1 Pessimistic game

Given VPG $G = (V, V_0, V_1, E, \Omega, \mathfrak{C}, \theta)$, we can create a PG for G that is pessimistic for player $\alpha \in \{0, 1\}$, denoted $G_{\triangleright\alpha}$. We have

$$G_{\triangleright\alpha} = \{V, V_0, V_1, E', \Omega\}$$

such that

$$E' = \{(v, w) \in E \mid v \in V_{\overline{\alpha}} \vee \theta(v, w) \supseteq \mathfrak{C}\}$$

Pessimistic games can have deadlocks.

Let G be a VPG with configurations $\mathfrak{C}$. Let W_0^c and W_1^c be the winning sets for G for some configuration $c \in \mathfrak{C}$.

Claim: Given winning set W_α for game $G_{\triangleright\alpha}$ we have $W_\alpha \subseteq W_\alpha^c$.

Given $v \in W_\alpha$, to prove: $v \in W_\alpha^c$. Every strategy played by player α in game $G_{\triangleright\alpha}$ can also be played in game G. Every strategy played by player $\overline{\alpha}$ in game G can also be played in game $G_{\triangleright\alpha}$.

2 Fixpoint iteration

Given VPG $G = (V, V_0, V_1, E, \Omega, \mathfrak{C}, \theta)$ we define subgame $G \cap \mathfrak{C}' = (V, V_0, V_1, E, \Omega, \mathfrak{C} \cap \mathfrak{C}', \theta)$.

Assume priorities are compressed such that the distinct priorities are exactly $0, \ldots, d-1$. We can solve VPG G by calling FIXPOINTITERMBR$(G, V, \emptyset)$.

Algorithm 1 FIXPOINTITERMBR($VPG\ G, I_\nu, I_\mu$)

1: **if** $|\mathfrak{C}| = 1$ **then**
2:     $W_0 \leftarrow$ FIXPOINTITER$(G, d-1, I_\nu, I_\mu)$
3:       **return** $(\mathfrak{C} \times W_0, \mathfrak{C} \times (V \backslash W_0))$
4: **end if**
5: $P_0 \leftarrow$ FIXPOINTITER$(G_{\triangleright 0}, d-1, I_\nu, I_\mu)$
6: $P_1 \leftarrow$ FIXPOINTITER$(G_{\triangleright 1}, d-1, I_\nu, I_\mu)$
7: $(\mathfrak{C}^a, \mathfrak{C}^b) \leftarrow$ partition $\mathfrak{C}$ in non-empty parts
8: $(W_0^a, W_1^a) \leftarrow$ FIXPOINTITERMBR$(G \cap \mathfrak{C}^a, P_1, P_0)$
9: $(W_0^b, W_1^b) \leftarrow$ FIXPOINTITERMBR$(G \cap \mathfrak{C}^b, P_1, P_0)$
10: $W_0 \leftarrow W_0^a \cup W_0^b$
11: $W_1 \leftarrow W_1^a \cup W_1^b$
12: **return** (W_0, W_1)

Algorithm 2 FIXPOINTITER($PG\ G, i, I_\nu, I_\mu$)

1: **if** $i = -1$ **then**
2:     **return** $\{v \in V_0 \mid \exists w \in V : (v, w) \in E \wedge w \in X_{\Omega(w)}\} \cup \{v \in V_1 \mid \forall w \in V : (v, w) \implies w \in X_{\Omega(w)}\}$
3: **else**
4:     $X_i \leftarrow I_\nu$ if i is even I_μ otherwise
5:     **repeat**
6:         $X_i' \leftarrow X_i$
7:         $X_i \leftarrow$ FIXPOINTITER$(G, i-1, I_\nu, I_\mu)$
8:     **until** $X_i = X_i'$
9:     **return** X_i
10: **end if**

$$F_0 = \{v \in V_0 \mid \exists w \in V : (v, w) \in E \wedge w \in X_{\Omega(w)}\} \cup \{v \in V_1 \mid \forall w \in V : (v, w) \implies w \in X_{\Omega(w)}\}$$

$$W_0 = \sigma X_{d-1} \ldots \mu X_1 . \nu X_0 . F_0$$

Every X_i has fixpoint operator ν if i is even and μ if i is odd.

Note that fixpoint iteration works for games with deadlocks.

Set P_α calculated by the algorithm is winning for player α for all configuration in $\mathfrak{C}$.

Claim: the following formula correctly calculates the winning sets for $G_{|c}$ with $c \in \mathfrak{C}$.

$$W_0 = \sigma X_{d-1} \ldots P_1 \cap \mu X_1.P_0 \cup \nu X_0.P_1 \cap F_0$$

From which we can conclude that it is correct to let least fixed point variables start at P_0 and greatest fixed point variables at P_1.