1 Pessimistic game

Given VPG $G = (V, V_0, V_1, E, \Omega, \mathfrak{C}, \theta)$, we can create a PG for G that is pessimistic for player $\alpha \in \{0, 1\}$, denoted $G_{\triangleright\alpha}$. We have

$$G_{\triangleright\alpha} = \{V, V_0, V_1, E', \Omega\}$$

such that

$$E' = \{(v, w) \in E \mid v \in V_{\overline{\alpha}} \vee \theta(v, w) \supseteq \mathfrak{C}\}$$

Pessimistic games can have deadlocks.

Let G be a VPG with configurations $\mathfrak{C}$. Let W_0^c and W_1^c be the winning sets for G for some configuration $c \in \mathfrak{C}$.

Claim: Given winning set W_α for game $G_{\triangleright\alpha}$ we have $W_\alpha \subseteq W_\alpha^c$.

Given $v \in W_\alpha$, to prove: $v \in W_\alpha^c$. Every strategy played by player α in game $G_{\triangleright\alpha}$ can also be played in game G. Every strategy played by player $\overline{\alpha}$ in game G can also be played in game $G_{\triangleright\alpha}$.

2 Fixpoint iteration

Given VPG $G = (V, V_0, V_1, E, \Omega, \mathfrak{C}, \theta)$ we define subgame $G \cap \mathfrak{C}' = (V, V_0, V_1, E, \Omega, \mathfrak{C} \cap \mathfrak{C}', \theta)$.

Assume priorities are compressed such that the distinct priorities are exactly $0, \dots, d-1$. We can solve VPG G by calling FixpointIterMBR$(G, V, \emptyset)$.

Algorithm 1 FixpointIterMBR(*VPG* G, I_ν, I_μ)

if $|\mathfrak{C}| = 1$ **then**
  $W_0 \leftarrow$ FixpointIter$(G, d-1, I_\nu, I_\mu)$
  return $(\mathfrak{C} \times W_0, \mathfrak{C} \times (V \backslash W_0))$
end if
$P_0 \leftarrow$ FixpointIter$(G_{\triangleright 0}, d-1, I_\nu, I_\mu)$
$P_1 \leftarrow$ FixpointIter$(G_{\triangleright 1}, d-1, I_\nu, I_\mu)$
$(\mathfrak{C}^a, \mathfrak{C}^b) \leftarrow$ partition $\mathfrak{C}$ in non-empty parts
$(W_0^a, W_1^a) \leftarrow$ FixpointIterMBR$(G \cap \mathfrak{C}^a, P_1, P_0)$
$(W_0^b, W_1^b) \leftarrow$ FixpointIterMBR$(G \cap \mathfrak{C}^b, P_1, P_0)$
$W_0 \leftarrow W_0^a \cup W_0^b$
$W_1 \leftarrow W_1^a \cup W_1^b$
return (W_0, W_1)

Algorithm 2 FixpointIter(*PG* G, i, I_ν, I_μ)

if $i = -1$ **then**
  return $\{v \in V_0 \mid \exists w \in V : (v, w) \in E \wedge w \in X_{\Omega(w)}\} \cup \{v \in V_1 \mid \forall w \in V : (v, w) \implies w \in X_{\Omega(w)}\}$
else
  $X_i \leftarrow I_\nu$ if i is even I_μ otherwise
  repeat
    $X_i' \leftarrow X_i$
    $X_i \leftarrow$ FixpointIter$(G, i-1, I_\nu, I_\mu)$
  until $X_i = X_i'$
  return X_i
end if

$$F_0 = \{v \in V_0 \mid \exists w \in V : (v, w) \in E \wedge w \in X_{\Omega(w)}\} \cup \{v \in V_1 \mid \forall w \in V : (v, w) \implies w \in X_{\Omega(w)}\}$$

$$W_0 = \sigma X_{d-1} \dots \mu X_1 . \nu X_0 . F_0$$

Every X_i has fixpoint operator ν if i is even and μ if i is odd.
Note that fixpoint iteration works for games with deadlocks.

Set P_α calculated by the algorithm is winning for player α for all configuration in $\mathfrak{C}$.

So the winning set calculation can be written as:

$$W_0 = (\sigma X_{d-1} \dots \mu X_1.\nu X_0.F_0) \cup P_0 \cap P_1$$

which is equal to

$$W_0 = \sigma X_{d-1} \dots \mu X_1.\nu X_0.(F_0 \cup P_0 \cap P_1)$$

which is equal to

$$W_0 = \sigma X_{d-1} \dots P_1 \cap \mu X_1.P_0 \cup \nu X_0.P_1 \cap F_0$$

From which we can conclude that it is correct to let least fixed point variables start at P_0 and greatest fixed point variables at P_1.