EFFECTIVE DENSITY OF SURFACES NEAR TEICHMÜLLER CURVES

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ABSTRACT. We study the dynamics of $\mathrm{SL}_2(\mathbb{R})$ on the stratum of translation surfaces $\mathcal{H}(2)$. Especially, we obtain effective density theorems on $\mathcal{H}(2)$ for orbits of the upper triangular subgroup P of $\mathrm{SL}_2(\mathbb{R})$ with the based surfaces near a small Teichmüller curve.

The proof is based on the use of McMullen's classification theorem, together with the effective equidistribution theorems in homogeneous dynamics. In particular, we compare the P-orbit of a surface, and the P-orbit of its absolute periods using the Lindenstrauss-Mohammadi-Wang's effective equidistribution theorem.

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1. Introduction

1.1. **Main results.** Let $\mathcal{H}(2)$ be the moduli space of translation surfaces of type 2. Thus, for $x \in \mathcal{H}(2)$, x is a translation surface of genus 2, and x has a double zero. In practice, we may identify x as a pair (M, ω), consisting of a compact Riemann surface M of genus 2, and a holomorphic 1-form ω on M. The space $\mathcal{H}(2)$ can be identified with the quotient of the Teichmüller space $\mathcal{TH}(2)$ and the mapping class group Mod:

$$\mathcal{H}(2) = \mathcal{TH}(2)/\operatorname{Mod}.$$

Let $\mathcal{H}_A(2)$ be the subset of translation surfaces of area A.

Given a translation surface $x = (M, \omega) \in \mathcal{H}_1(2)$, the *absolute period map* (abuse notation) $x : H_1(M; \mathbb{Z}) \to \mathbb{C}$ is defined by

$$x(\gamma) \coloneqq \int_\gamma \omega$$

whose image $x(H_1(M; \mathbb{Z}))$ is the group of *absolute periods* of x. Via the absolute period map, $\mathcal{H}(2)$ can be locally identified with $H^1(M; \mathbb{C})$.

In [McM07, §7], McMullen showed that each $x \in \mathcal{H}_1(2)$ can be presented as a connected sum of two tori. More precisely, for $x \in \mathcal{H}_1(2)$, there are a pair of lattices $\Lambda_1, \Lambda_2 \subset x(H_1(M; \mathbb{Z})) \subset \mathbb{C}$, and a vector $v \in \mathbb{C}^*$, such that

$$[0, v] \cap \Lambda_1 = \{0\}, \quad [0, v] \cap \Lambda_2 = \{0, v\}$$

or vice versa. Then $E_i = \mathbb{C}/\Lambda_i$ are forms of genus 1 (i.e. tori). The arcs $J_i = [0, v] \subset E_i$ are straight geodesics on E_i; in particular, we get a loop in E_2 (or vice versa). Slitting the two tori E_i open along J_i, and gluing corresponding edges using J_i, we obtain the *connected sum*

$$x = E_1 \underset{[0,v]}{\#} E_2.$$

We also write the connected sum as $x = \Lambda_1 \underset{[0,v]}{\#} \Lambda_2$, and call it a *splitting* of x.

Moreover, for $g \in \mathrm{GL}_2^+(\mathbb{R})$, we define

$$gx = g.\Lambda_1 \underset{g\cdot[0,v]}{\#} g.\Lambda_2.$$

It leads to a $\mathrm{GL}_2^+(\mathbb{R})$-action on $\mathcal{H}(2)$, and a $\mathrm{SL}_2(\mathbb{R})$-action on $\mathcal{H}_1(2)$. Further, there is a finite $\mathrm{SL}_2(\mathbb{R})$-measure, called *Masur-Veech measure*, on $\mathcal{H}_1(2)$. Thus, we study the dynamical properties of this action.

Let $G = \mathrm{SL}_2(\mathbb{R})$, and

$$a_t = \begin{bmatrix} e^{\frac{1}{2}t} & 0 \\ 0 & e^{-\frac{1}{2}t} \end{bmatrix}, \quad u_r = \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix}.$$

For any subset $I \subset \mathbb{R}$, denote $a_I = \{a_t : t \in I\}$, and $u_I = \{u_r : r \in I\}$. Let $P = a_{\mathbb{R}} u_{\mathbb{R}}$. The seminal work of Eskin and Mirzakhani in [EM18] shows that every P-orbit closure in $\mathcal{H}_1(2)$ is in fact G-invariant. Further, by work of McMullen [McM07], the G-orbit closure in $\mathcal{H}_1(2)$ are either Teichmüller curves or $\mathcal{H}_1(2)$ itself (see Theorem 3.1). In [McM05], a much more detailed description for Teichmüller curves in $\mathcal{H}_1(2)$ is available. In this paper, we are mainly interested in the Teichmüller curve ΩW_D generated by square-tiled surfaces (i.e. the discriminant D is a square, see Section 3.1).

For $\tilde{x} \in \mathcal{TH}(2)$, Let $\|\cdot\|_{\tilde{x}}$ be the *Avila-Gouëzel-Yoccoz norm* (or *AGY norm* for short) on $H^1(M;\mathbb{C})$, (see Definition 2.8). It induces a metric d on $\mathcal{H}_1(2)$.

For $x \in \mathcal{H}_1(2)$, let $\ell(x)$ be the shortest length of a saddle connection. For $\eta > 0$, define

$$\mathcal{H}_1^{(\eta)}(2) \coloneqq \{x \in \mathcal{H}_1(2) : \ell(x) \geq \eta\}$$

(see (2.3)). Similarly, for a pair of lattices $\Lambda = (\Lambda_1, \Lambda_2) \in X$, let $\ell(\Lambda)$ be the shortest length of a vector in Λ_1 and $\Lambda_2 \subset \mathbb{C}$, and let

$$X_\eta \coloneqq \{\Lambda \in X : \ell(\Lambda) \geq \eta\}.$$

In this paper, we will show:

Theorem 1.1. *For any $\eta > 0$, $L > 0$, there exists $\kappa_1 > 0$, $C_1 = C_1(\eta) > 0$ such that for $D \geq L^{\kappa_1}$, and*

$$t > C_1 D^{\kappa_1}, \tag{1.1}$$

the following holds: Suppose that $x \in \mathcal{H}_1(2)$ satisfying

$$\inf_{y_D \in (\Omega_1 W_D)_\eta} d(y_D, x) < e^{-t}.$$

Then we have

$$d(z, a_t u_{[0,1]} x) \leq L^{-1}$$

for every $z \in \mathcal{H}_1^{(L^{-1})}(2)$.

The speed of density (1.1) is in fact related to the spectral gaps of Teichmüller curves in $\mathcal{H}_1(2)$. McMullen made the following conjecture:

Conjecture 1.2 (McMullen's expansion conjecture)**.** *The family of graphs associated to arithmetic Veech groups in $\mathcal{H}(2)$ is expander. In other words, the spectral gaps of the arithmetic Veech groups possess a uniform lower bound.*

If Conjecture 4.4 is correct, then (1.1) in Theorem 1.1 can be improved to

$$t > C_1 \log D. \tag{1.2}$$

In other words, we would have that $a_t u_{[0,1]} x$ is dense with a polynomial error rate.

Roughly speaking, Theorem 1.1 indicates that if a point x is extremely close to a given Teichmüller curve $\Omega_1 W_D$, then its P-orbit Px is $D^{\frac{1}{\kappa_1}}$-dense in $\mathcal{H}_1(2)$.

Let $x \in \mathcal{H}_1(2)$ with a splitting $x = \Lambda_1 \underset{[0,v]}{\#} \Lambda_2$. Let $\operatorname{Area}(\Lambda_i)$ denote the covolume of Λ_i (i.e. the area of $E_i = \mathbb{C}/\Lambda_i$). Let $G = \mathrm{SL}_2(\mathbb{R})$, $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, $X = G/\Gamma \times G/\Gamma$. Locally speaking, $x = \Lambda_1 \underset{[0,v]}{\#} \Lambda_2 \in \mathcal{H}_1(2)$ is dominated by the ratio of areas $\operatorname{Area}(\Lambda_1)/\operatorname{Area}(\Lambda_2)$ and the shape of the lattices $(\Lambda_1, \Lambda_2) \in X$ (after forgetting the areas). By work of McMullen [McM07], the Teichmüller curves are only determined by the absolute periods (see Theorem 3.4). For an even square $D > 0$, let $Q_D = G.(\Lambda_1, \Lambda_2) \subset X$ be the space of absolute periods of $\Omega_1 W_D$, where $\Lambda_1 \underset{[0,v]}{\#} \Lambda_2 \in \Omega W_D$ is a splitting with $\operatorname{Area}(\Lambda_1) = \operatorname{Area}(\Lambda_2) = \frac{1}{2}$ (see Section 3.2).

Now given $D = 4d^2 > 0$, we consider a surface $x = \Lambda_1 \underset{[0,v]}{\#} \Lambda_2$ with $(\Lambda_1, \Lambda_2) \in Q_D$. Then $x \in \Omega_1 W_D$. It follows that the P-orbit $Px \subset \Omega_1 W_D$. In particular, it meets the requirement of Theorem 1.1. Note however that Px cannot be extremely close to another Teichmüller curve $\Omega_1 W_{D'}$ $(D' \neq D)$.

On the other hand, if $(\Lambda_1, \Lambda_2) \notin Q_D$ for any $D > 0$, then $\overline{Px} = \mathcal{H}_1(2)$. It means that $a_t u_{[0,1]} x$ can be extremely close to any Teichmüller curve $\Omega_1 W_D$, for sufficiently large t depending on D. Also, recall that for any surface $x \in \mathcal{H}_1(2)$ with $\overline{Px} = \mathcal{H}_1(2)$, there is a splitting $x = \Lambda_1 \underset{[0,v]}{\#} \Lambda_2$ so that $\overline{P.(\Lambda_1, \Lambda_2)} = X$.

Therefore, we observe that the absolute periods (Λ_1, Λ_2) of x connect to the behavior of Px. This enlightens us about using the effective results on X to study the density on $\mathcal{H}_1(2)$. In [LM23, LMW22], Lindenstrauss, Mohammadi, and Wang established the effective density and equidistribution of P-orbits in X. Let $\|\cdot\|$ be a norm (e.g. the maximum norm) on X. For $D = 4d^2 > 0$, $\varrho > 0$, $\eta > 0$, consider

$$J_{d,t}(\varrho) = \left\{ r \in [0,1] : \|(\tilde{\Lambda}_1, \tilde{\Lambda}_2) - a_t u_r (\Lambda_1, \Lambda_2)\| \leq \varrho,\ (\tilde{\Lambda}_1, \tilde{\Lambda}_2) \in (Q_D)_\eta \right\}.$$

Then the effective equidistribution (Theorem 4.5) implies that

$$|J_{d,t}(\varrho)| \geq \frac{1}{2} \varrho^3$$

for any sufficiently large t, then $a_t u_r(\Lambda_1, \Lambda_2)$ can be ϱ-close to $(Q_D)_\eta$ for $r \in J_{d,t}(\varrho)$.

We try to quantify the above observation. Let $\varkappa : \mathbb{R}^+ \to \mathbb{R}^+$ be a monotonic decreasing function so that

$$\lim_{t \to \infty} \varkappa(e^t) = 0.$$

Let $J \subset [0, e^t]$ be an interval, and

$$J'(\Lambda_1, \Lambda_2, D, t, \eta, \varrho, J) = \left\{ r \in J : \begin{array}{l} \|(\tilde{\Lambda}_1, \tilde{\Lambda}_2) - u_r a_t(\Lambda_1, \Lambda_2)\| \le \varrho, \\ \text{for some } (\tilde{\Lambda}_1, \tilde{\Lambda}_2) \in (Q_D)_\eta \end{array} \right\},$$

$$J''(x, \vartheta, D, t, \eta, \varkappa, J) = \left\{ r \in J : \begin{array}{l} \ell(u_r a_t x)^\vartheta \|(\tilde{\Lambda}_1, \tilde{\Lambda}_2) - u_r a_t x\|_{u_r a_t x} \le \varkappa(|J|), \\ \text{for some } (\tilde{\Lambda}_1, \tilde{\Lambda}_2) \in (Q_D)_\eta \end{array} \right\}.$$

Roughly speaking, J' indicates the moments when $u_r a_t(\Lambda_1, \Lambda_2)$ is close to Q_D in homogeneous dynamics, and J'' implies the moments when $u_r a_t x$ is close to W_D in Teichmüller dynamics.

Theorem 1.3. *There exists an absolute $\delta_0 > 0$, $C_1 > 0$ so that the following holds. Suppose that $x \in \mathcal{H}_1(2)$ has a splitting $x = \Lambda_1 \underset{[0,v]}{\#} \Lambda_2$ satisfying $\mathrm{Area}(\Lambda_1) = \mathrm{Area}(\Lambda_2)$. Then there exists $L_0 = L_0(\delta, \ell(x), \ell(\Lambda_1, \Lambda_2)) > 0$, and an $R > 0$ such that the following holds. For every $\delta \in (0, \delta_0)$, $L > L_0$, $\eta > 0$, $D = 4d^2 \ge L^{\kappa_1}$, $\varrho > 0$, $\vartheta > 0$, and function $\varkappa_\vartheta : \mathbb{R}^+ \to \mathbb{R}^+$ with*

$$\lim_{t \to \infty} \varkappa_\vartheta(e^t) = 0, \tag{1.3}$$

let $\xi = \kappa_1 + \vartheta$, and

$$\varkappa_\vartheta^\star : r \mapsto \underbrace{\varkappa_\vartheta(\cdots \varkappa_\vartheta(\varkappa_\vartheta(r)^{-\xi^2})^{-\xi^2} \cdots)^{\xi^2}}_{R^2 \text{ copies}}.$$

Then there exists an interval $J^{\star\star} \subset [0, e^t]$ with $|J^{\star\star}| \ge (\varkappa_\vartheta^\star(e^t))^{-1}$, such that for

- *$J' = J'(\Lambda_1, \Lambda_2, D, t, \eta^{R^2+1}, \varrho, J^{\star\star})$,*
- *$J'' = J''(x, \vartheta, D, t, \eta^{R^2+1}, \varkappa_\vartheta, J^{\star\star})$,*
- *$t > 0$ so that*

$$\log \varkappa_\vartheta^\star(e^t)^{-1} > C_1 D^{\kappa_1} \ge C_1 L^{\kappa_1^2}, \tag{1.4}$$

(or $\varkappa_\vartheta^\star(e^t)^{-1} > D^{C_1} \ge L^{C_1 \kappa_1}$ assuming Conjecture 4.4 is correct),

at least one of the following holds:

1. *$|J'| \ge \frac{1}{2}\varrho^3 |J^{\star\star}|$, and for every $z \in \mathcal{H}_1^{(L^{-1})}(2)$, we have*
$$d(z, a_{2t} u_{[0,2]} x) \le L^{-1}.$$
2. *There exists a pair of lattices $(\Lambda_1', \Lambda_2') \in X$ such that $G.(\Lambda_1', \Lambda_2')$ is periodic with $\mathrm{vol}(G.(\Lambda_1', \Lambda_2')) \le e^{\delta t}$ and*
$$\|(\Lambda_1', \Lambda_2') - (\Lambda_1, \Lambda_2)\| \le e^{-\frac{1}{2}t}.$$
3. *$|J'| \ge \frac{1}{2}\varrho^3 |J^{\star\star}|$, and*
$$\ell(u_r a_t x) \ge (\varkappa_\vartheta(|J^{\star\star}|))^\xi, \quad J' \cap J'' = \emptyset.$$

Next, we discuss the case (2) of Theorem 1.3. Let $y_1 = (N_1, \omega_1)$, $y_2 = (N_2, \omega_2)$ be two forms of genus 1 (i.e. tori). Then we say that $x = (M, \omega)$ can be presented as an *algebraic sum* $(N_1, \omega_1) + (N_2, \omega_2)$ if by a symplectic isomorphism

$$H_1(M; \mathbb{Z}) \cong H_1(N_1; \mathbb{Z}) \oplus H_1(N_2; \mathbb{Z}),$$

we have

$$x = y_1 + y_2$$

on passing to cohomology with coefficients in $\mathbb{C}$. Two algebraic sums of x are *equivalent* if they come from the same unordered splitting $H_1(M) \cong H_1(N_1) \oplus H_1(N_2)$.

Note that the 1-forms of an algebraic sum $(M, \omega) \cong (N_1, \omega_1) + (N_2, \omega_2)$ are uniquely determined by the splitting

$$H_1(M; \mathbb{Z}) = H_1(N_1; \mathbb{Z}) \oplus H_1(N_2; \mathbb{Z}).$$

In fact, we have $(N_i, \omega_i) = (\mathbb{C}/\Lambda_i, dz)$ where $\Lambda_i = \omega_i(H_1(N_i; \mathbb{Z}))$. Note also that any connected sum of $x = E_1 \underset{[0,v]}{\#} E_2$ gives rise to a natural isomorphism $H_1(M; \mathbb{Z}) = H_1(E_1; \mathbb{Z}) \oplus H_1(E_2; \mathbb{Z})$, and so an algebraic splitting $x = E_1 + E_2$.

Since the set of surfaces $x \in \mathcal{H}_1(2)$ having a splitting with equal areas is dense in $\mathcal{H}_1(2)$. With a bit more effort, we obtain an alternation of the case (2) of Theorem 1.3:

Theorem 1.4. *Let the notation be as in Theorem 1.3. Then the case (2) of Theorem 1.3 can be replaced by*

(2)′ There exists a surface $x'' \in \mathcal{H}_1(2)$ such that
- $d(x'', x) < e^{-\frac{1}{4}t}$,
- *it can be presented as an algebraic sum $x'' = \Lambda_1'' + \Lambda_2''$ such that* $\operatorname{Area}(\Lambda_1'') = \operatorname{Area}(\Lambda_2'')$, *and $G.(\Lambda_1'', \Lambda_2'')$ is periodic with* $\operatorname{vol}(G.(\Lambda_1'', \Lambda_2'')) \leq e^{\delta t}$.

Finally, we shall deduce a criterion for the Teichmüller curves in $\mathcal{H}(2)$ via Ratner's theorem. More precisely, Ratner's theorem gives us certain rigid information of periodic orbits in X. Suppose that there is a surface $x'' \in \mathcal{H}_1(2)$ that can be presented as an algebraic sum $x'' = \Lambda_1'' + \Lambda_2''$ such that $G.(\Lambda_1'', \Lambda_2'')$ is periodic. If we put some additional rationality on $\operatorname{Area}(\Lambda_1'')/\operatorname{Area}(\Lambda_2'')$, then by a criterion obtained in [McM07, §6], x'' generates a Teichmüller curve. Moreover, a detailed analysis about the quantities leads to a discriminant control of this Teichmüller curve.

Then together with Theorem 1.4, we obtain the following density theorem that is close to [LM23, Theorem 1.1] in homogeneous dynamics (see also the effective equidistribution (Theorem 4.5)).

Theorem 1.5. *Let the notation be as in Theorem 1.3. There exists an absolute constant $\kappa_2 > 0$ such that the following holds. For any $\delta \in (0, \delta_0)$, $x \in \mathcal{H}_1(2)$, $L > L_0(\delta, \frac{1}{2}\ell(x), \frac{1}{2}\ell(x))$, and sufficiently large $t > 0$ as in (1.4), we have $|J^{\star\star}| \geq (\varkappa_\vartheta^\star(e^t))^{-1}$, and at least one of the following holds:*

(1) $|J'| \geq \frac{1}{2}\varrho^3 |J^{\star\star}|$, *and for every* $z \in \mathcal{H}_1^{(L^{-1})}(2)$, *we have*

$$d(z, a_{2t}u_{[0,2]}x) \leq L^{-1}.$$

(2) There exists a surface $x''' \in \mathcal{H}_1(2)$ *such that*

- $d(x''', x) < e^{-\frac{1}{4}t}$,
- x''' *generates a Teichmüller curve with discriminant* D *so that*
  - *either* $D < e^{\kappa_2 \delta t}$,
  - *or* D *is a square and* $D < e^{\kappa_2 t}$.

(3) $|J'| \geq \frac{1}{2}\varrho^3 |J^{\star\star}|$, *and*

$$\ell(u_r a_t x) \geq (\varkappa_\vartheta(|J^{\star\star}|))^{\xi}, \quad J' \cap J'' = \emptyset.$$

It is interesting to know whether the case (3) really happens for any function $\varkappa_\vartheta : \mathbb{R}^+ \to \mathbb{R}^+$ with

$$\lim_{t\to\infty} \varkappa_\vartheta(e^t) = 0.$$

As shown in [For21], there exists a set $Z \subset \mathbb{R}$ of zero upper density such that

$$\lim_{t \notin Z} \int_0^1 f(a_t u_r x) dr = \int f d\mu \tag{1.5}$$

for $f \in C_c(\mathcal{H}_1(2))$, where μ is a G-invariant measure on $\mathcal{H}_1(2)$. And Forni conjectured that (1.5) holds with $Z = \emptyset$. The existence of $\varkappa_\vartheta$ so that Theorem 1.5(3) does not occur is linked to this problem.

Recently, Rached established the closing lemma for the P-action on $\mathcal{H}_1(2)$ [Rac24]. Note that the closing lemma in the homogeneous dynamics serves as the "initial dimension phase" for the proofs of effective density and equidistribution [LM23, LMW22]. It would be interesting to know if there is a way to improve the dimension to the unstable direction, and then apply the result of Sanchez [San23] to get effective equidistribution of Px provided the surface x not too close to a small Teichmüller curve, similar to the proofs in [LM23, LMW22].

1.2. **Outline of the proof of Theorem 1.1.** For simplicity, we assume that $z \in \mathcal{H}_1^{(L^{-1})}(2)$ has a splitting $z = \Lambda_1(z) \underset{I(z)}{\#} \Lambda_2(z)$ satisfies $\mathrm{Area}(\Lambda_1(z)) = \mathrm{Area}(\Lambda_2(z))$, so that we may ignore the error coming from the areas of tori.

Then for a sufficiently large discriminant D, the Teichmüller curve $\Omega_1 W_D$ is L^{-1}-close to z (see Figure 1). In fact, there exists $z_D = \Lambda_1(z_D) \underset{I(z_D)}{\#} \Lambda_2(z_D) \in \Omega_1 W_D$ with $\mathrm{Area}(\Lambda_1(z_D)) = \mathrm{Area}(\Lambda_2(z_D))$, such that

$$d(z, z_D) \asymp \|(\Lambda_1(z), \Lambda_2(z)) - (\Lambda_1(z_D), \Lambda_2(z_D))\|_z < L^{-1}. \tag{1.6}$$

Fix some $\eta > 0$. Then on this Teichmüller curve $\Omega_1 W_D$, for any $y \in (\Omega_1 W_D)_\eta$, we apply the effective equidistribution on $\Omega_1 W_D$ and obtain that for any s, there exists some $r' \in [0,1]$ such that

$$d(z_D, a_s u_{r'} y) \leq e^{-\aleph_1 s} \tag{1.7}$$

for some $\aleph_1 > 0$ (depending on D). Then for sufficiently large s (depending on D), the right hand side is $< L^{-1}$.

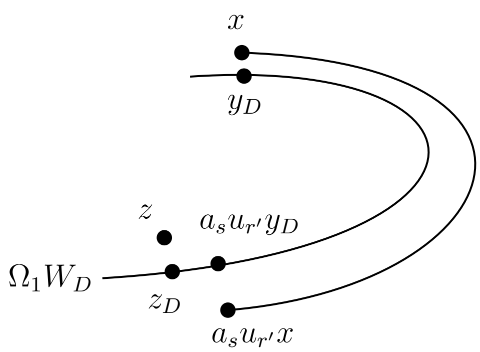

FIGURE 1. Outline of the proof of Theorem 1.1.

Now assume that for given surface $x \in \mathcal{H}_1(2)$, there is $y_D \in (\Omega_1 W_D)_\eta$ so that

$$d(y_D, x) < e^{-t}. \tag{1.8}$$

Combining (1.7) and (1.8), we obtain

$$\begin{aligned} d(z_D, a_s u_{r'} x) &\leq d(z_D, a_s u_{r'} y_D) + d(a_s u_{r'} y_D, a_s u_{r'} x) \\ &\leq d(z_D, a_s u_{r'} y_D) + e^{\aleph_2 s} d(y_D, x) \\ &\leq L^{-1} + e^{\aleph_2 s - t} \end{aligned}$$

for some $\aleph_2 > 0$. Then for sufficiently large $t > 0$ (depending on s), the right hand side is $< 2L^{-1}$.

Finally, by (1.6), we conclude that

$$d(z, a_s u_{r'} x) \leq d(z, z_D) + d(z_D, a_s u_{r'} x) < 3L^{-1}.$$

See Section 5 for more details.

1.3. **Outline of the proof of Theorem 1.3.** The idea of Theorem 1.3 is that when the absolute periods $a_t u_r \Lambda(x)$ of $a_t u_r x$ is closed to Q_D in the sense of a fixed norm $\|\cdot\|$, we require that $a_t u_r x$ is close to Q_D in the sense of AGY norm $\|\cdot\|$ for at least one time r.

In order to show that $a_t u_{[0,1]} x$ can be close to the Teichmüller curve $\Omega_1 W_D$ for at least one time r. We need that $a_t u_r x$ does not go to infinity. However, this does not always seem to be the case.

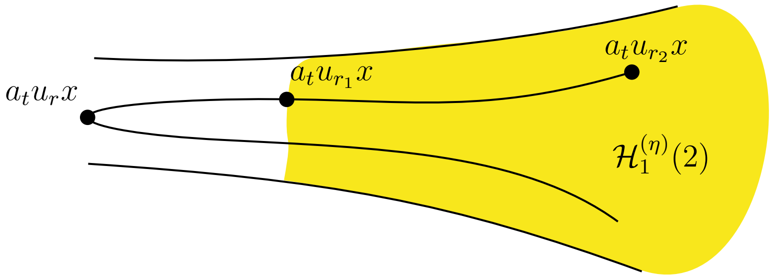

FIGURE 2. Quantitative nondivergence of horocycle flows on $\mathcal{H}_1(2)$.

To fix this, we observe the quantitative nondivergence of horocycle flows (Figure 2). In fact, if $\ell(a_t u_r x)$ is very small, then (by the nature of (C, α)-good functions) one may expect that there is a considerable interval $[r_1, r_2] \subset [0, 1]$ near r, so that $a_t u_{[r_1,r_2]} x$ does not go to infinity. Then we can apply the effective equidistribution to $[r_1, r_2]$, so that the absolute periods are close to the Teichmüller curve again. More precisely, for any $t > 0$, there is some $r^\star \in [r_1, r_2]$, and $(\Lambda_{1,D}, \Lambda_{2,D}) \in Q_D$ such that

$$\|(\Lambda_{1,D}, \Lambda_{2,D}) - a_t u_{r^\star} \tilde{x}\|_{a_t u_{r^\star} \tilde{x}} \leq e^{-\aleph_3 t}$$

for some $\aleph_3 > 0$. Eventually, we obtain a surface with sufficiently large injectivity radius, which guarantees a surface $y_D \in (\Omega_1 W_D)_\eta$ such that

$$\|\tilde{y}_D - a_t u_{r^\star} \tilde{x}\|_{a_t u_{r^\star} \tilde{x}} \leq e^{-\aleph_3 t}.$$

The details of this are in Section 6.

1.4. **Structure of the paper.** In Section 2 we recall basic definitions, including some basic material on the translation surfaces and Teichmüller curves (in Section 2.2, Section 2.3). In particular, we study the period map via triangulation (Section 2.6) and gives quantitative estimates in terms of Avila-Gouëzel-Yoccoz norm (Section 2.5).

In Section 3, we review the dynamics over $\mathcal{H}(2)$. In particular, we recall the McMullen's classification of Teichmüller curves in $\mathcal{H}(2)$. From this, we deduce some quantitative estimates of Teichmüller curves.

In Section 4, we review the effective results in homogenous dynamics, which shall serve as an effective estimate of absolute periods of surfaces in $\mathcal{H}(2)$.

In Section 5, we prove Theorem 1.1, as well as Theorems 1.3 and 1.4 by assuming certain nondivergence property. The proofs connect the effective estimates of absolute periods provided in Section 4, and the quantitative observations obtained in Section 3.

In Section 6, we show that the nondivergence assumption made in Section 5 can actually be removed. It relies on Minsky and Weiss's work on the nondivergence results of horocycle flows on $\mathcal{H}(2)$ [MW02]. In particular, we review the technique of sparse covers by (C, α)-good functions and obtain a large interval to apply the equidistribution again.

Finally, we present in Section 7 the proof of Theorem 1.5. More precisely, via Ratner's theorem and a criterion by McMullen, we conclude that if a surface has G-closed absolute periods and some additional rationality on the areas of tori, then it generates a Teichmüller curve.

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2. Preliminaries

2.1. **Notation.** We will denote the metric on all relevant metric spaces by $d(\cdot,\cdot)$; where this may cause confusion, we will give the metric space as a subscript, e.g. $d_X(\cdot,\cdot)$ etc. Next, $B(x,r)$ denotes the open ball of radius r in the metric space x belongs to; where needed, the space we work in will be given as a subscript, e.g. $B_T(x,r)$. We will assume implicitly that for any $x \in X$ (as well as any other locally compact metric space we will consider) and $r > 0$ the ball $B_X(x,r)$ is relatively compact.

We will use the asymptotic notation $A = O(B)$, $A \ll B$, or $A \gg B$, for positive quantities A, B to mean the estimate $|A| \leq CB$ for some constant C independent of B. In some cases, we will need this constant C to depend on a parameter (e.g. d), in which case we shall indicate this dependence by subscripts, e.g. $A = O_d(B)$ or $A \ll_d B$. We also use $A \asymp B$ as a synonym for $A \ll B \ll A$.

Let $G = \mathrm{SL}_2(\mathbb{R})$, $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Besides, we define

$$B_G(T) \coloneqq \{g \in G : \|g - e\| \leq T\}$$

where e is the identity of G and $\|\cdot\|$ is a fixed norm on the Euclidean space, e.g. $\|\cdot\|$ can be defined by

$$\|g\| \coloneqq \max_{ij}\{|g_{ij}|, |g_{ij}^{-1}|\}$$

where g_{ij} is the ij-th entry of g. Let $X = G/\Gamma \times G/\Gamma$. Let $d_X(\cdot,\cdot)$ be a right-invariant metric on $G \times G$ and the induced metric on X.

2.2. **Translation surfaces.** Let M be a compact oriented surface of genus g, and let $\Sigma \subset M$ be a nonempty finite set, called the set of zeroes. We make the convention that the points of Σ are labeled. Let $\alpha = \{\alpha_\sigma : \sigma \in \Sigma\}$ be a partition of $2g-2$, so $\sum_{\sigma\in\Sigma} \alpha_\sigma = 2g-2$.

Definition 2.1 (Translation surface)**.** A surface M is called a *translation surface of type* α if it has an affine atlas, i.e. a family of orientation preserving charts $\{(U_a, z_a)\}_a$ such that

- the $U_a \subset M \setminus \Sigma$ are open and cover $M \setminus \Sigma$,
- the transition maps $z_a \circ z_b^{-1}$ have the form $z \mapsto z + c$.

In addition, the planar structure of M in a neighborhood of each $\sigma \in \Sigma$ completes to a cone angle singularity of total cone angle $2\pi(\alpha_\sigma + 1)$.

There are many equivalent definitions of a translation surface, and a convenient one is a pair (M, ω) consisting of a compact Riemann surface and a holomorphic 1-form ω. We shall use these definitions interchangeably.

There is a natural $\mathrm{GL}_2^+(\mathbb{R})$ action on the translation surfaces. Let M be a translation surface with an atlas $\{(U_a, z_a)\}_a$. Since any matrix $h \in \mathrm{GL}_2^+(\mathbb{R})$ acts on $\mathbb{C} = \mathbb{R} + i\mathbb{R}$, we obtain a new atlas $\{(U_a, h \circ z_a)\}_a$, which induces a new translation surface hM.

An *affine isomorphism* is an orientation preserving homeomorphism $f : M_1 \to M_2$ which is affine in each chart. If $M_1 = M_2$, it is called an *affine automorphism* instead. Let $\mathrm{Aff}(M)$ denote the set of affine automorphisms of M. If an affine isomorphism whose linear part is $\pm\,\mathrm{Id}$ (for translation surfaces, Id), it is called a *translation equivalence*. Let $\mathcal{H}(\alpha) = \Omega\mathcal{M}_g(\alpha)$ denote the space of equivalence classes of translation surfaces of type α. We refer to $\mathcal{H}(\alpha)$ as the *moduli space of translation surfaces of type* α.

A *saddle connection* of M is a geodesic segment joining two zeroes in Σ or a zero to itself which has no zeroes in its interior.

We fix a compact surface (S, Σ) and refer to it as the model surface. A *marking map* of a surface M is a homeomorphism $\varphi : (S, \Sigma) \to (M, \Sigma_M)$ which preserves labels on Σ. (We sometimes drop the subscript and use the same symbol Σ to denote finite subsets of S and of M, if no confusion arises.) Two marking maps $\varphi_1 : (S, \Sigma) \to (M_1, \Sigma_{M_1})$ and $\varphi_2 : (S, \Sigma) \to (M_2, \Sigma_{M_2})$ are said to be *equivalent* if there is a translation equivalence $f : M_1 \to M_2$ such that

- $f \circ \varphi_1$ is isotopic to φ_2,
- f maps $\Sigma_{M_1} \to \Sigma_{M_2}$ respecting the labels.

An equivalent class of translation surfaces with marking maps is a *marked translation surface*. The space of marked translation surfaces of type α is denoted by $\mathcal{TH}(\alpha) = \Omega\mathcal{T}_g(\alpha)$. We refer to $\mathcal{TH}(\alpha)$ as the *Teichmüller space of marked translation surfaces of type* α. By forgetting the marking maps, we get a natural map $\pi : \mathcal{TH}(\alpha) \to \mathcal{H}(\alpha)$.

We can locally identify $\mathcal{TH}(\alpha)$ (and so $\mathcal{H}(\alpha)$) with $H^1(M, \Sigma; \mathbb{C})$. Let $\tilde{x} \in \mathcal{TH}(\alpha)$ be a marked translation surface with the marking $\varphi : (S, \Sigma) \to (M, \Sigma)$. Suppose M is equipped with a holomorphic 1-form ω. Then the *period map*

$$\omega' \mapsto \left(\gamma \mapsto \int_\gamma \omega'\right)$$

from a neighborhood of ω to $H^1(M,\Sigma;\mathbb{C})$ gives a local homeomorphism. Let $\tilde{x} \in \mathcal{TH}(\alpha)$ be a marked translation surface with the marking $\varphi : (S,\Sigma) \to (M,\Sigma)$. Suppose M is equipped with a holomorphic 1-form ω. Then after using the marking map φ to pullback ω, we get a distinguished element $\mathrm{hol}_{\tilde{x}} = \varphi^*(\omega) \in H^1(S,\Sigma;\mathbb{R}^2) \cong H^1(S,\Sigma;\mathbb{C})$. Thus, if $\gamma \in H_1(S,\Sigma;\mathbb{Z})$ is an oriented curve in S with endpoints in Σ, then

$$\mathrm{hol}_{\tilde{x}}(\gamma) = \tilde{x}(\gamma) \coloneqq \omega(\varphi(\gamma)).$$

We also refer to the map $\mathrm{hol} : \mathcal{TH}(\alpha) \to H^1(S,\Sigma;\mathbb{C})$ as the *developing map* or *period map*. It is a local homeomorphism (see Lemma 2.15). If we fix $2g + |\Sigma| - 1$ curves $\gamma_1, \ldots, \gamma_{2g+|\Sigma|-1}$ that form a basis for $H_1(S,\Sigma;\mathbb{Z})$, then it defines the *period coordinates* $\phi : \mathcal{TH}(\alpha) \to \mathbb{C}^{2g+|\Sigma|-1}$ by

$$\phi : \tilde{x} \mapsto (\mathrm{hol}_{\tilde{x}}(\gamma_i))_{i=1}^{2g+|\Sigma|-1}.$$

It is convenient to assume that the basis is obtained by fixing a triangulation τ of the surface by saddle connections of x (see Definition 2.14). Via the *Gauss-Manin connection*, period coordinates endow $\mathcal{TH}(\alpha)$ with a canonical complex affine structure.

Let $\mathrm{Mod}(M,\Sigma)$ be the group of isotopy classes of homeomorphisms M which fix Σ pointwise for a representative (M,Σ) of the stratum α. We will call this group the *mapping class group*. It acts on the right on $\mathcal{TH}(\alpha)$: letting $\tilde{x} \in \mathcal{TH}(\alpha)$ with a marking map $\varphi : (S,\Sigma) \to (M,\Sigma)$, $\gamma \in \mathrm{Mod}(M,\Sigma)$, we have the action

$$\gamma.\varphi = \varphi \circ \gamma.$$

The $\mathrm{Mod}(M,\Sigma)$-action on $\mathcal{TH}(\alpha)$ is properly discontinuous (e.g. [FM11, Theorem 12.2]). Hence, $\mathcal{H}(\alpha) = \mathcal{TH}(\alpha)/\mathrm{Mod}(M,\Sigma)$ has an orbifold structure. We choose a fundamental domain $\mathcal{D}$ on $\mathcal{TH}(\alpha)$ for the action of $\mathrm{Mod}(M,\Sigma)$. Note that $\mathrm{Mod}(M,\Sigma)$ also acts on the right by linear automorphisms on $H^1(S,\Sigma;\mathbb{R})$. Let $R : \mathrm{Mod}(M,\Sigma) \to \mathrm{Aut}(H^1(S,\Sigma;\mathbb{R})) \cong \mathrm{GL}(2g + |\Sigma| - 1, \mathbb{R})$. Since each element of $\mathrm{Mod}(M,\Sigma)$ is represented by an orientation-preserving homeomorphism of S, it follows that the image of R lies in $\mathrm{SL}(2g + |\Sigma| - 1, \mathbb{R})$.

Let $\tilde{x} \in \mathcal{D} \subset \mathcal{TH}(\alpha)$ and $h \in \mathrm{GL}_2^+(\mathbb{R})$. Then there is a unique element $\gamma \in \Gamma$ so that $h\tilde{x}\gamma \in \mathcal{D}$. The *Kontsevich-Zorich cocycle* is then defined by

$$R(h,\tilde{x}) \coloneqq R(\gamma).$$

Then for $\tilde{x} \in \mathcal{D} \subset \mathcal{TH}(\alpha)$, the G-action becomes

$$Dh_{\tilde{x}} : \begin{bmatrix} x_1 & \cdots & x_{2g+|\Sigma|-1} \\ y_1 & \cdots & y_{2g+|\Sigma|-1} \end{bmatrix} \mapsto h \begin{bmatrix} x_1 & \cdots & x_{2g+|\Sigma|-1} \\ y_1 & \cdots & y_{2g+|\Sigma|-1} \end{bmatrix} R(h,\tilde{x}). \tag{2.1}$$

See e.g. [Fil16, §2] for more details.

In the literature, we sometimes refer to $\mathcal{TH}(\alpha)$ and $\mathcal{H}(\alpha)$ as a stratum of $\mathcal{TH}^g$ and $\mathcal{H}^g$, namely the Teichmüller and moduli spaces of translation surfaces of genus

g, respectively. This is because we have the stratification

$$\mathcal{TH}^g = \bigsqcup_{\alpha_1+\cdots+\alpha_\sigma=2g-2} \mathcal{TH}(\alpha), \qquad \mathcal{H}^g = \bigsqcup_{\alpha_1+\cdots+\alpha_\sigma=2g-2} \mathcal{H}(\alpha).$$

On the other hand, let $\mathcal{T}_g$, $\mathcal{M}_g$ denote the Teichmüller and moduli spaces of Riemann surfaces of genus g respectively. Let $\Omega(M)$ denote the g-dimensional vector space of all holomorphic 1-forms of M. Then we may consider $\mathcal{TH}^g$ and $\mathcal{H}^g$ as vector bundles over $\mathcal{T}_g$, $\mathcal{M}_g$:

$$\mathcal{TH}^g = \Omega\mathcal{T}_g \to \mathcal{T}_g, \quad \mathcal{H}^g = \Omega\mathcal{M}_g \to \mathcal{M}_g$$

whose fiber over M is $\Omega(M) \setminus \{0\}$.

Suppose that $x = (M, \omega) \in \mathcal{H}(\alpha)$ is a translation surface of type α. Let $\mathrm{Area}(M, \omega)$ be the area of translation surface given by

$$\mathrm{Area}(M,\omega) \coloneqq \frac{i}{2}\int_M \omega \wedge \bar{\omega} = \frac{i}{2}\sum_{j=1}^{g}(A_j(\omega)\bar{B}_j(\omega) - B_j(\omega)\bar{A}_j(\omega))$$

where $A_j(\omega), B_j(\omega)$ form a canonical basis of absolute periods of ω, i.e.

$$A_j(\omega) = \int_{\alpha_j} \omega, \quad B_j(\omega) = \int_{\beta_j} \omega$$

and $\{\alpha_j, \beta_j\}_{j=1}^g$ is a symplectic basis of $H_1(M;\mathbb{R})$ (with respect to the intersection form). Let

$$\mathcal{H}_1(\alpha) \coloneqq \{(M,\omega) \in \mathcal{H}(\alpha) : \mathrm{Area}(M,\omega) = 1\}.$$

We see that the normalized stratum $\mathcal{H}_1(\alpha)$ resembles more a "unit hyperboloid". Note that $\mathcal{H}_1(\alpha)$ is a codimension one sub-orbifold of $\mathcal{H}(\alpha)$ but it is **not** an affine sub-orbifold. Let $\pi_1 : \mathcal{H}(\alpha) \to \mathcal{H}_1(\alpha)$ be the normalization of the area. We abuse notation and use the same symbol $\pi_1 : \mathcal{TH}(\alpha) \to \mathcal{H}_1(\alpha)$ to refer to the composition of the projection and normalization.

Let λ be the measure on $\mathcal{H}(\alpha)$ which is given by the pullback of the Lebesgue measure on $H^1(S, \Sigma; \mathbb{C}) \cong \mathbb{C}^{2g+|\Sigma|-1}$. We refer to λ as the *Lebesgue* or the *Masur-Veech measure* on $\mathcal{H}(\alpha)$. Let $\lambda_{(1)}$ be the $\mathrm{SL}(2,\mathbb{R})$-invariant Lebesgue (probability) measure on the "hyperboloid" $\mathcal{H}_1(\alpha)$ defined by the disintegration of the Lebesgue measure λ on $\mathcal{H}_1(\alpha)$, namely

$$d\lambda = r^{2g+|\Sigma|-2}dr \cdot d\lambda_{(1)}.$$

2.3. **Teichmüller curves.** As we have seen, there is a natural $G = \mathrm{SL}_2(\mathbb{R})$ action on $\mathcal{H}_1(\alpha)$. We are then interested in its smallest G-orbit closure:

Definition 2.2 (Teichmüller curve)**.** A *Teichmüller curve* $f : V \to \mathcal{M}_g$ is a finite volume hyperbolic Riemann surface V equipped with a holomorphic, totally geodesic, generically 1-1 immersion into moduli space.

Let $(M,\omega) \in \mathcal{H}^g$ be a translation surface. Recall that $\mathrm{Aff}(M)$ denotes the set of affine automorphisms. Consider the map $D : \mathrm{Aff}(M) \to G$ which assigns to an affine automorphism its linear part. It has a finite kernel Γ_M, consisting of translation equivalences of M. The image $\mathrm{SL}(M,\omega) \coloneqq D(\mathrm{Aff}(M))$ is called the *Veech group* of M. Then we have the short exact sequence:

$$0 \to \Gamma_M \to \mathrm{Aff}(M) \to \mathrm{SL}(M,\omega) \to 0. \tag{2.2}$$

The equivalent conditions for the lattice property of $\mathrm{SL}(M,\omega)$ has been studied by a vast literature (e.g. [SW10] and references therein):

Theorem 2.3. *For $x \in \Omega\mathcal{M}_g$, the following are equivalent:*

- *The group $\mathrm{SL}(x)$ is a lattice in $G = \mathrm{SL}_2(\mathbb{R})$.*
- *The orbit $G.x$ is closed in $\Omega\mathcal{M}_g$.*
- *The projection of the orbit to $\mathcal{M}_g$ is a Teichmüller curve.*

In this case, we say x *generates* the Teichmüller curve $V = \mathbb{H}/\,\mathrm{SL}(x) \to \mathcal{M}_g$. It follows that

$$\Omega V = \mathrm{GL}_2^+(\mathbb{R}).x \cong \mathrm{GL}_2^+(\mathbb{R})/\,\mathrm{SL}(x)$$

can be regarded as a bundle over V. Thus, we also abuse notation and refer to $f(V)$, or the $\mathbb{C}^*$-bundle ΩV (and the circle bundle $\Omega_1 V$) as a Teichmüller curve, if no confusion arise.

2.4. **Nondivergence.** First, we review the quantitative nondivergence in the homogeneous dynamics. Let $G = \mathrm{SL}_2(\mathbb{R})$, $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and $X = G/\Gamma \times G/\Gamma$. Note that any $\Lambda_1 \in G/\Gamma$ corresponds to a lattice in $\Lambda_1 \subset \mathbb{R}^2$. For $\Lambda_1 \in G/\Gamma$, define the systole function $\ell : G/\Gamma \to \mathbb{R}^+$ by

$$\ell(\Lambda_1) \coloneqq \min\{r : \Lambda_1 \subset \mathbb{R}^2 \text{ contains a vector of length} \le r\}.$$

Next, abuse notation and define $\ell : X \to \mathbb{R}^+$ by

$$\ell(\Lambda_1, \Lambda_2) \coloneqq \min\{\ell(\Lambda_1), \ell(\Lambda_2)\}.$$

Note that for all $\eta > 0$, the set

$$X_\eta \coloneqq \{x \in X : \ell(x) \ge \eta\}$$

is compact. In homogeneous dynamics, we have the following nondiverngence result, ultimately attributed to Margulis, Dani, and Kleinbock.

Theorem 2.4 ([LM23, Proposition 3.1])**.** *There exists $C \ge 1$ with the following property: Let $\epsilon, \eta \in (0,1)$, $(\Lambda_1, \Lambda_2) \in X$. Let $I \subset \mathbb{R}$ be an interval of length $|I| \ge \eta$. Then*

$$|\{r \in I : \ell(a_t u_r(\Lambda_1, \Lambda_2)) < \epsilon^2\}| < C\epsilon|I|$$

so long as $t \ge |\log(\eta^2 \operatorname{inj}(x))| + C$.

In the following, we shall develop a similar result in Teichmüller dynamics. Define the systole function $\ell : \mathcal{TH}(\alpha) \to \mathbb{R}^+$ by the shortest length of a saddle connection. Note that for all $\epsilon > 0$, the set

$$\mathcal{H}_1^{(\epsilon)}(\alpha) := \{x \in \mathcal{H}_1(\alpha) : \ell(x) \geq \epsilon\} \tag{2.3}$$

is compact. To say it differently, a sequence $x_n \in \mathcal{H}_1(\alpha)$ diverges to infinity iff $\ell(x_n) \to 0$. In addition, by the Siegel-Veech formula (see e.g. [EM01, AGY06]), we have

$$\lambda_{(1)}(\mathcal{H}_1(\alpha) \setminus \mathcal{H}_1^{(\epsilon)}(\alpha)) = \lambda_{(1)}\{x \in \mathcal{H}_1(\alpha) : \ell(x) < \epsilon\} \asymp O(\epsilon^2). \tag{2.4}$$

Now, we follow the idea in [EM01] and [Ath06] to discuss the non-divergence results. See also [AG13, §6] and [EMM22, §2].

Theorem 2.5 ([EM01, Ath06])**.** *There exist a continuous function $V : \mathcal{H}_1(\alpha) \to [2, \infty)$, a compact subset $K'_\alpha \subset \mathcal{H}_1(\alpha)$ and some $\kappa_3 > 0$ with the following property. For every t' and every $x \in \mathcal{H}_1(\alpha)$, there exist*

$$s \in [0, 1/2], \quad t' \leq t \leq \max\{2t', \kappa_3 \log V(x)\}$$

such that $a_t u_s x \in K'_\alpha$. Further, there exists a constant $C_2 > 1$ such that

$$C_2{}^{-1} \leq \frac{V(x)}{\max\{\ell(x)^{-5/4}, 1\}} \leq C_2.$$

where ℓ denotes the systole function.

We also need the following averaging nondivergence of horocyclic flows.

Theorem 2.6 ([MW02, Theorem 6.3])**.** *There are positive constants C_3, κ_4, ρ_0, depending only on α, such that if $\tilde{x} \in \mathcal{TH}_1(\alpha)$, an interval $I \subset \mathbb{R}$, and $\rho \in (0, \rho_0]$ satisfy:*

$$\sup_{s \in I} \ell(u_s \tilde{x}) \geq \rho,$$

then for any $\epsilon \in (0, \rho)$, we have

$$|\{s \in I : \ell(u_s \tilde{x}) < \epsilon\}| \leq C_3 \left(\frac{\epsilon}{\rho}\right)^{\kappa_4} |I|. \tag{2.5}$$

In Section 6, we shall revisit the technique in the proof of Theorem 2.6, in order to analyze the behavior of points going to infinity.

Now we are in the position to establish the desired nondivergence result. (See also Theorem 2.4.)

Corollary 2.7. *There are positive constants C_4, κ_5, depending only on α, with the following property. Let $\epsilon > 0$, $\eta > 0$, and $\tilde{x} \in \mathcal{TH}_1(\alpha)$. Let $I \subset [-10, 10]$ be an interval with $|I| \geq \eta$. Then we have*

$$|\{r \in I : \ell(a_t u_r \tilde{x}) < \epsilon\}| \leq C_4 \epsilon^{\kappa_4} |I|$$

whenever $t \geq \kappa_5 |\log \ell(x)| + 2|\log \eta| + C_4$.

Proof. Assume for simplicity that $I = [0,1]$. More general situation follows from a similar argument. Let $\epsilon_1 > 0$ satisfy $K'_\alpha \subset \mathcal{H}_1^{(\epsilon_1)}(\alpha)$. Let $\tilde{x} \in \mathcal{TH}_1(\alpha)$. Without loss of generality, we assume that $\ell(\tilde{x}) \ll 1$. Let $t' = \max\{1, \frac{\kappa_3}{2} \log V(x)\}$. Then applying Theorem 2.5 to x and t', there exist

$$s_0 \in [0, 1/2], \quad t_0 \in [1, \kappa_3 \log C_2 \ell(x)^{-5/4}]$$

such that $x_0 := a_{t_0} u_{s_0} x \in K'_\alpha$.

Next, let $t' = \max\{1, \frac{\kappa_3}{2} \log V(x_0)\}$, $C = \kappa_3 \log C_2 \epsilon_1^{-5/4}$. Applying Theorem 2.5 again to x_0 and t', we obtain that there exist

$$s_1 \in [0, 1/2], \quad t_1 \in [1, C]$$

such that $x_1 := a_{t_1} u_{s_1} x_0 \in K'_\alpha$. Repeating the argument, we further obtain that there exist

$$s_i \in [0, 1/2], \quad t_i \in [1, C]$$

such that $x_i := a_{t_i} u_{s_i} x_{i-1} \in K'_\alpha$ for all $i \in \mathbb{N}$. One calculates that $x_i = a_{t(i)} u_{s(i)} x$ where

$$t(i) := t_i + \cdots + t_0, \quad \text{and} \quad s(i) := \sum_{j=0}^{i} \frac{s_j}{e^{t_{j-1} + \cdots + t_0}} \leq \sum_{j=0}^{i} \frac{s_j}{e^j} \leq 1.$$

Then we see that

$$\sup_{s \in [0,1]} \ell(u_{e^{t(i)} s} a_{t(i)} \tilde{x}) = \sup_{s \in [0,1]} \ell(a_{t(i)} u_s \tilde{x}) \geq \epsilon_1.$$

Moreover, note that $t(i) - t(i-1) = t_i \leq C$. Then for any $t \geq t_0$, we have

$$\sup_{s \in [0,1]} \ell(u_{e^t s} a_t \tilde{x}) = \sup_{s \in [0,1]} \ell(a_t u_s \tilde{x}) \geq \epsilon_1 e^{-C}.$$

Now applying Theorem 2.6, we obtain that

$$|\{s \in [0,1] : \ell(a_t u_s \tilde{x}) < \epsilon\}| \leq C_3 \left(\epsilon_1 e^{-C}\right)^{-\kappa_4} \epsilon^{\kappa_4}$$

whenever $t \geq \kappa_3 \log C_2 \ell(x)^{-5/4}$. Letting $\kappa_5 = \frac{5}{4}\kappa_3$ and $C_4 = \kappa_3 \log C_2$, we establish (2.5). □

2.5. **Avila-Gouëzel-Yoccoz norm.** We now introduce the AGY norm, first defined in [AGY06], some properties of which were further developed in [AG13].

Definition 2.8 (AGY norm)**.** For $\tilde{x} \in \mathcal{TH}(\alpha)$ and any $c \in H^1(M, \Sigma; \mathbb{C})$, we define

$$\|c\|_{\tilde{x}} := \sup_\gamma \frac{|c(\gamma)|}{|\int_\gamma \omega|}$$

where γ is a saddle connection of $\tilde{x}$. We refer to $\|\cdot\|_{\tilde{x}}$ as the *Avila-Gouëzel-Yoccoz norm* or *AGY norm* for short.

Note first that by definition, for $w = a + ib \in H^1(M, \Sigma; \mathbb{C})$, we have

$$\max\{\|a\|_x, \|b\|_x\} \leq \|w\|_x \leq \|a\|_x + \|b\|_x. \tag{2.6}$$

By construction, the AGY norm is invariant under the action of the mapping class group Γ. Thus, it induces a norm on the moduli space $\mathcal{H}(\alpha)$.

It was shown in [AGY06, §2.2.2] that this defines a norm and the corresponding Finsler metric is complete. For $\tilde{x}, \tilde{y} \in \mathcal{TH}(\alpha)$, we define a distance

$$d(\tilde{x}, \tilde{y}) \coloneqq \inf_{\gamma} \int_0^1 \|\gamma'(r)\|_{\gamma(r)} dr$$

where γ ranges over smooth paths $\gamma : [0,1] \to \mathcal{TH}(\alpha)$ with $\gamma(0) = \tilde{x}$ and $\gamma(1) = \tilde{y}$. It also induces a quotient metric on $\mathcal{H}(\alpha)$.

Due to the splitting

$$H^1(M, \Sigma; \mathbb{C}) = H^1(M, \Sigma; \mathbb{R}) \oplus iH^1(M, \Sigma; \mathbb{R}),$$

we often write an element of $H^1(M, \Sigma; \mathbb{C})$ as $a + ib$ for $a, b \in H^1(M, \Sigma; \mathbb{R})$. Let $\tilde{x} \in \mathcal{TH}(\alpha)$. For every $r > 0$, define

$$R(\tilde{x}, r) \coloneqq \{\phi(\tilde{x}) + a + ib : a, b \in H^1(M, \Sigma; \mathbb{R}),\ \|a + ib\|_{\tilde{x}} \leq r\}.$$

Let $r > 0$ be so that ϕ^{-1} is a homeomorphism on $R_{\tilde{x}}(r)$. Let

$$B(\tilde{x}, r) \coloneqq \phi^{-1}(R(\tilde{x}, r)).$$

We call it a *period box* of radius r centered at $\tilde{x}$. Using [AG13, Proposition 5.3], $B(\tilde{x}, r)$ is well defined for all $r \in (0, 1/2]$ and all $\tilde{x} \in \mathcal{TH}(\alpha)$. Let $\mathrm{inj}(\tilde{x})$ be the injectivity radius of $\tilde{x}$ under the affine exponential map.

We have the following estimate:

Lemma 2.9. *Let $\tilde{x} \in \mathcal{TH}(\alpha)$. Then for all $\tilde{y}, \tilde{z} \in B(\tilde{x}, \mathrm{inj}(\tilde{x})/50)$, we have*

$$\frac{1}{2}\|\tilde{y} - \tilde{z}\|_{\tilde{y}} \leq \|\tilde{y} - \tilde{z}\|_{\tilde{z}} \leq 2\|\tilde{y} - \tilde{z}\|_{\tilde{y}},$$

and further

$$\frac{1}{4}\|\tilde{y} - \tilde{z}\|_{\tilde{x}} \leq d(\tilde{y}, \tilde{z}) \leq 4\|\tilde{y} - \tilde{z}\|_{\tilde{x}}.$$

Proof. It is actually a rephrasing of [AG13, Proposition 5.3]. See also [CKS23, Lemma 3.3]. □

Lemma 2.10. *For $g \in \mathrm{GL}_2(\mathbb{R})$, $w \in H^1(x)$, we have $\|gw - w\|_x \leq \|g - I\|\|w\|_x$. Moreover, we have $\|gx - x\|_x = \|g - I\|$.*

Proof. Clearly, one calculates,

$$\|gw - w\|_x = \sup_{\gamma} \frac{|(g - I)w(\gamma)|}{|x(\gamma)|} \leq \|g - I\| \sup_{\gamma} \frac{|w(\gamma)|}{|x(\gamma)|} = \|g - I\|\|w\|_x.$$

Moreover, since the slopes of saddle connections are dense in $\mathbb{R} \cup \{\infty\}$ (see [MT02, §4]), we have

$$\|g - I\| = \sup_{\gamma} \left| (g - I) \frac{x(\gamma)}{|x(\gamma)|} \right| = \sup_{\gamma} \frac{|(g - I)x(\gamma)|}{|x(\gamma)|} = \|gx - x\|_x.$$

This establishes the claim. □

Lemma 2.11 ([AG13, Lemma 5.1]). *For $x \in \mathcal{H}(\alpha)$, $g \in G$, we have*

$$d(x, gx) \leq d_G(e, g).$$

In particular, via the Cartan decomposition $g = kak'$, we have

$$d(x, gx) \leq \log \|a\| + 4\pi.$$

The following crude estimates are well known, e.g. [CSW20, Corollary 2.6].

Theorem 2.12. *For all $s, t \in \mathbb{R}$, $x \in \mathcal{H}(\alpha)$, we have*

$$\|u_s v\|_{u_s x} \leq \left(1 + \frac{s^2 + |s|\sqrt{s^2 + 4}}{2}\right) \|v\|_x$$

and

$$\|a_t v\|_{a_t x} \leq e^{2|t|} \|v\|_x.$$

Lemma 2.13 ([EMM22, Lemma 2.6]). *There exist $\kappa_6 = \kappa_6(\alpha) > 0$ and $C_5 > 1$ so that for all $x \in \mathcal{H}_1(\alpha)$, the following hold. For any $r \in (0, C_5 \ell(x)^{\kappa_6}]$, any lift $\tilde{x}$ of x, the restriction of the covering map $\pi : \mathcal{TH}(\alpha) \to \mathcal{H}(\alpha)$ to $B(\tilde{x}, r)$ is injective.*

2.6. **Triangulation.** In application, we usually choose a triangulation for the period coordinates, i.e. fix a triangulation τ of the surface and choose a sequence of saddle connections from τ which form a basis for $H_1(M, \Sigma; \mathbb{Z})$. In this section, we follow the idea in [MS91] to discuss the period coordinates, and find a lower bound of the non-degenerate deformations of a triangulation. In particular, this gives a lower bound of the injectivity radius of the period map $\phi : \mathcal{TH}(\alpha) \to H^1(M, \Sigma; \mathbb{C})$. We adopt the notation introduced in [CSW20].

Definition 2.14 (geodesic triangulation)**.** We say τ is a *geodesic triangulation* of x if it is a decomposition of the surface into triangles whose sides are saddle connections, and whose vertices are singular points, which need not be distinct.

In [MS91, §4], Masur and Smillie showed that every translation surface $x \in \mathcal{H}(\alpha)$ admits a *Delaunay triangulation* τ_x, which is a typical geodesic triangulation. By the construction, each triangle $\Delta \in \tau_x$ can be inscribed in a disk of radius not greater than the diameter $d(M)$ of M (cf. [MS91, Theorem 4.4]).

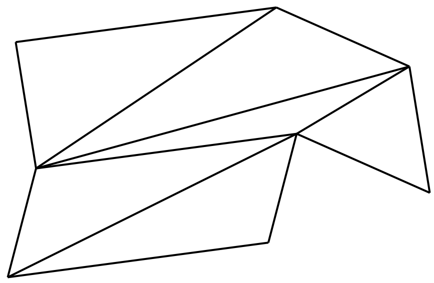

FIGURE 3. A Delaunay triangulation of a surface in $\mathcal{H}(2)$.

Let $\tilde{x} \in \mathcal{TH}(\alpha)$ be a marked translation surface with the marking map $\varphi : (S, \Sigma) \to (M, \Sigma)$, and $x = \pi(\tilde{x}) = (M, \omega) \in \mathcal{H}(\alpha)$. Let $\tau_{\tilde{x}}$ denote the pullback of the Delaunay triangulation with vertices in Σ, from (M, Σ) to (S, Σ).

Note that the period map $\mathrm{hol}_{\tilde{x}}(\gamma)$ can be thought of as giving a map from the triangles of $\tau_{\tilde{x}}$ to triangles in $\mathbb{C} \cong \mathbb{R}^2$ (well-defined up to translation). Moreover, we can define a local inverse of the period map as follows.

Let $U_{\tilde{x}} \subset H^1(S, \Sigma; \mathbb{C})$ be the collection of all cohomology classes which map each triangle of $\tau_{\tilde{x}}$ into a positively oriented non-degenerate triangle in $\mathbb{C}$. Each $\nu \in U_{\tilde{x}}$ gives a translation surface $M_{\tilde{x},\nu}$ built by gluing together the corresponding triangles in $\mathbb{C}$ along parallel edges, and a marking map $\varphi_{\tilde{x},\nu} : (S, \Sigma) \to (M_{\tilde{x},\nu}, \Sigma)$, by taking each triangle of the triangulation $\tau_{\tilde{x}}$ of S to the corresponding triangle of the triangulation of $M_{\tilde{x},\nu}$. Let $\tilde{y}_{\tilde{x},\nu} \in \mathcal{TH}(\alpha)$ denote the marked translation surface corresponding to the marking map $\varphi_{\tilde{x},\nu} : (S, \Sigma) \to (M_{\tilde{x},\nu}, \Sigma)$. Let

$$V_{\tilde{x}} := \{\tilde{y}_{\tilde{x},\nu} : \nu \in U_{\tilde{x}}\} \subset \mathcal{TH}(\alpha)$$

and $\psi_{\tilde{x}} : U_{\tilde{x}} \to V_{\tilde{x}}$ be defined by

$$\psi_{\tilde{x}} : \nu \mapsto \tilde{y}_{\tilde{x},\nu}.$$

Let $\phi : V_{\tilde{x}} \to U_{\tilde{x}}$ be the period map. By construction, $\nu \in U_{\tilde{x}}$ agrees with $\phi(\tilde{y}_{\tilde{x},\nu})$ on edges of $\tau_{\tilde{x}}$, and these edges generate $H_1(S, \Sigma; \mathbb{Z})$. Thus, the map $\psi_{\tilde{x}}$ is an inverse to ϕ. Thus, we obtain the following:

Lemma 2.15 ([MS91, Lemma 1.1])**.** *The map ϕ is injective and locally onto when restricted to $V_{\tilde{x}}$.*

Now let $x = (M, \omega) \in \mathcal{H}_1^{(\epsilon)}(\alpha)$; in other words, the shortest length of saddle connections in x is not smaller than ϵ. Let τ_x be the Delaunay triangulation of x. Then by the construction, each triangle $\Delta \in \tau_x$ can be inscribed in a disk of radius not greater than the diameter $d(M)$ of M (cf. [MS91, Theorem 4.4]). In addition, we can further control these quantities as follows.

Theorem 2.16 ([MS91, Theorem 5.3, Proposition 5.4])**.** *Let $x = (M, \omega) \in \mathcal{H}_1(\alpha)$. Then there exists a constant $C_6 > 0$, such that for any $p \in \Sigma$, $M \setminus B(p, C_6)$ is contained in a union of disjoint metric cylinders.*

Moreover, the Delaunay triangulation τ_x of x consists of edges which either have length $\leq C_6$ or which cross a cylinder $C \subset M$ whose height h is greater than its circumference c. If an edge crosses C, then its length l satisfies $h \leq l \leq \sqrt{h^2+c^2}$.

After calculating the area, we immediately obtain:

Corollary 2.17. *Let $\epsilon > 0$, $x = (M, \omega) \in \mathcal{H}_1^{(\epsilon)}(\alpha)$. Then the length l of any edge of the Delaunay triangulation τ_x of x is bounded above by $l \leq 2\epsilon^{-1}$. Also, the diameter $d(M) \leq 2\epsilon^{-1}$.*

Proof. Note that for any $x = (M, \omega) \in \mathcal{H}_1^{(\epsilon)}(\alpha)$, the circumference of a cylinder is not less than ϵ. Let $C \subset M$ be a cylinder with height h and circumference $c \geq \epsilon$. Then one can calculate the area $ch = \mathrm{Area}(C) \leq \mathrm{Area}(M) = 1$. The consequence follows from Theorem 2.16 immediately. $\square$

We shall also need certain elementary analysis of inscribed triangles. Let T_1 be the space of ordered triples of points in $\mathbb{C} \cong \mathbb{R}^2$ modulo the action of the group of translations. Let $T_2 \subset T_1$ be the set of triples with positive determinant. Let $T_3(\epsilon, d) \subset T_2$ be the set of isometry classes of triangles, with all edges of length not less than ϵ, which can be inscribed in circles of radius not greater than d.

Lemma 2.18 ([MS91, Lemma 6.7])**.** *There exists a constant $C_7 > 0$ satisfying the following property. Let $\epsilon, d > 0$, $\Delta \in T_3(\epsilon, d)$. Let $\Delta' \in T_3(\epsilon, d)$ be a triangle such that each vertex of Δ' differs from the corresponding vertex of Δ by at most $C_7\epsilon^2/d$. Then Δ' is a non-degenerate triangle with the same orientation as Δ.*

Corollary 2.19. *Let $\epsilon > 0$ be small enough, and let $x \in \mathcal{H}_1^{(\epsilon)}(\alpha)$. Let $\tilde{x} \in \mathcal{TH}(\alpha)$ satisfy $\pi_1(\tilde{x}) = x$. Let $\tilde{y} \in B(\tilde{x}, \epsilon^5)$. Suppose that $\Delta \in \tau_{\tilde{x}}$ is a triangle, and δ_1, δ_2 are two directed edges of Δ. Then $\tilde{y}(\delta_1), \tilde{y}(\delta_2)$ are not parallel.*

Proof. Recall that $\tau_{\tilde{x}}$ is the Delaunay triangulation of $\tilde{x}$. Then by Corollary 2.17, the lengths of edges of the triangle Δ satisfy $\epsilon \leq |\tilde{x}(\gamma)| \leq 4\epsilon^{-1}$. Then

$$|\tilde{y}(\gamma) - \tilde{x}(\gamma)| \leq \|\tilde{y} - \tilde{x}\|_{\tilde{x}} \cdot |\tilde{x}(\gamma)| \leq 4\epsilon^4 \ll \epsilon^2/d(x).$$

Then by Lemma 2.18, we get that $\Delta' \in \tau_{\tilde{y}}$ generated by $\tilde{y}(\delta_1), \tilde{y}(\delta_2)$ is a non-degenerate triangle with the same orientation as Δ. $\square$

Corollary 2.20. *Let $\kappa_7 = \kappa_7(\alpha) = \kappa_6 + 5 > 0$. Then for $x \in \mathcal{H}_1(\alpha)$, the composition of the affine exponential map and the covering map is injective on $R(\tilde{x}, \ell(x)^{\kappa_7})$.*

3. Dynamics over $\mathcal{H}(2)$

3.1. **McMullen's classification.** In this section, we shall recall the dynamics of $G = \mathrm{SL}_2(\mathbb{R})$ over $\mathcal{H}(2)$, and in particular the McMullen's classification of Teichmüller curves over $\mathcal{H}(2)$. We refer to $\mathcal{H}(0) = \Omega\mathcal{M}_1$ as the moduli space of holomorphic 1-forms of genus 1 with a marked point. We usually identify elements of $\mathcal{H}(0)$ with lattices $\Lambda \subset \mathbb{C}$, via the correspondence $(M, \omega) = (\mathbb{C}/\Lambda, dz)$; in other words, Λ is the image of the absolute periods $\omega(H_1(M;\mathbb{Z}))$. Thus, $\mathcal{H}(0) \cong \mathrm{GL}_2^+(\mathbb{R})/\Gamma = \mathbb{R}^+ \times G/\Gamma$. (Here G/Γ is identified with the space of tori of area 1.)

In the sequel, we shall study tori with different areas: $\Lambda_1 \in \mathcal{H}_A(0)$ and $\Lambda_2 \in \mathcal{H}_{1-A}(0)$. We consider $(\Lambda_1, \Lambda_2) \in G/\Gamma \times G/\Gamma$ as two corresponding tori with area 1, after rescaling.

For $(X, \omega) \in \mathcal{H}(2)$, we will be interested in presenting forms of genus 2 as connected sums of forms of genus 1,

$$(X,\omega) = (E_1,\omega_1) \underset{I}{\#} (E_2,\omega_2). \tag{3.1}$$

Here $(E_1,\omega_1),(E_2,\omega_2) \in \mathcal{H}(0)$, $v \in \mathbb{C}^*$ and $I = [0,v] \coloneqq [0,1]\cdot v$. We also say that $(E_1,\omega_1) \underset{I}{\#} (E_2,\omega_2)$ is a splitting of (X,ω).

It is straightforward to check that the connected sum operation commutes with the action of $\mathrm{GL}_2^+(\mathbb{R})$: we have

$$g.((Y_1,\omega_1) \underset{I}{\#} (Y_2,\omega_2)) = g.(Y_1,\omega_1) \underset{g\cdot I}{\#} g.(Y_2,\omega_2) \tag{3.2}$$

for all $g \in \mathrm{GL}_2^+(\mathbb{R})$.

Let $S(2)$ denote the splitting space, consisting of triples $(\Lambda_1, \Lambda_2, v) \in \mathcal{H}(0) \times \mathcal{H}(0) \times \mathbb{C}^*$ satisfying

$$[0,v] \cap \Lambda_1 = \{0\}, \quad [0,v] \cap \Lambda_2 = \{0,v\} \tag{3.3}$$

or vice versa. The group $\mathrm{GL}_2^+(\mathbb{R})$ acts on the space of triples (Λ_1,Λ_2,v), leaving $S(2)$ invariant. Clearly, given a triple (Λ_1,Λ_2,v), one defines

$$x = \Lambda_1 \underset{[0,v]}{\#} \Lambda_2 \in \mathcal{H}(2)$$

and we obtain a natural map $\Phi : S(2) \to \mathcal{H}(2)$. By [McM07, Theorem 7.2], the connected sum mapping Φ is a surjective, $\mathrm{GL}_2^+(\mathbb{R})$-equivariant local covering map.

In [McM07], McMullen classified the $\mathrm{SL}_2(\mathbb{R})$-orbit closures of $\mathcal{H}_1(2)$:

Theorem 3.1 ([McM07, Theorem 10.1]). *Let $Z = \overline{G.x}$ be a G-orbit closure of some $x \in \mathcal{H}_1(2)$. Then either:*

- *Z is a Teichmüller curve, or*
- *$Z = \mathcal{H}_1(2)$.*

We also have a simple criterion for Teichmüller curves in $\mathcal{H}(2)$:

Theorem 3.2 ([McM07, Corollary 5.6, Theorem 5.10]). *Whether a form $(X,\omega) \in \mathcal{H}(2)$ generates a Teichmüller curve or not, is completely determined by the absolute periods.*

In addition, in [McM05], McMullen provided a complete list of Teichmüller curves in $\mathcal{H}(2)$. We say $\Omega W_D \subset \Omega\mathcal{M}_2$ is a *Weierstrass curve* if it is the locus of Riemann surfaces $M \in \mathcal{M}_2$ such that

(i) $\mathrm{Jac}(M)$ admits real multiplication by $\mathcal{O}_D$, where $\mathcal{O}_D \cong \mathbb{Z}[x]/(x^2+bx+c)$ is a quadratic order with $b, c \in \mathbb{Z}$ and the *discriminant* $D = b^2 - 4c > 0$ (cf. [McM03]);
(ii) M carries an eigenform ω with a double zero at one of the six Weierstrass points of M.

Every **irreducible** component of W_D is a Teichmüller curve. When $D \equiv 1 \bmod 8$, one can also define a *spin invariant* $\epsilon(M,\omega) \in \mathbb{Z}/2\mathbb{Z}$ which is constant along the components of W_D. In [McM05], McMullen showed that each Teichmüller curve is uniquely determined by these two invariants:

Theorem 3.3 ([McM05, Theorem 1.1]). *For any integer $D \geq 5$ with $D \equiv 0$ or $1 \bmod 4$, either:*

- *The Weierstrass curve W_D is irreducible, or*
- *We have $D \equiv 1 \bmod 8$ and $D \neq 9$, in which case $W_D = W_D^0 \sqcup W_D^1$ has exactly two components, distinguished by their spin invariants.*

We are interested in the information provided by the absolute periods.

Theorem 3.4 ([McM05, Theorem 3.1]). *Let $(M,\omega) = (E_1,\omega_1) \underset{I}{\#} (E_2,\omega_2)$. Then the following are equivalent:*

(i) ω is an eigenform for real multiplication by $\mathcal{O}_D$ on $\mathrm{Jac}(M)$;
(ii) $\omega_1+\omega_2$ is an eigenform for real multiplication by $\mathcal{O}_D$ on $E_1 \times E_2$.

In particular, the discriminant D of a Weierstrass curve W_D is purely determined by the absolute period map $I_\omega : H_1(M;\mathbb{Z}) \to \mathbb{C}$. In addition, we have a complete list of all possible absolute period maps. More precisely, we define the locus

$$\Omega Q_D := \{(E_1 \times E_2, \omega) \in \Omega\mathcal{M}_1 \times \Omega\mathcal{M}_1 : \omega \text{ is an eigenform for real muliplication by } \mathcal{O}_D\}.$$

Besides, we define the splitting space

$$\Omega W_D^s := \{(X,\omega,I) : (X,\omega) \in \Omega W_D \text{ splits along } I\}.$$

Then by Theorem 3.4, there is a covering map

$$\Pi : \Omega W_D^s \to \Omega Q_D \tag{3.4}$$

which records the summands (E_i,ω_i) in (3.1).

In particular, suppose that $(\Lambda_1,\Lambda_2,I) \in \Omega W_D^s$ determines a surface $x = \Lambda_1 \underset{I}{\#} \Lambda_2 \in \Omega_1 W_D$ and satisfies $\mathrm{Area}(\Lambda_1) = \mathrm{Area}(\Lambda_2) = \frac{1}{2}$, and $(\Lambda_1,\Lambda_2) \in X_\eta$ for some $\eta > 0$. Then by Lemma 2.10 and (3.4), the injectivity radius of x in $\Omega_1 W_D$ is not less than η. In other words, we have

$$x \in (\Omega_1 W_D)_\eta \tag{3.5}$$

where

$$(\Omega_1 W_D)_\eta := \{y \in \Omega_1 W_D : \text{the injectivity radius of } y \text{ in } \Omega_1 W_D \text{ is not less than } \eta\}. \tag{3.6}$$

(See e.g. [EMV09, §3.6] for more discussion.)

3.2. **Prototypes of eigenforms.** Let us say a triple of integers (e, ℓ, m) is a *prototype* for real multiplication, with discriminant D, if

$$D = e^2 + 4\ell^2 m, \quad \ell, m > 0, \quad \gcd(e, \ell) = 1. \tag{3.7}$$

We can associate a prototype (e, ℓ, m) to each eigenform $(E_1 \times E_2, \omega) \in \Omega Q_D$. Moreover, we have

Theorem 3.5 ([McM05, Theorem 2.1])**.** *The space ΩQ_D decomposes into a finite union*

$$\Omega Q_D = \bigcup \Omega Q_D(e, \ell, m)$$

of closed $\mathrm{GL}_2^+(\mathbb{R})$-orbits, one for each prototype (e, ℓ, m). Besides, we have

$$\Omega Q_D(e, \ell, m) \cong \mathrm{GL}_2^+(\mathbb{R})/\Gamma_0(m)$$

where $\Gamma_0(m)$ is the Hecke congruence subgroup of level m:

$$\Gamma_0(m) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma : c \equiv 0 \bmod m \right\}.$$

Let $\lambda = (e + \sqrt{D})/2$. Define a pair of lattices in $\mathbb{C}$ by

$$\Lambda_1 = \mathbb{Z}(\ell m, 0) \oplus \mathbb{Z}(0, \ell), \quad \Lambda_2 = \mathbb{Z}(\lambda, 0) \oplus \mathbb{Z}(0, \lambda). \tag{3.8}$$

Let $(E_i, \omega_i) = (\mathbb{C}/\Lambda_i, dz)$ be the corresponding forms of genus 1, and let

$$(A, \omega) = (E_1 \times E_2, \omega_1 + \omega_2).$$

Then (A, ω) is an eigenform with invariant (e, ℓ, m), and we refer to it as the *prototypical example* of type (e, ℓ, m).

Corollary 3.6. *Every eigenform $(E_1 \times E_2, \omega) \in \Omega Q_D$ is equivalent, under the action of $\mathrm{GL}_2^+(\mathbb{R})$, to a unique prototypical example.*

3.3. **Prototypes of splittings.** Moreover, we can assign a prototypical splitting to a quadruple of integers (a, b, c, e). First, we say a quadruple of integers (a, b, c, e) is a *prototype* of discriminant D, if

$$\begin{aligned} &D = e^2 + 4bc, \quad 0 \le a < \gcd(b, c), \quad c + e < b, \\ &b > 0, \qquad\qquad c > 0, \qquad\qquad \gcd(a, b, c, e) = 1. \end{aligned} \tag{3.9}$$

We then assign a prototypical splitting to a prototype quadruple (a, b, c, e) as follows. Let

$$\Lambda_1 = \mathbb{Z}(b, 0) \oplus \mathbb{Z}(a, c), \quad \Lambda_2 = \mathbb{Z}(\lambda, 0) \oplus \mathbb{Z}(0, \lambda)$$

and $\lambda = (e + \sqrt{D})/2$ and $D = e^2 + 4bc$ (see Figure 4).

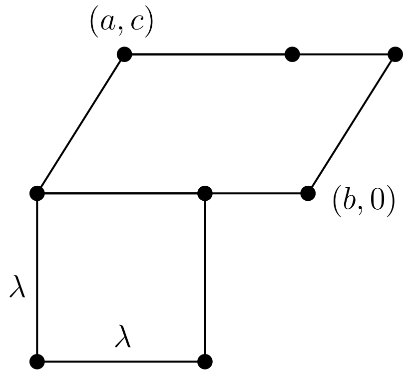

FIGURE 4. Prototypical splitting of type (a, b, c, e).

Then the splittings $(X, \omega) = \Lambda_1 \underset{[0,v]}{\#} \Lambda_2$ is said to be a *prototypical splitting of type* (a, b, c, e). In [McM05, §3], McMullen showed that all prototypical splittings are well-defined, and all other splittings of ΩW_D^s can be generated by these prototypes via $\mathrm{GL}_2^+(\mathbb{R})$-action.

Example 3.7 (L-shaped polygon)**.** For later use, we consider certain Teichmüller curves generated by L-shaped polygons. See also [McM03], [Cal04], [Wei14, §3].

Let $4 < D = d^2 \in \mathbb{N}$ be an even square. Define a pair of lattice in $\mathbb{C}$ by

$$\Lambda_{1,D} \coloneqq \mathbb{Z}(\sqrt{D}/2, 0) \oplus \mathbb{Z}(0, 1), \quad \Lambda_{2,D} \coloneqq \mathbb{Z}(1, 0) \oplus \mathbb{Z}(0, \sqrt{D}/2).$$

Let $v_D \coloneqq (1, 0)$. Then $(\Lambda_{1,D}, \Lambda_{2,D}, v_D) \in D(2)$ indicates an L-shaped polygon. In the notation of [McM03], it corresponds to the L-shaped polygon $P(1 + \sqrt{D}/2, \sqrt{D}/2)$. Let $y_D \in \mathcal{H}_1(2)$ correspond to the surface $\Phi(\Lambda_{1,D}, \Lambda_{2,D}, v_D)$ under the projection $\mathcal{H}(2) \to \mathcal{H}_1(2)$. Then it is easy to check that $\overline{G.(\Lambda_{1,D}, \Lambda_{2,D})} = G.(\Lambda_{1,D}, \Lambda_{2,D})$ and y_D generates a Teichmüller curve. Moreover, it corresponds to the prototype $(0, 1, D/4)$ in the sense of (3.8), and the prototype $(0, D/4, 1, 0)$ in the sense of (3.9). Thus, by Theorem 3.5, we have

$$Y_D \coloneqq G.(\Lambda_{1,D}, \Lambda_{2,D}) \cong G/\Gamma_0(D/4).$$

Note that for $n \geq 2$, the index is given by the formula:

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(n)] = n \cdot \prod_{\substack{p \mid n \\ p \text{ prime}}} \left(1 + \frac{1}{p}\right).$$

In particular, letting $n = D/4$, we have

$$D \ll \mathrm{vol}(G.(\Lambda_{1,D}, \Lambda_{2,D})) \ll D^2.$$

In what follows, we write

$$Q_D \coloneqq \Omega Q_D(0, 1, D/4).$$

On the other hand, note that D is even and so by Theorem 3.3, the Weierstrass curve $\Omega_1 W_D$ is irreducible, i.e. a Teichmüller curve. Now note that the L-shaped polygon can be tiled by squares. Thus, $\mathrm{SL}(y_D)$ is a lattice commensurable to $\mathrm{SL}_2(\mathbb{Z})$ ([GJ00, Theorem 5.5]). In addition, one can show that $\mathrm{SL}(y_D) \subset \mathrm{SL}_2(\mathbb{Z})$ ([Wei14, Proposition 3.13]).

Moreover, in [EMS03], Eskin, Masur, and Schmoll give the formula

$$[\mathrm{SL}_2(\mathbb{Z}) : \mathrm{SL}(y_D)] = \frac{3}{8}(d-2)d^2 \prod_{\substack{p\mid d \\ p \text{ prime}}} \left(1 - \frac{1}{p^2}\right).$$

It follows that

$$\mathrm{vol}(\Omega_1 W_D) = \mathrm{vol}(G/\,\mathrm{SL}(y_D)) \ll D^3.$$

3.4. **Quantitative estimates of Teichmüller curves.** In this section, we approach a given splitting through Teichmüller curves in terms of the areas of tori. With the help of the classification of Teichmüller curves, one obtains the following lemma by an elementary calculation.

Lemma 3.8. *Given a splitting $x = \Lambda_1 \underset{I}{\#} \Lambda_2 \in \mathcal{H}_1^{(\eta)}(2)$ induced by the Delaunay triangulation, there is a sufficiently large $D_\eta > 0$ so that for any $D \geq D_\eta$, the Teichmüller curve $\Omega_1 W_D$ has a prototypical splitting*

$$x_D = \Lambda_1(D) \underset{I(D)}{\#} \Lambda_2(D) \in \Omega_1 W_D$$

such that

$$\frac{\mathrm{Area}(\Lambda_1(D))}{\mathrm{Area}(\Lambda_2(D))} = \frac{\mathrm{Area}(\Lambda_1) + O(\eta^{-3}D^{-\frac{1}{2}})}{\mathrm{Area}(\Lambda_2) + O(\eta^{-3}D^{-\frac{1}{2}})}. \tag{3.10}$$

Here the implicit constant of $O(\eta^{-3}D^{-\frac{1}{2}})$ can be chosen to be < 3. Moreover, we have the volume estimate

$$D \ll \mathrm{vol}(G.(\Lambda_1(D), \Lambda_2(D))) \ll D^2. \tag{3.11}$$

Proof. Let

$$\lambda = \frac{\mathrm{Area}(\Lambda_1)}{\mathrm{Area}(\Lambda_2)} \in [\eta^4, \eta^{-4}].$$

Consider the prototype $(a, b, c, e) = (0, b, 1, e)$ with $e = \left\lfloor (\lambda - 1)\lambda^{-\frac{1}{2}} b^{\frac{1}{2}} \right\rfloor$. (Note that it is always possible when b is sufficiently large.) Then $e = (\lambda - 1)\lambda^{-\frac{1}{2}} b^{\frac{1}{2}} - \epsilon$ with $\epsilon \in [0, 1)$. Let

$$c_1 = \epsilon b^{-\frac{1}{2}}(-2(\lambda - 1)\lambda^{-\frac{1}{2}} + \epsilon b^{-\frac{1}{2}}) = O(\eta^{-2}b^{-\frac{1}{2}}).$$

Then one calculates

$$
\begin{aligned}
\frac{\operatorname{Area}(\Lambda_1(D))}{\operatorname{Area}(\Lambda_2(D))} =& \frac{(e^2+4b)^{\frac{1}{2}}+e}{(e^2+4b)^{\frac{1}{2}}-e} \\
=& \frac{((\lambda-1)^2\lambda^{-1}b+c_1b+4b)^{\frac{1}{2}}+e}{((\lambda-1)^2\lambda^{-1}b+c_1b+4b)^{\frac{1}{2}}-e} \\
=& \frac{((\lambda+1)^2\lambda^{-1}b+c_1b)^{\frac{1}{2}}+e}{((\lambda+1)^2\lambda^{-1}b+c_1b)^{\frac{1}{2}}-e} \\
=& \frac{\lambda^{-\frac{1}{2}}b^{\frac{1}{2}}((\lambda+1)^2+c_1\lambda)^{\frac{1}{2}}+e}{\lambda^{-\frac{1}{2}}b^{\frac{1}{2}}((\lambda+1)^2+c_1\lambda)^{\frac{1}{2}}-e} \\
=& \frac{\lambda^{-\frac{1}{2}}b^{\frac{1}{2}}((\lambda+1)+O(c_1\lambda))+e}{\lambda^{-\frac{1}{2}}b^{\frac{1}{2}}((\lambda+1)+O(c_1\lambda))-e} \\
=& \frac{2\lambda\lambda^{-\frac{1}{2}}b^{\frac{1}{2}}+\lambda^{-\frac{1}{2}}b^{\frac{1}{2}}\cdot O(c_1\lambda)-\epsilon}{2\lambda^{-\frac{1}{2}}b^{\frac{1}{2}}+\lambda^{-\frac{1}{2}}b^{\frac{1}{2}}\cdot O(c_1\lambda)+\epsilon} \\
=& \frac{\lambda+\frac{1}{2}\cdot O(c_1\lambda)-\frac{1}{2}\epsilon\lambda^{\frac{1}{2}}b^{-\frac{1}{2}}}{1+\frac{1}{2}\cdot O(c_1\lambda)+\frac{1}{2}\epsilon\lambda^{\frac{1}{2}}b^{-\frac{1}{2}}}.
\end{aligned} \tag{3.12}
$$

Here the implicit constant of $O(c_1\lambda)$ can be chosen to be < 3. Now the consequence follows from (3.12) and $D = e^2+4b = O(\eta^{-2}b)$.

Finally, the absolute periods of ΩW_D of type $(0,b,1,e)$ corresponds to ΩQ_D of type $(e,1,b)$. Then by Theorem 3.5, we get

$$
\operatorname{vol}(G.(\Lambda_1(D),\Lambda_2(D))) = \operatorname{vol}(G/\Gamma_0(b)) \asymp b\cdot \prod_{\substack{p\mid b\\ p \text{ prime}}}\left(1+\frac{1}{p}\right) \in [D,D^2].
$$

This establishes the lemma. $\square$

Let $x = \Lambda_1 \underset{[0,v]}{\#} \Lambda_2 \in \mathcal{H}_1(2)$ be a splitting induced by the Delaunay triangulation satisfying

$$
[0,v]\cap\Lambda_1 = \{0\}, \quad [0,v]\cap\Lambda_2 = \{0,v\}. \tag{3.13}
$$

Using the identification $H_1(M;\mathbb{Z}) = \Lambda_1\oplus\Lambda_2$, we define

$$
x_1(\lambda_1+\lambda_2) \coloneqq x(\lambda_1).
$$

Then for ϵ with $|\epsilon|$ sufficiently small, we see that $x+\epsilon x_1 = (1+\epsilon)\Lambda_1 \underset{[0,v]}{\#} \Lambda_2$ is a change of the areas:

$$
\frac{\operatorname{Area}((1+\epsilon)\Lambda_1)}{\operatorname{Area}(\Lambda_2)} = (1+\epsilon)\frac{\operatorname{Area}(\Lambda_1)}{\operatorname{Area}(\Lambda_2)}.
$$

Lemma 3.9. *For ϵ with $|\epsilon|$ sufficiently small, for $x \in \mathcal{H}_1^{(\eta)}(2)$, we have*

$$\|x - (x + \epsilon x_1)\|_x = \|\epsilon x_1\|_x \leq \eta^{-11}|\epsilon|.$$

Proof. For $x = \Lambda_1 \underset{[0,v]}{\#} \Lambda_2 \in \mathcal{H}_1^{(\eta)}(2)$, let $\Lambda_1 = \mathbb{Z}a_1 \oplus \mathbb{Z}b_1$, $\Lambda_2 = \mathbb{Z}a_2 \oplus \mathbb{Z}b_2$, for some $a_i, b_i \in \mathbb{C}^*$ with $a_2 = v$. Then we can identify $x = \Lambda_1 \underset{[0,v]}{\#} \Lambda_2$ as three parallelograms (see Figure 5):

$$P_1 = a_2 \times b_2, \quad P_2 = a_2 \times b_1, \quad P_3 = (a_1 - a_2) \times b_1.$$

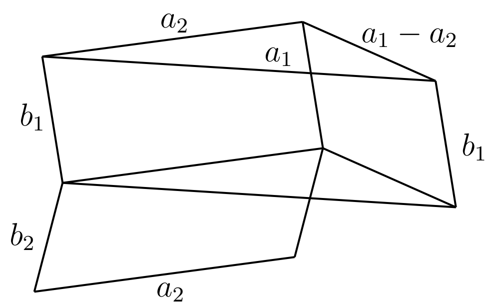

FIGURE 5. Pair of splittings.

See e.g. [McM05, §7] for more discussion about this identification.

Let $\|\cdot\|$ be a norm on $H_1(M;\mathbb{Z})$ define by:

$$\|z_1 a_1 + z_2 b_1 + z_3 a_2 + z_4 b_2\| := \max_i |z_i|.$$

Suppose that we have a long saddle connection γ passes through the parallelograms. We can use the sides of P_j to subdivide γ in n segments $[X_i, X_{i+1}]$ so that each of them lies in a single parallelogram $P_{j(i)}$ for $j(i) = 1, 2, 3$. Joining each X_i to the nearest singularity of the side, we get a decomposition in homology $[\gamma] = \sum_{i=1}^n [\gamma_i]$, where γ_i is a saddle connection in $P_{j(i)}$. Clearly, $\|[\gamma_i]\| \leq 1$. Further, we have:

Claim 3.10. *If $x([X_i X_{i+1}]) < \eta^{10}$, then $\|[\gamma_i]\| = 0$.*

Proof of Claim 3.10. Let Δ be a triangle in one of the parallelograms P_j, with three sides a, b, c and corresponding three angles α, β, γ. Suppose that

$$\sin\gamma = \max\{\sin\alpha, \sin\beta, \sin\gamma\} \geq \frac{1}{2}.$$

Since the sides of the parallelograms have the length within $[\eta, 2\eta^{-1}]$ (via Lemma 2.17), by the law of sines, we know that any angle α of the parallelograms has

$$\sin\alpha = \frac{a}{c}\sin\gamma \geq \epsilon^2 \sin\gamma \geq \frac{1}{2}\eta^2.$$

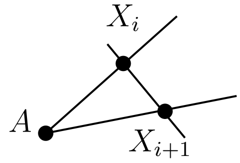

FIGURE 6. Pair of splittings.

Suppose that some $|x([X_iX_{i+1}])| < \eta^{10}$. Then together with the two sides of $P_{j(i)}$ that contain X_i and X_{i+1}, we obtain a triangle ΔX_iAX_{i+1}, where A is the endpoint of the two sides of $P_{j(i)}$. By using the law of sines again, we see that

$$|x([X_iA])| = \frac{\sin \angle AX_{i+1}X_i}{\sin \angle X_iAX_{i+1}}|x([X_iX_{i+1}])| \leq \frac{1}{\frac{1}{2}\eta^2} \cdot \eta^{10} \leq 2\eta^8.$$

Similarly, we have $|x([X_{i+1}A])| \leq 2\eta^8$. Since each side of P_j has the length at least η, we conclude that X_i and X_{i+1} are both joined by A. Thus, $x([\gamma_i]) = 0$. □

Thus, we obtain

$$\|[\gamma]\| \leq \sum_{\|[\gamma_i]\|\neq 0} \|[\gamma_i]\| \leq \eta^{-10} \sum_{\|[\gamma_i]\|\neq 0} |x([X_iX_{i+1}])| \leq \eta^{-10}|x([\gamma])|.$$

Then we have

$$\|\epsilon x_1\|_x = \sup_\gamma \frac{|\epsilon x_1([\gamma])|}{|x([\gamma])|} \leq \eta^{-10} \sup_\gamma \frac{|\epsilon x_1([\gamma])|}{\|[\gamma]\|} \leq \eta^{-11}|\epsilon|.$$

The consequence follows. □

Remark 3.11. The proof of Lemma 3.9 only uses the facts that $x \in \mathcal{H}_1^{(\eta)}(2)$, and the lengths of the sides of parallelograms (Λ_1, Λ_2) are controlled by η (more precisely, by $[\eta, 2\eta^{-1}]$). Thus, if there is another splitting $x = \Lambda_1' \underset{I'}{\#} \Lambda_2' \in \mathcal{H}_1^{(\eta)}(2)$ (not necessarily induced by the Delaunay triangulation) so that the lengths of the sides of parallelograms are controlled by some power of η, we can obtain a similar result to Lemma 3.9.

Therefore, for $x \in \mathcal{H}_1^{(\eta)}(2)$ satisfying (3.13), for $\epsilon > 0$, let

$$x(\epsilon) = (1+\epsilon)^{\frac{1}{2}}\Lambda_1 \underset{(1+\epsilon)^{-\frac{1}{2}}[0,v]}{\#} (1+\epsilon)^{-\frac{1}{2}}\Lambda_2. \tag{3.14}$$

Then by Lemma 3.9, we conclude that $x(\epsilon) \in \mathcal{H}_1(2)$,

$$\frac{\text{Area}((1+\epsilon)^{\frac{1}{2}}\Lambda_1)}{\text{Area}((1+\epsilon)^{-\frac{1}{2}}\Lambda_2)} = (1+\epsilon)\frac{\text{Area}(\Lambda_1)}{\text{Area}(\Lambda_2)}.$$

Now note that $x(\epsilon) = \begin{bmatrix} (1+\epsilon)^{-\frac{1}{2}} & 0 \\ 0 & (1+\epsilon)^{-\frac{1}{2}} \end{bmatrix} (x + \epsilon x_1)$, by Lemma 2.10, we get

$$\|x(\epsilon) - x\|_x \leq \|x(\epsilon) - (x+\epsilon x_1)\|_x + \|(x+\epsilon x_1) - x\|_x \leq |\epsilon| + \eta^{-11}|\epsilon|. \tag{3.15}$$

4. Effective results in homogeneous dynamics

4.1. **Spectral gaps.** Let M a n-dimensional locally symmetric space. Let Δ be the Laplace-Beltrami operator of M. Then Δ is self-adjoint and positive. Thus, its eigenvalues are real and nonnegative. Moreover, the constants are the only eigenfunctions with respect to the zero eigenvalue. Let $\lambda_1(M)$ be the smallest positive eigenvalue. For $\phi \in C_c^\infty(M)$, let $\mathcal{S}(\phi)$ be the Sobolev norm of ϕ with certain order (See e.g. [EMV09, §3.7], [Ven10, §2.9] for more discussion on Sobolev norms).

In [Rat87], Ratner proved that it is related to the rate of mixing for geodesic flows:

Theorem 4.1 (Rate of mixing, [Rat87, Theorem 1])**.** *Let $G = \mathrm{SL}_2(\mathbb{R})$, Γ' a lattice in G, $M = G/\Gamma'$, and μ the normalized Lebesgue measure on M. Then for any $\phi, \psi \in C_c^\infty(M)$, we have*

$$\left| \int \phi(a_t x)\psi(x) d\mu(x) - \int \phi d\mu \int \psi d\mu \right| \ll \mathcal{S}(\phi)\mathcal{S}(\psi) e^{-\kappa_8 t}$$

where $\kappa_8 = \kappa_8(M)$ can be chosen to be larger than $\frac{1}{2}(1 - \sqrt{1 - \lambda_1(M)})$.

The well known Cheeger's inequality provides a lower bound of λ_1 in terms of the Cheeger constant on Riemannian manifolds [Che70]. Let E be a $(n-1)$-submanifold which divides M into two disjoint submanifolds A and B. Let $\mu(E)$ be the area of E and $\mathrm{vol}(A)$ and $\mathrm{vol}(B)$ the volumes of A and B, respectively. The *Cheeger constant* of M is defined to be

$$h(M) = \inf_E \frac{\mu(E)}{\min\{\mathrm{vol}(A), \mathrm{vol}(B)\}}$$

where E runs over all possibilities for E as above.

Theorem 4.2 (Cheeger's inequality, [Che70])**.** *For a locally symmetric space M, we have*

$$\lambda_1(M) \geq \frac{1}{4} h^2(M).$$

It is convenient to calculate the Cheeger constant through the combinatorics. The *Cheeger constant* of a graph $\mathcal{G} = (V, E)$ is defined to be

$$h(\mathcal{G}) = \inf_{A,B \subset V} \frac{|E(A,B)|}{\min\{|A|, |B|\}}$$

where the infimum runs over all the disjoint partitions $V = A \sqcup B$, and $E(A,B)$ is the set of edges connecting vertices in A to vertices in B. The Cheeger constant $h(\mathcal{G})$ is related to the expander graphs. More precisely, if $\mathcal{G}$ is a k-regular graph, then $\mathcal{G}$ is an $(|V|, k, \frac{1}{k}h(\mathcal{G}))$-expander (cf. [Lub94, Proposition 1.1.4]).

Let $D = d^2$ be an even square. Let $y_D \in \mathcal{H}_1(2)$ be a surface that is tiled by d squares. Then $\Gamma_D = \mathrm{SL}(y_D)$ is a Veech group with discriminant D. Moreover, $\Gamma_D \subset \Gamma = \mathrm{SL}_2(\mathbb{Z})$ [Wei14, Proposition 3.13], and Γ_D is arithmetic [GJ00, Theorem

5.5]. Let $\Omega_1 W_D = G/\Gamma_D$. Then $\{\Omega_1 W_D\}$ is a sequence of locally symmetric spaces that are the finite sheeted coverings of $X = G/\Gamma$. In the context of the graph theory, the Γ-orbit of y_D is closed. This leads to a Schreier graph $\mathcal{G}_D = \mathcal{G}(\Gamma/\Gamma_D, S)$ with $S = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right\}$. In [Bro86], Brooks proved the following:

Lemma 4.3 ([Bro86, Lemma 2]). *There is a constant $C_8 = C_8(X) > 0$ such that*

$$h(\mathcal{G}_D) \leq C_8 h(\Omega_1 W_D).$$

Since Γ acts transitively on Γ/Γ_D, we have

$$h(\mathcal{G}_D) \geq \inf_{A,B \subset \Gamma/\Gamma_D} \frac{1}{\min\{|A|, |B|\}} \geq \frac{1}{[\Gamma : \Gamma_D]} \gg D^{-3}.$$

Then by Lemma 4.3 and Theorem 4.2, we have

$$\lambda_1(\Omega_1 W_D) \gg h^2(\mathcal{G}_D) \gg D^{-6}.$$

Moreover, McMullen made the following conjecture:

Conjecture 4.4 (McMullen's expansion conjecture). *The family of graphs $\mathcal{G}_D$ associated to arithmetic Veech groups is expander. In other words, $\lambda_1(\Omega_1 W_D)$ possess a uniform lower bound.*

Therefore, by Theorem 4.1, there exists a constant $C_9 > 0$ such that the rate of mixing

$$\kappa_8(\Omega_1 W_D) > C_9 D^{-6}. \tag{4.1}$$

If Conjecture 4.4 is correct, then $\lambda_1(\Omega_1 W_D)$ possess a uniform lower bound. Then we have

$$\kappa_8(\Omega_1 W_D) > C_9. \tag{4.2}$$

4.2. **Effective equidistribution in homogeneous dynamics.** In this section, we review the effective results of homogeneous dynamics which shall serve as the estimates of the absolute periods.

In [LMW22] (see also [LM23]), Lindenstrauss, Mohammadi, and Wang derive the following quantitative behavior of orbits in homogeneous dynamics.

Theorem 4.5 (Effective equidistribution on X). *Let $X = G/\Gamma \times G/\Gamma$. There exists $\delta_0 = \delta_0(X) > 0$, $\kappa_9 = \kappa_9(X) > 0$ such that for every $\Lambda \in X$, $\delta \in (0, \delta_0)$, there exists $t_0 = t_0(X, \ell(\Lambda)) > 0$ so that for any $t > \frac{1}{\delta} t_0$, at least one of the the following holds:*

(1) For every $\varphi \in C_c^\infty(X)$, we have

$$\left| \int_0^1 \varphi(a_t u_r \Lambda) dr - \int \varphi dm_X \right| \leq \mathcal{S}(\varphi) e^{-\kappa_9 \delta t}$$

where m_X is the Haar measure on X with $m_X(X) = 1$.

(2) There exists $\Lambda' \in X$ such that $G.\Lambda'$ is periodic with $\mathrm{vol}(G\Lambda') \le e^{\delta t}$ *and*

$$d_X(\Lambda', \Lambda) \le e^{-\frac{1}{2}t}.$$

We also need the effective equidistribution of unstable foliations on Teichmüller curves. $G/\Gamma \times G/\Gamma$. See also [KM96, Proposition 2.4.8], [LM23, Proposition 4.1].

- For even $D > 0$, let $\Omega_1 W_D := G/\mathrm{SL}(y_D)$ be the Teichmüller curve of discriminant D.
- Let $y_D \in \mathcal{H}(2)$ be a surface that generates a Teichmüller curve of discriminant D. In what follows, we shall mainly study the Teichmüller curve with even discriminant D. In this case, y_D determines a unique Veech group $\Gamma_D = \mathrm{SL}(y_D)$ and a unique Teichmüller curve $\Omega_1 W_D = G/\Gamma_D$.
- (Recall (3.6)) For $\eta > 0$, let

$$(\Omega_1 W_D)_\eta := \{y \in W_D : \text{the injectivity radius of } y \text{ in } \Omega_1 W_D \text{ is not less than } \eta\}.$$

- Let vol be the (G-invariant) volume measure on G/Γ. Note that any $\Gamma_D \subset \Gamma = \mathrm{SL}_2(\mathbb{Z})$ has finite index:

$$[\Gamma : \mathrm{SL}(y_D)] \le D^3$$

  (the precisely formula for the indexes is given in [EMS03]). Thus, it also induces a (G-invariant) volume measure on $\Omega_1 W_D = G/\mathrm{SL}(y_D)$.

Proposition 4.6 (Effective equidistribution on $\Omega_1 W_D$, [KM96, Proposition 2.4.8])**.** *For any even $D > 0$, there exist $\kappa_{10} > 0$, $\kappa_{11} > 0$, and $C_{10} = C_{10}(C_4, \kappa_4) > 0$ so that the following holds: Let $\eta \in (0,1)$, $t > 0$, and $x \in (\Omega_1 W_D)_\eta$. Then for every $f \in C_c^\infty(X) + \mathbb{C}\cdot 1$, we have*

$$\begin{aligned}\left|\int_0^1 f(a_t u_r x)dr - \frac{1}{\mathrm{vol}(\Omega_1 W_D)}\int f\, d\,\mathrm{vol}\right| &\le C_{10}(\mathrm{vol}(\Omega_1 W_D))^{\kappa_{10}}\eta^{-\frac{1}{\kappa_{11}}}\mathcal{S}(f)e^{-\kappa_8 t}\\ &\le C_{10}D^{3\kappa_{10}}\eta^{-\frac{1}{\kappa_{11}}}\mathcal{S}(f)e^{-\kappa_8 t}.\end{aligned}$$

where $\kappa_8 = \kappa_8(\Omega_1 W_D)$ is the rate of mixing given in Theorem 4.1.

Proof. We adopt the notation as in [LM23]. Let $\eta \in (0,1)$, $t > 0$, and $x \in (\Omega_1 W_D)_\eta$. By [LM23, Proposition 4.1], there exists $C > 0$, and an absolute constant κ_{11} such that for every $f \in C_c^\infty(X) + \mathbb{C}\cdot 1$, we have

$$\left|\int_0^1 f(a_t u_r x)dr - \frac{1}{\mathrm{vol}(\Omega_1 W_D)}\int f\, d\,\mathrm{vol}\right| \le C\eta^{-\frac{1}{\kappa_{11}}}\mathcal{S}(f)e^{-\kappa_8 t}.$$

Here the constant C is dominated by

$$C \le L(\eta_D^{-1}\,\mathrm{vol}(\Omega_1 W_D))^L$$

where L is absolute and η_D is so that the quantitative nondivergence

$$|\{r \in [0,1] : a_t u_r x \in (\Omega_1 W_D)_{\eta_D}\}| \ge 0.99$$

holds for $t \geq |\operatorname{inj}(x)| + \eta_D^{-1}$. However, since $\Omega_1 W_D \subset \mathcal{H}_1(2)$, by Corollary 2.7, one can choose $\eta_D = (0.01 C_4{}^{-1})^{\frac{1}{\kappa_4}}$; in particular, it does not depend on D. The consequence follows from letting $\kappa_{10} = L$ and $C_{10} = L\eta_D^{-L}$. ☐

Also, we recall the density of periodic G-orbit on the homogeneous space $X = G/\Gamma \times G/\Gamma$.

Theorem 4.7 (Density of periodic G-orbit on X, [LM23, Theorem 1.3])**.** *Let $G.\Lambda \subset X$ be a periodic G-orbit in X. Then there exist $\kappa_{12} = \kappa_{12}(X) > 0$ and $C_{11} = C_{11}(X) > 0$ such that for every $z^* \in X_{\operatorname{vol}(G\Lambda)^{-\kappa_{12}}}$, we have*

$$d_X(z^*, G\Lambda) \leq C_{11} \operatorname{vol}(G\Lambda)^{-\kappa_{12}}.$$

5. Proof of main theorems

Lemma 5.1. *For $L > C_{11}$, for even $D \geq L^{\frac{2(22+\kappa_7)}{\kappa_{12}}}$, there exists $z_D \in \Omega_1 W_D$ such that*

$$\|z - z_D\|_z \leq D^{-\frac{1}{2}\kappa_{12}}. \tag{5.1}$$

Proof. Let $\lambda = \operatorname{Area}(\Lambda_1)/\operatorname{Area}(\Lambda_2)$. By Lemma 3.8, for $D \geq D_{L^{-1}}$, the Teichmüller curve ΩW_D has a prototypical splitting

$$x_D = \Lambda_1(D) \underset{I(D)}{\#} \Lambda_2(D) \in \Omega W_D$$

such that

$$\begin{aligned} \frac{\operatorname{Area}(\Lambda_1(D))}{\operatorname{Area}(\Lambda_2(D))} &= \frac{\operatorname{Area}(\Lambda_1(z)) + O(L^3 D^{-\frac{1}{2}})}{\operatorname{Area}(\Lambda_2(z)) + O(L^3 D^{-\frac{1}{2}})} \\ &= \lambda \left(1 + \frac{O(L^3 D^{-\frac{1}{2}}) - \lambda O(L^3 D^{-\frac{1}{2}})}{\lambda(\operatorname{Area}(\Lambda_2) + O(L^3 D^{-\frac{1}{2}}))} \right). \end{aligned} \tag{5.2}$$

Note that $\operatorname{Area}(\Lambda_2) \in [L^{-2}, L^2]$, and $\lambda \in [L^{-4}, L^4]$. Let

$$\delta = \frac{O(L^3 D^{-\frac{1}{2}}) - \lambda O(L^3 D^{-\frac{1}{2}})}{\lambda(\operatorname{Area}(\Lambda_2) + O(L^3 D^{-\frac{1}{2}}))}.$$

Then $|\delta| \leq L^{10} D^{-\frac{1}{2}}$. Let $z(\delta)$ be as in (3.14). Then by (3.15), we have

$$\|z - z(\delta)\|_z \leq L^{12}|\delta| \leq L^{22} D^{-\frac{1}{2}}.$$

On the other hand, by Theorem 4.7 and (3.11), there exists $\tilde{\Lambda} \in G(\Lambda_1(D), \Lambda_2(D))$ such that

$$\|z(\delta) - \tilde{\Lambda}\|_z \leq C_{11} D^{-\kappa_{12}}.$$

Since $L > C_{11}$, $D \geq L^{\frac{2(22+\kappa_7)}{\kappa_{12}}}$, it follows that

$$\|z - \tilde{\Lambda}\|_z \leq L^{22} D^{-\frac{1}{2}} + C_{11} D^{-\kappa_{12}} \leq L^{22} D^{-\kappa_{12}} \leq D^{-\frac{\kappa_{12}}{2}}.$$

(To simplify the notation, we assume without loss of generality that $\kappa_{12} < \frac{1}{2}$.) By Corollary 2.20, if $L^{22}D^{-\kappa_{12}} < L^{-\kappa_7}$, then there exists $z_D \in \Omega_1 W_D$, such that the absolute period is $\tilde{\Lambda}$ and $\|z - z_D\|_z \leq D^{-\frac{\kappa_{12}}{2}}$. ☐

We are now in the position to prove the main theorems.

Proof of Theorem 1.1. We use the equidistribution of Teichmüller curves to approach the given surface $z = \Lambda_1(z) \underset{I(z)}{\#} \Lambda_2(z) \in \mathcal{H}_1^{(L^{-1})}(2)$.

- Let $\kappa_8 = \kappa_8(\Omega_1 W_D)$ be the rate of mixing given in Theorem 4.1.
- Let $\rho = e^{-\kappa_8 t/2N}$.
- Let $y_D \in (\Omega_1 W_D)_\eta$.
- Let z_D be as in Lemma 5.1.
- Let h_{ρ,z_D} be a function supported on $\mathsf{B}_G(\rho).z_D$ satisfying

$$\frac{1}{\mathrm{vol}(\Omega_1 W_D)} \int h_{\rho,z_D} d\,\mathrm{vol} = 1$$

and $\mathcal{S}(h_{\rho,z_D}) \leq \rho^{-N}$, where N is absolute.

Then by Proposition 4.6, for $s > 0$, we have

$$\left| \int_0^1 h_{\rho,z_D}(a_s u_r y_D) dr - 1 \right| \leq C_{10} D^{3\kappa_{10}} \eta^{-\frac{R^2}{\kappa_{11}}} e^{-\frac{\kappa_8}{2} s}.$$

Assuming s is large enough, depending on D, the right side of the above is $< 1/2$. Thus, $a_s u_{r'} y_D \in \mathrm{supp}(g_{\rho,z_D})$ for some $r' \in [0,1]$, i.e.

$$d(z_D, a_s u_{r'} y_D) \ll e^{-\frac{\kappa_8}{2N} s}. \tag{5.3}$$

Thus, let $x \in \mathcal{H}_1(2)$ and $\tilde{y}_D \in (\Omega_1 W_D)_\eta$ satisfy

$$\|\tilde{y}_D - \tilde{x}\|_{\tilde{x}} < e^{-t}.$$

Then by Theorem 2.12, we see that

$$\|a_s u_{r'} y_D - a_s u_{r'} \tilde{x}\|_{a_s u_{r'} \tilde{x}} < e^{2s - \kappa_7 t}. \tag{5.4}$$

Therefore, combining (5.1)(5.3)(5.3)(5.4), we conclude that

$$d(z, a_s u_{r'} x) \leq D^{-\frac{1}{2}\kappa_{12}} + e^{-\frac{\kappa_8}{4N} s} + e^{2s - t}.$$

Finally, we simplify the right hand side by taking

- D satisfying $D \geq L^{\frac{2(22+\kappa_7)}{\kappa_{12}}}$ and $D^{-\frac{1}{2}\kappa_{12}} < \frac{1}{3} L^{-1}$,
- s satisfying $C_{10} D^{3\kappa_{10}} \eta^{-\frac{R^2}{\kappa_{11}}} e^{-\frac{\kappa_8}{2} s} < 1/2$ and $e^{-\frac{\kappa_8}{4N} s} < \frac{1}{3} L^{-1}$,
- t satisfying $e^{2s-t} < \frac{1}{3} L^{-1}$.

Then by (4.1), one calculates that there exist $C_1 > 0$, and $\kappa_1 > 0$ such that if $L > L_0$, $D > L^{\kappa_1}$ and $t > C_1 D^{\kappa_1}$, then

$$d(z, a_t u_{[0,1]} x) \leq L^{-1} \tag{5.5}$$

Moreover, if Conjecture 4.4 is correct, then by (4.2), we only require $t > C_1 \log L$ to get (5.5). $\square$

Next, we discuss the proof of Theorem 1.3. Let $x = \Lambda_1 \underset{I}{\#} \Lambda_2 \in \mathcal{H}_1(2)$ with $\mathrm{Area}(\Lambda_1) = \mathrm{Area}(\Lambda_2)$.

- Let δ be small that will be determined later (see (6.51)).
- Let t be sufficiently large.

We apply Theorem 4.5 with $\Lambda(x) = (\Lambda_1, \Lambda_2), \delta, t$. Suppose that Theorem 4.5(1) occurs. Let $L > 0$, $z \in \mathcal{H}_1^{(L^{-1})}(2)$. Let $z = \Lambda_1(z) \underset{I(z)}{\#} \Lambda_2(z)$ be a splitting of z induced by the Delaunay triangulation.

Proof of Theorem 1.3. First, we apply Theorem 4.5 with $\Lambda(x) = (\Lambda_1, \Lambda_2), \delta, t$. Then we use the equidistribution of $x = \Lambda_1 \underset{I}{\#} \Lambda_2 \in \mathcal{H}_1(2)$ in absolute periods to approach a Teichmüller curve. More precisely,

- Let $d \in \mathbb{N}^*$ and $D = 4d^2$ satisfying $D \geq L^{\kappa_1}$.
- Let $\varrho = \varrho_d$ be as in (1.3).
- Let $\varphi_{\varrho,d}$ be a function so that

$$\chi_{\mathsf{B}_{G\times G}(\frac{9}{10}\varrho).(Q_D)_\eta} \leq \varphi_{\varrho,d} \leq \chi_{\mathsf{B}_{G\times G}(\varrho).(Q_D)_\eta}$$

with $\mathcal{S}(\varphi_{\varrho,d}) \leq \varrho^{-N}$, where $N \geq 3$ is absolute. In particular, we have

$$\int \varphi_{\varrho,d} dm_X \asymp \varrho^3.$$

Then at least one of the following holds:

(1) For any $\tilde{\Lambda} \in X_\eta$, we see that

$$\left| \int_0^1 \varphi_{\varrho,d}(a_t u_r(\Lambda_1, \Lambda_2)) dr - \int \varphi_{\varrho,d} dm_X \right| \leq \varrho^{-N} e^{-\kappa_9 \delta t} \leq e^{-\frac{\kappa_9 \delta}{2} t}.$$

(2) There exists $\Lambda' \in X$ such that $G.\Lambda'$ is periodic with $\mathrm{vol}(G\Lambda') \leq e^{\delta t}$ and

$$d_X(\Lambda', \Lambda) \leq e^{-\frac{1}{2}t}. \tag{5.6}$$

Suppose that the case (2) of Theorem 1.3 does not occur. Suppose also that t is large enough, so that

$$\varrho^3 - e^{-\frac{\kappa_9 \delta}{2} t} \ll \int_0^1 \varphi_{\varrho,d}(a_t u_r(\Lambda_1, \Lambda_2)) dr \ll \varrho^3 + e^{-\frac{\kappa_9 \delta}{2} t}.$$

Thus, there is a subset $J_d \subset [0,1]$ with $|J_d| \asymp \varrho^3$ such that $a_t u_r(\Lambda_1, \Lambda_2) \in \mathrm{supp}(\varphi_{\varrho,d})$ for $r \in J_d$. In other words, for

$$J_d = J_d(t) = \left\{ r \in [0,1] : \|\tilde{\Lambda} - a_t u_r(\Lambda_1, \Lambda_2)\| \leq \varrho,\ \tilde{\Lambda} \in (Q_D)_\eta \right\},$$

we have

$$|J_d(\varrho)| \asymp \varrho^3 \tag{5.7}$$

for $\rho_d \geq e^{-\frac{\kappa_9 \delta}{2N} t}$.

Suppose that the case (3) of Theorem 1.3 does not occur. Then there is $r_0 \in J_d$, and $\tilde{\Lambda}_d \in (Q_D)_\eta$ such that

$$\ell(a_t u_{r_0} \tilde{x})^{\vartheta} \|\tilde{\Lambda}_d - a_t u_{r_0} \tilde{x}\|_{a_t u_r \tilde{x}} < \varkappa_{\vartheta}(e^t). \tag{5.8}$$

Fix $\xi = \frac{1}{\kappa_7 + \vartheta}$. Suppose that $x_0 = a_t u_{r_0} x$ satisfies

$$\ell(x_0) > \varkappa_{\vartheta}(e^t)^{\xi}. \tag{5.9}$$

Then by Corollary 2.20, there exists a surface $y_d' \in (\Omega_1 W_D)_\eta$ with the absolute periods $\tilde{\Lambda}_d$ such that

$$\|y_d' - x_0\|_{x_0} < \ell(x_0)^{-\vartheta} \varkappa_{\vartheta}(e^t) \leq \ell(x_0)^{\kappa_7}. \tag{5.10}$$

In Section 6, we shall see that even if (5.9) fails, Proposition 6.13 indicates that there exists and a function $\varkappa_{\vartheta}^{\star} : \mathbb{R}^+ \to \mathbb{R}^+$ depending on $\varkappa_{\vartheta}$, κ_7, ϑ with $\lim_{t\to\infty} \varkappa_{\vartheta}^{\star}(e^t) = 0$, and an interval $J^{\star\star} \subset [0, e^t]$ with $|J^{\star\star}| \geq (\varkappa_{\vartheta}^{\star}(e^t))^{-1}$, such that for

$$J_d' = \left\{ r \in J^{\star\star} : \|\tilde{\Lambda} - u_r a_t \Lambda(x)\| \leq \varrho,\ \tilde{\Lambda} \in (Q_D)_{\eta R^2+1} \right\},$$

we have

$$|J_d'| \gg \varrho^3 |J^{\star\star}|,$$

and at least one of the following holds:

(i) For any $r \in J_d'$, and $\tilde{\Lambda}_d \in (Q_D)_{\eta R^2}$, we have

$$\ell(u_r a_t \tilde{x}) \geq \varkappa_{\vartheta}(|J^{\star\star}|)^{\kappa_7 + \vartheta}, \quad \ell(u_r a_t \tilde{x})^{\vartheta} \|\tilde{\Lambda}_d - u_r a_t \tilde{x}\|_{u_r a_t \tilde{x}} \geq \varkappa_{\vartheta}(|J^{\star\star}|).$$

(ii) There is a surface a time $r^\star \in [0,1]$, surface $x^\star = a_t u_{r^\star} x \in \mathcal{H}_1(2)$, and a surface $y_d^\star \in (\Omega_1 W_D)_{\eta R^2}$, such that

$$\|y_d^\star - x^\star\|_{x^\star} < \varkappa_{\vartheta}(|J^{\star\star}|) \leq \ell(x^\star)^{\kappa_7}.$$

Case (i) implies Theorem 1.3 (3). Case (ii) implies that the Teichmüller curve $\Omega_1 W_D$ is $\varkappa_{\vartheta}(|J^{\star\star}|)$-close to $a_t u_{[0,1]} x$.

Then by Theorem 1.1, we conclude that for any $L > 0$, if

$$\log \varkappa_{\vartheta}^{\star}(e^t)^{-1} > C_1 D^{\kappa_1} \geq C_1 L^{\kappa_1^2},$$

(or $\varkappa_{\vartheta}^{\star}(e^t)^{-1} > D^{C_1} \geq L^{C_1 \kappa_1}$ assuming Conjecture 4.4 is correct), then

$$d(z, a_{2t} u_{[0,2]} x) \leq L^{-1}$$

for every $z \in \mathcal{H}_1^{(L^{-1})}(2)$. □

As we have seen above, we can use Teichmüller curves to approximate a given surface. We can apply it to x and obtain Theorem 1.4.

Proof of Theorem 1.4. Let $\delta \in (0, \delta_0)$, $x \in \mathcal{H}_1(2)$, $L \geq L_0(\delta, \frac{1}{2}\ell(x), \frac{1}{2}\ell(x))$ and $t > C_1 L^{\kappa_1} \log L$. Discussing as in the proof of Theorem 1.3, one can find an arithmetic Teichmüller curve $\Omega_1 W_{D'}$ for sufficiently large D', such that there exists $x' \in \Omega_1 W_{D'}$ with $d(x', x) < e^{-100t}$. Then $\ell(x') > \frac{1}{2}\ell(x)$. In addition, x' has a splitting $x' = \Lambda_1' \underset{I'}{\#} \Lambda_2'$ satisfying $\mathrm{Area}(\Lambda_1') = \mathrm{Area}(\Lambda_2')$ (e.g. Example 3.7). Then by Lemma 2.10 and (3.4), we have $\ell(\Lambda_1', \Lambda_2') = \ell(x')$.

Now apply Theorem 1.3 to δ, x', (Λ_1', Λ_2'), L, and t. Note that although the G-orbit $G.(\Lambda_1', \Lambda_2') \subset X$ is closed, it is far from of small volume. Thus, Theorem 1.3 implies that either x' is effectively dense or (Λ_1', Λ_2') has a G-closed orbit of small volume nearby. If Theorem 1.3(2) occurs, then there is $(\Lambda_1'', \Lambda_2'') \in X$ such that $G.(\Lambda_1'', \Lambda_2'')$ is periodic with $\mathrm{vol}(G.(\Lambda_1'', \Lambda_2'')) \leq e^{\delta t}$ and

$$\|(\Lambda_1'', \Lambda_2'') - (\Lambda_1', \Lambda_2')\| \leq e^{-\frac{1}{2}t}.$$

Since the injectivity radius of x' is greater than $e^{-\frac{1}{2}t}$ by the choice of t, there is a surface $x'' \in \mathcal{H}_1(2)$ with absolute periods $(\Lambda_1'', \Lambda_2'')$. $\square$

6. Sparse cover argument

In Section 5, we have seen that in order to make the proof of Theorem 1.3 work, we require that certain surface of the form $a_t u_r x$ does not go to infinity (see (5.9)). More precisely, in the proof of Theorem 1.3, we use the equidistribution to get that the absolute periods of some $a_t u_r x$ are close to a Teichmüller curve $\Omega_1 W_D$. However, if $\ell(a_t u_r x)$ is too small, then $a_t u_r x$ fails to approach $\Omega_1 W_D$.

To fix this, we observe the nondivergence behavior of horocycle flows. For simplicity, we assume that $\{s \in I : \ell(u_s a_t x) < \eta\}$ is dominated by a single (C, α)-good function (cf. Theorem 6.9). Then by the nature of (C, α)-good functions, one observes that if $\ell(u_{e^t r} a_t x)$ is very small, say $< e^{-\xi^2 t}$, for some r and ξ, then there is an interval $J \subset [0, e^t]$ near $e^t r$ with the length $|J| > e^{\xi^2 t}$ so that $u_s a_t x \in \mathcal{H}_1^{(\eta)}(2)$ for all $s \in J$. Then we apply the equidistribution to $u_s a_t x$ for $s \in J$ to approach $\Omega_1 W_D$ again (with a weaker rate). Then we obtain the desired condition (see Proposition 6.13).

6.1. Sparse covers.

In this section, we review the sparse cover argument of horocycle flows in homogeneous and Teichmüller dynamics. See the references [KM98, MW02] therein.

For any $q \in \mathcal{TH}_1(\alpha)$, let $\mathscr{L}_q$ denote all the saddle connections for q. Recall that a saddle connection $\delta : [0, 1] \to M$ is a geodesic segment joining two zeroes in Σ or a zero to itself which has no zeroes in its interior. Then δ associates to a vector

$$q(\delta) = (x(\delta, q), y(\delta, q)) \in \mathbb{C}^*.$$

We say that two saddle connections δ_1, δ_2 are *disjoint* if $\delta_1(s_1) \neq \delta_2(s_2)$ for any $s_1, s_2 \in (0, 1)$. For $\delta \in \mathscr{L}_q$, define the length function

$$l_{q,\delta}(t) := \max\{|x(\delta, u_t q)|, |y(\delta, u_t q)|\}.$$

When q is clear from the context, we abbreviate $l_{q,\delta}(t)$ by $l_\delta(t)$.

Lemma 6.1 ([MW02, Lemma 4.4])**.** *For each $q \in \mathcal{TH}_1(\alpha)$, and each $\delta \in \mathscr{L}_q$, either $l_{q,\delta}(t)$ is a constant function of t, or there are t_0 and $c > 0$ such that*

$$l_{q,\delta}(t) = \max\{c, c|t - t_0|\}.$$

Let $\mathscr{F}$ be a collection of continuous functions $\mathbb{R} \to \mathbb{R}^+$. For $\theta > 0$, $f \in \mathscr{F}$ and $I \subset \mathbb{R}$ an interval, we let

$$\begin{aligned} I_f(\theta) &:= \{s \in I : f(s) < \theta\}, \\ I_{\mathscr{F}}(\theta) &:= \{s \in I : \exists f \in \mathscr{F},\ f(s) < \theta\} = \bigcup_{f \in \mathscr{F}} I_f(\theta), \\ \|f\|_I &:= \sup_{s \in I} f(s). \end{aligned}$$

Definition 6.2 ((C, α)-good function)**.** Let C, α, ρ be positive constants.

- We say that $\mathscr{F}$ is (C, α, ρ)-*good* if for every interval $I \subset \mathbb{R}$ and every $f \in \mathscr{F}$, we have

$$\frac{|I_f(\epsilon)|}{|I|} \leq C \left(\frac{\epsilon}{\|f\|_I} \right)^\alpha \tag{6.1}$$

  for $0 < \epsilon < \rho$.
- We say that $\mathscr{F}$ is (C, α)-*good* if it is (C, α, ρ)-good for every ρ.

Lemma 6.3 ([MW02, Lemma 4.5])**.** *The collection*

$$\mathscr{F}_1^{(\alpha)} := \{l_{q,\delta} : q \in \mathcal{H}(\alpha),\ \delta \in \mathscr{L}_q\}$$

is $(2, 1)$*-good.*

In particular, recall that $\mathcal{H}_1(0) = G/\Gamma$. Then $\mathscr{F}_1^{(0)}$ for G/Γ is also $(2, 1)$-good. Moreover, note that any $\Lambda_1 \in G/\Gamma$ with $\ell(\Lambda_1) < 1$ has a unique vector $v \in \Lambda_1 \subset \mathbb{R}^2$ such that

$$l_{\Lambda_1, v}(0) = \ell(\Lambda_1).$$

It follows that for any $\eta \in (0, 1)$, the collection

$$\mathscr{F}_2(\Lambda_1) := \{l_{\Lambda_1, v} \in \mathscr{F}_1^{(0)} : v \in \mathbb{R}^2 \text{ is a vector on } \Lambda_1\}$$

satisfies

$$\#\{f \in \mathscr{F}_2(\Lambda_1) : f(t) \leq \eta\} \leq 1.$$

On the other hand, one can easily deduce

$$\{s \in I : \ell(u_s \Lambda_1) < \eta\} \subset I_{\mathscr{F}_1^{(0)}}(\eta).$$

Now recall that for $(\Lambda_1, \Lambda_2) \in X$,

$$\ell(\Lambda_1, \Lambda_2) = \min\{\ell(\Lambda_1), \ell(\Lambda_2)\}.$$

Thus, we obtain:

Proposition 6.4 (Sparse cover for X). *For any $(\Lambda_1, \Lambda_2) \in X$, any interval $I \subset \mathbb{R}$, there exists a collection $\mathscr{F}''$ of $(2,1)$-good functions*

$$\mathscr{F}''(\Lambda_1, \Lambda_2) := \mathscr{F}_2(\Lambda_1) \cup \mathscr{F}_2(\Lambda_2)$$

such that for any $\eta \in (0,1)$, for any $t \in I$, we have

$$\#\{f \in \mathscr{F}''(\Lambda_1, \Lambda_2) : f(t) < 1\} \leq 2, \tag{6.2}$$

and

$$V_0(\eta) := \{r \in I : \ell(u_r(\Lambda_1, \Lambda_2)) < \eta\} \subset I_{\mathscr{F}''(\Lambda_1,\Lambda_2)}(\eta). \tag{6.3}$$

In the following, we shall develop a similar sparse cover for the Teichmüller spaces.

Proposition 6.5 ([MW02, Proposition 4.7]). *There exists R (depending only on α) such that for all $q \in \mathcal{TH}_1(\alpha)$, if $\delta_1, \dots, \delta_r \in \mathscr{L}_q$ are disjoint, then $r \leq R$.*

Let R be as in Proposition 6.5, let $1 \leq r \leq R$, and let $q \in \mathcal{TH}_1(\alpha)$. Define

$$\mathscr{E}_r := \{E \subset \mathscr{L}_q : E \text{ consists of } r \text{ disjoint segments}\}.$$

Define

$$l_{q,E}(t) := \max_{\delta \in E} l_{q,\delta}(t), \quad \alpha_r(t) = \alpha_{q,r}(t) := \min_{E \in \mathscr{E}_r} l_{q,E}(t).$$

For $E \in \mathscr{E}_r$, define $S(E)$ as the closure of the union of the simply connected components of $S \setminus \bigcup_{\delta \in E} \delta$.

Proposition 6.6 ([MW02, Lemma 6.1]). *There exists $\eta_0 = \eta_0(\alpha) > 0$ such that for every $q \in \mathcal{TH}_1(\alpha)$, every $r \in \{1, \dots, R\}$ and every $E \in \mathscr{E}_r$, if $S(E) = S$, then $l_{q,E}(0) \geq \eta_0$. In particular, $\alpha_{q,R}(0) \geq \eta_0$.*

Lemma 6.7 ([MW02, Lemma 6.2]). *Let $q \in \mathcal{TH}_1(\alpha)$, $1 \leq r \leq R-1$. Suppose that $E \in \mathscr{E}_r$ such that $l_{q,E}(0) < \frac{\theta}{3\sqrt{2}}$ and $\alpha_{r+1}(0) \geq \theta$ for some θ, and suppose that δ is a saddle connection on $\partial S(E)$. Then for any $\delta' \in \mathscr{L}_q$ such that $\delta' \neq \delta$ and $\delta' \cap \delta \neq \emptyset$, we have $l_{q,\delta'}(0) \geq \frac{\sqrt{2}}{3}\theta$.*

Lemma 6.8 ([MW02, Lemma 6.4]). *Let f and $\tilde{f}$ be two functions of the form $t \mapsto \max\{c, c|t - t_0|\}$. Suppose that for some $b > 0$ and $s \in \mathbb{R}$, we have $f(s) < b/3$ and $\tilde{f}(s) < b/3$. Then, possibly after exchanging f and $\tilde{f}$, $f(t) < b$ whenever $\tilde{f}(t) < b/3$.*

In [MW02], Minsky and Weiss implicitly gives a sparse cover for the set $V(\eta) := \{s \in I : \ell(u_s q) < \eta\} = \{t \in I : \alpha_1(t) < \eta\}$ (cf. Proposition 6.4):

Theorem 6.9 (Sparse cover for $\mathcal{H}_1(\alpha)$). *Let R be as in Proposition 6.5, and η_0 as in Proposition 6.6. For any $q \in \mathcal{TH}_1(\alpha)$, any interval $I \subset \mathbb{R}$, there exists an absolute constant $C_{12} > 0$ so that the following statement holds. For any $0 <$*

$C < \min\{\eta_0, C_{12}\}$, there are subcollections $\mathscr{F}_0(k,q) \subset \mathscr{F}_1^{(\alpha)}$ of length functions for $k = 1, \ldots, R$ such that for any $t \in I$, we have

$$\#\left\{f \in \mathscr{F}_0(k,q) : f(t) \leq \frac{\sqrt{2}}{9} C^{\frac{R-k-1}{R-1}} C\right\} \leq R, \tag{6.4}$$

and

$$V(C^2) = \{s \in I : \ell(u_s q) < C^2\} \subset \bigcup_{k=1}^{R} I_{\mathscr{F}_0(k,q)}(C^{\frac{R-k}{R-1}} C). \tag{6.5}$$

Theorem 6.9 has been implicitly given in the proof of Theorem 2.6 ([MW02, Theorem 6.3]). We provide a proof here for completion.

Proof of Theorem 6.9. Let C_{12} be a sufficiently small constant that will be determined below. Let $C \in (0, C_{12})$. For $k = 1, \ldots, R-1$, write

$$L_k \coloneqq C^{\frac{R-k}{R-1}} C.$$

Note that $L_1 = C^2$, $L_R = C$, and the L_k's increase by a constant multiplicative factor:

$$\frac{L_k}{L_{k+1}} = C^{\frac{1}{R-1}} < C_{12}{}^{\frac{1}{R-1}}. \tag{6.6}$$

For $t \in V(C^2)$, let

$$r(t) \coloneqq \max\{k : \alpha_k(t) < L_k\}.$$

Since $\alpha_R(t) \geq \eta_0 \geq C = L_R$ by Proposition 6.6, we have $r(t) \leq R-1$ and

$$\alpha_{r(t)}(t) < L_{r(t)}, \quad \alpha_{r(t)+1}(t) \geq L_{r(t)+1}.$$

Let $V_k \coloneqq \{t \in V(C^2) : r(t) = k\}$. Then we have

$$V(C^2) = \bigsqcup_{k=1}^{R-1} V_k. \tag{6.7}$$

Now, for $\delta \in \mathscr{L}_q$, let

$$H_k(\delta) \coloneqq \left\{t \in I : l_{q,\delta}(t) < L_k \text{ and } l_{q,\delta}(t) \geq \frac{\sqrt{2}}{3} L_{k+1} \text{ for } \delta \neq \delta' \in \mathscr{L}_q,\ \delta \cap \delta' \neq \emptyset\right\}.$$

Finally, define

$$\mathscr{F}_0(k,q) \coloneqq \{l_{q,\delta} \in \mathscr{F}_1^{(\alpha)} : V_k \cap H_k(\delta) \neq \emptyset\}.$$

Claim 6.10. *We have*

$$V_k \subset \bigcup_{\delta \in \mathscr{L}_q} H_k(\delta).$$

Proof of Claim 6.10. Let $t \in V_k$. Making C_{12} small enough, in (6.6), we get that $L_k < \frac{1}{3\sqrt{2}} L_{k+1}$. There is an $E \in \mathscr{E}_k$ such that

$$l_{q,E}(t) = \alpha_r(t) < L_k < \frac{1}{3\sqrt{2}} L_{k+1}.$$

By Proposition 6.6, $S(E) \neq S$, so let $\delta \in E$ be on the boundary of $S(E)$. Then for any $\delta' \in \mathscr{L}_q$, if $\delta' \neq \delta$ and $\delta' \cap \delta \neq \emptyset$, then by Lemma 6.7, we have

$$l_{q,\delta'}(t) \geq \frac{\sqrt{2}}{3} L_{k+1}.$$

Then $t \in H_k(\delta)$. ☐

Claim 6.11. *For all $t \in I$, we have*

$$\# \left\{ l_\delta \in \mathscr{F}_0(k,q) : l_\delta(t) \leq \frac{\sqrt{2}}{9} L_{k+1} \right\} \leq R.$$

Proof of Claim 6.11. Making C_{12} small enough, in (6.6), we get that $L_k < \frac{\sqrt{2}}{9} L_{k+1}$. Suppose $t \in I$ and $l_\delta, l_{\delta'} \in \mathscr{F}_0(k,q)$. Then $l_\delta(t), l_{\delta'}(t) \leq \frac{\sqrt{2}}{9} T_{k+1}$. By Lemmas 6.1 and 6.8, we obtain (possibly after exchanging δ and δ') that for any $s \in I$, if $l_\delta(s) < \frac{\sqrt{2}}{9} L_{k+1}$, then $l_{\delta'}(s) < \frac{\sqrt{2}}{3} L_{k+1}$.

Now for $s \in V_k \cap H_k(\delta)$, we must have $l_\delta(s), < L_k \leq \frac{\sqrt{2}}{9} L_{k+1}$. It follows that δ' is disjoint from δ. Since, by Proposition 6.5, the number of disjoint elements of $\mathscr{L}_q$ is at most R, the claim follows. ☐

Thus, (6.4) follows from Claim 6.11. Moreover, by Claim 6.10, we obtain that

$$\begin{aligned} V_k &= \bigcup_{\delta \in \mathscr{L}_q} V_k \cap H_k(\delta) \\ &= \bigcup_{l_\delta \in \mathscr{F}_0(k,q)} V_k \cap H_k(\delta) \\ &\subset \bigcup_{l_\delta \in \mathscr{F}_0(k,q)} H_k(\delta) \\ &\subset \bigcup_{l_\delta \in \mathscr{F}_0(k,q)} I_{l_\delta, L_k} = I_{\mathscr{F}_0(k,q), L_k}. \end{aligned} \tag{6.8}$$

By (6.7) and (6.8), we establish Theorem 6.9. ☐

6.2. **Existence of large intervals.** We shall use the following elementary lemma later to create large intervals as needed. For an interval $I = [a-b, a+b]$, $r > 0$, we write $rI := [a - rb, a + rb]$.

Lemma 6.12. *Let $A, B, D, E \geq 10$ be numbers so that $A \leq B$ and $D \geq 6BE/A$. Let $J \subset \mathbb{R}$ be an interval. Let I, I' be intervals so that the midpoints of I and I' coincide.*

(1). Suppose that we have

$$|J| \le B, \quad A \le |I| \le B, \quad J \cap I \neq \emptyset, \tag{6.9}$$

and

$$2D|I| \le |I'|. \tag{6.10}$$

Then there exists an interval J' such that

$$2J' \supset 2J, \quad |J'| \ge \frac{E}{2}(|I| + |J|), \quad \text{and} \quad J' \subset I' \setminus I. \tag{6.11}$$

(2). Suppose that we have (6.10), and

$$|J| \le B, \quad A \le |I|, \quad J \cap I \neq \emptyset, \tag{6.12}$$

and there is an endpoint $a \in J$ such that $a \notin I$. Then there exists an interval J' such that

$$2J' \supset 2J, \quad |J'| = E|J|, \quad \text{and} \quad J' \subset I' \setminus I. \tag{6.13}$$

Proof. This follows from the idea that I always sits in the middle of I'.

(1). Say $J = [a, a + |J|]$. Since $J \cap I \neq \emptyset$, we have $I \subset [a - B, a + |J| + B]$. Since by (6.9)(6.10), we see that

$$2DA \le 2D|I| \le |I'|.$$

Since the midpoints of I and I' coincide, let $x \in I \subset I'$ be the midpoint of I and I'. Then $x \in [a - B, a + |J| + B]$, and

$$[x, x + DA] \subset I'. \tag{6.14}$$

Now define

$$J' := [a + |J| + 2B, a + B + \frac{1}{2}DA].$$

First, one calculates the length

$$\begin{aligned} |J'| &= (B + \frac{1}{2}DA) - (|J| + 2B) \\ &\ge B + 3BE - 3B \ge BE \ge \frac{E}{2}(|I| + |J|). \end{aligned} \tag{6.15}$$

Then the left endpoint of $2J'$ is

$$(a + |J| + 2B) - \frac{1}{2}|J'| \le a + 3B - \frac{1}{2}BE < a - \frac{1}{2}|J|.$$

Since J' sits on the right of J, we conclude that $2J' \supset 2J$.

Next, note that

$$I \cap J' \subset [a - B, a + |J| + B] \cap J' = \emptyset. \tag{6.16}$$

To see $J' \subset I'$, we compare the right endpoints

$$a + B + \frac{1}{2}DA \le a - B + DA \le x + DA.$$

Thus, by (6.14), we have

$$J' \subset [x, x + DA] \subset I'. \tag{6.17}$$

(6.15), (6.16) and (6.17) establish (6.11). Note that one may similarly construct an interval J' that sits on the left of J.

(2). Say $J = [a, a+|J|]$ and suppose that $a+|J|$ is the endpoint so that $a+|J| \notin I$. By (6.12)(6.10), we see that

$$2DA \leq 2D|I| \leq |I'|.$$

Let $x \in I \subset I'$ be the midpoint of I and I'. Then

$$[x, x + DA] \subset [x, x + D|I|] \subset |I'|. \tag{6.18}$$

Since $J \cap I \neq \emptyset$, we have

$$I \subset [0, a + |J|), \quad \text{and} \quad a \leq x + \frac{1}{2}|I|. \tag{6.19}$$

Now define

$$J' := [a + |J|, a + |J| + E|J|].$$

First, one calculates the length

$$|J'| = E|J|. \tag{6.20}$$

Then the left endpoint of $2J'$ is

$$(a + |J|) - \frac{1}{2}|J'| \leq a + |J| - \frac{1}{2}E|J| < a - \frac{1}{2}|J|.$$

Since J' sits on the right of J, we conclude that $2J' \supset 2J$.

Next, note that

$$I \cap J' \subset (-\infty, a + |J|) \cap J' = \emptyset. \tag{6.21}$$

To see $J' \subset I'$, by (6.19), one observes

$$a + (|J| + E|J|) \leq (x + \frac{1}{2}|I|) + (B + \frac{1}{6}DA) \leq x + D|I|.$$

Thus, by (6.18), we have

$$J' \subset [x, x + D|I|] \subset I'. \tag{6.22}$$

(6.20), (6.21) and (6.22) establish (6.13). Note that one may similarly construct an interval J' that sits on the left of J if a is the endpoint so that $a \notin I$. $\square$

6.3. **New approximation.** From (5.9), we have seen that we fail to proceed if x_0 has a small injectivity radius. In the following, we assume that

$$\ell(x_0) < \varkappa_\vartheta(e^t)^\xi \tag{6.23}$$

and use sparse cover to find another qualified surface with weaker estimates.

Proposition 6.13. *Assume that (6.23) holds. Then there exists and a function $\varkappa^\star_\vartheta : \mathbb{R}^+ \to \mathbb{R}^+$ depending on $\varkappa_\vartheta$, κ_7, ϑ with $\lim_{t\to\infty} \varkappa^\star_\vartheta(e^t) = 0$, and an interval $J^{\star\star} \subset [0, e^t]$ with $|J^{\star\star}| \geq (\varkappa^\star_\vartheta(e^t))^{-1}$, such that for*

$$J'_d = \left\{ r \in J^{\star\star} : \|\tilde{\Lambda} - u_r a_t \Lambda(x)\| \leq \varrho,\ \tilde{\Lambda} \in (Q_D)_{\eta R^2+1} \right\},$$

we have

$$|J'_d| \gg \varrho^3 |J^{\star\star}|, \tag{6.24}$$

and at least one of the following holds:

(i) For any $r \in J'_d$, and $\tilde{\Lambda}_d \in (Q_D)_{\eta R^2}$, we have

$$\ell(u_r a_t \tilde{x}) \geq \varkappa_\vartheta(|J^{\star\star}|)^{\kappa_7+\vartheta}, \quad \ell(u_r a_t \tilde{x})^\vartheta \|\tilde{\Lambda}_d - u_r a_t \tilde{x}\|_{u_r a_t \tilde{x}} \geq \varkappa_\vartheta(|J^{\star\star}|).$$

(ii) There is a surface a time $r^\star \in [0, 1]$, surface $x^\star = a_t u_{r^\star} x \in \mathcal{H}_1(2)$, and a surface $y^\star_d \in (\Omega_1 W_D)_{\eta R^2}$, such that

$$\|y^\star_d - x^\star\|_{x^\star} < \varkappa_\vartheta(|J^{\star\star}|) \leq \ell(x^\star)^{\kappa_7}.$$

Let $y \coloneqq a_{s_1} x$. Then we see that there exists a $e^t r_0 \in [0, e^t]$ such that

$$\ell(u_{e^t r_0} y) < \varkappa_\vartheta(e^t)^\xi. \tag{6.25}$$

Now we apply the sparse cover. Let $C > 0$ be a small constant that will be determined later. Applying Theorem 6.9, set q, I, C in the statement of the theorem equal respectively to $y, [0, e^t], C$. Then we obtain a collection $\mathscr{F}_0(k)$ of $(2,1)$-functions for $k = 1, \ldots, R$ such that for any $t \in [0, e^t]$, we have

$$\#\left\{ f \in \mathscr{F}_0(k) : f(t) \leq \frac{\sqrt{2}}{9} C^{\frac{R-k-1}{R-1}} C \right\} \leq R, \tag{6.26}$$

and

$$V(C^2) = \{r \in [0, e^t] : \ell(u_r y) < C^2\} \subset \bigcup_{k=1}^{R} I_{\mathscr{F}_0(k)}(C^{\frac{R-k}{R-1}} C). \tag{6.27}$$

Finally, we define

$$F \coloneqq \bigcup_{k=1}^{R} \mathscr{F}_0(k).$$

Define $\phi : F \to \mathbb{R}^+$ by

$$\phi : f \mapsto C^{\frac{R-k}{R-1}} C, \quad \text{if } f \in \mathscr{F}_0(k),$$

and define $\psi : F \to \mathbb{R}^+$ by

$$\psi : f \mapsto \frac{\sqrt{2}}{9} C^{\frac{R-k-1}{R-1}} C, \qquad \text{if } f \in \mathscr{F}_0(k).$$

Then for any $f \in F$, one calculates

$$\frac{\phi(f)}{\psi(f)} \leq \frac{9\sqrt{2}}{2} C^{\frac{1}{R-1}}. \tag{6.28}$$

Also, it follows from the sparse cover (6.26)(6.27) that for any $t \in I$, we have

$$\# \{f \in F : f(t) \leq \psi(f)\} \leq R^2, \tag{6.29}$$

and

$$V(C^2) \subset \bigcup_{f \in F} I_f(\phi(f)). \tag{6.30}$$

For $\sigma > 0$, define

$$\begin{aligned} L_\sigma(r^*) &:= \min\{r < r^* : \max_{r' \in [r, r^*]} \ell(u_{r'} y) < \sigma\}, \\ R_\sigma(r^*) &:= \max\{r > r^* : \max_{r' \in [r^*, r]} \ell(u_{r'} y) < \sigma\}, \\ J_\sigma(r^*) &:= [L_\sigma(r^*), R_\sigma(r^*)]. \end{aligned}$$

Then by (6.25) and sufficiently large T, we have

$$|J_{\varkappa_\vartheta(e^t)^{-\frac{1}{2}\xi}}(e^t r_0)| \geq 1.$$

Since any length function is $(2,1)$-good (Lemma 6.3), we have

$$\frac{1}{2} C^2 \varkappa_\vartheta(e^t)^{-\frac{1}{2}\xi} \leq \frac{1}{2} C^2 \varkappa_\vartheta(e^t)^{-\frac{1}{2}\xi} |J_{\varkappa_\vartheta(e^t)^{\frac{1}{2}\xi}}(e^t r_0)| \leq |J_{C^2}(e^t r_0)|.$$

Write $J = J_{C^2}(e^t r_0)$. For simplicity, say $J \subset \frac{1}{3}[0, e^t]$ (one may adjust the coefficients and follow the same argument as below for other situations).

Then, $V(C^2)$ contains the interval J of length

$$|J| \geq \frac{1}{2} C^2 \varkappa_\vartheta(e^t)^{-\frac{1}{2}\xi} \geq \varkappa_\vartheta(e^t)^{-\xi^2}. \tag{6.31}$$

Let $D_1 > 1$ be sufficiently large that will be specified later. For $f \in F$, let $x \in I_f(\phi(f))$ be the midpoint of $I_f(\phi(f))$ and $I_f(\psi(f))$. Let $I = I_f(\phi(f))$, and

$$I' = [x - D_1|I|, x + D_1|I|]. \tag{6.32}$$

Then since f is $(2,1)$-good, by (6.28), we have

$$|I'| = 2D_1|I| \leq 4D_1 \cdot \frac{I_f(\phi(f))}{I_f(\psi(f))} |I_f(\psi(f))| \leq 18\sqrt{2} C^{\frac{1}{R-1}} D_1 |I_f(\psi(f))|. \tag{6.33}$$

Let C be sufficiently small so that $18\sqrt{2} C^{\frac{1}{R-1}} D_1 < 1$. Then $I' \subset I_f(\psi(f))$.

Lemma 6.14. *Let $J \subset \frac{1}{3}[0, e^t]$ be an interval in $V(C^2)$ with $|J| \geq 10R^2$. Suppose that for any $f \in F$, we have*

$$|I_f(\phi(f))| < \frac{1}{6}(2D_1)^{-1}e^t. \tag{6.34}$$

Then there exists an interval $J_1 \subset [0, e^t]$, and a function $f \in F$ such that

$$2J_1 \supset 2J, \tag{6.35}$$

$$|J_1| \geq \frac{D_1}{24R^2}|I_f(\phi(f))| + \frac{D_1}{24R^2}|J|, \tag{6.36}$$

$$J_1 \subset I_f(\psi(f)) \setminus I_f(\phi(f)). \tag{6.37}$$

Proof. By the sparse cover (6.30), we have

$$J \subset \bigcup_{f \in F} I_f(\phi(f)).$$

Claim 6.15. *There exists a function $f \in F$ such that*

$$I_f(\phi(f)) \cap J \neq \emptyset, \tag{6.38}$$

$$|I_f(\phi(f))| \geq R^{-2}|J| \geq 10. \tag{6.39}$$

Proof of Claim 6.15. Assume for contradiction that for any $f \in F$,

$$|I_f(\phi(f))| < R^{-2}|J|. \tag{6.40}$$

Since $100R^2\phi(f) \leq \psi(f)$, (6.40) indicates that

$$\frac{|I_f(\phi(f)) \cap J|}{|I_f(\psi(f)) \cap J|} < R^{-2}.$$

Note by (6.29) that

$$\sum_{f \in F} |I_f(\psi(f)) \cap J| = \int_J \# \{f \in F : f(r) \leq \psi(f)\}\, dr \leq R^2|J|.$$

It follows that

$$R^2|J| = R^2 \sum_{f \in F} |I_f(\phi(f)) \cap J| < \sum_{f \in F} |I_f(\psi(f)) \cap J| \leq R^2|J|.$$

This leads to a contradiction. $\square$

We choose f satisfying (6.38)(6.39) so that the length $|I_f(\phi(f))|$ is the maximal possible.

Let $x^\star$ be the midpoint of J, and

$$J^\star := [x^\star - R^2|I_f(\phi(f))|, x^\star + R^2|I_f(\phi(f))|].$$

Now we apply Lemma 6.12(1), setting A, B, D, E, I, I', J in the statement of the lemma equal to $|I_f(\phi(f))|, 2R^2|I_f(\phi(f))|, D_1, \frac{D_1}{12R^2}, I, I', J^\star$, respectively. To check the requirements, note that

$$D = D_1 = 6 \cdot 2R^2 \cdot \frac{D_1}{12R^2} = 6 \cdot \frac{B}{A} \cdot E. \tag{6.41}$$

On the other hand, recall from (6.32) that

$$2D_1|I| = |I'|. \tag{6.42}$$

This establishes (6.10). Also, by (6.38)(6.39), the interval $J^\star$ satisfies $J^\star \cap I_f(\phi(f)) \neq \emptyset$. Thus, we establishes (6.9).

Therefore, by (6.11), there exists $J' \subset \mathbb{R}$ such that

$$2J' \supset 2J,$$
$$|J'| \geq \frac{D_1}{24R^2}|I| + \frac{D_1}{24R^2}|J|,$$
$$J' \subset I' \setminus I.$$

Since $|J'| \leq |I'| = 2D_1|I| \leq \frac{1}{6}e^t$, $2J' \supset J$, and $J \subset [\frac{1}{3}e^t, \frac{2}{3}e^t]$, we conclude that $J' \subset [0, e^t]$. The lemma follows from letting $J_1 = J'$. ☐

Let $D_2 = \frac{D_1}{1000R^2 e}$. In particular, we have

$$\frac{1}{6}(2D_2)^{-1}\frac{D_1}{24R^2} > 1, \quad \text{and} \quad \frac{D_2 e}{D_1} < \frac{1}{6}. \tag{6.43}$$

Lemma 6.16. *Let the notation and assumptions be as in Lemma 6.14. Suppose that there exists $g \in F \setminus \{f\}$ such that*

$$I_g(\phi(g)) \cap J_1 \neq \emptyset, \quad \textit{and} \quad |I_g(\phi(g))| \geq \frac{1}{6}(2D_2)^{-1}|J_1|. \tag{6.44}$$

Then there exists an interval $J_2 \subset [0, e^t]$ such that

$$|J_2| = e|J_1|, \quad 2J_2 \supset 2J_1$$

and that for any

$$J_2 \subset I_g(\psi(g)) \setminus I_g(\phi(g)).$$

Proof. Now since by (6.44), we have

$$I_g(\phi(g)) \cap J_1 \neq \emptyset.$$

Also, by (6.36) and (6.43), we have

$$|I_g(\phi(g))| \geq \frac{1}{6}(2D_2)^{-1}|J_1| \geq \frac{1}{6}(2D_2)^{-1}\frac{D_1}{24R^2}|I_f(\phi(f))| > |I_f(\phi(f))|.$$

If $I_g(\phi(g)) \cap J \neq \emptyset$, then it violates the maximal choice of f (cf. the definition after Claim 6.15). Thus, we conclude that $I_g(\phi(g)) \cap J = \emptyset$. This, together with $2J_1 \supset 2J$ (by (6.35)) and $J_1 \subset I_g(\psi(g)) \setminus I_g(\phi(g))$ (by (6.37)), imply that there is an endpoint a of $2J_1$ such that $a \notin I_g(\phi(g))$.

Next, we apply Lemma 6.12(2), setting A, B, D, E, I, I', J in the statement of the lemma equal to $\frac{1}{6}(2D_2)^{-1}|J_1|, 2\cdot|J_1|, 6\cdot 24D_2e, e, I_g(\phi(g)), I_g(\psi(g)), 2J_1$, respectively. To check the requirements, since $D_2 < D_1$, by (6.33), we have

$$2D_2|I_g(\phi(g))| = 2D_2|I| \leq |I| = |I_g(\psi(g))| \tag{6.45}$$

This establishes (6.10). Also, (6.12) follows from (6.44).

Therefore, by Lemma 6.12(2), there exists $J_2 \subset \mathbb{R}$ such that

$$|J_2| = e|J_1|, \quad 2J_2 \supset 4J_1 \supset 2J_1, \quad J_2 \subset I_g(\psi(g)) \setminus I_g(\phi(g)).$$

Finally, note that by (6.34)(6.43), we have

$$|J_2| = e|J_1| \leq e\cdot 6(2D_2)|I_g(\phi(g))| < \frac{eD_2}{D_1} < \frac{1}{6}e^t,$$

$2J_2 \supset 2J_1 \supset J$, and $J \subset [\frac{1}{3}e^t, \frac{2}{3}e^t]$, we conclude that $J_2 \subset [0, e^t]$. □

Note that (6.44) cannot always occur. For instance, it does not hold for $J_1 = [0, e^t]$. Thus, we obtain:

Corollary 6.17. *Let the notation and assumptions be as in Lemma 6.14. There exists an interval $J_1^\star \subset [0, e^t]$, and a function $f^{(1)} \in F$ such that*

$$2J_1^\star \supset 2J, \quad J_1^\star \subset I_{f^{(1)}}(\psi(f^{(1)})) \setminus I_{f^{(1)}}(\phi(f^{(1)})).$$

Moreover, any $g \in F \setminus \{f^{(1)}\}$ with $I_g(\phi(g)) \cap J_1^\star \neq \emptyset$ satisfies

$$|I_g(\phi(g))| < \frac{1}{6}(2D_2)^{-1}|J_1^\star|. \tag{6.46}$$

In particular, for J as in (6.31), we have

$$|J_1^\star| \geq |J| \geq \varkappa_\vartheta(e^t)^{-\xi^2}.$$

Let $J_1^\star = [a, a + |J_1^\star|]$. For $i = 0, 1, \ldots, \lfloor \varkappa_\vartheta(e^t)^{\xi^2}|J_1^\star|\rfloor$, consider

$$J_i' := [a + i\varkappa_\vartheta(e^t)^{-\xi^2}, a + (i+1)\varkappa_\vartheta(e^t)^{-\xi^2}). \tag{6.47}$$

Lemma 6.18. *Let $D_3 = \frac{1}{2R^2}D_2$. Let J' be the union of J_i' so that any $g \in F \setminus \{f^{(1)}\}$ with $I_g(\phi(g)) \cap J_i' \neq \emptyset$ satisfies*

$$|I_g(\phi(g)) \cap J_i'| < \frac{1}{6}(2D_3)^{-1}|J_i'|. \tag{6.48}$$

Then we have $|J'| \geq \frac{1}{2}|J_1^\star|$.

Proof. Let J'' be the union of J_i' so that there is $g \in F \setminus \{f^{(1)}\}$ so that

$$|I_g(\phi(g)) \cap J_i'| \geq \frac{1}{6}(2D_3)^{-1}|J_i'|.$$

Now suppose that $|J''| \geq \frac{1}{2}|J_1^\star|$. Then, we have

$$\bigcup_{g\in F\setminus\{f^{(1)}\}} |I_g(\phi(g)) \cap J''| \geq \frac{1}{6}(2D_3)^{-1}|J''|$$
$$\geq \frac{D_2}{2R^2D_3}\cdot\frac{1}{6}R^2(2D_2)^{-1}\cdot 2|J''| \geq \frac{1}{6}R^2(2D_2)^{-1}|J_1^\star|.$$

However, by (6.29) and (6.46), we have that

$$\bigcup_{g\in F\setminus\{f^{(1)}\}} |I_g(\phi(g)) \cap J_1^\star| < \frac{1}{6}R^2(2D_2)^{-1}|J_1^\star|.$$

This leads to a contradiction. $\square$

Next, we shall pick a surface other than x_0 that is close to $\Omega_1 W_D$ in the sense of absolute periods again. For simplicity, we assume that $J_1^\star$ sits on the right of $\Lambda(x_0) = a_t u_{r_0}(\Lambda_1, \Lambda_2)$.

Proposition 6.19. *There exists an interval $J_1^{\star\star} \subset J'$ with $|J_1^{\star\star}| = \varkappa_\vartheta(e^t)^{-\xi^2}$ such that letting*

$$J_d'^{(1)} = \left\{r \in J_1^{\star\star} : \|\tilde{\Lambda} - u_r\Lambda(x_0)\| \leq \varrho,\ \tilde{\Lambda} \in (Q_D)_{\eta^2}\right\}, \tag{6.49}$$

we have

$$\left|J_d'^{(1)}\right| \gg |J_1^{\star\star}|^{1-\frac{3\kappa_9\delta}{2N}} = \varkappa_\vartheta(e^t)^{-\xi^2}\varrho^3.$$

Let

$$a_1(t) := a_{-\xi^2\log\varkappa_\vartheta(e^t)}.$$

First, $a_1(t)^{-1}\tilde{\Lambda}_d \in X_{2\eta}$. By (5.7), we have

$$\|a_1(t)^{-1}\tilde{\Lambda}_d - a_1(t)^{-1}\Lambda(x_0)\| < \varkappa_\vartheta(e^t)^{\frac{2}{3}\xi}. \tag{6.50}$$

In particular, $a_1(t)^{-1}\Lambda(x_0) \in X_\eta$. Moreover, by replacing a_t with $a_1(t)$, we see that $a_1(t)^{-1}\Lambda(x_0)$ is a point of second kind in Theorem 4.5.

The following is provided by the polynomial nature of unipotent flows:

Lemma 6.20. *For any periodic $G.\Lambda' \subset X$, the distance*

$$P_{G\Lambda'}(r) = d_X(u_r a_1(t)^{-1}\Lambda(x_0), G\Lambda')$$

is a polynomial with bounded degree (when the distance is small). Here d_X is the metric on X induced by $\|\cdot\|$.

Assume that $P_{G\Lambda_c}(r)$ is constant, i.e. there is $C > 0$ so that

$$d_X(u_r a_1(t)^{-1}\Lambda(x_0), Q_D) \equiv C < \varkappa_\vartheta(e^t)^{\frac{2}{3}\xi}$$

for all r. Then one calculates that $\Lambda(x) = (\Lambda_1, \Lambda_2)$ is also $\varkappa_\vartheta(e^t)^{\frac{2}{3}\xi}$-close to Q_D. Recall that by our assumption, $\Lambda(x)$ is a point of the first kind in Theorem 4.5. Thus, it is impossible as t being large. Thus, $P_{Q_D}(r)$ is a nonconstant polynomial.

Next, we recall the discreteness of G-closed orbits. It is ultimately attributed to the theory of heights and discriminants. The theory of heights of Lie groups have been used extensively. In particular, it measures the arithmetic complexity of Lie groups. See e.g. [ELMV09, EMV09, EMMV20] for more details.

Lemma 6.21 (Discreteness of periodic orbits, [ELMV09, §2.4])**.** *For any $\eta > 0$, there exists $\kappa_{13} > 0$, and $K_0 = K_0(\eta^2) > 0$ such that for any $\Lambda \in X_{\eta^2}$, any $K \geq K_0$, any periodic $\mathcal{O}_1, \mathcal{O}_2 \subset X$ of volumes $\mathrm{vol}(\mathcal{O}_1), \mathrm{vol}(\mathcal{O}_2) \leq K$, if $E_1 \subset B(x, \eta^2) \cap \mathcal{O}_1$, $E_2 \subset B(x, \eta^2) \cap \mathcal{O}_2$ are two distinct connected components of G-orbits, then*

$$d_X(E_1, E_2) \geq K^{-\kappa_{13}}.$$

Now for Theorem 4.5(2) (by replacing a_t with $a_1(t)$), we consider periodic $G.\Lambda'$, $G.\Lambda'' \subset X$ with $\mathrm{vol}(G\Lambda'), \mathrm{vol}(G\Lambda'') \leq (\varkappa_\vartheta(e^t)^{-\xi^2})^\delta$. Let δ be small so that

$$\kappa_{13}\delta < \frac{1}{3}. \tag{6.51}$$

Then by Lemma 6.21, for any $\Lambda \in X_{\eta^2}$, if $E_1 \subset B(x, \eta^2) \cap G.\Lambda'$, $E_2 \subset B(x, \eta^2) \cap G.\Lambda''$ are two distinct connected components of G-orbits, then

$$d_X(E_1, E_2) \geq \varkappa_\vartheta(e^t)^{\kappa_{13}\delta\xi^2} \geq \varkappa_\vartheta(e^t)^{\frac{1}{3}\xi^2}. \tag{6.52}$$

We say that $I \subset [0, e^t \varkappa_\vartheta(e^t)^{\xi^2}]$ is a *maximal interval* with respect to the periodic $G.\Lambda'$ if I is an interval so that for any $r \in I - e^t \varkappa_\vartheta(e^t)^{\xi^2} r_0$,

$$d_X(u_r a_1(t)^{-1} \Lambda(x_0), G\Lambda') \leq \varkappa_\vartheta(e^t)^{\frac{1}{2}\xi^2} \tag{6.53}$$

and for any interval $J \supsetneq I$, there is $r \in J$ so that (6.53) does not hold.

Lemma 6.22. *If $I \subset [0, \varkappa_\vartheta(e^t)^{\xi^2} e^t]$ is a maximal interval with respect to a periodic $G.\Lambda' \subset X$, then for any $r \in 1000I \setminus I$, $u_r a_1(t)^{-1} \Lambda(x_0)$ is a point of the first kind in Theorem 4.5 whenever $u_r a_1(t)^{-1} \Lambda(x_0) \in X_{\eta^2}$.*

Proof. Since $P_{G\Lambda'}(r) = d_X(u_r a_1(t)^{-1} \Lambda(x_0), G\Lambda')$ is a polynomial of bounded degree, there exists $N > 0$ (depending only on the degree) such that

$$\varkappa_\vartheta(e^t)^{\frac{1}{2}\xi^2} \leq d_X(u_r a_1(t)^{-1} \Lambda(x_0), G\Lambda') \leq N \varkappa_\vartheta(e^t)^{\frac{1}{2}\xi^2} \tag{6.54}$$

for any $r \in 1000I \setminus I$. Fix $r \in 1000I \setminus I$. Suppose that $u_r a_1(t)^{-1} \Lambda(x_0) \in X_{\eta^2}$, and there exists another periodic $G\Lambda''$ so that

$$d_X(u_r a_1(t)^{-1} \Lambda(x_0), G\Lambda'') \leq \varkappa_\vartheta(e^t)^{\frac{1}{2}\xi^2}. \tag{6.55}$$

Let $B = B(u_r a_1(t)^{-1} \Lambda(x_0), \eta^2)$, $E_1 = B \cap G\Lambda'$, and $E_2 = B \cap G\Lambda''$. Then by (6.54)(6.55), we have

$$d_X(E_1, E_2) \leq (N+1) \varkappa_\vartheta(e^t)^{\frac{1}{2}\xi^2}.$$

This contradicts (6.52). □

Proof of Proposition 6.19. Now for any interval $I=[a,b]$, $r>0$, we write $r\times I=[ra,rb]$. Write $\varkappa_\vartheta(e^t)^{\xi^2}\times J_1^\star=[a^\star,b^\star]$. Consider the interval

$$I^\star:=[0,b^\star-\varkappa_\vartheta(e^t)^{\xi^2}e^tr_0]\subset(\varkappa_\vartheta(e^t)^{\xi^2}\times 2J_1^\star)-\varkappa_\vartheta(e^t)^{\xi^2}e^tr_0.$$

First, since $a_1(t)^{-1}\Lambda(x_0)\in X_\eta$, by the sparse cover (Proposition 6.4), we have

$$|\{r\in I^\star:u_ra_1(t)^{-1}\Lambda(x_0)\notin X_{\eta^2}\}|<\frac{1}{1000}|I^\star|. \tag{6.56}$$

On the other hand, by Lemma 6.22, we have

$$|\{r\in I^\star:r\text{ satisfies (6.53) for periodic }G\Lambda'\neq Q_D\}|<\frac{1}{1000}|I^\star|. \tag{6.57}$$

Now for $i=0,1,\dots,\lfloor\varkappa_\vartheta(e^t)^{\xi^2}|J_1^\star|\rfloor$, let

$$J_i'':=[a^\star+i+\frac{45}{100},a^\star+i+\frac{55}{100}]\subset\varkappa_\vartheta(e^t)^{-\xi^2}\times\frac{1}{3}J_i'.$$

Then one directly calculates $|J_i''|=\frac{1}{5}|J_i'|$. Then by Lemma 6.18, we have

$$\begin{aligned}\sum_{J_i'\subset J'}|J_i''|&=\frac{1}{10}\varkappa_\vartheta(e^t)^{\xi^2}|J'|\\ &\geq\frac{1}{20}\varkappa_\vartheta(e^t)^{\xi^2}|J_1^\star|\geq\frac{1}{40}\varkappa_\vartheta(e^t)^{\xi^2}|2J_1^\star|\geq\frac{1}{40}|I^\star|. \end{aligned}\tag{6.58}$$

Finally, combining (6.56)(6.57)(6.58), there exists $i=0,1,\dots,\lfloor\varkappa_\vartheta(e^t)^{\xi^2}|J_1^\star|\rfloor$, and $r'\in J_i''-\varkappa_\vartheta(e^t)^{\xi^2}e^tr_0$ so that the point $u_{r'}a_1(t)^{-1}\Lambda(x_0)$ is of the first kind in Theorem 4.5 (by replacing a_t with $a_1(t)$).

We apply Theorem 4.5 to $u_{r'}a_1(t)^{-1}\Lambda(x_0)$:

- Let $J_1^{\star\star}=J_i'\subset J_1^\star$ (see (6.47)).
- Let $\varphi_{\varrho,d}$ be a function so that

$$\chi_{\mathsf{B}_{G\times G}(\frac{9}{10}\varrho).(Q_D)_\eta}\leq\varphi_{\varrho,d}\leq\chi_{\mathsf{B}_{G\times G}(\varrho).(Q_D)_\eta}$$

  with $\mathcal{S}(\varphi_{\varrho,d})\leq\varrho^{-N}\leq\varkappa_\vartheta(e^t)^{\frac{\kappa_9\delta}{2}\xi^2}$, where N is absolute. In particular, we have

$$\int\varphi_{\varrho,d}dm_X\asymp\varrho^3.$$

Then by the equidistribution, we see that

$$\left|\int_0^1\varphi_{\varrho,d}(a_1(t)u_{r''}\cdot u_{r'}a_1(t)^{-1}\Lambda(x_0))dr''-\varrho^3\right|\leq(\varkappa_\vartheta(e^t))^{\frac{\kappa_9\delta}{2}\xi^2}\asymp\varrho^N.$$

For large enough ϱ, we have

$$\varrho^3-\varkappa_\vartheta(e^t)^{\frac{\kappa_9\delta}{2}\xi^2}\ll\int_0^1\varphi_{\varrho,d}(a_1(t)u_{r''}\cdot u_{r'}a_1(t)^{-1}\Lambda(x_0))dr''\ll\varrho^3+\varkappa_\vartheta(e^t)^{\frac{\kappa_9\delta}{2}\xi^2}.$$

Thus, there is a subset $J_d'^{(1)}\subset[0,e^t]$ with $|J_d'^{(1)}|\asymp\varkappa_\vartheta(e^t)^{-\xi^2}\varrho^3$ such that

$$u_{\varkappa_\vartheta(e^t)^{-\xi^2}(r''+r')}\Lambda(x_0)\in\operatorname{supp}(\varphi_{\varrho,d})$$

for some $\varkappa_\vartheta(e^t)^{-\xi^2}(r''+r') \in J_d'^{(1)}$. In other words, for

$$J_d'^{(1)} = \left\{ r \in J_1^{\star\star} : \|\tilde{\Lambda} - u_r\Lambda(x_0)\| \leq \varrho,\ \tilde{\Lambda} \in (Q_D)_{\eta^2} \right\},$$

we have

$$|J_d'^{(1)}| \asymp \varrho^3 \varkappa_\vartheta(e^t)^{-\xi^2}. \tag{6.59}$$

This establishes the proposition. □

Proof of Proposition 6.13. We now apply the above argument inductively. By Corollary 6.17, Lemma 6.18 and Proposition 6.19, we conclude that

- There is a large $D_3 = D_3(D_1) > 0$ so that $D_3 \nearrow \infty$ as $D_1 \nearrow \infty$.
- There exists an interval $J_1^{\star\star} \subset I_{f^{(1)}}(\psi(f^{(1)})) \setminus I_{f^{(1)}}(\phi(f^{(1)}))$ with

$$|J_1^{\star\star}| = \varkappa_\vartheta(e^t)^{-\xi^2}$$

such that for any $g \in F \setminus \{f^{(1)}\}$ with $I_g(\phi(g)) \cap J_1^{\star\star} \neq \emptyset$, we have

$$|I_g(\phi(g)) \cap J_1^{\star\star}| < \frac{1}{6}(2D_3)^{-1}|J_1^{\star\star}|. \tag{6.60}$$

- We have

$$\left|J_d'^{(1)}\right| \gg \varrho^3 |J_1^{\star\star}|. \tag{6.61}$$

where $x_0' \in u_{J_1^{\star\star}} x_0$ and

$$J_d'^{(1)} = \left\{ r \in J_1^{\star\star} : \|\tilde{\Lambda} - u_r\Lambda(x_0)\| \leq \varrho,\ \tilde{\Lambda} \in (Q_D)_{\eta^2} \right\}.$$

Now let $\xi = \frac{1}{\kappa_7 + \vartheta}$. Suppose that

$$\ell(x_1) > \varkappa_\vartheta(|J_1^{\star\star}|)^\xi \tag{6.62}$$

for any $R_1 \in J_d'^{(1)} - e^t r_0$. If for any $R_1 \in J_d'^{(1)} - e^t r_0$, and $\tilde{\Lambda}_d \in (Q_D)_{\eta^2}$, we have

$$\ell(u_{R_1}\tilde{x}_0)^\vartheta \|\tilde{\Lambda}_d - u_{R_1}\tilde{x}_0\|_{u_{R_1}\tilde{x}_0} \geq \varkappa_\vartheta(|J_1^{\star\star}|),$$

then we obtain (i) of Proposition 6.13.

Now suppose that there is a $R_1 \in J_d'^{(1)} - e^t r_0$, and $\tilde{\Lambda}_d \in (Q_D)_{\eta^2}$, such that for $x_1 = u_{R_1} x_0$, we have

$$\ell(\tilde{x}_1)^\vartheta \|\tilde{x}_1 - \tilde{\Lambda}_d\|_{\tilde{x}_1} < \varkappa_\vartheta(|J_1^{\star\star}|) = \varkappa_\vartheta(\varkappa_\vartheta(e^t)^{-\xi^2}). \tag{6.63}$$

Then, we can follow the same idea as in (5.9). One may find a surface $y_d^{(1)} \in (\Omega_1 W_D)_{\eta^2}$ with the absolute periods $\tilde{\Lambda}_d$, so that

$$\|y_d^{(1)} - x_1\|_{x_1} < \ell(\tilde{x}_1)^{-\vartheta}\varkappa_\vartheta(|J_1^{\star\star}|) < \ell(x_1)^{\kappa_7}.$$

We obtain (ii) of Proposition 6.13.

Thus, assume that there exists $r_1 \in e^{-t}J_d'^{(1)} - r_0$ such that for $x_1 := a_t u_{r_0+r_1} x$,

$$\ell(x_1) < \varkappa_\vartheta(|J_1^{\star\star}|)^\xi = \varkappa_\vartheta(\varkappa_\vartheta(e^t)^{-\xi^2})^\xi. \tag{6.64}$$

Then by an argument similar to (6.31), one shows that

$$|J_{C^2}(e^t r_0 + R_1)| \geq \varkappa_\vartheta(|J_1^{\star\star}|)^{-\xi^2} = \varkappa_\vartheta(\varkappa_\vartheta(e^t)^{-\xi^2})^{-\xi^2}.$$

By (6.60), one can further deduce that $J_{C^2}(e^t(r_0+r_1)) \subset \frac{1}{3}J_1^{\star\star}$.

Now we set $[0,e^t], x_0, D_1, \eta, \Lambda_D, F, e^t r_0$ in the statement of this section to $J_1^{\star\star}, x_1$, D_3, η^2, $\Lambda_D^{(1)}$, $F \setminus \{f^{(1)}\}$, $e^t r_0 + R_1$ respectively. This completes an inductive step.

Note that the induction will stop in finite steps. More precisely, suppose that the inequality (6.64) keep occurring. The process will stop after at most R^2 times. In the end, we have

$$J_{R^2}^{\star\star} \subset \bigcap_{i=1}^{R^2} I_{f^{(i)}}(\psi(f^{(i)})) \setminus \bigcup_{i=1}^{R^2} I_{f^{(i)}}(\phi(f^{(i)})).$$

In particular, by Theorem 6.9, we have $J_{R^2}^{\star\star} \cap \{r \in J_{R^2}^{\star\star} : \ell(u_r y) < C^2\} = \emptyset$. Then (6.64) cannot occur anymore. To make sure the induction can proceed, we require that sufficiently large D_1 is sufficiently large (and so sufficiently small C), so that Lemma 6.12 can always be used. Thus, by (6.61)(6.62), we conclude that there exists some $k = 0, 1, \ldots, R^2$, times $r_1, \ldots, r_{k-1}$, a surface $x_{k-1} = a_t u_{r_0+r_1+\cdots+r_{k-1}} x \in \mathcal{H}_1(2)$ such that

$$\ell(u_{R_k} x_{k-1}) \geq \varkappa_\vartheta(|J_k^{\star\star}|)^{\xi}, \tag{6.65}$$

for any $R_k \in J_d'^{(k)} - e^t(r_0 + r_1 + \cdots + r_{k-1})$ and

$$\left|J_d'^{(k)}\right| \gg \varrho^3 |J_k^{\star\star}|, \tag{6.66}$$

where

$$J_d'^{(k)} = \left\{r \in J_k^{\star\star} : \|\tilde{\Lambda} - u_r \Lambda(x_{k-1})\| \leq \varrho,\ \tilde{\Lambda} \in (Q_D)_{\eta^{k+1}}\right\}.$$

Now let $J^{\star\star} = J_k^{\star\star}$. If for any $R_k \in J_d'^{(k)} - e^t(r_0+r_1+\cdots+r_{k-1})$, and $\tilde{\Lambda}_d \in (Q_D)_{\eta^2}$, we have

$$\ell(u_{R_k}\tilde{x}_{k-1})^{\vartheta} \|\tilde{\Lambda}_d - u_{R_k}\tilde{x}_{k-1}\|_{u_{R_k}\tilde{x}_{k-1}} \geq \varkappa_\vartheta(|J^{\star\star}|),$$

then we obtain (i) of Proposition 6.13.

Now suppose that there is a $r_k \in e^{-t} J_d'^{(k)} - (r_0 + r_1 + \cdots + r_{k-1})$, and $\tilde{\Lambda}_d \in (Q_D)_{\eta^2}$, such that for $x_k := a_t u_{r_0+r_1+\cdots+r_k} x$, we have

$$\ell(\tilde{x}_k)^{\vartheta} \|\tilde{x}_k - \tilde{\Lambda}_d\|_{\tilde{x}_k} < \varkappa_\vartheta(|J^{\star\star}|).$$

This guarantees the existence of a surface $y_d^{(k)} \in (\Omega_1 W_D)_{\eta^k}$ with the absolute periods $\tilde{\Lambda}_d^{(k)}$ (by (3.5)) such that

$$\ell(\tilde{x}_k)^{-\vartheta} \|\tilde{x}_k - y_d^{(k)}\|_{\tilde{x}_k} < \varkappa_\vartheta(|J_k^{\star\star}|) \leq \ell(x_k)^{\kappa_7}.$$

where $\varkappa_\vartheta^\star$ is given by

$$\varkappa_\vartheta^\star : r \mapsto \underbrace{\varkappa_\vartheta(\cdots \varkappa_\vartheta(\varkappa_\vartheta(r)^{-\xi^2})^{-\xi^2} \cdots)^{\xi^2}}_{R^2 \text{ copies}}.$$

Now Proposition 6.13 follows from letting $r^\star = r_0 + r_1 + \cdots + r_{k-1}$, $x^\star = x_k$, and $y_d^\star = y_d^{(k)}$. ☐

7. A criterion for Teichmüller curves

7.1. A criterion for the Teichmüller curves in $\mathcal{H}(2)$. In this section, we generate a class of Teichmüller curves via G-closed absolute periods and additional rationality.

Proposition 7.1. *Let $x'' \in \mathcal{H}(2)$ with an algebraic sum $x'' = \Lambda_1'' + \Lambda_2''$ that satisfies $\overline{G.(\Lambda_1'', \Lambda_2'')} = G.(\Lambda_1'', \Lambda_2'')$. Then there exists a square-free integer $k \in \mathbb{N}$ depending only on $(\Lambda_1'', \Lambda_2'')$ (regardless the areas of tori), such that if the areas of tori satisfy*

$$\sqrt{\frac{\mathrm{Area}(\Lambda_1'')}{\mathrm{Area}(\Lambda_2'')}} \cdot \sqrt{k} \in \mathbb{Q}, \tag{7.1}$$

then x'' generates a Teichmüller curve.

We start from the following lemma, which is a conclusion from *Ratner theorem*:

Lemma 7.2. *Let $(\Lambda_1, \Lambda_2) \in G/\Gamma \times G/\Gamma$ be a point with $\overline{G.(\Lambda_1, \Lambda_2)} = G.(\Lambda_1, \Lambda_2)$. Suppose that $v \in \mathbb{C}^*$ is a nonzero vector such that*

$$[0, v] \cap \Lambda_2 = \{0, v\}.$$

Then there exists a nonzero vector $w \in \mathbb{C}^*$ such that

$$[0, w] \cap \Lambda_1 = \{0, w\}$$

and $w = rv$ for some $r^2 \in \frac{\mathrm{Area}(\Lambda_1)}{\mathrm{Area}(\Lambda_2)}\mathbb{Q}$.

Proof. By replacing Λ_2 with a rescaling $\sqrt{\frac{\mathrm{Area}(\Lambda_1)}{\mathrm{Area}(\Lambda_2)}} \cdot \Lambda_2$ if necessary, we may assume that $\Lambda_1, \Lambda_2 \subset \mathbb{C}$ have the same area. Let $U_v < G$ be the unipotent subgroup stabilizing v. Let $\mathrm{SL}(\Lambda_2) := \{g \in G : g\Lambda_2 = \Lambda_2\}$ be the stabilizer of Λ_2. Let $U_v^{\mathbb{Z}} := U_v \cap \mathrm{SL}(\Lambda_2) \cong \mathbb{Z}$. It follows that

$$U_v^{\mathbb{Z}}.(\Lambda_1, \Lambda_2) = (U_v^{\mathbb{Z}}\Lambda_1, \Lambda_2).$$

Let $Z_1 := \overline{U_v^{\mathbb{Z}}\Lambda_1} \subset G/\Gamma$ be the orbit closure of the first coordinate. Then by Ratner theorem, we have $Z_1 = H\Lambda_1$ where $H = U_v^{\mathbb{Z}}$, U_v or G.

Assume first that no vectors $w \in \Lambda_1$ parallel to v. It means that $H = G$, and so

$$(G\Lambda_1, \Lambda_2) = \overline{U_v^{\mathbb{Z}}.(\Lambda_1, \Lambda_2)} \subset G.(\Lambda_1, \Lambda_2).$$

This leads to a contradiction since $G.(\Lambda_1, \Lambda_2)$ is given by diagonal G-action.

Thus, we see that there exist vectors of Λ_1 parallel to v. Let $w \in \mathbb{C}^*$ be one of them with shortest length. It follows that $w = rv$ for some $r \in \mathbb{R}$. Assume that r^2 is irrational. Then we have $H = U_v$ and so

$$(U_v\Lambda_1, \Lambda_2) = \overline{U_v^{\mathbb{Z}}.(\Lambda_1, \Lambda_2)} \subset G.(\Lambda_1, \Lambda_2).$$

It is again a contradiction. Therefore, we conclude that $r^2 \in \mathbb{Q}$. $\square$

Now assume that we have a surface $x'' \in \mathcal{H}(2)$ with an algebraic sum $x'' = \Lambda_1'' + \Lambda_2''$ that satisfies

$$\overline{G.(\Lambda_1'', \Lambda_2'')} = G.(\Lambda_1'', \Lambda_2'').$$

Now let $v^\star \in \Lambda_2''$ be a primitive vector, i.e.

$$[0, v^\star] \cap \Lambda_2'' = \{0, v^\star\}.$$

Denote $h := \sqrt{\frac{\mathrm{Area}(\Lambda_1'')}{\mathrm{Area}(\Lambda_2'')}}$. By Lemma 7.2, we have $r = hr' > 0$ with $r'^2 \in \mathbb{Q}$ so that

$$[0, rv^\star] \cap \Lambda_1'' = \{0, rv^\star\}.$$

Then we can write

$$\Lambda_1'' = \mathbb{Z}rv^\star \oplus \mathbb{Z}v, \quad \Lambda_2'' = \mathbb{Z}v^\star \oplus \mathbb{Z}w. \tag{7.2}$$

Now applying Lemma 7.2 to along v and w directions, we get that

$$tw = mrv^\star + nv, \quad sv = iv^\star + jw, \tag{7.3}$$

where $i, j, m, n \in \mathbb{Z}$, $t = ht'$, $s = h^{-1}s'$ with $t'^2, s'^2 \in \mathbb{Q}$. After counting the areas of Λ_1'' and Λ_2'', we obtain from (7.2) and (7.3) that

$$s' = jr', \quad n = r't'. \tag{7.4}$$

Now assume that the areas of tori satisfy

$$r = hr' \in \mathbb{Q}. \tag{7.5}$$

Then via (7.4), we get that

$$t, s \in \mathbb{Q}. \tag{7.6}$$

In what follows, we shall show that x'' generates a Teichmüller curve under the assumption (7.5). By replacing x'' with gx'' if necessary, we may further assume that v and $v^\star$ are orthogonal.

Next, let us recall that an *isogeny* between a pair of elliptic curves is a surjective holomorphic map $p : E_1 \to E_2$. A pair of 1-forms $(E_i, \omega_i) \in \Omega\mathcal{M}_1$ are said to be *isogenous* if there is a $\tau > 0$ and an isogenous $p : E_1 \to E_2$ such that $p^*(\omega_2) = \tau\omega_1$. This is equivalent to the condition that $\tau\Lambda_1 \subset \Lambda_2$, where $\Lambda_i \subset \mathbb{C}$ is the period lattice of (E_i, ω_i). In order to classify the $\mathrm{SL}_2(\mathbb{R})$-orbit closures of $\mathcal{H}_1(2)$, McMullen [McM07] showed the following:

Theorem 7.3. *Let $x \in \mathcal{H}(2)$ that can be presented, in more than one way, as an algebraic sum*

$$x \cong (E_1, \omega_1) + (E_2, \omega_2)$$

of isogenous forms of genus 1. *Then x generates a Teichmüller curve.*

Proof. This is just a restatement of [McM07, Theorem 5.10 & 6.1]. $\square$

Thus, we want to find two different algebraic sums of x'' to satisfy the requirements. Now by (7.3), for any $\tau > 0$, we have

$$\tau \cdot (rv^\star, v) = (v^\star, w) \begin{bmatrix} \tau r & \tau i/s \\ 0 & \tau j/s \end{bmatrix}.$$

Thanks to the fact that $r, s, i, j \in \mathbb{Q}$, we can choose certain $\tau > 0$ so that the 2×2 matrix on the right-hand side becomes an integer matrix. Then we conclude that $\tau\Lambda_1'' \subset \Lambda_2''$. Thus, the algebraic sum

$$x \cong (\mathbb{C}/\Lambda_1'', dz) + (\mathbb{C}/\Lambda_2'', dz)$$

is a pair of isogenous forms that meets our needs.

To find a different splitting, we consider a factor mix:

Lemma 7.4. *Let the notation and assumptions be as above. Then*

$$T_1 = \mathbb{Z}(rv^\star + w) \oplus \mathbb{Z}v, \quad T_2 = \mathbb{Z}(v^\star + v) \oplus \mathbb{Z}w$$

also define an algebraic sum $x'' = T_1 + T_2$.

Proof. Let $\{\alpha_1, \beta_1, \alpha_2, \beta_2\}$ span $\mathbb{Z}^4$. Then the map

$$(\alpha_1, \beta_1, \alpha_2, \beta_2) \mapsto (\alpha_1 + \beta_2, \beta_1, \alpha_2 + \beta_1, \beta_2)$$

is an element in $\mathrm{Sp}(4, \mathbb{Z})$ (in fact, it is a Burkhardt generator of $\mathrm{Sp}(4, \mathbb{Z})$). Thus, the new algebraic sum is just a shift of the old one by a mapping class. $\square$

Proof of Proposition 7.1. We want that T_1 and T_2 are isogenous. First, one calculates

$$\begin{aligned} &\tau \cdot (rv^\star + w, v) \\ =&(v^\star + v, w) \begin{bmatrix} \tau rs/(s+i) & \tau mr/(mr-n) \\ \tau(1 - rj/(s+i)) & \tau t/(n - mr) \end{bmatrix} \end{aligned}$$

for any $\tau > 0$. (Note that since $w \notin \mathbb{Z}(v^\star + v)$, for otherwise T_2 is not well-defined, we know from (7.3) that the denominators in the entries of the 2×2 matrix are nonzero.) Again, thanks to the fact that $r, s, t, i, j, m, n \in \mathbb{Q}$, we can choose certain $\tau > 0$ so that the 2×2 matrix on the right-hand side becomes an integer matrix. Thus, $\tau T_1 \subset T_2$ and the algebraic sum

$$x'' \cong (\mathbb{C}/T_1, dz) + (\mathbb{C}/T_2, dz)$$

is a pair of isogenous forms that meets our needs. Then, by Theorem 7.3, we conclude that x'' generates a Teichmüller curve. In other words, $\mathrm{SL}(x'')$ is a lattice and

$$G.x'' \cong G/\,\mathrm{SL}(x'').$$

Finally, since $r'^2 \in \mathbb{Q}$, one can write

$$r' = \frac{p}{q}\sqrt{k}$$

for some $p, q, k \in \mathbb{N}$ with k square-free. Thus, (7.5) reduces to (7.1). Therefore, we finish the proof of Proposition 7.1. $\square$

7.2. **Effective estimates of Teichmüller curves.** Now let $x'' \in \mathcal{H}(2)$ with an algebraic sum $x'' = \Lambda_1'' + \Lambda_2''$ that satisfies

$$\overline{G.(\Lambda_1'', \Lambda_2'')} = G.(\Lambda_1'', \Lambda_2'') \quad \text{and} \quad \sqrt{\frac{\operatorname{Area}(\Lambda_1'')}{\operatorname{Area}(\Lambda_2'')}} \cdot \sqrt{k} \in \mathbb{Q},$$

where k is the square-free integer provided by Proposition 7.1. It follows that x'' generates a Teichmüller curve. In other words, ω is an eigenform for real multiplication by $\mathcal{O}_D$ (cf. [McM07, Theorem 5.10 & 4.1]). Thus, by Theorem 3.4, it indicates that $(\mathbb{C}/\Lambda_1'' \times \mathbb{C}/\Lambda_2'', (dz)_1 + (dz)_2) \in \Omega Q_D$ is an eigenform with discriminant D. Now suppose that $(\mathbb{C}/\Lambda_1'' \times \mathbb{C}/\Lambda_2'', (dz)_1 + (dz)_2) \in \Omega Q_D$ is equivalent to the prototypical example of type (e, ℓ, m). A direct calculation shows the following:

Lemma 7.5. *Let the notation and assumptions be as in Proposition 7.1. Write*

$$\sqrt{\frac{\operatorname{Area}(\Lambda_1'')}{\operatorname{Area}(\Lambda_2'')}} = \frac{p}{q}\sqrt{k}.$$

Then prototype (e, ℓ, m) that attaches to x'' satisfies either

$$e = 0, \quad \ell = 1, \quad D = m/4$$

or D is a square and

$$k | m, \quad \ell | qp, \quad e^2 | m(q^2 - p^2 k)^2. \tag{7.7}$$

Proof. First, by (3.8), we get that

$$\frac{p}{q}\sqrt{k} = \sqrt{\frac{\operatorname{Area}(\Lambda_1)}{\operatorname{Area}(\Lambda_2)}} = \sqrt{\frac{\ell^2 m}{\lambda^2}} = \sqrt{\frac{D - e^2}{(\sqrt{D} + e)^2}} = \sqrt{\frac{\sqrt{D} - e}{\sqrt{D} + e}}.$$

If $\frac{p}{q}\sqrt{k} = 1$, then $e = 0$ and so $\ell = 1$ and $D = m/4$. If $\frac{p}{q}\sqrt{k} \neq 1$, one then solves from the above equation that

$$D = e^2 \frac{(q^2 + p^2 k)^2}{(q^2 - p^2 k)^2}.$$

In particular, D is a square. Because $D = e^2 + 4\ell^2 m$, we obtain

$$\ell^2 m \cdot (q^2 - p^2 k)^2 = e^2 \cdot q^2 p^2 k.$$

In particular, $\ell^2 | e^2 q^2 p^2$, $e^2 | \ell^2 m (q^2 - p^2 k)^2$ and

$$k | m$$

as k is square-free. On the other hand, since by (3.7), $\gcd(e, \ell) = 1$, we get that

$$\ell | qp \quad \text{and} \quad e^2 | m(q^2 - p^2 k)^2.$$

The consequence follows. $\square$

Given $x'' \in \mathcal{H}_1(2)$ with an algebraic sum $x'' = \Lambda_1'' + \Lambda_2''$. Then for $\epsilon > 0$, we change the area of x'' slightly and consider the surfaces with algebraic sums:

$$x''(\epsilon) \coloneqq (1+\epsilon)^{\frac{1}{2}}\Lambda_1'' + (1+\epsilon)^{-\frac{1}{2}}\Lambda_2'' \tag{7.8}$$

(cf. (3.14)). Note that $x''(\epsilon)$ is well defined if ϵ is sufficiently small by considering the period coordinates.

Lemma 7.6. *Let $T > 0$. Let $x'' \in \mathcal{H}_1(2)$ with an algebraic sum $x'' = \Lambda_1'' + \Lambda_2''$. Suppose that*

$$\mathrm{Area}(\Lambda_1'') = \mathrm{Area}(\Lambda_2''), \quad \overline{G.(\Lambda_1'', \Lambda_2'')} = G.(\Lambda_1'', \Lambda_2''), \quad \mathrm{vol}(G.(\Lambda_1'', \Lambda_2'')) < T. \tag{7.9}$$

Then for any $\eta_0 > 0$, there exists $\epsilon \in [0, \eta_0]$, and $C_{13}, \kappa_{14}, \kappa_{15} > 0$ such that if the surface $x''(\epsilon) \in \mathcal{H}_1(2)$ as in (7.9) is well defined, then $x''(\epsilon)$ generates a Teichmüller curve with discriminant D, and that

- *either $D < C_{13}T^{\kappa_{15}}$,*
- *or D is a square and $D < C_{13}\eta_0^{-\kappa_{14}}T^{\kappa_{15}}$.*

Proof. The idea is to change $\mathrm{Area}(\Lambda_1)/\mathrm{Area}(\Lambda_2)$ slightly to satisfy the rationality.

Let $x'' = \Lambda_1'' + \Lambda_2'' \in \mathcal{H}_1(2)$ satisfy (7.9). Then by Proposition 7.1, there exists a square-free integer $k \in \mathbb{N}$ corresponding to $(\Lambda_1'', \Lambda_2'') \in G/\Gamma \times G/\Gamma$.

Next, consider $x''(\epsilon)$ as in (7.8). Choose $0 < \epsilon < \eta_0$, $p, q \in \mathbb{N}$ so that

$$\sqrt{\frac{\mathrm{Area}((1+\epsilon)^{\frac{1}{2}}\Lambda_1'')}{\mathrm{Area}((1+\epsilon)^{-\frac{1}{2}}\Lambda_2'')}} = \sqrt{\frac{(1+\epsilon)^{\frac{1}{2}}}{(1+\epsilon)^{-\frac{1}{2}}}} = 1+\epsilon = \frac{p}{q}\sqrt{k}.$$

We want to pick a small q. Thus, by considering the rational numbers in $[\frac{1}{\sqrt{k}}, \frac{1+\eta_0}{\sqrt{k}}]$, we can pick

$$\frac{1}{q} \geq \frac{1}{2}\cdot\left(\frac{1+\eta_0}{\sqrt{k}} - \frac{1}{\sqrt{k}}\right) = \frac{\eta_0}{2\sqrt{k}}$$

and so,

$$q < 2\sqrt{k}\eta_0^{-1}. \tag{7.10}$$

With the choice of q, one can estimate

$$p \leq (1+\epsilon)\frac{q}{\sqrt{k}} < 4\eta_0^{-1}. \tag{7.11}$$

Under the assumption, we conclude from Proposition 7.1 that $x''(\epsilon)$ generates a Teichmüller curve, with some discriminant $D > 0$. Then by Theorem 3.5, we have

$$G.(\Lambda_1''(\delta), \Lambda_2''(\delta)) = \Omega Q_D(e, \ell, m)/\mathbb{R}^+ \cong G/\Gamma_0(m)$$

for some e, ℓ, m. Then the volume control (7.9) indicates that there exist $c_1, c_2 > 0$ such that

$$m < c_1 T^{c_2}. \tag{7.12}$$

If D is not a square, by Lemma 7.5, $D = m/4 \ll T^{c_2}$. Suppose D is a square. Then by (7.7), we immediately obtain $k < c_1 T^{c_2}$. Moreover, combining (7.7) (7.10) (7.11) (7.12), we can control e and ℓ by a polynomial of η_0^{-1} and T. Thus, there exists $C_{13}, \kappa_{14}, \kappa_{15} > 0$ such that

$$D = e^2 + 4\ell^2 m < C_{13}\eta_0^{-\kappa_{14}} T^{\kappa_{15}}$$

as what we needed. □

Now we are in the position to prove Theorem 1.5.

Proof of Theorem 1.5. Suppose that Theorem 1.4(2) holds. Then by the proof of Theorem 1.4. There exist $x', x'' \in \mathcal{H}_1(2)$ such that

- $d(x, x') < e^{-100t}$, $d(x', x'') < e^{-\frac{1}{2}t}$, $\ell(x') \geq \frac{1}{2}\ell(x)$,
- $x' = \Lambda_1' + \Lambda_2'$ belongs to a Teichmüller curve,
- $x'' = \Lambda_1'' + \Lambda_2''$ satisfies $\mathrm{Area}(\Lambda_1'') = \mathrm{Area}(\Lambda_2'')$, and $G.(\Lambda_1'', \Lambda_2'')$ is periodic with $\mathrm{vol}(G.(\Lambda_1'', \Lambda_2'')) \leq e^{\delta t}$.

Note in particular that $\ell(\Lambda_1', \Lambda_2') > \frac{1}{2}\ell(x)$, and that by considering the volumes, there are at most $O(\ell(x)^{-3})$ different surfaces y can be presented as $\Lambda_1' + \Lambda_2'$. One may deduce that x' has a splitting $x' = \Lambda_1' \underset{I}{\#} \Lambda_2'$ so that the lengths of the sides of parallelograms are controlled by some power of $\ell(x)$. Then by Lemma 3.9 (and Remark 3.11), we conclude that there exists $\kappa_{16} > 0$ such that

$$\|x'(\epsilon) - x'\|_x \leq \ell(x)^{-\kappa_{16}}|\epsilon|$$

for any $\epsilon \in [0, \frac{1}{2}\ell(x)]$. Now since $x_1'' = gx_1'$ for some $g \in B_G(e^{-\frac{1}{2}t})$, by Lemma 2.10, we have

$$\|x''(\epsilon) - x''\|_x \leq \ell(x)^{-\kappa_{16}}|\epsilon| \tag{7.13}$$

for any $\epsilon \in [0, \frac{1}{2}\ell(x)]$. In particular, (7.13) implies that $x''(\epsilon)$ is well defined for $\epsilon \in [0, \frac{1}{2}\ell(x)^{\kappa_{16}+\kappa_7}]$.

Finally, we apply Lemma 7.6, setting x'', T, η_0 in the statement of the lemma equal to x'', $e^{\delta t}$, $\ell(x)^{\kappa_{16}}e^{-t}$ respectively. Then $x''(\epsilon)$ generates a Teichmüller curve with discriminant D, and that

- either $D < C_{13}e^{\kappa_{15}\delta t} \leq e^{2\kappa_{15}\delta t}$,
- or D is a square and $D < C_{13}\ell(x)^{-\kappa_{14}\kappa_{16}}e^{\kappa_{14}t}e^{\kappa_{15}\delta t} \leq e^{2(\kappa_{14}+\kappa_{15})t}$.

(Here we use e^t for sufficiently large t to absorb the constants.) Theorem 1.5 now follows from letting $x''' = x''(\epsilon)$, and $\kappa_2 = 2(\kappa_{14} + \kappa_{15})$. □

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