M = \max\{M_1, M_2\}$ and then $L > M/\alpha$ and $\delta = \delta(\epsilon, M) > 0$ as obtained from the H-property (Theorem 3.3). Let $(x_1, y), (x_2, y) \in K_4$ with $d_X(x_1, x_2) < \delta$. Then replacing n by $(1+\alpha/10)L$ and applying (3.9), we know that

$$(\phi_1^U \times S)^s(x_1, y), (\phi_1^U \times S)^t(x_2, y) \in K_3$$

for some integers $s, t \in [L, (1+\alpha/10)L]$. Further, replacing the interval $[0,n]$ by $[s, (1+\alpha)L]$ (resp. $[t, (1+\alpha)L]$) and applying (3.8), we know that

$$\frac{1}{(1+\alpha)L - s}\left|\{k \in [s, (1+\alpha)L] : (\phi_1^U \times S)^k(x_1,y) \in K_2\}\right| > \frac{9}{10}$$

$$\frac{1}{(1+\alpha)L - t}\left|\{k \in [t, (1+\alpha)L] : (\phi_1^U \times S)^k(x_2,y) \in K_2\}\right| > \frac{9}{10}.$$

It follows that there exists $n \in [(1+\alpha/10)L, (1+\alpha)L]$ such that

$$(\phi_1^U \times S)^n(x_1, y),\ (\phi_1^U \times S)^n(x_2, y) \in K_2.$$

Then by (3.6), we have

$$d_X(\phi_n^U x_1, \phi_1^w \phi_n^U x_2) \geq \epsilon$$

which contradicts (3.7). ☐

Recall that via disintegration (cf. (3.1)), we have

$$\rho = \int_Y \rho_y \, d\nu(y).$$

Then by the ergodic theory, we have

Lemma 3.7. *Assume that $\rho \neq \mu \times \nu$. There exists a ν-conull set $\Theta \subset Y$ and $n > 0$ such that for any $y \in \Theta$, there exist $x_1^y, \dots, x_n^y$ in the support of ρ_y with*

$$\rho_y(C_G(u)x_i^y) = \frac{1}{n}$$

for $i = 1, \dots, n$.

Proof. Let $f : Y \to \mathbf{R}$ be defined by

$$f : y \mapsto \sup_{x \in X} \rho_y(C_G(u)x).$$

By Lemma 3.6, we know that for $y \in K_4^Y \coloneqq \{y \in Y : \rho_y\{x \in X : (x, y) \in K_4\} > 0\}$, $f(y) > 0$. Note also that $\nu(K_4^Y) > 0$ and f is S-invariant. By the ergodicity, f is a positive constant, say $f \equiv c$, on a ν-conull set $\Theta_1 \subset Y$.

Next, consider

$$D \coloneqq \{(x, y) \in X \times Y : y \in \Theta_1,\ \rho_y(C_G(u)x) = c\}.$$

Then D is $(\phi_1^U \times S)$-invariant and $\rho(D) > 0$. Thus, $\rho(D) = 1$. Next, define

$$\Theta \coloneqq \{y \in \Theta_1 : \rho_y\{x \in X : (x, y) \in D\} = 1\}.$$

Then $\Theta \subset Y$ is an S-invariant ν-conull set. Thus, for any $y \in \Theta$, we have

$$\rho_y(C_G(u)x) \equiv c$$

for any $x \in X$ with $(x, y) \in D$. It forces $n = 1/c$ to be an integer. Besides, for any $y \in \Theta$, there are only finitely many points $x_1^y, \dots, x_n^y$ with

$$\rho_y(C_G(u)x_i^y) = \frac{1}{n}$$

for $i = 1, \dots, n$. $\square$

Thus, by Lemma 3.7, we see that ρ_y supports on $\bigsqcup_{i=1}^n C_G(u)x_i^y$ whenever $y \in \Theta$. With a further effort, we observe that these ρ_y must have a compact support.

Proof of Theorem 3.5. For a Borel measurable subset $A \subset C_G(u)$, consider the map $f_A : X \times Y \to \mathbf{R}^+$ be defined by

$$f_A : (x, y) \mapsto \rho_y(Ax).$$

Note that since ρ is $(\phi_1^U \times S)$-invariant, we have

$$(\phi_1^U)_* \rho_y = \rho_{Sy}.$$

It follows that

$$f_A(x, y) = \rho_y(Ax) = \rho_{Sy}(\phi_1^U Ax) = \rho_{Sy}(A\phi_1^U x) = f_A(\phi_1^U x, Sy).$$

In other words, f_A is $(\phi_1^U \times S)$-invariant and therefore is ρ-a.e. a constant, say $m(A)$. Thus, for any $A \in \mathcal{B}(C_G(u))$, there exists a ρ-conull set $\Omega_A \subset X \times Y$, such that

$$\rho_y(Ax) \equiv m(A) \tag{3.10}$$

for $(x, y) \in \Omega_A$.

Next, we consider the fundamental domain, i.e. a Borel subset $F \subset C_G(u)$ such that the natural map $F \to C_G(u)/(C_G(u) \cap \Gamma)$ defined by $g \mapsto g\Gamma$ is bijective. Then since $\mathcal{B}(F)$ is countably generated, by *Carathéodory's extension theorem*, we know that $m : \mathcal{B}(F) \to \mathbf{R}^+$ is a measure. Besides, it follows from (3.10) that there exists a ρ-conull set $\Omega \subset X \times Y$, such that

$$\rho_y(Ax) \equiv m(A) \tag{3.11}$$

for $(x, y) \in \Omega$, $A \in \mathcal{B}(F)$.

Now assume that (3.11) holds for $(x, y), (gx, y) \in \Omega$ and $g \in C_G(u)$. Then

$$m(A) = \rho_y(Agx) = m(Ag)$$

for $A \in \mathcal{B}(F)$. In other words, m is g-(right) invariant and so is (right) Haar. Note that $C_G(u)$ is unimodular (since its Lie algebra $C_{\mathfrak{g}}(U)$ is a direct sum of a compact and a nilpotent Lie subalgebra). We conclude that m is also a (left) Haar measure, and therefore ρ_y is (left) Haar on $C_G(u)x$ for $(x, y) \in \Omega$.

Let C^ρ be the stabilizer of m. Then the above result shows that ρ is $(C^\rho \times \mathrm{id})$-invariant. Thus, according to Corollary 3.2, C^ρ must be compact. This finishes the proof of Theorem 3.5. ☐

Using Theorem 3.5, for any ergodic joining ρ of ϕ_1^U and S on $X \times Y$, we obtain an ergodic joining $\overline{\rho} := \pi_*\rho$ of ϕ_1^U and S on $C^\rho\backslash X \times Y$ under the natural projection $\pi : X \times Y \to C^\rho\backslash X \times Y$. Moreover, when $\overline{\rho} \neq \overline{\mu} \times \nu$ is not the product measure, it is a finite extension of ν, i.e. $\operatorname{supp} \overline{\rho}_y$ consists of exactly n points $\overline{x}_1^y, \ldots, \overline{x}_n^y$ for ν-a.e. $y \in Y$ (without loss of generality, we shall assume that it holds for all $y \in Y$). Note that $y \mapsto \overline{x}_i^y$ need not be measurable. However, this can be resolved by using *Kunugui's theorem* (see [Kun40], [Kal75]).

Therefore, let $\overline{X} := C^\rho\backslash X$, $\pi_X : X \times Y \to X$, $\pi_{\overline{X}} : \overline{X} \times Y \to \overline{X}$, $\pi_Y : \overline{X} \times Y \to Y$ be the natural projections. By Kunugui's theorem, we are able to find $\hat{\psi}_i : Y \to \overline{X} \times Y$ for $i = 1, \ldots, n$ such that $\pi_Y \circ \hat{\psi}_i = \mathrm{id}$ and $\hat{\psi}_i(Y) \cap \hat{\psi}_j(Y) = \emptyset$ whenever $i \neq j$. Let

$$\Omega_i := \hat{\psi}_i(Y), \quad \overline{\psi}_i := \pi_{\overline{X}} \circ \hat{\psi}_i. \tag{3.12}$$

Then $\rho(\Omega_i) = 1/n$, $\bigcup \Omega_i = \operatorname{supp} \overline{\rho}$, and $\Omega \cap \operatorname{supp} \overline{\rho}_y$ consists of exactly one point. Next, we can apply Kunugui's theorem again and obtain $\psi_i : Y \to X$ so that $P_X \circ \psi_i = \overline{\psi}_i$ where $P_X : X \to \overline{X}$.

4. Shearing property II, time changes of unipotent flows

We continue to study the shearing property of unipotent flows. More precisely, we shall study the shearing in directions different from Section 3.2 and deduce the following Proposition 4.19. In fact, in Section 3.2, we study the shearing between points of the form $(x, y), (gx, y) \in X \times Y$ for some $g \in G_X$ sufficiently close to the identity. Thus, the information basically comes from the X-factor. However, in this section, we shall study the shearing between points of the form $(\psi(y), y), (\psi(gy), gy) \in C^\rho\backslash X \times Y$ where $\psi : Y \to C^\rho\backslash X$ is a measurable map and

$g \in G_X$ is sufficiently close to the identity. Thus, the time-change on Y comes into play. The technique used in Proposition 4.19 generalizes the ideas in [Rat86] [Tan20], and provides us a quantitative estimate of a unipotent shearing on the double quotient space $C^\rho \backslash G_X/\Gamma_X$. Roughly speaking, Proposition 4.19 helps us better understand the non-shifting time under a unipotent shearing.

4.1. **Preliminaries.** We start with a combinatorial result. Let I be an interval in $\mathbf{R}$ and let J_i, J_j be disjoint subintervals of I, $J_i = [x_i, y_i]$, $y_i < x_j$ if $i < j$. Denote

$$d(J_i, J_j) \coloneqq \mathrm{Leb}[y_i, x_j] = x_j - y_i.$$

For a collection β of finitely many intervals, we define

$$|\beta| \coloneqq \mathrm{Leb}\left(\bigcup_{J\in\beta} J\right).$$

Besides, for a collection β of finitely many intervals, an interval I, let

$$\beta \cap I \coloneqq \{I \cap J : J \in \beta\}.$$

Proposition 4.1 (Existence of large intervals, Solovay [Rat79])**.** *Given $\eta \in (0,1)$, $\zeta \in (0,1)$, there is $\theta = \theta(\zeta, \eta) \in (0,1)$ such that if I is an interval of length $\lambda \gg 1$ and $\alpha = \{J_1, \dots, J_n\} = \mathcal{G} \cup \mathcal{B}$ is a partition of I into good and bad intervals such that*

(1) for any two good intervals $J_i, J_j \in \mathcal{G}$, we have

$$d(J_i, J_j) \geq [\min\{\mathrm{Leb}(J_i), \mathrm{Leb}(J_j)\}]^{1+\eta}, \tag{4.1}$$

(2) $\mathrm{Leb}(J) \leq \zeta\lambda$ for any good interval $J \in \mathcal{G}$,
(3) $\mathrm{Leb}(J) \geq 1$ for any bad interval $J \in \mathcal{B}$,

then the measure of bad intervals $\mathrm{Leb}(\bigcup_{J\in\mathcal{B}} J) \geq \theta\lambda$. More precisely, we can take

$$\theta = \theta(\zeta, \eta) = \prod_{n=0}^{\infty} (1 + C\zeta^{n\eta})^{-1}$$

for some constant $C > 0$ (independent of ζ, η).

Proof. Assume that $\zeta^{1-k} \leq \lambda \leq \zeta^{-k}$ for some $k \geq 1$. Let

$$\mathcal{G}_n \coloneqq \{J \in \mathcal{G} : \zeta^{n+1}\lambda \leq |J| \leq \zeta^n\lambda\},$$

$\mathcal{G}_{\leq n} \coloneqq \bigcup_{i=1}^n \mathcal{G}_i$, and $\mathcal{B}_{\leq n}$ be the collection of remaining intervals forming $I \setminus \bigcup_{J\in\mathcal{G}_{\leq n}} J$. Then for $n \in \mathbf{N}$, $J \in \mathcal{B}_{\leq n}$, by (4.1), we have

$$\begin{aligned}\frac{|\mathcal{B}_{\leq n+1} \cap J|}{\mathrm{Leb}(J)} &= \frac{|\mathcal{B}_{\leq n+1} \cap J|}{|\mathcal{G}_{n+1} \cap J| + |\mathcal{B}_{\leq n+1} \cap J|} = \left(1 + \frac{|\mathcal{G}_{n+1} \cap J|}{|\mathcal{B}_{\leq n+1} \cap J|}\right)^{-1} \\ &\geq \left(1 + \frac{l\zeta^{n+1}\lambda}{(l-1)\zeta^{(n+2)(1+\eta)}\lambda^{1+\eta}}\right)^{-1} = \left(1 + C\zeta^{(k-n)\eta}\right)^{-1}\end{aligned}$$

where $l \geq 2$ is the number of intervals in $\mathcal{G}_{n+1} \cap J$, and $C > 0$ is some constant depending on η and ζ. One can also show that when $k = 0, 1$, we have a similar relation. By summing over $J \in \mathcal{B}_{\leq n}$, we obtain

$$\frac{|\mathcal{B}_{\leq n+1}|}{|\mathcal{B}_{\leq n}|} \geq \left(1 + C\zeta^{(k-n)\eta}\right)^{-1}.$$

Note that by (2), $|\mathcal{B}_{\leq 0}| = \lambda$, and by (3), $\mathcal{B}_{\leq n} = \mathcal{B}_{\leq n+1}$ for all $n \geq k$. We calculate

$$|\mathcal{B}| = |\bigcap_{k\geq 0} \mathcal{B}_{\leq k}| = \lim_{k\to\infty} |\mathcal{B}_{\leq k}| = \prod_{n=0}^{\infty} \frac{|\mathcal{B}_{\leq n+1}|}{|\mathcal{B}_{\leq n}|} \cdot \lambda \geq \prod_{n=0}^{k} \left(1 + C\zeta^{(k-n)\eta}\right)^{-1} \cdot \lambda.$$

Now note that

$$\theta(\zeta, \eta) = \prod_{n=0}^{\infty} (1 + C\zeta^{n\eta})^{-1} \leq \prod_{n=0}^{k} \left(1 + C\zeta^{(k-n)\eta}\right)^{-1}$$

and the proposition follows. □

In light of (4.1), we make the following definition.

Definition 4.2 (Effective gaps between intervals)**.** We say that two intervals $I, J \subset \mathbf{R}$ have an *effective gap* if

$$d(I, J) \geq [\min\{\mathrm{Leb}(I), \mathrm{Leb}(J)\}]^{1+\eta}$$

for some $\eta > 0$. Later, we shall obtain some quantitative results relative to the effective gap.

Remark 4.3. It is worth noting that if $\mathcal{A}$ and $\mathcal{B}$ are collections of intervals with effective gaps, then the intersection $\mathcal{A} \cap \mathcal{B} := \{I \cap J : I \in \mathcal{A},\ J \in \mathcal{B}\}$ also have effective gaps. More generally, assume that $\mathcal{A}$ and $\mathcal{B}$ are collections of intervals with effective gaps. If $J_1, J_2 \in \mathcal{A} \cap \mathcal{B}$ have an effective gap, then there is a pair of intervals I_1, I_2, either in $\mathcal{A}$ or in $\mathcal{B}$, such that $J_1 \subset I_1$, $J_2 \subset I_2$ and I_1, I_2 have an effective gap.

In the following, we shall use the asymptotic notation:

- $A \ll B$ or $A = O(B)$ means there is a constant $C > 0$ such that $A \leq CB$ (we also write $A \ll_\kappa B$ if the constant $C(\kappa)$ depends on some coefficient κ);
- $A = o(B)$ means that $A/B \to 0$ as $B \to 0$;
- $A \asymp B$ means there is a constant $C > 1$ such that $C^{-1}B \leq A \leq CB$;
- $A \approx 0$ means $A \in (0, 1)$ close to 0, and $A \approx 1$ means $A \in (0, 1)$ close to 1.

Similar to [Tan20], we need to following quantitative property of polynomials.

Lemma 4.4. *Fix numbers $R_0 > 0, \kappa \in (0, 1]$, a real polynomial $p(x) = v_0 + v_1 x + \cdots + v_k x^k \in \mathbf{R}[x]$. Assume further that there exist intervals $[0, \bar{l}_1] \cup [l_2, \bar{l}_2] \cup \cdots \cup [l_m, \bar{l}_m]$ such that*

$$|p(t)| \ll \max\{R_0, t^{1-\kappa}\} \quad \textit{iff} \quad t \in [0, \bar{l}_1] \cup [l_2, \bar{l}_2] \cup \cdots \cup [l_m, \bar{l}_m] \tag{4.2}$$

Then $\bar{l}_1$ *has the lower bound* l *depending on* $\max_i |v_i|$, R_0, κ *and the implicit constant such that* $l \nearrow \infty$ *as* $\max_i |v_i| \searrow 0$ *for fixed* R_0, κ. *Besides,* $m \leq k$ *and we have*

(1) $|v_i| \ll_{k,\kappa} R_0 \bar{l}_1^{1-i-\kappa}$ *for all* $1 \leq i \leq k$;

(2) *Fix* $\eta \approx 0$. *Assume that for certain* $1 \leq j \leq m-1$, *sufficiently large* $\bar{l}_j$, *the intervals* $[0, \bar{l}_j]$ *and* $[l_{j+1}, \bar{l}_{j+1}]$ *do not have an effective gap:*

$$l_{j+1} - \bar{l}_j \leq \min\{\bar{l}_j, \bar{l}_{j+1} - l_{j+1}\}^{1+\eta}. \tag{4.3}$$

Then there exists $1 \approx \xi(\eta, k) \in (0,1)$ *with* $\xi(\eta, k) \to 1$ *as* $\eta \to 0$ *such that*

$$|v_i| \ll_{k,\kappa} \bar{l}_j^{\xi(\eta,k)(1-i-\kappa)}$$

for all $1 \leq i \leq k$.

Proof. The number m of intervals in (4.2) can be bounded by k via an elementary argument of polynomials.

(1) Let $F(x) := v_1(\bar{l}_1 x)^\kappa + \cdots + v_k(\bar{l}_1 x)^{k-1+\kappa}$ for $x \in [0,1]$. Then we have

$$\begin{pmatrix} v_1 \bar{l}_1^\kappa \\ v_2 \bar{l}_1^{1+\kappa} \\ \vdots \\ v_k \bar{l}_1^{k-1+\kappa} \end{pmatrix} = \begin{bmatrix} (1/k)^\kappa & (1/k)^{1+\kappa} & \cdots & (1/k)^{k-1+\kappa} \\ (2/k)^\kappa & (2/k)^{1+\kappa} & \cdots & (2/k)^{k-1+\kappa} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}^{-1} \begin{pmatrix} F(1/k) \\ F(2/k) \\ \vdots \\ F(1) \end{pmatrix}.$$

By (4.2), we know that $|F(1/k)|, |F(2/k)|, \cdots, |F(1)| \ll R_0$. Thus, we obtain $|v_i| \ll_{k,\kappa} R_0 \bar{l}_1^{1-i-\kappa}$ for all $1 \leq i \leq k$.

(2) This follows by induction. Assume that the statement holds for $j-1$. For j, the only difficult situation is when $\bar{l}_j \leq l_{j+1} - \bar{l}_j$ and $\bar{l}_{j+1} - l_{j+1} \leq l_{j+1} - \bar{l}_j$. If this is the case, then

$$\bar{l}_{j+1} = (\bar{l}_{j+1} - l_{j+1}) + (l_{j+1} - \bar{l}_j) + \bar{l}_j \leq 3\bar{l}_j^{1+\eta}.$$

Thus, by induction hypothesis, we get

$$|v_i| \ll \bar{l}_j^{\xi(\eta,j)(1-i-\kappa)} \ll \bar{l}_{j+1}^{\frac{\xi(\eta,j)}{1+\eta}(1-i-\kappa)}$$

for all $1 \leq i \leq k$. $\square$

4.2. **Effective estimates of shearing phenomena.** Now we begin to study the shearing between two nearby orbits of time-changes of unipotent flows. Let $G = SO(n,1)$. First, since all maximal compact subgroups of $C_G(U)$ are conjugate, we can assume without loss of generality that C^ρ is in the compact group generated by $\mathfrak{k}_C^\perp$. Thus, via (2.3) (2.4) and (2.8), we consider the decomposition

$$\mathfrak{g} = \mathfrak{sl}_2 \oplus V^{\perp\rho} \oplus \mathrm{Lie}(C^\rho), \quad V^{\perp\rho} = \sum_i V_i^{0\perp\rho} \oplus \sum_j V_j^2$$

$$\mathfrak{k}_C^\perp = \mathfrak{k}_C^{\perp\rho} \oplus \mathrm{Lie}(C^\rho)$$

where $\mathrm{Lie}(C^\rho)$ denotes the Lie algebra of C^ρ and note that $\mathrm{Lie}(C^\rho)$ consists of weight 0 spaces. Since C^ρ is compact, there is a G-right invariant metric $d_{C^\rho\backslash G}(\cdot,\cdot)$ on $C^\rho\backslash G$. Let $P : G \to C^\rho\backslash G$ be the natural projection

$$P : g \mapsto C^\rho g =: \overline{g}.$$

Then, for $g_x, g_y \in G$, we have

$$d_{C^\rho\backslash G}(\overline{g_x}, \overline{g_y}) = d_{C^\rho\backslash G}(C^\rho g_x, C^\rho g_y) = d_{C^\rho\backslash G}(C^\rho g_x g_y^{-1}, C^\rho) = d_{C^\rho\backslash G}(\overline{g_x g_y^{-1}}, \overline{e}).$$

Moreover, dP induces an isometry between $\mathfrak{sl}_2 + V^{\perp\rho}$ and $T_{\overline{e}}(C^\rho\backslash G)$. See for example [GQ19] for more details.

Assume $\overline{g} \in B_{C^\rho\backslash G}(e,\epsilon)$ for sufficiently small $0 < \epsilon$. Since C^ρ in fact commutes with $SO_0(2,1)$, we can identify

$$\overline{g} = C^\rho h \exp v \tag{4.4}$$

for some $h \in B_{SO_0(2,1)}(e,\epsilon)$ and $v \in B_{V^{\perp\rho}}(0,\epsilon)$. Besides, for $h = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in B_{SO_0(2,1)}(e,\epsilon)$, we must have $|b|, |c| < \epsilon$, $1-\epsilon < |a|, |d| < 1+\epsilon$.

Next, let $t(s) \in \mathbf{R}^+$ be a function of $s \in \mathbf{R}^+$. Then we want to study the difference $u^t\overline{g}u^{-s}$ of two nearby orbits of time-changes of unipotent flows. By (2.5), we have

$$\begin{aligned} u^t\overline{g}u^{-s} &= C^\rho u^t h \exp v u^{-s} = C^\rho(u^t h u^{-s})(u^s \exp(v) u^{-s}) \\ &= C^\rho(u^t h u^{-s}) \exp(\mathrm{Ad}\, u^s.v) = C^\rho(u^t h u^{-s}) \exp\left(\sum_{n=0}^{\varsigma}\sum_{i=0}^{n} b_i \binom{n}{i} s^{n-i} v_n\right). \end{aligned} \tag{4.5}$$

Then one may conclude that $u^t\overline{g}u^{-s} < \epsilon$ if and only if

$$u^t h u^{-s} \ll \epsilon, \quad \mathrm{Ad}\, u^s.v = \sum_{n=0}^{\varsigma}\sum_{i=0}^{n} b_i \binom{n}{i} s^{n-i} v_n \ll \epsilon \tag{4.6}$$

where $\overline{g} \ll \epsilon$ for $g \in G$ means $d_{C^\rho\backslash G}(\overline{g}, e) \ll \epsilon$. Therefore, later on, we shall split the elements closing to the identity into two parts, say the $SO(2,1)$-part and the $V^{\perp\rho}$-part.

As shown in (4.6), we consider the elements of the form $u^t h u^{-s} \in B_{SO(2,1)}(e,\epsilon)$. One may calculates

$$\begin{aligned} u^t h u^{-s} &= \begin{bmatrix} 1 & \\ t & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & \\ -s & 1 \end{bmatrix} \\ &= \begin{bmatrix} a - bs & b \\ c + (a-d)s - bs^2 + (t-s)(a-bs) & d + bt \end{bmatrix}. \end{aligned} \tag{4.7}$$

If we further impose the Hölder inequality $|s-t| \ll_\kappa \max\{R_0, s^{1-\kappa}\}$ for some $R_0 > \epsilon$ (see Section 2.3 or (4.33)), then we have the crude estimate

$$\begin{aligned} & |-bs^2 + (a-d)s + c + (-bs+a)(t-s)| < \epsilon \\ \Rightarrow\ & |-bs^2 + (a-d)s| - |c| - |(-bs+a)(t-s)| < \epsilon \\ \Rightarrow\ & |-bs^2 + (a-d)s| < 2\epsilon + 2|t-s| \\ \Rightarrow\ & |-bs^2 + (a-d)s| \ll_\kappa \max\{R_0, s^{1-\kappa}\}. \end{aligned}$$

By Lemma 4.4, we immediately obtain

Lemma 4.5 (Estimates for $SO_0(2,1)$-coefficients)**.** *Given $\kappa \approx 0$, $R_0 > 0$, $\epsilon \approx 0$, a matrix $h = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in B_{SO(2,1)}(e, \epsilon)$, then the solutions $s \in [0, \infty)$ of the following inequality*

$$|-bs^2 + (a-d)s| \ll_\kappa \max\{R_0, s^{1-\kappa}\} \tag{4.8}$$

consist of at most two intervals, say $[0, \bar{l}_1(h)] \cup [l_2(h), \bar{l}_2(h)]$, where $\bar{l}_1$ has the lower bound $l(\epsilon, R_0, \kappa)$ such that $l(\epsilon, R_0, \kappa) \nearrow \infty$ as $\epsilon \searrow 0$ for fixed R_0, κ. Moreover, we have

(1) $|b| \ll_\kappa \bar{l}_1^{-1-\kappa}$ and $|a-d| \ll_\kappa \bar{l}_1^{-\kappa}$;

(2) If we further assume that the intervals $[0, \bar{l}_1]$ and $[l_2, \bar{l}_2]$ do not have an effective gap (4.3), i.e. $l_2 - \bar{l}_1 \le \min\{\bar{l}_1, \bar{l}_2 - l_2\}^{1+\eta}$ for some $\eta \approx 0$, then

$$|b| \ll_\kappa \bar{l}_2^{\xi(\eta)(-1-\kappa)}, \quad |a-d| \ll_\kappa \bar{l}_2^{\xi(\eta)(-\kappa)}.$$

Next, we study the situation when $Adu^s.v \ll \epsilon$. Again by Lemma 4.4, we have

Lemma 4.6 (Estimates for $V^{\perp\rho}$-coefficients)**.** *Fix $v = b_0 v_0 + \cdots + b_\varsigma v_\varsigma \in B_{V_\varsigma}(0, \epsilon)$. Assume that*

$$Adu^s.v \ll \epsilon \quad \text{iff} \quad s \in [0, \bar{l}_1(v)] \cup \cdots \cup [l_m(v), \bar{l}_m(v)]$$

where $\bar{l}_1$ has the lower bound $l(\epsilon, R_0, \kappa)$ such that $l(\epsilon, R_0, \kappa) \nearrow \infty$ as $\epsilon \searrow 0$ for fixed R_0, κ. Then $m = m(v)$ is bounded by a constant depending on ς. Moreover, for $1 \le j \le \varsigma - 1$, the intervals $[0, \bar{l}_j]$ and $[l_{j+1}, \bar{l}_{j+1}]$ do not have an effective gap (4.3), i.e. $l_{j+1} - \bar{l}_j \le \min\{\bar{l}_j, \bar{l}_{j+1} - l_{j+1}\}^{1+\eta}$, then we have

$$|b_i| \ll_{\varsigma,\kappa} \bar{l}_j^{\xi(\eta,\varsigma)(-\varsigma+i)}.$$

Next, we shall combine the results of Lemma 4.5 and 4.6. The basic idea is to consider the intersection of the collections of intervals obtained from the above lemmas. For simplicity, we assume that "$V^{\perp\rho}$-part" consists of a single $\mathfrak{sl}_2$-irreducible representation. For the general case, we can repeat the argument for each $\mathfrak{sl}_2$-irreducible representation (cf. Section 2.2). First, for $\bar{g} = C^\rho h \exp(v) \in C^\rho \backslash G$, we write as in

Lemma 4.5 and 4.6

$$u^t h u^{-s} \ll \epsilon \text{ iff } s \in [0, \bar{l}_1(h)] \cup [l_2(h), \bar{l}_2(h)]$$
$$Adu^s.v \ll \epsilon \text{ iff } s \in [0, \bar{l}_1(v)] \cup \cdots \cup [l_{m(v)}(v), \bar{l}_{m(v)}(v)].$$

Write $l_1(h) = l_1(v) = 0$ and we shall consider the family of intervals

$$\{[l_k(g), \bar{l}_k(g)]\}_k \coloneqq \{[l_i(h), \bar{l}_i(h)] \cap [l_j(v), \bar{l}_j(v)]\}_{i,j} \tag{4.9}$$

where $\bar{l}_k(g) < l_{k+1}(g)$ for all k. Thus, in particular, $l_1(g) = 0$ and $[0, \bar{l}_1(g)] = [0, \bar{l}_1(h)] \cap [0, \bar{l}_1(v)]$.

Now assume that there exists k such that $[0, \bar{l}_k(g)]$ and $[l_{k+1}(g), \bar{l}_{k+1}(g)]$ do not have an effective gap (4.3), i.e.

$$l_{k+1}(g) - \bar{l}_k(g) \leq \min\{\bar{l}_k(g), \bar{l}_{k+1}(g) - l_{k+1}(g)\}^{1+\eta}.$$

Then by Remark 4.3, the corresponding "$SO(2,1)$-part" and "$V^{\perp\rho}$-part" should not have effective gaps either. More precisely, for the $SO(2,1)$-part, we define

$$i_{\geq k} \coloneqq \min\{i \in \{1,2\} : \bar{l}_k(g) \leq \bar{l}_i(h)\}, \quad i_{\leq k+1} \coloneqq \max\{i \in \{1,2\} : l_{k+1}(g) \geq l_i(h)\}.$$

Thus, we know

$$[0, \bar{l}_k(g)] \subset [0, \bar{l}_{i_{\geq k}}(h)], \quad [l_{k+1}(g), \bar{l}_{k+1}(g)] \subset [l_{i_{\leq k+1}}(h), \bar{l}_{i_{\leq k+1}}(h)]$$

and hence $[0, \bar{l}_{i_{\geq k}}(h)]$ and $[l_{i_{\leq k+1}}(h), \bar{l}_{i_{\leq k+1}}(h)]$ do not have an effective gap (4.3). Similarly, for the $V^{\perp\rho}$-part, we define

$$j_{\geq k} \coloneqq \min\{j : \bar{l}_k(g) \leq \bar{l}_j(v)\}, \quad j_{\leq k+1} \coloneqq \max\{j : l_{k+1}(g) \geq l_j(v)\}.$$

Then we know

$$[0, \bar{l}_k(g)] \subset [0, \bar{l}_{j_{\geq k}}(v)], \quad [l_{k+1}(g), \bar{l}_{k+1}(g)] \subset [l_{j_{\leq k+1}}(v), \bar{l}_{j_{\leq k+1}}(v)]$$

and hence $[0, \bar{l}_{j_{\geq k}}(v)]$ and $[l_{j_{\leq k+1}}(v), \bar{l}_{j_{\leq k+1}}(v)]$ do not have an effective gap (4.3). Further, one observes

$$\begin{aligned} [0, \bar{l}_k(g)] =& [0, \bar{l}_{i_{\geq k}}(h)] \cap [0, \bar{l}_{j_{\geq k}}(v)] \\ [l_{k+1}(g), \bar{l}_{k+1}(g)] =& [l_{i_{\leq k+1}}(h), \bar{l}_{i_{\leq k+1}}(h)] \cap [l_{j_{\leq k+1}}(v), \bar{l}_{j_{\leq k+1}}(v)]. \end{aligned}$$

Now recall by the definition that the number (4.9) of intervals in $\{[l_k(g), \bar{l}_k(g)]\}_k$ is bounded by a constant $c(\varsigma) > 0$ because the numbers of intervals $\{[l_i(h), \bar{l}_i(h)]\}_i$, $\{[l_j(v), \bar{l}_j(v)]\}_j$ are. Since $\varsigma \leq 2$ when $\mathfrak{g} = \mathfrak{so}(n,1)$, we see that $c(\varsigma)$ is uniformly bounded for all ς. Thus, we conclude that the number of intervals in $\{[l_k(g), \bar{l}_k(g)]\}_k$ is uniformly bounded for all $g \in G$. Then, combining Lemma 4.6 and 4.5, we obtain

Lemma 4.7 (Estimates for $C^\rho\backslash G$-coefficients)**.** *Let $\kappa \approx 0$, $R_0 > 0$, $\epsilon \approx 0$, $\overline{g} = C^\rho h \exp v \in B_{C^\rho\backslash G}(e, \epsilon)$ be as above, where*

$$h = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SO_0(2,1), \quad v = b_0 v_0 + \cdots + b_\varsigma v_\varsigma \in V_\varsigma.$$

Next, let $t(s) \in \mathbf{R}^+$ be a function of $s \in \mathbf{R}^+$ which satisfies the effectiveness

$$|s - t(s)| \ll_\kappa \max\{R_0, s^{1-\kappa}\}.$$

Then there exist intervals $\{[l_k(g), \bar{l}_k(g)]\}_k$ such that

$$u^t \overline{g} u^{-s} < \epsilon, \quad \textit{implies} \quad s \in \bigcup_k [l_k(g), \bar{l}_k(g)] \tag{4.10}$$

where $\bar{l}_1$ has the lower bound $l(\epsilon, R_0, \kappa)$ such that $l(\epsilon, R_0, \kappa) \nearrow \infty$ as $\epsilon \searrow 0$ for fixed R_0, κ. Besides, $k \leq c$ for some constant $c = c(\mathfrak{g}) > 0$, and

(1) $|b| \ll_\kappa \bar{l}_1(g)^{-1-\kappa}$, $|a - d| \ll_\kappa \bar{l}_1(g)^{-\kappa}$, $|b_i| \ll_{\varsigma,\kappa} \bar{l}_1(g)^{-\varsigma+i}$ for all $0 \leq i \leq \varsigma$;

(2) If we further assume that the intervals $[0, \bar{l}_k(g)]$ and $[l_{k+1}(g), \bar{l}_{k+1}(g)]$ do not have an effective gap (4.3). Then there exists $1 \approx \xi = \xi(\eta) \in (0, 1)$ with $\xi \to 1$ as $\eta \to 0$ such that

$$|b| \ll_\kappa \bar{l}_k(g)^{-\xi(1+\kappa)}, \quad |a - d| \ll_\kappa \bar{l}_k(g)^{-\xi\kappa}, \quad |b_i| \ll_{\varsigma,\kappa} \bar{l}_k(g)^{-\xi(\varsigma-i)}$$

for all $1 \leq i \leq \varsigma$.

In practical use, we consider two strictly increasing functions $t(r), s(r) \in \mathbf{R}^+$ of $r \in \mathbf{R}^+$ satisfying the effective estimates

$$|r - t(r)| \ll_\kappa \max\{R_0, r^{1-\kappa}\}, \quad |r - s(r)| \ll_\kappa \max\{R_0, r^{1-\kappa}\}. \tag{4.11}$$

It follows that t is also an increasing function of s and satisfies

$$|t(r) - s(r)| \leq |t(r) - r| + |r - s(r)| \ll_\kappa \max\{R_0, r^{1-\kappa}\} \ll_\kappa \max\{R_0, s(r)^{1-\kappa}\}.$$

Then by Lemma 4.7 and the monotonic nature, we deduce that

Corollary 4.8 (Change of variables)**.** *Let $\kappa \approx 0$, $R_0 > 0$, $\epsilon \approx 0$, $\overline{g} = C^\rho h \exp v \in B_{C^\rho \backslash G}(e, \epsilon)$ be as above, where*

$$h = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SO_0(2,1), \quad v = b_0 v_0 + \cdots + b_\varsigma v_\varsigma \in V_\varsigma.$$

Assume that we have (4.11). Then there exist intervals $\{[l_k(g), \bar{l}_k(g)]\}_k$ such that

$$u^{t(r)} \overline{g} u^{-s(r)} < \epsilon \quad \textit{implies} \quad r \in \bigcup_k [L_k(g), \overline{L}_k(g)] \tag{4.12}$$

where $\overline{L}_1$ has the lower bound $L(\epsilon, R_0, \kappa)$ such that $L(\epsilon, R_0, \kappa) \nearrow \infty$ as $\epsilon \searrow 0$ for fixed R_0, κ. Then we have $k \leq c$ for some constant $c = c(\mathfrak{g}) > 0$, and

(1) $|b| \ll_\kappa \overline{L}_1(g)^{-1-\kappa}$, $|a - d| \ll_\kappa \overline{L}_1(g)^{-\kappa}$, $|b_i| \ll_{\varsigma,\kappa} \overline{L}_1(g)^{-\varsigma+i}$ for all $0 \leq i \leq \varsigma$;

(2) If we further assume that the intervals $[0, \overline{L}_k(g)]$ and $[L_{k+1}(g), \overline{L}_{k+1}(g)]$ do not have an effective gap (4.3). Then there exists $1 \approx \xi = \xi(\eta) \in (0, 1)$ with $\xi \to 1$ as $\eta \to 0$ such that

$$|b| \ll_\kappa \overline{L}_k(g)^{-\xi(1+\kappa)}, \quad |a - d| \ll_\kappa \overline{L}_k(g)^{-\xi\kappa}, \quad |b_i| \ll_{\varsigma,\kappa} \overline{L}_k(g)^{-\xi(\varsigma-i)}$$

for all $1 \leq i \leq \varsigma$.

4.3. **ϵ-blocks and effective gaps.** Let $x \in \overline{X}$, $y \in B_{\overline{X}}(x, \epsilon)$. We say that $(\overline{g_x}, \overline{g_y}) \in C^\rho\backslash G \times C^\rho\backslash G$ *covers* (x, y) if $d_{C^\rho\backslash G}(\overline{g_x}, \overline{g_y}) < \epsilon$ and $\overline{P}(\overline{g_x}) = x$, $\overline{P}(\overline{g_y}) = y$, where $\overline{P} : C^\rho\backslash G \to C^\rho\backslash G/\Gamma$ is the projection. Since $\mathrm{Lie}(C^\rho\backslash G) \cong \mathfrak{sl}_2 + V^{\perp\rho}$, given a representative g_x of $\overline{g_x}$, we may choose $g_y \in G$ such that $P(g_y) = \overline{g_y}$ and

$$\log(g_y g_x^{-1}) \in \mathfrak{sl}_2 + V^{\perp\rho}.$$

We shall always make such a choice if no further explanation.

Definition 4.9 (ϵ-block)**.** Suppose that $x \in \overline{X}$, $y \in B_{\overline{X}}(x, \epsilon)$, $(\overline{g_x}, \overline{g_y})$ covers (x, y), and $R \in (0, \infty]$ satisfies

$$d_{C^\rho\backslash G}(u^{s(R)}\overline{g_x}, u^{t(R)}\overline{g_y}) < \epsilon.$$

Then we define the *ϵ-block of $\overline{g_x}, \overline{g_y}$ of length r* by

$$\mathrm{BL}(g_x, g_y) := \{(u^{s(r)}\overline{g_x}, u^{t(r)}\overline{g_y}) \in C^\rho\backslash G \times C^\rho\backslash G : 0 \leq r \leq R\}.$$

Similarly, we define the *ϵ-block of x, y of length r* by

$$\mathrm{BL}(x, y) := P(\mathrm{BL}(g_x, g_y)) = \{(u^{s(r)}\overline{g_x}, u^{t(r)}\overline{g_y}) \in \overline{X} \times \overline{X} : 0 \leq r \leq R\}.$$

In either case, we call $[0, R]$ the corresponding *time interval* and define the *length* $|\mathrm{BL}|$ of BL by

$$|\mathrm{BL}| := R.$$

We also write

$$\mathrm{BL}(x, y) = \{(x, y), (u^{s(R)}x, u^{t(R)}y)\} = \{(x, y), (\overline{x}, \overline{y})\}$$

emphasizing that (x, y) is the first and $(\overline{x}, \overline{y})$ is the last pair of the block $\mathrm{BL}(x, y)$.

For a pair of ϵ-blocks, a shifting problem may occur.

Definition 4.10 (Shifting)**.** Let $\overline{\mathrm{BL}}' = \{(x', y'), (\overline{x}', \overline{y}')\}$, $\overline{\mathrm{BL}}'' = \{(x'', y''), (\overline{x}'', \overline{y}'')\}$ be two ϵ-blocks. Then $x'' = u^s g_{x'}$, $y'' = u^t y'$ for some $s, t > 0$. Further, there is a unique $\gamma \in \Gamma$ such that

$$d_{C^\rho\backslash G}(\overline{g_{x''}}, \overline{g_{y''}}\gamma) < \epsilon \tag{4.13}$$

where $g_{x''} := u^s g_{x'}$, $g_{y''} := u^t g_{y'}$. We define

- (Shifting) $(x', y') \overset{\Gamma}{\sim} (x'', y'')$ if $\gamma \neq e$ in (4.13),
- (Non-shifting) $(x', y') \overset{e}{\sim} (x'', y'')$ if $\gamma = e$ in (4.13).

The key observation here is that whenever the difference of $\overline{g_x}, \overline{g_y}$ can be estimated by the length in an appropriate way, a shifting must lead to an effective gap between two ϵ-blocks. This follows from the natural renormalization of unipotent flows via diagonal flows.

Proposition 4.11 (Shiftings imply effective gaps)**.** *There are quantities $\eta_0 \approx 0$, $\sigma_0 \approx 0$, $\epsilon_0 \approx 0$, $r_0 > 0$ determined orderly such that for any*

- *$\eta \in (0, \eta_0)$,*
- *$\sigma \in (0, \sigma_0(\eta))$,*

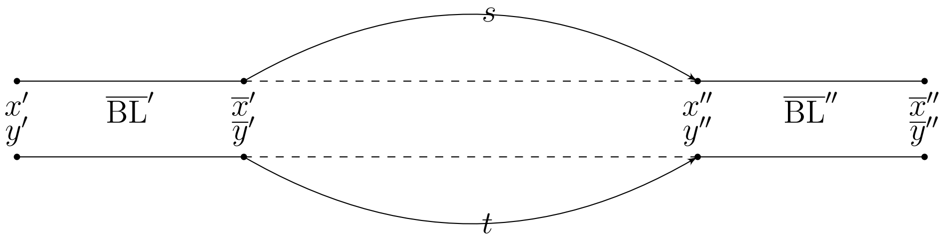

FIGURE 1. The solid straight lines are the unipotent orbits in the $\overline{\mathrm{BL}}'$ and $\overline{\mathrm{BL}}''$ respectively, and the dashed lines are the rest of the unipotent orbits. The bent curves indicate the length defined by the letters

- $\epsilon \in (0, \epsilon_0(\sigma))$,

there exists a compact set $K \subset \overline{X}$ *with* $\overline{\mu}(K) > 1 - \sigma$ *such that the following holds (see Figure 1):*

Assume that there are two ϵ*-blocks* $\overline{\mathrm{BL}}' = \{(x', y'), (\overline{x}', \overline{y}')\}$, $\overline{\mathrm{BL}}'' = \{(x'', y''), (\overline{x}'', \overline{y}'')\}$ *such that the* y*-endpoints lie in* K *(i.e.* $y', \overline{y}', y'', \overline{y}'' \in K$*) and satisfy*

$$g_{y'} = h' \exp(v') g_{x'}, \quad g_{y''} = h'' \exp(v'') g_{x''} \tag{4.14}$$

where $h', h'' \in SO_0(2,1)$, $v', v'' \in V_\varsigma$ *can be estimated by*

$$h', h'' = \begin{bmatrix} 1 + O(r^{-2\eta}) & O(r^{-1-2\eta}) \\ O(\epsilon) & 1 + O(r^{-2\eta}) \end{bmatrix}, \quad v', v'' = O(r^{-\xi\varsigma}) v_0 + \cdots + O(\epsilon) v_\varsigma \tag{4.15}$$

for some $r > r_0(\sigma, \epsilon_0)$, *where* $\xi = \xi(\eta) \approx 1$ *is given by Corollary 4.8. Assume further that* $x'' = u^s \overline{x}'$, $y'' = u^t \overline{y}'$ *and* $t \asymp s$. *If* $\overline{\mathrm{BL}}' \overset{\Gamma}{\sim} \overline{\mathrm{BL}}''$, *then*

$$s, t > r^{1+\eta}. \tag{4.16}$$

Proof. We only consider $\varsigma = 2$. Denote

$$g_{\overline{y}'} = \overline{h}' \exp(\overline{v}') g_{\overline{x}'} \tag{4.17}$$

for $\overline{h}' \in SO_0(2,1)$, $\overline{v}' \in V_2$. By Definition 4.9, we know that $g_{\overline{y}'}, g_{\overline{x}'}$ are obtained by the unipotent action on $g_{y'}, g_{x'}$, and the difference of $g_{\overline{y}'}, g_{\overline{x}'}$ is controlled by ϵ. Combining (4.15), we get that

$$\overline{h}' = \begin{bmatrix} 1 + O(\epsilon) & O(r^{-1-2\eta}) \\ O(\epsilon) & 1 + O(\epsilon) \end{bmatrix}, \quad \overline{v}' = O(r^{-2\xi}) v_0 + O(\epsilon) v_1 + O(\epsilon) v_2. \tag{4.18}$$

Since $\overline{\mathrm{BL}}' \overset{\Gamma}{\sim} \overline{\mathrm{BL}}''$ and $g_{x''} = u^s g_{\overline{x}'}$, we get that

$$g_{y''} = c u^t g_{\overline{y}'} \gamma \quad \text{for some } e \neq \gamma \in \Gamma,\ c \in C^\rho. \tag{4.19}$$

Then by (4.14) (4.17) (4.19), we have

$$
\begin{aligned}
g_{\overline{y}'} &= \overline{h}' \exp(\overline{v}') u^{-s} g_{x''} \\
g_{\overline{y}'}\gamma &= c^{-1} u^{-t} h'' \exp(v'') g_{x''}.
\end{aligned} \tag{4.20}
$$

Assume that one of s, t is not greater than $r^{1+\eta}$. Then since $s \asymp t$, we know

$$
0 < s, t \leq O(r^{1+\eta}). \tag{4.21}
$$

Next, we determine the quantities for the proposition.

- (Choice of η, δ (also η_0)) Choose a small $\eta \approx 0$ that satisfies

$$
1 + 2\delta < 1 + 2\eta < 2\xi(2\eta) \tag{4.22}
$$

  where $\xi(2\eta)$ was defined in Corollary 4.8, and $\delta := 3\eta/4$. Here $\eta_0 \approx 0$ can be defined to be the maximal η so that (4.22) holds.
- (Choice of σ) Then $\sigma = \sigma(\eta) > 0$ can be chosen as

$$
\sigma < \frac{3\eta}{4 + 6\eta}. \tag{4.23}
$$

- (Choice of ϵ_0, K_1; injectivity radius) Since Γ is discrete, there is a compact subset $K_1 \subset \overline{X}$, $\overline{\mu}(K_1) > 1 - \frac{1}{4}\sigma$ and $\epsilon_0 > 0$ such that for any $\overline{g_y} \in \overline{P}^{-1}(K_1)$ satisfying

$$
d_{C^\rho \backslash G}(\overline{g_y}, \overline{g_y}\gamma) < O(\epsilon_0) \tag{4.24}
$$

  for some $\gamma \in \Gamma$, then $\gamma = e$. Here the constants hidden in $O(\epsilon_0)$ will be determined after the estimate (4.28) (see also (4.29)).
- (Choice of K_2, K, T_0, r_0; ergodicity of a^T) Since the diagonal action a^T is ergodic on $(\overline{X}, \overline{\mu})$, there is a compact subset $K_2 \subset \overline{X}$, $\overline{\mu}(K_2) > 1 - \frac{1}{4}\sigma$ and $T_0 = T_0(K_2) > 0$ such that the relative length measure K_2 on $[y, a^T y]$ (and $[a^{-T} y, y]$) is greater than $1 - \sigma$ for any $y \in K_2$, $|T| \geq T_0$. Assume that

$$
K := K_1 \cap K_2, \quad r_0 > e^{(1+2\delta)^{-1} T_0}. \tag{4.25}
$$

  Note that $\overline{\mu}(K) > 1 - \sigma$. The quantity r_0 will be even larger and determined by ϵ_0 if necessary (see (4.29)).

Now we are in the position to apply the renomalization via the diagonal action a^w. Since $r > r_0 = e^{(1+2\delta)^{-1} T_0}$, let $e^{\omega_0} := r^{1+2\delta}$ and we know $\omega_0 > T_0$. Since $\overline{y}' \in K \subset K_2$, it follows from the choice of K_2 and T_0 that the relative length measure of K on $[\overline{y}', a^{\omega_0}\overline{y}']$ is greater than $1 - \sigma$. This implies that there is ω satisfying

$$
(1 - \sigma)\omega_0 < \omega \leq \omega_0
$$

such that $a^\omega \overline{y}' \in K$ and therefore

$$
a^\omega \overline{g_{\overline{y}'}} \in \overline{P}^{-1}(K). \tag{4.26}
$$

By (4.20), we have

$$
\begin{aligned}
a^\omega g_{\overline{y}'} &=(a^\omega \overline{h}' a^{-\omega}) \exp(\operatorname{Ad} a^\omega .\overline{v}')(a^\omega u^{-s} a^{-\omega}) a^\omega g_{x''} \\
a^\omega g_{\overline{y}'}\gamma &=c^{-1}(a^\omega u^{-t} a^{-\omega})(a^\omega h'' a^{-\omega}) \exp(\operatorname{Ad} a^\omega .v'') a^\omega g_{x''}
\end{aligned} \tag{4.27}
$$

Then by (4.18) (4.15) (4.21), we estimate

$$
\begin{aligned}
a^w \overline{h}' a^{-w} &= \begin{bmatrix} 1+O(\epsilon) & O(r^{2\delta-2\eta}) \\ O(\epsilon) & 1+O(\epsilon) \end{bmatrix} \\
a^w \overline{h}' a^{-w} &= \begin{bmatrix} 1+O(r^{-2\eta}) & O(r^{2\delta-2\eta}) \\ O(\epsilon) & 1+O(r^{-2\eta}) \end{bmatrix} \\
\operatorname{Ad} a^\omega .\overline{v}' &=O(r^{-2\xi+1+2\delta}) v_0 + O(\epsilon) v_1 + O(\epsilon) v_2 \\
\operatorname{Ad} a^\omega .v'' &=O(r^{-2\xi+1+2\delta}) v_0 + O(\epsilon) v_1 + O(r^{-(1-\sigma)(1+2\delta)}) v_2 \\
a^\omega u^{-t} a^{-\omega} &=u^{-te^{-\omega}} = u^{O(r^{1+\eta} r^{-(1-\sigma)(1+2\delta)})} \\
a^\omega u^{-s} a^{-\omega} &=u^{-se^{-\omega}} = u^{O(r^{1+\eta} r^{-(1-\sigma)(1+2\delta)})}.
\end{aligned} \tag{4.28}
$$

Notice that by the choice of σ, δ (see (4.22) (4.23)), we have

$$
1+\eta-(1-\sigma)(1+2\delta) = 1+\eta-(1-\sigma)(1+\frac{3}{2}\eta) < -\frac{1}{4}\eta.
$$

Also, by (4.22), we have

$$
2\delta - 2\eta < 0, \quad -2\xi+1+2\delta < 0.
$$

Thus, by enlarging r_0 if necessary, all terms of (4.28) can be quantitatively dominated by $O(\epsilon_0)$. Then by (4.27), we have

$$
d_{C^\rho\backslash G}(\overline{a^\omega g_{\overline{y}'}}\gamma, \overline{a^\omega g_{\overline{y}'}}) = d_{C^\rho\backslash G}(\overline{a^\omega g_{\overline{y}'}}\gamma (a^\omega g_{x''})^{-1}, \overline{a^\omega g_{\overline{y}'}}(a^\omega g_{x''})^{-1}) < O(\epsilon_0). \tag{4.29}
$$

Thus, by (4.24), we get $\gamma = e$, which contradicts our assumptions. □

4.4. **Construction of ϵ-blocks.** In light of Proposition 4.11, we try to construct a collection of ϵ-blocks based on the unipotent flows between two nearby points so that each pair of ϵ-blocks has an effective gap.

First, given $\eta_0 \approx 0$ as in Proposition 4.11, we fix a sufficiently small $\kappa \in (0, 2\eta_0)$, and then choose $\eta = \eta(\kappa) \approx 0$ such that

$$
\frac{1+2\eta}{\xi(2\eta)} < 1+\kappa < 1+2\eta_0 \tag{4.30}
$$

where $\xi(2\eta) \approx 1$ is given by Corollary 4.8. Then $\sigma_0 = \sigma_0(\eta) \approx 0$ given in Proposition 4.11 has been determined. Next, assume that there exist

- $\sigma \in (0, \sigma_0)$,
- $R_0 > 1$,

- $\epsilon_0 = \epsilon_0(\sigma) \approx 0$, $\epsilon = \epsilon(R_0) \in (0, \epsilon_0)$ so small that

$$\overline{L}_1(g) \geq L(\epsilon, R_0, \kappa) > \max\{r_0(\sigma, \epsilon_0), R_0\} \tag{4.31}$$

whenever $g \in B_G(e, \epsilon)$, where $\overline{L}_1, L$ are defined by Corollary 4.8,

such that for $K \subset \overline{X}$ with $\overline{\mu}(K) > 1 - \sigma$ given by Proposition 4.11, $x, y \in \overline{X}$, we have $A = A(x, y) \subset \mathbf{R}^+$ such that

(i) if $r \in A$, then

$$u^{t(r)}y \in K \quad \text{and} \quad d_{\overline{X}}(u^{s(r)}x, u^{t(r)}y) < \epsilon \tag{4.32}$$

for continuous increasing functions $t, s : [0, \infty) \to [0, \infty)$;

(ii) we have the Hölder inequalities:

$$\begin{aligned} |(t(r') - t(r)) - (r' - r)| &\ll |r' - r|^{1-\kappa} \\ |(s(r') - s(r)) - (r' - r)| &\ll |r' - r|^{1-\kappa} \end{aligned} \tag{4.33}$$

for all $r, r' \in A$ with $r' > r$, $r' - r \geq R_0$.

It is worth noting from (4.24) that points in K have injectivity radius at least ϵ_0. For simplicity, we shall assume that $0 \in A$ in what follows.

Remark 4.12. For the condition (i) (ii), the quantities s, t are symmetric. Thus, for instance, one can also consider s as an increasing function of t, and obtain similar Hölder inequalities. We have already made such a change of variables in Section 4.2, for notational simplicity.

On the other hand, the assumptions (4.32) (4.33) coincide with (4.11) (4.12). So Corollary 4.8 can apply.

Construction of β_1. For $\lambda \in A$ denote $A_\lambda := A \cap [0, \lambda]$. Now we construct a collection $\beta_1(A_\lambda)$ of ϵ-blocks. Let $x_1 := x$, $y_1 := y$. We follow the assumptions (4.32) (4.33). Suppose that $(\overline{g_{x_1}}, \overline{g_{y_1}}) \in C^\rho\backslash G \times C^\rho\backslash G$ covers (x_1, y_1) and

$$\overline{r}_1 := \sup\{r \in A_\lambda \cap [0, \overline{L}_1(g_{y_1} g_{x_1}^{-1})] : d_G(u^{t(r)}g_{y_1}, u^{s(r)}g_{x_1}) < \epsilon\}, \quad \overline{s}_1 := s(\overline{r}_1)$$

where $\overline{L}_1$ is defined by Corollary 4.8. Let BL_1 be the ϵ-block of x_1, y_1 of length $\overline{r}_1$, $\mathrm{BL}_1 = \{(x_1, y_1), (\overline{x}_1, \overline{y}_1)\}$. To define BL_2, we take

$$r_2 := \inf\{r \in A_\lambda : r > \overline{r}_1\}, \quad s_2 := s(r_2)$$

and apply the above procedure to

$$x_2 := u^{s(r_2)}x_1, \quad y_2 := u^{t(r_2)}y_1$$

(Note that by (4.12), $r_2 > \overline{r}_1$). This process defines a collection $\beta_1(A_\lambda) = \{\mathrm{BL}_1, \dots, \mathrm{BL}_n\}$ of ϵ-blocks on the orbit intervals $[x_1, u^{s(\lambda)}x_1]$, $[y_1, u^{t(\lambda)}y_1]$ (see Figure 2):

$$x_i = u^{s_i}x_1, \quad \overline{x}_i = u^{\overline{s}_i}x_1, \quad y_i = u^{t_i}y_1, \quad \overline{y}_i = u^{\overline{t}_i}y_1$$

$$s_i = s(r_i), \quad \overline{s}_i = s(\overline{r}_i), \quad t_i = t(r_i), \quad \overline{t}_i = t(\overline{r}_i).$$

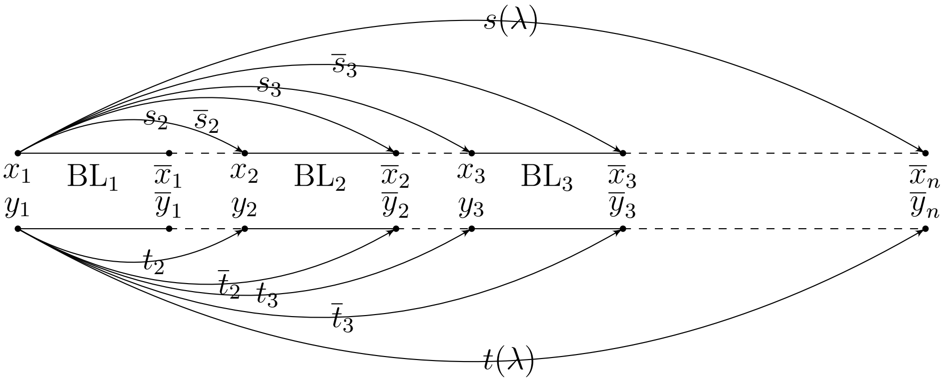

FIGURE 2. A collection of ϵ-blocks $\{\mathrm{BL}_1, \dots, \mathrm{BL}_n\}$. The solid straight lines are the unipotent orbits in the ϵ-blocks and the dashed lines are the rest of the unipotent orbits. The bent curves indicate the length defined by the letters

Note also that by the assumption of A, we have $x_i, \overline{x}_i \in K$ for all i, the corresponding time interval of BL_i is $[r_i, \overline{r}_i]$ and the length $|\,\mathrm{BL}_i\,|$ of BL_i is

$$|\,\mathrm{BL}_i\,| := \overline{r}_i - r_i.$$

Note that any $\mathrm{BL}_i = \{(x_i, y_i), (\overline{x}_i, \overline{y}_i)\} \in \beta_1(A_\lambda)$ has length $|\,\mathrm{BL}_i\,| \le \overline{L}_1(g_{y_i} g_{x_i}^{-1})$. By Corollary 4.8, we immediately obtain an estimate for the difference of g_{x_i} and g_{y_i} in terms of the length of ϵ-blocks.

Corollary 4.13 (Difference of $\beta_1(A_\lambda)$)**.** *Assume that* $\overline{g_{y_i} g_{x_i}^{-1}} = C^\rho h_i \exp(v_i)$, *where*

$$h_i = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SO_0(2,1), \quad v_i = b_0 v_0 + \dots + b_\varsigma v_\varsigma \in V_\varsigma.$$

Then we have

$$h_i = \begin{bmatrix} 1 + O(\mathbf{r}_i^{-\kappa}) & O(\mathbf{r}_i^{-1-\kappa}) \\ O(\epsilon) & 1 + O(\mathbf{r}_i^{-\kappa}) \end{bmatrix}, \quad v_i = O(\mathbf{r}_i^{-\varsigma}) v_0 + \dots + O(\epsilon) v_\varsigma$$

for some $\mathbf{r}_i \ge \max\{r_0, R_0, |\,\mathrm{BL}_i\,|\}$.

We then immediately conclude from Proposition 4.11 that for any $\mathrm{BL}', \mathrm{BL}'' \in \beta_1(A_\lambda)$ with $\mathrm{BL}' \overset{\Gamma}{\sim} \mathrm{BL}''$, there is an effective gap between them, i.e.

$$d(\mathrm{BL}', \mathrm{BL}'') \ge [\min\{|\,\mathrm{BL}'\,|, |\,\mathrm{BL}''\,|\}]^{1+\kappa/2}.$$

However, when $\mathrm{BL}' \overset{e}{\sim} \mathrm{BL}''$, they do not necessarily have an effective gap. This enlighten us to connect these ϵ-blocks and generate a new collection $\beta_2(A_\lambda)$.
Construction of β_2. Now we construct a new collection $\beta_2(A_\lambda) = \{\overline{\mathrm{BL}}_1, \dots, \overline{\mathrm{BL}}_N\}$ by the following procedure. The idea is to connect ϵ-blocks in $\beta_1(A_\lambda) = \{\mathrm{BL}_1, \dots, \mathrm{BL}_n\}$

so that each pair of new blocks must have an effective gap. Let $\mathrm{BL}_1 \in \beta_1(A_\lambda)$, $g_{y_1} = h\exp(v)g_{x_1}$ and

$$h = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SO(2,1), \quad v = b_0 v_0 + \cdots + b_\varsigma v_\varsigma \in V_\varsigma.$$

Then by Corollary 4.8, one can write $u^{t(r)} g u^{-s(r)} \in B_G(e,\epsilon)$ for

$$r \in \bigcup_k [L_k(g), \overline{L}_k(g)] \tag{4.34}$$

where $k \le c$ is uniformly bounded for all $g \in G$. Then consider the following two cases:

(i) There is no $j \in \{2,\dots,n\}$ such that $(x_1,y_1) \overset{e}{\sim} (x_j,y_j)$.
(ii) There is $j \in \{2,\dots,n\}$ such that $(x_1,y_1) \overset{e}{\sim} (x_j,y_j)$.

In case (i), we set $\overline{\mathrm{BL}_1} = \mathrm{BL}_1$. Then by Corollary 4.13, we have

$$|b| \ll \overline{L}_1(g_{y_1}g_{x_1}^{-1})^{-1-\kappa}, \quad |a-d| \le \overline{L}_1(g_{y_1}g_{x_1}^{-1})^{-\kappa} \tag{4.35}$$

In case (ii), suppose that $g_{x_j} = u^{s_j} g_{x_1}$, $g_{y_j} = u^{t_j} g_{y_1}$. Clearly, by the construction, $\overline{r}_j > \overline{L}_1(g_{y_1}g_{x_1}^{-1})$. On the other hand, by (4.34), we get

$$\overline{r}_j \in \bigcup_k [L_k(g_{y_1}g_{x_1}^{-1}), \overline{L}_k(g_{y_1}g_{x_1}^{-1})]$$

and $k \le C$ is uniformly bounded for all $g \in G$. Assume that $j_{\max}$ is the maximal j among $\overline{r}_j \in [L_2(g_{y_1}g_{x_1}^{-1}), \overline{L}_2(g_{y_1}g_{x_1}^{-1})]$. Whether $[0, \overline{L}_1(g_{y_1}g_{x_1}^{-1})]$ and $[L_2(g_{y_1}g_{x_1}^{-1}), \overline{L}_2(g_{y_1}g_{x_1}^{-1})]$ have an effective gap leads to a dichotomy of choices:

$$\overline{\mathrm{BL}_1} = \begin{cases} \text{remains unchange} & , \text{ if } L_2(g_{y_1}g_{x_1}^{-1}) - \overline{L}_1(g_{y_1}g_{x_1}^{-1}) > \overline{L}_1(g_{y_1}g_{x_1}^{-1})^{1+2\eta} \\ \{(x_1,y_1), (\overline{x}_{j_{\max}}, \overline{y}_{j_{\max}})\} & , \text{ otherwise} \end{cases}.$$

If the first case occurs, we will not change $\overline{\mathrm{BL}_1}$ anymore. If the second case occurs, i.e. we redefine $\overline{\mathrm{BL}_1} = \{(x_1,y_1), (\overline{x}_{j_{\max}}, \overline{y}_{j_{\max}})\}$, then we repeat the construction for the new $\overline{\mathrm{BL}_1}$ again:

Suppose that there is $\overline{r}_j > \overline{L}_2(g_{y_1}g_{x_1}^{-1})$. Then assume $j_{\max}$ to be the maximal j among $\overline{r}_j \in [L_3(g_{y_1}g_{x_1}^{-1}), \overline{L}_3(g_{y_1}g_{x_1}^{-1})]$. Then again, we set

$$\overline{\mathrm{BL}_1} = \begin{cases} \text{remains unchange} & , \text{ if } L_3(g_y g_x^{-1}) - \overline{L}_3(g_{y_1}g_{x_1}^{-1}) > \overline{L}_2(g_{y_1}g_{x_1}^{-1})^{1+2\eta} \\ \{(x_1,y_1), (\overline{x}_{j_{\max}}, \overline{y}_{j_{\max}})\} & , \text{ otherwise} \end{cases}$$

and so on.

The process will stop since the number of intervals is uniformly bounded for all $g \in G$. Now $\overline{BL_1} \in \beta_2(A_\lambda)$ has been constructed. By the choice of $\overline{\mathrm{BL}_1}$ and Corollary 4.8, we conclude that

$$|b| \ll_\kappa |\,\mathrm{BL}_1\,|^{-\xi(1+\kappa)}, \quad |a-d| \ll_\kappa |\,\mathrm{BL}_1\,|^{-\xi\kappa}, \quad |b_i| \ll_{\varsigma,\kappa} |\,\mathrm{BL}_1\,|^{-\xi(\varsigma-i)} \tag{4.36}$$

for $\xi = \xi(2\eta) \approx 1$ and for all $1 \le i \le \varsigma$.

Next, we repeat the above argument to construct $\overline{\mathrm{BL}}_{m+1}$. More precisely, suppose that $\overline{\mathrm{BL}}_m = \{(x_{j_{m-1}+1}, y_{j_{m-1}+1}), (\overline{x}_{j_m}, \overline{y}_{j_m})\} \in \beta_2(A_\lambda)$ has been constructed. To define $\overline{\mathrm{BL}}_{m+1}$, we repeat the above argument to $\mathrm{BL}_{j_m+1} \in \beta_1(A_\lambda)$. Thus, $\beta_2(A_\lambda)$ is completely defined. Further, one may conclude the difference of points of ϵ-blocks in $\beta_2(A_\lambda)$:

Lemma 4.14 (Difference of $\beta_2(A_\lambda)$)**.** *For any $\overline{\mathrm{BL}}_i = \{(x_i', y_i'), (\overline{x}_i', \overline{y}_i')\}$ in the collection $\beta_2(A_\lambda) = \{\overline{\mathrm{BL}}_1, \dots, \overline{\mathrm{BL}}_N\}$ of ϵ-blocks, we have*

$$\overline{g_{y_i'} g_{x_i'}^{-1}} = C^\rho h_i \exp(v_i)$$

where

$$h_i = \begin{bmatrix} 1 + O(\mathbf{r}_i^{-2\eta}) & O(\mathbf{r}_i^{-1-2\eta}) \\ O(\epsilon) & 1 + O(\mathbf{r}_i^{-2\eta}) \end{bmatrix}, \quad v_i = O(\mathbf{r}_i^{-\xi_\varsigma}) v_0 + \cdots + O(\epsilon) v_\varsigma \tag{4.37}$$

for some $\mathbf{r}_i \geq \max\{r_0, R_0, |\overline{\mathrm{BL}}_i|\}$.

Proof. (4.37) follows immediately from (4.35), (4.36), (4.30), (4.31). □

Then, recall that by the construction of $\beta_2(A_\lambda)$, for any $\overline{\mathrm{BL}}', \overline{\mathrm{BL}}'' \in \beta_2(A_\lambda)$ with $\overline{\mathrm{BL}}' \overset{e}{\sim} \overline{\mathrm{BL}}''$, there is an effective gap between them, i.e.

$$d(\overline{\mathrm{BL}}', \overline{\mathrm{BL}}'') \geq \left[\max\{r_0, R_0, \min\{|\overline{\mathrm{BL}}'|, |\overline{\mathrm{BL}}''|\}\}\right]^{1+2\eta}.$$

On the other hand, when $\overline{\mathrm{BL}}' \overset{\Gamma}{\sim} \overline{\mathrm{BL}}''$, by Proposition 4.11 and Lemma 4.14, we have

$$d(\overline{\mathrm{BL}}', \overline{\mathrm{BL}}'') \geq \left[\max\{r_0, R_0, \min\{|\overline{\mathrm{BL}}'|, |\overline{\mathrm{BL}}''|\}\}\right]^{1+\eta}.$$

Thus, we conclude from Proposition 4.1 that

Proposition 4.15 (Effective gaps of $\beta_2(A_\lambda)$)**.** *Let the notation and assumptions be as above. For any $\overline{\mathrm{BL}}', \overline{\mathrm{BL}}'' \in \beta_2(A_\lambda)$, we have*

$$d(\overline{\mathrm{BL}}', \overline{\mathrm{BL}}'') \geq \left[\max\{r_0, R_0, \min\{|\overline{\mathrm{BL}}'|, |\overline{\mathrm{BL}}''|\}\}\right]^{1+\eta}.$$

Thus, for any $\zeta \in [0, 1]$, if

$$\frac{1}{\lambda} \mathrm{Leb}(A_\lambda) \geq \overline{\theta}_\eta(\zeta) = 1 - \theta(\eta, \zeta) = 1 - \prod_{n=0}^{\infty} (1 + C\zeta^{n\eta})^{-1}$$

then there is an ϵ-block $\overline{\mathrm{BL}} \in \beta_2(A_\lambda)$ that has

$$|\overline{\mathrm{BL}}| \geq \zeta\lambda.$$

4.5. **Non-shifting time.** Now assume that for some $\lambda, \zeta > 0$, we know that

$$\mathrm{Leb}(A_\lambda) \geq \overline{\theta}_\eta(\zeta)\lambda.$$

Then Proposition 4.15 provides us an ϵ-block $\overline{\mathrm{BL}} = \{(x', y'), (\overline{x}', \overline{y}')\} \in \beta_2(A_\lambda)$ with $|\overline{\mathrm{BL}}| \geq \zeta\lambda$. In other words, if we write

$$x' = u^{s(R_1)}x, \quad \overline{x}' = u^{s(R_2)}x, \quad y' = u^{t(R_1)}y, \quad \overline{y}' = u^{t(R_2)}y, \tag{4.38}$$

then we can find $R_1, R_2 > 0$ with $R_2 - R_1 \geq \zeta\lambda$ such that

$$d_{C^\rho\backslash G}(u^{t(R_1)}.\overline{g_y}, u^{s(R_1)}.\overline{g_x}) < \epsilon, \quad d_{C^\rho\backslash G}(u^{t(R_2)}.\overline{g_y}, u^{s(R_2)}.\overline{g_x}) < \epsilon.$$

It is already quite surprising. However, it is still possible that

$$d_{C^\rho\backslash G}(u^{t(r)}.\overline{g_y}, u^{s(r)}.\overline{g_x}) > \epsilon$$

for some $r \in [R_1, R_2] \cap A$. Thus, define

$$\overline{A}_{R_1R_2} \coloneqq \{r \in [R_1, R_2] \cap A : d_{C^\rho\backslash G}(u^{t(r)}.\overline{g_y}, u^{s(r)}.\overline{g_x}) > \epsilon\}$$

and we want to show that $\mathrm{Leb}(\overline{A}_{R_1R_2})/\lambda$ has a upper bound in certain situations.

Remark 4.16. By (4.37), we can estimate the difference between x', y'; more precisely, we have

$$\overline{g_{y'}g_{x'}^{-1}} = C^\rho h \exp(v)$$

where

$$h = \begin{bmatrix} 1 + O((\zeta\lambda)^{-2\eta}) & O((\zeta\lambda)^{-1-2\eta}) \\ O(\epsilon) & 1 + O((\zeta\lambda)^{-2\eta}) \end{bmatrix}, \quad v = O((\zeta\lambda)^{-\xi_\varsigma})v_0 + \cdots + O(\epsilon)v_\varsigma.$$

Construction of $\widetilde{\beta}_1, \widetilde{\beta}_2$. Now we consider the shifting time of the ϵ-block $\overline{\mathrm{BL}} = \{(x', y'), (\overline{x}', \overline{y}')\} \in \beta_2(A_\lambda)$. Define a collection $\widetilde{\beta}_1(\overline{A}_{R_1R_2})$ of ϵ-blocks on the orbit intervals $[x', x'']$, $[y', y'']$ according to the following steps. Suppose that

$$r_1 \coloneqq \min\{r \in [R_1, R_2] : r \in \overline{A}_{R_1R_2}\}, \quad x_1 \coloneqq u^{s(R_1)}x', \quad y_1 \coloneqq u^{t(R_1)}y'$$

and that $(\overline{g_{x_1}}, \overline{g_{y_1}}) \in C^\rho\backslash G \times C^\rho\backslash G$ covers (x_1, y_1) and

$$\overline{r}_1 \coloneqq \sup\{R \in \overline{A}_{R_1R_2} : d_G(u^{t(r)}g_{y_1}, u^{s(r)}g_{x_1}) < \epsilon \text{ for any } r \in \overline{A}_{R_1R_2} \cap [0, R]\}.$$

Let $\mathrm{BL}_1 \in \widetilde{\beta}_1(\overline{A}_{R_1R_2})$ be the ϵ-block of x_1, y_1 of length $\overline{r}_1$, and write $\mathrm{BL}_1 = \{(x_1, y_1), (\overline{x}_1, \overline{y}_1)\}$. To define BL_2, we take

$$r_2 \coloneqq \inf\{r \in \overline{A}_{R_1R_2} : r > \overline{r}_1\}$$

and apply the above procedure to

$$x_2 \coloneqq u^{s(r_2)}x_1, \quad y_2 \coloneqq u^{t(r_2)}y_1.$$

This process defines a collection $\widetilde{\beta}_1(\overline{A}_{R_1R_2}) = \{\mathrm{BL}_1, \ldots, \mathrm{BL}_m\}$ of ϵ-blocks on the orbit intervals $[u^{s(r_1)}x', u^{s(\overline{r}_m)}x']$, $[u^{t(r_1)}y', u^{t(\overline{r}_m)}y']$. Completely similar to β_1, we can connect some of the ϵ-blocks in $\widetilde{\beta}_1(\overline{A}_{R_1R_2})$ and form a new collection $\widetilde{\beta}_2(\overline{A}_{R_1R_2})$ such that each pair of ϵ-blocks in $\widetilde{\beta}_2(\overline{A}_{R_1R_2})$ has an effective gap. Then, we conclude again from Proposition 4.1 that

Lemma 4.17 (Difference and effective gaps of $\widetilde{\beta}_2(\overline{A}_{R_1R_2})$)**.** *For any $\widetilde{\mathrm{BL}}_i = \{(\widetilde{x}_i', \widetilde{y}_i'), (\overline{\widetilde{x}}_i', \overline{\widetilde{y}}_i')\}$ in the collection $\widetilde{\beta}_2(\overline{A}_{R_1R_2}) = \{\widetilde{\mathrm{BL}}_1, \ldots, \widetilde{\mathrm{BL}}_M\}$ of ϵ-blocks, we have*

$$\overline{g_{\widetilde{y}_i'} g_{\widetilde{x}_i'}^{-1}} = C^\rho h_i \exp(v_i)$$

where

$$h_i = \begin{bmatrix} 1 + O(\mathbf{r}_i^{-2\eta}) & O(\mathbf{r}_i^{-1-2\eta}) \\ O(\epsilon) & 1 + O(\mathbf{r}_i^{-2\eta}) \end{bmatrix}, \quad v_i = O(\mathbf{r}_i^{-\xi_\varsigma}) v_0 + \cdots + O(\epsilon) v_\varsigma \tag{4.39}$$

for some $\mathbf{r}_i \geq \max\{r_0, R_0, |\widetilde{\mathrm{BL}}_i|\}$.

Moreover, for any $\widetilde{\mathrm{BL}}', \widetilde{\mathrm{BL}}'' \in \widetilde{\beta}_2(\overline{A}_{R_1R_2})$, we have

$$d(\widetilde{\mathrm{BL}}', \widetilde{\mathrm{BL}}'') \geq \left[\max\{r_0, R_0, \min\{|\widetilde{\mathrm{BL}}'|, |\widetilde{\mathrm{BL}}''|\}\}\right]^{1+\eta}.$$

Thus, for any $\widetilde{\zeta} \in [0, 1]$, if

$$\frac{1}{\lambda} \mathrm{Leb}(\overline{A}_{R_1R_2}) \geq \overline{\theta}_\eta(\widetilde{\zeta}) = 1 - \prod_{n=0}^{\infty} \left(1 + C\widetilde{\zeta}^{n\eta}\right)^{-1}$$

then there is an ϵ-block $\widetilde{\mathrm{BL}} \in \widetilde{\beta}_2(\overline{A}_{R_1R_2})$ that has

$$|\widetilde{\mathrm{BL}}| \geq \widetilde{\zeta}\lambda.$$

Thus, given $\widetilde{\zeta} \in (0, \zeta)$, we can apply Lemma 4.17 and obtain an ϵ-block $\widetilde{\mathrm{BL}} = \{(\widetilde{x}, \widetilde{y}), (\overline{\widetilde{x}}, \overline{\widetilde{y}})\} \in \widetilde{\beta}_2(\overline{A}_{R_1R_2})$ that has length $|\widetilde{\mathrm{BL}}| \geq \widetilde{\zeta}\lambda$. Then by (4.39), we get that

$$\overline{g_{\widetilde{y}} g_{\widetilde{x}}^{-1}} = C^\rho \widetilde{h} \exp(\widetilde{v})$$

where

$$\widetilde{h} = \begin{bmatrix} 1 + O((\widetilde{\zeta}\lambda)^{-2\eta}) & O((\widetilde{\zeta}\lambda)^{-1-2\eta}) \\ O(\epsilon) & 1 + O((\widetilde{\zeta}\lambda)^{-2\eta}) \end{bmatrix}, \quad \widetilde{v} = O((\widetilde{\zeta}\lambda)^{-\xi_\varsigma}) v_0 + \cdots + O(\epsilon) v_\varsigma.$$

Then combining Remark 4.16 and Proposition 4.11, we conclude that

$$r_1 > (\widetilde{\zeta}\lambda)^{1+\eta}.$$

Since $r_1 \in [R_1, R_2]$, we obtain $(\widetilde{\zeta}\lambda)^{1+\eta} \leq \zeta\lambda$ or

$$\widetilde{\zeta} \leq (\zeta\lambda^{-\eta})^{\frac{1}{1+\eta}}.$$

In other words, we obtain

Lemma 4.18 (Shifting is sparse in a big ϵ-block)**.** *Given $\lambda > 0, \zeta \in (0, 1), \eta \approx 0$, assume that*

$$\mathrm{Leb}(A_\lambda) \geq \overline{\theta}_\eta(\zeta)\lambda.$$

Then there is an ϵ-block $\overline{\mathrm{BL}} \in \beta_2(A_\lambda)$ with the corresponding time interval $[R_1, R_2]$ and $|\overline{\mathrm{BL}}| = R_2 - R_1 \geq \zeta\lambda$. Besides, denote the shifting time of $\overline{\mathrm{BL}}$ by

$$\overline{A}_{R_1R_2} \coloneqq \{r \in A \cap [R_1, R_2] : d_{C^\rho \backslash G}(u^{t(r)}.\overline{g_y}, u^{s(r)}.\overline{g_x}) > \epsilon\}.$$

Then we have

$$\mathrm{Leb}(\overline{A}_{R_1R_2})/\lambda \leq \overline{\theta}_\eta\left((\zeta\lambda^{-\eta})^{\frac{1}{1+\eta}}\right) = 1 - \prod_{n=0}^{\infty}\left(1 + C(\zeta\lambda^{-\eta})^{\frac{n\eta}{1+\eta}}\right)^{-1}.$$

In particular, $\mathrm{Leb}(\overline{A}_{R_1R_2})/\lambda = o(\lambda)$.

In the following, we present a key proposition below that will be used in the proof of Proposition 5.1. It basically says that non-shifting is always observable when the time scale is large.

Proposition 4.19 (Non-shifting time is not negligible)**.** *Given an integer* $n \geq 2$, $\kappa \in (0, 2\eta_0)$, *there exist* $\lambda_0 > 0$, $\sigma_0 \approx 0$, $\vartheta \approx 0$ *such that for any*

- *disjoint subsets* $A^1, \dots, A^n \subset [0, \infty)$ *that satisfy (4.32) (4.33),*
- $\lambda > \lambda_0$,
- $\sigma \in (0, \sigma_0)$ *satisfying*

$$\mathrm{Leb}\left(\coprod_{i=1}^{n} A^i \cap [0, \lambda]\right) > (1 - 2\sigma)\lambda,$$

there exists one $A^{i(\lambda)}$ *and* $[R_1'(\lambda), R_2'(\lambda)] \subset [0, \lambda]$ *such that there exists an* ϵ*-block* $\overline{\mathrm{BL}} \in \beta_2(A^{i(\lambda)} \cap [R_1', R_2'])$ *with the corresponding time interval* $[R_1, R_2]$ *such that*

$$R_2 - R_1 > \vartheta\lambda, \quad \mathrm{Leb}\left(A_\epsilon^{i(\lambda)} \cap [R_1, R_2]\right) > \vartheta\lambda$$

where $A_\epsilon^{i(\lambda)} := \{r \in A^{i(\lambda)} : d_{C^\rho\backslash G}(u^{t(r)}.\overline{g_y}, u^{s(r)}.\overline{g_x}) < \epsilon\}$ *is the non-shifting time of* $A^{i(\lambda)}$.

Proof. First, fix η satisfying (4.30), $\zeta_1 \in (0, 1)$ so that $\overline{\theta}_\eta(\zeta_1) = 1/(n+1)$ and choose $\zeta_2 \approx 0$ such that

$$\overline{\theta}_\eta(\zeta_2) < \frac{\zeta_1^{-1} - 1}{2(\zeta_1^{-n} - 1)} \tag{4.40}$$

and then $\lambda_0 > 0$ such that

$$\overline{\theta}_\eta(\zeta_2)\zeta_1 - \overline{\theta}_\eta\left((\zeta_2\lambda^{-\eta})^{\frac{1}{1+\eta}}\right) > \frac{1}{2}\overline{\theta}_\eta(\zeta_2)\zeta_1 \tag{4.41}$$

for $\lambda > \lambda_0$. Then choose

$$\sigma_0 = \min\left\{\frac{1}{4}\zeta_1^n, \frac{1}{2(n+1)}\right\}, \tag{4.42}$$

$$\vartheta = \frac{1}{2}\overline{\theta}_\eta(\zeta_2)\zeta_1^n. \tag{4.43}$$

Given $\sigma \in (0, \sigma_0)$, $\lambda > \lambda_0$, we write $[R_1^{(0)}, R_2^{(0)}] = [0, \lambda]$, $b_0 = 2\sigma$ and then apply the following algorithm on $k = 0, 1, \dots, n-1$ orderly:

First, assume that

- $i_1, \dots, i_k \in \{1, \dots, n\}$ have been chosen without repetition,
- $b_0, \dots, b_k > 0$ have been chosen,

and they satisfy

$$\operatorname{Leb}\left(\coprod_{i\notin\{i_1,\dots,i_k\}} A^i \cap [R_1^{(k)}, R_2^{(k)}]\right) / \operatorname{Leb}([R_1^{(k)}, R_2^{(k)}]) > 1 - b_k. \tag{4.44}$$

(Note that by the choice of ζ_1 and σ_0, (4.44) is possible for $k = 0$.) Then there is one $A^{i_{k+1}}$ for some $i_{k+1} \notin \{i_1, \dots, i_k\}$ with

$$\operatorname{Leb}\left(A^{i_{k+1}} \cap [R_1^{(k)}, R_2^{(k)}]\right) > \overline{\theta}(\zeta_1) \cdot \operatorname{Leb}([R_1^{(k)}, R_2^{(k)}]).$$

Applying Lemma 4.18 to $A^{i_{k+1}}$, we obtain an ϵ-block $\overline{\mathrm{BL}}_{k+1}$ with the corresponding time interval $[R_1^{(k+1)}, R_2^{(k+1)}] \subset [R_1^{(k)}, R_2^{(k)}]$ and

$$|\overline{\mathrm{BL}}_{k+1}| = R_2^{(k+1)} - R_1^{(k+1)} \geq \zeta_1 \cdot \operatorname{Leb}([R_1^{(k)}, R_2^{(k)}]) \geq \zeta_1^{k+1}\lambda > \vartheta\lambda. \tag{4.45}$$

It follows from (4.44) that

$$\begin{aligned}
&\operatorname{Leb}\left(\coprod_{i\notin\{i_1,\dots,i_k\}} A^i \cap [R_1^{(k+1)}, R_2^{(k+1)}]\right) \\
=&\operatorname{Leb}([R_1^{(k+1)}, R_2^{(k+1)}]) - \operatorname{Leb}\left(\left(\coprod_{i\notin\{i_1,\dots,i_k\}} A^i\right)^c \cap [R_1^{(k+1)}, R_2^{(k+1)}]\right) \\
\geq&\operatorname{Leb}([R_1^{(k+1)}, R_2^{(k+1)}]) - \operatorname{Leb}\left(\left(\coprod_{i\notin\{i_1,\dots,i_k\}} A^i\right)^c \cap [R_1^{(k)}, R_2^{(k)}]\right) \\
>&\operatorname{Leb}([R_1^{(k+1)}, R_2^{(k+1)}]) - b_k \cdot \operatorname{Leb}([R_1^{(k)}, R_2^{(k)}])
\end{aligned}$$

and so by (4.45), we obtain

$$\operatorname{Leb}\left(\coprod_{i\notin\{i_1,\dots,i_k\}} A^i \cap [R_1^{(k+1)}, R_2^{(k+1)}]\right) / \operatorname{Leb}([R_1^{(k+1)}, R_2^{(k+1)}]) > 1 - b_k\zeta_1^{-1}. \tag{4.46}$$

Then we face a dichotomy:

(1) $\operatorname{Leb}(A^{i_{k+1}} \cap [R_1^{(k+1)}, R_2^{(k+1)}]) / \operatorname{Leb}([R_1^{(k+1)}, R_2^{(k+1)}]) \geq \overline{\theta}_\eta(\zeta_2)$;
(2) $\operatorname{Leb}(A^{i_{k+1}} \cap [R_1^{(k+1)}, R_2^{(k+1)}]) / \operatorname{Leb}([R_1^{(k+1)}, R_2^{(k+1)}]) < \overline{\theta}_\eta(\zeta_2)$.

In the case (1), we take $i(\lambda) = i_{k+1}$, $[R_1'(\lambda), R_2'(\lambda)] = [R_1^{(k)}, R_2^{(k)}]$, $\overline{\mathrm{BL}} = \overline{\mathrm{BL}}_{k+1}$. By (4.41) (4.43) (4.45), we have

$$\begin{aligned}
&\mathrm{Leb}\left(A_\epsilon^{i(\lambda)} \cap [R_1^{(k+1)}, R_2^{(k+1)}]\right)\\
=&\mathrm{Leb}\left(A^{i(\lambda)} \cap [R_1^{(k+1)}, R_2^{(k+1)}]\right) - \mathrm{Leb}\left((A_\epsilon^{i(\lambda)})^c \cap A^{i(\lambda)} \cap [R_1^{(k+1)}, R_2^{(k+1)}]\right)\\
\geq&\overline{\theta}_\eta(\zeta_2)\cdot \mathrm{Leb}([R_1^{(k+1)}, R_2^{(k+1)}]) - \overline{\theta}_\eta\left((\zeta_2\lambda^{-\eta})^{\frac{1}{1+\eta}}\right)\cdot \mathrm{Leb}([R_1^{(k)}, R_2^{(k)}])\\
\geq&\left(\overline{\theta}_\eta(\zeta_2)\zeta_1 - \overline{\theta}_\eta\left((\zeta_2\lambda^{-\eta})^{\frac{1}{1+\eta}}\right)\right)\cdot \mathrm{Leb}([R_1^{(k)}, R_2^{(k)}])\\
>&\frac{1}{2}\overline{\theta}_\eta(\zeta_2)\zeta_1\cdot\zeta_1^k\lambda \geq \vartheta\lambda
\end{aligned} \tag{4.47}$$

and the consequence of Proposition 4.19 follows. In the case (2), by (4.46), we have

$$\mathrm{Leb}\left(\coprod_{i\notin\{i_1,\dots,i_{k+1}\}} A^i \cap [R_1^{(k+1)}, R_2^{(k+1)}]\right) / \mathrm{Leb}([R_1^{(k+1)}, R_2^{(k+1)}]) > 1 - b_k\zeta_1^{-1} - \overline{\theta}_\eta(\zeta_2). \tag{4.48}$$

Now note that

- $i_{k+1} \notin \{i_1, \dots, i_k\}$ has been chosen,
- choose $b_{k+1} = b_k\zeta_1^{-1} + \overline{\theta}_\eta(\zeta_2)$

and then (4.48) coincides with (4.44) by replacing k by $k+1$. Thus, we can apply the algorithm again by replacing k by $k+1$.

After applying the algorithm, we either stop in the middle and finish the proof, or we determine

- $i_1, \dots, i_{n-1} \in \{1, \dots, n\}$ without repetition,
- a sequence $\{b_k\}_{k=0}^{n-1}$ of positive numbers with $b_0 = 2\sigma$ and

$$b_{k+1} = b_k\zeta_1^{-1} + \overline{\theta}_\eta(\zeta_2). \tag{4.49}$$

Let $i(\lambda)$ be the only element in $\{1, \dots, n\} \setminus \{i_1, \dots, i_{n-1}\}$. Let $[R_1'(\lambda), R_2'(\lambda)] = [R_1^{(n-1)}, R_2^{(n-1)}]$. Besides, by (4.49) we calculate

$$b_{n-1} = 2\sigma\zeta_1^{-(n-1)} + \overline{\theta}_\eta(\zeta_2)\frac{\zeta_1^{-(n-1)} - 1}{\zeta_1^{-1} - 1}.$$

Now we try to do the algorithm one more time. Thus, we apply again Lemma 4.18 to $A^{i(\lambda)}$, and then we obtain an ϵ-block $\overline{\mathrm{BL}} = \overline{\mathrm{BL}}_n$ with the corresponding time interval $[R_1^{(n)}, R_2^{(n)}] \subset [R_1^{(n-1)}, R_2^{(n-1)}]$ satisfying (4.45) (4.46), i.e.

$$|\overline{\mathrm{BL}}_n| = \mathrm{Leb}([R_1^{(n)}, R_2^{(n)}]) \geq \zeta_1\cdot \mathrm{Leb}([R_1^{(n-1)}, R_2^{(n-1)}]) \geq \zeta_1^n\lambda > \vartheta\lambda, \tag{4.50}$$

$$\begin{aligned}
&\mathrm{Leb}\left(A^{i(\lambda)} \cap [R_1^{(n)}, R_2^{(n)}]\right) / \mathrm{Leb}([R_1^{(n)}, R_2^{(n)}])\\
&\qquad > 1 - b_{n-1}\zeta_1^{-1} = 1 - 2\sigma\zeta_1^{-n} - \overline{\theta}_\eta(\zeta_2)\frac{\zeta_1^{-n} - \zeta_1^{-1}}{\zeta_1^{-1} - 1} \geq \overline{\theta}_\eta(\zeta_2)
\end{aligned} \tag{4.51}$$

where the last inequality of (4.51) follows from (4.40) (4.42). Then, as in (4.47), we calculate

$$\mathrm{Leb}\left(A_\epsilon^{i(\lambda)} \cap [R_1^{(n)}, R_2^{(n)}]\right) \geq \left(\overline{\theta}_\eta(\zeta_2)\zeta_1 - \overline{\theta}_\eta\left((\zeta_2\lambda^{-\eta})^{\frac{1}{1+\eta}}\right)\right) \cdot \mathrm{Leb}([R_1^{(n-1)}, R_2^{(n-1)}]) > \vartheta\lambda$$

where the last inequality follows from (4.41) (4.43) (4.50). □

5. Invariance

Let $G_X = SO(n_X, 1)$ and $\Gamma_X \subset G_X$ be a lattice. Let (X, μ) be the homogeneous space $X = G_X/\Gamma_X$ equipped with the Lebesgue measure μ, and let $\phi_t^{U_X} = u_X^t$ be a unipotent flow on X as before. Besides, let G_Y be a Lie group and $\Gamma_Y \subset G_Y$ be a lattice. (Y, m_Y) be the homogeneous space $Y = G_Y/\Gamma_Y$ equipped with the Lebesgue measure m_Y and let $\phi_t^{U_Y} = u_Y^t$ be a unipotent flow on Y. Next, choose $\tau_Y \in \mathbf{K}_\kappa(Y)$ a positive integrable function τ_Y on Y such that τ_Y, τ_Y^{-1} are bounded and satisfies (2.10). Then define the measure $d\nu := \tau_Y dm_Y$ and so the time-change flow $\phi_t^{U_Y,\tau_Y} = \tilde{u}_Y^t$ preserves the measure ν by Remark 2.2. Also recall from (2.9) that

$$u_Y^t y = \phi_{z(y,t)}^{U_Y,\tau_Y}(y) = \tilde{u}_Y^{z(y,t)}(y).$$

We shall to study the joinings of (X, μ, u_X^t) and $(Y, \nu, \tilde{u}_Y^t)$. Let ρ be an ergodic *joining* of u_X^t and $\tilde{u}_Y^t$, i.e. ρ is a probability measure on $X \times Y$, whose marginals on X and Y are μ and ν respectively, and which is $(u_X^t \times \tilde{u}_Y^t)$-ergodic. As indicated at the end of Section 3, when ρ is not the product measure $\mu \times \nu$, we apply Theorem 3.5 and then obtain a compact subgroup $C^\rho \subset C_{G_X}(U_X)$ such that $\overline{\rho} := \pi_*\rho$ is an ergodic joining u_X^t and $\tilde{u}_Y^t$ on $C^\rho\backslash X \times Y$ under the natural projection $\pi : X \times Y \to C^\rho\backslash X \times Y$. Besides, it is a finite extension of ν, i.e. $\operatorname{supp}\overline{\rho}_y$ consists of exactly n points $\overline{\psi}_1(y), \ldots, \overline{\psi}_n(y)$ for ν-a.e. $y \in Y$ (without loss of generality, we shall assume that it holds for all $y \in Y$). By Kunugui's theorem, we obtain $\psi_i : Y \to X$ so that $P_X \circ \psi_i = \overline{\psi}_i$ where $P_X : X \to C^\rho\backslash X$.

5.1. **Central direction.** We want to study the behavior of $\overline{\psi}_p$ along the central direction $C_{G_Y}(U_Y)$ of U_Y. In the following, assume that ρ is a $(u_X^t \times \tilde{u}_Y^t)$-joining. Then by (3.12), we get that

$$\overline{\psi}_p(u_Y^t y) = \overline{\psi}_p(\tilde{u}_Y^{z(y,t)}(y)) = u_X^{z(y,t)}\overline{\psi}_{i_p}(y)$$

where the index $i_p = i_p(y, t) \in \{1, \ldots, n\}$ is determined by

$$(u_X^{-z(y,t)} \times \tilde{u}_Y^{-z(y,t)})(\overline{\psi}_p(\tilde{u}_Y^{z(y,t)}(y)), \tilde{u}_Y^{z(y,t)}(y)) \in \hat{\psi}_{i_p}(Y).$$

Now we orderly fix the following data so that the propositions in Section 4 can be used:

- fix $\kappa \in (0, 2\eta_0)$ satisfying (2.10), where $\eta_0 > 0$ comes from Proposition 4.11;
- fix $\sigma \in (0, \sigma_0)$, where $\sigma_0 \approx 0$ comes from both Proposition 4.11 and Proposition 4.19;
- fix $\epsilon \in (0, \epsilon_0)$ as in (4.31)

such that the following holds:

- (Effective ergodicity) By (2.11), there is $K_1 \subset Y$ with $\nu(K_1) > 1 - \sigma/6$ and $t_{K_1} > 0$ such that

$$|t - z(y,t)| = O(t^{1-\kappa}) \tag{5.1}$$

  for all $t \geq t_{K_1}$ and $y \in K_1$. Note that using ergodic theorem, we have

$$|t - z(y,t)| = o(t) \tag{5.2}$$

  for ν-almost all $y \in Y$.
- (Distinguishing $\overline{\psi}_p, \overline{\psi}_q$) There is $K_2 \subset Y$ with $\nu(K_2) > 1 - \sigma/6$ such that

$$d(\overline{\psi}_p(y), \overline{\psi}_q(y)) > 100\epsilon \tag{5.3}$$

  for $y \in K_2$, $1 \leq p < q \leq n$.
- (Lusin's theorem) There is $K_3 \subset Y$ such that $\nu(K_3) > 1 - \sigma/6$ and $\overline{\psi}_p|_{K_3}$ is uniformly continuous for all $p \in \{1, \dots, n\}$. Thus, there is $\delta > 0$ such that

$$d_{\overline{X}}(\overline{\psi}_p(y_1), \overline{\psi}_p(y_2)) < \epsilon \tag{5.4}$$

  for $p \in \{1, \dots, n\}$, $d_Y(y_1, y_2) < \delta$ and $y_1, y_2 \in K_3$.

Given $K \subset \overline{X}$ by Proposition 4.11, let

$$K^0 \coloneqq K_1 \cap K_2 \cap K_3 \cap \bigcap_{p=1}^{n} \overline{\psi}_p^{-1}(K). \tag{5.5}$$

Here we choose $\overline{\mu}(K)$ being so large that $m_Y(K^0) > 1 - \sigma/2$.

Fix $c \in C_{G_Y}(U_Y) \cap B_{G_Y}(e, \delta)$. We choose arbitrarily a representative $g_{\overline{\psi}_p(y)} \in G_X$ of $\overline{\psi}_p(y)$. Then there is a representative $g_{\overline{\psi}_p(cy)} \in G_X$ so that

- $\overline{g_{\overline{\psi}_p(y)}}$ and $\overline{g_{\overline{\psi}_p(cy)}}$ lie in the same fundamental domain;
- the difference $g(y) = g_{\overline{\psi}_p(cy)} g^{-1}_{\overline{\psi}_p(y)} = h^{(p)}(y) \exp(v^{(p)}(y))$ where

$$h^{(p)}(y) = \begin{bmatrix} a^{(p)}(y) & b^{(p)}(y) \\ c^{(p)}(y) & d^{(p)}(y) \end{bmatrix} \in SO_0(2,1), \quad v^{(p)} = b_0^{(p)}(y) v_0 + \dots + b_\varsigma^{(p)}(y) v_\varsigma \in V_\varsigma. \tag{5.6}$$

Further, applying the effectiveness of the unipotent flow, we shall show that the difference $g(y)$ has to lie in the centralizer $C_{G_X}(U_X)$.

Proposition 5.1. *Let the notation and assumptions be as above. For the quantities in (5.6), there is a measurable set $S(c) \subset Y$ with $\nu(S(c)) > 0$ such that*

$$b^{(p)}(y) = 0, \quad a^{(p)}(y) = d^{(p)}(y) = 1, \quad b_0^{(p)}(y) = \dots = b_{\varsigma-1}^{(p)}(y) = 0$$

for $y \in S(c)$, $p \in \{1, \dots, n\}$.

Proof. Consider the measure of the set

$$Y_l(c) \coloneqq \{y \in Y : |b^{(p)}(y)|, |a^{(p)}(y)-1|, |d^{(p)}(y)-1|, |b_0^{(p)}(y)|, \cdots, |b_{\varsigma-1}^{(p)}(y)| < 1/l, \text{ for any } p \in \{1,\dots,n\}\}$$

for $l \in \mathbf{Z}^+$. We shall show that $S(c) \coloneqq \bigcap_l Y_l(c)$ satisfies the requirement. By ergodic theorem, we have

$$m_Y(Y_l(c)) = \lim_{\lambda\to\infty} \frac{1}{\lambda}\int_0^\lambda \mathbf{1}_{Y_l(c)}(u_Y^r y)dr \tag{5.7}$$

for m_Y-a.e. $y \in Y$, where m_Y denotes the Lebesgue measure on Y.

On the other hand, by ergodic theorem, for m_Y-a.e. $y \in Y$, there is $A_{c,y} \subset \mathbf{R}^+$ and $\lambda_0(y) > 0$ such that

- for $r \in A_{c,y}$, we have
$$u_Y^r y, u_Y^r cy \in K^0;$$
- $\mathrm{Leb}(A_{c,y} \cap [0,\lambda]) \geq (1-2\sigma)\lambda$ whenever $\lambda \geq \lambda_0(y)$.

Then by the assumptions, we have

$$A_{c,y} \subset \left\{r \in [0,\infty) : d_{\overline{X}}(\overline{\psi}_p(u_Y^r y), \overline{\psi}_p(u_Y^r cy)) < \epsilon,\ p \in \{1,\dots,n\}\right\}. \tag{5.8}$$

It follows that for $r \in A_{c,y}$, we have

$$d_{\overline{X}}(u_X^{z(y,r)}\overline{\psi}_{i_p(y,r)}(y), u_X^{z(cy,r)}\overline{\psi}_{i_p(cy,r)}(cy)) < \epsilon \tag{5.9}$$

for any $p \in \{1,\dots,n\}$. Now we restrict our attention on $A_{c,y} \cap [0,\lambda]$ with $\lambda \geq \lambda_0(y)$. For simplicity, we assume that $0 \in A_{c,y}$. Let $I = ((p_1,p_2),\dots,(p_{2n-1},p_{2n})) \in \{1,\dots,n\}^{2n}$ be a sequence of indexes and

$$A_{c,y}^I \coloneqq \{r \in A_{c,y} : p_{2k-1} = i_k(y,r),\ p_{2k} = i_k(cy,r) \text{ for all } k \in \{1,\dots,n\}\}. \tag{5.10}$$

Then $A = A_{c,y}^I$, $R_0 = t_{K_1}$, $t(r) = z(cy,r)$, $s(r) = z(y,r)$ satisfy (4.32) (4.33) for points

$$\overline{\psi}_{p_{2k-1}}(y), \overline{\psi}_{p_{2k}}(cy) \in K$$

for all $k \in \{1,\dots,n\}$.

Since $A_{c,y} = \coprod_{I\in\{1,\dots,n\}^{2n}} A_{c,y}^I$ (is a disjoint union because of (5.3)), by Proposition 4.19, for any $\lambda \geq \lambda_0$, there exists one $A_{c,y}^{I(\lambda)}$ and $[R_1', R_2'] \subset [0,\lambda]$ such that there exists an ϵ-block $\overline{\mathrm{BL}} = \{(x',y'),(x'',y'')\} \in \beta_2(A_{c,y}^{I(\lambda)} \cap [R_1', R_2'])$ with the corresponding time interval $[R_1, R_2]$ such that

$$R_2 - R_1 > \vartheta\lambda, \quad \mathrm{Leb}\left(A_\epsilon^{I(\lambda)} \cap [R_1, R_2]\right) > \vartheta\lambda$$

where $A_\epsilon^{I(\lambda)}$ is the non-shifting time of $A_{c,y}^{I(\lambda)}$. Then by the definition of $A_\epsilon^{I(\lambda)}$, we know that

$$d_{C^\rho\backslash G}\left(u_X^{z(cy,r)}.\overline{g_{\overline{\psi}_{i_p(cy,r)}(cy)}}, u_X^{z(y,r)}.\overline{g_{\overline{\psi}_{i_p(y,r)}(y)}}\right) < \epsilon$$

for $r \in A_\epsilon^{I(\lambda)}$, $p \in \{1, \ldots, n\}$. Recall from (4.24) that points in K have injectivity radius at least ϵ_0. Thus, for $r \in A_\epsilon^{I(\lambda)}$,

$$u_X^{z(y,r)}.\overline{g_{\overline{\psi}_{ip(y,r)}(y)}} \quad \text{and} \quad u_X^{z(cy,r)}.\overline{g_{\overline{\psi}_{ip(cy,r)}(cy)}}$$

lie in the same fundamental domain. Thus, if $r \in A_\epsilon^{I(\lambda)}$ and

$$\overline{g_{\overline{\psi}_p(u_Y^r y)}} = u_X^{z(y,r)}.\overline{g_{\overline{\psi}_{ip(y,r)}(y)}}$$

then we get

$$\overline{g_{\overline{\psi}_p(u_Y^r cy)}} = u_X^{z(cy,r)}.\overline{g_{\overline{\psi}_{ip(cy,r)}(cy)}}.$$

Recall that the difference of $u_X^{z(y,r)}.\overline{g_{\overline{\psi}_{ip(y,r)}(y)}}$, $u_X^{z(cy,r)}.\overline{g_{\overline{\psi}_{ip(cy,r)}(cy)}}$ for $r \in A_\epsilon^{i(\lambda)} \cap [R_1, R_2]$ was estimated by (4.37) (see also (4.5) (4.6) (4.7)). In particular, for $r \in A_\epsilon^{i(\lambda)} \cap [R_1, R_2]$, the quantities of

$$g(u_Y^r y) = g_{\overline{\psi}_p(cu_Y^r y)} g_{\overline{\psi}_p(u_Y^r y)}^{-1} = u_X^{z(cy,r)} g_{\overline{\psi}_{ip(cy,r)}(cy)} \left(u_X^{z(y,r)} g_{\overline{\psi}_{ip(y,r)}(y)} \right)^{-1}$$

that need to estimate in $Y_l(c)$ are all decreasing as $\lambda \to \infty$. Then given $l \in \mathbf{Z}^+$, there is a sufficiently large λ such that

$$\int_0^\lambda \mathbf{1}_{Y_l(c)}(u_Y^r y) dr \geq \operatorname{Leb}\left(A_\epsilon^{i(\lambda)} \cap [R_1, R_2] \right) > \vartheta \lambda.$$

Thus, by (5.7), we have $m_Y(Y_l(c)) > \vartheta$. Now letting $\lambda \to \infty$ and then $l \to \infty$, we see that $m_Y(\bigcap_l Y_l(c)) > \vartheta$. Finally, by Remark 2.2 and $\tau_Y \in \mathbf{K}_\kappa(Y)$, we obtain $\nu(\bigcap_l Y_l(c)) > 0$. $\square$

Using Proposition 5.1, we immediately obtain

Corollary 5.2. *There is a measurable map $\varpi : C_{G_Y}(U_Y) \times X \times Y \to C_{G_X}(U_X)$ that induces a map $\widetilde{S}_c : \operatorname{supp}(\rho) \to \operatorname{supp}(\rho)$ by*

$$\widetilde{S}_c : (x, y) \mapsto (\varpi(c, x, y)x, cy) \tag{5.11}$$

for all $c \in C_{G_Y}(U_Y)$, ρ-a.e. $(x, y) \in X \times Y$. Moreover, we have

$$\varpi(c, x, y) = u_X^{-z(cy,t)} \varpi(c, (u_X^{z(y,t)} \times \widetilde{u}_Y^{z(y,t)}).(x, y)) u_X^{z(y,t)} \tag{5.12}$$

$$\varpi(c_1 c_2, x, y) = \varpi(c_1, \varpi(c_2, x, y)x, c_2 y) \varpi(c_2, x, y) \tag{5.13}$$

for $c, c_1, c_2 \in C_{G_Y}(U_Y)$, ρ-a.e. $(x, y) \in X \times Y$, $t \in \mathbf{R}$.

Remark 5.3. Note that when $c \in \exp(\mathbf{R}U_Y)$, ϖ reduces to an element in $\exp(\mathbf{R}U_X)$; in fact, we have

$$\varpi(u_Y^t, x, y) = u_X^{z(y,r)} = \exp(z(y, t)U_X)$$

for all $t \in \mathbf{R}$.

On the other hand, for distinct $q_1, q_2 \in \{1, \ldots, n\}$, any $c \in C_{G_Y}(U_Y)$, we have

$$w(c, \psi_{q_1}(y), y)\psi_{q_1}(y) \in C^\rho \psi_{p_1}(y), \quad w(c, \psi_{q_2}(y), y)\psi_{q_2}(y) \in C^\rho \psi_{p_2}(y) \tag{5.14}$$

for distinct $p_1, p_2 \in \{1, \ldots, n\}$; for otherwise it would lead to $\psi_{q_2}(y) \in C_{G_X}(U_X)\psi_{q_1}(y)$, which contradicts the definition of ψ (cf. Section 3.2).

Proof of Corollary 5.2. Fix $c \in C_{G_Y}(U_Y) \cap B(e, \delta)$. Proposition 5.1 provides us a subset $S(c) \subset Y$ with $\nu(S(c)) > 0$ such that

$$\psi_p(cy) = w_p(c, y)\psi_p(y) \tag{5.15}$$

for $y \in S(c)$, $w_p(c, y) \in C_{G_X}(U_X)$. Besides, for $y, u_Y^r y \in S(c)$, we know that

$$w_p(c, u_Y^r y)u_X^{z(y,r)}\psi_{i_p(y,r)}(y) = \psi_p(u_Y^r cy) = u_X^{z(cy,r)} w_{i_p(cy,r)}(c, y)\psi_{i_p(cy,r)}(y).$$

Thus, $\psi_{i_p(y,r)}(y) \in C_{G_X}(U_X)\psi_{i_p(cy,r)}(y)$ and so $i_p(y, r) = i_p(cy, r)$. It follows that

$$w_p(c, u_Y^r y)u_X^{z(y,r)} = u_X^{z(cy,r)} w_{i_p(cy,r)}(c, y) = u_X^{z(cy,r)} w_{i_p(y,r)}(c, y) \tag{5.16}$$

for $y, u_Y^r y \in S(c)$.

Thus, for $y \in S(c)$, we define

$$\varpi(c, \psi_p(y), y) \coloneqq w_p(c, y).$$

Let $\pi_Y : \mathrm{supp}(\rho) \to Y$ be the natural projection. Then for $(x, y) \in \pi_Y^{-1}(S(c))$, we know that $C^\rho x = C^\rho \psi_{p_x}(y)$ for some $p_x \in \{1, \ldots, n\}$. Thus, given $\psi_{p_x}(y) = k_x^\rho x$ for some $k_x^\rho \in C^\rho$, we define

$$\varpi(c, x, y) \coloneqq (k_x^\rho)^{-1} w_{p_x}(c, y)k_x^\rho. \tag{5.17}$$

Thus, we successfully define $\varpi(c, \cdot, \cdot)$ for $\pi_Y^{-1}(S(c))$. Then the $(u_X^t \times \tilde{u}_Y^t)$-flow helps us to define $\varpi(c, \cdot, \cdot)$ for all ρ-a.e. $(x, y) \in X \times Y$. More precisely, for $(x, y) \in X \times Y$ (in a ρ-conull set), we can choose $t = t(x, y) \in \mathbf{R}$ such that $(u_X^{z(y,t)} x, u_Y^t y) \in \pi_Y^{-1}(S(c))$. Then define

$$\begin{aligned} \varpi(c, x, y) \coloneqq& u_X^{-z(cy,t)}\varpi(c, u_X^{z(y,t)} x, u_Y^t y)u_X^{z(y,t)} \\ =& u_X^{-z(cy,t)}\varpi(c, (u_X^{z(y,t)} \times \tilde{u}_Y^{z(y,t)}).(x, y))u_X^{z(y,t)}. \end{aligned} \tag{5.18}$$

(Note that (5.16) tells us that (5.18) holds true for $y, u_Y^t y \in S(c)$ and thus ϖ is well-defined.) Finally, for general $c \in C_{G_Y}(U_Y)$, choose $k \in C_{G_Y}(U_Y) \cap B(e, \delta)$ such that $k^m = c$, and then define iteratively

$$\varpi(k^{i+1}, x, y) \coloneqq \varpi(k^i, \varpi(k, x, y)x, ky)\varpi(k, x, y)$$

and finally reach $c = k^m$. Then the map (5.11) is well defined on $\mathrm{supp}(\rho)$. ☐

In light of Corollary 5.2, we consider the decomposition (2.7) and write

$$\varpi(c, x, y) = u_X^{\alpha(c,x,y)}\beta(c, x, y) \tag{5.19}$$

where $\alpha(c, x, y) \in \mathbf{R}$ and $\beta(c, x, y) \in \exp V_{C_X}^\perp$. Then by (5.12), we have

$$z(cy, t) + \alpha(c, x, y) = \alpha(c, (u^{z(y,t)} \times \tilde{u}^{z(y,t)}).(x, y)) + z(y, t), \tag{5.20}$$

$$\beta(c, x, y) = \beta(c, (u^{z(y,t)} \times \tilde{u}^{z(y,t)}).(x, y)) \tag{5.21}$$

for all $t \in \mathbf{R}$.

First consider α. Recall that for fixed $y \in Y$, $\operatorname{supp}(\rho_y) = \bigsqcup_{p=1}^n C^\rho \psi_p(y)$. Then by (5.20), for ν-a.e. $y \in Y$, $x \in \operatorname{supp}(\rho_y)$, we have

$$\alpha(c, x, y) - \alpha(c, (u^{z(y,t)} \times \widetilde{u}^{z(y,t)}).(x, y)) = z(y, t) - z(cy, t) \tag{5.22}$$

for all $r \in \mathbf{R}$. Besides, by (5.17), we have

$$\alpha(c, x, y) = \alpha(c, kx, y) \tag{5.23}$$

for all $x \in \operatorname{supp}(\rho_y)$, $k \in C^\rho$. By (5.20), for any $(x_1, y), (x_2, y) \in \operatorname{supp}(\rho)$, we have

$$\alpha(c, x_1, y) - \alpha(c, x_2, y) = \alpha(c, (u^t \times \tilde{u}^t).(x_1, y)) - \alpha(c, (u^t \times \tilde{u}^t).(x_2, y)). \tag{5.24}$$

Define $\alpha_{\max} : C_{G_Y}(U_Y) \times X \times Y \to \mathbf{R}$ by

$$\alpha_{\max} : (c, x, y) \mapsto \max \{r \in \mathbf{R} : \rho_y\{x' \in X : \alpha(c, x', y) - \alpha(c, x, y) = r\} > 0\}.$$

Then by (5.24), we have

$$\alpha_{\max}(c, (x, y)) = \alpha_{\max}(c, (u_X^t \times \tilde{u}_Y^t).(x, y))$$

for any $t \in \mathbf{R}$, ρ-a.e. $(x, y) \in X \times Y$. Thus, $\alpha_{\max}(c, x, y) \equiv \alpha_{\max}(c)$. Now if $\alpha_{\max}(c) > 0$, then for ρ-a.e. (x, y), there is $x' \in X$ such that $\alpha(c, x', y) = \alpha(c, x, y) + \alpha_{\max}(c)$, which contradicts the fact that $\alpha_{\max}(c, x, y)$ take at most finitely many different values for fixed y (by (5.23)). Thus, we conclude that $\alpha_{\max}(c) \equiv 0$ and so

$$\alpha(c, x, y) \equiv \alpha(c, y)$$

for all $c \in C_{G_Y}(U_Y)$, ρ-a.e. $(x, y) \in X \times Y$.

On the other hand, via the ergodicity of the flow $u_X^t \times \tilde{u}_Y^t$, we conclude from (5.21) that

$$\beta(c, x, y) \equiv \beta(c)$$

for all $c \in C_{G_Y}(U_Y)$. In particular, we have

$$\varpi(c, x, y) = \varpi(c, y) = u_X^{\alpha(c,y)} \beta(y)$$

for all $c \in C_{G_Y}(U_Y)$, ρ-a.e. $(x, y) \in X \times Y$. Besides, we know from (5.13) that $\beta(c_1 c_2) = \beta(c_1)\beta(c_2)$ via the definition of β. Further, we always have $d\beta(U_Y) \equiv 0$. Therefore, we can restrict our attention to $V_C^\perp$ and conclude that $d\beta|_{V_C^\perp} : V_{C_Y}^\perp \to V_{C_X}^\perp$ is a Lie algebra homomorphism.

In sum, we obtain Theorem 1.2 for the centralizer $C_{G_Y}(U_Y)$.

Theorem 5.4 (Extra central invariance of ρ)**.** *For any $c \in C_{G_Y}(U_Y)$, the map $S_c : X \times Y \to X \times Y$ defined by*

$$S_c : (x, y) \mapsto (\beta(c)x, \tilde{u}_Y^{-\alpha(c,y)}(cy))$$

commutes with $u_X^t \times \tilde{u}_Y^t$, and is ρ-invariant. Besides, $S_{c_1 c_2} = S_{c_1} \circ S_{c_2}$ for any $c_1, c_2 \in C_{G_Y}(U_Y)$, and $S_{u_Y^t} = \mathrm{id}$ for $t \in \mathbf{R}$.

Proof. Clearly, S_c is well-defined:

$$S_c(x,y) = (u_X^{-\alpha(c,y)} \times \widetilde{u}_Y^{-\alpha(c,y)}).\widetilde{S}_c(x,y) \in \operatorname{supp}(\rho) \tag{5.25}$$

whenever $(x,y) \in \operatorname{supp}(\rho)$. Also, one may check that $S_{c_1c_2} = S_{c_1}S_{c_2}$ for any $c_1, c_2 \in C_{G_Y}(U_Y)$, and $S_{u_Y^t} = \mathrm{id}$ for $t \in \mathbf{R}$. Next, by (5.20), one verifies

$$(u_X^{z(y,r)} \times \widetilde{u}_Y^{z(y,r)}).S_c(x,y) = S_c(u_X^{z(y,r)} \times \widetilde{u}_Y^{z(y,r)}).(x,y)$$

for any $r \in \mathbf{R}$, $(x,y) \in \operatorname{supp}(\rho)$. That is, $(u_X^t \times \widetilde{u}_Y^t) \circ S_c = S_c \circ (u_X^t \times \widetilde{u}_Y^t)$.

Finally, let Ω be the set of $(u_X^t \times \widetilde{u}_Y^t)$-generic points, and we want to show that there is a point $(x_0, y_0) \in \Omega \cap S_c^{-1}\Omega$. By (5.25), it suffices to show that there is a point $(x_0, y_0) \in \Omega \cap \widetilde{S}_c^{-1}\Omega$. Fix $c \in C_{G_Y}(U_Y) \cap B(e, \delta)$. Recall that

$$1 = \rho(\Omega) = \int_Y \int_{C^\rho} \frac{1}{n} \sum_{p=1}^n \mathbf{1}_\Omega(k\psi_p(y), y) dm(k) d\nu(y).$$

Thus, there is $\Omega_Y \subset Y$ with $\nu(\Omega_Y) = 1$ such that

$$\int_{C^\rho} \frac{1}{n} \sum_{p=1}^n \mathbf{1}_\Omega(k\psi_p(y), y) dm(k) = 1 \tag{5.26}$$

for $y \in \Omega_Y$. Since ν and m_Y are equivalent, and $\Omega_Y \cap k^{-1}\Omega_Y$ is m_Y-conull, we get that $\Omega_Y \cap c^{-1}\Omega_Y$ is ν-conull. Choose $y_0 \in \Omega_Y \cap c^{-1}\Omega_Y \cap S(c)$, where $S(c)$ is given by Proposition 5.1 (cf. (5.15)). Then (5.26) leads to

$$\int_{C^\rho} \mathbf{1}_\Omega(k\psi_1(y_0), y_0) dm(k) = 1, \quad \int_{C^\rho} \mathbf{1}_\Omega(k\psi_1(cy_0), cy_0) dm(k) = 1.$$

Then we can choose $k_0 \in C^\rho$ such that $(k_0\psi_1(y_0), y_0), (k_0\psi_1(cy_0), cy_0) \in \Omega$. Let $x_0 := k_0\psi_1(y_0)$. Then by (5.15) (5.17), we have

$$\widetilde{S}_c(x_0, y_0) = (\varpi(c, y_0)x_0, cy_0) = (k_0 w_p(c, y_0)k_0^{-1}k_0\psi_1(y_0), cy_0) = (k_0\psi_1(cy_0), cy_0).$$

Thus, $(x_0, y_0) \in \Omega \cap \widetilde{S}_c^{-1}\Omega$.

Hence, since $u_X^t \times \widetilde{u}_Y^t$ is ρ-ergodic, by ergodic theorem, for any bounded continuous function f, we have

$$\begin{aligned} \int f d\rho &= \lim_{T\to\infty} \frac{1}{T} \int_0^T f((u_X^t \times \tilde{u}_Y^t).S_c(x_0, y_0)) dt \\ &= \lim_{T\to\infty} \frac{1}{T} \int_0^T f(S_c(u_X^t x_0, \tilde{u}_Y^t y_0)) dt = \int f \circ S_c d\rho \end{aligned}$$

and so $\rho = (S_c)_*\rho$. □

In particular, we obtain

Corollary 5.5 (Extra central invariance of ν)**.** *For any $c \in C_{G_Y}(U_Y)$, the map $S_c^Y : Y \to Y$ defined by*

$$S_c^Y : y \mapsto \tilde{u}_Y^{-\alpha(c,y)}(cy)$$

commutes with $\widetilde{u}^t$*, and is* ν*-invariant. Besides,* $S^Y_{c_1c_2} = S^Y_{c_1}S^Y_{c_2}$ *for any* $c_1, c_2 \in C_{G_Y}(U_Y)$*, and* $S^Y_{u^t_Y} = \mathrm{id}$ *for* $t \in \mathbf{R}$.

It is worth noting that (5.11) can be interpreted through the language of cohomology. More precisely, (5.11) implies the time change τ_Y and $\tau_Y \circ c$ are measurably cohomologous.

Theorem 5.6. *Let* $\tau_Y \in \mathbf{K}_\kappa(Y)$*. Suppose that there is a nontrivial ergodic joining* $\rho \in J(u^t_X, \phi^{U_Y,\tau_Y}_t)$ *. Then* $\tau_Y(y)$ *and* $\tau_Y(cy)$ *are (measurably) cohomologous along* u^t_Y *for all* $c \in C_{G_Y}(U_Y)$*. More precisely, the transfer function can be taken to be*

$$F_c(y) = \alpha(c, y).$$

Proof. By (5.20), for m_Y-a.e. $y \in Y$, $x \in \mathrm{supp}(\rho_y)$, we have

$$\begin{aligned}
&\int_0^t \tau_Y(u^s_Y y) - \tau_Y(u^s_Y cy)ds \\
=&\int_0^t \tau_Y(u^s_Y y)ds - \int_0^t \tau_Y(u^s_Y cy)ds \\
=&z(y, t) - z(cy, t) \\
=&\alpha(c, y) - \alpha(c, u^t_Y y).
\end{aligned}$$

Thus, we can take the transfer function as

$$F_c(y) \coloneqq \alpha(c, y).$$

Then $\tau_Y(y)$ and $\tau_Y(cy)$ are (measurably) cohomologous for all $c \in C_{G_Y}(U_Y)$. □

If $\tau_Y(y)$ and $\tau_Y(cy)$ are cohomologous with a L^1 transfer function, then we are able to do more via the *ergodic theorem.*

Lemma 5.7. *Given* $c \in C_{G_Y}(U_Y)$*, if*

- c *is* m_Y*-ergodic (as a left action on* Y*),*
- $\tau_Y(y)$ *and* $\tau_Y(cy)$ *are cohomologous with a* L^1 *transfer function* $F_c(y)$*,*

then for m_Y*-a.e.* $y \in Y$*, we have*

$$\lim_{t\to\infty} \frac{1}{t}\alpha(c^t, y) = \int \alpha(c, y)dm_Y(y).$$

Proof. By (5.13) (5.14), for $c_1, c_2 \in C_{G_Y}(U_Y)$, m_Y-a.e. $y \in Y$, we have the cocycle identity

$$\alpha(c_1c_2, y) = \alpha(c_1, c_2y) + \alpha(c_2, y).$$

Thus, if $F_c(\cdot) \in L^1(Y)$, then by the ergodicity, we get

$$\lim_{k\to\infty} \frac{1}{k}\alpha(c^k, y) = \lim_{k\to\infty} \frac{1}{k}\sum_{i=0}^{k} \alpha(c^i, y) = \int \alpha(c, y)dm_Y(y). \tag{5.27}$$

□

Remark 5.8. The results obtained in Section 5.1 also hold true for ρ being a finite extension of ν, when $(X, \phi_t^{U_X,\tau_X})$ is a time-change of the unipotent flow on $X = SO(n_X, 1)/\Gamma_X$. For example, we consider the case when $n_X = 2$, $\tau_X \in C^1(X)$, $\tau_Y \equiv 1$ (in other words, $\phi_t^{U_Y,\tau_Y} = \phi_t^{U_Y} = u_Y^t$ is the usual unipotent flow, and $\nu = m_Y$). First, [Rat87] shows that $(X, \phi_t^{U_X,\tau_X})$ has H-property. In particular, suppose that $\rho \in J(\phi_t^{U_X,\tau_X}, \phi_t^{U_Y})$ is not the product measure $\mu\times\nu$. Then H-property of $\tilde{u}_X^t := \phi_t^{U_X,\tau_X}$ deduces that ρ is a finite extension of ν (see Theorem 3, [Rat83]):

$$\int f(x,y)d\rho(x,y) = \int \frac{1}{n}\sum_{p=1}^{n} f(\psi_p(y),y)d\nu(y).$$

On the other hand, since $V_{C_X}^{\perp} = 0$, by Corollary 5.2 (and (5.19)), we again have a map $\widetilde{S}_c : \mathrm{supp}(\rho) \to \mathrm{supp}(\rho)$ given by

$$\widetilde{S}_c : (x,y) \mapsto (u_X^{\alpha(c,y)}x, cy) \tag{5.28}$$

In contrast to Theorem 5.4, $\widetilde{S}_c$ is ρ-invariant in this situation. We can further specify $\alpha(c,x,y)$ in certain situation as follows:

First, under the current setting, (5.20) changes to

$$\xi(\psi_p(cy),t) + \alpha(c,y) = \alpha(c,u_Y^t y) + \xi(\psi_p(y),t)$$

for $t \in \mathbf{R}$. It follows that

$$\begin{aligned}
0 &= \int_0^{\xi(\psi_p(y),t)} \tau(u_X^s\psi_p(y)) - \tau(u_X^s\psi_p(y))ds\\
&= \int_0^{\xi(\psi_p(cy),t)} \tau(u_X^s\psi_p(cy))ds - \int_0^{\xi(\psi_p(y),t)} \tau(u_X^s\psi_p(y))ds\\
&= \int_0^{\xi(\psi_p(cy),t)} \tau(u_X^{\alpha(c,y)+s}\psi_p(y))ds - \int_0^{\xi(\psi_p(y),t)} \tau(u_X^s\psi_p(y))ds\\
&= \int_0^{\alpha(c,y)+\xi(\psi_p(cy),t)} \tau(u_X^s\psi_p(y))ds - \int_0^{\alpha(c,y)} \tau(u_X^s\psi_p(y))ds - \int_0^{\xi(\psi_p(y),t)} \tau(u_X^s\psi_p(y))ds\\
&= \int_0^{\alpha(c,u_Y^t y)+\xi(\psi_p(y),t)} \tau(u_X^s\psi_p(y))ds - \int_0^{\xi(\psi_p(y),t)} \tau(u_X^s\psi_p(y))ds - \int_0^{\alpha(c,y)} \tau(u_X^s\psi_p(y))ds\\
&= \int_0^{\alpha(c,u_Y^t y)} \tau(u_X^s\tilde{u}_X^t(\psi_p(y)))ds - \int_0^{\alpha(c,y)} \tau(u_X^s\psi_p(y))ds.
\end{aligned}$$

In other words, we have

$$\int_0^{\alpha(c,u_Y^t y)} \tau(u_X^s\tilde{u}_X^t(x))ds = \int_0^{\alpha(c,y)} \tau(u_X^s x)ds$$

for ρ-a.e. $(x,y) \in X \times Y$ and therefore

$$\int_0^{\alpha(c,y)} \tau(u_X^s x)ds \equiv r_c$$

for some $r_c \in \mathbf{R}$. It follows that

$$\alpha(c,y) = \xi(x, r_c) \tag{5.29}$$

for ρ-a.e. $(x,y) \in X \times Y$. Moreover, we apply $\tilde{u}_X^{-r_c} \times u_Y^{-r_c}$ to (5.28), and get that

$$(x,y) \mapsto (u_X^{\alpha(c,y)} x, cy) \mapsto (x, u_Y^{-r_c} cy) \tag{5.30}$$

is ρ-invariant. In particular, suppose that G_Y is a semisimple Lie group with finite center and no compact factors and $\Gamma_Y \subset G_Y$ is a irreducible lattice. If the $\mathfrak{sl}_2$-weight decomposition $\mathfrak{g}_Y = \mathfrak{sl}_2 + V^\perp$ of $\mathfrak{g}_Y$ (see (2.3)) contains at least one $\mathfrak{sl}_2$-irreducible representation $V_\varsigma \subset V^\perp$ with a positive highest weight $\varsigma > 0$. Choosing $c = \exp(v_\varsigma)$, by *Moore's ergodicity theorem*, we must have $\rho = \mu \times \nu$ (cf. Lemma 3.1). Note that this coincides with the result obtained in [DKW20]. Besides, even if the highest weight of V_ς is $\varsigma = 0$ for any $V_\varsigma \subset V^\perp$, the only possible situation for $\rho \neq \mu \times \nu$ is that $\alpha(\exp v, y) \equiv 0$ for all $v \in V^\perp$. Thus, by (5.30), we conclude that ρ is $(\mathrm{id} \times \exp(v))$-invariant for any $v \in V^\perp$. In Section 6.2, we shall see that $\langle \exp(v) \rangle \subset G_Y$ is a normal subgroup, which leads to a contradiction. Thus, we conclude that $V^\perp = 0$ and so $\mathfrak{g}_Y = \mathfrak{sl}_2$.

5.2. **Normal direction.** Applying a similar argument in Section 5.1, we can study the behavior of $\overline{\psi}_p$ along the normal direction $N_{G_Y}(U_Y)$ of U_Y as well. Here we only study the diagonal action provided by the $\mathfrak{sl}_2$-triple. Thus, let

$$\mathrm{Span}\{U_Y, A_Y, \overline{U}_Y\} \subset \mathfrak{g}_Y, \quad \mathrm{Span}\{U_X, Y_n, \overline{U}_X\} \subset \mathfrak{g}_X$$

be $\mathfrak{sl}_2$-triples in $\mathfrak{g}_Y$, $\mathfrak{g}_X$ respectively, where Y_n is given in Section 2.1. Denote

$$a_Y^t := \exp(tA_Y), \quad a_X^t := \exp(tY_n).$$

We adopt the same notation and orderly fix the data as in Section 5.1; thus, $\sigma, \epsilon, t_{K_1}, \delta, K, K^0$ are chosen so that (5.1) (5.3) (5.4) hold. (Here we further assume $\delta < \epsilon$.) Fix $|t_0| < \delta$, $a_Y = a_Y^{t_0}$ and $a_X = a_X^{t_0}$. By ergodic theorem, there is $A_{a_Y,y} \subset \mathbf{R}^+$ and $\lambda_0 > 0$ such that

- for $r \in A_{a_Y,y}$, we have
$$u_Y^r y, a_Y u_Y^r y \in K^0;$$
- $\mathrm{Leb}(A_{a_Y,y} \cap [\lambda', \lambda'']) \geq (1-2\sigma)(\lambda''-\lambda')$ whenever $\lambda''-\lambda' \geq \lambda_0$ and $\lambda' \in A_{a_Y,y}$.

Then by the assumptions, we have

$$A_{a_Y,y} \subset \left\{ r \in [0,\infty) : d_{\overline{X}}(a_X \overline{\psi}_p(u_Y^r y), \overline{\psi}_p(a_Y u_Y^r y)) < 2\epsilon,\ p \in \{1,\dots,n\} \right\}. \tag{5.31}$$

It follows that for $r \in A_{a_Y,y}$, we have

$$\begin{aligned}2\epsilon >&d_{\overline{X}}(a_X\overline{\psi}_p(u_Y^r y), \overline{\psi}_p(a_Y u_Y^r y))\\ =&d_{\overline{X}}(a_X\overline{\psi}_p(u_Y^r y), \overline{\psi}_p(u_Y^{e^{-t_0}r} a_Y y))\\ =&d_{\overline{X}}\left(a_X u_X^{z(y,r)}\overline{\psi}_{i_p(y,r)}(y), u_X^{z(a_Y y, e^{-t_0}r)}\overline{\psi}_{i_p(a_Y y, e^{-t_0}r)}(a_Y y)\right)\\ =&d_{\overline{X}}\left(u_X^{e^{-t_0}z(y,r)} a_X\overline{\psi}_{i_p(y,r)}(y), u_X^{z(a_Y y, e^{-t_0}r)}\overline{\psi}_{i_p(a_Y y, e^{-t_0}r)}(a_Y y)\right)\end{aligned}$$

for any $p \in \{1, \dots, n\}$ (cf. (5.9)).

Assume that $0 \in A_{a_Y,y}$. Let $I = ((p_1, p_2), \dots, (p_{2n-1}, p_{2n})) \in \{1, \dots, n\}^{2n}$ be a sequence of indexes and

$$A^I_{a_Y,y} \coloneqq \{r \in A_{a_Y,y} : p_{2k-1} = i_k(y,r),\ p_{2k} = i_k(a_Y y, e^{-t_0}r) \text{ for all } k \in \{1, \dots, n\}\}.$$

Then $A = A^I_{a_Y,y}$, $R_0 = t_{K_1}$, $s(r) = e^{-t_0}z(y,r)$, $t(r) = z(a_Y y, e^{-t_0}r)$ satisfy (4.32) (4.33) for points

$$a_X\overline{\psi}_{p_{2k-1}}(y) \in \overline{X}, \quad \overline{\psi}_{p_{2k}}(a_Y y) \in K$$

for all $k \in \{1, \dots, n\}$. We can then apply Proposition 4.19 to $A_{a_Y,y} = \coprod_{I\in\{1,\dots,n\}^{2n}} A^I_{a_Y,y}$ for any $\lambda \geq \lambda_0$. Then we follow the same argument as in Proposition 5.1 (see also Corollary 5.2), and obtain

Proposition 5.9. *There is a measurable map $\varpi : \exp(\mathbf{R}A_Y) \times X \times Y \to C_{G_X}(U_X)$ that induces a map $\widetilde{S}_{a_Y^r} : \operatorname{supp}(\rho) \to \operatorname{supp}(\rho)$ by*

$$\widetilde{S}_{a_Y^r} : (x, y) \mapsto (\varpi(a_Y^r, x, y)a_X^r x, a_Y^r y) \tag{5.32}$$

for all $r \in \mathbf{R}$, ρ-a.e. $(x, y) \in X \times Y$. Moreover, we have

$$\varpi(a_Y^r, x, y) = u_X^{-z(a_Y y,t)}\varpi(a_Y^r, (u_X^{z(y,e^r t)} \times \tilde{u}_Y^{z(y,e^r t)}).(x, y))u_X^{e^{-r}z(y,e^r t)} \tag{5.33}$$

$$\varpi(a_Y^{r_1+r_2}, x, y) = \varpi(a_Y^{r_1}, \varpi(a_Y^{r_2}, x, y)a_X^{r_2}x, a_Y^{r_2}y)a_X^{r_1}\varpi(a_Y^{r_2}, x, y)a_X^{-r_1} \tag{5.34}$$

for $r, r_1, r_2 \in \mathbf{R}$, ρ-a.e. $(x, y) \in X \times Y$, $t \in \mathbf{R}$.

Similar to the discussion after Corollary 5.2, we consider the decomposition (2.7) and write

$$\varpi(a_Y^r, x, y) = u_X^{\alpha(a_Y^r,x,y)}\beta(a_Y^r, x, y) \tag{5.35}$$

where $\alpha(a_Y^r, x, y) \in \mathbf{R}$ and $\beta(a_Y^r, x, y) \in \exp V_{C_X}^{\perp}$. Then by (5.33), we have

$$z(a_Y^r y, t) + \alpha(a_Y^r, x, y) = \alpha(a_Y^r, (u_X^{z(y,e^r t)} \times \tilde{u}_Y^{z(y,e^r t)}).(x, y)) + e^{-r}z(y, e^r t), \tag{5.36}$$

$$\beta(a_Y^r, x, y) \equiv \beta(a_Y^r, (u_X^{z(y,e^r t)} \times \tilde{u}_Y^{z(y,e^r t)}).(x, y)) \tag{5.37}$$

for all $r, t \in \mathbf{R}$. The same argument then shows that

$$\alpha(a_Y^r, x, y) \equiv \alpha(a_Y^r, y), \quad \beta(a_Y^r, x, y) \equiv \beta(a_Y^r)$$

for all $r \in \mathbf{R}$, ρ-a.e. $(x, y) \in X \times Y$. Besides, following the same lines as in Theorem 5.4, we obtain Theorem 1.2:

Theorem 5.10 (Extra normal invariance of ρ)**.** *For any $a_Y \in \exp(\mathbf{R}A_Y)$, the map $S_{a_Y} : X \times Y \to X \times Y$ defined by*

$$S_{a_Y} : (x, y) \mapsto \left(\beta(a_Y)a_X x, \tilde{u}_Y^{-\alpha(a_Y,y)}(a_Y y)\right)$$

satisfies

$$S_{a_Y^r} \circ (u_X^t \times \tilde{u}_Y^t) = (u_X^{e^{-r}t} \times \tilde{u}_Y^{e^{-r}t}) \circ S_{a_Y^r}$$

and is ρ-invariant. Besides, $S_{a_Y^{r_1+r_2}} = S_{a_Y^{r_1}} S_{a_Y^{r_2}}$ for any $r_1, r_2 \in \mathbf{R}$. Also, we have

$$S_{a_Y} \circ S_c \circ S_{a_Y^{-1}} = S_{a_Y c a_Y^{-1}}$$

for any $a_Y \in \exp(\mathbf{R}A_Y)$, $c \in C_{G_Y}(U_Y)$.

Corollary 5.11 (Extra normal invariance of ν)**.** *For any $a_Y \in \exp(\mathbf{R}A_Y)$, the map $S_{a_Y}^Y : Y \to Y$ defined by*

$$S_{a_Y}^Y : y \mapsto \tilde{u}_Y^{-\alpha(a_Y,y)}(a_Y y)$$

satisfies

$$S_{a_Y^r}^Y \circ \tilde{u}_Y^t = \tilde{u}_Y^{e^{-r}t} \circ S_{a_Y^r}^Y$$

and is ν-invariant. Besides, $S_{a_Y^{r_1+r_2}}^Y = S_{a_Y^{r_1}}^Y S_{a_Y^{r_2}}^Y$ for any $r_1, r_2 \in \mathbf{R}$. Also, we have

$$S_{a_Y}^Y \circ S_c^Y \circ S_{a_Y^{-1}}^Y = S_{a_Y c a_Y^{-1}}^Y$$

for any $a_Y \in \exp(\mathbf{R}A_Y)$, $c \in C_{G_Y}(U_Y)$.

Theorem 5.12. *Let $\tau_Y \in \mathbf{K}_\kappa(Y)$. Suppose that there is an ergodic joining $\rho \in J(u_X^t, \phi_t^{U_Y,\tau_Y})$. Then $\tau_Y(y)$ and $\tau_Y(a_Y y)$ are (measurably) cohomologous along u_Y^t for all $a_Y^r \in \exp(\mathbf{R}A_Y)$. More precisely, the transfer function can be taken to be*

$$F_{a_Y^r}(y) = e^r \alpha(a_Y^r, y).$$

Proof. By (5.20), for m_Y-a.e. $y \in Y$, $x \in \operatorname{supp}(\rho_y)$, we have

$$\begin{aligned}
&e^{-r} \int_0^{e^r t} \tau(u_Y^s y) - \tau(a_Y^r u_Y^s y) ds \\
=&e^{-r} \int_0^{e^r t} \tau(u_Y^s y) ds - \int_0^t \tau(u_Y^s a_Y y) ds \\
=&e^{-r} z(y, e^r t) - z(a_Y^r y, t) \\
=&\alpha(a_Y^r, y) - \alpha(a_Y^r, \tilde{u}^{z(y,e^r t)}(y)) \\
=&\alpha(a_Y^r, y) - \alpha(a_Y^r, u_Y^{e^r t} y).
\end{aligned}$$

Thus, we can take the transfer function as

$$F_{a_Y^r}(y) \coloneqq e^r \alpha(a_Y^r, y).$$

Then $\tau(y)$ and $\tau(a_Y y)$ are (measurably) cohomologous for all $a_Y \in \exp(\mathbf{R}A_Y)$. $\square$

5.3. **Opposite unipotent direction.** Now we shall study the opposite unipotent direction $\overline{u}_Y^r = \exp(r\overline{U}_Y)$, $\overline{u}_X^r = \exp(r\overline{U}_X)$. Unlike previous sections, we cannot directly obtain ρ is invariant under the opposite unipotent direction. However, we compensate it by making the "a-adjustment". More precisely, by choosing appropriate coefficients $\lambda_k > 0$, set

$$\Psi_{k,p}(y) \coloneqq a_X^{\lambda_k}\overline{\psi}_p(a_Y^{-\lambda_k}y)$$

for a.e. $y \in Y$. Then we shall show that (see Theorem 5.15)

$$\lim_{n\to\infty} d_{\overline{X}}(\Psi_{k,p}(\overline{u}_Y^r y), \overline{u}_X^r \Psi_{k,p}(y)) = 0.$$

Here we adopt the argument given by Ratner [Rat87] and make a slight generalization. It is again convenient to consider $u, a, \overline{u} \in SL(2, \mathbf{R})$ as (2×2)-matrices. We first introduce a basic lemma by Ratner that estimates the time-difference of the $\phi_t^{U_Y,\tau}$-flow under the $\overline{u}_Y^r$-direction.

First of all, one directly calculates

$$\begin{aligned} (5.38) \quad u_Y^t\overline{u}_Y^r &= \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}\begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & r \\ t & 1+rt \end{bmatrix} \\ &= \begin{bmatrix} 1 & \frac{r}{1+rt} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{1+rt} & \\ & 1+rt \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \frac{t}{1+rt} & 1 \end{bmatrix} = \overline{u}_Y^{\frac{r}{1+rt}} a_Y^{-2\log(1+rt)} u_Y^{\frac{t}{1+rt}}. \end{aligned}$$

We are interested in the *fastest relative motion* of u_Y^t-shearing

$$(5.39) \quad \Delta_r(t) \coloneqq t - \frac{t}{1+rt} \quad \text{and} \quad \Delta_r^{\tau_Y}(y,t) \coloneqq \int_0^t \tau_Y(u_Y^s\overline{u}_Y^r y)ds - \int_0^{\frac{t}{1+rt}} \tau_Y(u_Y^s y)ds.$$

Lemma 5.13 ([Rat87] Lemma 1.2)**.** *Assume $\tau_Y \in C^1(Y)$. Then given sufficiently small $\epsilon > 0$, there are*

- $\delta = \delta(\epsilon) \approx 0$,
- $l = l(\epsilon) > 0$,
- $E = E(\epsilon) \subset Y$ *with* $\mu(E) > 1 - \epsilon$

such that if $y, \overline{u}_Y^r y \in E$ for some $|r| \le \delta/l$ then

$$(5.40) \qquad |\Delta_r^{\tau_Y}(y,t) - \Delta_r(t)| \le O(\epsilon)|\Delta_r(t)|$$

for all $t \in [l, \delta|r|^{-1}]$.

Proof. Denote

$$\tau_a(y) = \lim_{t\to 0}\frac{\tau_Y(a_Y^t y) - \tau_Y(y)}{t}, \qquad \tau_{\overline{u}}(y) = \lim_{t\to 0}\frac{\tau_Y(\overline{u}_Y^t y) - \tau_Y(y)}{t}.$$

The function τ_g, τ_k are continuous on Y and

$$(5.41) \qquad |\tau_Y(y)|, |\tau_a(y)|, |\tau_{\overline{u}}(y)| \le \|\tau_Y\|_{C^1(Y)}$$

for all $y \in Y$. Besides, we have

$$\int_Y \tau_a(y)dm_Y(y) = \int_Y \tau_{\overline{u}}(y)dm_Y(y) = 0.$$

Given $\epsilon > 0$, we fix the data as follows:

- Let $K \subset Y$ be an open subset of Y such that $\overline{K}$ is compact and
$$m_Y(K) > 1-\epsilon, \quad m_Y(\partial K) = 0$$
where ∂K denotes the boundary of K.
- Fix a sufficiently small $\delta' = \delta'(\epsilon) \approx 0$ such that
  (1) $\mu(B(\partial K, \delta')) \leq \epsilon$ where $B(\partial K, \delta')$ denotes the δ'-neighborhood of ∂K (It follows that $\mu(K \setminus B(\partial K, \delta')) \geq 1 - 2\epsilon$);
  (2) if $y_1, y_2 \in \overline{K}$, $d_Y(y_1, y_2) \leq \delta'$ then
$$|\tau_a(y_1) - \tau_a(y_2)| \leq \epsilon. \tag{5.42}$$
- Fix $\delta \in (0, \frac{1}{100}\delta')$ such that if $|rt| \leq \delta$ then for all $s \in [0,t]$
$$|\epsilon_{1,t}(s)| \leq \epsilon, \quad \text{where} \quad \epsilon_{1,t}(s) := \frac{\frac{1}{(1+rs)^2} - 1}{\frac{1}{t}\Delta_r(t)} - \frac{2s}{t}. \tag{5.43}$$
- Fix $t_1 = t_1(\epsilon) > 0$ and a subset $E = E(\epsilon) \subset Y$ with $m_Y(E) > 1-\epsilon$ such that if $y \in E$, $t \in [t_1, \infty)$, then the relative length measure of $K \setminus B(\partial K, \delta')$ on the orbit interval $[y, u_Y^t y]$ is at least $1 - 3\epsilon$ and $|\epsilon_2(t)| \leq \epsilon$, $|\epsilon_3(t)| \leq \epsilon$, where
$$\epsilon_2(t) := \frac{1}{t}\int_0^t \tau_Y(u_Y^s y)ds - 1, \quad \epsilon_3(t) := \frac{1}{t}\int_0^t \tau_a(u_Y^s y)ds. \tag{5.44}$$
- Fix $l = l(\epsilon) > t_1$ such that
$$t_1/l \leq \epsilon. \tag{5.45}$$

We shall show that if $y, \overline{u}_Y^r y \in E$ for some $|r| \leq \delta/l$, and $t \in [l, \delta|r|^{-1}]$ then (5.40) holds if ϵ is sufficiently small.

Now let us estimate $\Delta_r^{\tau_Y}(y,t)$. Recall that
$$\Delta_r^{\tau_Y}(y,t) = \int_0^t \tau_Y(u_Y^s \overline{u}_Y^r y)ds - \int_0^{\frac{t}{1+rt}} \tau_Y(u_Y^s y)ds.$$
Then by (5.38) and the mean value theorem, we have
$$\begin{aligned}
\int_0^{\frac{t}{1+rt}} \tau_Y(u_Y^s y)ds &= \int_0^t \tau_Y(u_Y^{\frac{s}{1+rs}} y) \cdot \frac{ds}{(1+rs)^2} \\
&= \int_0^t \tau_Y(a_Y^{2\log(1+rs)} \overline{u}_Y^{-\frac{r}{1+rs}} u_Y^s \overline{u}_Y^r y) \cdot \frac{ds}{(1+rs)^2} \\
&= \int_0^t \tau_Y(u_Y^s \overline{u}_Y^r y) \cdot \frac{ds}{(1+rs)^2} \\
&\quad - \int_0^t \frac{r}{1+rs} \tau_{\overline{u}}(\overline{u}_Y^{k_s} u_Y^s \overline{u}_Y^r y) \cdot \frac{ds}{(1+rs)^2} \\
&\quad + \int_0^t 2\log(1+rs) \tau_a(a_Y^{g_s} \overline{u}_Y^{-\frac{r}{1+rs}} u_Y^s \overline{u}_Y^r y) \cdot \frac{ds}{(1+rs)^2}
\end{aligned}$$

where $k_s \in \left[-\frac{r}{1+rs}, 0\right]$ and $g_s \in [0, 2\log(1+rs)]$. This implies

$$\begin{aligned}
\Delta_r^{\tau_Y}(y,t) =& \int_0^t \tau_Y(u_Y^s \overline{u}_Y^r y)\left(1 - \frac{1}{(1+rs)^2}\right) ds \\
&+ \int_0^t \frac{r}{1+rs}\tau_{\overline{u}}(\overline{u}_Y^{k_s} u_Y^s \overline{u}_Y^r y) \cdot \frac{ds}{(1+rs)^2} \\
&- \int_0^t 2\log(1+rs)\tau_a(a_Y^{g_s}\overline{u}_Y^{-\frac{r}{1+rs}} u_Y^s \overline{u}_Y^r y) \cdot \frac{ds}{(1+rs)^2} \\
=& J_1 + J_2 + J_3.
\end{aligned}$$

We estimate the integrals J_1, J_2, J_3 separately:

(1) Using (5.43) (5.44), we have

$$\begin{aligned}
J_1 =& 2\Delta_r(t)\frac{1}{t^2}\int_0^t s\tau_Y(u_Y^s \overline{u}_Y^r y) ds + \Delta_r(t)\frac{1}{t}\int_0^t \epsilon_{1,t}(s)\tau_Y(u_Y^s \overline{u}_Y^r y) ds \\
=& 2\Delta_r(t)\frac{1}{t^2}\int_0^t s\tau_Y(u_Y^s \overline{u}_Y^r y) ds + \Delta_r(t) O(\epsilon)
\end{aligned}$$

since $\overline{u}_Y^r y \in E$. Now by the integration by parts and (5.44), (5.41) (5.45), we have

$$\begin{aligned}
&\frac{1}{t^2}\int_0^t s\tau_Y(u_Y^s \overline{u}_Y^r y) ds \\
=& \frac{1}{t}\int_0^t \tau_Y(u_Y^s \overline{u}_Y^r y) ds - \frac{1}{t^2}\int_0^t \left(\int_0^s \tau_Y(u_Y^p \overline{u}_Y^r y) dp\right) ds \\
=& 1 + \epsilon_2(t) - \frac{1}{t^2}\left[\int_{t_1}^t + \int_0^{t_1}\right]\left(\int_0^s \tau_Y(u_Y^p \overline{u}_Y^r y) dp\right) ds \\
=& 1 + \epsilon_2(t) - \frac{1}{t^2}\int_{t_1}^t s\left(1 + \epsilon_2(s)\right) ds + O(\epsilon) = \frac{1}{2} + O(\epsilon).
\end{aligned}$$

It follows that

$$\left|\frac{J_1}{\Delta_r(t)} - 1\right| \leq O(\epsilon).$$

(2) For J_2, by (5.45), we have

$$|J_2| = \left|\int_0^t \frac{r}{1+rs}\tau_{\overline{u}}(\overline{u}_Y^{k_s} u_Y^s \overline{u}_Y^r y) \cdot \frac{ds}{(1+rs)^2}\right| \leq O\left(\frac{|\Delta_r(t)|}{t}\right) \leq O(\epsilon)|\Delta_r(t)|.$$

(3) Note that since $d_Y(a_Y^{g_s}\overline{u}_Y^{-\frac{r}{1+rs}} u_Y^s \overline{u}_Y^r y, u_Y^s \overline{u}_Y^r y) < \delta'$, we know $a_Y^{g_s}\overline{u}_Y^{-\frac{r}{1+rs}} u_Y^s \overline{u}_Y^r y \in \overline{K}$ if $u_Y^s \overline{u}_Y^r y \in K \setminus B(\partial K, \delta')$. Now set

$$I_y \coloneqq \{s \in [0,t] : u_Y^s \overline{u}_Y^r y \in K \setminus B(\partial K, \delta')\}.$$

Then by (5.44), one has $\mathrm{Leb}(I_y^c) < 3\epsilon t$. Then for J_3, using (5.41) and (5.42), we have

$$\begin{aligned}
&\left|J_3 - \left(-\int_0^t 2\log(1+rs)\tau_a(u_Y^s\overline{u}_Y^r y)\cdot\frac{ds}{(1+rs)^2}\right)\right| \\
\ll&\left|\log(1+rt)\right|\left[\int_{I_y}\left|\tau_a(a_Y^{g_s}\overline{u}_Y^{-\frac{r}{1+rs}}u_Y^s\overline{u}_Y^r y)-\tau_a(u_Y^s\overline{u}_Y^r y)\right|ds+\epsilon t\|\tau_Y\|_{C^1(Y)}\right] \\
\leq& t\left|\log(1+rt)\right|(\epsilon+\epsilon\|\tau_Y\|_{C^1(Y)}) \ll O(\epsilon)\left|\Delta_r(t)\right|.
\end{aligned}$$

We also have

$$\begin{aligned}
&\left|\int_0^t 2\log(1+rs)\tau_a(u_Y^s\overline{u}_Y^r y)\cdot\frac{ds}{(1+rs)^2}-\int_0^t 2\log(1+rs)\tau_a(u_Y^s\overline{u}_Y^r y)ds\right| \\
=&\left|\int_0^t 2\log(1+rs)\tau_a(u_Y^s\overline{u}_Y^r y)\cdot\left(\frac{1}{(1+rs)^2}-1\right)ds\right| \\
\ll&|\Delta_r(t)|\|\tau_Y\|_{C^1(Y)}\delta \ll O(\epsilon)|\Delta_r(t)|.
\end{aligned}$$

Finally, by using the integration by parts, we get

$$\begin{aligned}
&\left|\int_0^t \log(1+rs)\tau_a(u_Y^s\overline{u}_Y^r y)ds\right| \\
=&\left|\log(1+rt)\int_0^t\tau_a(u_Y^s\overline{u}_Y^r y)ds-\int_0^t\left(\int_0^s\tau_a(u_Y^p\overline{u}_Y^r y)dp\right)\frac{r}{1+rs}ds\right| \\
\ll&\frac{|\Delta_r(t)|}{t}\left|\int_0^t\tau_a(u_Y^s\overline{u}_Y^r y)ds\right|+\frac{|\Delta_r(t)|}{t^2}\left|\int_0^t\left(\int_0^s\tau_a(u_Y^p\overline{u}_Y^r y)dp\right)ds\right| \\
=&\epsilon_3(t)|\Delta_r(t)|+\frac{|\Delta_r(t)|}{t^2}\left|\left[\int_0^{t_1}+\int_{t_1}^t\right]\left(\int_0^s\tau_a(u_Y^p\overline{u}_Y^r y)dp\right)ds\right| \\
\ll&\epsilon_3(t)|\Delta_r(t)|+\frac{|\Delta_r(t)|}{t^2}\left|t_1^2\|\tau_Y\|_{C^1(Y)}+\int_{t_1}^t s\epsilon_3(s)ds\right| \ll O(\epsilon)|\Delta_r(t)|.
\end{aligned}$$

Thus, we conclude that $|J_3| \leq O(\epsilon)|\Delta_r(t)|$.

Therefore, combining the above estimates, we have

$$|\Delta_r^{\tau_Y}(y,t)-\Delta_r(t)| \leq O(\epsilon)\Delta_r(t).$$

This completes the proof of the lemma. □

The following lemma tells us that we only need to know the fastest relative motion at finitely many different time points to determine the difference of two nearby points.

Lemma 5.14 (Shearing comparison)**.** *Given $\epsilon > 0$, let $x, y, z \in \overline{X}$ be three ϵ-nearby points such that the fastest relative motions between the pairs (x, z) and (y, z) at* *time* $t > 0$ *are* $q_1(t)$ *and* $q_2(t)$ *respectively. Assume that there are* $s_1, s_2 > 0$ *with* $s_1 \in [\frac{1}{3}s_2, \frac{2}{3}s_2]$ *such that*

$$d_{\overline{X}}(u_X^{s_i}x, u_X^{s_i}q_1(s_i)z) < \epsilon, \quad d_{\overline{X}}(u_X^{s_i}y, u_X^{s_i}q_2(s_i)z) < \epsilon, \quad d_{G_X}(q_1(s_i), q_2(s_i)) < \epsilon$$

for $i \in \{1, 2\}$. *Then we have*

$$d_{\overline{X}}(u_X^t x, u_X^t y) < O(\epsilon) \tag{5.46}$$

for $t \in [0, s_2]$.

Proof. This is a direct consequence of Lemma 4.4. Assume that $x = gy$, $x = h_1 z$, $y = h_2 z$ for some $g, h_1, h_2 \in G_X$. Then by the definition (3.4), there are $\delta_1(t)$, $\delta_2(t) \in G_X$ with $d_{G_X}(\delta_1(t), e) < \epsilon$, $d_{G_X}(\delta_2(t), e) < \epsilon$ such that

$$u_X^t h_1 u_X^{-t} = \delta_1(t) q_1(t), \quad u_X^t h_2 u_X^{-t} = \delta_2(t) q_2(t)$$

for $t \in [0, s]$. By the assumption, we have

$$u_X^t g u_X^{-t} = u_X^t h_1 h_2^{-1} u_X^{-t} = \delta_1(t) q_1(t) q_2(t)^{-1} \delta_2(t)^{-1} \tag{5.47}$$

and

$$q_1(s_1) q_2(s_1)^{-1} < \epsilon, \quad q_1(s_2) q_2(s_2)^{-1} < \epsilon \tag{5.48}$$

Note that $q_1(t)q_2(t)^{-1} \in C_{G_X}(U_X)$ and so their corresponding vectors in the Lie algebra are polynomials of t with the degree at most 2 (see (2.5) (3.4)); Thus, we can write

$$h_1 h_2^{-1} = \exp\left(\sum_j \sum_{i=0}^{\varsigma(j)} b_j^i v_j^i\right), \quad q_1(t) q_2(t)^{-1} = \exp\left(\sum_j p_j(t) v_j^{\varsigma(j)}\right)$$

where $p_j(t) = \sum_{i=0}^{\varsigma(j)} b_j^{\varsigma(j)-i} \binom{\varsigma(j)}{i} t^i$ is a polynomial having the degree at most 2, $|b_i| < \epsilon$, $v_j^i \in V_j$ is the i-th weight vector of the $\mathfrak{sl}_2$-irreducible representation V_j. Then (5.48) and the proof of Lemma 4.4 (1) with $\kappa = 1$ imply that

$$|b_j^{\varsigma(j)-i}| < O(\epsilon) s_2^{-i}. \tag{5.49}$$

It follows that for $t \in [0, s_2]$

$$|p_j(t)| < O(\epsilon) \quad \text{and so} \quad q_1(t) q_2(t)^{-1} < O(\epsilon).$$

Then by (5.47), we obtain (5.46). $\square$

Next, we shall prove Theorem 5.15. The idea is to consider the fastest relative motion of the pairs $(\Psi_{k,p}(\overline{u}_Y^r y), \Psi_{k,p}(y))$ and $(\overline{u}_X^r \Psi_{k,p}(y), \Psi_{k,p}(y))$ at finitely many time points. And then apply Lemma 5.14. First, we orderly fix the following data:

- (Injectivity radius) Since Γ_X is discrete, there is a compact $K_1 \subset \overline{X}$ with $\nu(\overline{\psi}_p^{-1}(K_1)) > \frac{999}{1000}$ and $D_1 = D_1(K_1) > 0$ such that if $\overline{g} \in \overline{P}^{-1}(K_1)$, then D_1 is an isometry on the ball $B_{C^\rho \backslash G_X}(\overline{g}, D_1)$ of radius D_1 centered at $\overline{g}$. Here $\overline{P} : C^\rho \backslash G_X \to C^\rho \backslash G_X / \Gamma_X = \overline{X}$ is the projection

$$\overline{P} : C^\rho g \mapsto C^\rho g \Gamma_X.$$

- (Distinguishing $\overline{\psi}_p, \overline{\psi}_q$) There is $K_2 \subset Y$ with $\nu(K_2) > \frac{999}{1000}$ such that

$$d_{\overline{X}}(\overline{\psi}_p(y), \overline{\psi}_q(y)) > D_2 \tag{5.50}$$

  for $y \in K_2$, $1 \leq p < q \leq n$.
- Define $D = \min\{D_1, D_2, 1\}$.
- (Lemma 5.13) Let $\delta_k = \min\left\{\delta\left(\frac{1}{10}2^{-k}D\right), \frac{1}{10}2^{-k}D\right\}$, $l_k = l\left(\frac{1}{10}2^{-k}D\right)$ and $E_k = E\left(\frac{1}{10}2^{-k}D\right) \subset Y$ be as in Lemma 5.13 for τ_Y.
- (Lusin's theorem) There is $K'_k \subset Y$ such that $\nu(K'_k) > 1 - \frac{1}{10}2^{-k}$ and $\overline{\psi}_p|_{K'_k}$ is uniformly continuous for all $p \in \{1, \ldots, n\}$. Thus, for any $\epsilon > 0$, there is $\delta'(\epsilon) > 0$ such that for $p \in \{1, \ldots, n\}$, $d_Y(y_1, y_2) < \delta'(\epsilon)$ and $y_1, y_2 \in K'_k$, we have

$$d_{\overline{X}}(\overline{\psi}_p(y_1), \overline{\psi}_p(y_2)) < \epsilon. \tag{5.51}$$

  Let $\delta'_k = \min\left\{\delta'\left(\frac{1}{10}2^{-k}D\right), \frac{1}{10}2^{-k}D\right\}$.
- (Ergodicity) Fix $\tau_Y \in C^1(Y)$. By the ergodicity of unipotent flows, there are $T_k \geq \max\{l_k, 20\delta_k^{-1}, 20\delta_k'^{-1}\}$ and subsets $K''_k \subset Y$ with $\nu(K''_k) > 1 - \frac{1}{10}2^{-k}$ such that if $y \in K''_k$, $t \geq T_k$ then
  (1) the relative length measure of $K'_k \cap E_k \cap K_2 \cap \bigcap_p \overline{\psi}_p^{-1}(K_1)$ on the orbit interval $[y, u_Y^t y]$ is at least $\frac{998}{1000}$;
  (2) we have by the ergodic theorem

$$\left|\frac{1}{t}z(y,t) - 1\right| = \left|\frac{1}{t}\int_0^t \tau_Y(u_Y^s y)ds - 1\right| \leq \frac{1}{10}2^{-k}D. \tag{5.52}$$

- (Fastest relative motion)
  (1) For $r \in \mathbf{R}$, let $L_1^i(r)$ denote the first $t > 0$ with $\Delta_r(t) = i^2 D/10$ for $i \in \{1, 2\}$ where $\Delta_r(t)$ is defined in (5.39). Note that for sufficiently small r, one may calculate that

$$L_1^1(r) \in [\frac{9}{20}L_1^2(r), \frac{11}{20}L_1^2(r)]. \tag{5.53}$$

  (2) As in (4.4), for $\overline{x_1}, \overline{x_2} \in \overline{X}$ close enough, we can write $\overline{x_1} = \overline{gx_2}$ where $g = \exp(v)$ for $v \in \mathfrak{sl}_2 + V^{\rho\perp}$. Then the H-property (Remark 3.4) tells us that at time $t \in \mathbf{R}$, the *fastest relative motion* is given by

$$q(\overline{x_1}, \overline{x_2}, t) = \pi_{C_{\mathfrak{g}_{\overline{X}}}(U_X)} \operatorname{Ad}(u_X^t).v.$$

  Then let $L_2^i(\overline{x_1}, \overline{x_2})$ denote the first $t > 0$ with $\|q(\overline{x_1}, \overline{x_2}, t)\| = i^2 D/10$. For $y \in Y$, $i \in \{1, 2\}$, let

$$L^i(y, r) \coloneqq \min\left\{L_1^i(r), L_2^i(\overline{\psi}_1(\overline{u}^r y), \overline{\psi}_1(y)), \ldots, , L_2^i(\overline{\psi}_n(\overline{u}^r y), \overline{\psi}_n(y))\right\}. \tag{5.54}$$

  By applying Theorem 3.3 to $Q = B_{C_{G_Y}(U_Y)}(e, i^2 D/10)$ and $\epsilon = \frac{1}{10}2^{-k}$, we can choose small $0 < \omega_k \leq \min\{\delta_k, \delta'_k\}$ such that if $|r| \leq \omega_k$, $y, \overline{u}^r y \in K'_k$,

$i \in \{1,2\}$, then we have

$$L^i = L^i(y,r) \geq \max\left\{10T_k, \frac{10i^2D}{\delta_k'}\right\} \tag{5.55}$$

and for all $p \in \{1,\dots,p\}$

$$\|q_p^i\| \leq \frac{i^2D}{10}, \quad d_{\overline{X}}\left(u_X^L\overline{\psi}_p(\overline{u}^r y), u_X^L\overline{q_p^i(L)\psi_p(y)}\right) \leq \frac{1}{10}2^{-k}D \tag{5.56}$$

where $q_p^i = q(\overline{\psi}_p(\overline{u}^r y), \overline{\psi}_p(y), L^i)$.

Now let

$$K_k^0 := K_k' \cap K_k'' \cap E_k. \tag{5.57}$$

It follows that $\nu(K_k^0) > 1 - 2^{-k}$. Let

(5.58)

$$\lambda_k := 2\cdot\max\left\{\log\frac{10}{\omega_k}, \log T_k\right\}, \quad \Omega := \bigcup_{l\geq 1}\bigcap_{k\geq l} a_Y^{\lambda_k}(K_k^0), \quad \Psi_{k,p}(y) := a_X^{\lambda_k}\overline{\psi}_p(a_Y^{-\lambda_k}y).$$

It follows that $\nu(\Omega) > 1$.

Theorem 5.15. *Let the notation and assumption be as above. Then for $r \in \mathbf{R}$, $y \in \Omega$, we have*

$$\lim_{n\to\infty} d_{\overline{X}}(\Psi_{k,p}(\overline{u}_Y^r y), \overline{u}_X^r\Psi_{k,p}(y)) = 0.$$

Proof. Suppose that $y, \overline{u}_Y^r y \in \bigcup_{l\geq 1}\bigcap_{k\geq l} a_Y^{\lambda_k}(K_k^0)$. Then $y, \overline{u}_Y^r y \in a_Y^{\lambda_k}(K_k^0)$ for sufficiently large k. For $r \in \mathbf{R}$, let $r_k = e^{-\lambda_k}r$. Then for sufficiently large k,

$$a_Y^{-\lambda_k}\overline{u}_Y^r y = \overline{u}_Y^{r_k}a_Y^{-\lambda_k}y \quad \text{and} \quad |r_k| \leq |r|\omega_k^2 \leq \omega_k.$$

Thus, (5.55) holds true for $L^i(y,r_k)$ for any sufficient large k, $i \in \{1,2\}$. In the following, we fix $i = 1$ (for the case $i = 2$ is similar).

Next, since by (5.55) $L^1(y,r_k) > 10T_k$, there exists $t_k \in \left[\frac{98}{100}L^1(y,r_k), \frac{99}{100}L^1(y,r_k)\right]$ such that

$$u_Y^{t_k}a_Y^{-\lambda_k}\overline{u}_Y^r y,\ u_Y^{t_k'}a_Y^{-\lambda_k}y \in K_k' \cap K_2 \cap \bigcap_p \overline{\psi}_p^{-1}(K_1) \tag{5.59}$$

where $t_k' := \frac{t_k}{1+r_kt_k}$. Then by (5.38), we get

$$\begin{aligned} d_Y(u_Y^{t_k}a_Y^{-\lambda_k}\overline{u}_Y^r y, u_Y^{t_k'}a_Y^{-\lambda_k}y) &= d_Y(u_Y^{t_k}\overline{u}_Y^{r_k}a_Y^{-\lambda_k}y, u_Y^{t_k'}a_Y^{-\lambda_k}y) \\ &= d_Y\left(\begin{bmatrix} \frac{1}{1+r_kt_k} & r_k \\ 0 & 1+r_kt_k \end{bmatrix} u_Y^{t_k'}a_Y^{-\lambda_k}y, u_Y^{t_k'}a_Y^{-\lambda_k}y\right) \leq \min\{\delta_k, \delta_k'\} \end{aligned} \tag{5.60}$$

where the last inequality follows from (5.39)

$$|r_kt_k| \leq 2\frac{\Delta_{r_k}(t_k)}{t_k} \leq 4\frac{\Delta_{r_k}(L^1(y,r_k))}{T_k} \leq 4\frac{D}{10}\cdot\frac{\min\{\delta_k,\delta_k'\}}{20} \leq \min\{\delta_k,\delta_k'\}. \tag{5.61}$$

This implies via Lemma 5.13 that

$$|\Delta^{\tau_Y}_{r_k}(a_Y^{-\lambda_k}y,t_k)-\Delta_{r_k}(t_k)|\le\frac{1}{10}2^{-k}D \tag{5.62}$$

since $a_Y^{-\lambda_k}y,\overline{u}_Y^{r_k}a_Y^{-\lambda_k}y\in E_k$ and $t_k\in[T_k,\delta_k|r_k|^{-1}]\subset[l_k,\delta_k|r_k|^{-1}]$.

Next, consider

$$u_X^{s_k}\overline{\psi}_p(a_Y^{-\lambda_k}\overline{u}_Y^r y)=\overline{\psi}_{i(p,k)}(u_Y^{t_k}a_Y^{-\lambda_k}\overline{u}_Y^r y),\quad u_X^{h'_k}\overline{\psi}_p(a_Y^{-\lambda_k}y)=\overline{\psi}_{j(p,k)}(u_Y^{t'_k}a_Y^{-\lambda_k}y)$$

where s_k and h'_k are defined by

$$z(a_{Y,k}^{-1}\overline{u}^r y,t_k)=s_k,\quad z(a_{Y,k}^{-1}y,t'_k)=h'_k. \tag{5.63}$$

Then $\Delta^{\tau_Y}_{r_k}(a_{Y,k}^{-1}y,t_k)=s_k-h'_k$ and by (5.52), we have $s_k\in\left[\frac{97}{100}L^1(y,r_k),\frac{995}{1000}L^1(y,r_k)\right]$.

Claim 5.16. *For $p\in\{1,\dots,n\}$,*

$$d_G(q_p(s_k),u_X^{h'_k-s_k})\le\frac{2}{10}2^{-k}D$$

where $q_p(s_k)=q(\overline{\psi}_p(\overline{u}^{r_k}a_Y^{-\lambda_k}y),\overline{\psi}_p(a_Y^{-\lambda_k}y),s_k)$.

Proof. Since $|r_k|\le\omega_k$ and $a_Y^{-\lambda_k}y,\overline{u}_Y^{r_k}a_Y^{-\lambda_k}y\in K_k^0$, by (5.54) and Lemma 5.13, we know that

$$|\Delta^{\tau_Y}_{r_k}(a_{Y,k}^{-1}y,t_k)|\le\frac{11}{10}|\Delta_{r_k}(t_k)|\le\frac{11}{100}D. \tag{5.64}$$

It follows that

$$d_{\overline{X}}\left(u_X^{s_k}\overline{\psi}_p(a_Y^{-\lambda_k}y),u_X^{h'_k}\overline{\psi}_p(a_Y^{-\lambda_k}y)\right)<\frac{1}{3}D. \tag{5.65}$$

On the other hand, by (5.56), we have

$$\|q_p(s_k)\|\le\frac{D}{10},\quad d_{\overline{X}}\left(u_X^{s_k}\overline{\psi}_p(\overline{u}^{r_k}a_Y^{-\lambda_k}y),u_X^{s_k}\overline{q_p(s_k)\psi_p(a_Y^{-\lambda_k}y)}\right)\le\frac{1}{10}2^{-k}D \tag{5.66}$$

It follows that

$$d_{\overline{X}}\left(u_X^{s_k}\overline{\psi}_p(\overline{u}^{r_k}a_Y^{-\lambda_k}y),u_X^{s_k}\overline{\psi}_p(a_Y^{-\lambda_k}y)\right)<\frac{1}{3}D \tag{5.67}$$

for $p\in\{1,\dots,p\}$. Therefore, (5.65) and (5.67) tell us that

$$\begin{aligned}&d_{\overline{X}}\left(\overline{\psi}_{i(p,k)}(u_Y^{t_k}a_Y^{-\lambda_k}\overline{u}_Y^r y),\overline{\psi}_{j(p,k)}(u_Y^{t'_k}a_Y^{-\lambda_k}y)\right)\\=&d_{\overline{X}}\left(u_X^{s_k}\overline{\psi}_p(\overline{u}_Y^{r_k}a_Y^{-\lambda_k}y),u_X^{h'_k}\overline{\psi}_p(a_Y^{-\lambda_k}y)\right)<D.\end{aligned}$$

Then by (5.50), we must have $i(p,k)=j(p,k)$. Then by Lusin theorem (5.51) (5.60), we further obtain

$$\begin{aligned}&d_{\overline{X}}\left(u_X^{s_k}\overline{\psi}_p(\overline{u}_Y^{r_k}a_Y^{-\lambda_k}y),u_X^{h'_k}\overline{\psi}_p(a_Y^{-\lambda_k}y)\right)\\=&d_{\overline{X}}\left(\overline{\psi}_{i(p,k)}(u_Y^{t_k}a_Y^{-\lambda_k}\overline{u}_Y^r y),\overline{\psi}_{i(p,k)}(u_Y^{t'_k}a_Y^{-\lambda_k}y)\right)\le\frac{1}{10}2^{-k}D.\end{aligned} \tag{5.68}$$

Combining (5.66), we get

$$\begin{aligned} &d_{\overline{X}}\left(\overline{q_p(s_k)\cdot u_X^{s_k}\psi_p(a_Y^{-\lambda_k}y)}, \overline{u_X^{h'_k-s_k}\cdot u_X^{s_k}\psi_p(a_Y^{-\lambda_k}y)}\right)\\ =&d_{\overline{X}}\left(u_X^{s_k}\overline{q_p(s_k)\psi_p(a_Y^{-\lambda_k}y)}, u_X^{h'_k}\overline{\psi}_p(a_Y^{-\lambda_k}y)\right)\leq \frac{2}{10}2^{-k}D. \end{aligned}$$

Since by (5.59) $u_X^{h'_k}\overline{\psi}_p(a_Y^{-\lambda_k}y)\in K_1$, $\|q_p(s_k)\|\leq\frac{1}{10}D$, $|s_k-h'_k|=|\Delta_{r_k}^{\tau_Y}(a_{Y,k}^{-1}y,t_k)|\leq \frac{11}{100}D$, we conclude that

$$d_G(q_p(s_k),u_X^{h'_k-s_k})\leq\frac{2}{10}2^{-k}D$$

for any $p\in\{1,\dots,n\}$. ☐

It then follows from the definition of $L^1(y,r_k)$ (5.54) that

$$\|q_p^1(s_k)\|\geq\frac{9}{100}D,\quad |h'_k-s_k|\geq\frac{9}{100}D \tag{5.69}$$

for any $p\in\{1,\dots,n\}$.

On the other hand, denote $h_k=\frac{h'_k}{1-r_kh'_k}$.

Claim 5.17. *We have*

$$|h_k-s_k|<2^{1-k}D.$$

Proof. One can calculate via (5.62)

$$\begin{aligned} |h_k-s_k|=&|h_k-h'_k-(s_k-h'_k)|\\ =&|\Delta_{r_k}(h_k)-\Delta_{r_k}^{\tau_Y}(a_Y^{-\lambda_k}y,t_k)|\\ \leq&|\Delta_{r_k}(h_k)-\Delta_{r_k}(t_k)|+|\Delta_{r_k}(t_k)-\Delta_{r_k}^{\tau_Y}(a_Y^{-\lambda_k}y,t_k)|\\ \leq&|\Delta_{r_k}(h_k)-\Delta_{r_k}(t_k)|+\frac{1}{10}2^{-k}D. \end{aligned}\tag{5.70}$$

On the other hand, by the ergodicity (5.63) (5.52), we have

$$|h'_k-t'_k|\leq\frac{1}{10}2^{-k}D\cdot t'_k\leq\frac{2}{10}2^{-k}D\cdot t_k.$$

Then by (5.61) and $|\Delta_{r_k}(t_k)|\leq D/10$, we have

$$|h_k-t_k|=\left|\frac{h'_k}{1-r_kh'_k}-\frac{t'_k}{1-r_kt'_k}\right|=\left|\frac{h'_k-t'_k}{(1-r_kh'_k)(1-r_kt'_k)}\right|\leq\frac{4}{10}2^{-k}D\cdot t_k.$$

It follows that

$$\begin{aligned}|\Delta_{r_k}(h_k)-\Delta_{r_k}(t_k)| =&|r_k h_k h_k' - r_k t_k t_k'|\\ \le&|r_k h_k(h_k'-t_k')| + |r_k t_k'(h_k-t_k)|\\ \le&\frac{2}{10}2^{-k}D\cdot|r_k h_k t_k| + \frac{4}{10}2^{-k}D\cdot|r_k t_k' t_k|\\ \le&\frac{4}{10}2^{-k}D\cdot|\Delta(t_k)| + \frac{8}{10}2^{-k}D\cdot|\Delta(t_k)| \le \frac{12}{10}2^{-k}D.\end{aligned}$$

Then (5.70) is clearly not greater than $2^{1-k}D$. □

Now Claim 5.16 and 5.17 imply that $h_k \in \left[\frac{96}{100}L^1(y,r_k), \frac{999}{1000}L^1(y,r_k)\right]$, $|h_k' - h_k| \in [\frac{9}{100}D, \frac{11}{100}D]$ and

$$\begin{aligned}d_{\overline{X}}(u_X^{h_k}\overline{\psi}_p(\overline{u}_Y^{r_k}a_Y^{-\lambda_k}y), u_X^{h_k'}\overline{\psi}_p(a_Y^{-\lambda_k}y)) \le&\frac{2}{10}2^{1-k}D\\ d_{\overline{X}}(u_X^{h_k}\overline{u}_X^{r_k}\overline{\psi}_p(a_Y^{-\lambda_k}y), u_X^{h_k'}\overline{\psi}_p(a_Y^{-\lambda_k}y)) \le&\frac{2}{10}2^{1-k}D\\ d_{G_X}(q_p(h_k), u_X^{h_k'-h_k}) \le&\frac{2}{10}2^{1-k}D\end{aligned}$$

for $p \in \{1,\dots,n\}$.

Similarly, for $i=2$, there exists $h_{k,2} \in \left[\frac{96}{100}L^2(y,r_k), \frac{999}{1000}L^2(y,r_k)\right]$ and $h_{k,2}' \in \mathbf{R}$ with $|h_{k,2}' - h_{k,2}| \in [\frac{9}{100}2^2D, \frac{11}{100}2^2D]$ such that

$$\begin{aligned}d_{\overline{X}}(u_X^{h_{k,2}}\overline{\psi}_p(\overline{u}_Y^{r_k}a_Y^{-\lambda_k}y), u_X^{h_{k,2}'}\overline{\psi}_p(a_Y^{-\lambda_k}y)) \le&\frac{2}{10}2^{1-k}D\\ d_{\overline{X}}(u_X^{h_{k,2}}\overline{u}_X^{r_k}\overline{\psi}_p(a_Y^{-\lambda_k}y), u_X^{h_{k,2}'}\overline{\psi}_p(a_Y^{-\lambda_k}y)) \le&\frac{2}{10}2^{1-k}D\\ d_{G_X}(q_p^2(h_{k,2}), u_X^{h_{k,2}'-h_{k,2}}) \le&\frac{2}{10}2^{1-k}D\end{aligned}$$

for $p \in \{1,\dots,n\}$. Note that by (5.53), we have $h_k \in [\frac{1}{3}h_{k,2}, \frac{2}{3}h_{k,2}]$. Thus, we have met the requirement of Lemma 5.14 with pairs

$$(\overline{\psi}_p(\overline{u}_Y^{r_k}a_Y^{-\lambda_k}y), \overline{\psi}_p(a_Y^{-\lambda_k}y)) \quad \text{and} \quad (\overline{u}_X^{r_k}\overline{\psi}_p(a_Y^{-\lambda_k}y), \overline{\psi}_p(a_Y^{-\lambda_k}y))$$

at time $t = h_k, h_{k,2}$. Then Lemma 5.14 implies that

$$d_{\overline{X}}\left(u_X^t\overline{\psi}_p(\overline{u}^{r_k}a_Y^{-\lambda_k}y), u_X^t\overline{u}_X^{r_k}\overline{\psi}_p(a_Y^{-\lambda_k}y)\right) \le O\left(\frac{2}{10}2^{1-k}D\right) = O(2^{-k}).$$

for $t \in [0, h_{k,2}]$. Moreover, if we write $\overline{\psi}_p(\overline{u}_Y^{r_k}a_Y^{-\lambda_k}y) = g_{p,k}\overline{u}_X^{r_k}\overline{\psi}_p(a_Y^{-\lambda_k}y)$ and

$$g_{p,k} = \exp\left(\sum_j \sum_{i=0}^{\varsigma(j)} b_j^i v_j^i\right)$$

where v_j^i are the weight vectors of the $\mathfrak{sl}_2$-irreducible representation V_j, then by (5.49) we deduce

$$|b_j^{\varsigma(j)-i}| < O(2^{-k})h_{k,2}^{-i}.$$

Finally, one calculates via (2.6) (5.55) (5.58)

$$\begin{aligned} a_X^{\lambda_k} g_{p,k} a_X^{-\lambda_k} &\leq \exp\left(\sum_j \sum_{i=0}^{\varsigma(j)} O(2^{-k}) h_{k,2}^{\varsigma(j)-2i} \cdot h_{k,2}^{i-\varsigma(j)} v_j^i\right) \\ &= \exp\left(\sum_j \sum_{i=0}^{\varsigma(j)} O(2^{-k}) h_{k,2}^{-i} v_j^i\right) \leq O(2^{-k}). \end{aligned}$$

Therefore, we conclude that

$$d_{\overline{X}}(\Psi_{k,p}(\overline{u}_Y^r y), \overline{u}_X^r \Psi_{k,p}(y)) \leq O(2^{-k})$$

for $p \in \{1, \dots, n\}$. The theorem follows. □

Remark 5.18. Similar to Remark 5.8, Theorem 5.15 also holds true for ρ being a finite extension of ν, when $(X, \phi_t^{U_X,\tau_X})$ is a time-change of the unipotent flow on $X = SO(n_X,1)/\Gamma_X$: if for $f \in C(X \times Y)$

$$\int f(x,y) d\rho(x,y) = \int \frac{1}{n} \sum_{p=1}^{n} f(\psi_p(y), y) d\nu(y)$$

then we still have

$$\lim_{n\to\infty} d_X(\Psi_{k,p}(\overline{u}_Y^r y), \overline{u}_X^r \Psi_{k,p}(y)) = 0$$

for $p \in \{1, \dots, n\}$ and a.e. $y \in Y$.

6. Applications

In previous sections, we considered the measure of the form

$$\int f d\rho = \int \frac{1}{n} \sum_{p=1}^{n} f(\overline{\psi}_p(y), y) d\nu(y)$$

for some measurable functions $\overline{\psi}_p$. Besides, we studied the equivariant properties of $\overline{\psi}_p$. In this section, we use these results to develop the rigidity of ρ.

6.1. **Unipotent flows of $SO(n,1)$ vs. time-changes of unipotent flows.** In this section, we shall prove Theorem 1.3 and 1.6. Let $G_X = SO(n_X,1)$, G_Y be a semisimple Lie group with finite center and no compact factors and $\Gamma_X \subset G_X$, $\Gamma_Y \subset G_Y$ be irreducible lattices. Let (X,μ) be the homogeneous space $X = G_X/\Gamma_X$ equipped with the Lebesgue measure μ, and let $\phi_t^{U_X} = u_X^t$ be a unipotent flow on X. Suppose that

- Y is the homogeneous space $Y = G_Y/\Gamma_Y$,
- m_Y is the Lebesgue measure on Y,

- $u_Y \in G_Y$ is a unipotent element that $C_{\mathfrak{g}_Y}(u_Y)$ only contains vectors of weight at most 2,
- $\tau_Y \in \mathbf{K}_\kappa(Y) \cap C^1(Y)$ is a positive integrable and C^1 function on Y such that τ_Y, τ_Y^{-1} are bounded and satisfies (2.10),
- $\tilde{u}_Y^t = \phi_t^{U_Y,\tau_Y}$ of the unipotent flow u_Y,
- ν is a $\tilde{u}_Y^t$-invariant measure on Y,
- $\rho \in J(u_X^t, \phi_t^{U_Y,\tau_Y})$ is a nontrivial (i.e. not the product $\mu \times \nu$) ergodic joining.

Proposition 6.1. *$\tau_Y(y)$ and $\tau_Y(cy)$ are (measurably) cohomologous along u_Y^t for all $c \in C_{G_Y}(U_Y)$. Further, if $\tau_Y(y)$ and $\tau_Y(cy)$ are L^1-cohomologous, then after passing a subsequence if necessary,*

$$\Psi^*(y) := \lim_{n\to\infty} \Psi_k^*(y)$$

exists for ν-a.e. $y \in Y$, where $\Psi_k^*(y) := \{\Psi_{k,p}(y) : p \in \{1, \ldots, n\}\}$ and $\Psi_{k,p}(y)$ is given by (5.58).

Proof. The first consequence follows from Theorem 5.6. For the second one, we first apply Lemma 5.7 and obtain

$$\lim_{t\to\infty} \frac{1}{t}\alpha(c^t, y) = \int \alpha(c, y) dm_Y(y)$$

for m-a.e. $y \in Y$ whenever c is m_Y-ergodic. Note that $d\beta : C_{\mathfrak{g}_Y}(U_Y) \to V_{C_X}^{\perp}$ sends nilpotent elements to nilpotent elements. Thus, for weight vector $v \in C_{\mathfrak{g}_Y}(U_Y)$ of weight $\varsigma \leq 2$, ν-almost all $y \in Y$, we have

$$\Psi_k^*(\exp(v)y) = \begin{cases} u_X^{e^{-\lambda_k}\alpha(\exp(e^{\varsigma\lambda_k/2}v),y)} \beta(\exp(e^{\varsigma\lambda_k/2}v))^{e^{-\lambda_k}} \Psi_k^*(y) & , \text{ for } \varsigma \geq 1 \\ u_X^{e^{-\lambda_k}\alpha(\exp(v),y)} a_X^{\lambda_k} \beta(\exp(v)) a_X^{-\lambda_k} \Psi_k^*(y) & , \text{ for } \varsigma = 0 \end{cases}.$$

Thus, after passing to a subsequence if necessary, we have

$$\lim_{k\to\infty} \Psi_k^*(\exp(v)y) = \begin{cases} u_X^{\int \alpha(\exp(v),\cdot)} \beta(\exp(v)) \lim_{k\to\infty} \Psi_k^*(y) & , \text{ for } \varsigma = 2 \\ \lim_{k\to\infty} \Psi_k^*(y) & , \text{ for } \varsigma = 1 \\ \exp(v_0) \lim_{k\to\infty} \Psi_k^*(y) & , \text{ for } \varsigma = 0 \end{cases} \tag{6.1}$$

where $\beta(\exp(v)) = \exp(v_0 + v_2)$ for $v_0, v_2 \in V_{C_X}^{\perp}$ of weight 0 and 2 respectively. In particular, $\lim_{k\to\infty} \Psi_k^*(\exp(v)y)$ exists whenever $\lim_{k\to\infty} \Psi_k^*(y)$ exists. Besides, by Theorem 5.15, we have

$$\lim_{n\to\infty} d_{\overline{X}}(\Psi_k^*(\overline{u}_Y^r y), \overline{u}_X^r \Psi_k^*(y)) = 0$$

for $r \in \mathbf{R}$, ν-a.e. $y \in Y$.

It remains to show that for ν-almost all $y \in Y$, there exists a subsequence $\{k(y,l)\}_{l\in\mathbf{N}} \subset \mathbf{N}$ and $\Psi_p(y) \in \overline{X}$ such that

$$\lim_{l\to\infty} \Psi_{k(y,l),p}(y) = \Psi_p(y). \tag{6.2}$$

To do this, write $\overline{X} = \bigcup_{i=1} K_i$, where K_i are compact and $\overline{\mu}(K_i) \nearrow 1$ as $i \to \infty$. Let

$$\Omega \coloneqq \bigcup_{i\geq 1}\bigcap_{k\geq 1}\bigcup_{j\geq k}\bigcap_{p=1}^{n} \Psi_{j,p}^{-1}(K_i).$$

Claim 6.2. $\nu(\Omega) = 1$.

Proof. From a direct calculation (recall that $d\nu \coloneqq \tau dm_Y$), we know

$$m_Y\left(\bigcup_{i\geq 1}\bigcap_{k\geq 1}\bigcup_{j\geq k}\bigcap_{p=1}^{n} \Psi_{j,p}^{-1}(K_i)\right) \geq m_Y\left(\bigcap_{k\geq 1}\bigcup_{j\geq k}\bigcap_{p=1}^{n} \Psi_{j,p}^{-1}(K_i)\right)$$
$$= \lim_{k\to\infty} m_Y\left(\bigcup_{j\geq k}\bigcap_{p=1}^{n} \Psi_{j,p}^{-1}(K_i)\right) \geq m_Y(\psi_p^{-1}a^{-\lambda_j}K_i)$$

for any p, j and i. As $\overline{\mu}(K_i) \nearrow 1$ as $i \to \infty$, the claim follows. □

Then by Claim 6.2 for $y \in \Omega$, there exists $i \geq 1$ such that $\Psi_{j,p}(y) \in K_i$ for infinitely many j. Thus, we proved (6.2). Therefore, since the opposite unipotent and central directions generate the whole group $\langle \overline{u}_Y^r, C_{G_Y}(U_Y)\rangle = G_Y$, we conclude that after passing a subsequence if necessary,

$$\lim_{n\to\infty} \Psi_{k,p}(y)$$

exists for ν-a.e. $y \in Y$. □

Then, define a measure $\widetilde{\rho}$ on $\overline{X} \times Y$ by

$$\int f d\widetilde{\rho} \coloneqq \int_Y \frac{1}{n}\sum_{p=1}^{n} f(\Psi_p(y), y) dm_Y(y)$$

for $f \in C(\overline{X} \times Y)$ where $\Psi^*(y) = \{\Psi_1(y), \ldots, \Psi_n(y)\}$. Then $\widetilde{\rho}$ is a nontrivial $(u_X^t \times u_Y^t)$-invariant measure on $\overline{X} \times Y$ such that $(\pi_{\overline{X}})_*\widetilde{\rho} = \overline{\mu}$ and $(\pi_Y)_*\widetilde{\rho} = m_Y$. Then, *Ratner's theorem* [Rat90] asserts that $C^\rho = \{e\}$ and

$$\widetilde{\rho}(\mathrm{stab}(\widetilde{\rho}).(x_0, y_0)) = 1$$

for some $(x_0, y_0) \in X \times Y$, where $\mathrm{stab}(\widetilde{\rho}) \coloneqq \{(g_1, g_2) \in G_X \times G_Y : (g_1, g_2)_*\widetilde{\rho} = \widetilde{\rho}\}$. Then let

- $\mathrm{stab}_Y(\widetilde{\rho}) \coloneqq \{(e, g_2) \in G_X \times G_Y : (e, g_2)_*\widetilde{\rho} = \widetilde{\rho}\}$ (note that $\mathrm{stab}_Y(\widetilde{\rho}) \lhd G_Y$ is a normal subgroup of G_Y),
- $\Gamma_X^g \coloneqq \{\gamma : g^{-1}\gamma g \in \Gamma_X\}$ for $g \in G_X$.

Then *Ratner's theorem* [Rat90] further asserts that there is $g_0 \in G_Y$ and a continuous surjective homomorphism $\Phi : G_Y \to G_X$ with kernel $\mathrm{stab}_Y(\widetilde{\rho})$, $\Phi(g) = g$ for $g \in SL_2$ such that

$$\{\Psi_1(h\Gamma_Y), \ldots, \Psi_n(h\Gamma_Y)\} = \{\Phi(h)\gamma_1 g_0\Gamma_X, \ldots, \Phi(h)\gamma_n g_0\Gamma_X\} \tag{6.3}$$

for all $h \in G_Y$, where the intersection $\Gamma_0 \coloneqq \Phi(\Gamma_Y) \cap \Gamma_X^{g_0}$ is of finite index in $\Phi(\Gamma_Y)$ and in $\Gamma_X^{g_0}$, $n = |\alpha(\Gamma_Y)/\Gamma_0|$ and $\Phi(\Gamma_Y) = \{\gamma_p \Gamma_0 : p \in \{1, \ldots, n\}\}$.

Next, by using Proposition 6.1 and (6.3), for any $\sigma > 0$ $\epsilon > 0$, there exists a subset $K \subset Y$ with $\nu(K) > 1 - \sigma$ and $k_0 > 0$ such that

$$\max_p \min_q d_X \left(\Psi_{k,p}(h\Gamma_Y), \Phi(h)\gamma_q g_0 \Gamma_X\right) < \epsilon$$

for $h\Gamma_Y \in K$, $k \geq k_0$. In particular, by the ergodic theorem, we know that for ν-a.e. $y \in Y$, there is $A_y \subset \mathbf{R}^+$ and $\lambda_0(y) > 0$ such that

- for $r \in A_y$, we have $u_Y^r y \in K$;
- $\mathrm{Leb}(A_y \cap [0, \lambda]) \geq (1 - 2\sigma)\lambda$ whenever $\lambda \geq \lambda_0(y)$.

Therefore, one can repeat the same argument as in Section 5.1, and then conclude that there exists $c'(h\Gamma_Y) \in C_{G_Y}(U_Y)$, $q'(p, h\Gamma_Y) \in \{1, \ldots, n\}$ such that

$$\Psi_{k,p}(h\Gamma_Y) = c'(h\Gamma_Y)\Phi(h)\gamma_{q'(p,h\Gamma_Y)} g_0 \Gamma_X$$

for ν-a.e. $h\Gamma_Y \in Y$. We can then write

$$\psi_p(h\Gamma_Y) = c(h\Gamma_Y)\Phi(h)\gamma_{q(p,h\Gamma_Y)} g_0 \Gamma_X$$

for some $c(h\Gamma_Y) \in C_{G_Y}(U_Y)$, $q(p, h\Gamma_Y) \in \{1, \ldots, n\}$, ν-a.e. $h\Gamma_Y \in Y$. Thus, let $I = (q_1, q_2, \ldots, q_n)$ be a permutation of $\{1, \ldots, n\}$,

$$S_I \coloneqq \{y \in Y : q(1, y) = q_1, \ldots, q(n, y) = q_n\}$$

and let

$$\widetilde{\psi}_p(y) \coloneqq \psi_{q_p}(y) \quad \text{when} \quad y \in S_{(q_1,\ldots,q_n)}.$$

Then $\widetilde{\psi}_p(y)$ plays the same role as $\psi_p(y)$ and satisfies

$$\widetilde{\psi}_p(h\Gamma_Y) = c(h\Gamma_Y)\Phi(h)\gamma_p g_0 \Gamma_X. \tag{6.4}$$

for ν-a.e. $h\Gamma_Y \in Y$. Thus, without loss of generality, we assume that ψ_p satisfies (6.4). It follows that the map $\Upsilon : \mathrm{supp}(\rho) \to X \times Y$ defined by

$$\Upsilon : (\psi_p(h\Gamma_Y), h\Gamma_Y) \mapsto (\Phi(h)\gamma_p g_0 \Gamma_X, h\Gamma_Y) \quad \text{for} \quad p \in \{1, \ldots, n\}$$

is bijective and satisfies

$$\Upsilon(u_X^t x, \tilde{u}_Y^t(y)) = (u_X^{\xi(y,t)} \times u_Y^{\xi(y,t)}).\Upsilon(x, y) \tag{6.5}$$

for ρ-a.e. (x, y) and $t \in \mathbf{R}$. Equivalently, we obtain:

Proposition 6.3. *Assume that $\tau_Y(y)$ and $\tau_Y(cy)$ are L^1-cohomologous for all $c \in C_{G_Y}(U_Y)$. Then $\tau_X \equiv 1$ and τ_Y are joint cohomologous.*

Proof. By (6.5), we can write down the decomposition (2.7) for $c(y)$:

$$c(y) = u_X^{a(y)} b$$

and

$$a(y) + t = \xi(y, t) + a(u_Y^{\xi(y,t)} y).$$

It follows that

$$\int_0^{\xi(y,t)} \tau_Y(u_Y^s y) - 1 ds = t - \xi(y,t) = a(u_Y^{\xi(y,t)} y) - a(y).$$

Thus, 1 and τ_Y are joint cohomologous via $(\tilde{\rho}, a)$. $\square$

Recall (6.1) that when a weight vector $v \in C_{\mathfrak{g}_Y}(U_Y)$ of weight $\varsigma \geq 1$, we know that $\widetilde{\rho}$ is invariant under

$$(6.6) \qquad \begin{cases} u_X^{\int \alpha(\exp(v),\cdot)} \beta(\exp(v)) \times \exp(v) & , \text{ for } \varsigma = 2 \\ \mathrm{id} \times \exp(v) & , \text{ for } \varsigma = 1 \\ \exp(v_0) \times \exp(v) & , \text{ for } \varsigma = 0 \end{cases}$$

where $\beta(\exp(v)) = \exp(v_0 + v_2)$. Since $\widetilde{\rho}$ is also $(u_X^t \times u_Y^t)$-invariant, if $\beta(\exp(v)) = e$, then *Moore's ergodicity theorem* and Lemma 3.1 imply that $\langle \exp(v) \rangle \subset \ker \Phi$ is a compact normal subgroup of G_Y. It is a contradiction. Thus, we conclude

Proposition 6.4. *The map $d\beta|_{V_C^\perp} : V_{C_Y}^\perp \to V_{C_X}^\perp$ is an injective Lie algebra homomorphism.*

6.2. **Time-changes of unipotent flows of $SO(n,1)$ vs. unipotent flows.** In this section, we shall prove Theorem 1.8. Let $G_X = SO(n_X, 1)$, G_Y be a semisimple Lie group with finite center and no compact factors and $\Gamma_X \subset G_X$, $\Gamma_Y \subset G_Y$ be irreducible lattices. Let (Y, ν) be the homogeneous space $Y = G_Y/\Gamma_Y$ equipped with the Lebesgue measure ν, and let $\phi_t^{U_Y} = u_Y^t$ be a unipotent flow on Y. Suppose that

- X is the homogeneous space $X = G_X/\Gamma_X$,
- $u_X \in G_X$ is a unipotent element,
- $\tau_X \in \mathbf{K}_\kappa(X)$ is a positive integrable and C^1 function on Y such that τ_X, τ_X^{-1} are bounded and satisfies (2.10),
- $\tilde{u}_X^t = \phi_t^{U_X,\tau}$ of the unipotent flow u_X,
- μ is a $\tilde{u}_X^t$-invariant measure on X,
- $\rho \in J(\tilde{u}_X^t, u_Y^t)$ is an ergodic joining that is a compact extension of ν, i.e. has the form

$$\rho(f) = \int_Y \int_{C^\rho} \frac{1}{n} \sum_{p=1}^n f(k\psi_p(y), y) dm(k) d\nu(y)$$

for $f \in C(X \times Y)$ and compact $C^\rho \in C_{G_X}(U_X)$.

Recall that in Remark 5.8, for $c \in C_{G_Y}(U_Y)$, we know that ρ is invariant under the map

$$\widetilde{S}_c : (x, y) \mapsto (u_X^{\alpha(c,y)} \beta(c) x, cy)$$

(cf. (5.28)). Besides, α, β satisfy

$$\xi(\psi_p(cy), t) + \alpha(c, y) = \alpha(c, u_Y^t y) + \xi(\psi_p(y), t),$$

$$\alpha(c_1c_2, y) = \alpha(c_1, c_2y) + \alpha(c_2, y), \quad \beta(c_1c_2) = \beta(c_1)\beta(c_2) \tag{6.7}$$

where

$$t = \int_0^{\xi(x,t)} \tau_X(u_X^s x) ds.$$

Moreover, if $\beta(c) = e$ for some $c \in C_{G_Y}(U_Y)$, then we have (5.29):

$$\alpha(c, y) = \xi(x, r_c) \tag{6.8}$$

for some $r_c \in \mathbf{R}$. Note that (6.8) implies that

$$(x, y) \mapsto (u_X^{\alpha(c,y)} x, cy) \mapsto (x, u_Y^{-r_c} cy)$$

is ρ-invariant. Thus, *Moores ergodicity theorem* and Lemma 3.1 force

$$\alpha(\exp(v), y) \equiv 0 \quad \text{ and } \quad \langle \exp(v) \rangle \subset G_Y \tag{6.9}$$

is compact. In particular, we obtain (1.2):

$$d\beta|_{V_C^\perp}(v) \neq 0$$

for any weight vector $v \in V_{C_Y}^\perp$ of positive weight. Inspired by this, we deduce

Lemma 6.5. *For weight vectors $v \in C_{\mathfrak{g}_Y}(U_Y)$ of weight $\varsigma \neq 0, 2$, we must have*

$$d\beta(v) = 0.$$

Proof. Similar to Theorem 5.10, one can deduce that for $r \in \mathbf{R}$,

$$\widetilde{S}_{a_Y^r} : (x, y) \mapsto \left(u_X^{\alpha(a_Y^r, y)} \beta(a_Y^r) a_X^r x, a_Y^r y \right)$$

is ρ-invariant. Also, we have

$$\widetilde{S}_{a_Y} \circ \widetilde{S}_c \circ \widetilde{S}_{a_Y^{-1}} = \widetilde{S}_{a_Y c a_Y^{-1}}$$

for any $a_Y \in \exp(\mathbf{R}A_Y)$, $c \in C_{G_Y}(U_Y)$. In particular, one deduces

$$\beta(a_Y^r) a_X^r \beta(a_Y^{-r}) a_X^{-r} = e, \quad \beta(a_Y^r) a_X^r \beta(c) \beta(a_Y^{-r}) a_X^{-r} = \beta(a_Y^r c a_Y^{-r}).$$

Thus, suppose that $v \in C_{\mathfrak{g}_Y}(U_Y)$ is a weight vector of weight $\varsigma \neq 0, 2$. Then

$$\begin{aligned} \beta(\exp(v))^{e^{r\varsigma/2}} &= \beta(\exp(e^{r\varsigma/2} v)) = \beta(a_Y^r \exp(v) a_Y^{-r}) \\ &= \beta(a_Y^r) a_X^r \beta(\exp(v)) \beta(a_Y^{-r}) a_X^{-r} = \beta(a_Y^r) a_X^r \beta(\exp(v)) a_X^{-r} \beta(a_Y^r)^{-1}. \end{aligned} \tag{6.10}$$

Assume that $\beta(\exp(v)) = \exp(w)$ for some $w \in C_{\mathfrak{g}_X}(U_X)$. By the assumption, w has to be nilpotent and so

$$a_X^r \beta(\exp(v)) a_X^{-r} = a_X^r \exp(w) a_X^{-r} = \exp(e^r w). \tag{6.11}$$

Combining (6.10) and (6.11), we get

$$e^{r\varsigma/2} \|w\| = \|e^{r\varsigma/2} w\| = \| \operatorname{Ad} \beta(a_Y^r) . e^r w\| = \|e^r w\| = e^r \|w\|$$

which leads to a contradiction. □

Then by *Moores ergodicity theorem* and Lemma 3.1 (cf. Remark 5.8), we conclude

Corollary 6.6. *If $C_{\mathfrak{g}_Y}(U_Y)$ contains a weight vector of weight $\varsigma \neq 0, 2$, then*

$$\rho = \mu \times \nu.$$

Now we focus on the case $n_X = 2$ and $\tau_X \in \mathbf{K}(X) \cap C^1(X)$. Note that in this case, Ratner [Rat87] showed that $\tilde{u}_X^t$ also has H-property. Thus, we can repeat the same idea as in Section 6.1 to discuss the case when $C_{\mathfrak{g}_Y}(U_Y)$ consists only of weight vectors of weight $\varsigma = 0, 2$. Note that since $\beta \equiv 0$, by (6.8), we must have $\alpha(c, \cdot) \in L^\infty(Y)$ for any $c \in C_{G_Y}(U_Y)$. Then, similar to Proposition 6.1, we have

Proposition 6.7. *Assume that $C_{\mathfrak{g}_Y}(U_Y)$ consists only of weight vectors of weight $\varsigma = 0, 2$. Then after passing a subsequence if necessary,*

$$\Psi^*(y) \coloneqq \lim_{n\to\infty} \Psi_k^*(y) \tag{6.12}$$

exists for ν-a.e. $y \in Y$, where $\Psi_k^*(y) \coloneqq \{\Psi_{k,p}(y) : p \in \{1, \dots, n\}\}$ and $\Psi_{k,p}(y)$ is given by (5.58).

Remark 6.8. One nontrivial step of Proposition 6.7 is to obtain a similar version of Theorem 5.15. This requires that the time-change $\tilde{u}_X^t$ also has H-property. See [Rat87] Lemma 3.1 for further details.

Then by *Ratner's theorem* (cf. (6.4)), there exists $c(h\Gamma_Y) \in C_{G_X}(U_X) = \exp(\mathbf{R}U_X)$, a homomorphism $\Phi(h)$, $\gamma_p, g_0 \in G_X$ such that ψ_p can be written as

$$\psi_p(h\Gamma_Y) = c(h\Gamma_Y)\Phi(h)\gamma_p g_0 \Gamma_X \tag{6.13}$$

for $h\Gamma_Y \in Y$. Then as in Proposition 6.3, we get

Proposition 6.9. *τ_X and $\tau_Y \equiv 1$ are joint cohomologous.*

Finally, consider ρ is nontrivial$v \in C_{G_Y}(U_Y)$. Since $\beta(\exp(v)) = e$, (6.9) asserts that

$$\alpha(\exp(v), y) \equiv 0 \quad \text{and} \quad \langle \exp(v) \rangle \subset G_Y$$

is compact. However, *Ratner's theorem* implies that $\langle \exp(v) \rangle \subset \ker \Phi$ is a normal subgroup of G_Y. It is a contradiction. Thus, we conclude

$$V_{C_Y}^{\perp} = 0.$$

Therefore, we have proved Theorem 1.10.

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