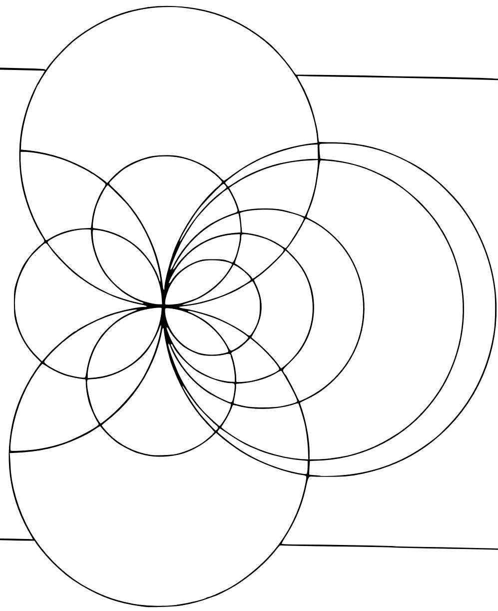

POLARIZATION IN ANTENNAS AND RADAR

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2

REPRESENTATION OF WAVE POLARIZATION

2.1. INTRODUCTION

The electric vector of a harmonic plane wave traces an ellipse in the transverse plane with time, as is well known. In this chapter we develop the equation of the ellipse for a general, nonplane wave and consider the ellipse and the behavior of the field vectors in detail for a plane wave. The parameters commonly used to describe wave polarization, namely, the linear and circular polarization ratios, the ellipse axial ratio, tilt angle, rotation sense, and the Stokes parameters, are introduced and related to each other. A polarization chart based on the familiar Smith chart of transmission line theory, first discussed by Rumsey, is used, and contours for some common polarization parameters are shown on the chart. The Poincaré sphere is utilized, and mapping from the sphere onto several complex planes is described. This process, which results in standard and nonstandard polarization charts, is illustrated.

2.2. THE GENERAL HARMONIC WAVE

In this section we will show that a general (nonplanar) harmonic wave is elliptically polarized and find the equation of the polarization ellipse [1]. A nonplanar single-frequency wave with components

$$\mathscr{E}_i(\mathbf{r}, t) = a_i(\mathbf{r}) \cos [\omega t - g_i(\mathbf{r})] \qquad i = 1, 2, 3 \tag{2.1}$$

where a_i and g_i are real, may be written as

$$\begin{aligned}\mathscr{E}(\mathbf{r}, t) &= \sum_1^3 \mathbf{u}_i a_i(\mathbf{r}) \cos [\omega t - g_i(\mathbf{r})] \\ &= \sum_1^3 \mathbf{u}_i a_i(\mathbf{r}) \operatorname{Re} [e^{j[\omega t - g_i(\mathbf{r})]}]\end{aligned} \tag{2.2}$$

and if we let

$$E_i(\mathbf{r}) = a_i(\mathbf{r}) e^{-jg_i(\mathbf{r})} \tag{2.3}$$

be the complex, time-invariant term associated with each real, time-varying electric field component, then

$$\mathscr{E}(\mathbf{r}, t) = \operatorname{Re}\left[\sum_1^3 \mathbf{u}_i E_i(\mathbf{r}) e^{j\omega t}\right] \tag{2.4}$$

where the $\mathbf{u}_i$ are real orthogonal unit vectors. For a plane wave the phase term is given by

$$g_i(\mathbf{r}) = \mathbf{k} \cdot \mathbf{r} - \delta_i \tag{2.5}$$

but at this point we will not restrict ourselves to plane waves.

If we define the complex vector

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}'(\mathbf{r}) + j\mathbf{E}''(\mathbf{r}) = \sum_1^3 \mathbf{u}_i E_i(\mathbf{r}) \tag{2.6}$$

then the harmonic vector field may be written as

$$\mathscr{E}(\mathbf{r}, t) = \operatorname{Re} [\mathbf{E}(\mathbf{r}) e^{j\omega t}] \tag{2.7}$$

Let us assume that $\mathbf{E}$ may be transformed to a new set of axes defined by the orthogonal real vectors $\mathbf{m}$ and $\mathbf{n}$, using the relation

$$\mathbf{E} = \mathbf{E}' + j\mathbf{E}'' = (\mathbf{m} + j\mathbf{n}) e^{j\theta} \tag{2.8}$$

Equating real and imaginary parts of this equation yields

$$\begin{aligned}\mathbf{E}' &= \mathbf{m} \cos \theta - \mathbf{n} \sin \theta \\ \mathbf{E}'' &= \mathbf{m} \sin \theta + \mathbf{n} \cos \theta\end{aligned} \tag{2.9}$$

and solving for $\mathbf{m}$ and $\mathbf{n}$ leads to

$$\begin{aligned}\mathbf{m} &= \mathbf{E}' \cos \theta + \mathbf{E}'' \sin \theta \\ \mathbf{n} &= -\mathbf{E}' \sin \theta + \mathbf{E}'' \cos \theta\end{aligned} \tag{2.10}$$

If we assume, without loss of generality, that $|\mathbf{m}| \geqq |\mathbf{n}|$, and require the orthogonality condition $\mathbf{m} \cdot \mathbf{n} = 0$, we find from (2.10) that

$$\tan 2\theta = \frac{2\mathbf{E}' \cdot \mathbf{E}''}{|\mathbf{E}'|^2 - |\mathbf{E}''|^2} \tag{2.11}$$

Since $\tan 2\theta$ as given by (2.11) is real, our assumed transformation may be carried out.

Next we substitute (2.8) into (2.7) to find the real field components. We obtain

$$\mathscr{E} = \mathrm{Re}\,[\mathbf{E}e^{j\omega t}] = \mathrm{Re}\,[(\mathbf{m} + j\mathbf{n})e^{j\theta}e^{j\omega t}] \tag{2.12}$$

and since $\mathbf{m}$ and $\mathbf{n}$ are real,

$$\mathscr{E} = \mathbf{m}\cos(\omega t + \theta) - \mathbf{n}\sin(\omega t + \theta) \tag{2.13}$$

If at each field point we now set up a local coordinate system with two of the axes directed along $\mathbf{m}$ and $\mathbf{n}$, the field components are

$$\begin{aligned} \mathscr{E}_m &= m\cos(\omega t + \theta) && \text{(a)} \\ \mathscr{E}_n &= -n\sin(\omega t + \theta) && \text{(b)} \\ \mathscr{E}_3 &= 0 && \text{(c)} \end{aligned} \tag{2.14}$$

where $m = |\mathbf{m}|$, $n = |\mathbf{n}|$. In (2.14) subscript 3 refers to the third of the three coordinates.

From (2.14) we see that

$$\frac{\mathscr{E}_m^2}{m^2} + \frac{\mathscr{E}_n^2}{n^2} = \cos^2(\omega t + \theta) + \sin^2(\omega t + \theta) = 1 \tag{2.15}$$

This is the equation of an ellipse, in the plane defined by $\mathbf{m}$ and $\mathbf{n}$, with semimajor and semiminor axes m and n. The field intensity ellipse is shown in Fig. 2.1. The field vector $\mathscr{E}$ terminates on the ellipse, since its components $\mathscr{E}_m$ and $\mathscr{E}_n$ are not independent but obey the ellipse equation, (2.15). The direction of $\mathscr{E}$ changes with time as its tip moves around the ellipse with a direction and velocity we will determine in a later section.

We see then that any harmonic wave, planar or nonplanar, is elliptically polarized. The plane of the ellipse and its shape and orientation in that plane are functions of the coordinates of the field point, but not of time.

This development has been concerned with the time-varying electric field, but it is clear that the magnetic field is also elliptically polarized. See problem 2.7 at the end of this chapter.

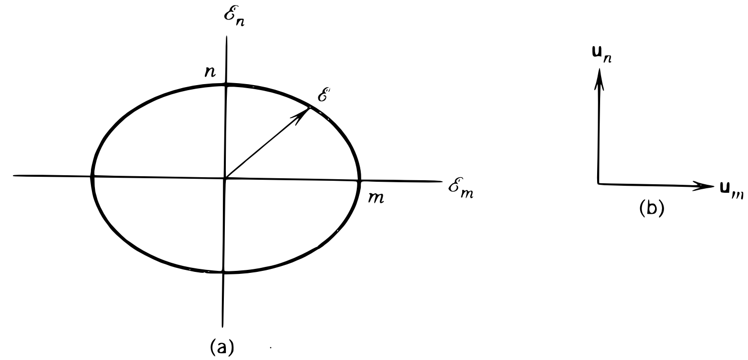


FIGURE 2.1. The polarization ellipse: (a) field intensity coordinates; (b) space coordinates.

2.3. POLARIZATION ELLIPSE FOR PLANE WAVES

A plane wave traveling in the z direction

$$\mathbf{E} = (E_x\mathbf{u}_x + E_y\mathbf{u}_y)e^{-jkz} \tag{2.16}$$

results if we use the phase term (2.5) and let the propagation constant be

$$\mathbf{k} = \mathbf{u}_z k \tag{2.17}$$

In (2.16) both E_x and E_y are complex and may be written as

$$E_x = |E_x|e^{j\phi_x} \qquad E_y = |E_y|e^{j\phi_y} \tag{2.18}$$

so that

$$\mathbf{E} = (\mathbf{u}_x|E_x|e^{j\phi_x} + \mathbf{u}_y|E_y|e^{j\phi_y})e^{-jkz} \tag{2.19}$$

and the time-varying field is

$$\mathscr{E} = \mathrm{Re}\,(\mathbf{E}e^{j\omega t}) = \mathbf{u}_x|E_x|\cos(\omega t - kz + \phi_x) + \mathbf{u}_y|E_y|\cos(\omega t - kz + \phi_y) \tag{2.20}$$

and if we set

$$\beta = \omega t - kz \tag{2.21}$$

the components of $\mathscr{E}$ become

$$\frac{\mathscr{E}_x}{|E_x|} = \cos\beta\cos\phi_x - \sin\beta\sin\phi_x \quad \text{(a)}$$
$$\frac{\mathscr{E}_y}{|E_y|} = \cos\beta\cos\phi_y - \sin\beta\sin\phi_y \quad \text{(b)} \qquad (2.22)$$

Multiplying and subtracting as indicated leads to

$$\frac{\mathscr{E}_x}{|E_x|}\sin\phi_y - \frac{\mathscr{E}_y}{|E_y|}\sin\phi_x = \cos\beta\sin(\phi_y - \phi_x) \quad \text{(a)}$$
$$\frac{\mathscr{E}_x}{|E_x|}\cos\phi_y - \frac{\mathscr{E}_y}{|E_y|}\cos\phi_x = \sin\beta\sin(\phi_y - \phi_x) \quad \text{(b)} \qquad (2.23)$$

Squaring and adding (2.23a) and (2.23b) gives

$$\frac{\mathscr{E}_x^2}{|E_x|^2} - 2\frac{\mathscr{E}_x}{|E_x|}\frac{\mathscr{E}_y}{|E_y|}\cos(\phi_y - \phi_x) + \frac{\mathscr{E}_y^2}{|E_y|^2} = \sin^2(\phi_y - \phi_x) \qquad (2.24)$$

This is the equation of a conic, and we have already seen in a more general case that it represents an ellipse. In (2.24) we set

$$\phi = \phi_y - \phi_x \qquad (2.25)$$

and the equation becomes

$$\frac{\mathscr{E}_x^2}{|E_x|^2} - 2\frac{\mathscr{E}_x}{|E_x|}\frac{\mathscr{E}_y}{|E_y|}\cos\phi + \frac{\mathscr{E}_y^2}{|E_y|^2} = \sin^2\phi \qquad (2.26)$$

We may see from (2.22), which can be rewritten as

$$\frac{\mathscr{E}_x}{|E_x|} = \cos(\beta + \phi_x) \quad \text{(a)} \qquad \frac{\mathscr{E}_y}{|E_y|} = \cos(\beta + \phi_y) \quad \text{(b)} \qquad (2.27)$$

that the greatest values of $\mathscr{E}_x$ and $\mathscr{E}_y$ are, respectively, $|E_x|$ and $|E_y|$. Then the ellipse of (2.26) can be inscribed in a rectangle with sides parallel to the x and y axes and dimensions $2|E_x|$ and $2|E_y|$ as shown in Fig. 2.2.

From (2.27) we see that for $\mathscr{E}_x$ to be maximum

$$\beta + \phi_x = 0$$

$$\beta + \phi_y = \beta + \phi_x + (\phi_y - \phi_x) = \phi$$

and for $\mathscr{E}_y$ maximum

$$\beta + \phi_y = 0 \qquad \beta + \phi_x = -\phi$$

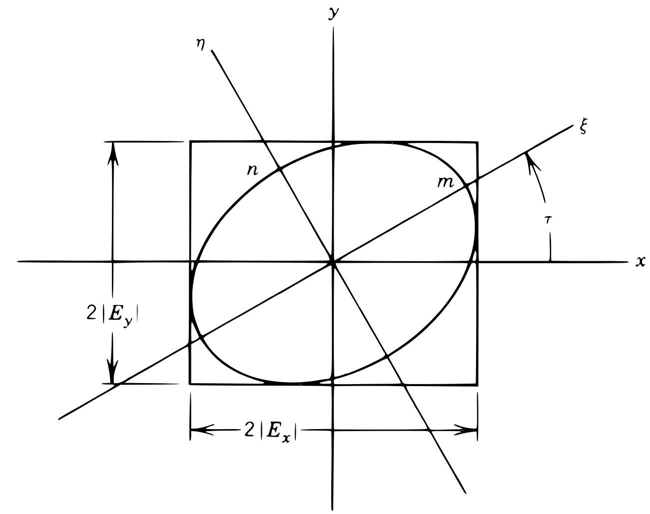


FIGURE 2.2. Tilted polarization ellipse.

and we find that the ellipse of Fig. 2.2 intersects the sides of the rectangle at $\pm|E_x|, \pm|E_y|\cos\phi$ and $\pm|E_x|\cos\phi, \pm|E_y|$.

The angle τ of Fig. 2.2, measured from the x axis, is called the *tilt angle* of the polarization ellipse. We define it between the limits

$$0 \leqq \tau \leqq \pi \qquad (2.28)$$

Let us find τ. From Fig. 2.3 we easily see that

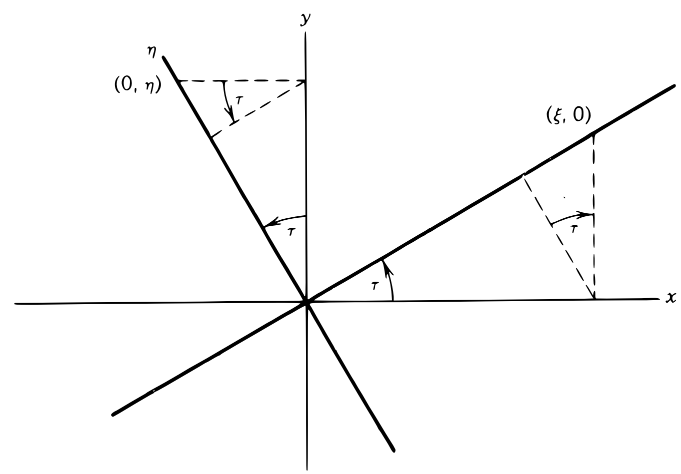


FIGURE 2.3. Coordinate transformations.

$$\begin{aligned} \xi &= x\cos\tau + y\sin\tau \quad &\text{(a)} \\ \eta &= y\cos\tau - x\sin\tau \quad &\text{(b)} \end{aligned} \tag{2.29}$$

and the field components transform as

$$\begin{aligned} \mathscr{E}_\xi &= \mathscr{E}_x\cos\tau + \mathscr{E}_y\sin\tau \quad &\text{(a)} \\ \mathscr{E}_\eta &= -\mathscr{E}_x\sin\tau + \mathscr{E}_y\cos\tau \quad &\text{(b)} \end{aligned} \tag{2.30}$$

Now the components $\mathscr{E}_\xi$ and $\mathscr{E}_\eta$ are also given by

$$\begin{aligned} \mathscr{E}_\xi &= m\cos(\beta + \phi_0) \quad &\text{(a)} \\ \mathscr{E}_\eta &= \pm n\sin(\beta + \phi_0) \quad &\text{(b)} \end{aligned} \tag{2.31}$$

where m and n are the positive semiaxes of Fig. 2.2, and ϕ_0 is some phase angle. That (2.31) is correct is easily seen by noting that it satisfies

$$\frac{\mathscr{E}_\xi^2}{m^2} + \frac{\mathscr{E}_\eta^2}{n^2} = 1$$

In (2.31) $\mathscr{E}_\eta$ carries the $\pm$ sign since we have not yet determined the rotation sense of $\mathscr{E}$. If we consider $\beta + \phi_0 = 0$, $\mathscr{E}_\xi = m$, and $\mathscr{E}_\eta = 0$ in (2.31), and then allow $\beta(=\omega t - kz)$ to increase infinitesimally, we see that the + sign corresponds to counterclockwise rotation of $\mathscr{E}$ (as we look at Fig. 2.2) as β (or time) increases, and the − sign to clockwise rotation.

We equate (2.31) to (2.30).

$$\begin{aligned} \mathscr{E}_\xi &= m\cos(\beta + \phi_0) = \mathscr{E}_x\cos\tau + \mathscr{E}_y\sin\tau \quad &\text{(a)} \\ \mathscr{E}_\eta &= \pm n\sin(\beta + \phi_0) = -\mathscr{E}_x\sin\tau + \mathscr{E}_y\cos\tau \quad &\text{(b)} \end{aligned} \tag{2.32}$$

Expanding the left side and using on the right the wave components as given by (2.22), we get

$$\begin{aligned} & m(\cos\beta\cos\phi_0 - \sin\beta\sin\phi_0) \\ &\quad = |E_x|(\cos\beta\cos\phi_x - \sin\beta\sin\phi_x)\cos\tau \\ &\qquad + |E_y|(\cos\beta\cos\phi_y - \sin\beta\sin\phi_y)\sin\tau \quad \text{(a)} \\ & \pm n(\sin\beta\cos\phi_0 + \cos\beta\sin\phi_0) \\ &\quad = -|E_x|(\cos\beta\cos\phi_x - \sin\beta\sin\phi_x)\sin\tau \\ &\qquad + |E_y|(\cos\beta\cos\phi_y - \sin\beta\sin\phi_y)\cos\tau \quad \text{(b)} \end{aligned} \tag{2.33}$$

Equating the coefficients of cos β and sin β in (2.33) leads to

$$\begin{aligned} m\cos\phi_0 &= |E_x|\cos\phi_x\cos\tau + |E_y|\cos\phi_y\sin\tau \quad &\text{(a)} \\ m\sin\phi_0 &= |E_x|\sin\phi_x\cos\tau + |E_y|\sin\phi_y\sin\tau \quad &\text{(b)} \\ \pm n\cos\phi_0 &= |E_x|\sin\phi_x\sin\tau - |E_y|\sin\phi_y\cos\tau \quad &\text{(c)} \\ \pm n\sin\phi_0 &= -|E_x|\cos\phi_x\sin\tau + |E_y|\cos\phi_y\cos\tau \quad &\text{(d)} \end{aligned} \tag{2.34}$$

Squaring and adding the four equations of (2.34) results in

$$m^2 + n^2 = |E_x|^2 + |E_y|^2 \tag{2.35}$$

Next we multiply the first and third equations of (2.34) and also the second and fourth, and add the products, obtaining

$$\pm mn = -|E_x|\,|E_y|\sin\phi \tag{2.36}$$

Dividing the third equation of (2.34) by the first, and the fourth by the second gives

$$\begin{aligned} \pm\frac{n}{m} &= \frac{|E_x|\sin\phi_x\sin\tau - |E_y|\sin\phi_y\cos\tau}{|E_x|\cos\phi_x\cos\tau + |E_y|\cos\phi_y\sin\tau} \\ &= \frac{-|E_x|\cos\phi_x\sin\tau + |E_y|\cos\phi_y\cos\tau}{|E_x|\sin\phi_x\cos\tau + |E_y|\sin\phi_y\sin\tau} \end{aligned} \tag{2.37}$$

Cross multiplying and collecting terms in (2.37) gives

$$(|E_x|^2 - |E_y|^2)\sin 2\tau = 2|E_x|\,|E_y|\cos 2\tau\cos\phi \tag{2.38}$$

If we define the auxiliary angle α by

$$\tan\alpha = \frac{|E_y|}{|E_x|} \qquad 0 \leqq \alpha \leqq \frac{\pi}{2} \tag{2.39}$$

then (2.38) becomes

$$\tan 2\tau = \tan 2\alpha\cos\phi \tag{2.40}$$

We have thus obtained the ellipse tilt angle in terms of the field component magnitudes and phase difference.

In order to find the axial ratio of the ellipse and the rotation sense of the $\mathscr{E}$ vector let us define another auxiliary angle δ by

$$\tan \delta = \mp \frac{n}{m} \qquad -\frac{\pi}{4} \leqq \delta \leqq \frac{\pi}{4} \tag{2.41}$$

From (2.41) we may obtain

$$\sin 2\delta = \mp \frac{2mn}{m^2 + n^2} \tag{2.42}$$

and the use of (2.35) and (2.36) leads to

$$\sin 2\delta = \frac{2|E_x||E_y|}{|E_x|^2 + |E_y|^2} \sin \phi \tag{2.43}$$

which will give us the axial ratio, n/m, from the field component magnitudes and phase difference.

Let us next determine the rotation sense of $\mathscr{E}$. The time-varying angle of $\mathscr{E}$, measured from the x axis, is

$$\psi = \tan^{-1} \frac{\mathscr{E}_y}{\mathscr{E}_x} = \tan^{-1} \frac{|E_y| \cos(\beta + \phi_y)}{|E_x| \cos(\beta + \phi_x)} \tag{2.44}$$

where $\beta = \omega t - kz$. Then

$$\frac{\partial \psi}{\partial \beta} = \frac{(|E_y|/|E_x|)[-\cos(\beta + \phi_x)\sin(\beta + \phi_y) + \sin(\beta + \phi_x)\cos(\beta + \phi_y)]}{[1 + |E_y|^2 \cos^2(\beta + \phi_y)/|E_x|^2 \cos^2(\beta + \phi_x)] \cos^2(\beta + \phi_x)} \tag{2.45}$$

and at some particular β, say $\beta = 0$,

$$\frac{\partial \psi}{\partial \beta} = -\frac{|E_x||E_y| \sin \phi}{|E_x|^2 \cos^2 \phi_x + |E_y|^2 \cos^2 \phi_y} \tag{2.46}$$

Thus we see that

$$\frac{\partial \psi}{\partial \beta} < 0, \quad 0 < \phi < \pi$$
$$> 0, \quad \pi < \phi < 2\pi \tag{2.47}$$

If we look in the direction of wave propagation, in this case the $+z$ direction, $\partial\psi/\partial\beta > 0$ corresponds to clockwise rotation of the $\mathscr{E}$ vector as β (or time) increases. By definition we call this right-handed rotation of the vector. Conversely, $\partial\psi/\partial\beta < 0$ corresponds to counterclockwise or left-handed rotation. We may see from (2.43) and (2.47) that

$$\sin 2\delta < 0, \quad \text{right-handed rotation}$$
$$> 0, \quad \text{left-handed rotation} \tag{2.48}$$

From (2.39) we can get

$$\sin 2\alpha = \frac{2|E_x||E_y|}{|E_x|^2 + |E_y|^2} \tag{2.49}$$

and, if this is used in (2.43), we get a simpler equation

$$\sin 2\delta = \sin 2\alpha \sin \phi \tag{2.50}$$

To summarize, from a knowledge of the field component amplitudes $|E_x|$ and $|E_y|$ and their phase difference $\phi = \phi_y - \phi_x$, we first find the auxiliary angle α from (2.39). Angle δ is next found from (2.50). The tilt angle of the polarization ellipse is then determined from (2.40) and the axial ratio and rotation sense from (2.41), where positive δ corresponds to right-handed rotation.

2.4. LINEAR AND CIRCULAR POLARIZATION

In the special cases of $|E_x| = 0$, or $|E_y| = 0$, or $\phi = 0$, the polarization ellipse degenerates to a straight line, and the wave is said to be linearly polarized. The axial ratio will of course be zero, and (2.39) and (2.40) may still be used to obtain the tilt angle.

If $|E_x| = |E_y|$ and $\phi = \pm \frac{1}{2}\pi$, the axial ratio as given by (2.41) becomes equal to one, the polarization ellipse degenerates to a circle, and the wave is said to be circularly polarized—right circular if $\phi = -\frac{1}{2}\pi$, and left circular if $\phi = +\frac{1}{2}\pi$.

2.5. POWER DENSITY

From

$$\mathbf{E} = (E_x \mathbf{u}_x + E_y \mathbf{u}_y) e^{-jkz} \tag{2.16}$$

and the Maxwell equations, we can find the magnetic field

$$\mathbf{H} = \frac{1}{Z_0} (-E_y \mathbf{u}_x + E_x \mathbf{u}_y) e^{-jkz} \tag{2.51}$$

where Z_0 is the characteristic impedance of the medium defined by

$$Z_0 = \sqrt{\frac{\mu}{\varepsilon}}$$

The complex Poynting vector is then

$$\mathbf{S}_c = \tfrac{1}{2}\mathbf{E} \times \mathbf{H}^* = \frac{|E_x|^2 + |E_y|^2}{2Z_0^*}\,\mathbf{u}_z \tag{2.52}$$

with the time-average Poynting vector given by

$$\mathbf{S} = \mathrm{Re}\,[\mathbf{S}_c] \tag{2.53}$$

2.6. ROTATION RATE OF THE FIELD VECTOR

In the $z = 0$ plane, the field given by (2.20) reduces to

$$\mathscr{E} = \mathbf{u}_x|E_x|\cos(\omega t + \phi_x) + \mathbf{u}_y|E_y|\cos(\omega t + \phi_y) \tag{2.54}$$

Then the angle ψ between $\mathscr{E}$ and the positive x axis is given as a function of time by

$$\psi = \tan^{-1}\left[\frac{|E_y|\cos(\omega t + \phi_y)}{|E_x|\cos(\omega t + \phi_x)}\right] \tag{2.55}$$

and the rate of increase of ψ with time is

$$\frac{\partial\psi}{\partial t} = \frac{-\omega|E_x|\,|E_y|\sin\phi}{|E_x|^2\cos^2(\omega t + \phi_x) + |E_y|^2\cos^2(\omega t + \phi_y)} \tag{2.56}$$

where, as before,

$$\phi = \phi_y - \phi_x \tag{2.25}$$

We see that in general the rotation rate of the field vector is not constant. If we take the special case of circular polarization,

$$|E_x| = |E_y| \qquad \phi = \pm\frac{\pi}{2}$$

where the upper sign corresponds to left circular rotation and the lower to right circular, (2.56) reduces to

$$\frac{\partial\psi}{\partial t} = \mp\omega \tag{2.57}$$

Figure 2.4 indicates that $-\omega$ is consistent with left circular polarization.

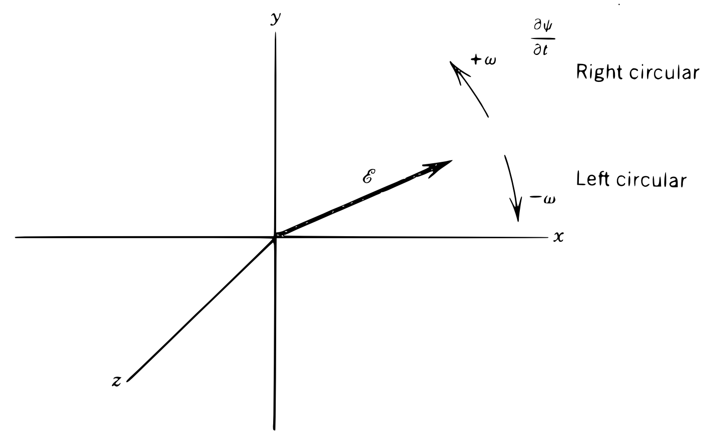


FIGURE 2.4. Rotation relationships for the polarization ellipse.

We can simplify (2.56) if we note that (2.54) gives, at $z = 0$,

$$|\mathscr{E}|^2 = \mathscr{E}\cdot\mathscr{E} = |E_x|^2\cos^2(\omega t + \phi_x) + |E_y|^2\cos^2(\omega t + \phi_y) \tag{2.58}$$

Using this, the rotation rate of the $\mathscr{E}$ vector becomes

$$\frac{\partial\psi}{\partial t} = \frac{-\omega|E_x|\,|E_y|\sin\phi}{|\mathscr{E}|^2} \tag{2.59}$$

On the major axis of the polarization ellipse, $|\mathscr{E}|$ is a maximum, given by m. Thus the rotation rate is a minimum, given by

$$\left.\frac{\partial\psi}{\partial t}\right|_{\min} = \frac{-\omega|E_x|\,|E_y|\sin\phi}{m^2} \tag{2.60}$$

On the minor axis, $|\mathscr{E}|$ is minimum $(=n)$, and therefore the maximum rotation rate occurs on the minor axis and is

$$\left.\frac{\partial\psi}{\partial t}\right|_{\max} = \frac{-\omega|E_x|\,|E_y|\sin\phi}{n^2} \tag{2.61}$$

2.7. AREA SWEEP RATE

The area swept by the $\mathscr{E}$ vector in a time dt as it moves through an angle $d\psi$ may be seen from Fig. 2.5 to be

$$dA = \tfrac{1}{2}|\mathscr{E}|^2\,d\psi \tag{2.62}$$

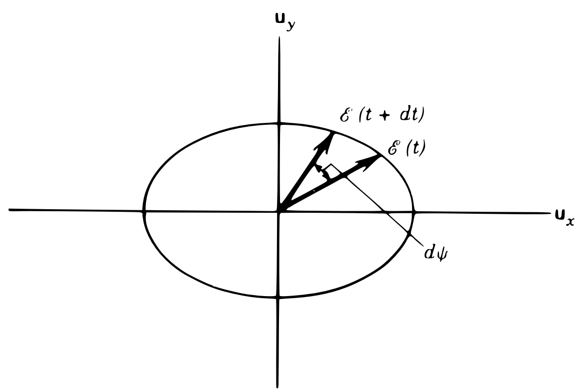


FIGURE 2.5. Area sweep of the ℰ vector.

Then the rate of increase of swept area is

$$\frac{\partial A}{\partial t} = \frac{1}{2}|\mathscr{E}|^2 \frac{d\psi}{\partial t} \tag{2.63}$$

and the use of (2.59) gives

$$\frac{\partial A}{\partial t} = -\frac{1}{2}\omega|E_x||E_y|\sin\phi \tag{2.64}$$

A negative value for the rate of area sweep is quite valid and indicates only that for right-handed rotation the rate of area sweep is positive, and for left-handed rotation it is negative.

Equation (2.64) shows that the rate of area sweep is not a function of time or of position of the tip of the electric field vector on the polarization ellipse. This may be considered a kind of Kepler's second law for electromagnetics. The laws are not precisely the same for electromagnetics and planetary motion, however, since the Kepler laws state that the planets move around the sun in ellipses with the sun at one focus, and the radius vector *from the sun* to a planet sweeps out equal areas in equal time intervals [2]. The electric field vector drawn from the ellipse origin, not a focus, sweeps out equal areas in equal intervals of time.

We note also that the $\mathscr{E}$ vector completes one rotation in the time

$$T = \frac{2\pi}{\omega} \tag{2.65}$$

2.8. ROTATION OF ℰ WITH DISTANCE

If we set $t = 0$ in (2.20), we can find the position angle of $\mathscr{E}$ as a function of distance z,

$$\psi = \tan^{-1}\frac{|E_y|\cos(-kz+\phi_y)}{|E_x|\cos(-kz+\phi_x)} \tag{2.66}$$

A comparison of (2.66) and (2.55) shows that we can find the rotation rate of $\mathscr{E}$ with distance at a fixed time if we replace ω in (2.59) by $-k$ and t by z. Then the rotation rate is

$$\frac{\partial\psi}{\partial z} = \frac{k|E_x||E_y|\sin\phi}{|E_x|^2\cos^2(-kz+\phi_x)+|E_y|^2\cos^2(-kz+\phi_y)} \tag{2.67}$$

and since the denominator is obviously $|\mathscr{E}|^2$ at $t = 0$,

$$\frac{\partial\psi}{\partial z} = \frac{k|E_x||E_y|\sin\phi}{|\mathscr{E}|^2} \tag{2.68}$$

This indicates that the rotation rate with z is minimum at the major axis of the polarization ellipse and maximum at the minor axis, just as it was with the time rotation rate.

If the rotation of $\mathscr{E}$ with increasing time in a fixed plane is clockwise, the fact that (2.59) and (2.68) have different signs shows that the rotation with increasing distance at a fixed time is counterclockwise. We may think of a right-handed circular wave at fixed time in space as looking like a *left-handed* screw. With increasing time the screw rotates in a clockwise direction as we look in the direction of wave motion. This is shown in Fig. 2.6.

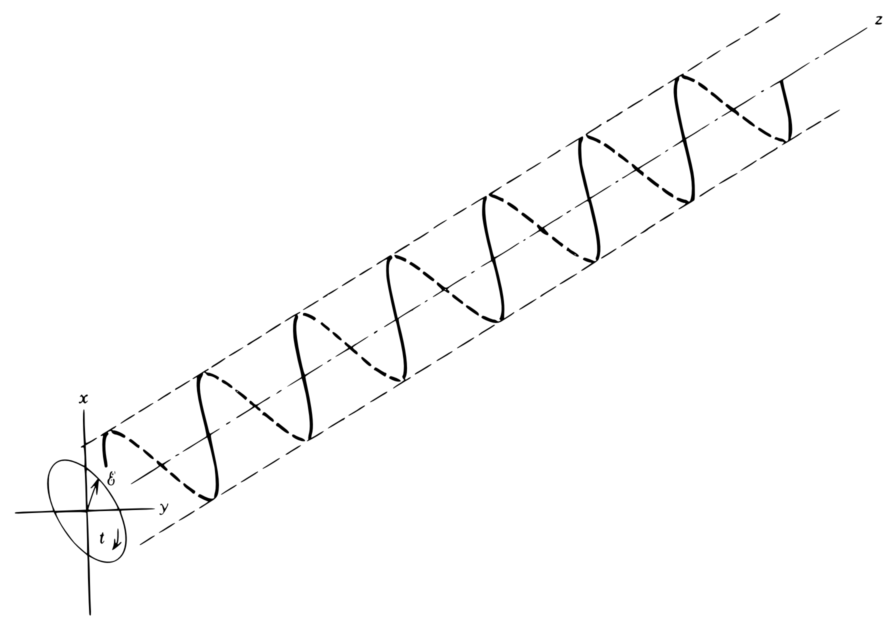


FIGURE 2.6. Rotation of ℰ with time and distance for a right-handed circular wave.

We may see from (2.66) that the distance between two points of the wave having parallel field vectors at constant time is

$$\Delta z = \frac{2\pi}{k} = \lambda \tag{2.69}$$

2.9. THE POLARIZATION RATIOS

A description of the elliptically polarized wave in terms of tilt angle, axial ratio, and rotation sense leads to a good physical understanding of the wave, but it is not convenient mathematically. In this and the following sections the wave will be characterized by more tractable mathematical terms.

The time-invariant **E** field of (2.16) may also be written as

$$\mathbf{E} = E_0(\mathbf{u}_x a + \mathbf{u}_y b e^{j\phi})e^{-jkz} \tag{2.70}$$

if we extract a common complex term E_0. For convenience we drop the distance phase term and write

$$\mathbf{E} = E_0(\mathbf{u}_x a + \mathbf{u}_y b e^{j\phi}) \tag{2.71}$$

Without loss of generality, we can choose E_0 and ϕ so that a and b are real and

$$a^2 + b^2 = 1 \tag{2.72}$$

Then E_0 has the same phase as E_x, and ϕ has the same meaning as in previous developments, the phase lead of E_y over E_x.

The value of E_0 does not affect the wave polarization in any way, and except in questions concerned with power, we will neglect it. We define a *polarization ratio P*, which alone carries all necessary polarization information, by

$$P = \frac{E_y}{E_x} = \frac{b}{a} e^{j\phi} \tag{2.73}$$

This is a commonly used definition, although we will see shortly that a slightly different definition is sometimes useful. Some special values of the polarization ratio are:

Wave	Characteristics	P
Linear vertical	$E_x = a = 0$	∞
Linear horizontal	$E_y = b = 0$	0
Right circular	$a = b,\ \phi = -\pi/2$	$-j$
Left circular	$a = b,\ \phi = +\pi/2$	$+j$

3

POLARIZATION MATCHING OF ANTENNAS

3.1. INTRODUCTION

It is obvious that when two antennas are used in a communication system, they should be matched in polarization so that the available power at the receiving antenna can be fully utilized. In this chapter a polarization match factor is developed and is given in terms of the standard polarization parameters. The relationship between the effective length of a receiving antenna and the field components of an incident wave necessary to yield maximum power is developed. In the final section a step-by-step process is outlined for obtaining the power received when two antennas are mismatched in polarization and do not have their main beam axes pointing at each other. It is interesting that this topic is not treated in most of the standard texts on antenna theory.

3.2. EFFECTIVE LENGTH OF AN ANTENNA

The electric field in the radiation zone of a dipole antenna, which is short compared to a free-space wavelength, as shown in Fig. 3.1, is given by

$$E_\theta(r, \theta, \phi) = \frac{jZ_0 I\ell}{2\lambda r} e^{-jkr} \sin\theta \tag{3.1}$$

where Z_0 is the intrinsic impedance of free space, k the free space propagation constant, λ the wavelength, and I the current into the antenna terminals of Fig. 3.1.

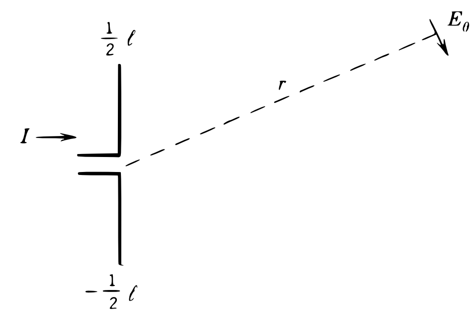


FIGURE 3.1. Short dipole antenna.

Equation (3.1) may be generalized to give the transmitted field of any antenna; thus [1]

$$\mathbf{E}^t(r, \theta, \phi) = \frac{jZ_0 I}{2\lambda r} e^{-jkr}\mathbf{h}(\theta, \phi) \tag{3.2}$$

where θ is the colatitude angle of Fig. 3.1 and ϕ is the azimuth angle. The current I is an input current at an arbitrary pair of terminals. Equation (3.2) describes a general antenna in terms of its *effective length* $\mathbf{h}(\theta, \phi)$. The effective length does not necessarily correspond to a physical length of the antenna, although there is a correspondence for the dipole. In fact, comparison of (3.1) and (3.2) shows that the effective length of the short dipole antenna is

$$\mathbf{h} = \mathbf{u}_\theta h_\theta = \mathbf{u}_\theta \ell \sin\theta \tag{3.3}$$

We see from this that $\mathbf{h}$ is not fixed for an antenna but depends on the angle θ (and more generally on ϕ) at which we measure the radiated field.

As mentioned, I is the current at an arbitrary pair of terminals, and it follows that the effective length $\mathbf{h}$ depends on the choice of terminal pair. Further, we note that if $\mathbf{E}^t$ is to describe an elliptically polarized field, it must be complex, and therefore $\mathbf{h}$ is a complex vector. With a proper choice of coordinate system, $\mathbf{E}^t$ and $\mathbf{h}$ will have only two components since in the radiation zone $\mathbf{E}^t$ has no radial component.

3.3. RECEIVED VOLTAGE

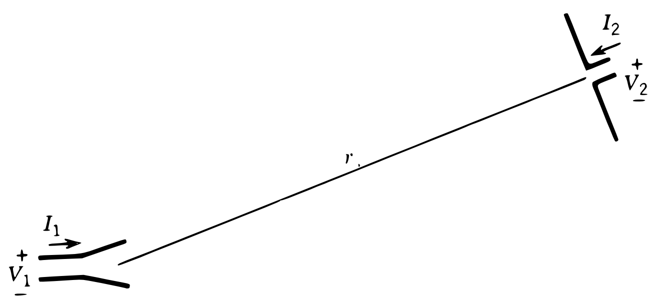
We defined the effective length of an antenna in terms of the radiation field produced by it. We will show in this section that the open-circuit voltage induced in the antenna by an externally produced field is proportional to this effective length; in fact, some authors *define* effective length in terms of the open-circuit voltage produced when the antenna is receiving a wave.

By the principle of reciprocity, if two antennas are fed by equal current sources, the open-circuit voltage produced across the terminals of antenna 1 by the current source feeding antenna 2 is the same as the open-circuit voltage produced across the terminals of antenna 2 by the current source feeding antenna 1.

We apply this principle to determine the open-circuit voltage across the terminals of our general antenna, whose transmitted field is given by (3.2), when it receives an incident wave. The general antenna is assumed to interact with a short dipole, as shown in Fig. 3.2, together with the coordinate system to be used and the assumed current directions and voltage polarities. Note that the same rectangular coordinate system is used for both antennas, although θ', ϕ' are not equal to θ, ϕ.

First, we let the general antenna fed by a current source of 1 A transmit a wave toward the short dipole. Its field at the dipole is

$$\mathbf{E}^i = \frac{jZ_0}{2\lambda r} e^{-jkr}\mathbf{h} \tag{3.4}$$



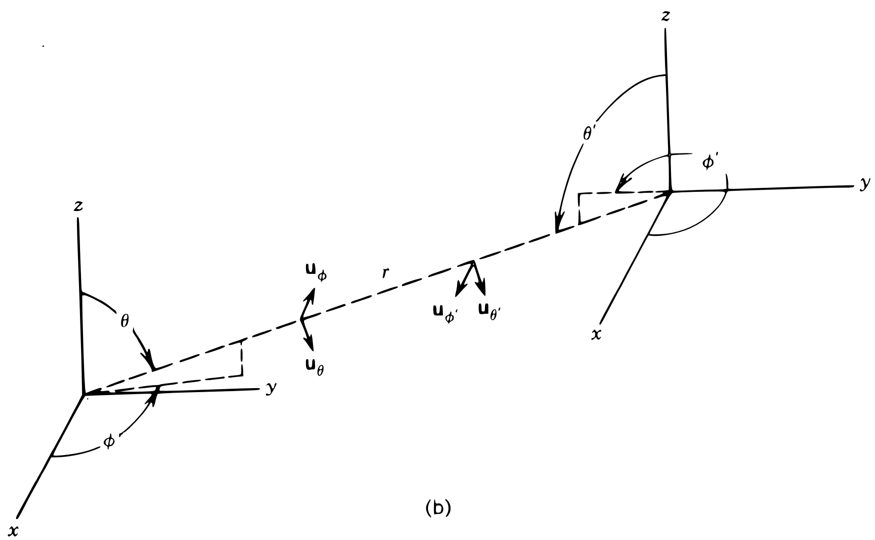


FIGURE 3.2. General antenna and short dipole: (a) antennas; (b) coordinates and unit vectors.

and the open-circuit voltage across the dipole terminals, with the polarity shown in Fig. 3.2, is

$$V_2 = \mathbf{E}^i \cdot \boldsymbol{\ell} \tag{3.5}$$

where $\boldsymbol{\ell}$ is the vector length of the dipole. Here we are considering that the dipole has infinitesimal length so that the incident field $\mathbf{E}^i$ is constant over the dipole length. The dipole may be arbitrarily oriented, but since $\mathbf{E}^i$ has no radial component, a radial component of the dipole length will not contribute to the received voltage. Then, still using the coordinate system of Fig. 3.2,

$$V_2 = E^i_\theta \ell_\theta + E^i_\phi \ell_\phi \tag{3.6}$$

where the dipole components ℓ_θ and ℓ_ϕ are given by

$$\ell_\theta = \mathbf{u}_\theta \cdot \boldsymbol{\ell} \qquad \ell_\phi = \mathbf{u}_\phi \cdot \boldsymbol{\ell} \tag{3.7}$$

Combining (3.4) and (3.5) gives the voltage induced across the open dipole terminals by the incident wave from the general antenna:

$$V_2 = \frac{jZ_0}{2\lambda r} e^{-jkr}\mathbf{h} \cdot \boldsymbol{\ell} \tag{3.8}$$

Next, suppose the dipole fed by a 1-A current source is transmitting, and the general antenna, with open terminals, is receiving. The field produced at the general antenna (1) is given by

$$E^i_{\theta'} = \frac{jZ_0}{2\lambda r} e^{-jkr}\ell_{\theta'} \quad \text{(a)}$$
$$E^i_{\phi'} = \frac{jZ_0}{2\lambda r} e^{-jkr}\ell_{\phi'} \quad \text{(b)} \tag{3.9}$$

where we continue to use the same coordinate system but note that θ', ϕ' differ from θ, ϕ. We note from Fig. 3.2 that although the angles just mentioned are different, we have

$$\mathbf{u}_\theta = \mathbf{u}_{\theta'} \qquad \mathbf{u}_\phi = -\mathbf{u}_{\phi'} \tag{3.10}$$

It follows that

$$E^i_{\theta'} = E^i_\theta \quad \text{(a)} \qquad E^i_{\phi'} = -E^i_\phi \quad \text{(c)}$$
$$\ell_{\theta'} = \ell_\theta \quad \text{(b)} \qquad \ell_{\phi'} = -\ell_\phi \quad \text{(d)} \tag{3.11}$$

and therefore the wave incident on antenna 1 is

$$\begin{aligned} E^i_\theta &= \frac{jZ_0}{2\lambda r} e^{-jkr} \ell_\theta \quad &(a) \\ -E^i_\phi &= \frac{jZ_0}{2\lambda r} e^{-jkr}(-\ell_\phi) \quad &(b) \end{aligned} \tag{3.12}$$

or

$$\mathbf{E}^i = \frac{jZ_0}{2\lambda r} e^{-jkr} \boldsymbol{\ell} \tag{3.13}$$

The open-circuit voltage induced in the general antenna (1) is V_1, which by the reciprocity theorem is equal to V_2, as given by (3.8). Then, from (3.8) and the reciprocity theorem, we get

$$V_1 = V_2 = \frac{jZ_0}{2\lambda r} e^{-jkr} \boldsymbol{\ell} \cdot \mathbf{h} \tag{3.14}$$

and if we recognize that the first part of this expression is the incident wave of (3.13), we get an expression for the received voltage across the open terminals of antenna 1 in terms of the incident field, $\mathbf{E}^i$, and its effective length $\mathbf{h}$. It is

$$V_1 = \mathbf{E}^i \cdot \mathbf{h} \tag{3.15}$$

It should be noted that in (3.15) both $\mathbf{E}^i$ and $\mathbf{h}$ are measured in the same coordinate system (in contrast to a situation to be discussed later). The voltage V_1 is in general a complex phasor voltage, since both $\mathbf{E}^i$ and $\mathbf{h}$ are complex. Finally, in specifying the effective length $\mathbf{h}$ of an antenna, a terminal pair at which input current is to be measured must be specified. Then V_1 is the open-circuit voltage measured across those terminals.

3.4. MAXIMUM RECEIVED POWER

It is reasonable to believe from looking at (3.15) that by proper selection of the effective length $\mathbf{h}$ of a receiving antenna, we can increase the open-circuit voltage and hence the received power. If we neglect the extraneous problems of, for example, impedance mismatch, the power received by the general antenna is proportional to the square of the magnitude of the open-circuit voltage; thus using an equality rather than a proportional symbol (an inconsequential action since we will later consider a power ratio), we have

$$W = VV^* = |\mathbf{E}^i \cdot \mathbf{h}|^2 = |h_\theta E^i_\theta + h_\phi E^i_\phi|^2 \tag{3.16}$$

where an appropriate coordinate system is used so that $\mathbf{h}$ has only two components and so only two are needed for $\mathbf{E}^i$.

Let us define

$$\begin{aligned} h_\theta &= |h_\theta| e^{j\alpha} \quad (a) \qquad & \frac{h_\phi}{|h_\phi|} &= \frac{h_\theta}{|h_\theta|} e^{j\delta_1} \quad (b) \\ \frac{E^i_\theta}{|E^i_\theta|} &= \frac{h_\theta}{|h_\theta|} e^{j\beta} \quad (c) \qquad & \frac{E^i_\phi}{|E^i_\phi|} &= \frac{E^i_\theta}{|E^i_\theta|} e^{j\delta_2} \quad (d) \end{aligned} \tag{3.17}$$

with δ_1 the phase angle by which h_ϕ leads h_θ, β the angle by which E^i_θ leads h_θ, and δ_2 the angle by which E^i_ϕ leads E^i_θ. Using these equations, the received power, (3.16), becomes

$$W = \big|\, |h_\theta|\,|E^i_\theta| + |h_\phi|\,|E^i_\phi| e^{j(\delta_1+\delta_2)} \big|^2 \tag{3.18}$$

where the angle $2\alpha + \beta$ has been removed as common to both terms in the sum.

Clearly W is a maximum, from (3.18), if

$$\delta_1 + \delta_2 = 0 \tag{3.19}$$

and has value

$$W_m = [|h_\theta|\,|E^i_\theta| + |h_\phi|\,|E^i_\phi|]^2 \tag{3.20}$$

Now W_m can be maximized further, for a fixed incident wave, $\mathbf{E}^i$, by varying $|h_\theta|$ or $|h_\phi|$. Certainly, however, there must be some constraint on $|h_\theta|$ and $|h_\phi|$; otherwise, W_m could be made as great as we please by increasing $|h_\theta|$ and $|h_\phi|$ arbitrarily. To determine this constraint, return to (3.2), which gives the transmitted field of an antenna in terms of its effective length. The transmitted Poynting vector, from (3.2), is obviously proportional to $\mathbf{h} \cdot \mathbf{h}^*$. Then a reasonable constraint on an antenna is that this Poynting vector remain constant as we vary $\mathbf{h}$. Therefore, we vary $\mathbf{h}$ to maximize W_m in (3.20) with the constraint

$$\mathbf{h} \cdot \mathbf{h}^* = |h_\theta|^2 + |h_\phi|^2 = C \tag{3.21}$$

Substituting (3.21) in (3.20), we get

$$W_m = [|h_\theta|\,|E^i_\theta| + (C - |h_\theta|^2)^{1/2} |E^i_\phi|]^2 \tag{3.22}$$

and differentiating with respect to $|h_\theta|$ in order to maximize W_m gives

$$\frac{\partial W_m}{\partial |h_\theta|} = 2[|h_\theta|\,|E^i_\theta| + (C - |h_\theta|^2)^{1/2} |E^i_\phi|]^2 \left[|E^i_\theta| - \frac{|h_\theta|\,|E^i_\phi|}{(C - |h_\theta|^2)^{1/2}} \right] = 0 \tag{3.23}$$

from which it is clear that

$$|E_\theta^i| - \frac{|h_\theta|\,|E_\phi^i|}{(C - |h_\theta|^2)^{1/2}} = |E_\theta^i| - \frac{|h_\theta|}{|h_\phi|}\,|E_\phi^i| = 0 \tag{3.24}$$

or

$$\frac{|h_\theta|}{|h_\phi|} = \frac{|E_\theta^i|}{|E_\phi^i|} \tag{3.25}$$

It seems quite reasonable that (3.25) will give maximum received power, rather than minimum, since if $\mathbf{E}^i$ has a large θ component, we would expect a large h_θ to give best reception. However, we will substitute (3.25) into (3.20) to see if the received power is maximum. We rewrite (3.20) as

$$W_m = |h_\theta|^2|E_\theta^i|^2 + |h_\phi|^2|E_\phi^i|^2 + |h_\theta|\,|E_\theta^i|\,|h_\phi|\,|E_\phi^i| + |h_\theta|\,|E_\theta^i|\,|h_\phi|\,|E_\phi^i| \tag{3.26}$$

In the third term of W_m, we make the substitution from (3.25) that

$$|E_\theta^i|\,|h_\phi| = |h_\theta|\,|E_\phi^i| \tag{3.27}$$

and in the fourth term of W_m we make the substitution in reverse. Thus

$$\begin{aligned} W_{mm} &= |h_\theta|^2|E_\theta^i|^2 + |h_\phi|^2|E_\phi^i|^2 + |h_\theta|^2|E_\phi^i|^2 + |h_\phi|^2|E_\theta^i|^2 \\ &= (|h_\theta|^2 + |h_\phi|^2)(|E_\theta^i|^2 + |E_\phi^i|^2) \end{aligned} \tag{3.28}$$

This is quite obviously maximum power rather than minimum. Finally, (3.28) may be written as

$$W_{mm} = (\mathbf{h}\cdot\mathbf{h}^*)(\mathbf{E}^i\cdot\mathbf{E}^{i*}) = |\mathbf{h}|^2|\mathbf{E}^i|^2 \tag{3.29}$$

There may be some concern on the part of the reader that (3.21) is a legitimate constraint. While we vary $\mathbf{h}$ for maximum *received* power, why should we apply a constraint that is meaningful only for the *transmitting* case? For this reason we return to (3.20), assume that $\mathbf{h}$ is fixed, and vary $\mathbf{E}^i$ in order to maximize the power. Now in this situation it is quite clear that we can cause only a fixed power density at the receiving antenna. Therefore

$$\mathbf{E}^i\cdot\mathbf{E}^{i*} = C \tag{3.30}$$

Using this equation makes W_m become

$$W_m = [|h_\theta|\,|E_\theta^i| + |h_\phi|(C - |E_\theta^i|^2)^{1/2}]^2 \tag{3.31}$$

and differentiation with respect to $|E_\theta^i|$ gives

$$\frac{\partial W_m}{\partial |E_\theta^i|} = 2[|h_\theta|\,|E_\theta^i| + |h_\phi|(C - |E_\theta^i|^2)^{1/2}]\left[|h_\theta| - \frac{|h_\phi|\,|E_\theta^i|}{(C - |E_\theta^i|^2)^{1/2}}\right] = 0 \tag{3.32}$$

from which it follows that

$$\frac{|h_\theta|}{|h_\phi|} = \frac{|E_\theta^i|}{|E_\phi^i|} \tag{3.33}$$

which is the same condition we arrived at previously.

Equation (3.33) gives one condition on $\mathbf{h}$ for maximum power reception. The other is given by

$$\delta_1 + \delta_2 = 0 \tag{3.19}$$

where δ_1 is the angle by which h_ϕ leads h_θ and δ_2 is the angle by which E_ϕ^i leads E_θ^i.

We rewrite (3.17b) as

$$\frac{h_\phi}{h_\theta} = \frac{|h_\phi|}{|h_\theta|}\,e^{j\delta_1} \tag{3.34}$$

and substitute (3.19) and (3.33) into it, obtaining

$$\frac{h_\phi}{h_\theta} = \frac{|E_\phi^i|}{|E_\theta^i|}\,e^{-j\delta_2} \tag{3.35}$$

From (3.17d) we recognize that the last term is $E_\phi^{i*}/E_\theta^{i*}$, so that the relationship between $\mathbf{h}$ and $\mathbf{E}^i$ for maximum received power is

$$\frac{h_\phi}{h_\theta} = \left(\frac{E_\phi^i}{E_\theta^i}\right)^* \tag{3.36}$$

3.5. POLARIZATION MATCH FACTOR

If we maintain the same degree of impedance matching for an antenna as we vary its polarization properties, then the ratio of actual power received to that received under the most favorable circumstances of matched polarization is, from (3.16) and (3.29),

$$\rho = \frac{|\mathbf{E}^i \cdot \mathbf{h}|^2}{|\mathbf{E}^i|^2 |\mathbf{h}|^2} \tag{3.37}$$

We will refer to ρ as the *polarization match factor*, although it is sometimes called the polarization efficiency. Its range is obviously

$$0 \le \rho \le 1$$

The polarization match factor shows how well a receiving antenna of effective length $\mathbf{h}$ is matched in polarization to an incoming wave. Now let us recognize that the incoming wave was transmitted by another antenna and so introduce the polarization properties of that antenna into the problem.

Figure 3.3 shows two antennas in a transmit–receive configuration. The transmitting antenna (1) will be described in its polarization properties by the right-handed coordinate system x, y, z adjacent to antenna 1 since the polarization of a wave is normally based on a right-handed coordinate system with one axis pointing in the direction of wave travel. The receiving antenna will be described by the right-handed ξ, η, ζ system. The antennas need not have their main beams pointed at each other, but the z and ζ axes are parallel and each points at the other antenna.

The incident wave from antenna 1 may be written as

$$\mathbf{E}_1^i = E_{01} a_1 \left(\mathbf{u}_x + j \frac{b_1}{a_1} e^{j\phi_1} e^{-j\pi/2} \mathbf{u}_y \right) = E_{01} a_1 (\mathbf{u}_x + p_1 e^{-j\pi/2} \mathbf{u}_y) \tag{3.38}$$

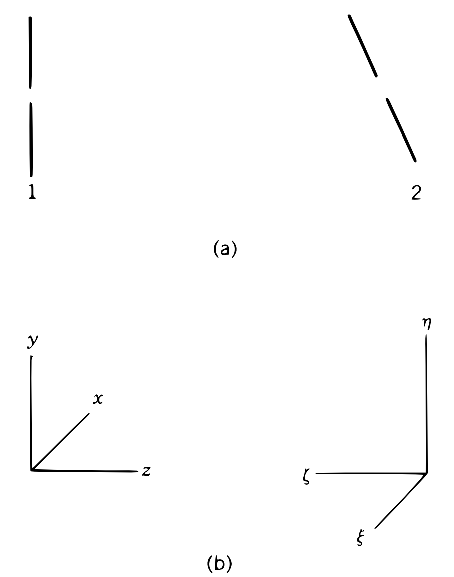


FIGURE 3.3. Antennas and coordinate systems used in development of polarization match factor: (a) antennas; (b) coordinate systems.

where p_1 is the modified polarization ratio [2] of the incident wave produced by antenna 1.

The *polarization ratio of an antenna* is defined as the polarization ratio of the field it transmits (far field). Therefore, p_1 is the modified polarization ratio of antenna 1. It is a function of θ and ϕ, the colatitude and azimuth angles measured for the transmission direction.

If antenna 2 were transmitting, its radiated wave could be written as

$$\mathbf{E}_2 = E_{02}(a_2 \mathbf{u}_\xi + b_2 e^{j\phi_2} \mathbf{u}_\eta) \tag{3.39}$$

where we use appropriate coordinates ξ, η, ζ for the wave propagating in the ζ direction, toward the first antenna. Equation (3.39) may be written in terms of the modified polarization ratio of antenna 2 as

$$\mathbf{E}_2 = E_{02} a_2 \left(\mathbf{u}_\xi + j \frac{b_2}{a_2} e^{j\phi_2} e^{-j\pi/2} \mathbf{u}_\eta \right) = E_{02} a_2 (\mathbf{u}_\xi + p_2 e^{-j\pi/2} \mathbf{u}_\eta) \tag{3.40}$$

where p_2 is the modified polarization ratio of antenna 2 in the ζ direction, using the appropriate right-handed coordinates at antenna 2.

Now the transmitted field (3.40) is related to the vector length of antenna 2 by

$$\mathbf{E}_2 = \frac{jZ_0 I_2}{2\lambda r} e^{-jkr} \mathbf{h}_2 \tag{3.41}$$

Therefore, $\mathbf{h}_2$ becomes, using (3.40) and (3.41),

$$\begin{aligned} \mathbf{h}_2 &= \frac{2\lambda r}{jZ_0 I_2} e^{jkr} \mathbf{E}_2 = \frac{2\lambda r}{jZ_0 I_2} e^{jkr} E_{02} a_2 (\mathbf{u}_\xi + p_2 e^{-j\pi/2} \mathbf{u}_\eta) \\ &= h_{02}(\mathbf{u}_\xi + p_2 e^{-j\pi/2} \mathbf{u}_\eta) \end{aligned} \tag{3.42}$$

where

$$h_{02} = \frac{2\lambda r}{jZ_0 I_2} e^{jkr} E_{02} a_2 \tag{3.43}$$

Let us return now to the situation where antenna 1 transmits and antenna 2 receives. The open-circuit voltage across the appropriate terminals of 2 is

$$V_2 = \mathbf{E}_1^i \cdot \mathbf{h}_2 = E_{1\xi}^i h_{2\xi} + E_{1\eta}^i h_{2\eta} \tag{3.44}$$

If we note from Fig. 3.3 that

$$E_{1\xi}^i = -E_{1x}^i \quad \text{(a)} \qquad E_{1\eta}^i = E_{1y}^i \quad \text{(b)} \tag{3.45}$$

and use the field and effective length components from (3.38) and (3.42), we get for the open-circuit voltage

$$V_2 = -E_{01}a_1h_{02} + E_{01}a_1p_1e^{-j\pi/2}h_{02}p_2e^{-j\pi/2}$$
$$= -E_{01}a_1h_{02}(1 + p_1p_2) \qquad (3.46)$$

We find also, from (3.38) and (3.42), that

$$|\mathbf{E}^i|^2 = |E_{01}a_1|^2(\mathbf{u}_x + p_1e^{-j\pi/2}\mathbf{u}_y)\cdot(\mathbf{u}_x + p_1^*e^{j\pi/2}\mathbf{u}_y)$$
$$= |E_{01}a_1|^2(1 + p_1p_1^*) \qquad (3.47)$$

and

$$|\mathbf{h}_2|^2 = |h_{02}|^2(\mathbf{u}_\xi + p_2e^{-j\pi/2}\mathbf{u}_\eta)\cdot(\mathbf{u}_x + p_2^*e^{j\pi/2}\mathbf{u}_\eta)$$
$$= |h_{02}|^2(1 + p_2p_2^*) \qquad (3.48)$$

If we substitute (3.44), (3.46), (3.47), and (3.48) into the polarization match factor (3.37), we obtain

$$\rho = \frac{|E_{01}a_1h_{02}|^2|1 + p_1p_2|^2}{|E_{01}a_1|^2(1 + p_1p_1^*)|h_{02}|^2(1 + p_2p_2^*)} = \frac{(1 + p_1p_2)(1 + p_1^*p_2^*)}{(1 + p_1p_1^*)(1 + p_2p_2^*)} \qquad (3.49)$$

It is worthwhile to repeat that the definition of p_1 uses wave components measured in a right-handed system with the z axis pointing away from antenna 1 and toward 2. In defining p_2, we used a right-handed system with the η axis parallel to and in the same direction as the y axis and with the ζ axis pointing toward antenna 1.

3.6. POLARIZATION MATCH FACTOR: SPECIAL CASES

Polarization-Matched Antennas

If we have two polarization-matched antennas in a transmit–receive system, the polarization match factor of (3.49) is equal to 1. (Note that it may change if the orientation of one of the antennas is changed.) Thus

$$\rho = \frac{(1 + p_1p_2)(1 + p_1^*p_2^*)}{(1 + p_1p_1^*)(1 + p_2p_2^*)} = 1 \qquad (3.50)$$

Cross multiplying, expanding, and canceling terms gives

$$(p_1^* - p_2)(p_2^* - p_1) = 0$$

which has a solution

$$p_1 = p_2^* \qquad (3.51)$$

In terms of the circular polarization ratio q,

$$q_1 = \frac{1 - p_1}{1 + p_1} = \frac{1 - p_2^*}{1 + p_2^*} = q_2^* \qquad (3.52)$$

and the axial ratios and tilt angles of the polarization ellipse of the two antennas are related by

$$\mathrm{AR}_1 = \left|\frac{1 + |q_1|}{1 - |q_1|}\right| = \left|\frac{1 + |q_2|}{1 - |q_2|}\right| = \mathrm{AR}_2 \quad \text{(a)}$$
$$\tau_1 = \frac{\theta_1}{2} = -\frac{\gamma_1}{2} = +\frac{\gamma_2}{2} = -\tau_2 \quad \text{(b)} \qquad (3.53)$$

Now, in (3.53), τ_1 and τ_2 are described in different coordinate systems, as shown in Fig. 3.3. It is obvious from Fig. 3.3 that the condition $\tau_1 = -\tau_2$ means that the major axes of the two polarization axes coincide. Equation (3.52) also shows that the rotation senses of the two polarization ellipses are the same when described in the appropriate coordinate systems. Having the same rotation sense, using the coordinate systems of Fig. 3.3, means that if we think of both antennas transmitting a right elliptic wave, for example, the two waves will appear to rotate in opposite directions at a point in space at which they "meet."

Cross-Polarized Antennas

Two antennas in a transmit–receive configuration that are so polarized that no signal is received are said to be cross-polarized. For this situation

$$\rho = 0 = \frac{(1 + p_1p_2)(1 + p_1^*p_2^*)}{(1 + p_1p_1^*)(1 + p_2p_2^*)} \qquad (3.54)$$

from which it follows that

$$p_1 = -\frac{1}{p_2} \qquad (3.55)$$

Solving for q, we obtain

$$q_1 = \frac{1 - p_1}{1 + p_2} = \frac{p_2 + 1}{p_2 - 1} = -\frac{1}{q_2} \qquad (3.56)$$

We see immediately that the rotation senses of the polarization ellipses of the antennas are opposite (so that if both antennas transmitted simultaneously, their field vectors would appear to rotate in the same direction).

The axial ratios are, from (3.56),

$$\mathrm{AR}_1 = \left| \frac{1 + |q_1|}{1 - |q_1|} \right| = \left| \frac{|q_2| + 1}{|q_2| - 1} \right| = \mathrm{AR}_2 \tag{3.57}$$

Also from (3.56)

$$Q_1 e^{j\gamma_1} = -\frac{1}{Q_2 \varepsilon^{j\gamma_2}} = \frac{1}{Q_2} e^{-j(\gamma_2 + \pi)}$$

so that

$$\gamma_1 = -\gamma_2 + \pi$$

and

$$\tau_1 = -\tfrac{1}{2}\gamma_1 = \tfrac{1}{2}\gamma_2 \mp \tfrac{1}{2}\pi = -\tau_2 \mp \tfrac{1}{2}\pi \tag{3.58}$$

Bearing in mind that τ_1 is measured from the x axis toward the y axis in Fig. 3.3, and τ_2 is measured from the ξ axis toward the η axis in Fig. 3.3, we see that (3.58) means that the major axis of one polarization ellipse coincides with the minor axis of the other.

Identical, Polarization-Matched Antennas

It would seem to be quite easy to define identical antennas, but surprisingly there is a degree of arbitrariness involved. When placed side by side and oriented similarly, identical antennas are indistinguishable except by position. Although not overly precise, this definition is quite clear. Now we make the assumption that they are placed into a transmit–receive configuration by rotating one of them by π radians around a *vertical* axis (the y axis of Fig. 3.3). We might also consider a rotation about a horizontal axis or the major or minor axis of the polarization ellipse—hence the arbitrariness mentioned—but we will rotate first about the vertical axis.

For identical antennas, before one is rotated into a receiving position,

$$\begin{gathered} h_{x1} = h'_{x2} \qquad h_{y1} = h'_{y2} \\ p_1 = j\,\frac{h_{y1}}{h_{x1}} = p'_2 = j\,\frac{h'_{y2}}{h'_{x2}} \end{gathered} \tag{3.59}$$

where the primes are used with the parameters of the antenna to be rotated.

After antenna 2 is rotated 180° about the y axis of Fig. 3.3, its new length components are

$$h_{x2} = -h'_{x2} \qquad h_{y2} = h'_{y2} \tag{3.60}$$

Changing these components to the ξ, η, ζ coordinates, which are now appropriate for antenna 2, we have

$$\begin{aligned} h_{\xi 2} &= -h_{x2} = h'_{x2} \quad \text{(a)} \\ h_{\eta 2} &= h_{y2} = h'_{y2} \quad \text{(b)} \end{aligned} \tag{3.61}$$

Then the new value for the polarization ratio p_2 is

$$p_2 = j\,\frac{h_{\eta 2}}{h_{\xi 2}} = j\,\frac{h'_{y2}}{h'_{x2}} = p'_2 \tag{3.62}$$

and thus p_2 is unchanged by rotation about a vertical axis. A little thought will show that, in general, the major axes of the ellipses no longer coincide.

Let the antennas be not only identical but polarization matched. Then they must satisfy (3.51), (3.59), and (3.62), or

$$p_1 = p_2^* \quad \text{(a)} \qquad p_1 = p_2 \quad \text{(b)} \tag{3.63}$$

and also

$$q_1 = q_2^* \quad \text{(a)} \qquad q_1 = q_2 \quad \text{(b)} \tag{3.64}$$

We conclude then that identical, polarization-matched antennas must have

$$p_1 = p_2 = \text{real quantity} \qquad q_1 = q_2 = \text{real quantity} \tag{3.65}$$

from which it follows that

$$\gamma_1 = \gamma_2 = 0,\ \pi \qquad \tau_1 = \tau_2 = 0,\ -\tfrac{1}{2}\pi \tag{3.66}$$

We see that the major axis of the polarization ellipse must be either vertical or horizontal if the antennas are to be identical (in our sense of rotation about a vertical axis) and matched. This does not exclude circularly polarized antennas for which the concept of major axis is not meaningful.

Antennas that are identical and cross-polarized must satisfy

$$\begin{gathered} p_1 = -\frac{1}{p_2} \qquad q_1 = -\frac{1}{q_2} \\ p_1 = p_2 \qquad q_1 = q_2 \end{gathered} \tag{3.67}$$

which gives

$$p_1 = q_1 = \pm j1 \tag{3.68}$$

from which we find that

$$\begin{aligned} &\text{AR} \to \infty \\ &\tau = \tfrac{1}{4}\pi \qquad \tfrac{3}{4}\pi \end{aligned} \tag{3.69}$$

which describe linearly polarized antennas with tilt angles of 45° or 135°.

When we rotated one of the antennas about a vertical axis, we found that the two antennas would be matched if their major axes were vertical. Perhaps then if we started with identical, side-by-side antennas and rotated one of them about its major axis, we would obtain polarization matching.

Let the polarization parameters *before* rotation be $p_1, p_2', \ldots$, where the primes are used with the parameters of the antenna to be rotated. Then for identical antennas

$$\begin{aligned} p_1 &= p_2' \qquad & q_1 &= q_2' \\ \text{AR}_1 &= \text{AR}_2' \qquad & \tau_1 &= \tau_2' \end{aligned} \tag{3.70}$$

After antenna 2 is rotated about its major axis, we recognize that the axial ratio and rotation sense are unchanged, that is,

$$\begin{aligned} \text{AR}_2 &= \text{AR}_2' \\ |q_2| &= |q_2'| \end{aligned} \tag{3.71}$$

Since the rotation takes place about the major axis, obviously the major axis does not change, but as Fig. 3.3 shows, the tilt angle in the ξ, η, ζ system is measured oppositely from that in x, y, z. Therefore, after rotation the new tilt angle is given by

$$\tau_2 = -\tau_2' \tag{3.72}$$

Equations (3.71) and (3.72) lead to

$$q_2 = q_2'^{*} \tag{3.73}$$

and from (3.70)

$$q_2 = q_1^{*} \tag{3.74}$$

Now, from (3.52), this is the condition for polarization matching. We conclude then that identical antennas (indistinguishable when placed side by side and similarly oriented) will be matched in polarization if one of them is rotated 180° about its major polarization axis to bring it to a receive position.

3.7. MATCH FACTOR IN OTHER FORMS

Since the modified polarization ratio p is not always the most convenient parameter for an antenna, we need the equation for ρ in terms of other parameters. If we make the substitution

$$p = \frac{1-q}{1+q} \tag{2.91}$$

in (3.49), the match factor is found in terms of q to be

$$\rho = \frac{(1+q_1q_2)(1+q_1^*q_2^*)}{(1+q_1q_1^*)(1+q_2q_2^*)} \tag{3.75}$$

It is not surprising that ρ has the same form in q as in p, since the form for q in terms of p is the same as for p in terms of q.

Now (3.75) is valid for any value of q, but nonetheless if we treat left elliptic polarizations by means of the parameter w ($|q| > 1$, $|w| < 1$), we might wish ρ in terms of w. Substituting

$$p = \frac{w^* - 1}{w^* + 1} \tag{2.138}$$

into (3.49) leads to

$$\rho = \frac{(1+w_1w_2)(1+w_1^*w_2^*)}{(1+w_1w_1^*)(1+w_2w_2^*)} \tag{3.76}$$

an equation that also has the same form as (3.49).

A mixed form in terms of q_1 and w_2 or q_2 and w_1 might also be useful. Replacing q_2 in (3.75) by $1/w_2^*$ leads to

$$\rho = \frac{(w_2^*+q_1)(w_2+q_1^*)}{(1+q_1q_1^*)(1+w_2w_2^*)} \tag{3.77}$$

and interchanging subscripts in (3.77) gives

$$\rho = \frac{(w_1^*+q_2)(w_1+q_2^*)}{(1+q_2q_2^*)(1+w_1w_1^*)} \tag{3.78}$$

All four forms (3.75)–(3.78) are valid for any value of q and w, but it would be natural to use (3.75) for both antennas right handed, (3.76) for both left handed, and (3.77) or (3.78) for one left and the other right handed.

We may find ρ in terms of axial ratios and tilt angles of the polarization ellipses, but here we must be careful about the rotation sense of the ellipses, since axial ratio and tilt alone are not sufficient to describe the antenna polarization. Consider first that both antennas are right handed. We have, from (2.107), if $|q| < 1$,

$$\mathrm{AR} = \frac{1 + |q|}{1 - |q|} \tag{3.79}$$

In (3.75) we write

$$q = |q| e^{-j2\tau} \tag{3.80}$$

and ρ becomes

$$\rho = \frac{1 + 2|q_1 q_2| \cos 2(\tau_1 + \tau_2) + |q_1 q_2|^2}{(1 + |q_1|^2)(1 + |q_2|^2)} \tag{3.81}$$

and if $|q|$ from (3.79) is substituted into (3.81), there results, after some manipulation,

$$\rho = \frac{(\mathrm{AR}_1\mathrm{AR}_2 + 1)^2 + (\mathrm{AR}_1 + \mathrm{AR}_2)^2 + (\mathrm{AR}_1^2 - 1)(\mathrm{AR}_2^2 - 1)\cos 2(\tau_1 + \tau_2)}{2(\mathrm{AR}_1^2 + 1)(\mathrm{AR}_2^2 + 1)} \tag{3.82}$$

If both antennas are left handed, $|w| < 1$, we find from (2.147) that

$$\mathrm{AR} = \frac{1 + |w|}{1 - |w|} \tag{3.83}$$

We also have

$$w = |w| e^{-j2\tau} \tag{3.84}$$

If we substitute (3.83) and (3.84), which have the same forms as (3.79) and (3.80), into (3.76), which has the same form as (3.75), it is obvious that (3.82) will result. Therefore, (3.82) holds if both antennas are right handed or if both are left handed.

If antenna 1 is right handed and 2 is left handed, we substitute

$$\begin{aligned} \mathrm{AR}_1 &= \frac{1 + |q_1|}{1 - |q_1|} \qquad q_1 = |q_1| e^{-j2\tau_1} \quad \text{(a)} \\ \mathrm{AR}_2 &= \frac{1 + |w_2|}{1 - |w_2|} \qquad w_2 = |w_2| e^{-j2\tau_2} \quad \text{(b)} \end{aligned} \tag{3.85}$$

into the mixed form (3.77) and obtain

$$\rho = \frac{(\mathrm{AR}_1\mathrm{AR}_2 - 1)^2 + (\mathrm{AR}_1 - \mathrm{AR}_2)^2 + (\mathrm{AR}_1^2 - 1)(\mathrm{AR}_2^2 - 1)\cos 2(\tau_1 + \tau_2)}{2(\mathrm{AR}_1^2 + 1)(\mathrm{AR}_2^2 + 1)} \tag{3.86}$$

for the match factor in terms of axial ratios and tilt angles.

If antenna 1 is left handed and 2 is right handed, we could make the appropriate substitutions in (3.78), and (3.86) would again result.

In terms of axial ratios and tilt angles, the polarization match factor may be found from (3.82) if both antennas have the same polarization rotation sense and from (3.86) if they are of opposite sense.

Finally, we note that ρ may be written in terms of a, b, and ϕ of (2.70) as

$$\rho = \frac{1 - 2(b_1/a_1)(b_2/a_2)\cos(\phi_1 + \phi_2) + [(b_1/a_1)(b_2/a_2)]^2}{[1 + (b_1/a_1)^2][1 + (b_2/a_2)^2]} \tag{3.87}$$

in terms of left and right circular components as

$$\rho = \frac{1 + 2(L_1/R_1)(L_2/R_2)\cos(\theta_1 + \theta_2) + [(L_1/R_1)(L_2/R_2)]^2}{[1 + (L_1/R_1)^2][1 + (L_2/R_2)^2]} \tag{3.88}$$

and in terms of the common polarization ratio P $(= -jp)$ as

$$\rho = \frac{(1 - P_1 P_2)(1 - P_1^* P_2^*)}{(1 + P_1 P_1^*)(1 + P_2 P_2^*)} \tag{3.89}$$

3.3. Show that the polarization match factor between two antennas can be written as

$$\rho = \frac{|\mathbf{h}_1 \cdot \mathbf{h}_2|^2}{|\mathbf{h}_1|^2 |\mathbf{h}_2|^2}$$

where $\mathbf{h}_1$ and $\mathbf{h}_2$ are the effective lengths of the two antennas measured in the same coordinate system.

8.4. THE POLARIZATION PATTERN

Equation (3.82) for the polarization match factor of two antennas with the same rotation sense and (3.86) for antennas of opposite sense both reduce to

$$\rho = \frac{AR_1^2 \cos^2(\tau_1 + \tau_2) + \sin^2(\tau_1 + \tau_2)}{AR_1^2 + 1} \tag{8.4}$$

if antenna 2 is linearly polarized, with $AR_2 \rightarrow \infty$. This equation leads to a widely used method for obtaining the polarization ellipse of an antenna experimentally.

Antenna 2 is rotated around a line drawn between the two antennas, say the z axis of Fig. 3.3. Further, antenna 2 is so oriented that it cannot receive any z-directed wave components as it is rotated. For a dipole the rotation axis is perpendicular to the dipole.

At $\tau_2 = -\tau_1$, which corresponds to coincidence between the major axes of the ellipses for the two antennas,

$$\rho = \frac{AR_1^2}{AR_1^2 + 1} \tag{8.5}$$

which is a maximum. At $\tau_2 = -\tau_1 \pm \frac{1}{2}\pi$, which corresponds to the major axis of the linearly polarized antenna coinciding with the minor axis of the antenna being tested.

$$\rho = \frac{1}{AR_1^2 + 1} \tag{8.6}$$

which is a minimum.

The open-circuit voltage is proportional to the square root of ρ, so the ratio of maximum to minimum open-circuit voltage, in magnitude, is

$$\frac{|V_{max}|}{|V_{min}|} = AR_1 \tag{8.7}$$

We thus have the axial ratio of the antenna undergoing test, and of course we have its tilt angle from the known rotation angle of the linear antenna when maximum power is received (or better yet, from the angle for minimum power plus 90°, since the minimum power angle is more sharply defined than the maximum power angle).

A plot of the square root of ρ from (8.4) is called the *polarization pattern* of the antenna whose polarization is being measured. Figure 8.1 shows the polar form of the pattern for (a) an antenna with an axial ratio of 2 and (b) a linearly polarized antenna. Since the ratio of maximum to minimum values of the polarization pattern is the axial ratio of the antenna under test, the

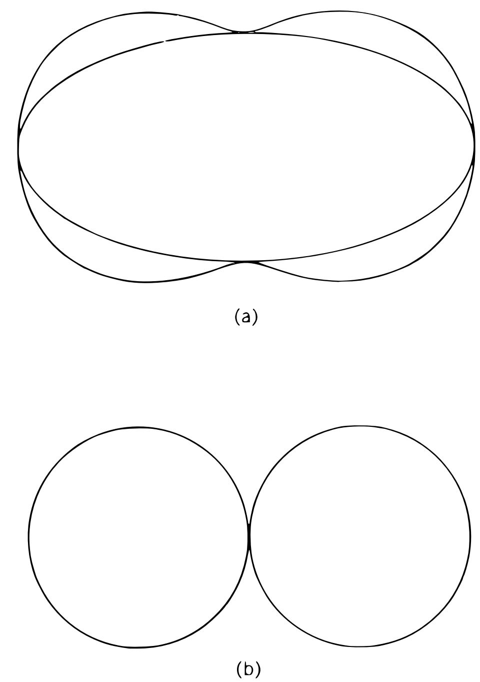


FIGURE 8.1. Polarization patterns for measuring axial ratio and tilt angle of the polarization ellipse: (a) AR = 2, with inscribed polarization ellipse; (b) $AR \rightarrow \infty$.

polarization ellipse can be inscribed in the polarization pattern, and Fig. 8.1(a) shows this.

The polarization pattern method lends itself well to the rapid testing of an antenna's polarization properties as a function of angle from beam maximum. The linear sampling antenna is rotated rapidly while the antenna under test is scanned slowly. A recording of the received voltage shows the antenna pattern with a rapid cyclic variation on it caused by the spinning of the sampling antenna. The ratio of amplitudes of adjacent maxima and minima will yield the axial ratio of the antenna being tested if the antenna pattern does not change significantly while the sampling antenna rotates through one-half revolution [2]. This automated process will clearly be more effective for antennas almost circularly polarized than for linearly polarized antennas.

An obvious deficiency of the polarization pattern method is its failure to give the rotation sense of the antenna under test. This information sometimes may be inferred from the antenna construction. It may also be obtained by making additional measurements with two equal-gain, opposite-sense, circularly polarized antennas.

Since the sampling antenna is mechanically rotated, care must be taken that the received power is not affected by the motion. In particular, rotary joints must have a constant output or be calibrated.