Eindhoven University of Technology
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Verifying SPLs using parity games expressing variability

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Abstract

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1. Introduction

Model verification techniques can be used to improve the quality of software. These techniques require the behaviour of the software to be modelled, these models can then be checked to verify that it behaves conforming to some formally specified requirement. These verification techniques are well studied, specifically techniques to verify a single software product.

Software product lines (SPLs) are systems that can be configured to result in different variants of the same system [10, 29]. So SPLs describe *families* of software products where the products originate from the same system and often times have a lot of commonalities. The difference between the products in a family is called the *variability* of the family [37]. A family of products can be verified by using traditional verification techniques to verify every single product independently. However, verifying models is expensive in term of computing costs and the number of products in an SPL can grow large, therefore having to verify every single product independently is undesirable [9]. In this thesis we explore techniques to verify a family of software products in a collective manner that exploits commonalities between the different products.

A common way of modelling the behaviour of software is by using *labelled transition systems* (LTSs). [19] While LTSs can model behaviour well they cannot model variability. Efforts to also model variability include modal transition systems [12, 13, 35], I/O automata [24, 21] and *featured transition systems* (FTSs) [9, 6]. Specifically the latter is well suited to model all the different behaviours of the software products as well as the variability of the entire system in a single model. FTSs use *features* to express variability; a feature is an option that can be turned on or off for the system. In the context of FTSs, a set of features is synonymous with a software product; an FTS describes the behaviour of a software product by enabling and disabling parts of the system based on the which features are enabled.

There are numerous temporal logics that can be used to formally express requirements. Examples include LTL, CTL, CTL* and modal μ-calculus [28, 1, 19]. Of the different temporal logics, the modal μ-calculus is the most expressive one; it subsumes the other temporal logics [25]. In this thesis we aim to efficiently verify requirements expressed as modal μ-calculus features for all the products in an SPL. We partition the products, described by an FTS, such that all products in one of the partition satisfy the requirements and the products in the other partition do not.

Parity games can be used to determine if an LTS behaves according to a modal μ-calculus formula. Parity games are directed graphs that express a game played by two players [3]. Every vertex in the graph is won by exactly one of the players and a parity game is *solved* when it is determined for every vertex who the winner is. A parity game can be constructed from an LTS and a modal μ-calculus formula such that solving the parity game provides the information needed to determine if the LTS behaves according to the formula.

We introduce a variation of parity games, called *variability parity games* (VPGs). A VPG expresses variability similar to how an FTS expresses variability. However, instead of using features a VPG expresses variability through *configurations.* Parity games have a winner for every vertex, VPGs have a winner for every vertex configuration combination. A VPG is solved when it is determined for every vertex configuration combination who the winner is. We introduce a way of constructing a VPG from an FTS and a modal μ-calculus formula such that solving the VPG provides the information needed to determine which products, described by the FTS, behave according to the formula.

We introduce numerous algorithms to solve VPGs. We also introduce algorithms to partially solve VPGs, in this case we only determine the winner of the vertex configuration combinations that are needed to determine which products, described by the FTS, behave according to the formula. This technique is called *locally* solving a VPG. We can also locally solve a parity game, where we only determine the winner of the vertex that is needed to determine if the LTS behaves according to the formula.

Finally we implement the algorithms and compare their performances. We use two SPLs to create a number of VPGs. We compare the time it takes the algorithms to solve the VPGs to the time it takes to verify every product in the SPLs independently. For the independent verification approach we create parity games for all the products and requirements; these parity games are solved using existing parity game algorithms.

2. Related work

Previous work has been done to verify SPL, we discuss four notable contributions.

In [9] a method is introduced to verify for which products in an FTS an LTL property holds. It does so by constructing a Büchi automaton representing the complement of the LTL property and checking if the synchronous product of the automaton and the transition system has an empty language [38]. When applying this method to verify LTSs the reachability of the synchronous product is explored. For FTSs [9] introduces a reachability definition that determines for every product if a state is reachable. It is observed that a symbolic representation of the sets of products is advantageous when keeping track of the reachability. The paper uses the minepump [23] example to perform an experimental evaluation and find a substantial gain using verifying a family of products collectively as opposed to independently.

This work is expended upon in [6]. The performance of such an approach is further elaborated upon and it is confirmed that a collective approach indeed outperforms an independent approach. Furthermore an extension to LTL is presented, namely featured LTL (fLTL). fLTL parametrizes LTL to be able to express properties in terms of features. Using this language one can distinguish between products when expressing temporal requirements.

In [8]

Todo:

- Featured languages: fLTL [6], fCTL [8] and feature μ-calculus [33] and associated verification work.
- Modal transition systems
- I/O automata

3. Preliminaries

3.1 Fixed-point theory

A fixed-point of a function is an element in the domain of that function such that the function maps to itself for that element. Fixed-points are used in model verification as well as in some parity game algorithms.

Fixed-point theory goes hand in hand with lattice theory which we introduce first.

3.1.1 Lattices

We introduce definitions for ordering and lattices taken from [2].

Definition 3.1 ([2]). *A partial order is a binary relation $x \leq y$ on set S where for all $x, y, z \in S$ we have:*

- *$x \leq x$. (Reflexive)*
- *If $x \leq y$ and $y \leq x$, then $x = y$. (Antisymmetric)*
- *If $x \leq y$ and $y \leq z$, then $x \leq z$. (Transitive)*

Definition 3.2 ([2]). *A partially ordered set is a set S and a partial order $\leq$ for that set, we denote a partially ordered set by $\langle S, \leq \rangle$.*

Definition 3.3 ([2]). *Given partially ordered set $\langle P, \leq \rangle$ and subset $X \subseteq P$. An upper bound to X is an element $a \in P$ such that $x \leq a$ for every $x \in X$. A least upper bound to X is an upper bound $a \in P$ such every other upper bound is larger or equal to a.*

The term least upper bound is synonymous with the term supremum, we write $\sup\{S\}$ to denote the supremum of set S.

Definition 3.4 ([2]). *Given partially ordered set $\langle P, \leq \rangle$ and subset $X \subseteq P$. A lower bound to X is an element $a \in P$ such that $a \leq x$ for every $x \in X$. A greatest lower bound to X is a lower bound $a \in P$ such that every other lower bound is smaller or equal to a.*

The term greatest lower bound is synonymous with the term infimum, we write $\inf\{S\}$ to denote the infimum of set S.

Definition 3.5 ([2]). *A lattice is a partially ordered set where any two of its elements have a supremum and an infimum.*

Definition 3.6 ([2]). *A complete lattice is a partially ordered set in which every subset has a supremum and an infimum.*

Definition 3.7 ([2]). *Given a lattice $\langle D, \leq \rangle$, function $f : D \to D$ is monotonic if and only if for all $x \in D$ and $y \in D$ it holds that if $x \leq y$ then $f(x) \leq f(y)$.*

3.1.2 Fixed-points

Fixed-points are formally defined as follows:

Definition 3.8. *Given function $f : D \to D$ the value $x \in D$ is a fixed-point for f if and only if $f(x) = x$. Furthermore x is the least fixed-point for f if every other fixed-point for f is greater or equal to x and dually x is the greatest fixed-point for f if every other fixed-point f is less or equal to x.*

The Knaster-Tarski theorem states that least and greatest fixed-points exist for some domain and function given that a few conditions hold.

Theorem 3.1 (Knaster-Tarski[32]). *Let*

- *$\langle A, \leq \rangle$ be a complete lattice,*
- *f be a monotonic function on A to A,*
- *P be the set of all fixed-points of f.*

Then the set P is not empty and the system $\langle P, \leq \rangle$ is a complete lattice; in particular we have

$$\sup P = \sup\{x \mid f(x) \geq x\} \in P$$

and

$$\inf P = \inf\{x \mid f(x) \leq x\} \in P$$

3.2 Model verification

It is difficult to develop correct software, one way to improve reliability of software is through model verification; the behaviour of software is specified in a model and formal verification techniques are used to show that the behaviour adheres to certain requirements. In this section we inspect how to model behaviour and how to specify requirements.

Behaviour can be modelled as a *labelled transition system* (LTS). An LTS consists of states in which the system can find itself and transitions between states. Transitions represent the possible state change of the system. Transitions are labelled with actions that indicate what kind of change is happening. Formally we define an LTS as follows.

Definition 3.9 ([19]). *A labelled transition system (LTS) is a tuple $M = (S, Act, trans, s_0)$, where:*

- *S is a finite set of states,*
- *Act a finite set of actions,*
- *$trans \subseteq S \times Act \times S$ is the transition relation with $(s, a, s') \in trans$ denoted by $s \xrightarrow{a} s'$,*
- *$s_0 \in S$ is the initial state.*

An LTS is usually depicted as a graph where the vertices represent the states, the edges represent the transitions, edges are labelled with actions and an edge with no origin state indicates the initial state. Such a representation is depicted in the example below.

Example 3.1 ([34])**.** *Consider the behaviour of a coffee machine that accepts a coin, after which it serves a standard coffee, this can be repeated infinitely often.*

The behaviour can be modelled as an LTS that has two states: in the initial state it is ready to accept a coin and in the second state it is ready to serve a standard coffee. We introduce two actions: ins, which represents a coin being inserted, and std, which represents a standard coffee being served. We get the following LTS which is also depicted in Figure 3.1.

$$(\{s_1, s_2\}, \{std, ins\}, \{(s_1, ins, s_2), (s_2, std, s_1)\}, s_1)$$

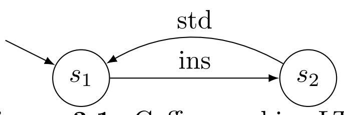


Figure 3.1: Coffee machine LTS

LTSs might be non-deterministic, meaning that from a state there might be multiple transitions that can be taken. Moreover multiple transitions with the same action can be taken. This is depicted in the example below.

Example 3.2. *We extend the coffee machine example such that at some point the coffee machine can be empty and needs to be filled before the system is ready to receive a coin again. This LTS is depicted in Figure 3.2. When the std transition is taken from state s_2 it is non-determined in which states the system ends.*

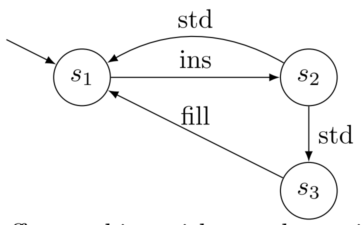


Figure 3.2: Coffee machine with non-deterministic behaviour

A system can be verified by checking if its behaviour adheres to certain requirements. The behaviour can be modelled in an LTS. Requirements can be expressed in a temporal logic; with a temporal logic we can express certain propositions with a time constraint such as *always*, *never* or *eventually.* For example (relating to the coffee machine example) we can express the following constraint: "After a coin is inserted the machine always serves a standard coffee immediately afterwards". The most expressive temporal logic is the modal μ-calculus. A modal μ-calculus formula is expressed over a set of actions and a set of variables.

We define the syntax of the modal μ-calculus below. Note that the syntax is in positive normal form, i.e. no negations.

Definition 3.10 ([19])**.** *A modal μ-calculus formula over the set of actions Act and a set of variables $\mathcal{X}$ is defined by*

$$\varphi = \top \mid \bot \mid X \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \langle a \rangle \varphi \mid [a]\varphi \mid \mu X.\varphi \mid \nu X.\varphi$$

with $a \in Act$ and $X \in \mathcal{X}$.

The modal μ-calculus contains boolean constants $\top$ and $\bot$, propositional operators $\vee$ and $\wedge$, modal operators $\langle\,\rangle$ and $[\,]$ and fixed-point operators μ and ν.

A variable $X \in \mathcal{X}$ *occurs free* in formula ϕ if and only if X occurs in ϕ such that X is not a sub-formula of $\mu X.\phi'$ or $\nu X.\phi'$ in ϕ. A formula is *closed* if and only if there are no variables that occurs free.

A formula can be interpreted in the context of an LTS, such an interpretation results in a set of states in which the formula holds. Given formula φ we define the interpretation of φ as $[\![\varphi]\!]^\eta \subseteq S$ where $\eta : \mathcal{X} \to 2^S$ maps a variable to a set of states. We can assign $S' \subseteq S$ to variable X in η by writing $\eta[X := S']$, i.e. $(\eta[X := S'])(X) = S'$.

Definition 3.11 ([19])**.** *For LTS $(S, Act, trans, s_0)$ we inductively define the interpretation of a modal μ-calculus formula φ, notation $[\![\varphi]\!]^\eta$, where $\eta : \mathcal{X} \to 2^S$ is a variable valuation, as a set of states where φ is valid, by:*

$$\begin{aligned}
[\![\top]\!]^\eta &= S\\
[\![\bot]\!]^\eta &= \emptyset\\
[\![\varphi_1 \wedge \varphi_2]\!]^\eta &= [\![\varphi_1]\!]^\eta \cap [\![\varphi_2]\!]^\eta\\
[\![\varphi_1 \vee \varphi_2]\!]^\eta &= [\![\varphi_1]\!]^\eta \cup [\![\varphi_2]\!]^\eta\\
[\![\langle a\rangle\varphi]\!]^\eta &= \{s \in S | \exists_{s' \in S}\ s \xrightarrow{a} s' \wedge s' \in [\![\varphi]\!]^\eta\}\\
[\![[a]\varphi]\!]^\eta &= \{s \in S | \forall_{s' \in S}\ s \xrightarrow{a} s' \implies s' \in [\![\varphi]\!]^\eta\}\\
[\![\mu X.\varphi]\!]^\eta &= \bigcap\{f \subseteq S | f \supseteq [\![\varphi]\!]^{\eta[X:=f]}\}\\
[\![\nu X.\varphi]\!]^\eta &= \bigcup\{f \subseteq S | f \subseteq [\![\varphi]\!]^{\eta[X:=f]}\}\\
[\![X]\!]^\eta &= \eta(X)
\end{aligned}$$

Since there are no negations in the syntax we find that every modal μ-calculus formula is monotone, i.e. if we have for $U \subseteq S$ and $U' \subseteq S$ that $U \subseteq U'$ holds then $[\![\varphi]\!]^{\eta[X:=U]} \subseteq [\![\varphi]\!]^{\eta[X:=U']}$ holds for any variable $X \in \mathcal{X}$. Using the Knaster-Tarski theorem (Theorem 3.1) we find that the least and greatest fixed-points always exist.

Given closed formula φ, LTS $M = (S, Act, trans, s_0)$ and $s \in S$ we say that M satisfies formula φ in state s, and write $(M, s) \models \varphi$, if and only if $s \in [\![\varphi]\!]^\eta$. If and only if M satisfies φ in the initial state do we say that M satisfies formula φ and write $M \models \varphi$.

Example 3.3 ([34])**.** *Consider the coffee machine example from Figure 3.1, which we call C, and formula $\varphi = \nu X.\mu Y([ins]Y \wedge [std]X)$ which states that action std must occur infinitely often over all infinite runs. Obviously this holds for the coffee machine, therefore we have $C \models \varphi$.*

3.3 Parity games

A *parity game* is a game played by two players: player 0 (also called player *even*) and player 1 (also called player *odd*). We write $\alpha \in \{0, 1\}$ to denote an arbitrary player and $\overline{\alpha}$ to denote α's opponent, i.e. $\overline{0} = 1$ and $\overline{1} = 0$. A parity game is played on a playing field which is a directed graph where every vertex is owned by either player 0 or player 1. Furthermore every vertex has a natural number, called its *priority*, associated with it.

Definition 3.12 ([3])**.** *A parity game is a tuple (V, V_0, V_1, E, Ω), where:*

- *V is a finite set of vertices partitioned in sets V_0 and V_1, containing vertices owned by player 0 and player 1 respectively,*
- *$E \subseteq V \times V$ is the edge relation,*
- *$\Omega : V \to \mathbb{N}$ is the priority assignment function.*

Parity games are usually represented as a graph where vertices owned by player 0 are shown as diamonds and vertices owned by player 1 are shown as boxes. Furthermore the priorities are depicted as numbers inside the vertices. Such a representation is shown in the example below.

Example 3.4. *Figure 3.3 shows the parity game:*

$$V_0 = \{v_1, v_4, v_5\}, V_1 = \{v_2, v_3\}, V = V_0 \cup V_1$$

$$E = \{(v_1, v_2), (v_2, v_1), (v_1, v_3), (v_2, v_4), (v_3, v_4), (v_3, v_5), (v_4, v_4)\}$$

$$\Omega = \{v_1 \mapsto 2, v_2 \mapsto 3, v_3 \mapsto 0, v_4 \mapsto 0, v_5 \mapsto 1\}$$

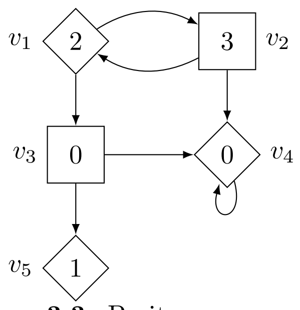


Figure 3.3: Parity game example

A parity game can be played for a vertex $v \in V$, we start by placing a token on vertex v. The player that owns vertex v can choose to move the token along an edge to a vertex $w \in V$ such that $(v, w) \in E$. Again the player that owns vertex w can choose where to move the token next. This is repeated either infinitely often or until a player cannot make a move, i.e. the token is on a vertex with no outgoing edges. Playing in this manner gives a sequence of vertices, called a *path*, starting from vertex v. For path π we write π_i to denote i^{th} vertex in path π. Every path is associated with a winner (either player 0 or 1). If a player α cannot move at some point we get a finite path and player $\overline{\alpha}$ wins the path. If we get an infinite path π then the winner is determined by the parity of the highest priority that occurs infinitely often in the path. Formally we determine the highest priority occurring infinitely often by the following formula.

$$\max\{p \mid \forall_j \exists_i j < i \wedge p = \Omega(\pi_i)\}$$

If the highest priority is odd then player 1 wins the path, if it is even player 0 wins the path.

A path is *valid* if and only if for every $i > 0$ such that π_i exists we have $(\pi_{i-1}, \pi_i) \in E$.

Example 3.5. *Again consider the example in Figure 3.3. If we play the game for vertex v_1 we start by placing a token on v_1. Consider the following exemplary paths where $(w_1 \dots w_m)^\omega$ indicates an infinite repetition of vertices $w_1 \dots w_m$.*

- $\pi = v_1 v_3 v_5$ *is won by player* 1 *since player* 0 *cannot move at* v_5.
- $\pi = (v_1 v_2)^\omega$ *is won by player* 1 *since the highest priority occurring infinitely often is 3.*
- $\pi = v_1 v_3 (v_4)^\omega$ *is won by player* 0 *since the highest priority occurring infinitely often is* 0.

The moves that the players make are determined by their *strategies*. A strategy σ_α determines for a vertex in V_α where the token goes next. We can define a strategy for player α as a partial function $\sigma_\alpha : V^* V_\alpha \to V$ that maps a series of vertices ending with a vertex owned by player α to the next vertex such that for any $\sigma_\alpha(w_0 \dots w_m) = w$ we have $(w_m, w) \in E$. A path π *conforms to* strategy σ_α if for every $i > 0$ such that π_i exists and $\pi_{i-1} \in V_\alpha$ we have $\pi_i = \sigma_\alpha(\pi_0 \pi_1 \dots \pi_{i-1})$.

A strategy is *winning* for player α from vertex v if and only if α is the winner of every valid path starting in v that conforms to σ_α. If such a strategy exists for player α from vertex v we say that vertex v is winning for player α.

Example 3.6. *In the parity game seen in Figure 3.3 vertex v_1 is winning for player 1. Player 1 has a strategy that plays every vertex sequence ending in v_2 to v_1 and plays every vertex sequence ending in v_3 to v_5. Regardless of the strategy for player 0 the path will either end up in v_5 or will pass v_2 infinitely often. In the former case player 1 wins the path because player 0 can not move at v_5. In the latter case the highest priority occurring infinitely often is 3.*

Parity games are known to be positionally determined [3], meaning that every vertex in a parity game is winning for exactly one of the two players. Also every player has a *positional strategy* that is winning starting from each of his/her winning vertices. A positional strategy is a strategy that only takes the current vertex into account to determine the next vertex, it does not look at already visited vertices. Therefore we can consider a strategy for player α as a function $\sigma_\alpha : V_\alpha \to V$. Finally it is decidable for each of the vertices in a parity who the winner is [3].

A parity game is *solved* if the vertices are partitioned in two sets, namely W_0 and W_1, such that every vertex in W_0 is winning for player 0 and every vertex in W_1 is winning for player 1. We call these sets the *winning sets* of a parity game. Solving parity games is in complexity class UP $\cap$ CO-UP and NP $\cap$ CO-NP [20]. No polynomial algorithms are known, however finding a polynomial algorithm does not prove P=NP.

Finally parity games are considered *total* if and only if every vertex has at least one outgoing edge. Playing a total parity game always results in an infinite path. We can make a non-total parity game total by adding two sink vertices: l_0 and l_1. Each sink vertex has only one outgoing edge, namely to itself. Vertex l_0 has priority 1 and vertex l_1 has priority 0. Clearly if the token ends up in l_α then player α looses the game because with only one outgoing edge we only get a single priority that occurs infinitely often, namely priority $\overline{\alpha}$. For every vertex $v \in V_\alpha$ that does not have an outgoing edge we create an edge from v to l_α. In the original game player α lost when the token was in vertex v because he/she could not move any more. In the total game player α can only play to l_α from v where he/she still looses. So using this method vertices in the total game have the same winner as they had in the original game (except for l_0 and l_1 which did not exist in the original game). In general we try to only work with total games because no distinction is required between finite paths and infinite paths when reasoning about them, however we will encounter some scenario's where non-total games are still considered.

3.3.1 Relation between parity games and model checking

Verifying LTSs against a modal μ-calculus formula can be done by solving a parity game. This is done by translating an LTS in combination with a formula to a parity game, the solution of the parity game provides the information needed to conclude if the model satisfies the formula. This relation is depicted in Figure 3.4.

$$\begin{tikzcd}
\text{LTS } M \arrow[dr] & & \text{Modal } \mu\text{-calculus formula } \varphi \arrow[dl] \\
& \text{PG} \arrow[d] & \\
& M \overset{?}{\models} \varphi &
\end{tikzcd}$$

Figure 3.4: LTS verification using parity games

We consider a method of creating parity games from an LTS and a modal μ-calculus formula such that there is a special vertex w in the parity game that indicates if the LTS satisfies the formula; if and only if w is won by player 0 is the formula satisfied.

First we introduce the notion of unfolding. A fixed-point formula $\mu X.\varphi$ can be unfolded, resulting in formula φ where every occurrence of X is replaced by $\mu X.\varphi$, denoted by $\varphi[X := \mu X.\varphi]$. Interpreting a fixed-point formula results in the same set as interpreting its unfolding as shown in [3]; i.e. $[\![\mu X.\varphi]\!]^\eta = [\![\varphi[X := \mu X.\varphi]]\!]^\eta$. The same holds for the fixed-point operator ν.

Next we define the Fischer-Ladner closure for a closed μ-calculus formula [31, 14]. The Fischer-Ladner closure of φ is the set $FL(\varphi)$ of closed formulas containing at least φ. Furthermore for every formula ψ in $FL(\varphi)$ it holds that for every direct subformula ψ' of ψ there is a formula in $FL(\varphi)$ that is equivalent to ψ'.

Definition 3.13. *The Fischer-Ladner closure of closed μ-calculus formula φ is the smallest set $FL(\varphi)$ satisfying the following constraints:*

- $\varphi \in FL(\varphi)$,
- *if* $\varphi_1 \vee \varphi_2 \in FL(\varphi)$ *then* $\varphi_1, \varphi_2 \in FL(\varphi)$,
- *if* $\varphi_1 \wedge \varphi_2 \in FL(\varphi)$ *then* $\varphi_1, \varphi_2 \in FL(\varphi)$,
- *if* $\langle a \rangle \varphi' \in FL(\varphi)$ *then* $\varphi' \in FL(\varphi)$,
- *if* $[a]\varphi' \in FL(\varphi)$ *then* $\varphi' \in FL(\varphi)$,
- *if* $\mu X.\varphi' \in FL(\varphi)$ *then* $\varphi'[X := \mu X.\varphi'] \in FL(\varphi)$ *and*
- *if* $\nu X.\varphi' \in FL(\varphi)$ *then* $\varphi'[X := \nu X.\varphi'] \in FL(\varphi)$.

We also define the alternation depth of a formula.

Definition 3.14 ([3]). *The dependency order on bound variables of φ is the smallest partial order such that $X \leq_\varphi Y$ if X occurs free in $\sigma Y.\psi$. The alternation depth of a μ-variable X in formula φ is the maximal length of a chain $X_1 \leq_\varphi \cdots \leq_\varphi X_n$ where $X = X_1$, variables $X_1, X_3, \ldots$ are μ-variables and variables $X_2, X_4, \ldots$ are ν-variables. The alternation depth of a ν-variable is defined similarly. The alternation depth of formula φ, denoted $adepth(\varphi)$, is the maximum of the alternation depths of the variables bound in φ, or zero if there are no fixed-points.*

Example 3.7. *Consider the formula $\varphi = \nu X.\mu Y.([ins]Y \wedge [std]X)$ which states that for an LTS with $Act = \{ins, std\}$ the action std must occur infinitely often over all infinite runs. Since X occurs free in $\mu Y.([ins]Y \wedge [std]X)$ we have $adepth(Y) = 1$ and $adepth(X) = 2$.*

As shown in [3] it holds that formula $\mu X.\psi$ has the same alternation depth as its unfolding $\psi[X := \mu X.\psi]$. Similarly for the greatest fixed-point.

Next we define the transformation from an LTS and a formula to a parity game.

Definition 3.15 ([3]). *LTS2PG(M, φ) converts LTS $M = (S, Act, trans, s_0)$ and closed formula φ to a parity game (V, V_0, V_1, E, Ω).*

Vertices in the parity game are represented as pairs of states and sub-formulas. A vertex is created for every state with every formula in the Fischer-Ladner closure of φ. We define the set of vertices:

$$V = S \times FL(\varphi)$$

Vertices have the following owners, successors and priorities:

Vertex	*Owner*	*Successor(s)*	*Priority*
$(s, \bot)$	*0*		*0*
$(s, \top)$	*1*		*0*
$(s, \psi_1 \vee \psi_2)$	*0*	(s, ψ_1) *and* (s, ψ_2)	*0*
$(s, \psi_1 \wedge \psi_2)$	*1*	(s, ψ_1) *and* (s, ψ_2)	*0*
$(s, \langle a \rangle \psi)$	*0*	(s', ψ) *for every* $s \xrightarrow{a} s'$	*0*
$(s, [a]\psi)$	*1*	(s', ψ) *for every* $s \xrightarrow{a} s'$	*0*
$(s, \mu X.\psi)$	*1*	$(s, \psi[X := \mu X.\psi])$	$2\lfloor adepth(X)/2 \rfloor + 1$
$(s, \nu X.\psi)$	*1*	$(s, \psi[X := \nu X.\psi])$	$2\lfloor adepth(X)/2 \rfloor$

Since the Fischer-Ladner formula's are closed we never get a vertex (s, X).

Example 3.8. *Consider LTS M in Figure 3.5 and formula $\varphi = \mu X.([a]X \vee \langle b \rangle \top)$ expressing that on any path reached by a's we can eventually do a b action.*

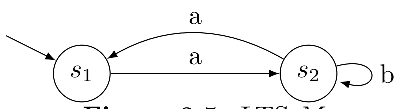


Figure 3.5: LTS M

The resulting parity game is depicted in Figure 3.6. Let V denote the set of vertices of this parity game. There are two vertices with more than one outgoing edge. From vertex $(s_1, [a](\mu X.\phi) \vee$

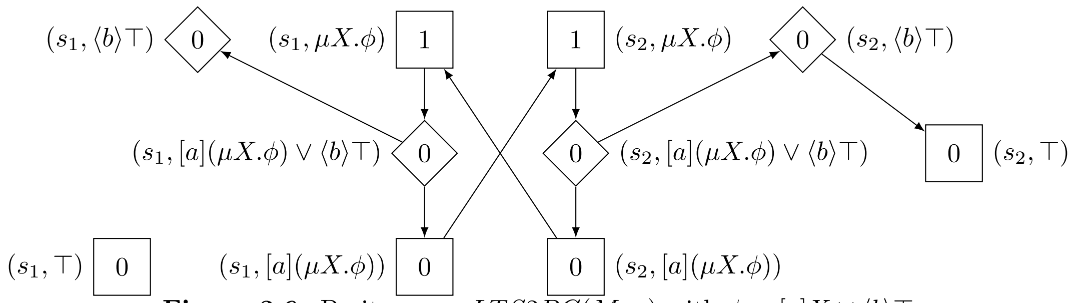
$\langle b\rangle\top)$ player 0 does not want to play to $(s_1, \langle b\rangle\top)$ because he/she will not be able to make another move and looses the path. From vertex $(s_2, [a](\mu X.\phi) \vee \langle b\rangle\top)$ player 0 can play to $(s_2, \langle b\rangle\top)$ to bring the play in $(s_2, \top)$ to win the path. We get the following winning sets:

$$W_1 = \{(s_1, \langle b\rangle\top)\}$$
$$W_0 = V\backslash W_1$$

With the strategies σ_0 for player 0 and σ_1 for player 1 being (vertices with one outgoing edge are omitted):

$$\begin{aligned}\sigma_0 = \{&(s_1, [a](\mu X.\phi) \vee \langle b\rangle\top) \mapsto (s_1, [a](\mu X.\phi)),\\ &(s_2, [a](\mu X.\phi) \vee \langle b\rangle\top) \mapsto (s_2, \langle b\rangle\top)\}\\ \sigma_1 = \{&\}\end{aligned}$$

Note that the choice where to go from $(s_2, [a](\mu X.\phi) \vee \langle b\rangle\top)$ does not matter for the winning sets.



$(s_1, \langle b\rangle\top)$
0
$(s_1, \mu X.\phi)$
1
1
$(s_2, \mu X.\phi)$
0
$(s_2, \langle b\rangle\top)$
$(s_1, [a](\mu X.\phi) \vee \langle b\rangle\top)$
0
0
$(s_2, [a](\mu X.\phi) \vee \langle b\rangle\top)$
0
$(s_2, \top)$
$(s_1, \top)$
0
$(s_1, [a](\mu X.\phi))$
0
0
$(s_2, [a](\mu X.\phi))$

Figure 3.6: Parity game $LTS2PG(M, \varphi)$ with $\phi = [a]X \vee \langle b\rangle\top$

Parity games created in this manner relate back to the model verification question; state s in LTS M satisfies φ if and only if player 0 wins vertex (s, φ). This is formally stated in the following theorem which is proven in [3].

Theorem 3.2 ([3])**.** *Given LTS $M = (S, Act, trans, s_0)$, modal μ-calculus formula φ and state $s \in S$ it holds that $(M, s) \models \varphi$ if and only if $(s, \varphi) \in W_0$ for the game $LTS2PG(M, \varphi)$.*

3.3.2 Globally and locally solving parity games

Parity games can be solved *globally* or *locally*; globally solving a parity game means that for every vertex in the game it is determined who the winner is. Locally solving a parity game means that for a specific vertex in the game it is determined who the winner is. For some applications of parity games, including model checking, there is a specific vertex that needs to be solved to solve the original problem. Locally solving the parity game is sufficient in such cases to solve the original problem.

Most parity game algorithms (including the two considered next) are concerned with globally solving. When talking about solving a parity game we talk about globally solving it unless stated otherwise.

3.3.3 Parity game algorithms

Various algorithms for solving parity games are known, we introduce two of them. First Zielonka's recursive algorithm which is well studied and generally considered to be one of the best performing parity game algorithms in practice [36, 16]. We also inspect the fixed-point iteration algorithm which tends to perform well for model-checking problems with a low number of distinct priorities [30].

Zielonka's recursive algorithm

First we consider Zielonka's recursive algorithm [41, 26], which solves total parity games. Pseudo code is presented in Algorithm 1. Zielonka's recursive algorithm has a worst-case time complexity of $O(e * n^d)$ where e is the number of edges, n the number of vertices and d the number of distinct priorities [15].

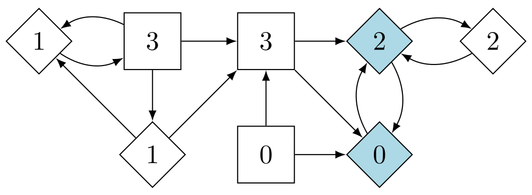
Algorithm 1 RecursivePG(*parity game* $G = (V, V_0, V_1, E, \Omega)$)

```
1:  if $V = \emptyset$ then
2:      return $(\emptyset, \emptyset)$
3:  end if
4:  $h \leftarrow \max\{\Omega(v) \mid v \in V\}$
5:  $\alpha \leftarrow 0$ if $h$ is even and 1 otherwise
6:  $U \leftarrow \{v \in V \mid \Omega(v) = h\}$
7:  $A \leftarrow \alpha\text{-}Attr(G, U)$
8:  $(W'_0, W'_1) \leftarrow$ RecursivePG($G \backslash A$)
9:  if $W'_{\overline{\alpha}} = \emptyset$ then
10:     $W_\alpha \leftarrow A \cup W'_\alpha$
11:     $W_{\overline{\alpha}} \leftarrow \emptyset$
12: else
13:     $B \leftarrow \overline{\alpha}\text{-}Attr(G, W'_{\overline{\alpha}})$
14:     $(W''_0, W''_1) \leftarrow$ RecursivePG($G \backslash B$)
15:     $W_\alpha \leftarrow W''_\alpha$
16:     $W_{\overline{\alpha}} \leftarrow W''_{\overline{\alpha}} \cup B$
17: end if
18: return $(W_0, W_1)$
```

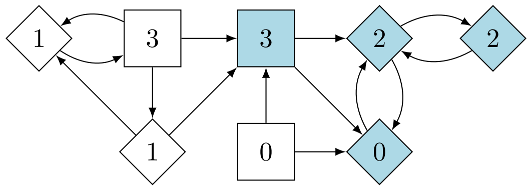
The algorithm solves G by taking the set of vertices with the highest priority and choosing player α such that α has the same parity as the highest priority. Next the algorithm finds set A such that player α can force the play to one of these high priority vertices. Next this set of vertices is removed from G and the resulting subgame G' is solved recursively.

If G' is entirely won by player α then we distinguish three cases for any path played in G. Either the path eventually stays in G', A is infinitely often visited or the path eventually stays in A. In the first case player α wins because game G' was entirely won by player α. In the second and third case player α can play to the highest priority from A. The highest priority, which has parity α, is visited infinitely often and player α wins.

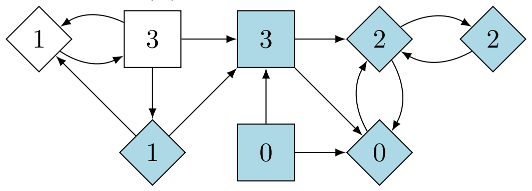
If G' is not entirely won by player α we consider winning sets (W'_0, W'_1) of subgame G'. Vertices in set $W'_{\overline{\alpha}}$ are won by player $\overline{\alpha}$ in G' but are also won by player $\overline{\alpha}$ in G. The algorithm tries to find all the vertices in G such that player $\overline{\alpha}$ can force the play to a vertex in $W'_{\overline{\alpha}}$ and therefore



(a) Set $U = U_0$ highlighted



(b) Set U_1 highlighted



(c) Set $U_2 = 0\text{-}\mathit{Attr}(G, U)$ highlighted

Figure 3.7: Game G showing the attractor calculation for $0\text{-}\mathit{Attr}(G, U)$

winning the game. We now have a set of vertices that are definitely won by player $\overline{\alpha}$ in game G. In the rest of the game player α can keep the play from $W'_{\overline{\alpha}}$ so the algorithm solves the rest of the game recursively to find the complete winning sets for game G.

A complete explanation of the algorithm can be found in [41], we do introduce definitions for the attractor set and for subgames.

An attractor set is a set of vertices $A \subseteq V$ calculated for player α given set $U \subseteq V$ where player α has a strategy to force the play starting in any vertex in $A \backslash U$ to a vertex in U. Such a set is calculated by adding vertices owned by player α that have an edge to the attractor set and adding vertices owned by player $\overline{\alpha}$ that only have edges to the attractor set.

Definition 3.16 ([41]). *Given parity game $G = (V, V_0, V_1, E, \Omega)$ and a non-empty set $U \subseteq V$ we define α-Attr(G, U) such that*

$$U_0 = U$$

For $i \geq 0$:

$$\begin{aligned} U_{i+1} = U_i &\cup \{v \in V_\alpha \mid \exists v' \in V : v' \in U_i \wedge (v, v') \in E\} \\ &\cup \{v \in V_{\overline{\alpha}} \mid \forall v' \in V : (v, v') \in E \implies v' \in U_i\} \end{aligned}$$

Finally:

$$\alpha\text{-}\mathit{Attr}(G, U) = \bigcup_{i \geq 0} U_i$$

Example 3.9. *Figure 3.7 shows an example parity game in which an attractor set is calculated for player 0. For set U_2 no more vertices can be attracted so we found the complete attractor set.*

The algorithm also creates subgames, where a set of vertices is removed from a parity game to create a new parity game.

Definition 3.17 ([41])**.** *Given a parity game $G = (V, V_0, V_1, E, \Omega)$ and $U \subseteq V$ we define the subgame $G \backslash U$ to be the game $(V', V_0', V_1', E', \Omega)$ with:*

- $V' = V \backslash U$,
- $V_0' = V_0 \cap V'$,
- $V_1' = V_1 \cap V'$ *and*
- $E' = E \cap (V' \times V')$.

Note that a subgame is not necessarily total, however the recursive algorithm always creates subgames that are total [41].

Fixed-point iteration algorithm

Parity games can be solved by solving an alternating fixed-point formula [39]. Consider parity game $G = (V, V_0, V_1, E, \Omega)$ with d distinct priorities. We can apply *priority compression* to make sure every priority in G maps to a value in $\{0, \dots, d-1\}$ or $\{1, \dots, d\}$ [17, 4]. We assume without loss of generality that the priorities map to $\{0, \dots, d-1\}$ and that $d-1$ is even.

Consider the following formula

$$S(G) = \nu Z_{d-1}.\mu Z_{d-2}.\dots.\nu Z_0.F_0(G, Z_{d-1}, \dots, Z_0)$$

with

$$\begin{aligned} F_0(G = (V, V_0, V_1, E, \Omega), Z_{d-1}, \dots, Z_0) =& \{v \in V_0 \mid \exists_{w \in V}\ (v, w) \in E \wedge w \in Z_{\Omega(w)}\} \\ & \cup \{v \in V_1 \mid \forall_{w \in V}\ (v, w) \in E \implies w \in Z_{\Omega(w)}\} \end{aligned}$$

where $Z_i \subseteq V$. The formula $\nu X.f(X)$ solves the greatest fixed-point of X in f, similarly $\mu X.f(X)$ solves the least fixed-point of X in f. As shown in [39] formula $S(G)$ calculates the set of vertices winning for player 0 in parity game G.

To understand the formula we consider sub-formula $\nu Z_0.F_0(Z_{d-1}, \dots, Z_0)$. This formula holds for vertices from which player 0 can either force the play into a node with priority $i > 0$ for which Z_i holds or the player can stay in vertices with priority 0 indefinitely. The formula $\mu Z_0.F_0(Z_{d-1}, \dots, Z_0)$ holds for vertices from which player 0 can force the play into a node with priority $i > 0$, for which Z_i holds, in finitely many steps. By alternating fixed-points the formula allows infinitely many consecutive stays in even vertices and finitely many consecutive stays in odd vertices. For an extensive treatment we refer to [39].

We further inspect formula S. Given game G, consider the following sub-formulas:

$$S^{d-1}(Z_{d-1}) = \mu Z_{d-2}.S^{d-2}(Z_{d-2})$$

$$S^{d-2}(Z_{d-2}) = \nu Z_{d-3}.S^{d-3}(Z_{d-3})$$

$$\dots$$

$$S^0(Z_0) = F_0(Z_{d-1}, \dots, Z_0)$$

The fixed-point variables are all elements of 2^V, therefore we have for every sub-formula the following type:

$$S^i(Z_i) : 2^V \to 2^V$$

Furthermore, since V is finite, the partially ordered set $\langle 2^V, \subseteq \rangle$ is a complete lattice; for every subset $X \subseteq 2^V$ we have infimum $\bigcap_{x \in X} x$ and supremum $\bigcup_{x \in X} x$. Finally every sub-formula $S^i(Z_i)$ is monotonic, i.e. if $S^i(Z_i) \geq S^i(Z_i')$ then $Z_i \geq Z_i'$.

Fixed-point formula's can be solved by *fixed-point iteration.* As shown in [11] we can calculate $\mu X.f(X)$, where f is monotonic in X and $X \in 2^V$, by iterating X:

$$\mu X.f(X) = \bigcup_{i \geq 0} X^i$$

where $X^i = f(X^{i-1})$ for $i > 0$ and $X^0 \subseteq \mu X.f(X)$. So picking the smallest value possible for X_0 will always correctly calculate $\mu X.f(X)$.

Similarly we can calculate fixed-point $\nu X.f(X)$ when f is monotonic in X by iterating X:

$$\nu X.f(X) = \bigcap_{i \geq 0} X^i$$

where $X^i = f(X^{i-1})$ for $i > 0$ and $X^0 \supseteq \nu X.f(X)$. So picking the largest value possible for X_0 will always correctly calculate $\nu X.f(X)$.

Since every subformula is monotonic and maps from a value in 2^V to another value in 2^V we can apply fixed-point iteration to solve the subformula's, we choose initial values $\emptyset$ for least fixed-point variables and V for greatest fixed-point variables.

An algorithm to perform the iteration is presented in [4] and shown in Algorithm 2. This algorithm has a worst-case time complexity of $O(e * n^d)$ where e is the number of edges, n the number of vertices and d the number of distinct priorities.

Algorithm 2 Fixed-point iteration

```
function FPIter($G = (V, V_0, V_1, E, \Omega)$)
    for $i \leftarrow d-1, \ldots, 0$ do
        Init($i$)
    end for
    repeat
        $Z'_0 \leftarrow Z_0$
        $Z_0 \leftarrow$ Diamond() $\cup$ Box()
        $i \leftarrow 0$
        while $Z_i = Z'_i \wedge i < d-1$ do
            $i \leftarrow i+1$
            $Z'_i \leftarrow Z_i$
            $Z_i \leftarrow Z_{i-1}$
            Init($i-1$)
        end while
    until $i = d-1 \wedge Z_{d-1} = Z'_{d-1}$
    return $(Z_{d-1}, V \backslash Z_{d-1})$
end function
```

```
function Init($i$)
    $Z_i \leftarrow \emptyset$ if $i$ is odd, $V$ otherwise
end function
```

```
function Diamond
    return $\{v \in V_0 \mid \exists_{w \in V}(v, w) \in E \wedge w \in Z_{\Omega(w)}\}$
end function
```

```
function Box
    return $\{v \in V_1 \mid \forall_{w \in V}(v, w) \in E \implies w \in Z_{\Omega(w)}\}$
end function
```

3.4 Symbolically representing sets

A set can straightforwardly be represented by a collection containing all the elements that are in the set. We call this an *explicit* representation of a set. We can also represent sets *symbolically* in which case the set of elements is represented by some sort of formula. A typical way to represent a set symbolically is through a boolean formula encoded in a *binary decision diagram* [40, 5].

Example 3.10. *The set $S = \{2, 4, 6, 7\}$ can be expressed by boolean formula:*

$$F(x_2, x_1, x_0) = (\neg x_2 \wedge x_1 \wedge \neg x_0) \vee (x_2 \wedge (x_1 \vee \neg x_0))$$

where x_0, x_1 and x_2 are boolean variables. The formula gives the following truth table:

$\mathbf{x_2x_1x_0}$	$\mathbf{F(x_2, x_1, x_0)}$
000	*0*
001	*0*
010	*1*
011	*0*
100	*1*
101	*0*
110	*1*
111	*1*

The function F defines set S' in the following way: $S' = \{x_2x_1x_0 \mid F(x_2, x_1, x_0) = 1\}$. As we can see set S' contains the same numbers as S, represented in binary format.

We can perform set operations on sets represented as boolean functions by performing logical operations on the functions. For example, given boolean formula's f and g representing sets V and W the formula $f \wedge g$ represents set $V \cap W$.

Given a set S with arbitrary elements we can represent subsets $S' \subseteq S$ as boolean formula's by assigning a number to every element in S and creating a boolean formula that maps boolean variables to true if and only if they represent a number such that the element associated with this number in S is also in S'.

3.4.1 Binary decision diagrams

A boolean function can efficiently be represented as a binary decision diagram (BDD). For a comprehensive treatment of BDDs we refer to [40, 5].

BDDs represent boolean formula's as a directed graph where every vertex represents a boolean variable and has two outgoing edges labelled 0 and 1. Furthermore the graph contains special vertices 0 and 1 that have no outgoing edges. We decide if a boolean variable assignment satisfies the formula by starting in the initial vertex of the graph and following a path until we get to either vertex 0 or 1. Since every vertex represents a boolean formula, we can create a path from the initial vertex by choosing edge 0 at a vertex if the boolean variable represented by that vertex is false in the variable assignment and choosing edge 1 if is true. Eventually we end up in either vertex 0 or 1. In the former case the boolean variable assignment does not satisfy the formula, in the latter it does.

Example 3.11. *Consider the boolean formula in Example 3.10. This formula can be represented as the BDD shown in Figure 3.8. The vertices representing boolean variables are shown as circles and the boolean variables they represent are indicated inside them. The special vertices are represented as squares and the initial vertex is represented by an edge with that has no origin vertex.*

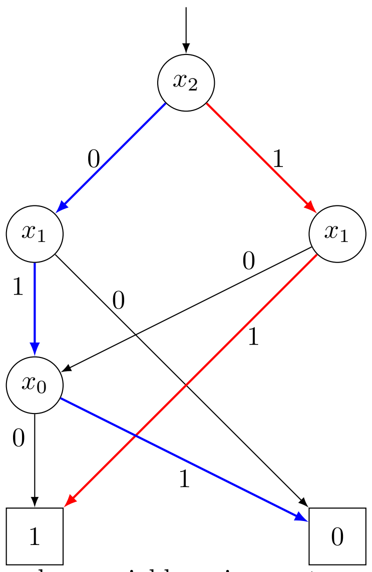


Figure 3.8: BDD highlighting boolean variable assignment $x_2x_1x_0 = 011$ in blue and $x_2x_1 = 11$ in red

The path created from variable assignment $x_2x_1x_0 = 011$ is highlighted in blue in the diagram *and shows that this assignment is indeed not satisfied by the boolean formula. The red path shows the variable assignments* 110 *and* 111*. Determining the path and the outcome for every variable assignment results in the same truth table as seen in Example 3.10.*

Given n boolean variables and two boolean functions encoded as BDDs we can perform binary operations $\vee, \wedge$ on the BDDs in $O(N_a * N_b)$, where N_a and N_b are the number of nodes in the decision diagrams of the two functions. A decision diagram is a tree with n levels, so $N_a = O(2^n)$ and $N_b = O(2^n)$. Therefore with n boolean variables we can perform binary operations $\vee$ and $\wedge$ on them in $O(2^{2n}) = O(m^2)$ where $m = 2^n$ is the maximum set size that can be represented using n variables [40, 5]. The running time specifically depends on the size of the decision diagrams, in general if the boolean functions are simple then the size of the decision diagram is also small and operations can be performed quickly.

4. Problem statement

If we have an SPL with certain requirements that must hold for every product then we want to apply verification techniques to formally verify that indeed every product satisfies the requirements. We could verify every product individually, however verification is expensive in terms of computing time and the number of different products can grow large. Differences in behaviour between products might be very small; large parts of the different products might behave similar. In this thesis we aim to exploit commonalities between products to find a method that verifies an SPL in a more efficient way than verifying every product independently.

First we take a look at a method of modelling the behaviour of the different products in an SPL, namely *featured transition systems* (FTSs). An FTS extends the definition an LTS to express variability, it does so by introducing *features* and *products*. Features are options that can be enabled or disabled for the system. A product is a feature assignments, i.e. a set of features that is enabled for that product. Not all products are valid; some features might be mutually exclusive for example. To express the relation between features one can use feature diagrams as explained in [7]. Feature diagrams offer a nice way of expressing which feature assignments are valid, however for simplicity we represent the collection of valid products simply as a set of feature assignments.

An FTS models the behaviour of multiple products by guarding transitions with boolean expressions over the features such that the transition is only enabled for products that satisfy the guard.

Let $\mathbb{B}(A)$ denote the set of all boolean expressions over the set of boolean variables A, a boolean expression is a function that maps a boolean assignment to either true of false. A boolean expression over a set of features is called a feature expression, it maps a feature assignment, i.e. a product, to either true or false. Given boolean expression f and boolean variable assignment p we write $p \models f$ if and only if f is true for p and write $p \not\models f$ otherwise. Boolean expression $\top$ denotes the boolean expression that is satisfied by all boolean assignments.

Definition 4.1 ([7])**.** *A featured transition system (FTS) is a tuple $M = (S, Act, trans, s_0, N, P, \gamma)$, where:*

- *$S, Act, trans, s_0$ are defined as in an LTS,*
- *N is a non-empty set of features,*
- *$P \subseteq 2^N$ is a non-empty set of products, i.e. feature assignments, that are valid,*
- *$\gamma : trans \to \mathbb{B}(N)$ is a total function, labelling each transition with a feature expression.*

A transition $s \xrightarrow{a} s'$ and $\gamma(s, a, s') = f$ is denoted by $s \xrightarrow{a \mid f} s'$. FTSs are presented similarly as LTSs, the labels of the transition are expanded to represent both the action and the feature expression associated with it.

Example 4.1 ([34])**.** *Consider a coffee machine that has two variants: in the first variant it takes a single coin and serves a standard coffee, in the second variant the machine either serves a standard coffee after a coin is inserted or it takes another coin after which it serves an xxl coffee. Note that there is no variant that only serves xxl coffees. We introduce two features: \$ which, if enabled, allows the coffee machine to serve xxl coffees and € which, if enabled, allows the coffee machine to serve standard coffees. The valid products are: $\{\{€\}, \{€, \$\}\}$. This FTS is depicted in Figure 4.1.*

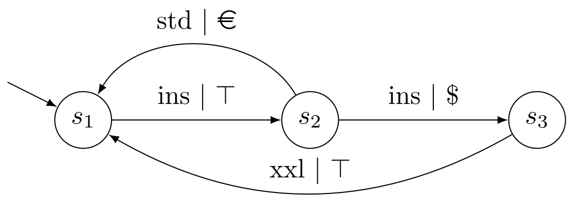
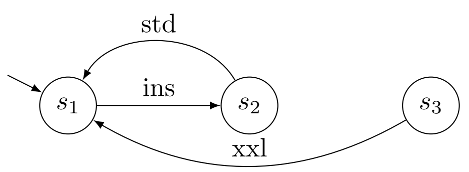


Figure 4.1: Coffee machine FTS C

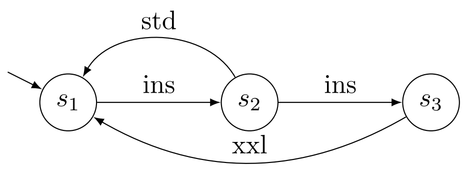
An FTS expresses the behaviour of multiple products. The behaviour of a single product can be derived by simply removing all the transitions from the FTS for which the product does not satisfy the feature expression guarding the transition. We call this a *projection*.

Definition 4.2 ([7])**.** *The projection of FTS $M = (S, Act, trans, s_0, N, P, \gamma)$ onto product $p \in P$, denoted by $M_{|p}$, is the LTS $(S, Act, trans', s_0)$, where $trans' = \{t \in trans \mid p \models \gamma(t)\}$.*

Example 4.2 ([34])**.** *The coffee machine example can be projected to its two products, which results in the LTSs in Figure 4.2.*



(a) C projected to the *euro* product: $C_{|\{€\}}$



(b) C projected to the *euro and dollar* product: $C_{|\{€,\$\}}$

Figure 4.2: Projections of the coffee machine FTS

Problem statement Given an FTS M, that models the behaviour of an SPL, with products P and modal μ-calculus formula φ we want to verify that φ holds for every product in P. Formally we want to find set P_s such that:

- for every $p \in P_s$ we have $M_{|p} \models \varphi$ and
- for every $p \in P \backslash P_s$ we have $M_{|p} \not\models \varphi$.

We aim to find P_s in a way that utilizes the commonalities in behaviour between the different products.

5. Variability parity games

In the preliminaries we have seen how parity games can be used to verify if a modal μ-calculus formula is satisfied by an LTS. A parity game can be constructed such that it contains the information needed to determine if an LTS satisfies a modal μ-calculus formula. We have also seen how an LTS can be extended with transition guards to model the behaviour of multiple LTSs. In this section we introduce *variability parity games* (VPGs); a VPG extends the definition of a parity game much like an FTS extends the definition of an LTS. Similar as to how an FTS expresses multiple LTSs does a VPG express multiple parity games. Moreover we introduce a way of creating VPGs such that every parity game it expresses contains the information needed to determine if a product in an FTS satisfies a modal μ-calculus formula.

We extend parity games such that edges in the game are guarded. Instead of using features, feature expressions and products we choose a syntactically simpler representation and introduce *configurations*. A VPG has a set of configurations and is played for a single configuration. Edges are guarded by sets of configurations; if the VPG is played for a configuration that is in the guard set then the edge is enabled, otherwise it is disabled.

Definition 5.1. *A variability parity game (VPG) is a tuple $(V, V_0, V_1, E, \Omega, \mathfrak{C}, \theta)$, where:*

- *V,V_0,V_1, E and Ω are defined as in a parity game,*
- *$\mathfrak{C}$ is a non-empty finite set of configurations,*
- *$\theta : E \to 2^{\mathfrak{C}} \setminus \emptyset$ is a total function mapping every edge to a set of configurations guarding that edge.*

VPGs are considered total when every vertex has at least one outgoing edge. Since edges are guarded with sets of configurations we also require that for every configuration $c \in \mathfrak{C}$ every vertex has at least one outgoing edge that admits configuration c. Formally a VPG is total if and only if for all $v \in V$:

$$\bigcup\{\theta(v, w) \mid (v, w) \in E\} = \mathfrak{C}$$

VPGs are depicted as parity games with labelled edges that represent the sets of configurations guarding them.

Example 5.1. *Figure 5.1 shows an example of a total VPG with configuration $\mathfrak{C} = \{c_1, c_2, c_3\}$.*

A VPG can be played for a vertex-configuration pair. When playing a VPG for $v \in V$ and $c \in \mathfrak{C}$ we start by placing a token on vertex v. We proceed with the game similar as with a parity game, however player α can only move the token from $v \in V_\alpha$ to $w \in V$ if $(v, w) \in E$ and $c \in \theta(v, w)$. Similar as in a parity game this results in a path. Again the winner is determined by the highest priority occurring infinitely often in the path or, in case of a finite path, the winner is the opponent of the player that cannot make a move any more. Paths might be valid for some configurations but not valid for others, we call a path valid for configuration c if and only if for every $i > 0$ such that π_i exists we have $(\pi_{i-1}, \pi_i) \in E$ and $c \in \theta(\pi_{i-1}, \pi_i)$.

Moves made by the players can again be determined by strategies, however for different configurations for which the game is played different strategies might be needed. So we define a strategy not only for a player but also for a configuration. We define a strategy for player α and configuration $c \in \mathfrak{C}$ as a partial function $\sigma_\alpha^c : V^*V_\alpha \to V$ that maps a series of vertices ending with a vertex owned by player α to the next vertex such that for any $\sigma_\alpha^c(w_0 \dots w_m) = w$ we have

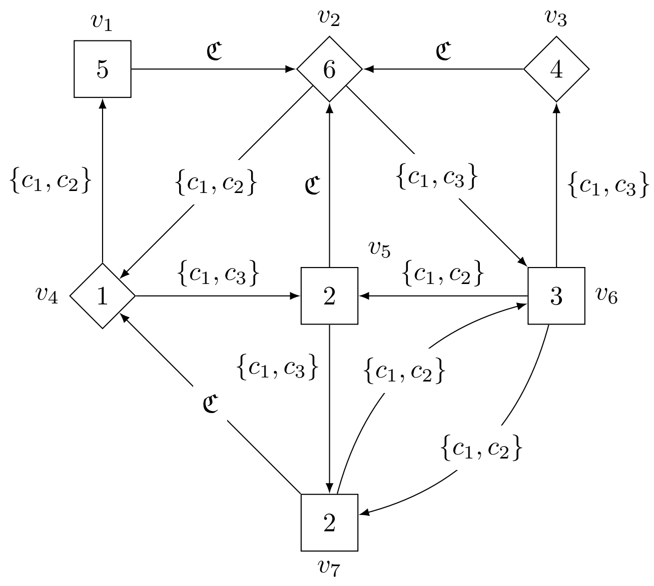


Figure 5.1: VPG with configurations $\mathfrak{C} = \{c_1, c_2, c_3\}$

$(w_m, w) \in E$ and $c \in \theta(w_m, w)$. A path π conforms to strategy σ^c_α if for every $i > 0$ such that π_i exists and $\pi_{i-1} \in V_\alpha$ we have $\pi_i = \sigma^c_\alpha(\pi_0\pi_1 \dots \pi_{i-1})$.

A strategy σ^c_α is winning in configuration c for player α from vertex v if and only if α is the winner of every valid path for c starting in v that conforms to σ^c_α. If such a strategy exists for player α and configuration c starting from vertex v then vertex v is winning for player α and configuration c.

Example 5.2. *Consider the VPG in Figure 5.1. When playing the game for vertex v_5 and configuration c_1 we can define strategy*

$$\sigma^{c_1}_1 = \{v_5 \mapsto v_7, v_7 \mapsto v_6, v_6 \mapsto v_7, \dots\}$$

This always results in the path $v_5(v_7v_6)^\omega$ where the highest priority occurring infinitely often is 3, so player 1 wins. Since this is the only valid path vertex v_5 is won by player 1 in configuration c_1.

If the game is played for vertex v_5 and configuration c_2 the strategy $\sigma^{c_1}_1$ is not valid because the edge (v_5, v_7) is not enabled. For player 0 *we can define strategy*

$$\sigma^{c_2}_0 = \{v_2 \mapsto v_4, v_4 \mapsto v_1, \dots\}$$

Player 1 can only play from v_5 to v_2 so the only path that conforms to $\sigma^{c_2}_0$ is $v_5(v_2v_4v_1)^\omega$ which is winning for player 0. So vertex v_5 is won by player 0 in configuration c_2.

If the game is played for vertex v_5 and configuration c_3 then player 0 *can win v_5 using the strategy*

$$\sigma^{c_3}_0 = \{v_2 \mapsto v_6, v_3 \mapsto v_2, v_4 \mapsto v_5\}$$

Player 1 can play to v_2 or v_7. If the play goes to v_2 then player 0 plays to v_6 from where play can only go to v_3 and v_2 next. We get path $v_5(v_2v_6v_3)^\omega$, which is won by player 0. If player 1 *plays to v_7 then play can only go to v_4 where player 0 plays to v_5. If play stays in this loop then player 0 wins because the highest priority occurring infinitely often is 2. If play eventually goes to v_2 then player 0 wins as well.*

A VPG is solved if for every configuration c the vertices are partitioned in two sets, namely W_0^c and W_1^c, such that every vertex in W_α^c is winning for player α in configuration c. We call these sets the winning sets of a VPG.

We can create a parity game from a VPG by simply choosing configuration c and removing all the edges that do not have c in their guard set. We call this a projection.

Definition 5.2. *The projection of VPG $G = (V, V_0, V_1, E, \Omega, \mathfrak{C}, \theta)$ onto configuration $c \in \mathfrak{C}$, denoted by $G_{|c}$, is the parity game $(V, V_0, V_1, E', \Omega)$ where $E' = \{e \in E \mid c \in \theta(e)\}$.*

If a VPG is total then there is at least one outging edge for every vertex that admits configuration $c \in \mathfrak{C}$. This edge will be in the projection $G_{|c}$ so clearly when the VPG is total then its projections are also total.

A VPG contains multiple parity games, in fact playing a VPG G for configuration c is the same as playing the parity game $G_{|c}$ which we show in the following lemma's and theorem.

Lemma 5.1. *Path π is valid in $G = (V, V_0, V_1, E, \Omega, \mathfrak{C}, \theta)$ for configuration c if and only if path π is valid in $G_{|c} = (V, V_0, V_1, E', \Omega)$.*

Proof. Consider path π that is valid in G for configuration c. For every $i > 0$ we have $(\pi_{i-1}, \pi_i) \in E$ and $c \in \theta(\pi_{i-1}, \pi_i)$. Using the projection definition (Definition 5.2) we can conclude that $(\pi_{i-1}, \pi_1) \in E'$ making the path valid in $G_{|c}$.

Consider path π that is valid in $G_{|c}$. For every $i > 0$ we have $(\pi_{i-1}, \pi_i) \in E'$. Given the projection definition we find that because $(\pi_{i-1}, \pi_i) \in E'$ we must have $(\pi_{i-1}, \pi_i) \in E$ and $c \in \theta(\pi_{i-1}, \pi_i)$. This makes the path valid in G for configuration c. □

Lemma 5.2. *Any strategy σ_α^c for player α and configuration $c \in \mathfrak{C}$ in VPG $G = (V, V_0, V_1, E, \Omega, \mathfrak{C}, \theta)$ is also a strategy in $G_{|c}$ for player α and any strategy σ_α for player α in $G_{|c}$ is also a strategy in G for player α and configuration c.*

Proof. The same reasoning as in Lemma 5.1 can be applied to prove this lemma. □

Theorem 5.3. *Winning sets (W_0^c, W_1^c) of VPG $G = (V, V_0, V_1, E, \Omega, \mathfrak{C}, \theta)$ played for configuration $c \in \mathfrak{C}$ are equal to winning sets (Q_0, Q_1) of parity game $G_{|c}$.*

Proof. Let $v \in W_\alpha^c$ for some $\alpha \in \{0, 1\}$. There exists a strategy σ_α^c in VPG G for player α and configuration c such that any valid path starting in v and conforming to σ_α^c is winning for player α. As shown in Lemma 5.2, σ_α^c is also a strategy for player α in $G_{|c}$. Any valid path starting in v in game G played for configuration c is also valid in game G as shown in Lemma 5.1, additionally any path valid in $G_{|c}$ is also valid in G played for c. Assume there is a valid path in $G_{|c}$ that conforms to σ_α^c starting from v that is not won by player α. This path is also valid in G played for configuration c and conforming to σ_α^c which contradicts $v \in W_\alpha^c$, therefore no such path exists and strategy σ_α^c is winning for player α from v in parity game $G_{|c}$, hence $v \in Q_\alpha$.

Let $v \in Q_\alpha$ for some $\alpha \in \{0,1\}$. There exists a strategy σ_α in parity game $G_{|c}$ for player α such that any valid path starting in v and conforming to σ_α is winning for player α. Using Lemma 5.2 we find that σ_α is a strategy in game G for player α and configuration c. Assume there is a valid path in G for c that conforms to σ_α starting from v that is not won by player α. This path is also valid in $G_{|c}$ and conforming to σ_α which contradicts $v \in Q_\alpha$, therefore no such path exists and strategy σ_α is winning for player α and configuration c from v in VPG G, hence $v \in W^c_\alpha$. □

Parity games have a unique winner for every vertex, from Theorem 5.3 we can conclude that a VPG player for a configuration also has a unique winner for every vertex. Moreover since it is decidable who wins a vertex in a parity game it is also decidable who wins a vertex in a VPG for configuration c. Finally, in a parity game there exists a positional strategy for player α that is winning for all the vertices won by player α in the game. In Theorem 5.3 we argued that a strategy that is winning for player α starting in vertex v in a projection of G onto c is also winning in G for player α and configuration c starting in vertex v. So we can conclude that VPGs are also positionally determined and we can consider a strategy for player α and configuration c as a function $\sigma^c_\alpha : V_\alpha \to V$.

5.1 Verifying featured transition systems

Given an LTS and a modal μ-calculus formula we can construct a parity game such that solving this parity game tells us if the LTS satisfies the formula. Similarly we can construct a VPG from an FTS and a modal μ-calculus in such a way that solving the VPG tells us what products satisfy the formula.

We create a VPG from an FTS by choosing the set of configurations to be equal to the set of products in the FTS. The game graph is created similar as to how a parity game is created from an LTS. Finally transition guards from the FTS are translated into guard sets for the VPG.

Definition 5.3. *FTS2VPG(M, φ) converts FTS $M = (S, Act, trans, s_0, N, P, \gamma)$ and closed formula φ to VPG $(V, V_0, V_1, E, \Omega, \mathfrak{C}, \theta)$.*

The set of configurations is equal to the set of products, i.e. $\mathfrak{C} = P$.

Vertices are created for every state with every formula in the Fischer-Ladner closure of φ. We define the set of vertices:

$$V = S \times FL(\varphi)$$

The following table shows the owners, successors with edge guards and priorities of vertices. We write $w \mid C$ as a successor of v to denote that there is an edge $(v, w) \in E$ such that the edge is guarded by set $C \subseteq \mathfrak{C}$, i.e. $\theta(v, w) = C$.

Vertex	*Owner*	*Successor* | *guard set*	*Priority*
$(s, \bot)$	*0*		*0*
$(s, \top)$	*1*		*0*
$(s, \psi_1 \vee \psi_2)$	*0*	$(s, \psi_1) \mid \mathfrak{C}$ *and* $(s, \psi_2) \mid \mathfrak{C}$	*0*
$(s, \psi_1 \wedge \psi_2)$	*1*	$(s, \psi_1) \mid \mathfrak{C}$ *and* $(s, \psi_2) \mid \mathfrak{C}$	*0*
$(s, \langle a \rangle \psi)$	*0*	$(s', \psi) \mid \{c \in \mathfrak{C} \mid c \models g\}$ *for every* $s \xrightarrow{a \mid g} s'$	*0*
$(s, [a]\psi)$	*1*	$(s', \psi) \mid \{c \in \mathfrak{C} \mid c \models g\}$ *for every* $s \xrightarrow{a \mid g} s'$	*0*
$(s, \mu X.\psi)$	*1*	$(s, \psi[X := \mu X.\psi]) \mid \mathfrak{C}$	$2\lfloor adepth(X)/2 \rfloor + 1$
$(s, \nu X.\psi)$	*1*	$(s, \psi[X := \nu X.\psi]) \mid \mathfrak{C}$	$2\lfloor adepth(X)/2 \rfloor$

Since the Fischer-Ladner formula's are closed we never get a vertex (s, X).

Similar to a parity game, a VPG can be made total by creating sink vertices l_0 and l_1 with priority 1 and 0 respectively and each having an edge to itself with guard set $\mathfrak{C}$. When the VPG is played for configuration c and the token ends up in l_α then clearly player α looses. We make a VPG total by adding vertices l_0 and l_1 and adding an edge from every vertex $v \in V_\alpha$ that has $\bigcup\{\theta(v, w) \mid (v, w) \in E\} \neq \mathfrak{C}$ to l_α with guard set $\mathfrak{C}\backslash \bigcup\{\theta(v, w) \mid (v, w) \in E\}$. Any vertex v_α where player α could not have made a move in the original game played for configuration c now has an edge admitting c to l_α where player α still looses. An edge admitting c is only added if there was no outgoing edge admitting c so the winner of vertex v for configuration c in the original game is the same as in the total game.

Example 5.3. *Consider FTS M, as shown in Figure 5.2, that has features f and g and products $\{\emptyset, \{f\}, \{f, g\}\}$. Modal μ-calculus formula $\varphi = \mu X.([a]X \vee \langle b\rangle\top)$ expresses that on any path reached by a's we can eventually do a b action. This holds true for product $\{\emptyset\}$ because s_1 can only go to s_2 where b can always be done. For product $\{f\}$ this does not hold because once in s_1 it is possible to stay in s_1 indefinitely through an a transition. For product $\{f, g\}$ the formula does hold because we can indeed stay in s_1 indefinitely, however from s_1 we can always do a b step.*

Figure 5.3 shows the VPG resulting from FTS2VPG(M, φ) made total using sink vertices l_0 and l_1. Products $\{\emptyset\}, \{f\}, \{f, g\}$ are depicted as configurations c_1, c_2, c_3 respectively.

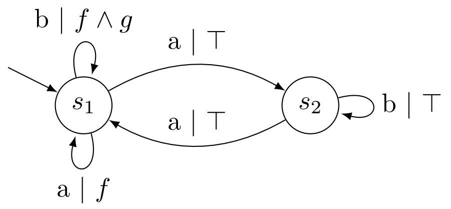


Figure 5.2: FTS M

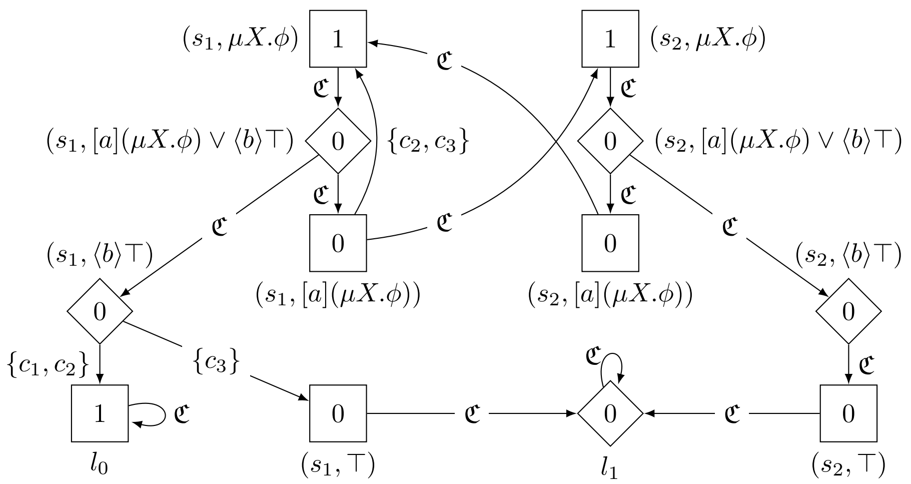


Figure 5.3: Total VPG created by *FTS2VPG* with $\phi = [a]X \vee \langle b\rangle\top$

In order to prove that solving a VPG created by *FTS2VPG* can be used to model check an FTS we inspect the relations we have seen between FTSs, LTSs, parity games and VPGs. Given FTS M and formula φ we can project M onto a product to create an LTS and create a parity game from the resulting LTS and φ using *LTS2PG*. Alternatively we can create a VPG from M and φ using *FTS2VPG* which can be projected onto a configuration to get a parity game. These different transformations are shown in the following diagram, where Π_p depicts a projection onto product p or configuration p:

$$\begin{tikzcd}
\text{FTS } M \arrow[rr, "\mathit{FTS2VPG}(M,\varphi)"] \arrow[d, dashed, "\Pi_p"'] & & \text{VPG } \hat{G} \arrow[d, dashed, "\Pi_p"] \\
\text{LTS } M_{|p} \arrow[r, "\mathit{LTS2PG}(M_{|p},\varphi)"] & \text{PG } G & \text{PG } \hat{G}_{|p}
\end{tikzcd}$$

In the following lemma we prove that in fact parity game G and $\hat{G}_{|p}$ are identical.

Lemma 5.4. *Given FTS $M = (S, Act, trans, s_0, N, P, \gamma)$, closed modal μ-calculus formula φ and product $p \in P$ it holds that parity games $LTS2PG(M_{|p}, \varphi)$ and $FTS2VPG(M, \varphi)_{|p}$ are identical.*

Proof. Let $\hat{G} = (\hat{V}, \hat{V}_0, \hat{V}_1, \hat{E}, \hat{\Omega}, \mathfrak{C}, \theta)$ be the VPG created from $FTS2VPG(M, \varphi)$ and let $G = (V, V_0, V_1, E, \Omega)$ be the parity game created from $LTS2PG(M_{|p}, \varphi)$. Let the projection of $\hat{G}$ onto p (using Definition 5.2) be the parity game $G_{|p} = (\hat{V}, \hat{V}_0, \hat{V}_1, \hat{E}', \hat{\Omega})$. We will prove that $\hat{G}_{|p} = G$.

First observe that when an FTS is projected onto a product (using Definition 4.2) the FTS has the same states as the projection, we find that M has the same states as $M_{|p}$. The vertices created by *LTS2PG* and *FTS2VPG* rely only on the formula and the states in the LTS and FTS respectively. Similarly the owner and priority of these vertices is only determined by the states and the formula. Given that M and $M_{|p}$ have the same states we find that $\hat{V} = V$, $\hat{V}_0 = V_0$, $\hat{V}_1 = V_1$ and $\hat{\Omega} = \Omega$.

We are left with showing $\hat{E}' = E$ in order to conclude $\hat{G}_{|p} = G$. Consider vertex v, we distinguish two cases.

Let $v = (s, \langle a \rangle \psi)$ or $v = (s, [a]\psi)$. If v has a successor to (s', ψ) in G then we have $s \xrightarrow{a} s'$ in $M_{|p}$ and therefore $s \xrightarrow{a \mid f} s'$ with $p \models f$ in M. Using the *FTS2VPG* definition we find that vertex v in $\hat{G}$ has successor (s', ψ) with a guard containing p. Since p is in the guard set we also find this successor in the projection $\hat{G}_{|p}$.

If v has a successor to (s', ψ) in $\hat{G}_{|p}$ then in $\hat{G}$ the edge from v to (s', ψ) also exists and the set guarding it contains p. In M we find $s \xrightarrow{a \mid g} s'$ with $p \models g$, therefore we find $s \xrightarrow{a} s'$ in $M_{|p}$. Using the *LTS2PG* definition we find that vertex v in G has successor (s', ψ).

Let $v \neq (s, \langle a \rangle \psi)$ and $v \neq (s, [a]\psi)$. Any successor of v created by *LTS2PG* does not depend on the LTS but only on the formula. Similarly any successor of v created by *FTS2VPG* does not depend on the FTS and has guard set $\mathfrak{C}$. The two definitions create the same successors for v, so the successors in games G and $\hat{G}$ are the same. Since the guard sets of these successors are always $\mathfrak{C}$ the successors are also the same in $\hat{G}_{|p}$.

We have proven $\hat{E}' = E$ and therefore $\hat{G}_{|p} = G$. $\square$

Using this lemma we get the following diagram showing the relation between FTSs, LTSs, parity games and VPGs.

$$\begin{tikzcd}
\text{FTS } M \arrow[r, "\mathit{FTS2VPG}(M,\varphi)"] \arrow[d, dashed, "\Pi_p"'] & \text{VPG } \hat{G} \arrow[d, dashed, "\Pi_p"] \\
\text{LTS } M_{|p} \arrow[r, "\mathit{LTS2PG}(M_{|p},\varphi)"] & \text{PG } \hat{G}_{|p}
\end{tikzcd}$$

We know from existing theory that solving a parity game constructed using *LTS2PG* can be used to model check an LTS, furthermore we have seen that the winning sets of a VPG for configuration c are equal to the winning sets of that VPG projected onto c. Given these facts and the lemma above we can prove that VPGs can be used to model check FTSs.

Theorem 5.5. *Given:*

- *FTS $M = (S, Act, trans, s_0, N, P, \gamma)$,*
- *closed modal μ-calculus formula φ,*
- *product $p \in P$ and*
- *state $s \in S$*

it holds that $(M_{|p}, s) \models \varphi$ if and only if $(s, \varphi) \in W_0^p$ in FTS2VPG(M, φ).

Proof. Assume $(M_{|p}, s) \models \varphi$, using the relation between LTSs and parity games (Theorem 3.2) we find that vertex (s, φ) in parity game *LTS2PG*$(M_{|p}, \varphi)$ is won by player 0. Using Lemma 5.4 we find that vertex (s, φ) is also in game *FTS2VPG*$(M, \varphi)_{|p}$ and is also won by player 0. Using Theorem 5.3 we find that the winning sets of a VPG for configuration c are the same as the winning sets of the projection of the VPG onto c. We find that vertex (s, φ) is winning in game *FTS2VPG*(M, φ) for configuration p, hence $(s, \varphi) \in W_0^p$.

Similarly if $(M_{|p}, s) \not\models \varphi$ vertex (s, φ) is won by player 1 in parity game *LTS2PG*$(M_{|p}, \varphi)$ and we get $(s, \varphi) \notin W_0^p$. □

Example 5.4. *Again consider Example 5.3. We argued that M satisfies φ for products $\{\emptyset\}$ and $\{f, g\}$. We see, in the VPG in Figure 5.3, that $(s_1, \mu X.([a]X \vee \langle b \rangle \top))$ is indeed winning for player 0 when played for $\{\emptyset\} = c_1$ using the strategy*

$$\sigma_0^{c_1} = \{(s_1, [a](\mu X.\phi) \vee \langle b \rangle \top) \mapsto (s_1, [a](\mu X.\phi)), (s_2, [a](\mu X.\phi) \vee \langle b \rangle \top) \mapsto (s_2, \langle b \rangle \top), \dots\}$$

Using this strategy play always ends up in l_1, which is winning for player 0.

For product $\{f\} = c_2$ player 1 wins using strategy

$$\sigma_1^{c_2} = \{(s_1, [a](\mu X.\phi)) \mapsto (s_1, \mu X.\phi), \dots\}$$

Using this strategy we either infinitely often visit $(s_1, \mu X.\psi)$ in which case player 1 wins or player 0 can decide to play to $(s_1, \langle b \rangle \top)$ in which case play ends in l_0 and player 1 wins.

For product $\{f, g\} = c_3$ player 0 wins using strategy

$$\sigma_0^{c_3} = \{(s_1, [a](\mu X.\phi) \vee \langle b \rangle \top) \mapsto (s_1, \langle b \rangle \top), (s_1, \langle b \rangle \top) \mapsto (s_1, \top), \dots\}$$

Using this strategy player 0 can prevent the path from infinitely often visiting $(s_1, \mu X.\psi)$ by playing to $(s_1, \langle b \rangle \top)$ and to $(s_1, \top)$ next, which brings the play in l_1 winning it for player 0.

We conclude by visualizing the verification of an FTS in Figure 5.4.

$$\begin{tikzcd}
\text{FTS } M \arrow[dr] & & \text{Modal } \mu\text{-calculus formula } \varphi \arrow[dl] \\
& \text{VPG} \arrow[d] & \\
& M_{|p} \overset{?}{\models} \varphi &
\end{tikzcd}$$

for every valid product p in M

Figure 5.4: FTS verification using VPG

6. Solving variability parity games

In this section we inspect methods so solve VPGs, for convenience we only consider total VPGs. We distinguish two general approaches for solving VPGs. The first approach is to simply project the VPG to the different configurations and solve all the resulting parity games independently; we call this *independently* solving a VPG. Existing parity game algorithms can be used in this approach. Alternatively, we solve the VPG *collectively* where a VPG is solved in its entirety and similarities between the configurations are used to improve performance.

We aim to solve VPGs originating from model verification problems, such VPGs generally have certain properties that a random VPG might not have. In general (V)PGs originating from model verification problems have a relatively low number of distinct priorities compared to the number of vertices, this is because new priorities are only introduced when fixed points are nested in the μ-calculus formula. Furthermore the transition guards of FTSs are boolean formula's over features. In general these formula's will be quite simple, specifically excluding or including a small number of features.

When reasoning about time complexities of parity games or VPGs we use n to denote the number of vertices, e the number of edges, and d the number of distinct priorities. When analysing a VPG then we also use c to indicate the number of configurations.

6.1 Recursive algorithm for variability parity games

We can use the original Zielonka's recursive algorithm to solve VPGs by creating one big parity game of a VPG through a process we introduce called *unification.* This parity game can be solved using the original recursive algorithm. However, we introduce a way of representing this parity game that potentially increases performance and exploits commonalities between different configurations in the VPG.

6.1.1 Unified parity games

We can create a parity game from a VPG by taking all the projections of the VPG, which are parity games, and combining them into one parity games by taking the union of them. We call the resulting parity games the *unification* of the VPG. A parity game that is the result of a unification is called a *unified parity game.* Also any subgame of it will be called a unified parity games. A unified parity game always has a VPG from which it originated.

Definition 6.1. *Given VPG $\hat{G} = (\hat{V}, \hat{V}_0, \hat{V}_1, \hat{E}, \hat{\Omega}, \mathfrak{C}, \theta)$ we define the unification of $\hat{G}$, denoted by $\hat{G}_\downarrow$, as*

$$\hat{G}_\downarrow = \biguplus_{c \in \mathfrak{C}} \hat{G}_{|c}$$

where the disjoint union of two parity games is defined as

$$(V, V_0, V_1, E, \Omega) \uplus (V', V_0', V_1', E', \Omega') = (V \uplus V', V_0 \uplus V_0', V_1 \uplus V_1', E \uplus E', \Omega \uplus \Omega')$$

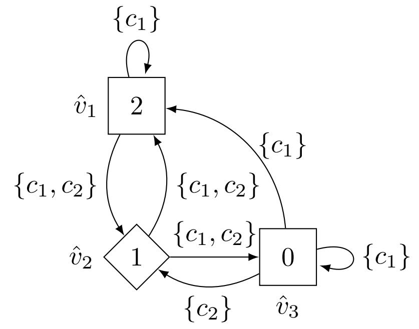
and the disjoint union of functions $\Omega : V \to \mathbb{N}$ and $\Omega' : V' \to \mathbb{N}$ is defined as

$$(\Omega \uplus \Omega')(v) = \begin{cases} \Omega(v) & \textit{if } v \in V \\ \Omega'(v) & \textit{if } v \in V' \end{cases}$$

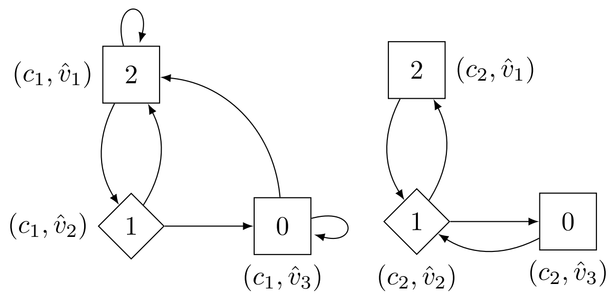
In this section we use the hat decoration ($\hat{G}, \hat{V}, \hat{E}, \hat{\Omega}, \hat{W}$) when referring to a VPG and use no hat decoration when referring to a (unified) parity game.

Every vertex in unified parity game $\hat{G}_\downarrow$ originates from a configuration and an original vertex. Therefore we can consider every vertex in a unification as a vertex-configuration pair, i.e. $V = \mathfrak{C} \times \hat{V}$. We can consider edges in a unification similarly, so $E \subseteq (\mathfrak{C} \times \hat{V}) \times (\mathfrak{C} \times \hat{V})$. Note that edges do not cross configurations, so for every $((c, \hat{v}), (c', \hat{v}')) \in E$ we have $c = c'$. We call set $\hat{V}$ the *origin vertices* of a unified parity game.

Example 6.1. *Figure 6.1 shows a VPG and its the unification.*



(a) VPG with 2 configurations



(b) Unified parity game, created by unifying the two projections

Figure 6.1: A VPG with its corresponding unified parity game

Clearly solving unified parity game $\hat{G}_\downarrow$ solves all the projections of VPG $\hat{G} = (\hat{V}, \hat{V}_0, \hat{V}_1, \hat{E}, \hat{E}, \mathfrak{C}, \theta)$. Theorem 5.3 shows that if we solve all the projections of a VPG we have solved the VPG. So solving $\hat{G}_\downarrow$ also solves $\hat{G}$. Consider winning sets (W_0^c, W_1^c) for $\hat{G}$ played for configuration c and winning sets (Q_0, Q_1) for $\hat{G}_\downarrow$. Using Theorem 5.3 we find the following relation:

$$W_\alpha^c = \{\hat{v} \mid (c, \hat{v}) \in Q_\alpha\}$$

Projections and totality

A unified parity game can be projected onto a configuration to get one of the parity games from which it is the union. This is very similar to the projection of a VPG onto a configuration. Specifically we have for VPG $\hat{G}$ and configuration c that $\hat{G}_{|c} = (\hat{G}_\downarrow)_{|c}$. Eventhough these definitions are so similar we do need to introduce the projection of unified parity games to be able to reason about projections of subgames of unified parity games.

Definition 6.2. *The projection of unified parity game $G = (V, V_0, V_1, E, \Omega)$ to configuration c, denoted as $G_{|c}$, is the parity game $(V', V_0', V_1', E', \Omega)$ such that:*

- $V' = \{\hat{v} \mid (c, \hat{v}) \in V\}$,
- $V_0' = \{\hat{v} \mid (c, \hat{v}) \in V_0\}$,
- $V_1' = \{\hat{v} \mid (c, \hat{v}) \in V_1\}$ *and*
- $E' = \{(\hat{v}, \hat{w}) \mid ((c, \hat{v}), (c, \hat{w})) \in E\}$

One of the properties of a parity game is its totality; a game is total if every vertex has at least one outgoing vertex. The VPGs we consider are also total, meaning that every vertex has, for every configuration $c \in \mathfrak{C}$, at least one outgoing edge admitting c. Because VPGs are total their unifications are also total. Since edges in a unified parity game do not cross configurations the projection of a total unified parity game is also total.

6.1.2 Solving unified parity games

Since unified parity games are total they can be solved using Zielonka's recursive algorithm. The recursive algorithm revolves around the attractor operation. Consider the example presented in Figure 6.1. Vertices with the highest priority are

$$\{(c_1, \hat{v}_1), (c_2, \hat{v}_1)\}$$

attracting these for player 0 gives the set

$$\begin{aligned}\{(c_1, \hat{v}_1),&(c_2, \hat{v}_1),\\ (c_1, \hat{v}_2),&(c_2, \hat{v}_2),\\ &(c_2, \hat{v}_3)\}\end{aligned}$$

The algorithm tries to attract vertices $(c_1, \hat{v}_2)$ and $(c_2, \hat{v}_2)$ because they have edges to $\{(c_1, \hat{v}_1), (c_2, \hat{v}_1)\}$. So the algorithm, in this case, asks the questions: "Can vertices $(c_1, \hat{v}_2)$ and $(c_2, \hat{v}_2)$ be attracted?" We could also ask the question: "For which configurations can we attract origin vertex $\hat{v}_2$?" Since the vertices in unified parity games are pairs of configurations and origin vertices we can, instead of considering vertices individually, consider origin vertices and try to attract as many configurations as possible for each origin vertex. This is the idea of the collective recursive algorithm for VPGs we present next. We introduce a way of efficiently representing unified parity games and an algorithm that behaves the same as the original recursive algorithm but uses the modified representation. Using this representation we can create an attractor set algorithm that tries to attract as many configurations per origin vertex as possible instead of trying to attract each vertex individually.

In VPGs originating from FTSs a large number of edges admit all configurations (as is evident from Definition 5.3). Furthermore the sets that do not admit all configurations originate from the boolean formulas guarding transitions in the FTS. As argued before, these sets will most likely admit many configurations, because in many cases the boolean function will simply include or exclude a small number of features. Because of these two facts we hypothesise that VPGs originating from FTSs have edge guard sets that are relatively large (i.e. admit many of the configurations) and therefore we can attract many configurations at the same time per origin vertex.

6.1.3 Representing unified parity games

Unified parity games have a specific structure because they are the union of parity games that have the same vertices with the same owner and priority. Because they have the same priority we do not actually need to create a new function that is the unification of all the projections, we can simply use the original priority assignment function because the following relation holds:

$$\Omega(c,\hat{v}) = \hat{\Omega}(\hat{v})$$

Similarly we can use the original partition sets $\hat{V}_0$ and $\hat{V}_1$ instead of having the new partition V_0 and V_1 because the following relations hold:

$$(c,\hat{v}) \in V_0 \iff \hat{v} \in \hat{V}_0$$
$$(c,\hat{v}) \in V_1 \iff \hat{v} \in \hat{V}_1$$

So instead of considering unified parity game (V, V_0, V_1, E, Ω) we consider $(V, \hat{V}_0, \hat{V}_1, E, \hat{\Omega})$.

Next we consider how we represent vertices and edges in a unified parity game. A set $X \subseteq (\mathfrak{C} \times \hat{V})$ can be represented as a total function $X^\lambda : \hat{V} \to 2^{\mathfrak{C}}$. The set X and function X^λ are equivalent, denoted by the operator $=_\lambda$: $2^{\mathfrak{C}\times\hat{V}} \times (\hat{V} \to 2^{\mathfrak{C}}) \to \mathbb{B}$, such that

$$X =_\lambda X^\lambda \text{ if and only if } (c,\hat{v}) \in X \iff c \in X^\lambda(\hat{v}) \text{ for all } c \in \mathfrak{C} \text{ and } \hat{v} \in \hat{V}$$

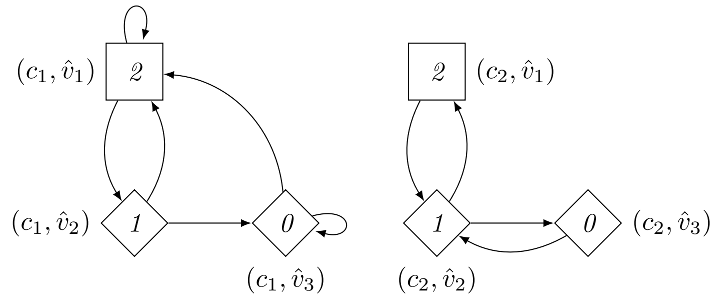
We can also represent edges as a total function $E^\lambda : \hat{E} \to 2^{\mathfrak{C}}$. The set E and function E^λ are equivalent, denoted by the operator $=_\lambda$ $2^{(\mathfrak{C}\times\hat{V})\times(\mathfrak{C}\times\hat{V})} \times (\hat{E} \to 2^{\mathfrak{C}}) \to \mathbb{B}$, such that:

$$E =_\lambda E^\lambda \text{ if and only if } ((c,\hat{v}),(c,\hat{v}')) \in E \iff c \in f(\hat{v},\hat{v}') \text{ for all } c \in \mathfrak{C} \text{ and } \hat{v},\hat{v}' \in \hat{V}$$

We use the $=_\lambda$ operator to indicate that a set and a function represent the same vertices or edges. For convenience of notation we denote equality for edges and vertices both using the $=_\lambda$ operator. We define $\lambda^\emptyset$ to be the function that maps every element to $\emptyset$, clearly $\lambda^\emptyset =_\lambda \emptyset$.

We call using a set of pairs to represent vertices and edges a *set-wise* representation and using functions a *function-wise* representation.

Example 6.2. *We consider a few examples of (sub)games and show their set-wise and function-wise representation. First reconsider the following unified parity game.*



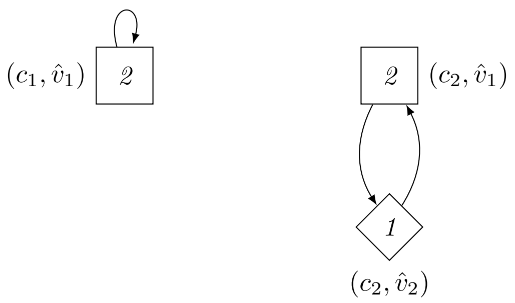
This game can be represented set-wise:

$$\begin{aligned}V = \{&(c_1,\hat{v}_1),(c_2,\hat{v}_1),(c_1,\hat{v}_2),(c_2,\hat{v}_2),(c_1,\hat{v}_3),(c_2,\hat{v}_3)\}\\ E = \{&((c_1,\hat{v}_1),(c_1,\hat{v}_1)),((c_1,\hat{v}_1),(c_1,\hat{v}_2)),((c_1,\hat{v}_2),(c_1,\hat{v}_1)),\\ &((c_1,\hat{v}_2),(c_1,\hat{v}_3)),((c_1,\hat{v}_3),(c_1,\hat{v}_1)),((c_1,\hat{v}_3),(c_1,\hat{v}_3)),\\ &((c_2,\hat{v}_1),(c_2,\hat{v}_2)),((c_2,\hat{v}_2),(c_2,\hat{v}_1)),((c_2,\hat{v}_2),(c_2,\hat{v}_3)),((c_2,\hat{v}_3),(c_2,\hat{v}_2))\}\end{aligned}$$

and function-wise:

$$\begin{aligned}V^\lambda &= \{\hat{v}_1 \mapsto \{c_1, c_2\}, \hat{v}_2 \mapsto \{c_1, c_2\}, \hat{v}_3 \mapsto \{c_1, c_2\}\}\\ E^\lambda &= \{(\hat{v}_1, \hat{v}_2) \mapsto \{c_1, c_2\}, (\hat{v}_2, \hat{v}_1) \mapsto \{c_1, c_2\}, (\hat{v}_2, \hat{v}_3) \mapsto \{c_1, c_2\},\\ &\quad (\hat{v}_1, \hat{v}_1) \mapsto \{c_1\},\\ &\quad (\hat{v}_3, \hat{v}_1) \mapsto \{c_1\},\\ &\quad (\hat{v}_3, \hat{v}_2) \mapsto \{c_2\},\\ &\quad (\hat{v}_3, \hat{v}_3) \mapsto \{c_1\}\}\end{aligned}$$

Consider the following subgame:



This subgame can be represented set-wise:

$$\begin{aligned}V &= \{(c_1, \hat{v}_1), (c_2, \hat{v}_1), (c_2, \hat{v}_2)\}\\ E &= \{((c_1, \hat{v}_1), (c_1, \hat{v}_1)),\\ &\qquad ((c_2, \hat{v}_1), (c_2, \hat{v}_2)), ((c_2, \hat{v}_2), (c_2, \hat{v}_1))\}\end{aligned}$$

and function-wise:

$$\begin{aligned}V^\lambda &= \{\hat{v}_1 \mapsto \{c_1, c_2\}, \hat{v}_2 \mapsto \{c_2\}, \hat{v}_3 \mapsto \emptyset\}\\ E^\lambda &= \{(\hat{v}_1, \hat{v}_2) \mapsto \{c_2\}, (\hat{v}_2, \hat{v}_1) \mapsto \{c_2\}, (\hat{v}_2, \hat{v}_3) \mapsto \emptyset,\\ &\quad (\hat{v}_1, \hat{v}_1) \mapsto \{c_1\},\\ &\quad (\hat{v}_3, \hat{v}_1) \mapsto \emptyset,\\ &\quad (\hat{v}_3, \hat{v}_2) \mapsto \emptyset\}\end{aligned}$$

Finally consider an empty subgame which we can represent set-wise:

$$V = \emptyset, E = \emptyset$$

and function-wise:

$$V^\lambda = \lambda^\emptyset, E^\lambda = \lambda^\emptyset$$

We define the union of two functions $X^\lambda : \hat{V} \to 2^{\mathfrak{C}}$ and $Y^\lambda : \hat{V} \to 2^{\mathfrak{C}}$ as

$$(X^\lambda \cup Y^\lambda)(\hat{v}) = X^\lambda(\hat{v}) \cup Y^\lambda(\hat{v})$$

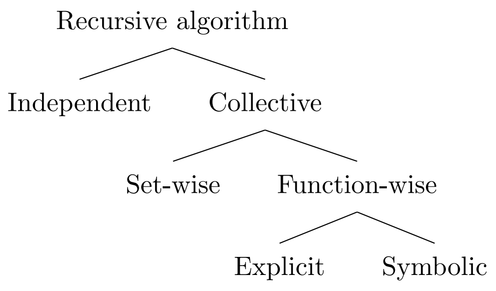
Given $X^\lambda =_\lambda X$ and $Y^\lambda =_\lambda Y$, then clearly $X^\lambda \cup Y^\lambda =_\lambda X \cup Y$. We also define the subset or equal operator for two functions $X^\lambda : \hat{V} \to 2^{\mathfrak{C}}$ and $Y^\lambda : \hat{V} \to 2^{\mathfrak{C}}$ as

$$X^\lambda \subseteq Y^\lambda \text{ if and only if } X^\lambda(\hat{v}) \subseteq Y^\lambda(\hat{v}) \text{ for all } \hat{v} \in \hat{V}$$

Given $X^\lambda =_\lambda X$ and $Y^\lambda =_\lambda Y$, then clearly $X^\lambda \subseteq Y^\lambda$ if and only if $X \subseteq Y$.

6.1.4 Algorithms

Using the recursive algorithm as a basis we can solve a VPG in numerous ways. First of all we can solve the projections, i.e. solve the VPG independently. Alternatively we can solve it collectively using a set-wise representation or a function-wise representation. For the function-wise representation we are working with functions mapping vertices and edges to sets of configurations. These sets of configurations can either be represented explicitly or symbolically. The following diagram shows the different algorithms:



The independent approach uses the original algorithm repeatedly; once for every projection. The collective set-wise approach also uses the original algorithm, applied to a unified parity game. The function-wise representation requires modifications to the algorithm, as we try to attract multiple configurations at the same time. As we will discuss later, this modified algorithm relies heavily on set operations over sets of configurations.

Symbolically representing sets of configurations

For VPGs originating from an FTS, the configuration sets guarding the edges either admit all configurations or originate from boolean functions over the features. These boolean functions will most likely be relatively simple and are therefore specifically appropriate to represent as BDDs.

Set operations $\cap, \cup, \setminus$ over two explicit sets can be performed in $O(m)$ where m is the maximum size of the sets. This is better than the time complexity of a set operation using BDDs, which is $O(m^2)$ (as explained in preliminary section 3.4.1). However if the BDDs are small then the set size can still be large but the set operations are performed very quickly. This is a trade-off between worst-case time complexity and actual running time; using a symbolic representation might yield better results if the sets are structured in such a way that the BDDs are small, however if the sets are not structures in a way that the BDDs are small then the running time is worse than with an explicit representation.

We hypothesize that since the collective function-wise symbolic recursive algorithm relies heavily on set operations over sets of configurations this algorithm will perform well when solving VPGs originating from FTSs.

A note on symbolically solving games

The function-wise algorithm has two variants: an explicit and a symbolic variant. In the explicit variant both the game graph and the sets of configurations are represented explicitly. In the symbolic variant the sets of configurations are represented symbolically, however the graph is still represented explicitly, so the algorithm is partially symbolic and partially explicit. Alternatively an algorithm could completely work symbolically by representing both the graph and the sets of configurations symbolically.

Solving parity games symbolically has been studied in [30]. The obstacle is that representing graphs with a large number of nodes can make the corresponding BDDs very complex if no underlying structure is known for the graph. In such a case performance decreases rapidly. For model verification problems a game graph can conceivably be represented as a BDD by using the structure of the original model to build the BDD. However this is not trivial as argued in [30]. As to not repeat work done in [30] we only consider algorithms where we represent the graph explicitly.

6.1.5 Recursive algorithm using a function-wise representation

The recursive algorithm can be modified to work with the function-wise representation of vertices and edges. The algorithm behaves the same as the original; operations are modified to work with the different representation. Pseudo code for the modified algorithm is presented in Algorithm 3. Note that for this pseudo code no distinction is needed between explicit and symbolic representations of sets of configurations.

```
Algorithm 3 RecursiveUPG(unified parity game $G = ($
$V^\lambda : \hat{V} \to 2^{\mathfrak{C}}$,
$\hat{V}_0 \subseteq \hat{V}$,
$\hat{V}_1 \subseteq \hat{V}$,
$E^\lambda : \hat{E} \to 2^{\mathfrak{C}}$,
$\hat{\Omega} : \hat{V} \to \mathbb{N}))$
1: if $V^\lambda = \lambda^\emptyset$ then
2:     return $(\lambda^\emptyset, (\lambda^\emptyset)$
3: end if
4: $h \leftarrow \max\{\hat{\Omega}(\hat{v}) \mid V^\lambda(\hat{v}) \neq \emptyset\}$
5: $\alpha \leftarrow 0$ if $h$ is even and 1 otherwise
6: $U^\lambda \leftarrow \lambda^\emptyset$, $U^\lambda(\hat{v}) \leftarrow V^\lambda(\hat{v})$ for all $\hat{v}$ with $\hat{\Omega}(\hat{v}) = h$
7: $A^\lambda \leftarrow \alpha\text{-}FAttr(G, U^\lambda)$
8: $(W_0^{\lambda\prime}, W_1^{\lambda\prime}) \leftarrow$ RecursiveUPG$(G \backslash A^\lambda)$
9: if $W_{\overline{\alpha}}^{\lambda\prime} = \lambda^\emptyset$ then
10:     $W_\alpha^\lambda \leftarrow A^\lambda \cup W_\alpha^{\lambda\prime}$
11:     $W_{\overline{\alpha}}^\lambda \leftarrow \lambda^\emptyset$
12: else
13:     $B^\lambda \leftarrow \overline{\alpha}\text{-}FAttr(G, W_{\overline{\alpha}}^{\lambda\prime})$
14:     $(W_0^{\lambda\prime\prime}, W_1^{\lambda\prime\prime}) \leftarrow$ RecursiveUPG$(G \backslash B^\lambda)$
15:     $W_\alpha^\lambda \leftarrow W_\alpha^{\lambda\prime\prime}$
16:     $W_{\overline{\alpha}}^\lambda \leftarrow W_{\overline{\alpha}}^{\lambda\prime\prime} \cup B^\lambda$
17: end if
18: return $(W_0^\lambda, W_1^\lambda)$
```

We introduce a modified attractor definition to work with the function-wise representation.

Definition 6.3. *Given unified parity game $G = (V^\lambda, \hat{V}_0, \hat{V}_1, E^\lambda, \hat{\Omega})$ and a non-empty set $U^\lambda \subseteq V^\lambda$, both represented function-wise, we define α-FAttr(G^λ, U^λ) such that*

$$U_0^\lambda(\hat{v}) = U^\lambda(\hat{v})$$

For $i \geq 0$:

$$U_{i+1}^\lambda(\hat{v}) = U_i^\lambda(\hat{v}) \cup \begin{cases} V^\lambda(\hat{v}) \cap \bigcup_{\hat{v}'}(E^\lambda(\hat{v},\hat{v}') \cap U_i^\lambda(\hat{v}')) & \text{if } \hat{v} \in \hat{V}_\alpha \\ V^\lambda(\hat{v}) \cap \bigcap_{\hat{v}'}((\mathfrak{C} \backslash E^\lambda(\hat{v},\hat{v}')) \cup U_i^\lambda(\hat{v}')) & \text{if } \hat{v} \in \hat{V}_{\overline{\alpha}} \end{cases}$$

Finally:

$$\alpha\text{-}FAttr(G^\lambda, U^\lambda)(\hat{v}) = \bigcup_{i \geq 0} U_i^\lambda(\hat{v})$$

This attractor definition relies heavily on performing set operations on sets of configurations. We will show later that this definition is equal to the original attractor set definition (Definition 3.16).

We also introduce a modified subgame definition to work with the function-wise representation.

Definition 6.4. *For unified parity game $G = (V^\lambda, \hat{V}_0, \hat{V}_1, E^\lambda, \hat{\Omega})$ and set $X^\lambda \subseteq V^\lambda$, both represented function-wise, we define the subgame $G \backslash X^\lambda = (V^{\lambda\prime}, \hat{V}_0, \hat{V}_1, E^{\lambda\prime}, \hat{\Omega})$ such that:*

- $V^{\lambda\prime}(\hat{v}) = V^\lambda(\hat{v}) \backslash X^\lambda(\hat{v})$
- $E^{\lambda\prime}(\hat{v},\hat{v}') = E^\lambda(\hat{v},\hat{v}') \cap V^{\lambda\prime}(\hat{v}) \cap V^{\lambda\prime}(\hat{v}')$

Note that we can omit the modification to the partition (V_0 and V_1) because, as we have seen, we can use the partitioning from the VPG in the representation of unified parity games. As we will show later, this definition is equal to the original subgame definition (Definition 3.17).

Example 6.3. *Consider unified parity game $G = (V^\lambda, \hat{V}_0, \hat{V}_1, E^\lambda, \hat{\Omega})$, originating from a VPG with configurations $\mathfrak{C} = \{c_1, c_2, c_3\}$, represented function-wise in Figure 6.2. We depict every (v, w) for which $E^\lambda(v, w) \neq \emptyset$ with an edge annotated by the set $E^\lambda(v, w)$. All the origin vertices are depicted and for every origin vertex $\hat{v}$ we annotate the square or diamond with a label $\hat{v} \mid C$ where $C = V^\lambda(\hat{v})$. We calculate the function-wise attractor set for player 0 from origin vertex $\hat{v}_2$ with all configuration, we have*

$$U_0^\lambda = U^\lambda = \{\hat{v}_1 \mapsto \emptyset, \hat{v}_2 \mapsto \mathfrak{C}, \hat{v}_3 \mapsto \emptyset, \hat{v}_4 \mapsto \emptyset, \hat{v}_5 \mapsto \emptyset, \hat{v}_6 \mapsto \emptyset, \hat{v}_7 \mapsto \emptyset\}$$

After the first iteration we find

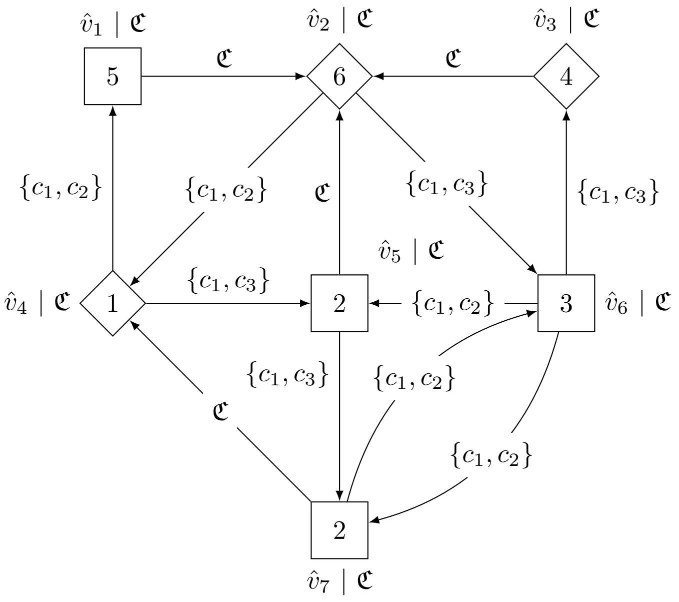
$$U_1^\lambda = \{\hat{v}_1 \mapsto \mathfrak{C}, \hat{v}_2 \mapsto \mathfrak{C}, \hat{v}_3 \mapsto \mathfrak{C}, \hat{v}_4 \mapsto \emptyset, \hat{v}_5 \mapsto \{c_2\}, \hat{v}_6 \mapsto \emptyset, \hat{v}_7 \mapsto \emptyset\}$$

Note that $\hat{v}_5$ can be attracted for configuration $\{c_2\}$ because $\mathfrak{C} \backslash E^\lambda(\hat{v}_5, \hat{v}_7) = \{c_2\}$, $U_0^\lambda(\hat{v}_3) = \mathfrak{C}$ and for any other origin vertex $\hat{v}$ we have $\mathfrak{C} \backslash E^\lambda(\hat{v}_5, \hat{v}) = \mathfrak{C}$.

In the next iteration we find

$$U_2^\lambda = \{\hat{v}_1 \mapsto \mathfrak{C}, \hat{v}_2 \mapsto \mathfrak{C}, \hat{v}_3 \mapsto \mathfrak{C}, \hat{v}_4 \mapsto \{c_1, c_2\}, \hat{v}_5 \mapsto \{c_2\}, \hat{v}_6 \mapsto \{c_3\}, \hat{v}_7 \mapsto \emptyset\}$$

Next iterations result in the same function, so 0*-FAttr$(G, U^\lambda) = U_2^\lambda$. We create subgame $G \backslash U_2^\lambda$ depicted in Figure 6.3.*



$\hat{v}_1 \mid \mathfrak{C}$
5
$\hat{v}_2 \mid \mathfrak{C}$
6
$\hat{v}_3 \mid \mathfrak{C}$
4
$\mathfrak{C}$
$\mathfrak{C}$
$\{c_1, c_2\}$
$\{c_1, c_2\}$
$\mathfrak{C}$
$\{c_1, c_3\}$
$\{c_1, c_3\}$
$\hat{v}_5 \mid \mathfrak{C}$
$\hat{v}_4 \mid \mathfrak{C}$
1
$\{c_1, c_3\}$
2
$\{c_1, c_2\}$
3
$\hat{v}_6 \mid \mathfrak{C}$
$\{c_1, c_3\}$
$\{c_1, c_2\}$
$\mathfrak{C}$
$\{c_1, c_2\}$
2
$\hat{v}_7 \mid \mathfrak{C}$

Figure 6.2: Unified parity game originating from a VPG with configuration $\mathfrak{C} = \{c_1, c_2, c_3\}$

In the next two lemma's we show that the function-wise attractor and subgame operators give results that are equal to the original attractor and subgame operators.

Lemma 6.1. *Given:*

- *unified parity game* $G = (V, \hat{V}_0, \hat{V}_1, E, \hat{\Omega})$,
- *set* $U \subseteq V$, *and*
- *function* U^λ *such that* $U =_\lambda U^\lambda$

it holds that the function-wise attractor α*-FAttr*(G, U^λ) *is equivalent to the set-wise attractor* α*-Attr*(G, U) *for any* $\alpha \in \{0, 1\}$.

Proof. Let V and E be the set-wise representation of the vertices and edges for game G. Let V^λ and E^λ be the function-wise representation of the vertices and edges for game G.

The following properties hold by definition:

$$(c, \hat{v}) \in V \iff c \in V^\lambda(\hat{v})$$

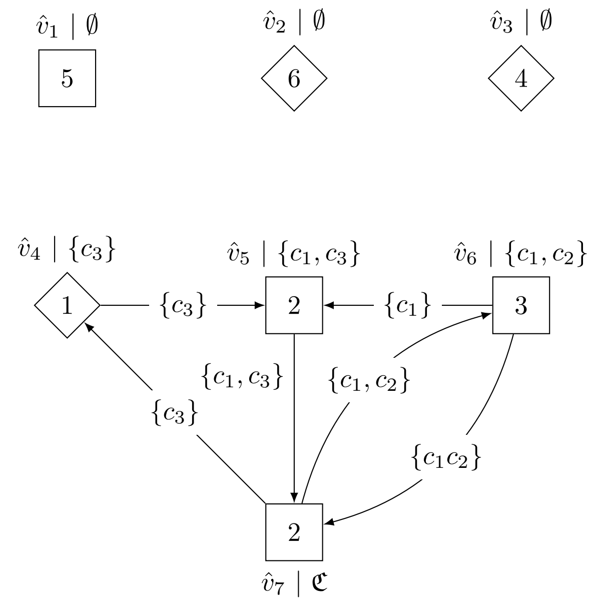
$$(c, \hat{v}) \in U \iff c \in U^\lambda(\hat{v})$$

$$((c, \hat{v}), (c, \hat{v}')) \in E \iff c \in E^\lambda(\hat{v}, \hat{v}')$$

Since the attractors are inductively defined and $U_0 =_\lambda U_0^\lambda$ (because $U =_\lambda U^\lambda$) we have to prove that for some $i \geq 0$, with $U_i =_\lambda U_i^\lambda$, we have $U_{i+1} =_\lambda U_{i+1}^\lambda$, which holds iff:

$$(c, \hat{v}) \in U_{i+1} \iff c \in U_{i+1}^\lambda(\hat{v})$$

Let $(c, \hat{v}) \in V$ (and therefore $c \in V^\lambda(\hat{v})$), we consider 4 cases.



$\hat{v}_1 \mid \emptyset$
5
$\hat{v}_2 \mid \emptyset$
6
$\hat{v}_3 \mid \emptyset$
4
$\hat{v}_4 \mid \{c_3\}$
1
$\hat{v}_5 \mid \{c_1, c_3\}$
2
$\hat{v}_6 \mid \{c_1, c_2\}$
3
$\{c_3\}$
$\{c_1\}$
$\{c_1, c_3\}$
$\{c_1, c_2\}$
$\{c_3\}$
$\{c_1 c_2\}$
2
$\hat{v}_7 \mid \mathfrak{C}$

Figure 6.3: Unified parity game $G \backslash U_2^\lambda$

- Case: $\hat{v} \in \hat{V}_\alpha$ and $(c, \hat{v}) \in U_{i+1}$:
  To prove: $c \in U_{i+1}^\lambda(\hat{v})$.

  If $(c, \hat{v}) \in U_i$ then $c \in U_i^\lambda(\hat{v})$ and therefore $c \in U_{i+1}^\lambda(\hat{v})$. If $(c, \hat{v}) \notin U_i$ then we have $c \notin U_i^\lambda(\hat{v})$.

  Because $\hat{v} \in \hat{V}_\alpha$ and $c \in V^\lambda(\hat{v})$ we get

  $$U_{i+1}^\lambda = \bigcup_{\hat{v}'} (E^\lambda(\hat{v}, \hat{v}') \cap U_i^\lambda(\hat{v}'))$$

  There exists an $(c', \hat{v}') \in V$ such that $(c', \hat{v}') \in U_i$ and $((c, \hat{v}), (c', \hat{v}')) \in E$. Because edges do not cross configurations we can conclude that $c' = c$. Due to equivalence we have $c \in U_i^\lambda(\hat{v}')$ and $c \in E^\lambda(\hat{v}, \hat{v}')$. If we fill this in in the above formula we can conclude that $c \in U_{i+1}^\lambda(\hat{v})$.

- Case: $\hat{v} \in \hat{V}_\alpha$ and $(c, \hat{v}) \notin U_{i+1}$:
  To prove: $c \notin U_{i+1}^\lambda(\hat{v})$.

  First we observe that since $(c, \hat{v}) \notin U_{i+1}$ we get $(c, \hat{v}) \notin U_i$ and therefore $c \notin U_i^\lambda(\hat{v})$.

  Because $\hat{v} \in \hat{V}_\alpha$ and $c \in V^\lambda(\hat{v})$ we get

  $$U_{i+1}^\lambda = \bigcup_{\hat{v}'} (E^\lambda(\hat{v}, \hat{v}') \cap U_i^\lambda(\hat{v}'))$$

  Assume $c \in U_{i+1}^\lambda(\hat{v})$. There must exist a $\hat{v}'$ such that $c \in E^\lambda(\hat{v}, \hat{v}')$ and $c \in U_i^\lambda(\hat{v}')$. Due to equivalence we have a vertex $((c, \hat{v}), (c, \hat{v}')) \in E$ and $(c, \hat{v}') \in U_i$. In which case $(c, \hat{v})$ would be attracted and would be in U_{i+1} which is a contradiction.

- Case: $\hat{v} \in \hat{V}_{\overline{\alpha}}$ and $(c, \hat{v}) \in U_{i+1}$:
  To prove: $c \in U^\lambda_{i+1}(\hat{v})$.

  If $(c, \hat{v}) \in U_i$ then $c \in U^\lambda_i(\hat{v})$ and therefore $c \in U^\lambda_{i+1}(\hat{v})$. If $(c, \hat{v}) \notin U_i$ then we have $c \notin U^\lambda_i(\hat{v})$.

  Because $\hat{v} \in \hat{V}_{\overline{\alpha}}$ we get

  $$U^\lambda_{i+1} = V^\lambda(\hat{v}) \cap \bigcap_{\hat{v}'} ((\mathfrak{C} \backslash E^\lambda(\hat{v}, \hat{v}')) \cup U^\lambda_i(\hat{v}')$$

  Assume $c \notin U^\lambda_{i+1}(\hat{v})$. Because $c \in V^\lambda(\hat{v})$ there must exist an $\hat{v}$ such that

  $$c \notin ((\mathfrak{C} \backslash E^\lambda(\hat{v}, \hat{v}')) \text{ and } c \notin U^\lambda_i(\hat{v}')$$

  which is equal to

  $$c \in E^\lambda(\hat{v}, \hat{v}') \text{ and } c \notin U^\lambda_i(\hat{v}')$$

  By equivalence we have $((c, \hat{v}), (c, \hat{v}')) \in E$ and $(c, \hat{v}') \notin U_i$. Which means that $(c, \hat{v})$ will not be attracted and $(c, \hat{v}) \notin U_{i+1}$ which is a contradiction.

- Case: $\hat{v} \in \hat{V}_{\overline{\alpha}}$ and $(c, \hat{v}) \notin U_{i+1}$:
  To prove: $c \notin U^\lambda_{i+1}(\hat{v})$.

  First we observe that since $(c, \hat{v}) \notin U_{i+1}$ we get $(c, \hat{v}) \notin U_i$ and therefore $c \notin U^\lambda_i(\hat{v})$.

  Because $\hat{v} \in \hat{V}_{\overline{\alpha}}$ we get

  $$U^\lambda_{i+1} = V^\lambda(\hat{v}) \cap \bigcap_{\hat{v}'} ((\mathfrak{C} \backslash E^\lambda(\hat{v}, \hat{v}')) \cup U^\lambda_i(\hat{v}')$$

  Since $(c, \hat{v})$ is not attracted there must exist a $(c, \hat{v}') \in V$ such that

  $$((c, \hat{v}), (c, \hat{v}')) \in E \text{ and } (c, \hat{v}') \notin U_i$$

  By equivalence we have

  $$c \in E^\lambda(\hat{v}, \hat{v}') \text{ and } c \notin U^\lambda_i(\hat{v}')$$

  Which is equal to

  $$c \notin (\mathfrak{C} \backslash E^\lambda(\hat{v}, \hat{v}')) \text{ and } c \notin U^\lambda_i(\hat{v}')$$

  From which we conclude

  $$c \notin ((\mathfrak{C} \backslash E^\lambda(\hat{v}, \hat{v}')) \cup U^\lambda_i(\hat{v}')))$$

  Therefore we have $c \notin U^\lambda_{i+1}(\hat{v})$.

□

Lemma 6.2. *Given:*

- *unified parity game* $G = (V, \hat{V}_0, \hat{V}_1, \mathcal{E}, \hat{\Omega})$,
- *set* $U \subseteq V$ *and*
- *function* U^λ *such that* $U =_\lambda U^\lambda$

it holds that the subgame $G \backslash U = (V', \hat{V}_0, \hat{V}_1, E', \hat{\Omega})$ represented set-wise is equal to the subgame $G \backslash U^\lambda$ represented function-wise.

Proof. Let $V^\lambda, V'^\lambda, E^\lambda, E^{\lambda'}$ the function-wise representations of V, V', E, E' respectively. We know $V =_\lambda V^\lambda$, $E =_\lambda E^\lambda$ and $U =_\lambda U^\lambda$. To prove: $V' =_\lambda V'^\lambda$ and $E' =_\lambda E^{\lambda'}$.

1. Let $(c, \hat{v}) \in V$.

   If $(c, \hat{v}) \in U$ then $c \in U^\lambda(\hat{v})$, also $(c, \hat{v}) \notin V'$ (by Definition 3.17) and $c \notin V'^\lambda(\hat{v})$ (by Definition 6.4).

   If $(c, \hat{v}) \notin U$ then $c \notin U^\lambda(\hat{v})$, also $(c, \hat{v}) \in V'$ (by Definition 3.17) and $c \in V'^\lambda(\hat{v})$ (by Definition 6.4).

   We conclude that $V' =_\lambda V'^\lambda$.

2. Let $((c, \hat{v}), (c, \hat{w})) \in E$.

   If $(c, \hat{v}) \in U$ then $(c, \hat{v}) \notin V'$ and $c \notin V'^\lambda(\hat{v})$ (as shown above). We get $((c, \hat{v}), (c, \hat{w})) \notin V' \times V'$ so $((c, \hat{v}), (c, \hat{w})) \notin E'$ (by Definition 3.17). Also $c \notin E^{\lambda'}(\hat{v}, \hat{w})$ (by Definition 6.4).

   If $(c, \hat{w}) \in U$ then we apply the same logic.

   If neither is in U then both are in V' and in $V' \times V'$ and therefore the $((c, \hat{v}), (c, \hat{w})) \in E'$. Also we get $c \in V'^\lambda(\hat{v})$ and $c \in V'^\lambda(\hat{w})$ so we get $c \in E^{\lambda'}(\hat{v}, \hat{w})$ (by Definition 6.4).

   We conclude that $E' =_\lambda E^{\lambda'}$.

□

Next we prove the correctness of the algorithm by showing that the winning sets of the function-wise algorithm are equal to the winning sets of the set-wise algorithm.

Theorem 6.3. *Given unified parity game $G = (V, \hat{V}_0, \hat{V}_1, E, \hat{\Omega})$ and $G^\lambda = (V^\lambda, \hat{V}_0, \hat{V}_1, E^\lambda, \hat{\Omega})$ which is the functional representation of G. It holds that the winning sets resulting from* RECURSIVEUPG(G^λ) *are equal to the winning sets resulting from* RECURSIVEPG(G).

Proof. Proof by induction on G.

Base: When there are no vertices then RECURSIVEUPG(G^λ) returns $(\lambda^\emptyset, \lambda^\emptyset)$ and RECURSIVEPG(G) returns $(\emptyset, \emptyset)$, these two results are equal therefore the theorem holds in this case.

Step: Player α gets the same value in both algorithms since the highest priority is equal for both algorithms.

Let $U = \{(c, \hat{v}) \in V \mid \hat{\Omega}(\hat{v}) = h\}$ (as calculated by RECURSIVEPG) and $U^\lambda(\hat{v}) = V^\lambda(\hat{v})$ for all $\hat{v}$ with $\hat{\Omega}(\hat{v}) = h$ (as calculated by RECURSIVEUPG). We will show that $U =_\lambda U^\lambda$.

Let $(c, \hat{v}) \in U$. Then $\hat{\Omega}(\hat{v}) = h$ and therefore $U^\lambda(\hat{v}) = V^\lambda(\hat{v})$. Since $U \subseteq V$ we have $(c, \hat{v}) \in V$ and because the equality between V and V^λ we get $c \in V^\lambda(\hat{v})$ and $c \in U^\lambda(\hat{v})$.

Let $c \in U^\lambda(\hat{v})$, since $U^\lambda(\hat{v})$ is not empty we have $\hat{\Omega}(\hat{v}) = h$, furthermore $c \in V^\lambda(\hat{v})$ and therefore $(c, \hat{v}) \in V$. We can conclude that $(c, \hat{v}) \in U$ and $U =_\lambda U^\lambda$.

For the rest of the algorithm it is sufficient to see that attractor sets are equal if the game and input set are equal (as shown in Lemma 6.1) and that the created subgames are equal (as shown in Lemma 6.2). Since the subgames are equal we can apply the theorem on it by induction and conclude that the winning sets are also equal. □

Theorem 5.3 shows that solving a unified parity game solves the VPG, furthermore the algorithm RecursiveUPG correctly solves a unified parity game. Therefore, we can conclude that for VPG $\hat{G}$ vertex $\hat{v}$ is won by player α for configuration c if and only if $c \in W_\alpha^\lambda(\hat{v})$ with $(W_0^\lambda, W_1^\lambda) =$ RecursiveUPG$(G_\downarrow)$.

Function-wise attractor set

Next we present an algorithm to calculate the function-wise attractor, the pseudo code is presented in Algorithm 4. The algorithm considers vertices that are in the attractor set for some configuration. For every such vertex the algorithm tries to attract vertices that are connected by an incoming edge. If a vertex is attracted for some configuration then the incoming edges of that vertex will also be considered.

Algorithm 4 α-FAttractor$(G, U^\lambda : \hat{V} \to 2^{\mathfrak{C}})$

```
1: $A^\lambda \leftarrow U^\lambda$
2: Queue $Q \leftarrow \{\hat{v} \in \hat{V} \mid U^\lambda(\hat{v}) \neq \emptyset\}$
3: while $Q$ is not empty do
4:     $\hat{v}' \leftarrow Q.pop()$
5:     for every $\hat{v}$ such that $E^\lambda(\hat{v}, \hat{v}') \neq \emptyset$ do
6:         if $\hat{v} \in \hat{V}_\alpha$ then
7:             $a \leftarrow V^\lambda(\hat{v}) \cap E^\lambda(\hat{v}, \hat{v}') \cap A^\lambda(\hat{v}')$
8:         else
9:             $a \leftarrow V^\lambda(\hat{v})$
10:            for every $\hat{v}''$ such that $E^\lambda(\hat{v}, \hat{v}'') \neq \emptyset$ do
11:                $a \leftarrow a \cap ((\mathfrak{C} \backslash E^\lambda(\hat{v}, \hat{v}'')) \cup A^\lambda(\hat{v}''))$
12:            end for
13:        end if
14:        if $a \backslash A^\lambda(\hat{v}) \neq \emptyset$ then
15:            $A^\lambda(\hat{v}) \leftarrow A^\lambda(\hat{v}) \cup a$
16:            $Q.push(\hat{v})$
17:        end if
18:    end for
19: end while
20: return $A^\lambda$
```

We prove that the result calculated by α-FAttractor is equal to the definition of α-*FAttr* (Definition 6.3).

Theorem 6.4. *Given unified parity game $G = (V^\lambda, \hat{V}_0, \hat{V}_1, E^\lambda, \hat{\Omega})$, represented function-wise, and $U^\lambda \subseteq V^\lambda$, the algorithm α-*FAttractor(G, U^λ) *correctly calculates α-FAttr(G, U^λ).*

Proof.

Termination. First note that the algorithm terminates. This follows from the fact that only vertices are added to Q (line 16) when something is added to A^λ (lines 14 - 15). This can only happen finitely many times before $A^\lambda(\hat{v}) = \mathfrak{C}$ for every $\hat{v}$, so Q is empty after finitely many steps.

Soundness. To prove the soundness of the algorithm we must show that at the end of the algorithm we have for every $c \in A^\lambda(\hat{w})$ that $c \in \alpha$-FAttractor$(G, U^\lambda)(\hat{w})$. This property actually holds throughout the entire algorithm. Before the while loop (line 3) we have $A^\lambda = U^\lambda$ and the property holds trivially. Consider the beginning of a while loop iteration (the algorithm is on line 4) and assume that the property holds. The algorithm considers a number of vertices in the first for loop (line 5), let $\hat{v}$ be such a vertex. The algorithm calculates $a \subseteq \mathfrak{C}$, which is added to $A^\lambda(\hat{v})$ on line 15. Note that this is the only place in the while loop where A^λ is modified. The value calculated for a on lines 6-13 exactly reflects the definition of α*-FAttr* (Definition 6.3). Because we assumed that the property holds at the beginning of the while loop iteration we can conclude that $a \subseteq \alpha$*-FAttr*$(G, U^\lambda)(\hat{v})$. We conclude that the property is maintained during the while loop and that it holds at the end of the algorithm.

Completeness. To prove completeness we consider $c \in \alpha$*-FAttr*$(G, U^\lambda)(\hat{w})$ and show that $c \in \alpha$-FAttractor$(G, U^\lambda)(\hat{w})$.

Consider the values for U_i^λ for α*-FAttr*(G, U^λ) as defined in Definition 6.3. It is clear that for some $i \geq 0$ we have $c \in U_i^\lambda(\hat{w})$, however not necessarily for all $i \geq 0$. Let $i \geq 0$ such that $c \in U_i^\lambda(\hat{w})$ and either $i = 0$ or $c \notin U_{i-1}^\lambda(\hat{w})$.

We apply induction on i, to show that at some point during the algorithm we have $c \in A^\lambda(\hat{w})$ and $\hat{w} \in Q$. Note that $A^\lambda(\hat{w})$ never decreases during the algorithm, so by proving this we get completeness.

Base $i = 0$: In this case $c \in U^\lambda(\hat{w})$ and in the first two lines of the algorithm we find $c \in A^\lambda(\hat{w})$ and $\hat{w}$ in Q.

Step $i > 0$: We distinguish two cases:

- If $\hat{w} \in \hat{V}_\alpha$ then there exists a $\hat{w}'$ such that $c \in E^\lambda(\hat{w}, \hat{w}')$ and $c \in U_{i-1}^\lambda(\hat{w}')$. We apply induction on w' to find that at some point $c \in A^\lambda(\hat{w}')$ and $\hat{w}'$ is in Q. So at some point, the algorithm pops $\hat{w}'$ from Q (line 4) and we have $c \in A^\lambda(\hat{w}')$. Because $c \in E^\lambda(\hat{w}, \hat{w}')$ the first for loop (line 5) considers vertex $\hat{w}$ and on line 7 we get $c \in a$.

  If at this point we do not already have $c \in A^\lambda(\hat{w})$ then we get $c \in A^\lambda(\hat{w})$ on line 15 and $\hat{w}$ is added to Q on line 16. If we already had $c \in A^\lambda(\hat{w})$ then c had to be added to $A^\lambda(\hat{w})$ on line 15 (because $i > 0$ this could not have been done in the first two lines) and $\hat{w}$ would also be added to Q on line 16. So the induction hypothesis holds in this case.

- If $\hat{w} \in \hat{V}_{\overline{\alpha}}$ then for every $\hat{w}'$ with $c \in E^\lambda(\hat{w}, \hat{w}')$ we get $c \in U_{i-1}^\lambda(\hat{w}')$, we call these vertices the successors of $\hat{w}$ for configuration c. We apply induction on every successor $\hat{w}'$ of $\hat{w}$ for c to find that at some point $c \in A^\lambda(\hat{w}')$ and $\hat{w}'$ is in Q. So at some point, the algorithm pops successor $\hat{w}''$ from Q (line 4) and we have $c \in A^\lambda(\hat{w}')$ for all successor $\hat{w}'$ of $\hat{w}$ for c. Because $c \in E^\lambda(\hat{w}, \hat{w}'')$ the first for loop (line 5) considers vertex $\hat{w}$ and on lines 9-12 we get $c \in a$.

  If at this point we do not already have $c \in A^\lambda(\hat{w})$ then we get $c \in A^\lambda(\hat{w})$ on line 15 and $\hat{w}$ is added to Q on line 16. If we already had $c \in A^\lambda(\hat{w})$ then c had to be added to $A^\lambda(\hat{w})$ on line 15 (because $i > 0$ this could not have been done in the first two lines) and $\hat{w}$ would also be added to Q on line 16. So the induction hypothesis holds in this case.

$\square$

6.1.6 Running time

We consider the running time for solving VPG $\hat{G} = (\hat{V}, \hat{V}_0, \hat{V}_1, \hat{E}, \hat{\Omega}, \mathfrak{C}, \theta)$ independently and collectively using the different types of representations. We use n to denote the number of vertices, e the number of edges, d the number of distinct priorities and c the number of configurations.

The original algorithm runs in $O(e * n^d)$ [15], if we run c parity games independently we get $O(c * e * n^d)$. We can also apply the original algorithm to a unified parity game (represented set-wise) for a collective approach, in this case we get a parity game with $c * n$ vertices and $c * e$ edges. which gives a time complexity of $O(c * e * (c * n)^d)$. However, as we show next, this upper bound can be improved by using the property that a unified parity game consists of c disconnected graphs.

We have introduced three types of collective algorithms: set-wise, function-wise with explicit configuration sets and function-wise with symbolic configuration sets. In all three algorithms the running time of the attractor set dominates the other operations performed, so we need three things: analyse the running time of the base cases, analyse the running time of the attractor set and analyse the recursion.

Base cases

In the base cases the algorithm needs to check if there are no more vertices in the game. For the set-wise variant this is done in $O(1)$. For the function wise algorithms this is done in $O(n)$ since we have to check $V(\hat{v}) = \emptyset$ for every $\hat{v}$. Note that in a symbolic representation using BDDs we can check if a set is empty in $O(1)$ because the decision diagram contains a single node when representing an empty set.

Attractor sets

For the set-wise collective approach we can use the attractor calculation from the original algorithm which has a time complexity of $O(e)$ [22]. So for a unified parity game having $c * e$ edges we have $O(c * e)$.

The function-wise variants use a different attractor algorithm. First we consider the variant where sets of configurations are represented explicitly.

Consider Algorithm 4. A vertex will be added to the queue when this vertex is attracted for some configuration, this can only happen $c * n$ times, once for every vertex-configuration combination.

During an iteration of the while loop, the first for loop considers all vertices with an edge to the vertex under consideration by the while loop. We note that during one iteration of the while loop the first for loop never considers a vertex twice. Because of this we can also conclude that during one iteration of the while loop the second for loop considers no edge twice. Since the while loop runs at most $c * n$ times and in every iteration the second for loop considers at most e edges, we conclude that the second for loop runs at most $c * n * e$ times.

The second for loop performs set operations on the set of configurations which can be done in $O(c)$ using an explicit representation. This gives a total time complexity for the attractor set of $O(n * c^2 * e)$.

Symbolic set operations can be done in $O(c^2)$ so we get a time complexity of $O(n * c^3 * e)$.

This gives the following time complexities

	Base	Attractor set
Set-wise	$O(1)$	$O(c * e)$
Function-wise explicit	$O(n)$	$O(n * c^2 * e)$
Function-wise symbolic	$O(n)$	$O(n * c^3 * e)$

Recursion

The three algorithms behave the same way with regards to their recursion, so we analyse the recursion for all three algorithms at the same time. Let O_B denote the time complexity of the base case for the algorithm and let O_A denote the time complexity of the attractor set. For all variants of the algorithm we have $O_B \leq O_A$ and $O_A + O_B = O_A$.

The algorithm has two recursions. The first recursion lowers the number of distinct priorities by 1. The second recursion removes at least one vertex. However the game is comprised of disjoint projections. We can use this fact in the analyses. Consider unified parity game G and set A as specified by the algorithm. Now consider the projection of G to an arbitrary configuration q, $G_{|q}$. If $(G\backslash A)_{|q}$ contains a vertex that is won by player $\overline{\alpha}$ then this vertex is removed in the second recursion step. If there is no vertex won by player $\overline{\alpha}$ then the game is won in its entirety and the only vertices won by player $\overline{\alpha}$ are in different projections. We can conclude that for every configuration q the second recursion either removes a vertex or $(G\backslash A)_{|q}$ is entirely won by player α. Let $\bar{w}$ denote the maximum number of vertices that are won by player $\overline{\alpha}$ in game $(G\backslash A)_{|q}$. Since every projection has at most n vertices the value for $\bar{w}$ can be at most n. Furthermore since $\bar{w}$ depends on A, which depends on the maximum priority, the value $\bar{w}$ gets reset when the top priority is removed in the first recursion. We can now write down the recursion of the algorithm:

$$T(d, \bar{w}) \leq T(d-1, n) + T(d, \bar{w}-1) + O_A$$

When $\bar{w} = 0$ we will get $W_{\overline{\alpha}} = \emptyset$ as a result of the first recursion. In such a case there will be only 1 recursion.

$$T(d, 0) \leq T(d-1, n) + O_A$$

Finally we have the base cases. If $d = 0$ then there are no vertices and we have the base time complexity.

$$T(0, \overline{w}) \leq O_B$$

If $d = 1$ then all the vertices have the same priority, therefore the first subgame created is empty and entirely won by player α. So we never go in the second recursion.

$$T(1, \overline{w}) \leq T(0, n) + O_A \leq O_B + O_A = O_A$$

Expanding the second recursion gives

$$\begin{aligned} &T(d) \leq (n+1)T(d-1) + (n+1)O_A \\ &T(1) \leq O_A \end{aligned}$$

We prove that $T(d) \leq (n+d)^d O_A$ by induction on d.

Base $d = 1$: $T(1) \leq O_A \leq (n+1)^1 O_A$

Step $d > 1$:

$$\begin{aligned} T(d) &\leq (n+1)T(d-1) + (n+1)O_A \\ &\leq (n+1)(n+d-1)^{d-1}O_A + (n+1)O_A \end{aligned}$$

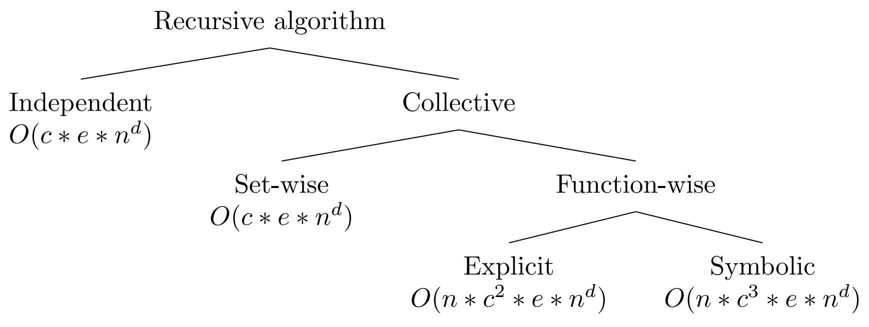
Since $n + 1 \leq n + d - 1$ we get:

$$\begin{aligned} T(d) &\leq (n+d-1)(n+d-1)^{d-1}O_A + (n+1)O_A \\ &\leq ((n+d-1)(n+d-1)^{d-1} + n + 1)O_A \\ &\leq (n(n+d-1)^{d-1} + d(n+d-1)^{d-1} - (n+d-1)^{d-1} + n + 1)O_A \\ &\leq (n(n+d)^{d-1} + d(n+d)^{d-1} - (n+d-1)^{d-1} + n + 1)O_A \end{aligned}$$

Because $(n + d - 1)^{d-1} \geq n + 1$ we have $-(n + d - 1)^{d-1} + n + 1 \leq 0$, therefore:

$$\begin{aligned} T(d) &\leq (n(n+d)^{d-1} + d(n+d)^{d-1})O_A \\ &\leq (n+d)(n+d)^{d-1}O_A \\ &\leq (n+d)^d O_A \end{aligned}$$

This gives a time complexity of $O(O_A * (n + d)^d) = O(O_A * n^d)$ because $n \geq d$. Filling in values for O_A gives the following time complexities:



Running time in practice

Earlier we hypothesized that the symbolic function-wise algorithm could have the best performance of the 4 algorithms, however it has the worst time complexity. Our hypothesis is based on the notion that VPGs most likely have a lot of commonalities and that sets of configurations in the VPG can be represented efficiently symbolically. Next we argue why the worst-case time complexity might not represent the running time in practice.

We dissect the running time of the function-wise algorithms. The running time complexities of the collective algorithms consist of two parts: the time complexity of the attractor set times n^d. The function-wise attractor set time complexity consists of three parts: the number of edges times the maximum number of vertices in the queue ($c * n$) times the time complexity for set operations ($O(c)$ for the explicit variant, $O(c^2)$ for the symbolic variant).

The number of vertices in the queue during attracting is at most $c * n$, however this number will only be large if we attract a very small number of configurations per time we evaluate an edge. As argued earlier we can most likely attract multiple configurations at the same time. This will decrease the number of vertices in the queue.

The time complexity of set operations is $O(c)$ when using an explicit representation and $O(c^2)$ when using a symbolic one. However, as shown in [40], we can implement BDDs to keeps a table of already computed results. This allows us to get already calculated results in sublinear time. In total there are 2^c possible sets and therefore 2^{2c} possible set combinations and $O(2^c)$ possible set operations that can be computed. However when solving a VPG originating from an FTS there will most likely be a relatively small number of different edge guards, in which case the number of unique sets considered in the algorithm will be small and we can often retrieve a set calculation from the computed table.

We can see that even though the running time of the collective symbolic algorithm is the worse, its practical running time might be good when we are able to attract multiple configurations at the same time and have a small number of different edge guards.

6.2 Incremental pre-solve algorithm

Next we explore a collective algorithm that tries to solve the VPG for all configurations as much as possible, then split the configurations in two sets, create subgames using those two configuration sets and recursively repeat the process. Specifically, we try to find vertices that are won by the same player for all configurations in $\mathfrak{C}$. If we find a vertex that is won by the same player for all configurations we call such a vertex *pre-solved*. The algorithm tries to recursively increase the set of pre-solved vertices until all vertices are either pre-solved or a single configuration remains. Pseudo code is presented in Algorithm 5. The algorithm is based around finding sets P_0 and P_1. We want to find these sets in an efficient manner such that the algorithm does not spend time finding vertices that are already pre-solved. Finally, when there is only a single configuration left we want an algorithm that solves the parity game $G_{|c}$ in an efficient manner by using the vertices that are pre-solved.

The subgames created are based on a set of configurations. We define the subgame operator as follows:

Definition 6.5. *Given VPG $G = (V, V_0, V_1, E, \Omega, \mathfrak{C}, \theta)$ and non-empty set $\mathfrak{X} \subseteq \mathfrak{C}$ we define the subgame $G \cap \mathfrak{X} = (V, V_0, V_1, E', \Omega, \mathfrak{C}', \theta')$ such that*

- $\mathfrak{C}' = \mathfrak{C} \cap \mathfrak{X}$,
- $\theta'(e) = \theta(e) \cap \mathfrak{C}'$ *and*
- $E' = \{e \in E \mid \theta'(e) \neq \emptyset\}$.

VPGs we consider are total, meaning that for every configuration and every vertex there is an outgoing edge from that vertex admitting that configuration. In subgames the set of configurations is restricted and only edge guards and edges are removed for configurations that fall

Algorithm 5 IncPreSolve(*VPG* $G = (V, V_0, V_1, E, \Omega, \mathfrak{C}, \theta), P_0, P_1$)

```
 1: if $|\mathfrak{C}| = 1$ then
 2:     Let $\{c\} = \mathfrak{C}$
 3:     $(W'_0, W'_1) \leftarrow$ solve $G_{|c}$ using $P_0$ and $P_1$
 4:     return $(\mathfrak{C} \times W'_0, \mathfrak{C} \times W'_1)$
 5: end if
 6: $P'_0 \leftarrow$ find vertices won by player 0 for all configurations in $\mathfrak{C}$
 7: $P'_1 \leftarrow$ find vertices won by player 1 for all configurations in $\mathfrak{C}$
 8: if $P'_0 \cup P'_1 = V$ then
 9:     return $(\mathfrak{C} \times P'_0, \mathfrak{C} \times P'_1)$
10: end if
11: $\mathfrak{C}^a, \mathfrak{C}^b \leftarrow$ partition $\mathfrak{C}$ in non-empty parts
12: $(W^a_0, W^a_1) \leftarrow$ IncPreSolve($G \cap \mathfrak{C}^a, P'_0, P'_1$)
13: $(W^b_0, W^b_1) \leftarrow$ IncPreSolve($G \cap \mathfrak{C}^b, P'_0, P'_1$)
14: $W_0 \leftarrow W^a_0 \cup W^b_0$
15: $W_1 \leftarrow W^a_1 \cup W^b_1$
16: return $(W_0, W_1)$
```

outside the restricted set, therefore we still have totality. Furthermore it is trivial to see that every projection $G_{|c}$ is equal to $(G \cap \mathfrak{X})_{|c}$ for any $c \in \mathfrak{C} \cap \mathfrak{X}$.

Finally a subsubgame of two configuration sets is the same as the subgame of the intersection of these configuration sets, i.e. $(G \cap \mathfrak{X}) \cap \mathfrak{X}' = G \cap (\mathfrak{X} \cap \mathfrak{X}') = G \cap \mathfrak{X} \cap \mathfrak{X}'$.

6.2.1 Finding P_0 and P_1

We can find P_0 and P_1 using *pessimistic* parity games; a pessimistic parity game is a parity game created from a VPG for a player $\alpha \in \{0, 1\}$ such that the parity game allows all edges that player $\overline{\alpha}$ might take but only allows edges for α when that edge admits all the configurations in $\mathfrak{C}$.

Definition 6.6. *Given VPG* $G = (V, V_0, V_1, E, \Omega, \mathfrak{C}, \theta)$*, we define pessimistic parity game* $G_{\triangleright\alpha}$ *for player* $\alpha \in \{0, 1\}$*, such that*

$$G_{\triangleright\alpha} = (V, V_0, V_1, E', \Omega)$$

with

$$E' = \{(v, w) \in E \mid v \in V_{\overline{\alpha}} \vee \theta(v, w) = \mathfrak{C}\}$$

Note that pessimistic parity games are not necessarily total. A parity game that is not total might result in a finite path, in which case the player that cannot make a move loses the path.

When solving a pessimistic parity game $G_{\triangleright\alpha}$ we get winning sets (W_0, W_1). Every vertex in W_α is winning for player α in G played for any configuration, as shown in the following theorem.

Theorem 6.5. *Given:*

- *VPG* $G = (V, V_0, V_1, E, \Omega, \mathfrak{C}, \theta)$*,*
- *configuration* $c \in \mathfrak{C}$*,*
- *winning sets* (W^c_0, W^c_1) *for game* G*,*

- *player $\alpha \in \{0,1\}$ and*
- *pessimistic parity game $G_{\triangleright\alpha}$ with winning sets (P_0, P_1)*

we have $P_\alpha \subseteq W^c_\alpha$.

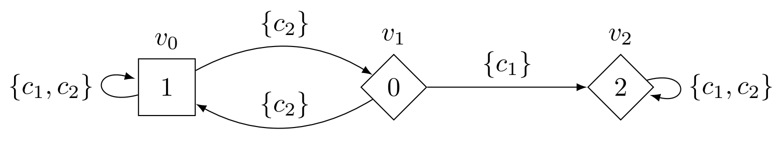
Proof. Player α has a strategy in game $G_{\triangleright\alpha}$ such that vertices in P_α are won. We show that this strategy can also be applied to game $G_{|c}$ to win the same or more vertices.

First we observe that any edge that is taken by player α in game $G_{\triangleright\alpha}$ can also be taken in game $G_{|c}$ so player α can play the same strategy in game $G_{|c}$.

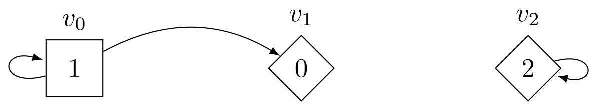
For player $\overline{\alpha}$ there are possibly edges that can be taken in $G_{\triangleright\alpha}$ but cannot be taken in $G_{|c}$. In such a case player $\overline{\alpha}$'s choices are limited in game $G_{|c}$ compared to $G_{\triangleright\alpha}$ so if player $\overline{\alpha}$ cannot win a vertex in $G_{\triangleright\alpha}$ then he/she cannot win that vertex in $G_{|c}$.

We can conclude that applying the strategy from game $G_{\triangleright\alpha}$ in game $G_{|c}$ for player α wins the same or more vertices. Note that this strategy might be incomplete for game $G_{|c}$, it could be the case that a vertex owned by player α in game $G_{\triangleright\alpha}$ has no successor while the same vertex has successors in $G_{|c}$. In such a case the vertex is never in P_α so it is not relevant to the theorem who would win this vertex in $G_{|c}$. $\square$

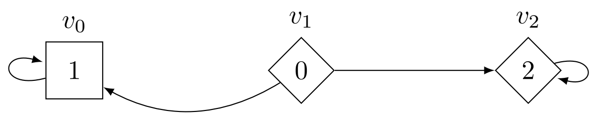
Example 6.4. *Figure 6.4 shows an example VPG with corresponding pessimistic parity games. After solving the pessimistic parity games we find $P_0 = \{v_2\}$ and $P_1 = \{v_0\}$.*



(a) VPG G consisting of 2 configurations



(b) Pessimistic parity game $G_{\triangleright 0}$ with winning sets $(P_0, -)$



(c) Pessimistic parity game $G_{\triangleright 1}$ with winning sets $(-, P_1)$

Figure 6.4: A VPG with its corresponding pessimistic parity games

Pessimistic subgames

Vertices in winning set P_α for $G_{\triangleright\alpha}$ are also winning for player α in pessimistic subgames of G, as shown in the following lemma.

Lemma 6.6. *Given:*

- *VPG $G = (V, V_0, V_1, E, \Omega, \mathfrak{C}, \theta)$,*

- *P_0 being the winning set of pessimistic parity game $G_{\triangleright 0}$ for player 0,*
- *P_1 being the winning set of pessimistic parity game $G_{\triangleright 1}$ for player 1,*
- *non-empty set $\mathfrak{X} \subseteq \mathfrak{C}$,*
- *player $\alpha \in \{0,1\}$ and*
- *winning sets (Q_0, Q_1) for pessimistic parity game $(G \cap \mathfrak{X})_{\triangleright\alpha}$*

we have

$$P_0 \subseteq Q_0$$
$$P_1 \subseteq Q_1$$

Proof. Let edge (v,w) be an edge in game $G_{\triangleright\alpha}$ with $v \in V_\alpha$. Edge (v,w) admits all configuration in $\mathfrak{C}$ so it also admits all configuration in $\mathfrak{C} \cap \mathfrak{X}$, therefore we can conclude that edge (v,w) is also an edge of game $(G \cap \mathfrak{X})_{\triangleright\alpha}$.

Let edge (v,w) be an edge in game $(G \cap \mathfrak{X})_{\triangleright\alpha}$ with $v \in V_{\overline{\alpha}}$. The edge admits some configuration in $\mathfrak{C} \cap \mathfrak{X}$, this configuration is also in $\mathfrak{C}$ so we can conclude that edge (v,w) is also an edge of game $G_{\triangleright\alpha}$.

We have concluded that game $(G \cap \mathfrak{X})_{\triangleright\alpha}$ has the same or more edges for player α as game $G_{\triangleright\alpha}$ and the same or fewer edges for player $\overline{\alpha}$. Therefore we can conclude that any vertex won by player α in $G_{\triangleright\alpha}$ is also won by α in game $(G \cap \mathfrak{X})_{\triangleright\alpha}$, i.e. $P_\alpha \subseteq Q_\alpha$.

Let $v \in P_{\overline{\alpha}}$, using Theorem 6.5 we find that v is winning for player $\overline{\alpha}$ in $G_{|c}$ for any $c \in \mathfrak{C}$. Because projections of subgames are the same as projections of the original game we can conclude that v is winning for player $\overline{\alpha}$ in $(G \cap \mathfrak{X})_{|c}$ for any $c \in \mathfrak{C} \cap \mathfrak{X}$. Assume $v \notin Q_{\overline{\alpha}}$. Then $v \in Q_\alpha$ and using Theorem 6.5 we find that v is winning for player α in $(G \cap \mathfrak{X})_{|c}$ for any $c \in \mathfrak{C} \cap \mathfrak{X}$. This is a contradiction so we can conclude $v \in Q_{\overline{\alpha}}$ and therefore $P_{\overline{\alpha}} \subseteq Q_{\overline{\alpha}}$. □

6.2.2 Algorithm

In order to find P_0 and P_1 we need to solve pessimistic parity games. Specifically we want a parity game algorithm that uses the vertices that are already pre-solved to efficiently solve the parity game. Note that when there is a single configuration left we also need a parity game algorithm that uses the vertices that are already pre-solved. In Algorithm 6 we present the IncPreSolve algorithm using pessimistic parity games. The algorithm uses a Solve algorithm for solving parity games using the pre-solved vertices. First we show the correctness of the IncPreSolve algorithm while assuming the correctness of the Solve algorithm. Later we explore an appropriate Solve algorithm.

A Solve algorithm is correct when it correctly solves a parity game using sets P_0 and P_1, as long as P_0 and P_1 are in fact vertices that are won by player 0 and 1 respectively. We assume that the Solve algorithm is correct and prove that the values for P_0 and P_1 are always correct in IncPreSolve.

Lemma 6.7. *Given VPG $\hat{G}$ and assuming the correctness of* Solve*. For every* Solve(G, P_0, P_1) *that is invoked during* IncPreSolve$(\hat{G}, \emptyset, \emptyset)$ *we have winning sets (W_0, W_1) for game G for which the following holds:*

$$P_0 \subseteq W_0$$

Algorithm 6 $\textsc{IncPreSolve}(G = (V, V_0, V_1, E, \Omega, \mathfrak{C}, \theta), P_0, P_1)$

```
1: if $|\mathfrak{C}| = 1$ then
2:     Let $\{c\} = \mathfrak{C}$
3:     $(W'_0, W'_1) \leftarrow \textsc{Solve}(G_{|c}, P_0, P_1)$
4:     return $(\mathfrak{C} \times W'_0, \mathfrak{C} \times W'_1)$
5: end if
6: $(P'_0, -) \leftarrow \textsc{Solve}(G_{\triangleright 0}, P_0, P_1)$
7: $(-, P'_1) \leftarrow \textsc{Solve}(G_{\triangleright 1}, P_0, P_1)$
8: if $P'_0 \cup P'_1 = V$ then
9:     return $(\mathfrak{C} \times P'_0, \mathfrak{C} \times P'_1)$
10: end if
11: $\mathfrak{C}^a, \mathfrak{C}^b \leftarrow$ partition $\mathfrak{C}$ in non-empty parts
12: $(W_0^a, W_1^a) \leftarrow \textsc{IncPreSolve}(G \cap \mathfrak{C}^a, P'_0, P'_1)$
13: $(W_0^b, W_1^b) \leftarrow \textsc{IncPreSolve}(G \cap \mathfrak{C}^b, P'_0, P'_1)$
14: $W_0 \leftarrow W_0^a \cup W_0^b$
15: $W_1 \leftarrow W_1^a \cup W_1^b$
16: return $(W_0, W_1)$
```

$$P_1 \subseteq W_1$$

Proof. When $P_0 = \emptyset$ and $P_1 = \emptyset$ the theorem holds trivially. So we will start the analyses after the first recursion.

After the first recursion the game is $\hat{G} \cap \mathfrak{X}$ with $\mathfrak{X}$ being either $\mathfrak{C}^a$ or $\mathfrak{C}^b$. The set P_0 is the winning set for player 0 for game $\hat{G}_{\triangleright 0}$ and the set P_1 is the winning set for player 1 for game $\hat{G}_{\triangleright 1}$. In the next recursion the game is $\hat{G} \cap \mathfrak{X} \cap \mathfrak{X}'$ with P_0 being the winning set for player 0 in game $(\hat{G} \cap \mathfrak{X})_{\triangleright 0}$ and P_1 being the winning set for player 1 in game $(\hat{G} \cap \mathfrak{X})_{\triangleright 1}$. In general, after the kth recursion, with $k > 0$, the game is of the form $(\hat{G} \cap \mathfrak{X}^1 \cap \cdots \cap \mathfrak{X}^{k-1}) \cap \mathfrak{X}^k$. Furthermore P_0 is the winning set for player 0 for game $(\hat{G} \cap \mathfrak{X}^1 \cap \cdots \cap \mathfrak{X}^{k-1})_{\triangleright 0}$ and P_1 is the winning set for player 1 for game $(\hat{G} \cap \mathfrak{X}^1 \cap \cdots \cap \mathfrak{X}^{k-1})_{\triangleright 1}$.

Next we inspect the three places where Solve is invoked:

1. Consider the case where there is only one configuration in $\mathfrak{C}$ (line 1-5). Because P_0 is the winning set for player 0 for game $(\hat{G} \cap \mathfrak{X}^1 \cap \cdots \cap \mathfrak{X}^{k-1})_{\triangleright 0}$ the vertices in P_0 are won by player 0 in game $G_{|c}$ for all $c \in \mathfrak{X}^1 \cap \cdots \cap \mathfrak{X}^{k-1}$ (using Theorem 6.5). This includes the one element in $\mathfrak{C}$. So we can conclude $P_0 \subseteq W_0$ where W_0 is the winning set for player 0 in game $G_{|c}$ where $\{c\} = \mathfrak{C}$.

   Similarly for player 1 we can conclude $P_1 \subseteq W_1$ and the lemma holds in this case.

2. On line 6 the game $G_{\triangleright 0}$ is solved with P_0 and P_1. Because $G = \hat{G} \cap \mathfrak{X}^1 \cap \cdots \cap \mathfrak{X}^{k-1} \cap \mathfrak{X}^k$ and P_0 is the winning set for player 0 for game $(\hat{G} \cap \mathfrak{X}^1 \cap \cdots \cap \mathfrak{X}^{k-1})_{\triangleright 0}$ and P_1 is the winning set for player 1 for game $(\hat{G} \cap \mathfrak{X}^1 \cap \cdots \cap \mathfrak{X}^{k-1})_{\triangleright 1}$ we can apply Lemma 6.6 to conclude that the lemma holds in this case.

3. On line 7 we apply the same reasoning and lemma to conclude that the lemma holds in this case.

□

Next we prove the correctness of the algorithm, assuming the correctness of the SOLVE algorithm.

Theorem 6.8. *Given VPG $G = (V, V_0, V_1, E, \Omega, \mathfrak{C}, \theta)$ with winning sets (W_0^c, W_1^c) and $(W_0, W_1) =$ INCPRESOLVE$(G, \emptyset, \emptyset)$. For every configuration $c \in \mathfrak{C}$ it holds that:*

$$(c, v) \in W_0 \iff v \in W_0^c$$
$$(c, v) \in W_1 \iff v \in W_1^c$$

Proof. We assumed that SOLVE(G', P_0, P_1) correctly solves G' as long as vertices in P_0 and P_1 are won by player 0 and 1 respectively. Lemma 6.7 shows that this is always the case when invoking INCPRESOLVE$(G, \emptyset, \emptyset)$. We therefore find that SOLVE(G', P_0, P_1) always correctly solves G' during the algorithm.

We prove the theorem by applying induction on $\mathfrak{C}$.

Base $|\mathfrak{C}| = 1$: When there is only one configuration, being c, then the algorithm solves game $G_{|c}$. The product of the winning sets and $\{c\}$ is returned, so the theorem holds.

Step: Consider P_0' and P_1' as calculated in the algorithm (line 6-7). By Theorem 6.5 all vertices in P_0' are won by player 0 in game $G_{|c}$ for any $c \in \mathfrak{C}$, similarly for P_1' and player 1.

If $P_0' \cup P_1' = V$ then the algorithm returns $(\mathfrak{C} \times P_0', \mathfrak{C} \times P_1')$. In which case the theorem holds because there are no configuration vertex combinations that are not in either winning set and Theorem 6.5 proves the correctness.

If $P_0' \cup P_1' \neq V$ then we have winning sets (W_0^a, W_1^a) for which the theorem holds (by induction) for game $G \cap \mathfrak{C}^a$ and (W_0^b, W_1^b) for which the theorem holds (by induction) for game $G \cap \mathfrak{C}^b$. The algorithm returns $(W_0^a \cup W_0^b, W_1^a \cup W_1^b)$. Since $\mathfrak{C}^a \cup \mathfrak{C}^b = \mathfrak{C}$ and $\mathfrak{C}^a \cap \mathfrak{C}^b = \emptyset$ all vertex configuration combinations are in the winning sets and the correctness follows from induction. □

6.2.3 A parity game algorithm using P_0 and P_1

We can modify the fixed-point iteration algorithm to solve parity games using pre-solved vertices. Recall that the fixed-point iteration algorithm calculates an alternating fixed-point formula to find the winning set for player 0. When iterating fixed-point formula $\mu X.f(X)$ we choose some initial value for X and keep iterating $f(X)$ until we find $X = f(X)$. The original fixed-point iteration algorithm chooses $\emptyset$ as the initial value. In this section we show that given P_0 and P_1 we can use the fixed-point iteration algorithm, but instead of choosing initial value $\emptyset$ we choose initial value P_0. This will most likely decrease the number of iterations needed before we find $X = f(X)$. Moreover we show that we can ignore vertices in P_0 in parts of the calculation because we already know these vertices are winning. Similarly, we find that we can choose initial value $V \backslash P_1$ instead of V (where V is the set of vertices) when iterating a greatest fixed-point formula and ignore vertices in P_1.

We choose to use the fixed-point parity game algorithm because the modified version using pre-solved vertices is very similar to the original version. When experimenting with the incremental pre-solve algorithm we can compare its performance with the performance of independently solving the projections using the fixed-point iteration algorithm to get a good idea of how well the incremental pre-solve idea performs.

First recall the fixed-point formula to calculate W_0:

$$S(G) = \nu Z_{d-1}.\mu Z_{d-2}.\ldots.\nu Z_0.F_0(G, Z_{d-1}, \ldots, Z_0)$$

with

$$\begin{aligned}F_0(G = (V, V_0, V_1, E, \Omega), Z_{d-1}, \dots, Z_0) =& \{v \in V_0 \mid \exists_{w \in V}\ (v,w) \in E \wedge w \in Z_{\Omega(w)}\} \\ & \cup \{v \in V_1 \mid \forall_{w \in V}\ (v,w) \in E \implies w \in Z_{\Omega(w)}\}\end{aligned}$$

Also recall that we can calculate a least fixed-point as follows:

$$\mu X.f(X) = \bigcup_{i \geq 0} X^i$$

where $X^i = f(X^{i-1})$ for $i > 0$ and $X^0 \subseteq \mu X.f(X)$. So picking the smallest value possible for X_0 will always correctly calculate $\mu X.f(X)$. Similarly we can calculate fixed-point a greatest fixed-point as follows:

$$\nu X.f(X) = \bigcap_{i \geq 0} X^i$$

where $X^i = f(X^{i-1})$ for $i > 0$ and $X^0 \supseteq \nu X.f(X)$. So picking the largest value possible for X_0 will always correctly calculate $\nu X.f(X)$.

Let G be a parity game and let sets P_0 and P_1 be such that vertices in P_0 are won by player 0 and vertices in P_1 are won by player 1. We can fixed-point iterate $S(G)$ to calculate W_0, we know that W_0 is bounded by P_0 and P_1, specifically we have

$$P_0 \subseteq W_0 \subseteq V \backslash P_1$$

We will prove that formula

$$S^P(G) = \nu Z_{d-1}.\mu Z_{d-2} \dots \nu Z_0.(F_0(G, Z_{d-1}, \dots, Z_0) \cap (V \backslash P_1) \cup P_0)$$

also solves W_0 for G. Note that the formula $F_0(G, Z_{d-1}, \dots, Z_0) \cap (V \backslash P_1) \cup P_0$ is still monotonic, as shown in Lemma 6.9.

Lemma 6.9. *Given lattice $\langle 2^D, \subseteq \rangle$, monotonic function $f : 2^D \to 2^D$ and $A \subseteq D$. The functions $f^\cup(x) = f(x) \cup A$ and $f^\cap(x) = f(x) \cap A$ are also monotonic.*

Proof. Let $x, y \subseteq D$ and $x \subseteq y$ then $f(x) \subseteq f(y)$.

Let $e \in f(x) \cup A$. If $e \in f(x)$ then $e \in f(y)$ and $e \in f(y) \cup A$. If $e \in A$ then $e \in f(y) \cup A$. We find $f^\cup(x) \subseteq f^\cup(y)$.

Let $e \in f(x) \cap A$. We have $e \in f(x)$ and $e \in A$. Therefore $e \in f(y)$ and $e \in f(y) \cap A$. We find $f^\cap(x) \subseteq f^\cap(y)$. □

Fixed-point iteration index

We introduce the notion of fixed-point *iteration indices* to help with the proof of S^P.

Consider the following alternating fixed-point formula:

$$\nu X_{m-1} \dots . \nu X_0.f(X_{m-1}, \dots, X_0)$$

Using fixed-point iteration to solve this formula results in a number of intermediate values for the iteration variables $X_{m-1}, \dots X_0$. We define an iteration index that, intuitively, indicates where in the iteration process we are. For an alternating fixed-point formula with m fixed-point variables we define an iteration index $\zeta \subseteq \mathbb{N}^m$.

When applying iteration to formula $\nu X_j.f(X)$ we start with some value for X_j^0 and calculate $X_j^{i+1} = f(X_j^i)$. So we get a list of values for X_j, however when we have alternating fixed-point formulas we might iterate X_j multiple times but get different lists of values because the values for $X_{m-1}, \dots, X_{j-1}$ are different. We use the iteration index to distinguish between these different lists.

Iteration index $\zeta = (k_{m-1}, \dots, k_0)$ indicates where in the iteration process we are. We start at $\zeta = (0, 0, \dots, 0)$ and first iterate X_0. When we calculate X_0^1 we are at iteration index $\zeta = (0, 0, \dots, 1)$, when we calculate X_0^2 we are at iteration index $\zeta = (0, 0, \dots, 2)$ and so on. In general when we calculate a value for X_j^i then $k_j = i$ in ζ. This induces the lexicographical order

$$\begin{gathered}(0,\dots,0,0,0)\\(0,\dots,0,0,1)\\(0,\dots,0,0,2)\\\vdots\\(0,\dots,0,1,0)\\(0,\dots,0,1,1)\\(0,\dots,0,1,2)\\\vdots\end{gathered}$$

We define $\{k_{m-1}, \dots, k_0\} - 1 = \{k_{m-1}, \dots, k_0 - 1\}$ and $\{k_{m-1}, \dots, k_0\} + 1 = \{k_{m-1}, \dots, k_0 + 1\}$ for convenience of notation.

We write X_j^ζ to indicate the value of variable X_j at moment ζ of the iteration process. Variable X_j does not change values when a variable X_l with $j > l$ changes values, there we have for indexes $\zeta = (k_{m-1}, \dots, k_j, k_{j-1}, \dots, k_0)$ and $\zeta' = (k_{m-1}, \dots, k_j, k'_{j-1}, \dots, k'_0)$ that $X_j^\zeta = X_j^{\zeta'}$.

We use the fixed-point iteration definition to define the values for X_j^ζ. Let $\zeta = (k_{m-1}, \dots, k_0)$, we have:

$$X_0^{\zeta+1} = f(X_{m-1}^\zeta, \dots, X_0^\zeta)$$

and for any even $0 < j < m$

$$X_j^{(\dots,k_j+1,\dots)} = \mu X_{j-1} \cdots = \bigcup_{i \geq 0} X^{(\dots,k_j,i,\dots)}$$

and for any odd $0 < j < m$

$$X_j^{(\dots,k_j+1,\dots)} = \nu X_{j-1} \cdots = \bigcap_{i \geq 0} X^{(\dots,k_j,i,\dots)}$$

Γ-games We define Γ, which transforms a parity game, to help with the proof. The Γ operator removes the pre-solved vertices from a game and modifies it such that the winners of the remaining vertices are preserved.

Definition 6.7. *Given parity game $G = (V, V_0, V_1, E, \Omega)$ with winning set W_0 such that $P_0 \subseteq W_0 \subseteq V \backslash P_1$. We define $\Gamma(G, P_0, P_1) = (V', V_0', V_1', E', \Omega')$ such that*

$$\begin{aligned}
&V' = (V \backslash P_0 \backslash P_1) \cup \{s_0, s_1\}\\
&V_0' = (V_0 \cap V') \cup \{s_1\}\\
&V_1' = (V_1 \cap V') \cup \{s_0\}\\
&E' = (E \cap (V' \times V')) \cup \{(v, s_\alpha) \mid (v, w) \in E \wedge w \in P_\alpha\}\\
&\Omega'(v) = \begin{cases} 0 & \textit{if } v \in \{s_0, s_1\} \\ \Omega(v) & \textit{otherwise} \end{cases}
\end{aligned}$$

Parity game $\Gamma(G, P_0, P_1)$ contains vertices s_0 and s_1 such that they have no outgoing edges and s_α is owned by player $s_{\overline{\alpha}}$. Clearly if the token ends in s_α then player α wins. Vertices that had edges to a vertex in P_α now have an edge to s_α.

Note that because s_0 and s_1 do not have successors, their priorities do not matter for winning sets of G'. Also note that this parity game is not total, as shown in [39] the formula $S(G)$ also solves non-total games.

Next we show that vertices in $V \backslash P_0 \backslash P_1$ have the same winner in games G and G'.

Lemma 6.10. *Given parity game $G = (V, V_0, V_1, E, \Omega)$ with winning set W_0 such that $P_0 \subseteq W_0 \subseteq V \backslash P_1$ and parity game $G' = \Gamma(G, P_0, P_1)$ with winning set Q_0 we have $W_0 \backslash P_0 \backslash P_1 = Q_0 \backslash \{s_0, s_1\}$.*

Proof. Let vertex $v \in V \backslash P_0 \backslash P_1$. Assume v is won by player α in G using strategy $\sigma_\alpha : V_\alpha \to V$. We construct a strategy $\sigma_\alpha' : V_\alpha' \to V'$ for game G' as follows:

$$\sigma_\alpha'(w) = \begin{cases} s_\beta & \text{if } \sigma_\alpha(w) \in P_\beta \text{ for some } \beta \in \{0, 1\} \\ \sigma_\alpha(w) & \text{otherwise} \end{cases}$$

This strategy maps the vertices to the same successors except when a vertex is mapped to a vertex in P_β, in which case σ_α' maps the vertex to s_β.

Let π' be a valid path in G', starting from v and conforming to σ_α'. Since vertices s_0 and s_1 do not have any successors we distinguish three cases for π':

- Assume π' ends in $s_{\overline{\alpha}}$. Let $\pi' = (x_0 \dots x_m s_{\overline{\alpha}})$ with $v = x_0$. Because s_0 and s_1 do not have successors no x_i is s_0 or s_1; we find $x_i \in V \backslash P_0 \backslash P_1$. For every $x_i x_{i+1}$ we have $(x_i, x_{i+1}) \in E'$, any such edge is also in E because the edges between vertices in $V \backslash P_0 \backslash P_1$ were left intact when creating G'. Finally we find that $(x_m, y) \in E$ with $y \in P_{\overline{\alpha}}$. There must exist a valid path $\pi = (x_0 \dots x_m y \dots)$ in game G. Moreover this path conforms to σ_α because σ_α' and σ_α map to the same vertices for all $x_0 \dots x_{m-1}$ and x_m maps to a vertex in $P_{\overline{\alpha}}$. Player $\overline{\alpha}$ has a winning strategy from y so we conclude that π is won by $\overline{\alpha}$ in game G. Because π exists and conforms to σ_α we find that σ_α is not winning for α from v in G. This is a contradiction so we conclude that π' never ends in $s_{\overline{\alpha}}$.
- Assume π' ends in s_α. In this case player α wins the path.
- Assume π' never visits s_α or $s_{\overline{\alpha}}$. Assume the path is won by player $\overline{\alpha}$, as we argued above we find that this path is also valid in game G, conforms to σ_α and is winning for $\overline{\alpha}$. Therefore σ_α is not winning for player α from v in game G, this is a contradiction so we conclude that player α wins the path π'.

We find that π' is always won by player α in game G'. We conclude that any vertex $v \in V\backslash P_0\backslash P_1$ won by player α in game G is also won by player α in G'.

Let $v \in V'\backslash\{s_0, s_1\}$. Let v be won by player α in game G'. Assume that v is not won by α in game G then v is won by $\overline{\alpha}$ in game G. Clearly $v \in V\backslash P_0\backslash P_1$ so we conclude that v is won by player $\overline{\alpha}$ in game G'. This is a contradiction so v is won by player α in game G. □

Proof Using the Γ operator and the iteration indices we can now prove the correctness of S^P.

Theorem 6.11. *Given parity game $G = (V, V_0, V_1, E, \Omega)$ with winning set W_0 such that $P_0 \subseteq W_0 \subseteq V\backslash P_1$. The formula*

$$S^P(G) = \nu Z_{d-1}.\mu Z_{d-2} \dots \nu Z_0.(F_0(G, Z_{d-1}, \dots, Z_0) \cap (V\backslash P_1) \cup P_0)$$

correctly solves W_0 for G.

Proof. Let $G' = (V', V_0', V_1', E', \Omega') = \Gamma(G, P_0, P_1)$. We consider $S(G')$, which calculates the winning set for player 0 for game G'. Formula $F_0(G', Z_{d-1}, \dots, Z_0)$ will always include s_0 and never include s_1, regardless of the values for $Z_{d-1} \dots Z_0$. Clearly any $\nu Z_i \dots$ or $\mu Z_i \dots$ contains s_0 and does not contain s_1. As shown in [11] we can start the iteration of least fixed-point formula $\mu X.f(X)$ at any value $X^0 \subseteq \mu X.f(X)$. Similarly, we can start the iteration of greatest fixed-point formula $\nu X.f(X)$ at any value $X^0 \supseteq \nu X.f(X)$. So we can calculate $S(G')$ using fixed-point iteration, starting least fixed-point variables at $\{s_0\}$ and greatest fixed-point variables at $V'\backslash\{s_1\}$.

We can also calculate $S^P(G)$ using fixed-point iteration starting at P_0 and $V\backslash P_1$ because clearly any $\nu Z_i \dots$ or $\mu Z_i \dots$ contains all vertices from P_0 and none from P_1.

We prove the theorem by going through the iteration process of $S^P(G)$ and $S(G')$ simultaneously. We write Z_i to denote variables in $S(G')$ and Y_i to denote variables in $S^P(G)$. We will show that for any iteration index ζ any iteration variable Z_i^ζ is equal to Y_i^ζ for vertices $V\backslash P_0\backslash P_1$, that is $Y_i^\zeta\backslash P_0\backslash P_1 = Z_i^\zeta\backslash\{s_0, s_1\}$. We only prove that this is the case when we start iteration of $S^P(G)$ at P_0 and $V\backslash P_1$ and start iteration of $S(G')$ at $\{s_0\}$ and $V'\backslash\{s_1\}$. As argued above, starting at these values correctly calculates $S^P(G)$ and $S(G')$.

Trivially, for any ζ and $i \in [0, d-1]$ we have $P_0 \subseteq Y_i^\zeta \subseteq V\backslash P_1$ and $\{s_0\} \subseteq Z_i^\zeta \subseteq V\backslash\{s_1\}$.

We define operator $\simeq: V \times V' \to \mathbb{B}$ such that for $Y \subseteq V$ and $Z \subseteq V'$ we have $Y \simeq Z$ if and only if:

$$Y\backslash P_0\backslash P_1 = Z\backslash\{s_0, s_1\}$$

We prove, by induction on ζ, that for any $\zeta = (k_{d-1}, \dots, k_0)$ we have $Y_i^\zeta \simeq Z_i^\zeta$ for every $i \in [0, d-1]$.

Base $\zeta = (0, 0, \dots, 0)$: we have for least fixed-point variables Z_i^ζ and Y_i^ζ the values $\{s_0\}$ and P_0, clearly $Y_i^\zeta \simeq Z_i^\zeta$.

For greatest fixed-point variables Z_j^ζ and Y_j^ζ we have $Z_j^\zeta\backslash\{s_0, s_1\} = V\backslash P_1\backslash P_0$. So we find $Y_j^\zeta \simeq Z_j^\zeta$.

Step: Consider $\zeta = (k_{d-1}, \dots, k_0)$. Let $d-1 \geq j \geq 0$. If $k_j = 0$ then $Z_j^\zeta = Z_j^{(0,0,\dots,0)}$ and $Y_j^\zeta = Y_j^{(0,0,\dots,0)}$, furthermore $Z_j^{(0,0,\dots,0)} \simeq Y_j^{(0,0,\dots,0)}$ so we find $Y_j^\zeta \simeq Z_j^\zeta$. If $k_j > 0$ then we distinguish three cases for j to show that $Y_j^\zeta \simeq Z_j^\zeta$:

- Case $j = 0$: We have the following equations:

  $$Y_0^\zeta = F_0(G, Y_{d-1}^{\zeta-1}, \dots, Y_0^{\zeta-1}) \cap (V \backslash P_1) \cup P_0$$

  and

  $$Z_0^\zeta = F_0(G', Z_{d-1}^{\zeta-1}, \dots, Z_0^{\zeta-1})$$

  By induction we find $Y_i^{\zeta-1} \simeq Z_i^{\zeta-1}$ for all $i \in [0, d-1]$.

  Consider vertex $v \in V \backslash P_0 \backslash P_1$. We distinguish two cases:

  – Assume $v \in V_0$.

    If $v \in Y_0^\zeta$ then v must have an edge in game G to w such that $w \in Y_{\Omega(w)}^{\zeta-1}$. We find $w \notin P_1$ because vertices from P_1 are never in the iteration variable. If $w \in P_0$ then it follows from the way we created G' that in G' there exists an edge from v to s_0 and since s_0 is always in the iteration variable we find $v \in Z_0^\zeta$. If $w \notin P_0$ then because $Y_{\Omega(w)}^{\zeta-1} \simeq Z_{\Omega(w)}^{\zeta-1}$ we find $w \in Z_{\Omega(w)}^{\zeta-1}$ and therefore $v \in Z_0^\zeta$.

    If $v \in Z_0^\zeta$ then v must have an edge in game G' to w such that $w \in Z_{\Omega(w)}^{\zeta-1}$. We find $w \neq s_1$ because w is never in the iteration variable. If $w = s_0$ then it follows from the way we created G' that in G there exists an edge from v to a vertex in P_0 and since any vertex in P_0 is always in the iteration variable we find $v \in Y_0^\zeta$. If $w \neq s_0$ then because $Y_{\Omega(w)}^{\zeta-1} \simeq Z_{\Omega(w)}^{\zeta-1}$ we find $w \in Y_{\Omega(w)}^{\zeta-1}$ and therefore $v \in Y_0^\zeta$.

  – Assume $v \in V_1$.

    If $v \in Y_0^\zeta$ then for any successor w of v in game G it holds that $w \in Y_{\Omega(w)}^{\zeta-1}$. Consider successor x of v in game G'. We distinguish three cases:

    * $x = s_0$: In this case $x \in Z_{\Omega(x)}^{\zeta-1}$ because s_0 is always in the iteration variables.
    * $x = s_1$: Because of the way G' is constructed we find vertex v must have a successor w in P_1 in game G. However we found $w \in Y_{\Omega(w)}^{\zeta-1}$. This is a contradiction because vertices in P_1 are never in the iteration variables. So this case can not happen.
    * $x \notin \{s_0, s_1\}$: We have $x \in V' \backslash \{s_0, s_1\}$ and therefore x is also a successor of v in game G. We find $x \in Y_{\Omega(x)}^{\zeta-1}$ and because $Y_{\Omega(x)}^{\zeta-1} \simeq Z_{\Omega(x)}^{\zeta-1}$ we have $x \in Z_{\Omega(x)}^{\zeta-1}$.

    We always find $x \in Z_{\Omega(x)}^{\zeta-1}$, therefore $v \in Z_0^\zeta$.

    If $v \in Z_0^\zeta$ then for any successor w of v in game G' it holds that $w \in Z_{\Omega(w)}^{\zeta-1}$. Consider successor x of v in game G. We distinguish three cases:

    * $x \in P_0$: In this case $x \in Y_{\Omega(x)}^{\zeta-1}$ because vertices in P_0 are always in the iteration variables.
    * $x \in P_1$: Because of the way G' is constructed we find vertex v must have successor s_1 in game G', however we found that for any successor w of v in game G' we have $w \in Z_{\Omega(w)}^{\zeta-1}$. This is a contradiction because s_1 is never in the iteration variable. So this case can not happen.

  - $*$ $x \in V \backslash P_0 \backslash P_1$: We find that x is also a successor of v in game G'. We find $x \in Z^{\zeta-1}_{\Omega(w)}$ and because $Y^{\zeta-1}_{\Omega(x)} \simeq Z^{\zeta}_{\Omega(x)}$ we have $x \in Y^{\zeta}_{\Omega(x)}$.

  We always find $x \in Y^{\zeta-1}_{\Omega(x)}$, therefore $v \in Y^{\zeta}_0$.

- Case $j > 0$ being even: We have

$$Z^{\zeta}_j = \mu Z_{j-1} \cdots = \bigcup_{i \geq 0} Z^{\{k_{d-1},\ldots,k_j-1,i,\ldots\}}_{j-1}$$

and

$$Y^{\zeta}_j = \mu Y_{j-1} \cdots = \bigcup_{i \geq 0} Y^{\{k_{d-1},\ldots,k_j-1,i,\ldots\}}_{j-1}$$

Let $v \in V \backslash P_0 \backslash P_1$.

If $v \in Z^{\zeta}_j$ then there exists some i such that $v \in Z^{\{k_{d-1},\ldots,k_j-1,i,\ldots\}}_{j-1}$. Since $\{k_{d-1},\ldots,k_j - 1, i, \ldots\} < \zeta$ we apply induction to find $Y^{\{k_{d-1},\ldots,k_j-1,l,\ldots\}}_{j-1} \simeq Z^{\{k_{d-1},\ldots,k_j-1,l,\ldots\}}_{j-1}$. Because $v \in V \backslash P_0 \backslash P_1$ we find $v \in Y^{\{k_{d-1},\ldots,k_j-1,i,\ldots\}}_{j-1}$ and therefore $v \in Y^{\zeta}_j$.

If $v \in Y^{\zeta}_j$ then we apply symmetrical reasoning to find $v \in Z^{\zeta}_j$.

- Case $j > 0$ being odd: We have

$$Z^{\zeta}_j = \nu Z_{j-1} \cdots = \bigcap_{i \geq 0} Z^{\{k_{d-1},\ldots,k_j-1,i,\ldots\}}_{j-1}$$

and

$$Y^{\zeta}_j = \nu Y_{j-1} \cdots = \bigcap_{i \geq 0} Y^{\{k_{d-1},\ldots,k_j-1,i,\ldots\}}_{j-1}$$

Let $v \in V \backslash P_0 \backslash P_1$.

If $v \in Z^{\zeta}_j$ then for all $i \geq 0$ we have $v \in Z^{\{k_{d-1},\ldots,k_j-1,i,\ldots\}}_{j-1}$. Assume $v \notin Y^{\zeta}_j$, there must exist an $l \geq 0$ such that $v \notin Y^{\{k_{d-1},\ldots,k_j-1,l,\ldots\}}_j$. Since $\{k_{d-1},\ldots,k_j - 1, l, \ldots\} < \zeta$ we apply induction to find $Y^{\{k_{d-1},\ldots,k_j-1,l,\ldots\}}_{j-1} \simeq Z^{\{k_{d-1},\ldots,k_j-1,l,\ldots\}}_{j-1}$. Because $v \in V \backslash P_0 \backslash P_1$ we find $v \notin Z^{\{k_{d-1},\ldots,k_j-1,i,\ldots\}}_{j-1}$ which is a contradiction so we have $v \in Y^{\zeta}_j$.

If $v \in Y^{\zeta}_j$ then we apply symmetrical reasoning to find $v \in Z^{\zeta}_j$.

This proves that for any ζ we have $Y^{\zeta}_i \simeq Z^{\zeta}_i$ for every $i \in [0, d-1]$.

We have shown that when starting the iteration of $S(G')$ and $S^P(G)$ at specific values we get identical results for vertices in $V \backslash P_0 \backslash P_1$. We chose these values such that they solve the formula's correctly, so we conclude that $S(G') \backslash \{s_0, s_1\} = S^P(G) \backslash P_0 \backslash P_1$. Lemma 6.10 shows that $S(G')$ correctly solves vertices in $V \backslash P_0 \backslash P_1$ for game G. So $S^P(G)$ also correctly solves vertices $V \backslash P_0 \backslash P_1$ for game G.

Moreover any vertex in P_0 is in $S^P(G)$, which is correct because P_0 vertices are winning for player 0. Any vertex in P_1 is not in $S^P(G)$, which is correct because P_1 vertices are winning for player 1. We conclude that all vertices are correctly solved by $S^P(G)$. $\square$

Algorithm

We use the original fixed-point algorithm presented in [4] and modify it such that its starts in iteration at P_0 and $V\backslash P_1$. Moreover we ignore vertices in P_0 or P_1 in the diamond and box calculation. Finally we always add vertices in P_0 to the results of the diamond and box operator. The correctness follow from Theorem 6.11 and [4, 39].

Note that in [4] total games are used. However, it is argued that the algorithm correctly solves the formula presented in [39]. The only property of parity games that is used in this argumentation is that parity games have a unique owner and priority. Clearly this is still the case for total parity games so the algorithm correctly solves the formula presented in [39]. In [39] it is shown that the formula also correctly solves non-total parity games.

Algorithm 7 Fixed-point iteration with P_0 and P_1

```
1: function FPIter($G = (V, V_0, V_1, E, \Omega)$, $P_0 \subseteq V, P_1 \subseteq V$)
2:     for $i \leftarrow d-1, \ldots, 0$ do
3:         Init($i$)
4:     end for
5:     repeat
6:         $Z'_0 \leftarrow Z_0$
7:         $Z_0 \leftarrow P_0 \cup$ Diamond() $\cup$ Box()
8:         $i \leftarrow 0$
9:         while $Z_i = Z'_i \wedge i < d-1$ do
10:            $i \leftarrow i+1$
11:            $Z'_i \leftarrow Z_i$
12:            $Z_i \leftarrow Z_{i-1}$
13:            Init($i-1$)
14:        end while
15:    until $i = d-1 \wedge Z_{d-1} = Z'_{d-1}$
16:    return $(Z_{d-1}, V\backslash Z_{d-1})$
17: end function
```

```
1: function Init($i$)
2:     $Z_i \leftarrow P_0$ if $i$ is odd, $V\backslash P_1$ otherwise
3: end function
```

```
1: function Diamond
2:     return $\{v \in V_0\backslash P_0\backslash P_1 \mid \exists_{w\in V}\ (v,w) \in E \wedge w \in Z_{\Omega(w)}\}$
3: end function
```

```
1: function Box
2:     return $\{v \in V_1\backslash P_0\backslash P_1 \mid \forall_{w\in V}\ (v,w) \in E \implies w \in Z_{\Omega(w)}\}$
3: end function
```

This algorithm can be used as a Solve algorithm in IncPreSolve since it solves parity games using P_0 and P_1.

6.2.4 Running time

We consider the running time for solving VPG $G = (V, V_0, V_1, E, \Omega, \mathfrak{C}, \theta)$ independently and collectively. We use n to denote the number of vertices, e the number of edges, d the number of distinct priorities and c the number of configurations.

The fixed-point iteration algorithm without P_0 and P_1 runs in $O(e * n^d)$ [4]. We can use this algorithm to solve G independently, i.e. solve all the projections of G. This gives a time complexity of $O(c * e * n^d)$.

Next consider the IncPreSolve algorithm for a collective approach, observe that in the worst case we have to split the set of configurations all the way down to individual configurations. We can consider the recursion as a tree where the leafs are individual configurations and at every internal node the set of configurations is split in two. In the worst case there are c leaves so there are at most $c-1$ internal nodes. At every internal node the algorithm solves two games and at every leaf the algorithm solves 1 game, so we get $O(c+2c-2)=O(c)$ games that are being solved by IncPreSolve. In the worst case the values for P_0 and P_1 are empty. In this case the FPIte algorithm behaves the same as the original algorithm and has a time complexity of $O(e*n^d)$. This gives an overall time complexity of $O(c*e*n^d)$, which is equal to an independent solving approach.

Running time in practice

The incremental pre-solve algorithm will, most likely, need to solve more (pessimistic) parity games than an independent approach would need to solve. However, the algorithm keeps trying to increase the number of pre-solved vertices which might speeds up the solving of these games. This would cause the algorithm to solve the (pessimistic) increasingly quickly. Therefore, we hypothesize that, even though more games are solved, the increment pre-solve algorithm performs better than an independent approach.

7. Locally solving (variability) parity games

As discussed in the preliminaries, parity games can be solved either globally or locally. Similar to parity games we can solve VPGs either globally or locally. When locally solving a VPG for vertex $\hat{v}_0$ we determine for every configuration the winner of vertex $\hat{v}_0$. When globally solving a VPG we determine this for every vertex in the VPG.

When solving a VPG globally we might encounter significant differences in parts of the game or intermediate results between configurations that we perhaps do not encounter when solving it locally because we can terminate earlier. Therefore we hypothesize that the increase in performance between globally-collectively solving VPGs and locally-collectively solving VPGs is greater than the increase in performance between globally-independently solving VPGs and locally-independently solving VPGs.

The algorithms we have seen thus far are global algorithms, in this section we introduce local variants for the parity game algorithms we have seen: Zielonka's recursive algorithm and fixed-point iteration algorithm. Furthermore we introduce local variants for the VPG algorithms we have seen: the recursive algorithm for VPGs and the incremental pre-solve algorithm.

7.1 Locally solving parity games

The two parity game algorithms introduced in the preliminaries (Zielonka's recursive algorithm and the fixed-point iteration algorithm) can be turned into local variants. These local variants can be used to solve VPGs locally and independently.

7.1.1 Local recursive algorithm for parity games

The recursive algorithm has two recursion steps. The first recursion step gives two winning sets which are used to find set B such that all the vertices in B won are by a particular player. The second recursion step solves the remaining part of the game. When locally solving a parity game we can sometimes not go into the second recursion when we already found the vertex we are interested in to be in set B. In this section we introduce an algorithm that utilizes this idea to create a local variant of the recursive algorithm.

First we inspect the notion of *traps* [41]. Traps are used in the original proof of the recursive algorithm and we will use them again to reason about a local variant of the recursive algorithm. Consider total parity game (V, V_0, V_1, E, Ω) in the next definition and two lemma's.

Definition 7.1. *[41] Set $X \subseteq V$ is an α-trap in G if and only if player $\overline{\alpha}$ can play in such a way that once the token is in X, it will not leave X.*

Lemma 7.1. *[41] Set $V \backslash \alpha$-Attr(G, X) is an α-trap in G for any non-empty $X \subseteq V$.*

Lemma 7.2. *[41] Let $X \subseteq V$ be an α-trap in G. Then $\overline{\alpha}$-Attr(G, X) is also an α-trap in G.*

Observe that a winning set W_α of parity game G is an $\overline{\alpha}$-trap in G. If $\overline{\alpha}$ could play to $W_{\overline{\alpha}}$ from a vertex $v \in W_\alpha$ then v would be winning for $\overline{\alpha}$.

We show that if a vertex in the first recursion is won by player $\overline{\alpha}$, as calculated in the recursive algorithm, then this vertex is also won by player $\overline{\alpha}$ in the game itself.

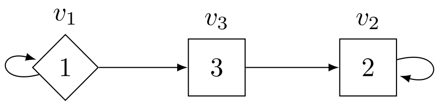
Lemma 7.3. *Given total parity game $G = (V, V_0, V_1, E, \Omega)$, player $\alpha \in \{0, 1\}$ and non-empty set $X \subseteq V$ it holds that the winning set $W_{\overline{\alpha}}$ for player $\overline{\alpha}$ in $G' = G \backslash \alpha$-Attr(G, X) is an α-trap in G and all vertices in $W_{\overline{\alpha}}$ are winning for $\overline{\alpha}$ in G.*

Proof. Using Lemma 7.1 we find that $V' = V\backslash\alpha\text{-}Attr(G,X)$ is a an α-trap in G. Set $W_{\overline{\alpha}}$ is an α-trap in G because if α could escape to $V\backslash V'$ then V' would not be an α-trap in G and if α could escape to $V'\backslash W_{\overline{\alpha}}$ then $W_{\overline{\alpha}}$ would not be an α-trap in G'. Moreover keeping the token in $W_{\overline{\alpha}}$ causes player $\overline{\alpha}$ to win the path because the strategy that was winning in G' can also be applied in G. □

Let v_0 be the vertex we are trying to solve locally. We could argue that if the algorithm finds v_0 to be winning for player $\overline{\alpha}$ in the first recursion of the algorithm then we can terminate and report v_0 to be winning for $\overline{\alpha}$. Using the lemma above we find that indeed v_0 is winning for player $\overline{\alpha}$ in game G when v_0 is winning for $\overline{\alpha}$ in the first recursion. However game G itself might be the subgame of some game H. Vertex v_0 is winning for $\overline{\alpha}$ in G and in the subgame created in the first recursion, however if we want to terminate early then v_0 must also be winning for $\overline{\alpha}$ in game H. If v_0 is not winning for $\overline{\alpha}$ in game H and game G is the first subgame created from game H then in order to correctly solve game H we need the complete winning sets of G. In the conjecture below we express this property. If the conjecture holds we can terminate when we find v_0 in the first recursion to be winning for player $\overline{\alpha}$. However, as is shown below, the conjecture does not hold.

Conjecture 7.4 (Disproven)**.** *For any* RecursivePG$(G\backslash A)$*, with winning sets* (W'_0, W'_1)*, that is invoked during* RecursivePG(G) *it holds that any vertex* $v \in W'_{\overline{\alpha}}$ *is won by player* $\overline{\alpha}$ *in game* G*.*

Counterexample. Consider the following parity game G:



All vertices are won by player 0 (v_1 plays to v_3, v_3 must play to v_2 and v_2 must play to itself; we always get an infinite path of v_2's).

We solve this game using RecursivePG and write down the values of relevant variables below. We use the tilde decoration to indicate values for variables in the first recursion:

```
RecursivePG($G$):
$h = 3, \alpha = 1$
$A = \{v_3\}$
    RecursivePG($G\backslash A$):
    $\tilde{h} = 2, \tilde{\alpha} = 0$
    $\tilde{A} = \{v_2\}$
        RecursivePG($G\backslash A\backslash\tilde{A}$)
    $\tilde{W}'_0 = \emptyset$
    $\tilde{W}'_1 = \{v_1\} = \tilde{W}'_{\overline{\tilde{\alpha}}}$
    **Vertex $v_1$ is in $\tilde{W}'_{\overline{\tilde{\alpha}}}$ however in $G$ the vertex is won by player $\tilde{\alpha}$.**
    $\tilde{B} = \{v_1\}$
        RecursivePG($G\backslash A\backslash\tilde{B}$)
    $\tilde{W}''_0 = \{v_2\}, \tilde{W}''_1 = \emptyset$
$W'_0 = W'_{\overline{\alpha}} = \{v_2\}$
```

```
| $W'_1 = W'_\alpha = \{v_1\}$
| $B = V$
| | RecursivePG($G \backslash B$)
| $W_0 = W_{\overline{\alpha}} = V$
```

This counterexample disproves the conjecture. □

When v_0 is not winning for player $\overline{\alpha}$ in the first recursion then we need the complete winning sets to calculate B and go in the next recursion. We extend the recursive algorithm with a variable $\Delta \subseteq \{0, 1\}$. The algorithm either returns partial winning sets, containing v_0, when v_0 is won by player $\beta \in \Delta$ or the algorithm returns complete winning sets. This solves the problem, that Conjecture 7.4 does not hold, by only allowing the algorithm to terminate before the second recursion when $\overline{\alpha}$ is in Δ. Pseudo code for the algorithm using Δ is provided in Algorithm 8.

Algorithm 8 RecursivePGLocal(*parity game* $G = (V, V_0, V_1, E, \Omega), v_0, \Delta$)

```
if $V = \emptyset$ then
    return $(\emptyset, \emptyset)$
end if
$h \leftarrow \max\{\Omega(v) \mid v \in V\}$
$\alpha \leftarrow 0$ if $h$ is even and 1 otherwise
$U \leftarrow \{v \in V \mid \Omega(v) = h\}$
$A \leftarrow \alpha\text{-}Attr(G, U)$
if $\overline{\alpha} \in \Delta$ then
    $(W'_0, W'_1) \leftarrow$ RecursivePGLocal($G \backslash A, v_0, \{\overline{\alpha}\}$)
else
    $(W'_0, W'_1) \leftarrow$ RecursivePGLocal($G \backslash A, v_0, \emptyset$)
end if
if $W'_{\overline{\alpha}} = \emptyset$ then
    $W_\alpha \leftarrow A \cup W'_\alpha$
    $W_{\overline{\alpha}} \leftarrow \emptyset$
else
    if $\overline{\alpha} \in \Delta \wedge v_0 \in W'_{\overline{\alpha}}$ then
        $W_\alpha \leftarrow \emptyset$
        $W_{\overline{\alpha}} \leftarrow W'_{\overline{\alpha}}$
    else
        $B \leftarrow \overline{\alpha}\text{-}Attr(G, W'_{\overline{\alpha}})$
        if $\overline{\alpha} \in \Delta \wedge v_0 \in B$ then
            $W_\alpha \leftarrow \emptyset$
            $W_{\overline{\alpha}} \leftarrow B$
        else
            $(W''_0, W''_1) \leftarrow$ RecursivePGLocal($G \backslash B, v_0, \Delta$)
            $W_\alpha \leftarrow W''_\alpha$
            $W_{\overline{\alpha}} \leftarrow W''_{\overline{\alpha}} \cup B$
        end if
    end if
end if
return $(W_0, W_1)$
```

To prove the correctness we show any vertex in the winning set W_γ resulting from RecursivePGLocal is indeed winning for player γ and that either the winning sets completely partition the graph or that vertex v_0 is in winning set W_β such that $\beta \in \Delta$.

The proof is in many ways similar to the proof given in [41]. We repeat part of the original reasoning in the following lemma.

Lemma 7.5. *Given:*

- *total parity game $G = (V, V_0, V_1, E, \Omega)$,*
- *non-empty set $X \subseteq V$ such that X is an α-trap in G and all vertices in X are winning for player $\overline{\alpha}$ in game G,*
- *subgame $G' = G \backslash \overline{\alpha}\text{-}Attr(G, X)$*

it holds that the winner of any vertex in G' is also the winner of that vertex in G.

Proof. Let (W'_0, W'_1) be the winning sets of game G'. Using Lemma 7.2 we find that $\overline{\alpha}\text{-}Attr(G, X)$ is an α-trap in G. Using lemma 7.1 we find that $V' = V \backslash \overline{\alpha}\text{-}Attr(G, X)$ is an $\overline{\alpha}$-trap in G.

Consider winning set W'_α for game G'. Set W'_α is an $\overline{\alpha}$-trap in G'. In game G we find that player $\overline{\alpha}$ can not escape W'_α by going to $V \backslash V'$ because V' is an $\overline{\alpha}$-trap in G. Furthermore player $\overline{\alpha}$ can not escape to $W'_{\overline{\alpha}}$ because W'_α is an $\overline{\alpha}$-trap in G'. We find that W_α is an $\overline{\alpha}$-trap in G. Finally we know that the strategy for player α used in game G' is still applicable in game G to win the vertices in W'_α for game G.

Consider winning set $W'_{\overline{\alpha}}$ for game G'. Set $W'_{\overline{\alpha}}$ is an α-trap in G'. In game G we find that player α can not escape $W'_{\overline{\alpha}}$ by going to W'_α, however he/she might escape by going to $V \backslash V' = \overline{\alpha}\text{-}Attr(G, X)$. When play goes to $\overline{\alpha}\text{-}Attr(G, X)$ then player $\overline{\alpha}$ can get the play into X which is an α-trap in G and is winning for player $\overline{\alpha}$ in G. So when the token is in $W'_{\overline{\alpha}}$ either the play stays there and $\overline{\alpha}$ uses the strategy from game G' to win or the token goes to X where player $\overline{\alpha}$ can keep the play and win. □

Theorem 7.6. *Given total parity game $G = (V, V_0, V_1, E, \Omega)$, vertex v_0 (which is not necessarily in V), $\Delta \subseteq \{0, 1\}$ and winning sets (Q_0, Q_1) for game G. We have for sets $(W_0, W_1) =$ RecursivePGLocal(G, v_0, Δ) that at least one of the following statements hold:*

(I) For some $\beta \in \Delta$ we have $v_0 \in W_\beta$, $W_0 \subseteq Q_0$ and $W_1 \subseteq Q_1$.

(II) $W_0 = Q_0$ and $W_1 = Q_1$.

Proof. First note that both statements require the vertices in the winning sets to be in the correct winning sets. Statement (I) only allows winning sets to be incomplete, it does not allow vertices to be in a winning set when that vertex is not actually won by that player. Furthermore note that statement (II) simply states that the game is solved completely.

Proof by induction on G.

Base: When G is empty then the algorithm returns $(\emptyset, \emptyset)$ in which case statement (II) holds trivially.

Step: The algorithm considers the highest priority in the game and assigns the parity of this priority to α. The set U contains all vertices with this priority and A contains all vertices from where player α can force the play into U.

The first recursion removes vertices in A from the game, since A is non-empty we can apply induction to find that at least one of the two statements hold for $G\backslash A$ and winning sets (W'_0, W'_1).

If $W'_{\overline{\alpha}} = \emptyset$ (line 13) then statement (II) must be true for $G\backslash A$ because in the first recursion α is never in Δ. We find that indeed all vertices in $G\backslash A$ are won by player α, moreover player α has a strategy σ_α for $G\backslash A$ that is winning for all vertices in $V\backslash A$. Clearly this strategy can also be applied game to G. Consider valid path π in game G conforming to σ_α. When this path eventually stays in $V\backslash A$ then player α wins because σ_α is winning here. Otherwise the path visits A infinitely often, in which case player α can force the play infinitely often into U and therefore the highest priority occurring infinitely often has parity α. So player α wins all vertices in V and the algorithm returns winning sets accordingly; statement (II) holds.

Let $W'_{\overline{\alpha}} \neq \emptyset$. If $\overline{\alpha} \in \Delta$ and $v_0 \in W'_{\overline{\alpha}}$ (line 17) then the algorithm returns $W'_{\overline{\alpha}}$ to be winning for player $\overline{\alpha}$ in game G. Using induction on game $G\backslash A$ we find that every vertex in $W'_{\overline{\alpha}}$ is won by player $\overline{\alpha}$ in $G\backslash A$ (regardless of whether statement (I) or (II) holds). Using Lemma 7.3 we find that all these vertices are also won for player $\overline{\alpha}$ in game G. Because $v_0 \in W'_{\overline{\alpha}}$ and $\overline{\alpha} \in \Delta$, statement (I) holds for game G.

Otherwise statement (II) holds for $G\backslash A$ and the algorithm continues to calculate set B (line 21). If $\overline{\alpha} \in \Delta$ and $v_0 \in B$ (line 22) then the algorithm returns all vertices in B to be winning for player $\overline{\alpha}$. As argued, all vertices in $W'_{\overline{\alpha}}$ are winning for player $\overline{\alpha}$ in game G. Clearly any vertex where player $\overline{\alpha}$ can force the play to $W'_{\overline{\alpha}}$ is also winning for player $\overline{\alpha}$. So all vertices in B are winning for player $\overline{\alpha}$ in G. Because $v_0 \in B$ and $\overline{\alpha} \in \Delta$, statement (I) holds for game G.

Otherwise the algorithm goes into the second recursion (line 26). Using induction we find that any vertex $v \in W''_\beta$ is indeed won by player β in game $G\backslash B$. The algorithm returns v to be winning for player β in game G, using Lemma 7.5 we find this to be correct. Note that we can apply Lemma 7.5 because statement (II) holds for $G\backslash A$ and using Lemma 7.3 we find that $W'_{\overline{\alpha}}$ is an α-trap in G. The algorithm also returns B to be winning for $\overline{\alpha}$, which is correct because it contains vertices such that player $\overline{\alpha}$ can play to $W'_{\overline{\alpha}}$ where player $\overline{\alpha}$ wins. If statement (II) holds for $G\backslash B$ then statement (II) also holds for G. If statement (I) holds for $G\backslash B$ then statement (I) also holds for G because we pass Δ into the recursion unmodified. $\square$

Calling RecursivePGLocal$(G, v_0, \{0,1\})$ with v_0 in G either solves the full game (statement (II)) or correctly puts v_0 in either winning set (statement (I)). In both cases v_0 is in the correct winning set and the game is solved locally.

The worst-case time complexity of the local variant is the same as the original algorithm: $O(e * n^d)$. If vertex v_0 is not winning for a player in Δ then the algorithm behaves the same as the original so its worst-case time complexity is the same.

7.1.2 Local fixed-point iteration algorithm

The fixed-point iteration algorithm can be modified to locally solve a game for vertex v_0 by distinguishing two cases:

1. If $d - 1$ is even then the outermost fixed-point variable is a greatest fixed-point variable.

When at some point $v_0 \notin Z_{d-1}$ then we know v_0 is never won by player 0 and we are done.

2. If $d-1$ is odd then the outermost fixed-point variable is a least fixed-point variable. When at some point $v_0 \in Z_{d-1}$ then we know v_0 is won by player 0 and we are done.

If vertex v_0 is won by player 0 in the first case or won by player 1 in the second case then the algorithm never terminates early. So in the worst-case the local algorithm behaves the same as the global algorithm, therefore we have identical worst-case time complexities of $O(e * n^d)$.

7.2 Locally solving variability parity games

We consider the two collective VPG algorithms we have seen thus far and create local variants of them.

7.2.1 Local recursive algorithm for variability parity games

In the previous section we have seen a local variant of Zielonka's recursive algorithm for parity games that uses $\Delta \subseteq \{0,1\}$ to indicate for which player we are trying to find the specific vertex.

Consider VPG $\hat{G}$ with configuration set $\mathfrak{C}$ and origin vertex $\hat{v}_0$ which we are trying to solve locally. Unified parity game $\hat{G}_\downarrow$ contains vertices $\mathfrak{C} \times \{\hat{v}_0\}$, if we find the winning player for each of these vertices we have solved the VPG locally. We are going to solve a unified parity game locally, but instead of finding the winner of a single vertex we are finding the winners for a set of vertices, specifically vertices: $\mathfrak{C} \times \{\hat{v}_0\}$.

When we locally solve a parity game using the recursive algorithm we can sometimes not go into the second recursion because we already found $\hat{v}_0$. However when locally solving a unified parity game we might find $\hat{v}_0$ for some configuration but not for all. When we find $\hat{v}_0$ to be won by player $\overline{\alpha} \in \Delta$ for configurations $C \subseteq \mathfrak{C}$ then we remove all vertices with configurations C from the game, i.e. we remove vertices $C \times \hat{V}$. For the remaining vertices we do go into the second recursion. Pseudo code is presented in Algorithm 9, we introduce function LocalConfs which returns the configurations for which we have found the local solution.

The algorithm uses definitions to reason about projections of unified parity games and sets to configuration(s). Previously we introduced a simple projection definition that projects a unified parity game to a configuration (Definition 5.2). This is possible because vertices in a unified parity game consist of pairs of configurations and origin vertices. We define a similar projection for sets of vertices consisting of pairs of configurations and origin vertices.

Definition 7.2. *Given set $X \subseteq (\mathfrak{C} \times \hat{V})$ we define the projection of X to $c \in \mathfrak{C}$, denoted by $X_{|c}$, as*

$$X_{|c} = \{\hat{v} \mid (c, \hat{v}) \in X\}$$

Furthermore we need to be able to reason about projections not only to a single vertex but to a group of vertices. To this end we introduce the following two definitions.

Definition 7.3. *Given set $X \subseteq (\mathfrak{C} \times \hat{V})$ we define the projection of X to $C \subseteq \mathfrak{C}$, denoted by $X_{||C}$, as*

$$X_{||C} = X \cap (C \times \hat{V})$$

Definition 7.4. *Given set $X \subseteq (\mathfrak{C} \times \hat{V})$ we define the complementary projection of X to $C \subseteq \mathfrak{C}$, denoted by $X_{|\backslash C}$, as*

$$X_{|\backslash C} = X \backslash (C \times \hat{V})$$

We prove the correctness of the algorithm by showing that every projection of the unified parity game is either solved globally or locally. We first prove the following auxiliary lemma to reason about projections.

Lemma 7.7. *Given unified parity game $G = (V, \hat{V}_0, \hat{V}_1, E, \hat{\Omega})$, configuration $c \in \mathfrak{C}$ and non-empty set $X \subseteq (\mathfrak{C} \times \hat{V})$ such that $X_{|c} \neq \emptyset$, it holds that α-Attr$(G, X)_{|c} = \alpha$-Attr$(G_{|c}, X_{|c})$.*

Proof. This lemma follows immediately from the fact that a unified parity game is the union of its projections. Furthermore, edges in unified parity games do not cross configurations, i.e. for any $((c, v), (c', v')) \in E$ we get $c = c'$. □

Theorem 7.8. *Given:*

- *VPG $\hat{G} = (\hat{V}, \hat{V}_0, \hat{V}_1, \hat{E}, \hat{\Omega}, \mathfrak{C}, \theta)$,*
- *origin vertex $\hat{v}_0 \in \hat{V}$,*
- *total unified parity game $G = (V, \hat{V}_0, \hat{V}_1, E, \hat{\Omega})$ that is a subgame of, or equal to, unified parity game $\hat{G}_\downarrow$,*
- *configuration $c \in \mathfrak{C}$,*
- *winning sets (Q_0, Q_1) for game $G_{|c}$,*
- *a set of players $\Delta \subseteq \{0, 1\}$ and*
- *winning sets $(W_0, W_1) = \textsc{RecursiveUPGLocal}(G, \hat{v}_0, \Delta)$*

at least one of the following statements hold:

(I) For some $\beta \in \Delta$ we have $(c, \hat{v}_0) \in W_\beta$, $(W_0)_{|c} \subseteq Q_0$ and $(W_1)_{|c} \subseteq Q_1$.

(II) $(W_0)_{|c} = Q_0$ and $(W_1)_{|c} = Q_1$.

Proof. Proof by induction on G.

Base: If G is empty then the algorithm returns $(\emptyset, \emptyset)$ in which case statement (II) holds trivially.

Step: When $G_{|c}$ is empty then $(W_0)_{|c} = \emptyset$ and $(W_0)_{|c} = \emptyset$ because the algorithm only returns vertices in the winning sets that are in V. In this case statement (II) holds trivially. Assume for the remainder of the proof that $G_{|c}$ is not empty, that is $V_{|c} \neq \emptyset$.

The algorithm considers the highest priority in the game and assign the parity of this priority to α. The set U contains all vertices with this priority and A contains all vertices from where player α can force the play into U.

The first recursion removes vertices in A from the game. Since A is non-empty we can apply induction to find that at least one of the two statements hold for $G \backslash A$ and winning sets (W'_0, W'_1).

Algorithm 9 RecursiveUPGLocal(*parity game* $G = ($
$V \subseteq \mathfrak{C} \times \hat{V}$,
$\hat{V}_0 \subseteq \hat{V}$,
$\hat{V}_1 \subseteq \hat{V}$,
$E \subseteq (\mathfrak{C} \times \hat{V}) \times (\mathfrak{C} \times \hat{V})$,
$\hat{\Omega} : \hat{V} \to \mathbb{N})$,
$\hat{v}_0 \in \hat{V}$,
$\Delta \subseteq \{0, 1\})$

```
1: if $V = \emptyset$ then
2:     return $(\emptyset, \emptyset)$
3: end if
4: $h \leftarrow \max\{\hat{\Omega}(\hat{v}) \mid (c, \hat{v}) \in V\}$
5: $\alpha \leftarrow 0$ if $h$ is even and 1 otherwise
6: $U \leftarrow \{(c, \hat{v}) \in V \mid \hat{\Omega}(\hat{v}) = h\}$
7: $A \leftarrow \alpha\text{-}Attr(G, U)$
8: if $\overline{\alpha} \in \Delta$ then
9:     $(W_0', W_1') \leftarrow$ RecursiveUPGLocal$(G \backslash A, \hat{v}_0, \{\overline{\alpha}\})$
10: else
11:     $(W_0', W_1') \leftarrow$ RecursiveUPGLocal$(G \backslash A, \hat{v}_0, \emptyset)$
12: end if
13: if $W_{\overline{\alpha}}' = \emptyset$ then
14:     $W_\alpha \leftarrow A \cup W_\alpha'$
15:     $W_{\overline{\alpha}} \leftarrow \emptyset$
16: else
17:     $C_W \leftarrow$ LocalConfs$(W_{\overline{\alpha}}')$
18:     if $(W_{\overline{\alpha}}')_{|\backslash C_W} = \emptyset$ then
19:         $W_{\overline{\alpha}} \leftarrow W_{\overline{\alpha}}'$
20:         $W_\alpha \leftarrow (W_\alpha' \cup A)_{|\backslash C_W}$
21:     else
22:         $B \leftarrow \overline{\alpha}\text{-}Attr(G, (W_{\overline{\alpha}}')_{|\backslash C_W})$
23:         $C_B \leftarrow$ LocalConfs$(B)$
24:         $(W_0'', W_1'') \leftarrow$ RecursiveUPGLocal$(G \backslash B \backslash (V_{||C_W \cup C_B}), \hat{v}_0, \Delta)$
25:         $W_\alpha \leftarrow W_\alpha''$
26:         $W_{\overline{\alpha}} \leftarrow W_{\overline{\alpha}}'' \cup B \cup (W_{\overline{\alpha}}')_{||C_W}$
27:     end if
28: end if
29: return $(W_0, W_1)$
```

```
1: function LocalConfs$(X \subseteq V)$
2:     if $\overline{\alpha} \in \Delta$ then
3:         return $\{c \in \mathfrak{C} \mid (c, \hat{v}_0) \in X\}$
4:     else
5:         return $\emptyset$
6:     end if
7: end function
```

If $W'_{\overline{\alpha}} = \emptyset$ (line 13) then no vertex is won by player $\overline{\alpha}$ in $G\backslash A$ and therefore no vertex in $(G\backslash A)_{|c}$ is won by player $\overline{\alpha}$. Therefore statement (II) holds for $G\backslash A$ and indeed all vertices in $(G\backslash A)_{|c}$ are won by player α, moreover player α has a strategy σ_α for $(G\backslash A)_{|c}$ that is winning for all vertices. Clearly this strategy can also be applied in game $G_{|c}$. Consider valid path π in game $G_{|c}$ conforming to σ_α. When this path eventually stays in $(V\backslash A)_{|c}$ then player α wins because σ_α is winning here. Otherwise the path visits $A_{|c}$ infinitely often, in which case player α can force the play into $U_{|c}$ infinitely often and therefore the highest priority occurring infinitely often has parity α. So player α wins all vertices in $V_{|c}$ and the algorithm returns winning sets accordingly; statement (II) holds.

Otherwise the algorithm continues by calculating configuration set C_W (line 17). For the remainder of the proof numerous case distinction need to be made. These distinctions will be presented in a Fitch-like style to improve readability.

First we distinguish two cases for $A_{|c}$.

Assume $A_{|c} = \emptyset$

Clearly $G_{|c} = (G\backslash A)_{|c}$, so all vertices in $(W'_{\overline{\alpha}})_{|c}$ are won by player $\overline{\alpha}$ in game $G_{|c}$.

Assume $A_{|c} \neq \emptyset$

We use Lemma's 7.7 and 7.3 to find that the vertices in $(W'_{\overline{\alpha}})_{|c}$ are won by player $\overline{\alpha}$ in $G_{|c}$.

In either case we find that all vertices in $(W'_{\overline{\alpha}})_{|c}$ are won by player $\overline{\alpha}$ in $G_{|c}$.

Assume $c \in C_W$

We find $(c, \hat{v}_0) \in W'_{\overline{\alpha}}$ and $\overline{\alpha} \in \Delta$.

Assume $(W'_{\overline{\alpha}})_{|\backslash C_W} = \emptyset$ (line 18)

The algorithm returns all vertices in $W'_{\overline{\alpha}}$ to be winning for player $\overline{\alpha}$ in game G, this is correct because we found that all vertices in $(W'_{\overline{\alpha}})_{|c}$ are won by player $\overline{\alpha}$ in $G_{|c}$. For player α the set $(W'_\alpha \cup A)_{|\backslash C_W}$ is returned, this set contains no vertices of configuration c (because $c \in C_W$). We conclude that statement (I) holds.

Assume $(W'_{\overline{\alpha}})_{|\backslash C_W} \neq \emptyset$ (line 18)

The algorithm continues with calculating set B (line 23). Set B, as calculated by the algorithm, contains no vertices with configuration c because the attractor set is calculated over $(W'_{\overline{\alpha}})_{|\backslash C_W}$.

The second subgame that is created by the algorithm (line 24) does not contain any vertices with configuration c because $V_{||C_W}$ is removed from the game. Therefore W''_0 and W''_1 do not contain any vertices with configuration c.

We find that the only vertices with configuration c that are returned by the algorithm are in the set $(W'_{\overline{\alpha}})_{||C_W}$. Earlier we found that indeed the vertices in $(W'_{\overline{\alpha}})_{|c}$ are won by player $\overline{\alpha}$ in $G_{|c}$. We conclude that statement (I) holds.

Assume $c \notin C_W$

If statement (I) holds for $G\backslash A$ then we would find $c \in C_W$, since this is not the case we find that statement (II) holds for $G\backslash A$.

Assume $(W'_{\overline{\alpha}})_{|\backslash C_W} = \emptyset$ (line 18)

Because $c \notin C_W$ we find that $(W'_{\overline{\alpha}})_{|c} = \emptyset$. Because statement (II) holds for $G\backslash A$ we find that all vertices in game $(G\backslash A)_{|c}$ are won by player α. Earlier we argued that when all vertices in game $(G\backslash A)_{|c}$ are won by player α then all vertices in game $G_{|c}$ are won by player α. The algorithm returns all vertices in $(W'_\alpha \cup A)_{|c}$ to be winning for player α, these are all the vertices from $G_{|c}$. No vertices with configuration c are returned to be winning for player $\overline{\alpha}$. We conclude that statement (II) holds.

Assume $(W'_{\overline{\alpha}})_{|\backslash C_W} \neq \emptyset$ (line 18)

The algorithm continues with calculating B (line 22) and C_B.

Assume $c \in C_B$

The second subgame that is created by the algorithm (line 24) does not contain any vertices with configuration c because $V_{||C_B}$ is removed from the game. Therefore W''_0 and W''_1 do not contain any vertices with configuration c. We find that the only vertices with configuration c that are returned by the algorithm are in the set B and $W'_{\overline{\alpha}}$.
For c to be in C_B we must have $(W'_{\overline{\alpha}})_{|c} \neq \emptyset$. We can apply Lemma 7.7 to find that set $B_{|c}$ contains all vertices such that player $\overline{\alpha}$ can force the play to $(W'_{\overline{\alpha}})_{|c}$ in game $G_{|c}$. Earlier we found that all vertices in $(W'_{\overline{\alpha}})_{|c}$ are won by player $\overline{\alpha}$ in game $G_{|c}$ so clearly all vertices in $B_{|c}$ are also won by player $\overline{\alpha}$.
The algorithm returns vertices $B_{|c}$ to be winning for player $\overline{\alpha}$. Because $c \in C_B$ we find that $\overline{\alpha} \in \Delta$ and $\hat{v}_0 \in B_{|c}$. We conclude that statement (I) holds.

Assume $c \notin C_B$

Assume $(W'_{\overline{\alpha}})_{|c} = \emptyset$

In this case $B_{|c} = \emptyset$ and the second subgame G' created (line 24) projected onto c is identical to $G_{|c}$. Using induction we find that statement (I) or (II) hold for G'. The algorithm returns W''_0 and W''_1 for game G' so the same statement that holds for the subgame holds for G. Note that B and $(W'_{\overline{\alpha}})_{||C_W}$ do not contain vertices with configuration c.

Assume $(W'_{\overline{\alpha}})_{|c} \neq \emptyset$

We apply Lemma 7.7 to find that $B_{|c} = \overline{\alpha}\text{-}Attr(G_{|c}, (W'_{\overline{\alpha}})_{|c})$. For the second subgame G' created (line 24) we have $(G')_{|c} = G_{|c}\backslash B_{|c}$ because $V_{||C_W \cup C_B}$ contains no vertices with configuration c.
The algorithm returns any vertex $\hat{v}$ in $(W''_\beta)_{|c}$ to be winning for player β in game $G_{|c}$. Using induction and Lemma 7.5 we find that indeed $\hat{v}$ is won by player β in game $G_{|c}$. Furthermore the algorithm returns $B_{|c}$ to be winning for player $\overline{\alpha}$ as well as $(W'_{\overline{\alpha}})_{|c}$. Earlier we found that all vertices in $(W'_{\overline{\alpha}})_{|c}$ are won by player $\overline{\alpha}$ in game $G_{|c}$, so clearly all vertices in $B_{|c}$ are also won by player $\overline{\alpha}$.
The vertices with configuration c that are returned by the algorithm are in the correct winning set. If statement (I) holds for the subgame G' then statement (I) also holds for G because we use Δ unmodified in the recursion. If statement (II) holds for the subgame G' then statement (II) also holds for G.

□

The pseudo code presented for the algorithm uses a set-wise representation of unified parity games. As we have seen previously in the RecursiveUPG algorithm, we can modify the recursive algorithm to use a function-wise representation with a function-wise attractor set calculation. The RecursiveUPGLocal algorithm can be transformed in the same way. The RecursiveUPGLocal algorithm introduces defintions for projections to sets of configurations as well as the LocalConfs subroutine. We introduce function-wise variants for these definitions and subroutine.

Definition 7.5. *Given function $X : \hat{V} \to 2^{\mathfrak{C}}$ we define the projection of X to $C \subseteq \mathfrak{C}$, denoted by $X_{||C}$, as*

$$X_{||C}(\hat{v}) = X(\hat{v}) \cap C$$

Definition 7.6. *Given function $X : \hat{V} \to 2^{\mathfrak{C}}$ we define the complementary projection of X to $C \subseteq \mathfrak{C}$, denoted by $X_{|\backslash C}$, as*

$$X_{|\backslash C}(\hat{v}) = X(\hat{v}) \backslash C$$

Algorithm 10 shows a function wise implementation of the LocalConfs subroutine. It is trivial to see that this algorithm and the projection definitions are equal under the $=_\lambda$ operator to their set-wise counterparts.

Algorithm 10 Function-wise LocalConfs subroutine

```
function FLocalConfs($X : \hat{V} \to 2^{\mathfrak{C}}$)
    if $\overline{\alpha} \in \Delta$ then
        return $X(\hat{v}_0)$
    else
        return $\emptyset$
    end if
end function
```

We can solve a VPG locally using this local recursive algorithm for unified parity games. We can either represent the parity games set-wise or we can present them function-wise, in which case we can represent the sets of configurations explicitly or symbolically. In all three cases the time complexities are identical to their global counterparts because in the worst case the vertex we are searching for is never won by player $\overline{\alpha} \in \Delta$ at any recursion level. Furthermore the added projection operations are subsumed in worst-case time complexity by the attractor set calculation so the worst-case time complexity argumentation presented for the global variants is also valid for the local variants.

7.2.2 Local incremental pre-solve algorithm

The incremental pre-solve algorithm is particularly appropriate for local solving; if we find v_0 in P_α then we know that v_0 is won by player α for every configuration, therefore we are done for that particular recursion. This can potentially reduce the recursion depth of the algorithm and therefore reduce the number of (pessimistic) games solved.

Furthermore, when there is only a single configuration left the incremental pre-solve algorithm solves the corresponding parity game. When taking a local approach it is sufficient to solve this parity game locally using the local fixed-point iteration algorithm. Note that the pessimistic games still must be solved globally to find as much assistance as possible for further recursions.

If v_0 is not found in either P_0 or P_1 and is only solved when there is one configuration left, then the local algorithm behaves the same as the global algorithm; we have identical worst-case time complexities: $O(c * e * n^d)$.

8. Experimental evaluation

The algorithms proposed to solve VPGs collectively all have the same or a worse time complexity than the independent approach. The aim of the collective algorithms is to solve VPGs effectively when there are a lot of commonalities between configurations. A worst-case time complexity analyses does not say much about the performance in case there are many commonalities. In order to evaluate actual running time the algorithms are implemented and a number of test VPGs are created to test the performance on. In this section we discuss the implementation and look at the results.

During the previous sections we put forth a number of hypotheses about the performance of the algorithms introduced. In this section we evaluate these hypotheses, specifically we hypothesised

- that the recursive algorithm for VPGs can attract a large number of configurations per origin vertex at the same time,
- that the recursive symbolic algorithm for VPGs performs well when solving VPGs originating from FTSs,
- that the incremental pre-solve algorithm outperforms independent approaches and
- that the increase in performance between a global-collective and local-collective approach is greater than the increase in performance between a global-independent and local-independent approach.

8.1 Implementation

The algorithms are implemented in C++ version 14 and use BuDDy[1] as a BDD library. The complete source is hosted on github[2].

The implementation is split in three phases: parsing, solving and solution printing. The solving part contains the implementations of the algorithms presented. The parsing and solution printing parts are implemented trivially and hardly optimized, their running times are not considered in the experimental evaluation.

The parsing phase of the algorithm creates BDDs from the input file, in doing so parts of the BDD cache gets filled. After parsing the BDD cache is cleared to make sure that the work done in the solving phase corresponds with the algorithms presented and no work to assist it has been done prior to this phase. Creating BDDs is not a trivial task, however one could argue that an FTS should already express its transition guards as BDDs. In any case, we leave the creation of BDDs out of scope.

8.1.1 Game representation

The graph is represented using adjacency lists for incoming and outgoing edges, furthermore every edge maps to a set of configurations representing the θ value for the edge. Sets of configurations are either represented symbolically or explicitly. In the former case we use BDDs, in the latter case we use bit-vectors. For independent algorithms the edges are not mapped to sets of configurations. Finally sets of vertices are represented using bit-vectors.

[1]https://sourceforge.net/projects/buddy/
[2]https://github.com/SjefvanLoo/VariabilityParityGames/tree/master/implementation/VPGSolver

Note that only the representation of the games used during the algorithm is relevant, since we do not evaluate the parsing phase it is not relevant how the games are stored in a file.

8.1.2 Independent algorithms

Four independent algorithms are implemented, i.e. algorithms that solve parity games. A global and local variant is implemented of the following algorithms:

- Zielonka's recursive algorithm and
- fixed-point iteration algorithm.

We implement the fixed-point iteration algorithm to use pre-solved vertices P_0 and P_1. When using the algorithm for an independent approach we use $\emptyset$ for P_0 and P_1, in this case the algorithm behaves the same as the original fixed-point iteration algorithm.

A few optimizations are applied to the fixed-point iteration algorithm. The following three are described in [4]:

- For fixed-point variable Z_i its value is only ever used to check if a vertex with priority i is in Z_i. So instead of storing all vertices in Z_i we only have to store the vertices that have priority i. We can store all fixed-point variables in a single bit-vector, named Z, of size n.
- The algorithm only reinitializes a certain range of fixed-point variables. So the diamond and box operations can use the previous result and only reconsider vertices that have an edge to a vertex that has a priority for which its fixed-point variable is reset.
- The algorithm updates variables Z_0 to Z_m and reinitializes Z_0 to Z_{m-1}, however if Z_m is a least fixed-point variable then Z_m has just increased and due to monotonicity the other least fixed-point formula's, i.e. $Z_{m-2}, Z_{m-4}, \dots$, will also increase so there is no need to reset them. Similarly for greatest fixed-point variables. So we only to reset half of the variables instead of all of them.

Furthermore, the vertices in the game are reordered such that they are sorted by parity first and by priority second. Using the above optimizations the algorithm needs to reset variables $Z_m, Z_{m-2}, \dots$, these variables are stored in a single bit-vector Z. By reordering the variables to be sorted by parity and priority these vertices that need to be reset are always consecutively stored in Z, so resetting this sequence can be done by a memory copy instead of iterating all the different vertices. Note that when the algorithm is used by the pre-solve algorithm the variables are not reset to simply $\emptyset$ and V but are reset to two specific bit-vectors that are given by the pre-solve algorithm. These bit-vectors have the same order and resetting can be done by copying a part of them into Z.

The advantage of using a memory copy as opposed to iterating all the different vertices is due to the fact that a bit vector uses integers to store its boolean values. A 64-bit integer can store 64 boolean values. Iterating and writing every boolean value individually causes the integer to be written 64 times. However with a memory copy we can simply copy the entire integer value and the integer is only written once.

Finally, priority compression is applied when using the fixed-point iteration algorithm. Priority compression makes sure the lowest priority is 0 or 1 and for every priority p that is lower or equal to the highest priority occurring in the game we have p being either the lowest priority in the game or there is a vertex in the game with priority $p - 1$ [4, 15].

8.1.3 Collective algorithms

Six collective algorithms are implemented, i.e. algorithms for solving VPGs. A global and local variant is implemented of the following algorithms:

- Zielonka's recursive algorithm for VPGs with explicit configuration set representation,
- Zielonka's recursive algorithm for VPGs with symbolic configuration set representation and
- incremental pre-solve algorithm.

The incremental pre-solve algorithms use the fixed-point iteration algorithm as described above to solve the (pessimistic) parity games. When using the incremental pre-solve algorithm we apply priority compression once, directly on the VPG. Since the (pessimistic) parity games that are created during the algorithm have the same vertices as the VPG we do not have to apply priority compression again when using the fixed-point iteration algorithm to solve them.

The incremental pre-solve algorithm creates subgames by splitting the set of configurations. The games we evaluate are based on features, we simply split the set of configurations by choosing a feature arbitrarily and requiring this feature in one set of configurations and excluding this feature in the other set of configurations.

8.1.4 Random verification

In order to prevent implementation mistakes 200 VPGs are created randomly, every VPG is projected to all its configurations to get a set of parity games. These parity games are solved using the PGSolver tool [18]. All algorithms implemented are used to solve the 200 VPGs independently and collectively, the solutions are compared to the solutions created by the PGSolver.

8.2 Test cases

We evaluate the performance of the algorithms on numerous test cases. We have two model checking problems as well as random VPGs. The model checking VPGs are created as described in chapter 5, with the exception that only vertices are added when they are reachable from the initial vertex. So these games are never disjointed, random games can be disjointed.

8.2.1 Model checking games

We use two SPL examples, first the minepump example as described in [23] and implemented in the mCRL2 toolset as described in [34]. The minepump example models the behaviour of controllers for a pump that pumps water out of a mineshaft. There are 10 different features that change the way the sensors/actors behave. In total there are 128 valid feature assignments, i.e. products.

The mCRL2 implementation creates an LTS with parametrized actions, the parameters describe the boolean formula's guarding the transitions, effectively making it an FTS consisting of 582 states and 1376 transitions. This FTS is interpreted in combination with nine different μ-calculus formulas to create nine VPGs.

We choose to represent the sets of configurations using 10 boolean variables even though 128 configurations could be represented using only 7 boolean variables. By using the same number of variables as there are features the boolean formula's from the FTS are left intact when using them in the VPG. Table 8.1 shows the different formula's, as well as the result of the verification and the size of the resulting games. All the properties can be expressed in the modal μ-calculus we introduced in Definition 3.10,. However, for readability, we present them using action formula's, regular formula's and universal quantifiers [19]. Table 8.1 is largely identical to the table with results from [34].

	formula	**t/f**	n	d
φ_1	*Absence of deadlock* $[\mathbf{true}^*]\langle\mathbf{true}\rangle\top$	128/0	3494	2
φ_2	*The controller cannot infinitely often receive water level readings* $\mu X.[(\neg levelMsg^*.levelMsg]X$	0/128	3004	3
φ_3	*The controller cannot fairly receive each of the three message types* $\mu X.([\mathbf{true}^*.commandMsg]X \vee [\mathbf{true}^*.alarmMsg]X \vee$ $[\mathbf{true}^*.levelMsg]X)$	0/128	9156	3
φ_4	*The pump cannot be switched on infinitely often* $(\mu X.\nu Y.([pumpStart.(\neg pumpStop)^*.pumpStop]X \wedge$ $[\neg pumpStart]Y)) \wedge ([\mathbf{true}^*.pumpStart]\mu Z.[\neg pumpStop]Z)$	96/32	6236	4
φ_5	*The system cannot be in a situation in which the pump runs indefinitely in the presence of methane* $[\mathbf{true}^*]((([pumpStart.(\neg pumpStop)^*.methaneRise]\mu X.[R]X) \wedge$ $([methaneRise.(\neg methaneLower)^*.pumpStart]\mu X.[R]X))$ for $R = \neg(pumpStop + methaneLower)$	96/32	7096	3
φ_6	*Assuming fairness (φ_3), the system cannot be in a situation in which the pump runs indefinitely in the presence of methane (φ_5)* $[\mathbf{true}^*]((([pumpStart.(\neg pumpStop)^*.methaneRise]\Psi) \wedge$ $([methaneRise.(\neg methaneLower)^*.pumpStart]\Psi)$ for $\Psi = \mu X.([R^*.commandMsg]X \vee [R^*.alarmMsg]X \vee [R^*.levelMsg]X)$ and $R = \neg(pumpStop + methaneLower)$	112/16	9224	4
φ_7	*The controller can always eventually receive/read a message, i.e. it can return to its initial state from any state* $[\mathbf{true}^*]\langle\mathbf{true}^*.receiveMsg\rangle\top$	128/0	5285	3
φ_8	*Invariantly the pump is not started when the low water level signal fires* $[\mathbf{true}^*.lowLevel.(\neg(normalLevel + highLevel))^*.pumpStart]\bot$	128/0	3902	2
φ_9	*Invariantly, when the level of methane rises, it inevitably decreases* $[\mathbf{true}^*.methaneRise]\mu X.[\neg methaneLower]X \wedge \langle\mathbf{true}\rangle\top$	0/128	5418	3

Table 8.1: Minepump properties with their partitioning and the size of the resulting VPG. In the **t/f** columns the first number shows for how many products the property holds. Columns n and d shows the number of vertices and distinct priorities in the resulting VPG. A large part of the table is taken from [34]

Next we have the elevator example, described in [27]. This example models the behaviour of an elevator where five different features modify the behaviour of the model. All feature assignments are valid. Therefore, we have $2^5 = 32$ feature assignments, i.e. products. Again an mCRL2 implementation[3] (created by T.A.C. Willemse) is used to create seven VPGs. The FTS consists 33738 states and 206290 transitions. Table 8.2 shows the different formula's, as well as the result of the verification and the size of the resulting games.

	formula	**t/f**	n	d
φ_1	*If a landing button is pressed at Level i, the lift will inevitably open its doors on Level i* $[\mathbf{true}^*]\forall i \in [1,5].[\mathit{landingButton}(i)](\mu X.([\neg \mathit{open}(i)]X \wedge \langle \mathbf{true}\rangle\top)$ $)$	2/30	1379959	3
φ_2	*If a lift button is pressed at Level i, the lift will inevitably open its doors on Level i* $[\mathbf{true}^*]\forall i \in [1,5].[\mathit{liftButton}(i)](\mu X.([\neg \mathit{open}(i)]X \wedge \langle \mathbf{true}\rangle\top))$	4/28	1381390	3
φ_3	*If the lift is travelling up while there are calls in that direction it will not change the direction it is travelling* $[\mathbf{true}^*]($ $[\mathit{direction}(\mathit{up}).(\neg(\mathit{direction}(\mathit{down}) + \exists k \in [1,5].\mathit{open}(k)))^*]$ $\forall i \in [1,5].[\mathit{open}(i)]\forall j \in [i+1,5].$ $[\mathit{liftButton}(j)]\mu Y.($ $[\neg \mathit{open}(j)]Y \wedge [\mathit{direction}(\mathit{down})]\mathbf{false} \wedge \langle \mathbf{true}\rangle\top))$	4/28	1778065	3
φ_4	*If the lift is travelling down while there are calls in that direction it will not change the direction it is travelling* $[\mathbf{true}^*]($ $[\mathit{direction}(\mathit{down}).(\neg(\mathit{direction}(\mathit{up}) + \exists k \in [1,5].\mathit{open}(k)))^*]$ $\forall i :\in [1,5].[\mathit{open}(i)]\forall j \in [1,i-1].$ $[\mathit{liftButton}(j)]\mu Y.($ $[\neg \mathit{open}(j)]Y \wedge [\mathit{direction}(\mathit{up})]\mathbf{false} \wedge \langle \mathbf{true}\rangle\top))$	4/28	1853633	3
φ_5	*If the lift is idling on Level i, it can remain at Level i* $(\forall i \in [1,5].\langle \mathbf{true} * .\mathit{idling}(i)\rangle\top)\wedge$ $[\mathbf{true}^*]\forall i \in [1,5].[\mathit{idling}(i)]\nu Y.\langle \mathit{idling}(i)\rangle Y$	16/16	1282147	2
φ_6	*The lift may stop at Levels 2,3 and 4 for landing calls when travelling upwards* $\forall i \in [2,4].(\langle(\neg \mathit{liftButton}(i))^*.\mathit{direction}(\mathit{up}).$ $(\neg(\mathit{liftButton}(i) + \mathit{direction}(\mathit{down})))^*.\mathit{open}(i)\rangle\top)$	32/0	443352	2
φ_7	*The lift may stop at Levels 2,3 and 4 for landing calls when travelling downwards* $\forall i \in [2,4].(\langle(\neg \mathit{liftButton}(i))^*.\mathit{direction}(\mathit{down}).$ $(\neg(\mathit{liftButton}(i) + \mathit{direction}(\mathit{up})))^*.\mathit{open}(i)\rangle\top)$	32/0	443012	2

Table 8.2: Elevator properties with their partitioning and the size of the resulting VPG. In the **t/f** columns the first number shows for how many products the property holds. Columns n and d shows the number of vertices and distinct priorities in the resulting VPG.

[3]https://github.com/SjefvanLoo/VariabilityParityGames/blob/master/implementation/Elevator.tar.gz

8.2.2 Random games

Random VPGs can be created by creating a random parity game and create sets of configurations that guard the edges. For these sets we need to consider two factors: how large are the sets guarding the edges and how are they constructed.

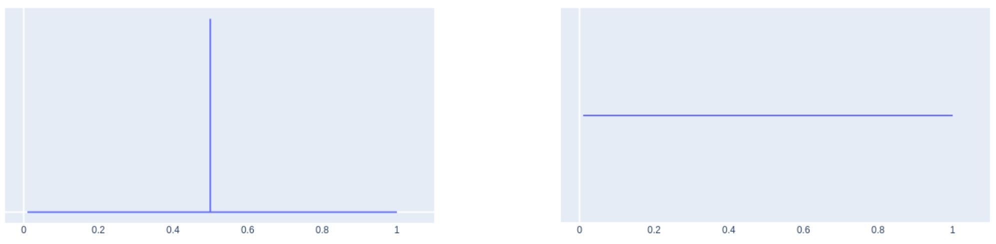
The guard sets in the minepump and elevator games have a very specific distribution where nearly all of the sets admit either 100% or 50% of the configurations. This is because an edge requiring the presence or absence of one specific feature results in a set admitting 50%. On average the edges in the examples admit 92% of the configurations. Most likely VPGs originating from FTSs will have such a distribution.

We use λ to denote the average relative size of guard sets in a VPG. So for every guard set in a VPG we divide its size by the total number of configurations to get the relative size of the guard set. Taking the average of all these relative sizes calculates λ.

We create a random game with a specific λ by using a probabilistic distribution ranging from 0 to 1 to determine the size of an individual guard set. Such a distribution must have a mean equal to λ. We consider two distributions:

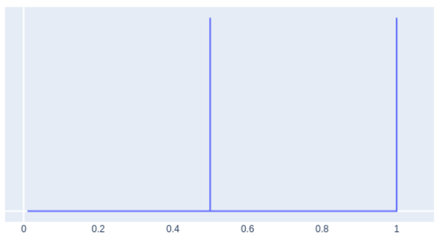
- A modified Bernoulli distribution; in a Bernoulli distribution there is a probability of p to get an outcome of 1 and a probability of $1-p$ to get an outcome of 0. We modify this such that there is a probability of p to get 1 and a probability of $1-p$ to get 0.5. This gives a mean of $1p + 0.5(1-p) = 0.5p + 0.5$. So to get a mean of λ we choose $p = 2\lambda - 1$. Note that we cannot use this distribution when $\lambda < 0.5$ because p becomes less than 0.
- A beta distribution; a beta distribution ranges from 0 to 1 and is curved such that it has a specific mean. The beta distribution has two parameters: α and β and a mean of $\frac{\alpha}{\alpha+\beta}$. We pick $\beta = 1$ and $\alpha = \frac{\lambda\beta}{1-\lambda}$ to get a mean of λ.

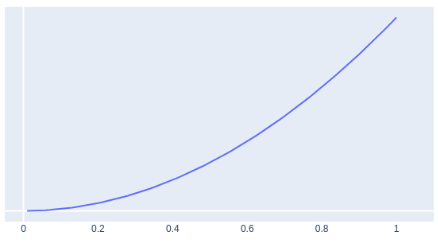
Figures 8.1, 8.2 and 8.3 show the shapes of the distribution for different values for λ.



(a) Modified Bernoulli distribution with $p = 0$ **(b)** Beta distribution with $\beta = 1$ and $\alpha = 1$

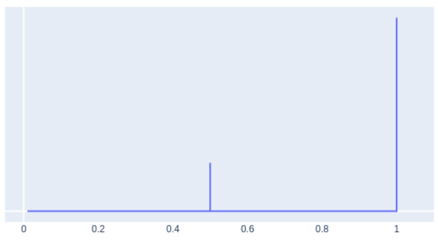
Figure 8.1: Edge guard size distribution for $\lambda = 0.5$

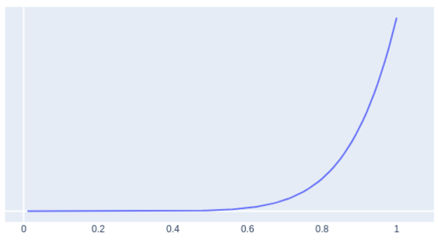




(a) Modified Bernoulli distribution with $p = 0.5$ **(b)** Beta distribution with $\beta = 1$ and $\alpha = 3$

Figure 8.2: Edge guard size distribution for $\lambda = 0.75$





(a) Modified Bernoulli distribution with $p = 0.8$ **(b)** Beta distribution with $\beta = 1$ and $\alpha = 9$

Figure 8.3: Edge guard size distribution for $\lambda = 0.9$

Once we have determined the size for a configuration set we need to consider how the set is constructed. We can simply create a random set of configurations without any notion of features, we call this a *configuration based* approach. Alternatively we can use a *feature based* approach where we create sets by looking at features. Consider features $f_0, \dots, f_m$, we can create a boolean function that is the conjunction of k features where every feature in the conjunction has probability $\frac{1}{2}$ of being negated. For example when using $k = 3$ and $m = 5$ we might get boolean formula $f_1 \wedge \neg f_2 \wedge \neg f_4$. Such a boolean formula corresponds to a set of configurations of size 2^{m-k} and a relative size $\frac{2^{m-k}}{2^m} = 2^{-k}$, so when creating a set with relative size r we choose $k = \min(m, \lfloor -\log_2 r \rfloor)$. When using a feature based approach we can only create sets that have a relative size of 2^{-i} for some $i \in \mathbb{N}$.

We can create 4 types of games:

1. Bernoulli distributed and feature based. These games are most similar to the model verification games.
2. Bernoulli distributed and configuration based. These games do have the characteristics of a model verification game in terms of set size but have unstructured sets guarding the

Category	# vertices	Maximum # successors	# distinct priorities	# confs	λ
Type 1, scale in λ	$100 - 600$	$3 - 20$	$1 - 10$	$2^4 - 2^{12}$	$\frac{game\ nr}{100}$
Type 2, scale in λ					
Type 3, scale in λ					
Type 1, scale in # confs	$100 - 600$	$3 - 20$	$1 - 10$	$2^{game\ nr}$	0.92

Table 8.3: Categories of random games

edges. Furthermore with a configuration based approach less guard sets will be identical than with a feature based approach.

3. Beta distributed and configuration based. These games are most different from the model verification games.

4. Beta distributed and feature based. Using a feature based approach we can only create sets of size 2^{-i} for any $\lambda \geq \frac{1}{2}$. So using a beta distribution we must round to such a size. Almost all the sets will get a relative size of either $\frac{1}{2}$ or 1. So this creates almost the exact same games as using the Bernoulli distribution, therefore we will not consider this category of games.

We create four sets of random games. For random games of type 1,2 and 3 we create 25 games: game 75 to game 99, where game i has $\lambda = \frac{i}{100}$ and a random number of features, nodes, edges and maximum priority. Furthermore we create 52 games to evaluate how the algorithm scales when the number of features becomes larger. For every $i \in [2, 15]$ we create random games ia, ib, ic and id of type 1 with $\lambda = 0.92$, i features and a random number of nodes, edges and maximum priority.

Besides the number of configurations and the value for λ we need to choose the number of vertices for a game, the minimum number of successors of a vertex, the maximum number of successors of a vertex and the number of distinct priorities in the game. The number of minimum and maximum successor is decides per game. So if we pick l and h as the number of minimum and maximum then for every vertex of the game we uniformly pick its number of successors between l and h.

Table 8.3 shows the different categories of games and the corresponding parameters. The minimum number of successors per vertex is always 1 so this value is omitted from the table. The games that scale in λ share the same random configuration per game number. So game i of type 1 that scales in λ has the same number of vertices, maximum successors, distinct priorities and configurations as game i of type 2 and 3 that scale in λ.

8.3 Results

In this section the experimental results are presented. We evaluate the performance on six sets of games:

- the minepump games,
- the elevator games,
- random games of type 1 with an increasing λ,

- random games of type 2 with an increasing λ,
- random games of type 3 with an increasing λ and
- random games of type 1 with an increasing number of configuration.

We present the times it took to solve a VPG. For an independent approach this means the sum of the times it taken to solve every projection of the VPG. For a collective approach this means simply means the solve time for the VPG. In either case we only measure the solve time; parsing, projecting and solution printing is excluded from the evaluation.

The exact times can be found in appendix A, in this section the results are visualized and presented in a way such that we can easily compare independent and collective approaches. We have four independent approaches:

- Recursive algorithm (global),
- Recursive algorithm (local),
- fixed-point iteration (global) and
- fixed-point iteration (local).

For every set of problems we present four charts; for every independent approaches we present a chart where its performance is compared to one or two collective algorithms. We compare the performance of the recursive algorithm for VPGs with the performance of the original recursive algorithm and the performance of the incremental pre-solve algorithm with the performance of the fixed-point iteration algorithm. In every chart the solve times are divided by the independent solve times to visualize how much better or worse the collective variants perform.

The following legend holds for all charts presented in this section:

Independent approaches:

🟦━━ Recursive algorithm for parity games (global)
🟪━━ Fixed-point iteration algorithm for parity games (global)
🟦┅┅ Recursive algorithm for parity games (local)
🟪┅┅ Fixed-point iteration algorithm for parity games (local)

Collective approaches:

🟩━━ Recursive algorithm for VPGs with a symbolic representation of configurations (global)
🟥━━ Recursive algorithm for VPGs with an explicit representation of configurations (global)
🟧━━ Incremental pre-solve algorithm (global)
🟩┅┅ Recursive algorithm for VPGs with a symbolic representation of configurations (local)
🟥┅┅ Recursive algorithm for VPGs with an explicit representation of configurations (local)
🟧┅┅ Incremental pre-solve algorithm (local)

All the experiments are ran on a Linux x64 operating system with an Intel i5-4570 @ 3.20 GHz processor and 8GB of DDR3 RAM.

8.3.1 Model checking examples

Figures 8.4 and 8.5 show the solving times of the algorithms when applied to the model checking examples.

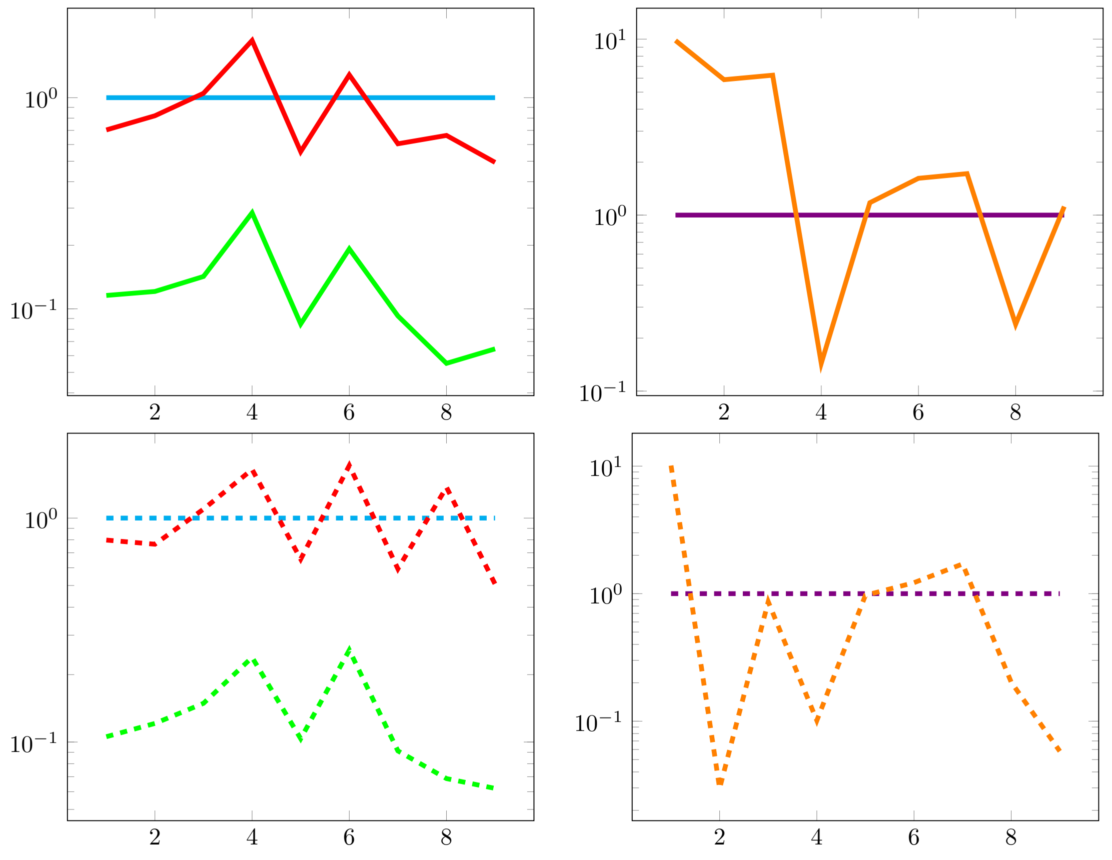


Figure 8.4: Running times on the minepump games, times are normalized

For the minepump example we see that the recursive algorithm for VPGs using a symbolic representation of configurations performs particularly well. For the elevator example we find a similar result, however for these games we also find a good performance for the incremental pre-solve algorithm. There is no clear difference between the relative performances of the global algorithms and the local algorithms.

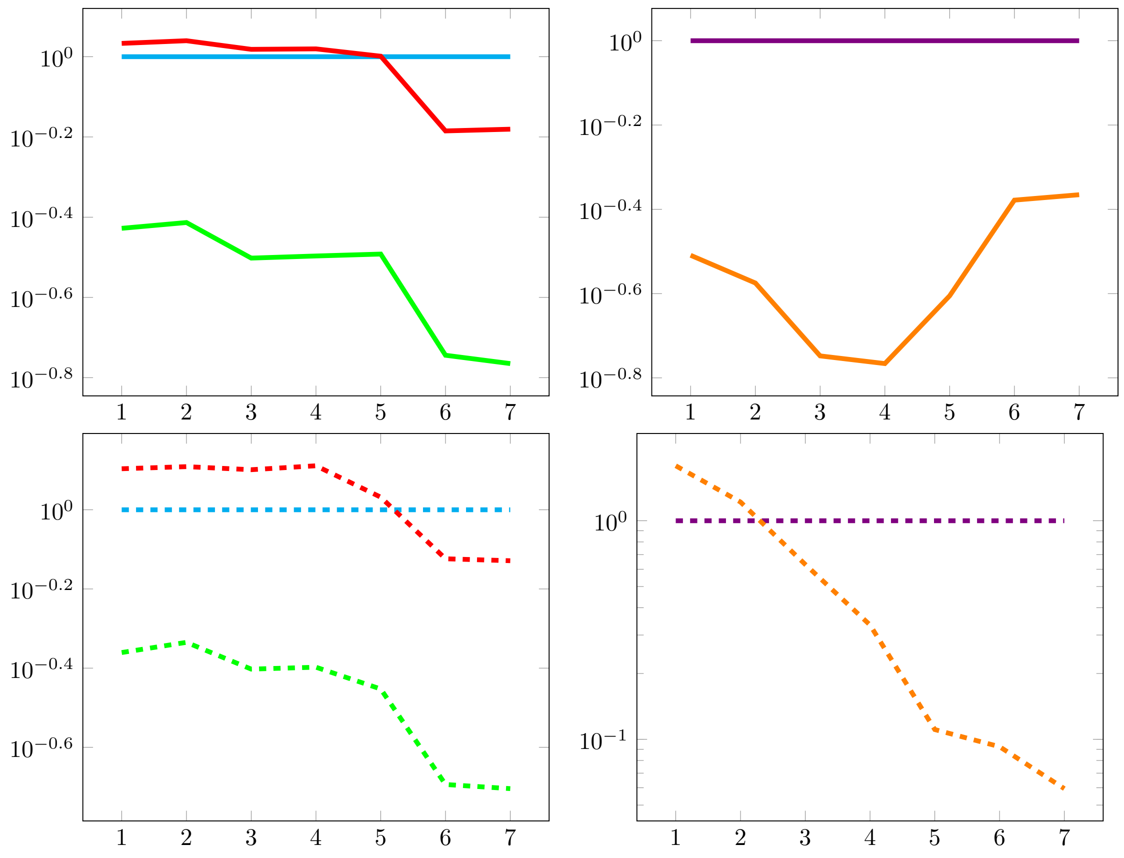


Figure 8.5: Running times on the elevator games, times are normalized

8.3.2 Random games

Figure 8.6 shows the performance of the algorithms on type 1 games. The recursive algorithms for VPGs performh quite well, even though there are a few instances where the performance is worse that the independent approach. The symbolic variant performs quite a bit better than the explicit variant. The relative performance of the local variants of the recursive algorithms is about the same as the relative performance of the global variants.

For the incremental pre-solve algorithm we do see a big difference between a local and global approach. The global variant performs well only for games 90 and up. The local variant, however, performs well for nearly all the games. Furthermore, even for the games where the global variant performs well does the local variant perform even better.

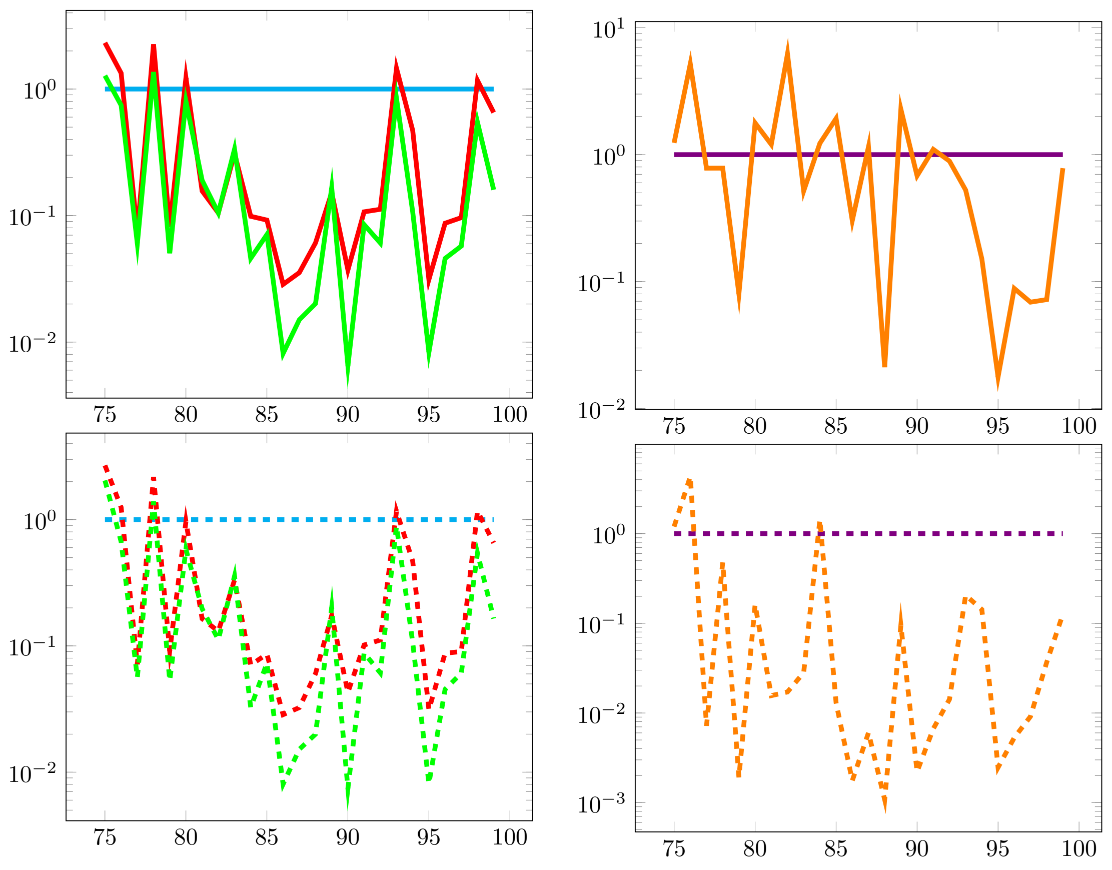


Figure 8.6: Running times on random games of type 1 with $\lambda = \frac{game\ nr}{100}$, times are normalized

Figure 8.7 shows the performance of the algorithms on type 1 games. For the recursive algorithms we see that the explicit variant takes over from the symbolic variant. This is to be expected since these games have edge guards that are not created from features but created by picking random configurations. Both variants still perform somewhat better than the independent approach. Again we do not find a significant difference between the global and local approach.

For the incremental pre-solve algorithm we find a similar result as with type 1 games. The global variant performs well only when λ is high. The local variant performs significantly better and performs well for almost all games.

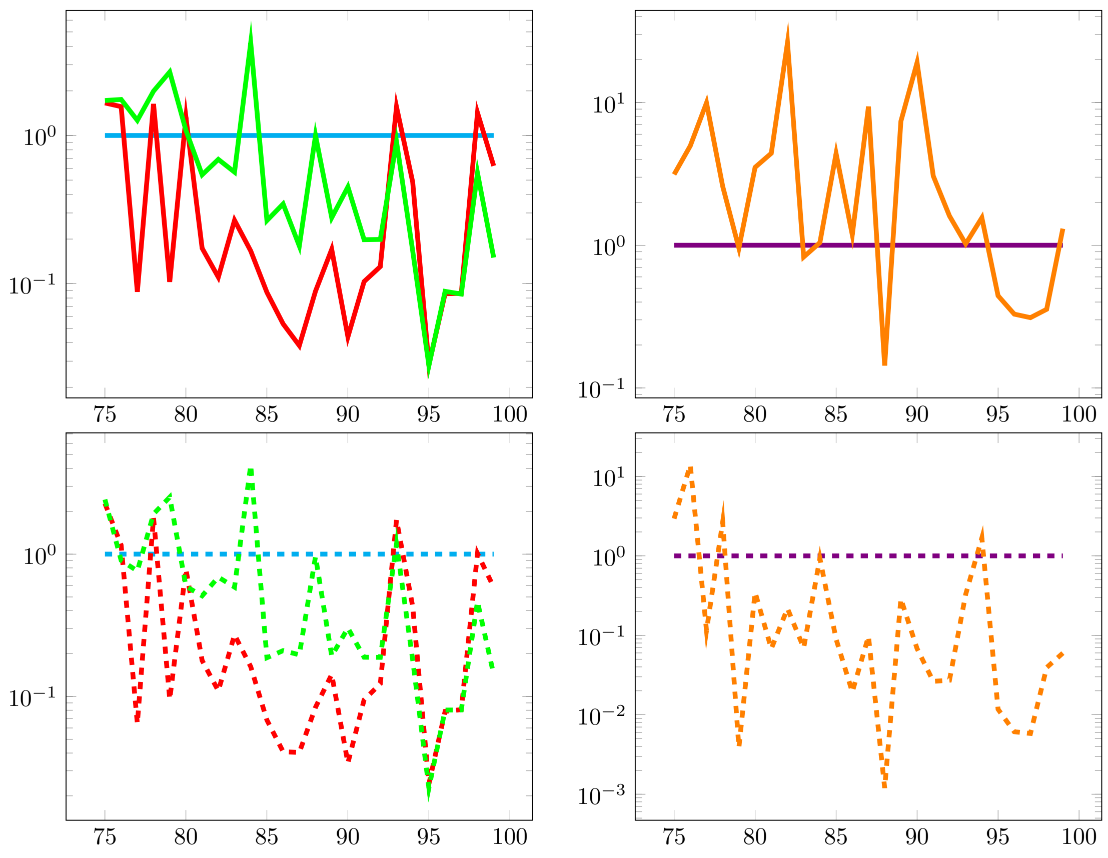


Figure 8.7: Running times on random games of type 2 with $\lambda = \frac{game\ nr}{100}$, times are normalized

Figure 8.8 shows the performance of the algorithms on type 3 games. For these games we see the symbolic variant of the recursive algorithm performing worse that the independent approach for almost all games. The explicit variant still performs significantly better that the symbolic variant and performs somewhat better than the independent approach. Again we do not find a significant difference between the global and local approach.

The global incremental pre-solve algorithm performs worse than the independent approach for almost all games. Notably for these games we do not find a significant increase in relative performance when using a local variant.

Notably, the explicit recursive algorithm seems to be the only algorithm unaffected by the fact that the guard sets of type 3 games vary wildly (they are distributed using a beta distribution). Maybe surprisingly, the incremental pre-solve algorithm is affected heavily by this. This is most likely because there are a lot fewer edges that admit all configurations and therefore player α will probably win less vertices in a pessimistic game for player α.

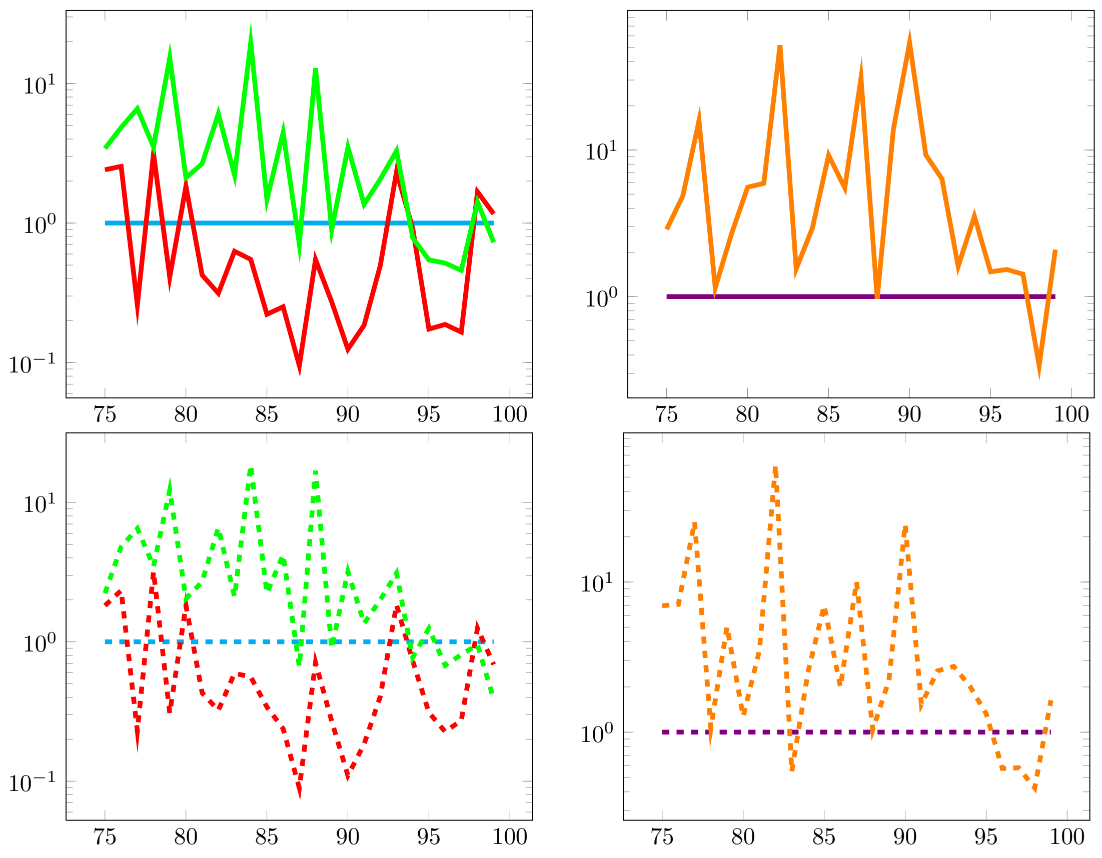


Figure 8.8: Running times on random games of type 3 with $\lambda = \frac{game\ nr}{100}$, times are normalized

8.3.3 Scaling

Figure 8.9 shows the performance of the algorithms on type 1 games where the number of configurations increase exponentially in the x-axis of the charts. For the recursive algorithm we see that the collective approach starts outperforming the independent approach around 2^4 configurations. As the number of configurations grow we see that the symbolic variant keeps increasing in relative performance while the explicit variants relative performance starts to flatten. This is to be expected because the performance of the explicit variant always scales linearly in the number of configurations. In the worst case the symbolic variant scales quadratically in the number of configurations, however when the sets of configurations can be represented efficiently it scales much better and in this case sublinear (since the performance of the local variant keeps increasing relative to the explicit variant).

The global incremental pre-solve algorithms does not increase notably in relative performance when the number of configurations increases. However, the relative performance of the local variant does increase in performance when the number of configurations increases. The recursion of the incremental pre-solve algorithm can be conceptualized as a tree where at every node the algorithm tries to increase the pre-solved vertices. The local variant can terminate when at some node the vertex that is being locally solved is found. In such a case the whole subtree of that node is longer computed. When the number of configurations grow then potentially the size of this subtree also grows. The fact that the local incremental pre-solve algorithm scales well in the number of configurations is most likely because the algorithm can terminate early for a relatively large set of configurations.

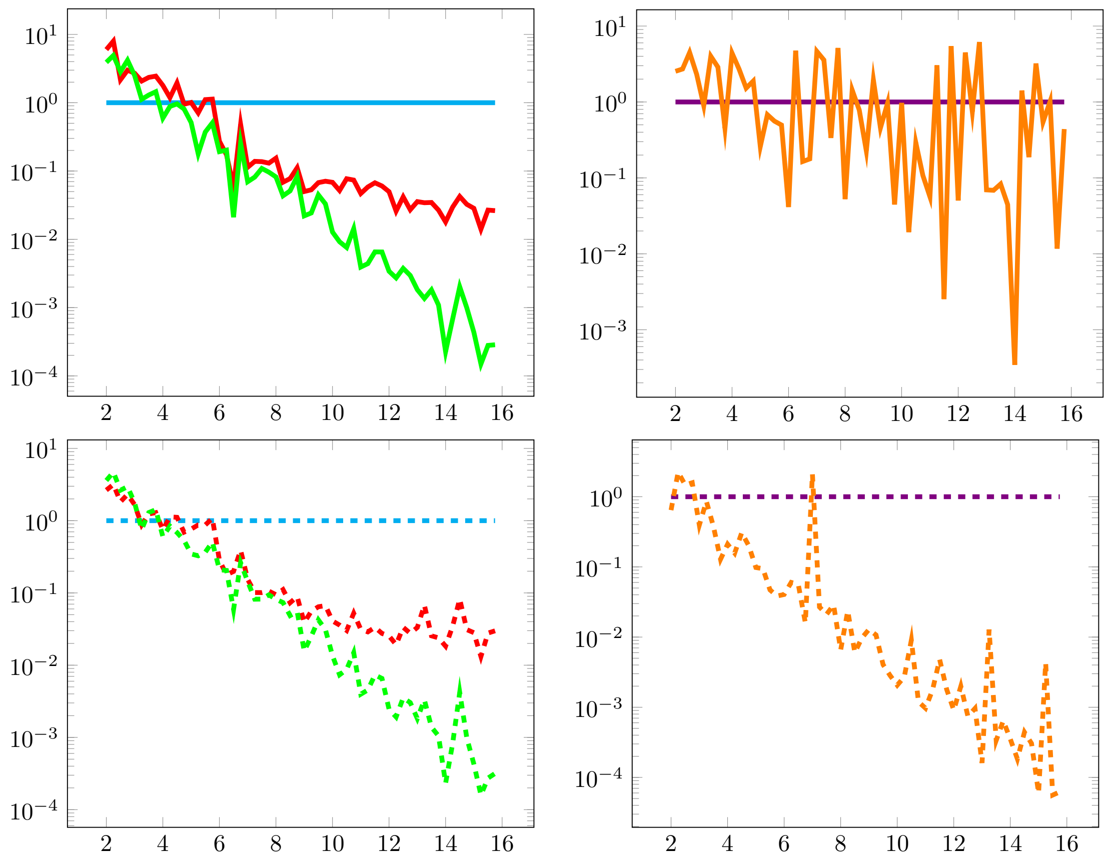


Figure 8.9: Running times on random games of type 1 with $\lambda = 0.92$ and the number of features equal to the *game nr*, times are normalized

8.3.4 Comparison

From the experimental results we observe that the symbolic variant of the recursive algorithms for VPGs performs well for the model verification problems. For type 1 random games it also performs well and particularly scales very well in the number of configurations. For type 2 and 3 games the sets of configurations can no longer be efficiently represented symbolically. Earlier we hypothesised that the recursive algorithm for VPGs could perform well if we can attract many configurations simultaneously. For every VPG we measure the average number of configurations that were attracted simultaneously. We measure this relative to the total number of configurations in the VPG. This gives a number for every VPG. For every set of VPGs we average this number to get an average set size for every problem set. These values indicate how many configuration were attracted simultaneously, they are presented in Table 8.4. We see that this number somewhat predicts the performance of the recursive algorithms.

After comparing independent and collective approaches we compare the performances of the

Minepump	46%
Elevator	51%
Type 1, scaling in λ	61%
Type 2, scaling in λ	55%
Type 3, scaling in λ	22%
Type 1, scaling in # confs	72%

Table 8.4: Relative size of attracted sets

algorithms overall.

First we compare the independent algorithms. Table 8.5 shows for every set of VPGs how long on average it took each algorithm to solve a VPG in that set. We observe that the recursive

	Recursive global	Recursive local	Fixed-point global	Fixed-point local
Minepump	113 ms	97 ms	1852 ms	1775 ms
Elevator	11374 ms	9420 ms	1061198 ms	301363 ms
Type 1, scaling in λ	75 ms	69 ms	2147 ms	2056 ms
Type 2, scaling in λ	75 ms	71 ms	2986 ms	2900 ms
Type 3, scaling in λ	74 ms	77 ms	2496 ms	2048 ms
Type 1, scaling in # confs	590 ms	529 ms	7857 ms	5359 ms

Table 8.5: Comparison of independent algorithms

algorithm performs significantly better than the fixed-point algorithm across all problems. We also see that the local variants perform somewhat better across the board. Notably the elevator problem seems to lend itself well for local solving.

In table 8.6 we compare the performance of the collective algorithms. We observe that the re-

	Recursive explicit global	Recursive explicit local	Recursive symbolic global	Recursive symbolic local	Incremental pre-solve global	Incremental pre-solve local
Minepump	112 ms	103 ms	16 ms	15 ms	658 ms	359 ms
Elevator	11542 ms	11327 ms	3706 ms	3701 ms	230301 ms	183465 ms
Type 1, scaling in λ	7 ms	6 ms	4 ms	4 ms	324 ms	541 ms
Type 2, scaling in λ	9 ms	8 ms	109 ms	104 ms	2695 ms	833 ms
Type 3, scaling in λ	26 ms	26 ms	624 ms	630 ms	7742 ms	4131 ms
Type 1, scaling in # confs	20 ms	18 ms	2 ms	2 ms	967 ms	12 ms

Table 8.6: Comparison of collective algorithms

cursive symbolic variant performs the best for model-checking problems and for type 1 games. Furthermore, most likely the algorithm will scale well for models with a large number of features. The local variant of the incremental pre-solve algorithm also performs well relative to its independent counterpart. However, because the fixed-point iteration is heavily outperformed by the recursive algorithm its overall performance is worse that the recursive variants.

Furthermore, we observe that the explicit variant of the recursive algorithm performs decent across most games. We conclude from this that the efficiency of the symbolic algorithm does not only come from representing sets of configurations efficiently; using a collective approach even without this representation seems to be efficient.

Earlier we hypothesized that a local-collective approach would increase performance more compared to a global-collective approach than a local-independent approach would compared to a global-independent approach. We observe that this is very much the case for the incremental pre-solve algorithm but not at all the case for the recursive algorithms. We conclude that local solving has the potential to greatly increase performance, however this is not a given for just any algorithm.

9. Conclusion

An SPL that models its behaviour in an FTS can be verified using traditional model-checking techniques. These techniques check every product described in SPL independently. However, the number of products potentially scales exponentially in the number of features. So we could end up with a large number of products which makes independent checking undesirable.

We extended parity games to express variability, these games are called variability parity games (VPGs). VPGs express variability through configurations; a VPG describes a parity game for every configuration. We have shown that we can construct a VPG from an FTS and modal μ-calculus formula such that solving the FTS gives the information needed to decide which products satisfy the formula.

VPGs can be solved independently where we solve every parity game described by the VPG, this is however similar to independently model-checking all the products in the SPL. We introduced numerous collective algorithms that solve a VPG as a whole and try to exploit commonalities between the different configurations.

First we introduced a variant of Zielonka's recursive algorithm that solves VPGs. The algorithm views a VPG as a collection of parity games; a parity game for every configuration. We can represent such a collection with a single game graph and for every vertex and edge we have a set of configurations indicating if this vertex or edge is part of the parity game of that configuration. We modified the recursive algorithm to use such a representation. Specifically, we modified the attractor algorithm to try and attract multiple configurations per vertex at the same time. This modified attractor algorithm relies heavily on set operations over the sets of configurations associated with the vertices and edges. These sets can be either represented symbolically or explicitly, this gives two variants of the recursive algorithm for VPGs.

Next we introduced the incremental pre-solve algorithm for VPGs. This algorithm tries to find vertices that are won by one of the players for all configurations, if such a vertex is found it is said to be pre-solved. The algorithm tries to find these vertices and then splits the configurations in two sets and goes into recursion for both of them. In the recursion the configuration set has decreased in size so potentially more vertices can be pre-solved. The algorithm finds these vertices through solving pessimistic parity games. Pessimistic parity games are created from a VPG and for a player α, they have the property that any vertex won by player α is also won by player α in the VPG played for any configuration. The incremental pre-solve algorithm creates two pessimistic parity games (for player 0 and 1) and solves them using the fixed-point iteration algorithm. The fixed-point algorithm is modified to use vertices that already were pre-solved to increase its performance. The algorithm incrementally builds up the set of pre-solved vertices until either all vertices are pre-solved or a single configuration remains. In the worst case the algorithm solves linearly more (pessimistic) parity game than we would using an independent approach. However by increasing the number of pre-solved vertices the algorithm tries to outperform the independent approach by solving the (pessimistic) parity games increasingly quicker.

We introduced local variants of the recursive algorithm for parity games and the fixed-point algorithm for parity games. A local parity game algorithm tries to only determine the winner of a single vertex instead of all the vertices, potentially increasing its performance. We also introduced local variants of the collective algorithms mentioned above.

The incremental pre-solve algorithm has the same time complexity as independently solving a VPG using the fixed-point iteration algorithm. The recursive algorithms for VPGs have a worse worst-case time complexity than independently solving a VPG using the recursive algorithm for parity games. However, the aim of the algorithms is to solve VPGs originating from FTSs efficiently. Often times these VPGs will have a lot of commonalities between the configurations. The algorithms are implemented and their actual performance is compared to independent approaches. The minepump and elevator SPLs are used to evaluate the performances. Furthermore a collection of random games is created with different characteristics ranging from very similar to VPGs originating from SPLs to completely different from VPGs originating from SPLs.

We observed that for the SPL VPGs and random VPGs created to be similar to SPL VPGs the symbolic variant of the incremental pre-solve algorithm performs best. The incremental pre-solve algorithm generally also outperforms its independent counterpart, i.e. independently solving a VPG using the fixed-point iteration algorithm. However, for the SPL VPGs, it does so less significantly and consistently than the symbolic recursive algorithm does. It also scales poorer in the number of features than the symbolic recursive algorithm does. Furthermore the independent approach using the recursive algorithm greatly outperforms the independent approach using the fixed-point algorithm. Because the incremental pre-solve algorithm uses the fixed-point algorithm its absolute performance is significantly worse than the recursive algorithm.

Notably, the difference between the local and global variants of the recursive algorithms for VPGs is very little. However, the difference between the local and global variant of the incremental pre-solve algorithm is very large across most types of VPGs. Furthermore, the local variant of the incremental pre-solve algorithm does seem to scale well in the number of features. This that local algorithms for VPGs can greatly increase performance compared to global algorithms, more so that locally solving parity games increases performance compared to globally solving parity games.

Future work Even though the incremental pre-solve algorithms performance was not the best, it did outperform its independent counterpart. Therefore it would be interesting to study the incremental pre-solve algorithm using a different way of solving pessimistic parity games; for example, using a variant of the recursive algorithm that can work with pre-solved vertices. This would potentially yield an algorithm that is more *robust* than the symbolic recursive algorithm in the sense that it performs well across different VPGs and not only for VPGs that originate from SPLs. Note that the local variant of the incremental pre-solve algorithm terminates early when the vertex we are looking for is pre-solved. This does not depend on the parity game algorithm that is used to solve pessimistic parity games. So, most likely, in an incremental pre-solve algorithm using a different parity game algorithm for its pessimistic parity games we will still see a very large increase in performance when running the algorithm locally as opposed to globally.

Furthermore, many optimizations are known for solving parity games. It would be interesting to study if these improvements are applicable to VPGs and if they increase the performance of VPG solving more than they increase the performance of parity game solving.

Finally, the creation of VPGs is left unstudied in this thesis, it would be interesting to study how one could efficiently create VPGs from FTSs including the creation of BDDs.

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A. Running time results