

Verifying Featured Transition Systems using Variability Parity Games

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Part I

Verifying featured transition systems

1 Introduction

Model verification techniques can be used to improve the quality of software. These techniques require the behaviour of the software to be modelled, after which the model can be checked to verify that it behaves conforming to some requirement. Different languages are proposed and well studied to express these requirements, examples are LTL, CTL, CTL* and μ -calculus (TODO: cite). Once the behaviour is modelled and the requirement is expressed in some language we can use modal checking techniques to determine if the model satisfies the requirement.

These techniques are well suited to model and verify the behaviour of a single software product. However software systems can be designed to have certain parts enabled or disabled. This gives rise to many software products that all behave very similar but not identical, such a collection is often called a *product family*. The differences between the products in a product family is called the *variability* of the family. A family can be verified by using the above mentioned techniques to verify every single product independently. However this approach does not use the similarities in behaviour of these different products, an approach that would make use of the similarities could potentially be a lot more efficient.

Labelled transition systems (LTSs) are often used to model the behaviour of a system, while it can model behaviour well it can't model variability. Efforts to also model variability include I/O automata, modal transition systems and *featured transition systems* (FTSs) (TODO: cite). Specifically the latter is well suited to model all the different behaviours of the software products as well as the variability of the entire system in a single model.

Efforts have been made to verify requirements for entire FTSs, as well as to be able to reason about features. Notable contributions are fLTL, fCTL and fNuSMV (TODO: cite). However, as far as we know, there is no technique to verify an FTS against a μ -calculus formula. Since the modal μ -calculus is very expressive, it subsumes other temporal logics like LTL, CTL and CTL*, this is desired. In this thesis we will introduce a technique to do this. We first look at LTSs, the modal μ -calculus and FTSs. Next we will look at an existing technique to verify an LTS, namely solving *parity games*, as well as show how this technique can be used to verify an FTS by verifying every software product it describes independently. An extension to this technique is then proposed, namely solving *variability parity games*. We will formally define variability parity games and prove that solving them can be used to verify FTSs.

2 Verifying transition systems

We first look at labelled transition systems (LTSs) and the modal μ -calculus and what it means to verify an LTS. The definitions below are derived from [1].

Definition 2.1. A labelled transition system (LTS) is a tuple $M = (S, Act, trans, s_0)$, where:

- S is a set of states,
- Act a set of actions,
- $trans \subseteq S \times Act \times S$ is the transition relation with $(s, a, s') \in trans$ denoted by $s \xrightarrow{a} s'$,
- $s_0 \in S$ is the initial state.

Consider the example in figure 1 (directly taken from [2]) of a coffee machine where we have two actions: ins (insert coin) and std (get standard sized coffee).

Definition 2.2. A modal μ -calculus formula over the set of actions Act and a set of variables \mathcal{X} is defined by

$$\varphi = \top \mid \perp \mid X \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \langle a \rangle \varphi \mid [a] \varphi \mid \mu X. \varphi \mid \nu X. \varphi$$

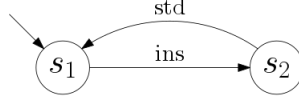


Figure 1: Coffee machine LTS C

with $a \in Act$ and $X \in \mathcal{X}$.

We don't include negations in the language because negations can be pushed inside to the propositions, ie. the \top and \perp elements.

The modal μ -calculus contains boolean constants \top and \perp , propositional operators \vee and \wedge , modal operators $\langle \rangle$ and $[]$ and fixpoint operators μ and ν . A formula is closed when variables only occur in the scope of a fixpoint operator for that variable.

A modal μ -calculus formula can be interpreted with an LTS, this results in a set of states for which the formula holds.

Definition 2.3. For LTS $(S, Act, trans, s_0)$ we inductively define the interpretation of a modal μ -calculus formula φ , notation $\llbracket \varphi \rrbracket^\eta$, where $\eta : \mathcal{X} \rightarrow \mathcal{P}(S)$ is a logical variable valuation, as a set of states where φ is valid, by:

$$\begin{aligned}
 \llbracket \top \rrbracket^\eta &= S \\
 \llbracket \perp \rrbracket^\eta &= \emptyset \\
 \llbracket \varphi_1 \wedge \varphi_2 \rrbracket^\eta &= \llbracket \varphi_1 \rrbracket^\eta \cap \llbracket \varphi_2 \rrbracket^\eta \\
 \llbracket \varphi_1 \vee \varphi_2 \rrbracket^\eta &= \llbracket \varphi_1 \rrbracket^\eta \cup \llbracket \varphi_2 \rrbracket^\eta \\
 \llbracket \langle a \rangle \varphi \rrbracket^\eta &= \{s \in S \mid \exists s' \in S s \xrightarrow{a} s' \wedge s' \in \llbracket \varphi \rrbracket^\eta\} \\
 \llbracket [a] \varphi \rrbracket^\eta &= \{s \in S \mid \forall s' \in S s \xrightarrow{a} s' \implies s' \in \llbracket \varphi \rrbracket^\eta\} \\
 \llbracket \mu X. \varphi \rrbracket^\eta &= \bigcap_{f \subseteq S} \{f \mid f = \llbracket \varphi \rrbracket^{\eta[X:=f]}\} \\
 \llbracket \nu X. \varphi \rrbracket^\eta &= \bigcup_{f \subseteq S} \{f \mid f = \llbracket \varphi \rrbracket^{\eta[X:=f]}\} \\
 \llbracket X \rrbracket^\eta &= \eta(X)
 \end{aligned}$$

Given closed formula φ , LTS $M = (S, Act, trans, s_0)$ and $s \in S$ iff $s \in \llbracket \varphi \rrbracket^\eta$ for M we say that formula φ holds for M in state s and write $(M, s) \models \varphi$. Iff formula φ holds for M in the initial state we say that formula φ holds for M and write $M \models \varphi$.

Again consider the coffee machine example (figure 1) and formula $\varphi = \nu X. \mu Y. ([ins]Y \wedge [std]X)$ (taken from [2]) which states that action std must occur infinitely often over all runs. Obviously this holds for the coffee machine, therefore we have $C \models \varphi$.

2.1 Featured transition systems

A *featured transition system* (FTS) extends the LTS definition to express variability. It does so by introducing *features* and *products* into the definition. Features are options that can be enabled or disabled for the system. A product is a feature assignments, ie. a set of features that is enabled for that product. Not all products are valid, some features might be mutually exclusive while others features might always be required. To express the relation between features one can use feature diagrams as explained in [3]. Feature diagrams offer a nice way of expressing which feature assignments are valid, however for simplicity we will represent the collection of valid products simply with a set of feature assignments. Finally FTSs guard every transition with a boolean expression over the set of features.

Definition 2.4. [3] A *featured transition system* (FTS) is a tuple $M = (S, Act, trans, s_0, N, P, \gamma)$, where:

- $S, Act, trans, s_0$ are defined as in an LTS,

- N is a non-empty set of features,
 - $P \subseteq \mathcal{P}(N)$ is a non-empty set of products, ie. feature assignments, that are valid,
 - $\gamma : \text{trans} \rightarrow \mathbb{B}(N)$ is a total function, labelling each transition with a boolean expression over the features. A product $p \in \mathcal{P}(N)$ satisfying the boolean expression of transition t is denoted by $p \models \gamma(t)$. The boolean expression that is satisfied by any feature assignment is denoted by \top , ie $p \models \top$ for any p .
- A transition $s \xrightarrow{a} s'$ and $\gamma(s, a, s') = f$ is denoted by $s \xrightarrow{a|f} s'$.

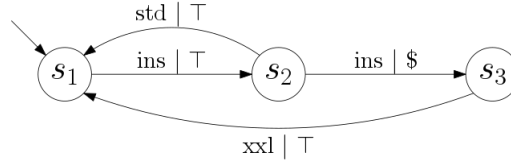


Figure 2: Coffee machine FTS C

Consider the example in figure 2 (directly taken from [2]) which shows an FTS for a coffee machine. For this example we have two features $N = \{\$, \text{€}\}$ and two valid products $P = \{\{\$\}, \{\text{€}\}\}$.

An FTS expresses the behaviour of multiple products, we can derive the behaviour of a single product by simply removing all the transitions from the FTS for which the product doesn't satisfy the feature expression guarding the transition. We call this a *projection*.

Definition 2.5. [3] The projection of an FTS $M = (S, \text{Act}, \text{trans}, s_0, N, P, \gamma)$ to a product $p \in P$, noted $M|_p$, is the LTS $M' = (S, \text{Act}, \text{trans}', s_0)$, where $\text{trans}' = \{t \in \text{trans} \mid p \models \gamma(t)\}$.

The coffee machine example can be projected to its two products, which results in the LTSs in figure 3.

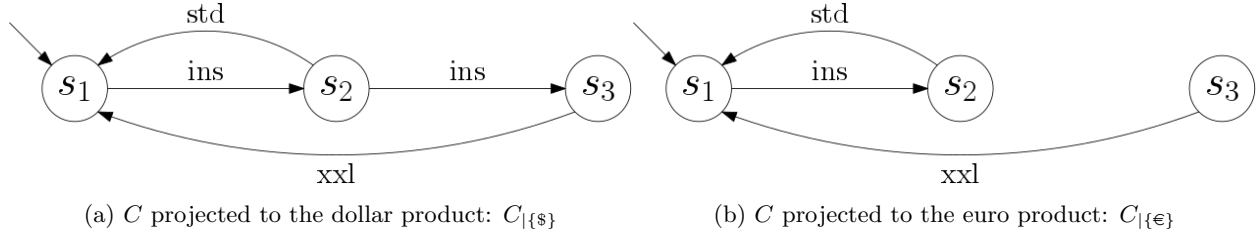


Figure 3: Projections of the coffee machine FTS

2.2 FTS verification question

When verifying an FTS against a modal μ -calculus formula φ , we are trying to answer the question: For which products in the FTS does its projection satisfy φ ? Formally, given FTS $M = (S, \text{Act}, \text{trans}, s_0, N, P, \gamma)$ and modal μ -calculus formula φ we want to find $P_s \subseteq P$ such that:

- for every $p \in P_s$ we have $M|_p \models \varphi$ and
- for every $p \in P \setminus P_s$ we have $M|_p \not\models \varphi$.

Furthermore a counterexample for every $p \in P \setminus P_s$ is preferred.

3 Verification using parity games

Verifying LTSs against a modal μ -calculus formula can be done by solving a *parity game*. This is done by translating an LTS in combination with a formula to a parity game, the solution of the parity game provides

the information needed to conclude if the model satisfies the formula. This relation is depicted in figure 4. This technique is well known and well studied, in this section we will first look at parity games, the translation from LTS and formula to a parity game and finally what we can do with this technique to verify FTS.

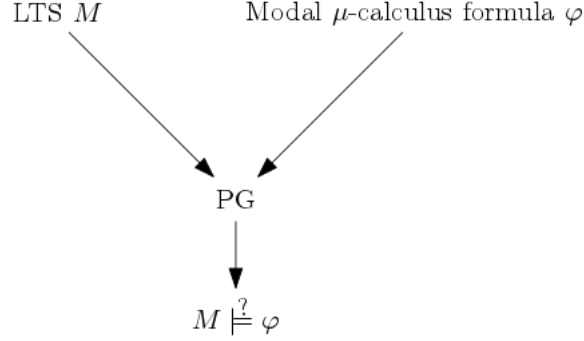


Figure 4: LTS verification using PG

3.1 Parity games

Definition 3.1. [4] A parity game (PG) is a tuple (V, V_0, V_1, E, Ω) , where:

- $V = V_0 \cup V_1$ and $V_0 \cap V_1 = \emptyset$,
- V_0 is the set of vertices owned by player 0,
- V_1 is the set of vertices owned by player 1,
- $E \subseteq V \times V$ is the edge relation,
- $\Omega : V \rightarrow \mathbb{N}$ is a priority assignment.

A parity game is played by players 0 and 1. We write $\alpha \in \{0, 1\}$ to denote an arbitrary player. We write $\bar{\alpha}$ to denote α 's opponent, ie. $\bar{0} = 1$ and $\bar{1} = 0$.

A play starts with placing a token on vertex $v \in V$. Player α moves the token if the token is on a vertex owned by α , ie. $v \in V_\alpha$. The token can be moved to $w \in V$, with $(v, w) \in E$. A series of moves results in a sequence of vertices, called a path. For path π we write π_i to denote the i^{th} vertex in path π . A play ends when the token is on vertex $v \in V_\alpha$ and α can't move the token anywhere, in this case player $\bar{\alpha}$ wins the play. If the play results in an infinite path π then we determine the highest priority that occurs infinitely often in this path, formally

$$\max\{p \mid \forall_j \exists_i j < i \wedge p = \Omega(\pi_i)\}$$

If the highest priority is odd then player 1 wins, if it is even player 0 wins.

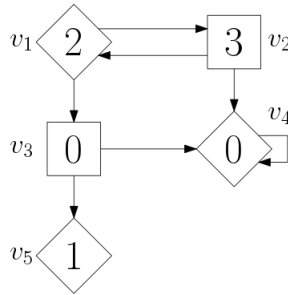


Figure 5: Parity game example

Figure 5 shows an example of a parity game. We usually depict the vertices owned by player 0 by diamonds and vertices owned by player 1 by boxes, the priority is depicted inside the vertices. If the game starts by placing a token on v_1 we can consider the following exemplary paths:

- $\pi = v_1 v_3 v_5$ is won by player 1 since player 0 can't move at v_5 .
- $\pi = (v_1 v_2)^\omega$ is won by player 1 since the highest priority occurring infinitely often is 3.
- $\pi = v_1 v_3 (v_4)^\omega$ is won by player 0 since the highest priority occurring infinitely often is 0.

A strategy for player α is a function $\sigma : V^* V_\alpha \rightarrow V$ that maps a path ending in a vertex owned by player α to the next vertex. Parity games are positionally determined [4], therefore a strategy $\sigma : V_\alpha \rightarrow V$ that maps the current vertex to the next vertex is sufficient.

A strategy σ for player α is winning from vertex v iff any play that results from following σ results in a win for player α . The graph can be divided in two partitions $W_0 \subseteq V$ and $W_1 \subseteq V$, called winning sets. Iff $v \in W_\alpha$ then player α has a winnigns strategy from v . Every vertex in the graph is either in W_0 or W_1 [4]. Furthermore finite parity games are decidable [4].

3.2 Creating parity games

A parity game can be created from a combination of an LTS and a modal μ -calculus formula. To do this we introduce some auxiliary definitions regarding the modal μ -calculus.

First we introduce the notion of unfolding, a fixpoint formula $\mu X.\varphi$ can be unfolded resulting in formula φ where every occurrence of X is replaced by $\mu X.\varphi$, denoted by $\varphi[X := \mu X.\varphi]$. A fixpoint formula is equivalent to its unfolding [4], ie. for some LTS $\llbracket \mu X.\varphi \rrbracket^\eta = \llbracket \varphi[X := \mu X.\varphi] \rrbracket^\eta$. The same holds for the fixpoint operator ν .

Next we define the Fischer-Ladner closure for a closed μ -calculus formula [5, 6]. The Fischer-Ladner closure of φ is the set $FL(\varphi)$ of closed formula's containing at least φ . Furthermore for every formula ψ in $FL(\varphi)$ it holds that for every direct subformula ψ' of ψ there is a formula in $FL(\varphi)$ that is equivalent to ψ' .

Definition 3.2. *The Fischer-Ladner closure of closed μ -calculus formula φ is the smallest set $FL(\varphi)$ satisfying the following constraints:*

- $\varphi \in FL(\varphi)$,
- if $\varphi_1 \vee \varphi_2 \in FL(\varphi)$ then $\varphi_1, \varphi_2 \in FL(\varphi)$,
- if $\varphi_1 \wedge \varphi_2 \in FL(\varphi)$ then $\varphi_1, \varphi_2 \in FL(\varphi)$,
- if $\langle a \rangle \varphi' \in FL(\varphi)$ then $\varphi' \in FL(\varphi)$,
- if $[a] \varphi' \in FL(\varphi)$ then $\varphi' \in FL(\varphi)$,
- if $\mu X.\varphi' \in FL(\varphi)$ then $\varphi'[X := \mu X.\varphi'] \in FL(\varphi)$ and
- if $\nu X.\varphi' \in FL(\varphi)$ then $\varphi'[X := \nu X.\varphi'] \in FL(\varphi)$.

Finally we define alternating depth.

Definition 3.3. [4] *The dependency order on bound variables of φ is the smallest partial order such that $X \leq_\varphi Y$ if X occurs free in $\sigma Y.\psi$. The alternation depth of a μ -variable X in formula φ is the maximal length of a chain $X_1 \leq_\varphi \dots \leq_\varphi X_n$ where $X = X_1$, variables X_1, X_3, \dots are μ -variables and variables X_2, X_4, \dots are ν -variables. The alternation depth of a ν -variable is defined similarly. The alternation depth of formula φ , denoted $adepth(\varphi)$, is the maximum of the alternation depths of the variables bound in φ , or zero if there are no fixpoints.*

Consider the example formula $\varphi = \nu X.\mu Y.([ins]Y \wedge [std]X)$ which states that for an LTS with $Act = \{ins, std\}$ the action std must occur infinitely often over all runs. Since X occurs free in $\mu Y.([ins]Y \wedge [std]X)$ we have $adepth(Y) = 1$ and $adepth(X) = 2$. As shown in [4] it holds that formula $\mu X.\psi$ has the same alternation depth as its unfolding $\psi[X := \mu X.\psi]$. Similarly for the greatest fixpoint.

We can now define the transformation from an LTS and a formula to a parity game.

Definition 3.4. [4] $LTS2PG(M, \varphi)$ converts LTS $M = (S, Act, trans, s_0)$ and closed formula φ to a PG (V, V_0, V_1, E, Ω) .

A vertex in the parity game is represented by a pair (s, ψ) where $s \in S$ and ψ is a modal μ -calculus formula. We will create a vertex for every state with every formula in the Fischer-Ladner closure of φ . We define the set of vertices:

$$V = S \times FL(\varphi)$$

We create the parity game with the smallest set E such that:

- $V = V_0 \cup V_1$,
- $V_0 \cap V_1 = \emptyset$ and
- for every $v = (s, \psi) \in V$ we have:
 - If $\psi = \top$ then $v \in V_1$.
 - If $\psi = \perp$ then $v \in V_0$.
 - If $\psi = \psi_1 \vee \psi_2$ then:
 - $v \in V_0$,
 - $(v, (s, \psi_1)) \in E$ and
 - $(v, (s, \psi_2)) \in E$.
 - If $\psi = \psi_1 \wedge \psi_2$ then:
 - $v \in V_1$,
 - $(v, (s, \psi_1)) \in E$ and
 - $(v, (s, \psi_2)) \in E$.
 - If $\psi = \langle a \rangle \psi'$ then $v \in V_0$ and for every $s \xrightarrow{a} s'$ we have $(v, (s', \psi')) \in E$.
 - If $\psi = [a] \psi'$ then $v \in V_1$ and for every $s \xrightarrow{a} s'$ we have $(v, (s', \psi')) \in E$.
 - If $\psi = \mu X. \psi'$ then $(v, (s, \psi'[X := \mu X. \psi'])) \in E$.
 - If $\psi = \nu X. \psi'$ then $(v, (s, \psi'[X := \nu X. \psi'])) \in E$.

Since the Fischer-Ladner formula's are closed we never get the case $\psi = X$.

$$\text{Finally we have } \Omega(s, \psi) = \begin{cases} 2 \lfloor \text{adepth}(X)/2 \rfloor & \text{if } \psi = \nu X. \psi' \\ 2 \lfloor \text{adepth}(X)/2 \rfloor + 1 & \text{if } \psi = \mu X. \psi' \\ 0 & \text{otherwise} \end{cases}$$

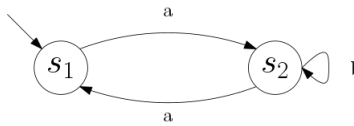


Figure 6: LTS M

Consider LTS M in figure 6 and formula $\varphi = \mu X. ([a]X \vee \langle b \rangle \top)$ expressing that on any path reached by a 's we can eventually do a b action. We will use this as a working example in the next few sections. The

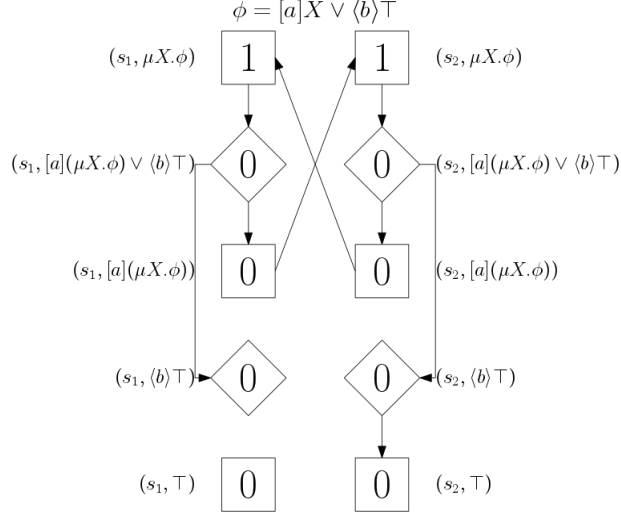


Figure 7: Parity game $LTS2PG(M, \varphi)$

resulting parity game is depicted in figure 7. Solving this parity game results in the following winning sets:

$$\begin{aligned}
 W_0 = \{ & (s_1, \mu X.\phi), \\
 & (s_1, [a](\mu X.\phi) \vee \langle b \rangle \top), \\
 & (s_1, [a](\mu X.\phi)), \\
 & (s_1, \top), \\
 & (s_2, \mu X.\phi), \\
 & (s_2, [a](\mu X.\phi) \vee \langle b \rangle \top), \\
 & (s_2, [a](\mu X.\phi)), \\
 & (s_2, \langle b \rangle \top), \\
 & (s_2, \top) \} \\
 W_1 = \{ & (s_1, \langle b \rangle \top) \}
 \end{aligned}$$

With the strategies σ_0 for player 0 and σ_1 for player 1 being (vertices with one outgoing edge are omitted):

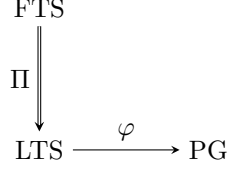
$$\begin{aligned}
 \sigma_0 = \{ & (s_1, [a](\mu X.\phi) \vee \langle b \rangle \top) \mapsto (s_1, [a](\mu X.\phi)), \\
 & (s_2, [a](\mu X.\phi) \vee \langle b \rangle \top) \mapsto (s_2, \langle b \rangle \top) \} \\
 \sigma_1 = \{ & \}
 \end{aligned}$$

State s in LTS M only satisfies φ iff player 0 has a winning strategy from vertex (s, φ) . This is formally stated in the following theorem which is proven in [4].

Theorem 3.1. *Given LTS $M = (S, Act, trans, s_0)$, modal μ -calculus formula φ and state $s \in S$ it holds that $(M, s) \models \varphi$ iff $s \in W_0$ for the game $LTS2PG(M, \varphi)$.*

3.3 FTSs and parity games

Using the theory we have seen thus far we can verify FTSs by verifying every projection of the FTS to a valid product. This relation is depicted in the following diagram where Π indicates a projection:



As mentioned before verifying products dependently is potentially more efficient. In the next two sections we define an extension to parity games, namely *variability parity games* (VPGs) which can be used to verify an FTS. We will translate an FTS and a formula into a VPG which solution will provide the information needed to conclude for which products the FTS satisfies the formula.

4 Featured parity games

Before we can define variability parity games we first define *featured parity games* (FPG), featured parity games extend the definition of parity games to capture the variability represented in an FTS. It uses the same concepts as FTSs: features, products and a function that guards edges. In this section we will introduce the definition of FPGs and show that solving them answers the verification questions for FTS: For which products in the FTS does its projection satisfy φ ?

First we introduce the definition of an FPG:

Definition 4.1. A featured parity game (FPG) is a tuple $(V, V_0, V_1, E, \Omega, N, P, \gamma)$, where:

- $V = V_0 \cup V_1$ and $V_0 \cap V_1 = \emptyset$,
- V_0 is the set of vertices owned by player 0,
- V_1 is the set of vertices owned by player 1,
- $E \subseteq V \times V$ is the edge relation,
- $\Omega : V \rightarrow \mathbb{N}$ is a priority assignment,
- N is a non-empty set of features,
- $P \subseteq \mathcal{P}(N)$ is a non-empty set of products, ie. feature assignments, for which the game can be played,
- $\gamma : E \rightarrow \mathbb{B}(N)$ is a total function, labelling each edge with a Boolean expression over the features.

An FPG is played similarly to a PG, however the game is played for a specific product $p \in P$. Player α can only move the token from $v \in V_\alpha$ to $w \in V$ if $(v, w) \in E$ and $p \models \gamma(v, w)$.

A game played for product $p \in P$ results in winnings sets W_0^p and W_1^p , which are defined similar to the W_0 and W_1 winning sets for parity games.

An FPG can simply be projected to a product p by removing the edges that are not satisfied by p .

Definition 4.2. The projection from FPG $G = (V, V_0, V_1, E, \Omega, N, P, \gamma)$ to a product $p \in P$, noted $G|_p$, is the parity game $(V, V_0, V_1, E', \Omega)$ where $E' = \{e \in E \mid p \models \gamma(e)\}$.

Playing FPG G for a specific product $p \in P$ is the same as playing the PG $G|_p$. Any path that is valid in G for p is also valid in $G|_p$ and vice versa. Therefore the strategies are also interchangeable, furthermore the winning sets W_α for $G|_p$ and W_α^p for G are identical. Since parity games are positionally determined so are FPGs. Similarly, since finite parity games are decidable, so are finite FPGs.

We say that an FPG is solved when the winning sets for every valid product in the FPG are determined.

4.1 Creating featured parity games

An FPG can be created from an FTS in combination with a model μ -calculus formula. We translate an FTS to an FPG by first creating a PG from the transition system as if there were no transition guards, next we apply the same guards to the FPG as are present in the FTS for edges that originate from transitions. The features and valid products in the FPG are identical to those in the FTS.

Definition 4.3. $FTS2FPG(M, \varphi)$ converts FTS $M = (S, Act, trans, s_0, N, P, \gamma)$ and closed formula φ to FPG $(V, V_0, V_1, E, \Omega, N, P, \gamma')$.

We have $(V, V_0, V_1, E, \Omega) = LTS2PG((S, Act, trans, s_0), \varphi)$ and

$$\gamma'((s, \psi), (s', \psi')) = \begin{cases} \gamma(s, a, s') & \text{if } \psi = \langle a \rangle \psi' \text{ or } \psi = [a] \psi' \\ \top & \text{otherwise} \end{cases}$$

Consider our working example which we extend to an FTS depicted in figure 8, for this example we have features $N = \{f, g\}$ and products $P = \{\emptyset, \{f\}, \{f, g\}\}$. We can translate this FTS with formula

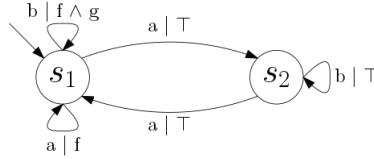


Figure 8: FTS M

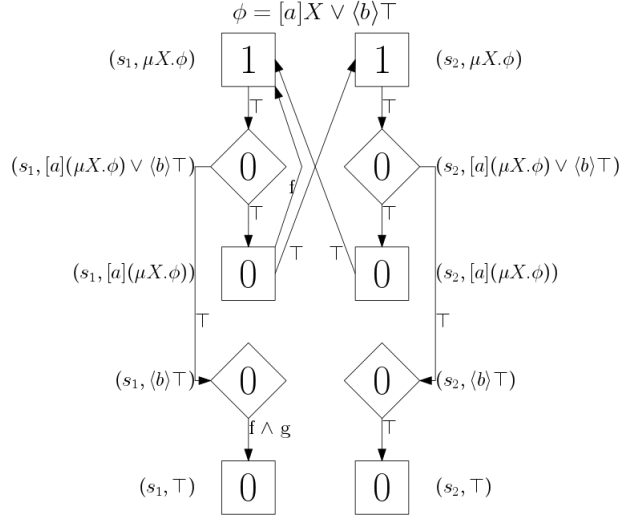


Figure 9: FPG for M and φ

$\varphi = \mu X.([a]X \vee \langle b \rangle \top)$ to an FPG depicted in figure 9. As we can see from the FTS if feature f is enabled and g is disabled then we have an infinite path of a 's where b is never enabled, therefore φ doesn't hold for $M_{\{f\}}$. If g is enabled however we can always do a b so φ holds for $M_{\{f, g\}}$. As we have seen φ does hold for

$M|_{\emptyset}$. For the product \emptyset we have the same winning set as before:

$$\begin{aligned}
W_0^{\emptyset} = \{ & (s_1, \mu X.\phi), \\
& (s_1, [a](\mu X.\phi) \vee \langle b \rangle \top), \\
& (s_1, [a](\mu X.\phi)), \\
& (s_1, \top), \\
& (s_2, \mu X.\phi), \\
& (s_2, [a](\mu X.\phi) \vee \langle b \rangle \top), \\
& (s_2, [a](\mu X.\phi)), \\
& (s_2, \langle b \rangle \top), \\
& (s_2, \top) \} \\
W_1^{\emptyset} = \{ & (s_1, \langle b \rangle \top) \}
\end{aligned}$$

In the FPG we can see that if f is enabled and g is disabled then player 1 can move the token from $(s_1, [a]X)$ to (s_1, X) . This results in player 0 either moving the token to $(s_1, \langle b \rangle \top)$ and losing or an infinite path where 1 occurs infinitely often which is also player 1 wins. For product $\{f\}$ we have winning sets:

$$\begin{aligned}
W_0^{\{f\}} = \{ & (s_1, \top), \\
& (s_2, \mu X.\phi), \\
& (s_2, [a](\mu X.\phi) \vee \langle b \rangle \top), \\
& (s_2, \langle b \rangle \top), \\
& (s_2, \top) \} \\
W_1^{\{f\}} = \{ & (s_1, \mu X.\phi), \\
& (s_1, [a](\mu X.\phi) \vee \langle b \rangle \top), \\
& (s_1, [a](\mu X.\phi)), \\
& (s_1, \langle b \rangle \top), \\
& (s_2, [a](\mu X.\phi)) \}
\end{aligned}$$

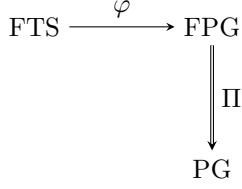
However if g is also enabled then player 0 wins in $(s_1, \langle b \rangle \top)$, thus giving the following winning sets:

$$\begin{aligned}
W_0^{\{f,g\}} = \{ & (s_1, \mu X.\phi), \\
& (s_1, [a](\mu X.\phi) \vee \langle b \rangle \top), \\
& (s_1, [a](\mu X.\phi)), \\
& (s_1, \langle b \rangle \top), \\
& (s_1, \top), \\
& (s_2, \mu X.\phi), \\
& (s_2, [a](\mu X.\phi) \vee \langle b \rangle \top), \\
& (s_2, [a](\mu X.\phi)), \\
& (s_2, \langle b \rangle \top), \\
& (s_2, \top) \} \\
W_1^{\{f,g\}} = \{ & \}
\end{aligned}$$

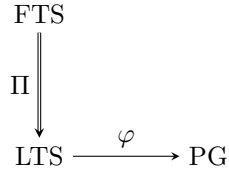
In the next section we will show how the winning sets relate to the model verification question.

4.2 FTS verification using FPG

We can create an FPG from an FTS and project it to a product, resulting in a PG, this is shown in the following diagram:



Earlier we saw that we could also derive a PG by projecting the FTS to a product and then translation the resulting LTS to a PG, depicted by the following diagram:



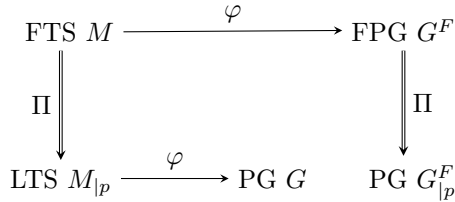
We will now show that the resulting parity games are identical.

Theorem 4.1. *Given:*

- $\text{FTS } M = (S, \text{Act}, \text{trans}, s_0, N, P, \gamma)$,
- a closed modal mu-calculus formula φ ,
- a product $p \in P$

it holds that the parity games $\text{LTS2PG}(M|_p, \varphi)$ and $\text{FTS2FPG}(M, \varphi)|_p$ are identical.

Proof. Let $G^F = (V^F, V_0^F, V_1^F, E^F, \Omega^F, N, P, \gamma') = \text{FTS2FPG}(M, \varphi)$, using definition 4.3, and $G_{|p}^F = (V^F, V_0^F, V_1^F, E^{F'}, \Omega^F)$, using definition 4.2. Furthermore we have $M|_p = (S, \text{Act}, \text{trans}', s_0)$ and we let $G = (V, V_0, V_1, E, \Omega) = \text{LTS2PG}(M|_p, \varphi)$. We depict the different transition systems and games in the following diagram.



We will prove that $G = G_{|p}^F$. We first note that game G is created by

$$(V, V_0, V_1, E, \Omega) = \text{LTS2PG}((S, \text{Act}, \text{trans}', s_0), \varphi)$$

and the vertices, edges and priorities of game G^F are created by

$$(V^F, V_0^F, V_1^F, E^F, \Omega^F) = \text{LTS2PG}((S, \text{Act}, \text{trans}, s_0), \varphi)$$

Using the definition of LTS2PG (3.4) we find that the vertices and the priorities only depend on the states in S and the formula φ , since these are identical in the above two statements we immediately get $V = V^F$, $V_0 = V_0^F$, $V_1 = V_1^F$ and $\Omega = \Omega^F$. The vertices and priorities don't change when an FTS is projected, therefore $G_{|p}^F$ has the same vertices and priorities as G^F .

Now we are left with showing that $E = E^{F'}$ in order to conclude that that $G = G_{|p}^F$. We will do this by showing $E \subseteq E^{F'}$ and $E \supseteq E^{F'}$.

First let $e \in E$. Note that a vertex in the parity game is represented by a pair of a state and a formula. So we can write $e = ((s, \psi), (s', \psi'))$. To show that $e \in E^{F'}$ we distinguish two cases:

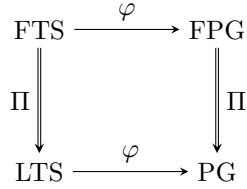
- If $\psi = \langle a \rangle \psi'$ or $\psi = [a] \psi'$ then there exists an $a \in Act$ such that $(s, a, s') \in trans'$. Using the FTS projection definition (2.5) we get $(s, a, s') \in trans$ and $p \models \gamma(s, a, s')$. Using the FTS2FPG definition (4.3) we find that $\gamma'((s, \psi), (s', \psi')) = \gamma(s, a, s')$ and therefore $p \models \gamma'((s, \psi), (s', \psi'))$. Now using the FPG projection definition (4.2) we find $((s, \psi), (s', \psi')) \in E^{F'}$.
- Otherwise the existence of the edge does not depend on the $trans$ parameter and therefore $((s, \psi), (s', \psi')) \in E^{F'}$ if $(s, \psi) \in V^F$, since $V^F = V$ we have $(s, \psi) \in V^F$.

We can conclude that $E \subseteq E^{F'}$, next we will show $E \supseteq E^{F'}$. Let $e = ((s, \psi), (s', \psi')) \in E^{F'}$. We distinguish two cases:

- If $\psi = \langle a \rangle \psi'$ or $\psi = [a] \psi'$ then there exists an $a \in Act$ such that $(s, a, s') \in trans$. Using the FPG projection definition (4.2) we get $p \models \gamma'(s, a, s')$. Using the FTS2FPG definition (4.3) we get $p \models \gamma(s, a, s')$. Using the FTS projection definition (2.5) we get $(s, a, s') \in trans'$ and therefore $((s, \psi), (s', \psi')) \in E$.
- Otherwise the existence of the edge does not depend on the $trans$ parameter and therefore $((s, \psi), (s', \psi')) \in E$ if $(s, \psi) \in V$, since $V^F = V$ we have $(s, \psi) \in V$.

□

Having proven this we can visualize the relation between the different games and transition systems in the following diagram:



Finally we prove that solving an FTS, ie. finding winning sets for all products, answers the verification question.

Theorem 4.2. *Given:*

- *FTS* $M = (S, Act, trans, s_0, N, P, \gamma)$,
- *closed modal mu-calculus formula* φ ,
- *product* $p \in P$ and
- *state* $s \in S$

it holds that $(M|_p, s) \models \varphi$ *if and only if* $(s, \varphi) \in W_0^p$ *in* $FTS2FPG(M, \varphi)$.

Proof. The winning set W_α^p is equal to winning set W_α in $FTS2FPG(M, \varphi)|_p$, for any $\alpha \in \{0, 1\}$, using the FPG definition (4.1). Using theorem 4.1 we find that the game $FTS2FPG(M, \varphi)|_p$ is equal to the game $LTS2PG(M|_p, \varphi)$, obviously their winning sets are also equal. Using the well studied relation between parity games and LTS verification, stated in theorem 3.1, we know that $(M|_p, s) \models \varphi$ if and only if $(s, \varphi) \in W_0$ in game $LTS2PG(M|_p, \varphi)$. Winning set W_α^p is equal to W_α , therefore the theorem holds. □

Revisiting our prior example we can see the theorem in action by noting that $M|_\emptyset \models \varphi$, $M|_{\{f\}} \not\models \varphi$ and $M|_{\{f, g\}} \models \varphi$. This is reflected by the vertex $(s_1, \mu X. [a]X \vee \langle b \rangle \top)$ being present in W_0^\emptyset and $W_0^{\{f, g\}}$ but not in $W_0^{\{f\}}$.

5 Variability parity games

Next we will introduce *variability parity games* (VPGs). VPGs are very similar to FPGs, however VPGs use configurations instead of features and products to express variability. This gives a syntactically more pleasant representation that is not solely tailored for FTSs. Furthermore in VPGs deadlocks are removed, by doing so VPG plays can only result in infinite paths and no longer in finite paths.

Later we will show the relation between VPGs and FTS verification, which is similar to the relation between FPGs and FTS verification. First we introduce VPGs.

Definition 5.1. A *variability parity game* (VPG) is a tuple $(V, V_0, V_1, E, \Omega, \mathfrak{C}, \theta)$, where:

- $V = V_0 \cup V_1$ and $V_0 \cap V_1 = \emptyset$,
- V_0 is the set of vertices owned by player 0,
- V_1 is the set of vertices owned by player 1,
- $E \subseteq V \times V$ is the edge relation; we assume that E is total, i.e. for all $v \in V$ there is some $w \in V$ such that $(v, w) \in E$,
- $\Omega : V \rightarrow \mathbb{N}$ is a priority assignment,
- \mathfrak{C} is a non-empty finite set of configurations,
- $\theta : E \rightarrow \mathcal{P}(\mathfrak{C}) \setminus \emptyset$ is the configuration mapping, satisfying for all $v \in V$, $\bigcup \{\theta(v, w) \mid (v, w) \in E\} = \mathfrak{C}$.

A VPG is played similarly to a PG, however the game is played for a specific configuration $c \in \mathfrak{C}$. Player α can only move the token from $v \in V_\alpha$ to $w \in V$ if $(v, w) \in E$ and $c \in \theta(v, w)$. Furthermore VPGs don't have deadlocks, therefore every play results in an infinite path.

A game played for configuration $c \in \mathfrak{C}$ results in winning sets W_0^c and W_1^c , which are defined similar to the W_0 and W_1 winning sets for parity games.

Solving a VPG means determining winning sets for every configuration in the VPG.

Definition 5.2. The *projection from VPG $G = (V, V_0, V_1, E, \Omega, \mathfrak{C}, \theta)$ to a configuration $c \in \mathfrak{C}$* , noted $G|_c$, is the parity game $(V, V_0, V_1, E', \Omega)$ where $E' = \{e \in E \mid c \in \theta(e)\}$.

Playing VPG G for a specific configuration $c \in \mathfrak{C}$ is the same as playing the PG $G|_c$. Any path that is valid in G for c is also valid in $G|_c$ and vice versa. Therefore the strategies are also interchangeable, furthermore the winning sets W_α for $G|_c$ and W_α^c for G are identical. Since parity games are positionally determined so are VPGs. Similarly, since finite parity games are decidable, so are finite VPGs.

5.1 Creating variability parity games

We will define a translation from an FPG to a VPG. To do so we use the set of valid products as the set of configurations. Furthermore we make the FPG deadlock free, this is done by creating two losing vertices l_0 and l_1 such that player α loses when the token is in vertex l_α . Any vertex that can't move for a configuration will get an edge that is admissible for that configuration towards one of the losing vertices.

Definition 5.3. $FPG2VPG(G^F)$ converts $FPG G^F = (V^F, V_0^F, V_1^F, E^F, \Omega^F, N, P, \gamma)$ to $VPG G = (V, V_0, V_1, E, \Omega, \mathfrak{C}, \theta)$.

We define $\mathfrak{C} = P$. We create vertices l_0 and l_1 and define $V_0 = V_0^F \cup \{l_0\}$, $V_1 = V_1^F \cup \{l_1\}$ and $V = V_0 \cup V_1$.

We construct E by first making $E = E^F$ and adding edges (l_0, l_0) and (l_1, l_1) to E . Simultaneously we construct θ by first making $\theta(e) = \{p \in \mathfrak{C} \mid p \models \gamma(e)\}$ for every $e \in E^F$. Furthermore $\theta(l_0, l_0) = \theta(l_1, l_1) = \mathfrak{C}$.

Next, for every vertex $v \in V_\alpha$ with $\alpha = \{0, 1\}$, we have $C = \mathfrak{C} \setminus \{\theta(v, w) \mid (v, w) \in E\}$. If $C \neq \emptyset$ then we add (v, l_α) to E and make $\theta(v, l_\alpha) = C$. Finally we have

$$\Omega(v) = \begin{cases} 1 & \text{if } v = l_0 \\ 0 & \text{if } v = l_1 \\ \Omega^F(v) & \text{otherwise} \end{cases}$$

Again considering our previous working example we can translate the FPG shown in figure 9 to the VPG shown in figure 10. Where c_0 is product \emptyset , c_1 is $\{f\}$ and c_2 is $\{f, g\}$.

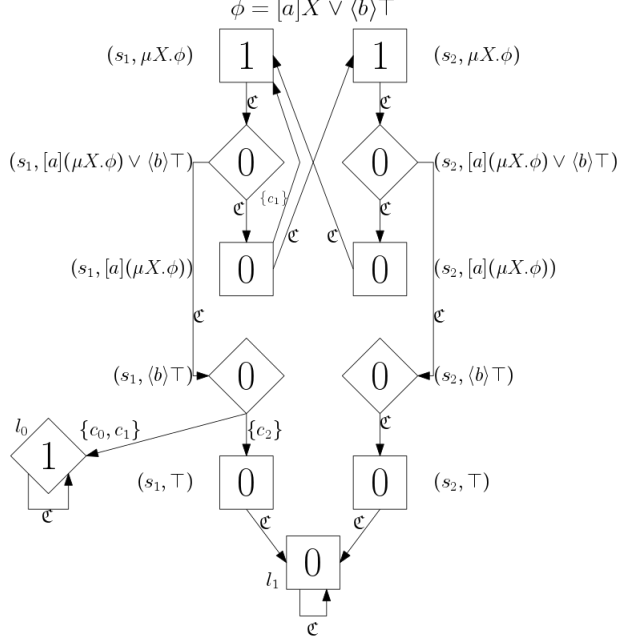


Figure 10: VPG

5.2 FTS verification using VPG

We have shown in theorem 4.2 that we can use an FPG to verify an FTS. Next we will show that a winning set in the FPG M is the subset of the winning set in the VPG $FPG2VPG(M)$.

Theorem 5.1. *Given:*

- $FPG\ G^F = (V^F, V_0^F, V_1^F, E^F, \Omega^F, N, P, \gamma)$,
- product $p \in P$

we have for winning sets Q_α^p in G^F and W_α^p in $FPG2VPG(G^F)$ that $Q_\alpha^p \subseteq W_\alpha^p$ for any $\alpha \in \{0, 1\}$.

Proof. Let $G = (V, V_0, V_1, E, \Omega, \mathfrak{C}, \theta) = FPG2VPG(G^F)$. Consider finite play π that is valid in game G^F for product p . We have for every (π_i, π_{i+1}) in π that $(\pi_i, \pi_{i+1}) \in E^F$ and $p \models \gamma(\pi_i, \pi_{i+1})$. From the $FPG2VPG$ definition (5.3) it follows that $(\pi_i, \pi_{i+1}) \in E$ and $p \in \theta(\pi_i, \pi_{i+1})$. So we can conclude that path π is also valid in game G for configuration p . Since the play is finite the winner is determined by the last vertex v in π , player α wins such that $v \in V_{\bar{\alpha}}$. Furthermore we know, because the play is finite, that there exists no $(v, w) \in E^F$ with $p \models \gamma(v, w)$. From this we can conclude that $(v, l_{\bar{\alpha}}) \in E$ and $p \in \theta(v, l_{\bar{\alpha}})$. Vertex $l_{\bar{\alpha}}$ has one outgoing edge, namely to itself. So finite play π will in game G^F results in an infinite play $\pi(l_{\bar{\alpha}})^\omega$. Vertex $l_{\bar{\alpha}}$ has a priority with the same parity as player α , so player α wins the infinite play in G for configuration p .

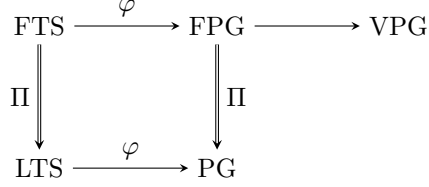
Consider infinite play π that is valid in game G^F for product p . As shown above this play is also valid in game G for configuration p . Since the win conditions of both games are the same the play will result in the same winner.

Consider infinite play π that is valid in game G for configuration p . We distinguish two cases:

- If l_α doesn't occur in π then the path is also valid for game G^F with product p and has the same winner.
- If $\pi = \pi'(l_\alpha)^\omega$ with no occurrence of l_α in π' then the winner is player $\bar{\alpha}$. The path π' is valid for game G^F with product p . Let vertex v be the last vertex of π' . Since $(v, l_\alpha) \in E$ and $p \in \theta(v, l_\alpha)$ we know that there is no $(v, w) \in E^F$ with $p \models \gamma(v, w)$ and that vertex v is owned by player α . So in game G^F player α can't move at vertex v and therefore loses the game (in which case the winner is also $\bar{\alpha}$).

We have shown that every path (finite or infinite) in game G^F with product p can be played in game G with configuration p and that they have the same winner. Furthermore every infinite path in game G with configuration p can be either played as an infinite path or the first part of the path can be played in G^F with product p and they have the same winner. From this we can conclude that the theorem holds. \square

We can conclude the diagram depicting the relation between the different games and transition systems:



Finally we show that solving VPGs, ie. finding the winning sets for all configurations, can be used to verify FTSs.

Theorem 5.2. *Given:*

- $\text{FTS } M = (S, \text{Act}, \text{trans}, s_0, N, P, \gamma)$,
- *closed modal mu-calculus formula φ ,*
- *product $p \in P$ and*
- *state $s \in S$*

it holds that $(M|_p, s) \models \varphi$ if and only if $(s, \varphi) \in W_0^p$ in $\text{FPG2VPG}(\text{FTS2FPG}(M, \varphi))$.

Proof. Let W_0^p and W_1^p denote the winning sets for game $\text{FPG2VPG}(\text{FTS2FPG}(M, \varphi))$. And Q_0^p and Q_1^p denote the winning sets for game $\text{FTS2FPG}(M, \varphi)$.

Using theorem 4.2 we find that $(M|_p, s) \models \varphi$ if and only if $(s, \varphi) \in Q_0^p$. If $(s, \varphi) \in Q_0^p$ then we find by using theorem 5.1 that $(s, \varphi) \in W_0^p$. If $(s, \varphi) \notin Q_0^p$ then $(s, \varphi) \in Q_1^p$ and therefore $(s, \varphi) \in W_1^p$ and $(s, \varphi) \notin W_0^p$. \square

Using this theorem we can visualize verification of an FTS in figure 11.

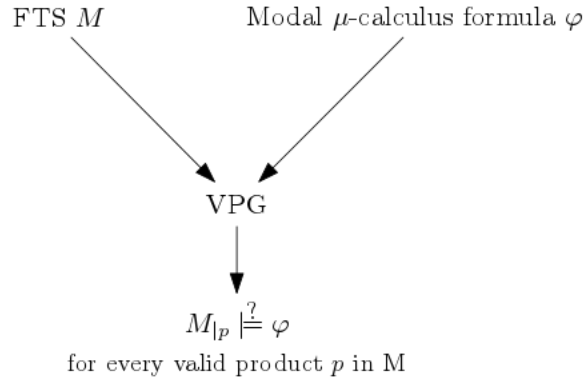


Figure 11: FTS verification using VPG

Part II

Solving variability parity games

6 Introduction

For solving VPGs we distinguish two general approaches, the first approach is to simply project the VPG to the different configurations and solve all the resulting parity games independently. We call this *independently* solving a VPG. Alternatively we solve the VPG *collectively*, where a VPG is solved in its entirety and similarities between the configurations are used to improve performance.

In the next sections we explore collective algorithms and analyse their time complexity. We aim to solve VPGs originating from model verification problems, such VPGs generally have certain properties that a completely random VPG might not have. In general parity games originating from model verification problems have a relatively low number of distinct priorities compared to the number of vertices because new priorities are only introduced when fixed points are nested in the μ -calculus formula. Furthermore the transition guards of featured transition systems are expressed over features. In general these transition guards will be quite simple, specifically excluding or including a small number of features.

7 Preliminary concepts

7.1 Set representation

A set can straightforwardly be represented by a collection containing all the elements that are in the set. We call this an *explicit* representation of a set. We can also represent sets *symbolically* in which case the set of elements is represented by some sort of formula. A typical way to represent a set symbolically is through a boolean formula encoded in a *binary decision diagram* [7, 8]. For example the set $S = \{0, 1, 2, 4, 5, 7\}$ can be expressed by boolean formula:

$$F(x_2, x_1, x_0) = (x_2 \vee \neg x_1 \vee \neg x_0) \wedge (\neg x_2 \vee \neg x_1 \vee x_0)$$

where x_0, x_1 and x_2 are boolean variables. The formula gives the following truth table:

$\mathbf{x_2x_1x_0}$	$\mathbf{F(x_2, x_1, x_0)}$
000	1
001	1
010	1
011	0
100	1
101	1
110	0
111	1

The function F defines set S' in the following way: $S' = \{x_2x_1x_0 \mid F(x_2, x_1, x_0) = 1\}$. As we can see set S' and S represent the same numbers. We can perform set operations on sets represented as boolean functions by performing logical operations on the functions. For example, given boolean formula's f and g representing sets V and W the formula $f \wedge g$ represents set $V \cap W$.

Boolean functions can efficiently be represented in BDDs, for a comprehensive treatment of BDDs we refer to [7, 8]. We will note here that given x boolean variables and two boolean functions encoded as BDDs we can perform binary operations \vee, \wedge on them in $O(2^{2^x}) = O(n^2)$ where $n = 2^x$ is the maximum set size that can be represented by x variables [9, 8]. The running time specifically depends on the size of the decision diagrams, in general if the boolean functions are simple then the size of the decision diagram is also small and operations can be performed quickly.

7.1.1 Symbolically representing sets of configurations

For VPGs originating from an FTS the configuration sets are already boolean functions over the features. The formula's guarding the edges in the VPG will generally have relatively simple boolean functions, therefore they are specifically appropriate to represent as BDDs.

A set operation over two explicit sets can be performed in $O(n)$ where n is the maximum size of the sets, this is better than the time complexity of a set operation using BDDs ($O(n^2)$). However if the BDDs are small then the set size can still be large but the set operations can be performed very quickly. This is a trade-off between worst case running time complexity and average actual running time; using a symbolic representation might yield better results if the sets are structured in such a way that the BDDs are small, however its worse case running time complexity will be worse.

7.2 Fixed-point preliminaries

7.2.1 Lattices

The following definition regarding ordering and lattices are taken from [10].

Definition 7.1. A partial order is a binary relation $x \leq y$ on set S where for all $x, y, z \in S$ we have:

- $x \leq x$. (Reflexive)
- If $x \leq y$ and $y \leq x$, then $x = y$. (Antisymmetric)
- If $x \leq y$ and $y \leq z$, then $x \leq z$. (Transitive)

Definition 7.2. A partially ordered set is a set S and a partial order \leq for that set, we denote a partially ordered set by $\langle S, \leq \rangle$.

Definition 7.3. Given partially ordered set $\langle P, \leq \rangle$ and subset $X \subseteq P$. An upper bound to X is an element $a \in P$ such that $x \leq a$ for every $x \in X$. A least upper bound to X is an upper bound $a \in P$ such that $a' \leq a$ for every upper bound $a' \in P$ to X .

The term least upper bound is synonymous with the term supremum.

Definition 7.4. Given partially ordered set $\langle P, \leq \rangle$ and subset $X \subseteq P$. A lower bound to X is an element $a \in P$ such that $a \leq x$ for every $x \in X$. A greatest lower bound to X is a lower bound $a \in P$ such that $a \leq a'$ for every lower bound $a' \in P$ to X .

The term greatest lower bound is synonymous with the term infimum.

Definition 7.5. A lattice is a partially ordered set where any two of its elements have a supremum and an infimum.

Definition 7.6. A complete lattice is a partially ordered set in which every subset has a supremum and an infimum.

Definition 7.7. A function $f : D \rightarrow D'$ is monotonic, also called order preserving, if for all $x \in D$ and $y \in D$ it holds that if $x \leq y$ then $f(x) \leq f(y)$.

7.2.2 Fixed-points

Definition 7.8. Given function $f : D \rightarrow D$ the value $x \in D$ is a fixed point for f if and only if $f(x) = x$.

Definition 7.9. Given function $f : D \rightarrow D$ the value $x \in D$ is the least fixed point for f if and only if x is a fixed point for f and every other fixed point for f is greater or equal to x .

Definition 7.10. Given function $f : D \rightarrow D$ the value $x \in D$ is the greatest fixed point for f if and only if x is a fixed point for f and every other fixed point for f is less or equal to x .

The Knaster-Tarski theorem states that least and greatest fixed points exist for some domain and function given that a few conditions hold. The theorem, as written down by Tarski [11], states:

Theorem 7.1 (Knaster-Tarski[11]). *Let*

- $\langle A, \leq \rangle$ be a complete lattice,
- f be an increasing function on A to A ,
- P be a the set of all fixpoints of f .

Then the set P is not empty and the system $\langle P, \leq \rangle$ is a complete lattice; in particular we have

$$\sup P = \sup\{x \mid f(x) \geq x\} \in P$$

and

$$\inf P = \inf\{x \mid f(x) \leq x\} \in P$$

8 Parity game algorithms

We inspect two existing parity game algorithms which will be used in VPG algorithms.

8.1 Zielonka's recursive algorithm

First we consider Zielonka's recursive algorithm, created from the constructive proof given in [12], which solves total PGs. Pseudo code is presented in algorithm 1.

Algorithm 1 RECURSIVEPG($PG\ G = (V, V_0, V_1, E, \Omega)$)

```

1:  $m \leftarrow \min\{\Omega(v) \mid v \in V\}$ 
2:  $h \leftarrow \max\{\Omega(v) \mid v \in V\}$ 
3: if  $h = m$  or  $V = \emptyset$  then
4:   if  $h$  is even or  $V = \emptyset$  then
5:     return  $(V, \emptyset)$ 
6:   else
7:     return  $(\emptyset, V)$ 
8:   end if
9: end if
10:  $\alpha \leftarrow 0$  if  $h$  is even and 1 otherwise
11:  $U \leftarrow \{v \in V \mid \Omega(v) = h\}$ 
12:  $A \leftarrow \alpha\text{-Attr}(G, U)$ 
13:  $(W'_0, W'_1) \leftarrow \text{RECURSIVEPG}(G \setminus A)$ 
14: if  $W'_{\bar{\alpha}} = \emptyset$  then
15:    $W_{\alpha} \leftarrow A \cup W'_{\alpha}$ 
16:    $W_{\bar{\alpha}} \leftarrow \emptyset$ 
17: else
18:    $B \leftarrow \bar{\alpha}\text{-Attr}(G, W'_{\bar{\alpha}})$ 
19:    $(W''_0, W''_1) \leftarrow \text{RECURSIVEPG}(G \setminus B)$ 
20:    $W_{\alpha} \leftarrow W''_{\alpha}$ 
21:    $W_{\bar{\alpha}} \leftarrow W''_{\bar{\alpha}} \cup B$ 
22: end if
23: return  $(W_0, W_1)$ 

```

An exhaustive explanation of the algorithm can be found in [12], we do introduce the definitions used in the algorithm. First we introduce the notion of an attractor set. An attractor set is a set of vertices $A \subseteq V$ calculated for player α given set $U \subseteq V$ where player α has a strategy to force the play starting in any vertex in $A \setminus U$ to a vertex in U .

Definition 8.1. [12] Given parity game $G = (V, V_0, V_1, E, \Omega)$ and a non-empty set $U \subseteq V$ we define $\alpha\text{-Attr}(G, U)$ such that

$$U_0 = U$$

For $i \geq 0$:

$$U_{i+1} = U_i \cup \{v \in V_\alpha \mid \exists v' \in V : v' \in U_i \wedge (v, v') \in E\} \\ \cup \{v \in V_{\bar{\alpha}} \mid \forall v' \in V : (v, v') \in E \implies v' \in U_i\}$$

Finally:

$$\alpha\text{-Attr}(G, U) = \bigcup_{i \geq 0} U_i$$

Next we present the definition of a subgame, where a PG and a set of vertices which are removed from the game are given.

Definition 8.2. [12] Given a parity game $G = (V, V_0, V_1, E, \Omega)$ and $U \subseteq V$ we define the subgame $G \setminus U$ to be the game $(V', V'_0, V'_1, E', \Omega)$ with:

- $V' = V \setminus U$,
- $V'_0 = V_0 \cap V'$,
- $V'_1 = V_1 \cap V'$ and
- $E' = E \cap (V' \times V')$.

Note that a subgame is not necessarily total, however the recursive algorithm always creates subgames that are total (shown in [12]).

8.1.1 Zielonka's recursive algorithm local

We can modify Zielonka's algorithm to solve parity games locally. We use the property that vertices won by player $\bar{\alpha}$ in game $G \setminus A$ are also won by player $\bar{\alpha}$. We use this property to, in some cases, not go into the second recursion because we already found the vertex. We extend the algorithm with two parameters, v_0 and Δ . Vertex v_0 represents the vertex that we try to solve and is used unmodified in recursive calls. Parameter $\Delta \subseteq \{0, 1\}$ contains a set of players for which we try to find vertex v_0 , if we find v_0 to be won by a player that is in Δ we are done. However if we find vertex v_0 to be won by a player that is not in Δ we need to solve the entire game. The algorithm can make use of this to tell a recursive call it needs to find the vertex to be won by player $\bar{\alpha}$ in the subgame because then this vertex is also won by player $\bar{\alpha}$ in the game itself. If the vertex is not won by $\bar{\alpha}$ the entire game needs to be solved because the winning sets are required in the second recursion. Pseudo code for the algorithm is provided in algorithm 2.

Algorithm 2 RECURSIVEPGLOCAL($PG\ G = (V, V_0, V_1, E, \Omega), v_0, \Delta$)

```

1:  $m \leftarrow \min\{\Omega(v) \mid v \in V\}$ 
2:  $h \leftarrow \max\{\Omega(v) \mid v \in V\}$ 
3: if  $h = m$  or  $V = \emptyset$  then
4:   if  $h$  is even or  $V = \emptyset$  then
5:     return  $(V, \emptyset)$ 
6:   else
7:     return  $(\emptyset, V)$ 
8:   end if
9: end if
10:  $\alpha \leftarrow 0$  if  $h$  is even and 1 otherwise
11:  $U \leftarrow \{v \in V \mid \Omega(v) = h\}$ 
12:  $A \leftarrow \alpha\text{-Attr}(G, U)$ 
13: if  $\bar{\alpha} \in \Delta$  then
14:    $(W'_0, W'_1) \leftarrow \text{RECURSIVEPGLOCAL}(G \setminus A, v_0, \{\bar{\alpha}\})$ 
15: else
16:    $(W'_0, W'_1) \leftarrow \text{RECURSIVEPGLOCAL}(G \setminus A, v_0, \emptyset)$ 
17: end if
18: if  $W'_\alpha = \emptyset$  then
19:    $W_\alpha \leftarrow A \cup W'_\alpha$ 
20:    $W_{\bar{\alpha}} \leftarrow \emptyset$ 
21: else
22:   if  $\bar{\alpha} \in \Delta \wedge v_0 \in W'_{\bar{\alpha}}$  then
23:      $(W_0, W_1) \leftarrow (W'_0, W'_1)$ 
24:   else
25:      $B \leftarrow \bar{\alpha}\text{-Attr}(G, W'_{\bar{\alpha}})$ 
26:     if  $\bar{\alpha} \in \Delta \wedge v_0 \in B$  then
27:        $W_\alpha \leftarrow \emptyset$ 
28:        $W_{\bar{\alpha}} \leftarrow B$ 
29:     else
30:        $(W''_0, W''_1) \leftarrow \text{RECURSIVEPGLOCAL}(G \setminus B, v_0, \Delta)$ 
31:        $W_\alpha \leftarrow W''_\alpha$ 
32:        $W_{\bar{\alpha}} \leftarrow W''_{\bar{\alpha}} \cup B$ 
33:     end if
34:   end if
35: end if
36: return  $(W_0, W_1)$ 

```

Theorem 8.1. *Given total parity game $G = (V, V_0, V_1, E, \Omega)$, vertex v_0 (which is not necessarily in V), $\Delta \subseteq \{0, 1\}$ and winning sets Q_0 and Q_1 for game G . We have for sets $(W_0, W_1) = \text{RECURSIVEPGLOCAL}(G, v_0, \Delta)$ such that either or both of the following statements hold:*

(I) For some $\beta \in \{0, 1\}$ we have $v_0 \in W_\beta$, $v_0 \in Q_\beta$ and $\beta \in \Delta$.

(II) $W_0 = Q_0$ and $W_1 = Q_1$.

Proof. To prove the correctness we compare the behaviour of `RECURSIVEPGLOCAL` with the behaviour of the original algorithm `RECURSIVEPG` for which we know that it is correct.

Prove by induction on G :

Base: When there are no vertices in V or there is only one priority then the local algorithm behaves the same as the original algorithm so statement (II) holds.

Step: We start by distinguishing two cases, first we assume that $v_0 \notin V$. Vertex v_0 is also not in any subgame of G . In the first recursion we know by induction that either or both of the statements are true for $G \setminus A$, since v_0 is not in game $G \setminus A$ only statement (II) can be true. So the first recursion returns the same result as the original algorithm would. The if statements guarding the second recursion are always false because $v_0 \notin B$ and $v_0 \notin W'_\alpha$ and similarly as with game $G \setminus A$ the vertex v_0 is not in $G \setminus B$. So the second recursion also returns the same values as the original algorithm would. We can conclude that when $v_0 \notin V$ the local algorithm behaves the same as the original and therefore statement (II) holds.

Now consider $v_0 \in V$. Again we distinguish two cases, starting with $\bar{\alpha} \notin \Delta$. In the first recursive call we get $\Delta = \emptyset$, by induction either or both statements hold for $G \setminus A$, however with $\Delta = \emptyset$ statement (II) must be true. Therefore the first recursive call behaves the same as the original algorithm. The if statements guarding the second recursion are always false since $\bar{\alpha} \notin \Delta$ so the second recursive call is always performed. Up until this second recursive call the local algorithm behaves the same as the original algorithm. From the correctness of the original algorithm we can conclude that vertices won by some player in game $G \setminus B$ are also won by that player in game G . This holds true in the local algorithm since it has behaved identical up to this point. By induction we know either or both statements are true, if statement (II) is true for $G \setminus B$ then the algorithm behaves the same as the original and statement (II) is also true for game G . If statement (I) is true for $G \setminus B$ and v_0 is won by player β in $G \setminus B$ then we know v_0 is also won by player β in game G and thus statement (I) holds for G .

Next consider $\bar{\alpha} \in \Delta$ (and $v_0 \in V$). The first recursive call is made with $\Delta = \{\bar{\alpha}\}$. If $W'_\alpha = \emptyset$ then statement (II) must be true for game $G \setminus A$ and the algorithm behaves the same as the original in which case statement (II) is true for G . If $W'_\alpha \neq \emptyset$ then we consider v_0 being in W'_α , by induction we can conclude that v_0 is indeed won by player $\bar{\alpha}$ in game $G \setminus A$. Furthermore from the correctness of the original algorithm we can conclude that any vertex won by player $\bar{\alpha}$ in game $G \setminus A$ is also won by player $\bar{\alpha}$ in game G . So returning winning sets W'_0 and W'_1 make statement (I) true for game G .

Consider v_0 not being in W'_α but in B . We can conclude that statement (II) is true for subgame $G \setminus A$ so the algorithm behaves the same as the original algorithm up until this point. Using the original algorithm we can see that any vertex in B is won by player $\bar{\alpha}$ in game G so if we find v_0 in B we can return B for player $\bar{\alpha}$ which includes v_0 and \emptyset for player α to make statement (I) true for game G .

If v_0 is also not in B then we still conclude that statement (II) is true for subgame $G \setminus A$ and that the algorithm has behaved the same as the original algorithm up until this point. From the correctness of the original algorithm we can conclude that vertices won by some player in game $G \setminus B$ are also won by that player in game G . This holds true in the local algorithm since it has behaved identical up to this point. By induction we know either or both statements are true, if statement (II) is true for $G \setminus B$ then the algorithm behaves the same as the original and statement (II) is also true for game G . If statement (I) is true for $G \setminus B$ and v_0 is won by player β in $G \setminus B$ then we know v_0 is also won by player β in game G and thus statement (I) holds for G . \square

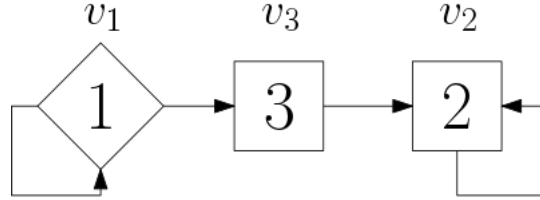
Calling `RECURSIVEPGLOCAL`($G, v_0, \{0, 1\}$) with v_0 in G either solves the full game (statement (II)) or correctly puts v_0 in either winning set (statement (I)). In both cases v_0 is in the correct winning set and the game is solved locally.

The algorithm uses the property that vertices won by player $\bar{\alpha}$ in a subgame during the algorithm are also won by player $\bar{\alpha}$ in the game itself. However this property only goes one recursion deep, meaning that during the solving of a game there can be a subgame where a vertex is won by player $\bar{\alpha}$ however in the original game we try to solve this vertex is won by player α . For this reason the local algorithm can't simply claim to be done when v_0 is won by $\bar{\alpha}$, but we need to pass down a value for Δ .

We disprove the following conjecture to show that there indeed can be a vertex that is won by player $\bar{\alpha}$ at some point but be won by player α in a higher recursion level.

Conjecture 8.2. *For any $\text{RECURSIVEPG}(G')$ that is invoked during $\text{RECURSIVEPG}(G)$ it holds that any vertex $v \in W'_\alpha$ is won by player $\bar{\alpha}$ in game G .*

We disprove this conjecture using the following counter example G :



All vertices are won by player 0 (v_1 plays to v_3 , v_3 must play to v_2 and v_2 must play to itself therefore we always get an infinite path of v_2 's).

We solve this game using RECURSIVEPG , we write down the values of relevant variables below. We use the hat decoration to indicate values for variables in the first recursion:

```

RECURSIVEPG(G):
  h = 3, α = 1
  A = {v3}
  RECURSIVEPG(G \ A):
    ĥ = 2, â = 0
    Â = {v2}
    RECURSIVEPG(G \ A \ Â):
      Ŵ'_0 = ∅
      Ŵ'_1 = {v1} = Ŵ'_α
      Vertex v1 is in Ŵ'_α however in G the vertex is won by player â.
      B̂ = {v1}
      RECURSIVEPG(G \ A \ B̂):
        Ŵ''_0 = {v2}, Ŵ''_1 = ∅
        W'_0 = W'_α = {v2}
        W'_1 = W'_α = {v1}
        B = V
        W_0 = V

```

8.2 Fixed-point iteration algorithm

Parity games can be solved by solving an alternating fixed point formula, as shown in [13]. We will consider PG $G = (V, V_0, V_1, E, \Omega)$ with d distinct priorities. We can apply *priority compression* to make sure every priority in G maps to a value in $\{0, \dots, d-1\}$ or $\{1, \dots, d\}$ [14, 15]. We assume without loss of generality that the priorities map to $\{0, \dots, d-1\}$ and that $d-1$ is even.

Consider the following formula

$$S(G = (V, V_0, V_1, E, \Omega)) = \nu Z_{d-1}. \mu Z_{d-2}. \dots. \nu Z_0. F_0(Z_{d-1}, \dots, Z_0)$$

with

$$F_0(Z_{d-1}, \dots, Z_0) = \{v \in V_0 \mid \exists w \in V (v, w) \in E \wedge Z_{\Omega(w)}\} \cup \{v \in V_1 \mid \forall w \in V (v, w) \in E \implies Z_{\Omega(w)}\}$$

where $Z_i \subseteq V$. The formula $\nu X.f(X)$ solves the greatest fixed-point of X in f , similarly $\mu X.f(X)$ solves the least fixed-point of X in f .

To understand the formula we consider sub-formula $\nu Z_0.F_0(Z_{d-1}, \dots, Z_0)$. This formula holds for vertices from which player 0 can either force the play into a node with priority $i > 0$ for which Z_i holds or the player

can stay in vertices with priority 0 indefinitely. The formula $\mu Z_0.F_0(Z_{d-1}, \dots, Z_0)$ holds for vertices from which player 0 can force the play into a node with priority $i > 0$ for which Z_i holds in finitely many steps.

As shown in [13], solving $S(G)$ gives the winning set for player 0 in game G . A concrete algorithm is introduced in [15], note that this algorithm can solve finite games. We will extend this algorithm such that it calculates $S(G)$ in an efficient manner by using P_0 and P_1 where it is known beforehand that vertices in P_0 are won by player 0 and vertices in P_1 are won by player 1.

8.2.1 Fixed-point approximation

As shown in [16] we can calculate fixed-point $\mu X.f(X)$ when f is monotonic in X by approximating X .

$$\mu X.f(X) = \bigcup_{i \geq 0} X^i$$

where $X^i = f(X^{i-1})$ for $i > 0$ and $X^0 \subseteq \mu X.f(X)$. So picking the smallest value possible for X_0 will always correctly calculate $\mu X.f(X)$.

Similarly we can calculate fixed-point $\nu X.f(X)$ when f is monotonic in X by approximating X .

$$\nu X.f(X) = \bigcap_{i \geq 0} X^i$$

where $X^i = f(X^{i-1})$ for $i > 0$ and $X^0 \supseteq \nu X.f(X)$. So picking the largest value possible for X_0 will always correctly calculate $\nu X.f(X)$.

Clearly the formula $F_0(Z_{d-1}, \dots, Z_0)$ is monotonic in any Z_j , so we can calculate Z_{d-1} by approximating every Z_j starting at V for νZ_j and starting at \emptyset for μZ_j .

An algorithm to perform the approximation is presented in [15] and shown in algorithm 3.

Algorithm 3 Fixed-point approximation

<pre> 1: function FPITER($G = (V, V_0, V_1, E, \Omega)$) 2: for $i \leftarrow d - 1, \dots, 0$ do 3: INIT(i) 4: end for 5: repeat 6: $Z'_0 \leftarrow Z_0$ 7: $Z_0 \leftarrow \text{DIAMOND}() \cup \text{BOX}()$ 8: $i \leftarrow 0$ 9: while $Z_i = Z'_i \wedge i < d - 1$ do 10: $i \leftarrow i + 1$ 11: $Z'_i \leftarrow Z_i$ 12: $Z_i \leftarrow Z_{i-1}$ 13: INIT($i - 1$) 14: end while 15: until $i = d - 1 \wedge Z_{d-1} = Z'_{d-1}$ 16: return ($Z_{d-1}, V \setminus Z_{d-1}$) 17: end function </pre>	<pre> 1: function INIT(i) 2: $Z_i \leftarrow \emptyset$ if i is odd, V otherwise 3: end function 1: function DIAMOND 2: return $\{v \in V_0 \mid \exists w \in V (v, w) \in E \wedge w \in Z_{\Omega(w)}\}$ 3: end function 1: function BOX 2: return $\{v \in V_1 \mid \forall w \in V (v, w) \in E \implies w \in Z_{\Omega(w)}\}$ 3: end function </pre>
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9 Recursive algorithm for VPGs

We can use Zielonka's recursive algorithm to solve VPGs, to do so we will first consider a method of creating a parity game from a VPG, called *unification*.

9.1 Unified parity games

We can create a PG from a VPG by taking all the projections of the VPG, which are PGs, and combining them into one PG by taking the union of them. We call the resulting PG the *unification* of the VPG. A parity game that is the result of a unification is called a *unified PG*, also any total subgame of it will be called a unified PG. A unified PG always has a VPG from which it originated.

Definition 9.1. Given VPG $\hat{G} = (\hat{V}, \hat{V}_0, \hat{V}_1, \hat{E}, \hat{\Omega}, \mathfrak{C}, \theta)$ we define the unification of \hat{G} , denoted as \hat{G}_\downarrow , as

$$\hat{G}_\downarrow = \bigsqcup_{c \in \mathfrak{C}} \hat{G}|_c$$

where the union of two PGs is defined as

$$(V, V_0, V_1, E, \Omega) \uplus (V', V'_0, V'_1, E', \Omega') = (V \uplus V', V_0 \uplus V'_0, V_1 \uplus V'_1, E \uplus E', \Omega \uplus \Omega')$$

We will use the hat decoration $(\hat{G}, \hat{V}, \hat{E}, \hat{\Omega}, \hat{W})$ when referring to a VPG and use no hat decoration when referring to a PG.

Every vertex in game \hat{G}_\downarrow originates from a configuration and an original vertex. Therefore we can consider every vertex in a unification as a pair consisting of a vertex and a configuration, ie. $V = \mathfrak{C} \times \hat{V}$. We can consider edges in a unification similarly, so $E \subseteq (\mathfrak{C} \times \hat{V}) \times (\mathfrak{C} \times \hat{V})$. Note that edges don't cross configurations so for every $((c, \hat{v}), (c', \hat{v}')) \in E$ we have $c = c'$. Figure 12 shows an example of a unification.

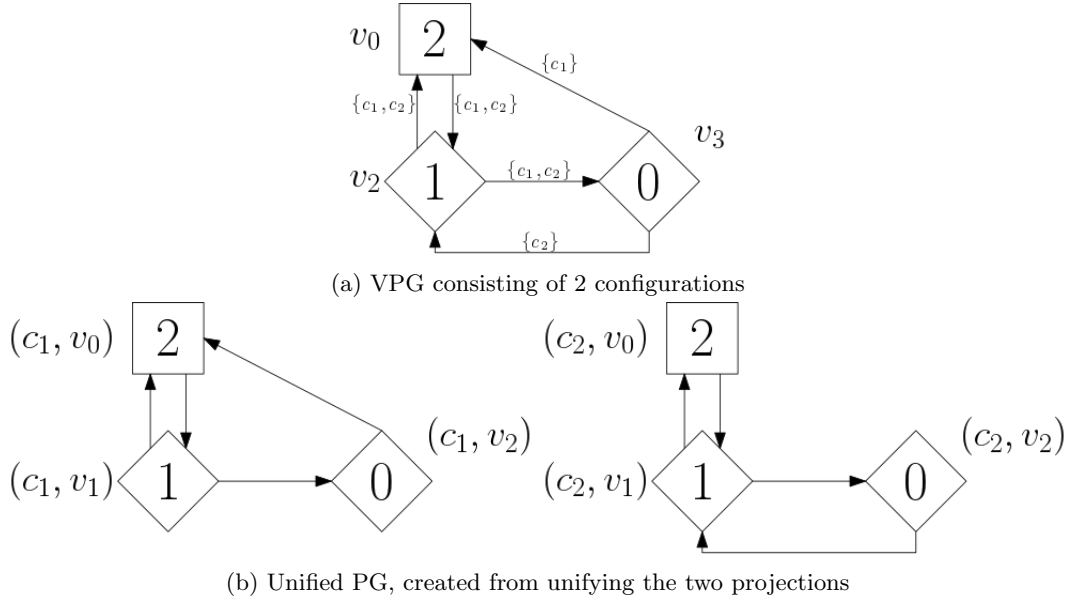


Figure 12: A VPG with its corresponding unified PG

If we solve the PG that is the unification of a VPG we have solved the VPG, as shown in the next theorem.

Theorem 9.1. Given

- VPG $\hat{G} = (\hat{V}, \hat{V}_0, \hat{V}_1, \hat{E}, \hat{\Omega}, \mathfrak{C}, \theta)$,
- some configuration $c \in \mathfrak{C}$,
- winning sets \hat{W}_0^c and \hat{W}_1^c for game \hat{G} and
- winning sets W_0 and W_1 for game \hat{G}_\downarrow

it holds that

$$(c, \hat{v}) \in W_\alpha \iff \hat{v} \in \hat{W}_\alpha^c, \text{ for } \alpha \in \{0, 1\}$$

Proof. The bi-implication is equal to the following to implications.

$$(c, \hat{v}) \in W_\alpha \implies \hat{v} \in \hat{W}_\alpha^c, \text{ for } \alpha \in \{0, 1\}$$

and

$$(c, \hat{v}) \notin W_\alpha \implies \hat{v} \notin \hat{W}_\alpha^c, \text{ for } \alpha \in \{0, 1\}$$

Since the winning sets partition the game we have $\hat{v} \notin \hat{W}_\alpha^c \implies \hat{v} \in \hat{W}_\alpha^c$ (similar for set W). Therefore it is sufficient to prove only the first implication.

Let $(c, \hat{v}) \in W_\alpha$, player α has a strategy to win game \hat{G}_\downarrow from vertex (c, \hat{v}) . Since \hat{G}_\downarrow is the union of all the projections of \hat{G} we can apply the same strategy to game $\hat{G}_{|c}$ to win vertex \hat{v} as player α . Because we can win \hat{v} in the projection of \hat{G} to c we have $\hat{v} \in \hat{W}_\alpha^c$. \square

9.1.1 Projections and totality

A unified PG can be projected back to one of the games from which it is the union.

Definition 9.2. *The projection of unified PG $G = (V, V_0, V_1, E, \Omega)$ to configuration c , denoted as $G_{|c}$, is the parity game $(V', V_0, V_1, E', \Omega)$ such that $V' = \{v \mid (c, v) \in V\}$ and $E' = \{(v, w) \mid ((c, v), (c, w)) \in E\}$.*

One of the properties of a PG is its totality; a game is total if every vertex has at least 1 outgoing vertex. VPGs are also total, meaning that every vertex has, for every configuration $c \in \mathfrak{C}$, at least 1 outgoing vertex admitting c . Because VPGs are total their unifications are also total. Since edges in a unified PG don't cross configurations we can conclude that a unified PG is total iff every projection is total.

9.2 Solving unified PGs

We can solve a unified PG using the Zielonka's recursive algorithm. The algorithm revolves around the attractor operation and the creation of subgames. Consider the example presented in figure 12. Vertices with the highest priority are

$$\{(c_1, v_0), (c_2, v_0)\}$$

attracting these for player 0 gives the set

$$\begin{aligned} &\{(c_1, v_0), (c_2, v_0), \\ &\quad (c_1, v_1), (c_2, v_1), \\ &\quad (c_2, v_2)\} \end{aligned}$$

The algorithm tries to attract all vertices starting from v_0 with any configuration, in this case v_1 can be attracted for all configurations, v_2 can only be attracted for c_2 . When edge guards are permissive, ie. they admit many configurations, then there is a good chance that vertices can be attracted for multiple configurations at the same time during the algorithm. This is the idea for the collective recursive VPG algorithm we represent next, we will use the recursive algorithm to solve unified parity games such that it tries to attract multiple configurations simultaneously.

9.3 Representing unified parity games

In order to create an algorithm based around attracting multiple configurations we need to take a look at how unified parity games can be represented.

Unified PGs have a specific structure because they are the union of PGs that have the same vertices with the same owner and priority. Because they have the same priority we don't actually need to create a new function that is the unification of all the projections, we can simply use the original priority assignment function because the following relation holds:

$$\Omega(c, \hat{v}) = \hat{\Omega}(\hat{v})$$

Similarly we can use the original partition sets \hat{V}_0 and \hat{V}_1 instead of having the new partition V_0 and V_1 because the following relations holds:

$$\begin{aligned}(c, \hat{v}) \in V_0 &\iff \hat{v} \in \hat{V}_0 \\ (c, \hat{v}) \in V_1 &\iff \hat{v} \in \hat{V}_1\end{aligned}$$

So instead of considering unified PG (V, V_0, V_1, E, Ω) we will consider $(V, \hat{V}_0, \hat{V}_1, E, \hat{\Omega})$.

Next we consider how we represent vertices and edges in a unified PG. A set $X \subseteq (\mathfrak{C} \times \hat{V})$ can be represented as a complete function $f : \hat{V} \rightarrow 2^{\mathfrak{C}}$. The set X and function f are equivalent, denoted by the operator $=_{\lambda}$, iff the following relation holds:

$$(c, \hat{v}) \in X \iff c \in f(\hat{v})$$

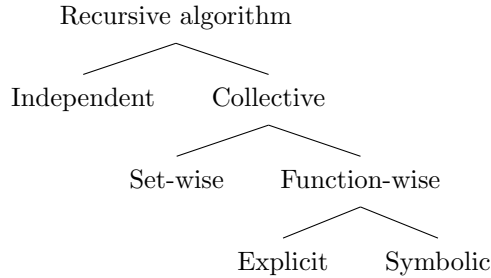
We can also represent edges as a complete function $f : \hat{E} \rightarrow 2^{\mathfrak{C}}$. The set E and function f are equivalent, denoted by the operator $=_{\lambda}$, iff the following relation holds:

$$((c, \hat{v}), (c, \hat{v}')) \in E \iff c \in f(\hat{v}, \hat{v}')$$

We write λ^{\emptyset} to denote the function that maps every element to \emptyset , clearly $\lambda^{\emptyset} =_{\lambda} \emptyset$. We call using a set of pairs to represent vertices and edges a *set-wise* representation and using functions a *function-wise* representation.

9.4 Algorithms

Using the recursive algorithm as a basis we can solve a VPG independently, alternatively we can solve it collectively using a set-wise presentation or a function-wise presentation. For the function-wise presentation we can either use an explicit or a symbolic representation of sets of configurations. The following diagram shows the different algorithms:



9.5 Recursive algorithm using a function-wise representation

We can modify the recursive algorithm to work with the function-wise representation of vertices and edges. Pseudo code for the modified algorithm is presented in algorithm 4.

We introduce a modified attractor definition to work with the function-wise representation.

Definition 9.3. Given unified PG $G = (V, \hat{V}_0, \hat{V}_1, E, \hat{\Omega})$ and a non-empty set $U \subseteq V$, both represented function-wise, we define $\alpha\text{-FAttr}(G, U)$ such that

$$U_0 = U$$

For $i \geq 0$:

$$U_{i+1}(\hat{v}) = U_i(\hat{v}) \cup \begin{cases} V(\hat{v}) \cap \bigcup_{\hat{v}'} (E(\hat{v}, \hat{v}') \cap U_i(\hat{v}')) & \text{if } \hat{v} \in \hat{V}_{\alpha} \\ V(\hat{v}) \cap \bigcap_{\hat{v}'} ((\mathfrak{C} \setminus E(\hat{v}, \hat{v}')) \cup U_i(\hat{v}')) & \text{if } \hat{v} \in \hat{V}_{\bar{\alpha}} \end{cases}$$

Finally:

$$\alpha\text{-FAttr}(G, U) = \bigcup_{i \geq 0} U_i$$

We will now prove that the function-wise attractor definition gives a result equal to the original definition.

Algorithm 4 RECURSIVEUVPG($PG\ G = ($

$V : \hat{V} \rightarrow 2^{\mathcal{C}},$

$\hat{V}_0 \subseteq \hat{V},$

$\hat{V}_1 \subseteq \hat{V},$

$E : \hat{E} \rightarrow 2^{\mathcal{C}},$

$\hat{\Omega} : \hat{V} \rightarrow \mathbb{N}$

```

1:  $m \leftarrow \min\{\hat{\Omega}(\hat{v}) \mid V(\hat{v}) \neq \emptyset\}$ 
2:  $h \leftarrow \max\{\hat{\Omega}(\hat{v}) \mid V(\hat{v}) \neq \emptyset\}$ 
3: if  $h = m$  or  $V = \lambda^\emptyset$  then
4:   if  $h$  is even or  $V = \lambda^\emptyset$  then
5:     return  $(V, \lambda^\emptyset)$ 
6:   else
7:     return  $(\lambda^\emptyset, V)$ 
8:   end if
9: end if
10:  $\alpha \leftarrow 0$  if  $h$  is even and 1 otherwise
11:  $U \leftarrow \lambda^\emptyset, U(\hat{v}) \leftarrow V(\hat{v})$  for all  $\hat{v}$  with  $\hat{\Omega}(\hat{v}) = h$ 
12:  $A \leftarrow \alpha\text{-FAttr}(G, U)$ 
13:  $(W'_0, W'_1) \leftarrow \text{RECURSIVEUVPG}(G \setminus A)$ 
14: if  $W'_{\bar{\alpha}} = \lambda^\emptyset$  then
15:    $W_\alpha \leftarrow A \cup W'_\alpha$ 
16:    $W_{\bar{\alpha}} \leftarrow \lambda^\emptyset$ 
17: else
18:    $B \leftarrow \bar{\alpha}\text{-FAttr}(G, W'_{\bar{\alpha}})$ 
19:    $(W''_0, W''_1) \leftarrow \text{RECURSIVEUVPG}(G \setminus B)$ 
20:    $W_\alpha \leftarrow W''_\alpha$ 
21:    $W_{\bar{\alpha}} \leftarrow W''_{\bar{\alpha}} \cup B$ 
22: end if
23: return  $(W_0, W_1)$ 

```

Lemma 9.2. *Given unified PG $G = (\mathcal{V}, \hat{V}_0, \hat{V}_1, \mathcal{E}, \hat{\Omega})$ and set $\mathcal{U} \subseteq \mathcal{V}$ the function-wise attractor α -FAttr(G, \mathcal{U}) is equivalent to the set-wise attractor α -Attr(G, \mathcal{U}) for any $\alpha \in \{0, 1\}$.*

Proof. Let V, E, U be the set-wise representation and $V^\lambda, E^\lambda, U^\lambda$ be the function-wise representation of $\mathcal{V}, \mathcal{E}, \mathcal{U}$ respectively.

The following properties hold by definition:

$$\begin{aligned} (c, \hat{v}) \in V &\iff c \in V^\lambda(\hat{v}) \\ (c, \hat{v}) \in U &\iff c \in U^\lambda(\hat{v}) \\ ((c, \hat{v}), (c, \hat{v}')) \in E &\iff c \in E^\lambda(\hat{v}, \hat{v}') \end{aligned}$$

Since the attractors are inductively defined and $U_0 =_\lambda U_0^\lambda$ (because $U =_\lambda U^\lambda$) we have to prove that for some $i \geq 0$, with $U_i =_\lambda U_i^\lambda$, we have $U_{i+1} =_\lambda U_{i+1}^\lambda$, which holds iff:

$$(c, \hat{v}) \in U_{i+1} \iff c \in U_{i+1}^\lambda(\hat{v})$$

Let $(c, \hat{v}) \in V$ (and therefore $c \in V^\lambda(\hat{v})$), we consider 4 cases.

- Case: $\hat{v} \in \hat{V}_\alpha$ and $(c, \hat{v}) \in U_{i+1}$:

To prove: $c \in U_{i+1}^\lambda(\hat{v})$.

If $(c, \hat{v}) \in U_i$ then $c \in U_i^\lambda(\hat{v})$ and therefore $c \in U_{i+1}^\lambda(\hat{v})$. If $(c, \hat{v}) \notin U_i$ then we have $c \notin U_i^\lambda(\hat{v})$.

Because $\hat{v} \in \hat{V}_\alpha$ and $c \in V^\lambda(\hat{v})$ we get

$$U_{i+1}^\lambda = \bigcup_{\hat{v}'} (E^\lambda(\hat{v}, \hat{v}') \cap U_i^\lambda(\hat{v}'))$$

There exists an $(c', \hat{v}') \in V$ such that $(c', \hat{v}') \in U_i$ and $((c, \hat{v}), (c', \hat{v}')) \in E$. Because edges don't cross configurations we can conclude that $c' = c$. Due to equivalence we have $c \in U_i^\lambda(\hat{v}')$ and $c \in E^\lambda(\hat{v}, \hat{v}')$. If we fill this in in the above formula we can conclude that $c \in U_{i+1}^\lambda(\hat{v})$.

- Case: $\hat{v} \in \hat{V}_\alpha$ and $(c, \hat{v}) \notin U_{i+1}$:

To prove: $c \notin U_{i+1}^\lambda(\hat{v})$.

First we observe that since $(c, \hat{v}) \notin U_{i+1}$ we get $(c, \hat{v}) \notin U_i$ and therefore $c \notin U_i^\lambda(\hat{v})$.

Because $\hat{v} \in \hat{V}_\alpha$ and $c \in V^\lambda(\hat{v})$ we get

$$U_{i+1}^\lambda = \bigcup_{\hat{v}'} (E^\lambda(\hat{v}, \hat{v}') \cap U_i^\lambda(\hat{v}'))$$

Assume $c \in U_{i+1}^\lambda(\hat{v})$. There must exist a \hat{v}' such that $c \in E^\lambda(\hat{v}, \hat{v}')$ and $c \in U_i^\lambda(\hat{v}')$. Due to equivalence we have a vertex $((c, \hat{v}), (c, \hat{v}')) \in E$ and $(c, \hat{v}') \in U_i$. In which case (c, \hat{v}) would be attracted and would be in U_{i+1} which is a contradiction.

- Case: $\hat{v} \in \hat{V}_{\bar{\alpha}}$ and $(c, \hat{v}) \in U_{i+1}$:

To prove: $c \in U_{i+1}^\lambda(\hat{v})$.

If $(c, \hat{v}) \in U_i$ then $c \in U_i^\lambda(\hat{v})$ and therefore $c \in U_{i+1}^\lambda(\hat{v})$. If $(c, \hat{v}) \notin U_i$ then we have $c \notin U_i^\lambda(\hat{v})$.

Because $\hat{v} \in \hat{V}_{\bar{\alpha}}$ we get

$$U_{i+1}^\lambda = V^\lambda(\hat{v}) \cap \bigcap_{\hat{v}'} ((\mathcal{C} \setminus E^\lambda(\hat{v}, \hat{v}')) \cup U_i^\lambda(\hat{v}'))$$

Assume $c \notin U_{i+1}^\lambda(\hat{v})$. Because $c \in V^\lambda(\hat{v})$ there must exist an \hat{v}' such that

$$c \notin ((\mathcal{C} \setminus E^\lambda(\hat{v}, \hat{v}')) \text{ and } c \notin U_i^\lambda(\hat{v}'))$$

which is equal to

$$c \in E^\lambda(\hat{v}, \hat{v}') \text{ and } c \notin U_i^\lambda(\hat{v}')$$

By equivalence we have $((c, \hat{v}), (c, \hat{v}')) \in E$ and $(c, \hat{v}') \notin U_i$. Which means that (c, \hat{v}) will not be attracted and $(c, \hat{v}) \notin U_{i+1}$ which is a contradiction.

- Case: $\hat{v} \in \hat{V}_{\bar{\alpha}}$ and $(c, \hat{v}) \notin U_{i+1}$:
To prove: $c \notin U_{i+1}^\lambda(\hat{v})$.

First we observe that since $(c, \hat{v}) \notin U_{i+1}$ we get $(c, \hat{v}) \notin U_i$ and therefore $c \notin U_i^\lambda(\hat{v})$.

Because $\hat{v} \in \hat{V}_{\bar{\alpha}}$ we get

$$U_{i+1}^\lambda = V^\lambda(\hat{v}) \cap \bigcap_{\hat{v}'} ((\mathcal{C} \setminus E^\lambda(\hat{v}, \hat{v}')) \cup U_i^\lambda(\hat{v}'))$$

Since (c, \hat{v}) is not attracted there must exist a $(c, \hat{v}') \in V$ such that

$$((c, \hat{v}), (c, \hat{v}')) \in E \text{ and } (c, \hat{v}') \notin U_i$$

By equivalence we have

$$c \in E^\lambda(\hat{v}, \hat{v}') \text{ and } c \notin U_i^\lambda(\hat{v}')$$

Which is equal to

$$c \notin (\mathcal{C} \setminus E^\lambda(\hat{v}, \hat{v}')) \text{ and } c \notin U_i^\lambda(\hat{v}')$$

From which we conclude

$$c \notin ((\mathcal{C} \setminus E^\lambda(\hat{v}, \hat{v}')) \cup U_i^\lambda(\hat{v}'))$$

Therefore we have $c \notin U_{i+1}^\lambda(\hat{v})$.

□

We also introduce a modified subgame definition to work with the function-wise representation.

Definition 9.4. For unified PG $G = (V, \hat{V}_0, \hat{V}_1, E, \hat{\Omega})$, represented function-wise, and set $X \subseteq V$ we define the subgame $G \setminus X = (V', \hat{V}_0, \hat{V}_1, E', \hat{\Omega})$ such that:

- $V'(\hat{v}) = V(\hat{v}) \setminus X(\hat{v})$
- $E'(\hat{v}, \hat{v}') = E(\hat{v}, \hat{v}') \cap V'(\hat{v}) \cap V'(\hat{v}')$

We will now prove that this new subgame definition gives a result equal to the original subgame definition. Note that when using the original subgame definition for unified PGs we can omit the modification to the partition because, as we have seen, we can use the partitioning from the VPG in the representation of unified PGs.

Lemma 9.3. Given unified PG $G = (\mathcal{V}, \hat{V}_0, \hat{V}_1, \mathcal{E}, \hat{\Omega})$ and set $\mathcal{U} \subseteq \mathcal{V}$ the subgame $G \setminus \mathcal{U} = (\mathcal{V}', \hat{V}_0, \hat{V}_1, \mathcal{E}', \hat{\Omega})$ represented set-wise is equal to the subgame represented function-wise.

Proof. Let V, V', E, E', U be the set-wise and $V^\lambda, V^{\lambda'}, E^\lambda, E^{\lambda'}, U^\lambda$ the function-wise representations of $\mathcal{V}, \mathcal{V}', \mathcal{E}, \mathcal{E}', \mathcal{U}$ respectively. We know $V =_\lambda V^\lambda$, $E =_\lambda E^\lambda$ and $U =_\lambda U^\lambda$. To prove: $V' =_\lambda V^{\lambda'}$ and $E' =_\lambda E^{\lambda'}$.

Let $(c, \hat{v}) \in V$.

If $(c, \hat{v}) \in U$ then $c \in U^\lambda(\hat{v})$, also $(c, \hat{v}) \notin V'$ (by definition 8.2) and $c \notin V^{\lambda'}(\hat{v})$ (by definition 9.4).

If $(c, \hat{v}) \notin U$ then $c \notin U^\lambda(\hat{v})$, also $(c, \hat{v}) \in V'$ (by definition 8.2) and $c \in V^{\lambda'}(\hat{v})$ (by definition 9.4).

Let $((c, \hat{v}), (c, \hat{w})) \in E$.

If $(c, \hat{v}) \in U$ then $(c, \hat{v}) \notin V'$ and $c \notin V^{\lambda'}(\hat{v})$ (as shown above). We get $((c, \hat{v}), (c, \hat{w})) \notin V' \times V'$ so $((c, \hat{v}), (c, \hat{w})) \notin E'$ (by definition 8.2). Also $c \notin E^{\lambda'}(\hat{v}, \hat{w})$ (by definition 9.4).

If $(c, \hat{w}) \in U$ then we apply the same logic.

If neither is in U then both are in V' and in $V' \times V'$ and therefore the $((c, \hat{v}), (c, \hat{w})) \in E'$. Also we get $c \in V^{\lambda'}(\hat{v})$ and $c \in V^{\lambda'}(\hat{w})$ so we get $c \in E^{\lambda'}(\hat{v}, \hat{w})$ (by definition 9.4). □

Next we prove the correctness of the algorithm by showing that the winning sets of the function-wise algorithm are equal to the winning sets of the set-wise algorithm.

Theorem 9.4. *Given unified PG $\mathcal{G} = (V, \hat{V}_0, \hat{V}_1, \mathcal{E}, \hat{\Omega})$ the winning sets resulting from $\text{RECURSIVEUVPG}(\mathcal{G})$ ran over the function-wise representation of \mathcal{G} is equal to the winning sets resulting from $\text{RECURSIVEPG}(\mathcal{G})$ ran over the set-wise representation of \mathcal{G} .*

Proof. Let $G = (V, \hat{V}_0, \hat{V}_1, E, \hat{\Omega})$ be the set-wise representation of \mathcal{G} and $G^\lambda = (V^\lambda, \hat{V}_0, \hat{V}_1, E^\lambda, \hat{\Omega})$ be the function-wise representation of \mathcal{G} .

Proof by induction on \mathcal{G} .

Base When there are no vertices or only one priority $\text{RECURSIVEUVPG}(G^\lambda)$ returns λ^\emptyset and $\text{RECURSIVEPG}(G)$ returns \emptyset , these two results are equal therefore the theorem holds in this case.

Step Player α gets the same value in both algorithms since the highest priority is equal for both algorithms.

Let $U = \{(c, \hat{v}) \in V \mid \hat{\Omega}(\hat{v}) = h\}$ (as calculated by RECURSIVEPG) and $U^\lambda(\hat{v}) = V^\lambda(\hat{v})$ for all \hat{v} with $\hat{\Omega}(\hat{v}) = h$ (as calculated by RECURSIVEUVPG). We will show that $U =_\lambda U^\lambda$.

Let $(c, \hat{v}) \in U$ then $\hat{\Omega}(\hat{v}) = h$, therefore $U^\lambda(\hat{v}) = V^\lambda(\hat{v})$. Since $U \subseteq V$ we have $(c, \hat{v}) \in V$ and because the equality between V and V^λ we get $c \in V^\lambda(\hat{v})$ and $c \in U^\lambda(\hat{v})$.

Let $c \in U^\lambda(\hat{v})$, since $U^\lambda(\hat{v})$ is not empty we have $\hat{\Omega}(\hat{v}) = h$, furthermore $c \in V^\lambda(\hat{v})$ and therefore $(c, \hat{v}) \in V$. We can conclude that $(c, \hat{v}) \in U$ and $U =_\lambda U^\lambda$.

For the rest of the algorithm it is sufficient to see that attractor sets are equal if the game and input set are equal (as shown in lemma 9.2) and that the created subgames are equal (as shown in lemma 9.3). Since the subgames are equal we can apply the theorem on it by induction and conclude that the winning sets are also equal. \square

We have seen in theorem 9.1 that solving a unified PG solves the VPG, furthermore the algorithm RECURSIVEUVPG correctly solves a unified PG therefore we can now conclude that for VPG \hat{G} vertex \hat{v} is won by player α for configuration c iff $c \in W_\alpha(\hat{v})$ with $(W_0, W_1) = \text{RECURSIVEUVPG}(G_\downarrow)$.

9.5.1 Function-wise attractor set

Next we present an algorithm to calculate the function-wise attractor, the pseudo code is presented in algorithm 5. The algorithm considers vertices that are in the attracted set for some configuration, for every such vertex the algorithm tries to attract vertices that are connected by an incoming edge. If a vertex is attracted for some configuration then the incoming edges of that vertex will also be considered. We prove the correctness of the algorithm in the following lemma and theorem.

Algorithm 5 $\alpha\text{-FATTRACTOR}(G, A : \hat{V} \rightarrow 2^{\mathcal{C}})$

```

1: Queue  $Q \leftarrow \{\hat{v} \in \hat{V} \mid A(\hat{v}) \neq \emptyset\}$ 
2: while  $Q$  is not empty do
3:    $\hat{v}' \leftarrow Q.pop()$ 
4:   for  $E(\hat{v}, \hat{v}') \neq \emptyset$  do
5:     if  $\hat{v} \in \hat{V}_\alpha$  then
6:        $a \leftarrow V(\hat{v}) \cap E(\hat{v}, \hat{v}') \cap A(\hat{v}')$ 
7:     else
8:        $a \leftarrow V(\hat{v})$ 
9:       for  $E(\hat{v}, \hat{v}'') \neq \emptyset$  do
10:         $a \leftarrow a \cap (\mathcal{C} \setminus E(\hat{v}, \hat{v}'') \cup A(\hat{v}''))$ 
11:      end for
12:    end if
13:    if  $a \setminus A(\hat{v}) \neq \emptyset$  then
14:       $A(\hat{v}) \leftarrow A(\hat{v}) \cup a$ 
15:       $Q.push(\hat{v})$ 
16:    end if
17:  end for
18: end while
19: return  $A$ 

```

Lemma 9.5. *Vertex \hat{v} and configuration c , with $c \in V(\hat{v})$, can only be attracted if there is a vertex \hat{v}' such that $c \in E(\hat{v}, \hat{v}')$ and $c \in U_i(\hat{v}')$.*

Proof. We first observe that if $\hat{v} \in \hat{V}_\alpha$ then this property follows immediately from definition the function-wise attractor definition (9.3). If $\hat{v} \in \hat{V}_{\bar{\alpha}}$ we note that unified PGs are total and therefore all of their projections are also total. So vertex \hat{v} has at least one outgoing edge for c , we have \hat{w} such that $c \in E(\hat{v}, \hat{w})$. For \hat{v} with c to be attracted we must have $c \in U_i(\hat{w})$. \square

Theorem 9.6. *Set A calculated by α -FATTRACTOR(G, U) satisfies $A = \alpha$ -FAttr(G, U).*

Proof. We will prove two loop invariants over the while loop of the algorithm.

IV1: For every $\hat{w} \in \hat{V}$ and $c \in \mathfrak{C}$ with $c \in A(\hat{w})$ we have $c \in \alpha$ -FAttr(G, U)(\hat{w}).

IV2: For every $\hat{w} \in \hat{V}$ and $c \in \mathfrak{C}$ that can be attracted to A either $c \in A(\hat{w})$ or there exists a $\hat{w}' \in Q$ such that $c \in E(\hat{w}, \hat{w}')$.

Base: Before the loop starts we have $A = U$, therefore IV1 holds. Furthermore all the vertices that are in A for some c are also in Q so IV2 holds.

Step: Consider the beginning of an iteration and assume IV1 and IV2 hold. To prove: IV1 and IV2 hold at the end of the iteration.

Set A only contains vertices with configurations that are in α -FAttr(G, U). The set is only updated through lines 5-12 and 14 of the algorithm which reflects the exact definition of the attractor set therefore IV1 holds at the end of the iteration.

Consider $\hat{w} \in \hat{V}$ and $c \in \mathfrak{C}$, we distinguish three cases to prove IV2:

- \hat{w} with c can be attracted by the beginning of the iteration but not by the end.

This case can't happen because $A(\hat{w})$ only increases during the algorithm and the values for E and V are not changed throughout the algorithm.

- \hat{w} with c can't be attracted by the beginning of the iteration but can by the end.

For \hat{w} with c to be able to be attracted at the end of the iteration there must be some \hat{w}' with c such that during the iteration c was added to $A(\hat{w}')$ (lemma 9.5). Every \hat{w}' for which $A(\hat{w}')$ is updated is added to the queue (lines 13-16). Therefore we have $\hat{w}' \in Q$ with $c \in E(\hat{w}, \hat{w}')$ and IV1 holds.

- \hat{w} with c can be attracted by the beginning of the iteration and also by the end.

Since IV2 holds at the beginning of the iteration we have either $c \in A(\hat{w})$ or we have some $\hat{w}' \in Q$ such that $c \in E(\hat{w}, \hat{w}')$. In the former case IV2 holds trivially by the end of the iteration since $A(\hat{w})$ can only increase. For the latter case we distinguish two scenario's.

First we consider the scenario where vertex \hat{v}' that is considered during the iteration (line 3 of the algorithm) is \hat{w}' . There is a vertex $c \in E(\hat{w}, \hat{w}')$ by IV2. Therefore we can conclude that \hat{w} is considered in the for loop starting at line 4 and will be attracted in lines 5-12 and added to $A(\hat{w})$ in line 14. Therefore IV2 holds by the end of the iteration.

Next we consider the scenario where $\hat{v}' \neq \hat{w}'$. In this case by the end of the iteration \hat{w}' will still be in Q and IV2 holds.

Vertices are only added to the queue when something is added to A (if statement on line 13). This can only finitely often happen because $A(\hat{v})$ can never be larger than $V(\hat{v})$ so we can conclude that the while loop terminates after a finite number of iterations.

When the while loop terminates IV1 and IV2 hold so for every $\hat{w} \in \hat{V}$ and $c \in \mathfrak{C}$ that can be attracted to A we have $c \in A(\hat{w})$. Since we start with $A = U$ we can conclude the soundness of the algorithm. IV1 shows the completeness. \square

9.6 Running time

We will consider the running time for solving VPG $G = (V, V_0, V_1, E, \Omega, \mathfrak{C}, \theta)$ independently and collectively using the different types of representations. We will use n to denote the number of vertices, e the number of edges, c the number of configurations and d the number of distinct priorities.

The original algorithm runs in $O(e * n^d)$, if we run c parity games independently we get $O(c * e * n^d)$. We can also apply the original algorithm to a unified PG (represented set-wise) for a collective approach, in this case we get a parity game with $c * n$ vertices and $c * e$ edges. which gives a running time $O(c * e * (c * n)^d)$. However this running time can be improved by using the property that a unified PG consists of c disconnected graphs as we shown next.

We have introduced three types of collective algorithms: set-wise, function-wise with explicit configuration sets and function-wise with symbolic configuration sets. In all three algorithms the running time of the attractor set is dominant, so we need three things: analyse the running time of the base cases, analyse the running time of the attractor set and analyse the recursion.

Base cases. In the base cases the algorithm needs to do two things: find the highest and lowest priority and check if there are no more vertices in the game. For the set-wise variant we find the highest and lowest priorities by iterating all vertices, which takes $O(c * n)$. Checking if there are no more vertices is done in $O(1)$. For the function wise algorithms we can find the highest and lowest priority in $O(n)$ and checking if there are no vertices is also done in $O(n)$ since we have to check $V(\hat{v}) = \emptyset$ for every \hat{v} . Note that in a symbolic representation using BDDs we can check if a set is empty in $O(1)$ because the decision diagram contains a single node.

Attractor sets. For the set-wise collective approach we can use the attractor calculation from the original algorithm which has a time complexity of $O(e)$, so for a unified PG having $c * e$ edges we have $O(c * e)$.

The function-wise variants use a different attractor algorithm. First we consider the variant where sets of configurations are represented explicitly.

Consider algorithm 5. A vertex will be added to the queue when this vertex is attracted for some configuration, this can only happen $c * n$ times, once for every vertex configuration combination.

The first for loop considers all the incoming edges of a vertex. When we consider all vertices the for loop will have considered all edges, since we consider every vertex at most c times the for loop will run at most $c * e$ times in total.

The second for loop considers all outgoing edges of a vertex. The vertices that are considered are the vertices that have an edge going to the vertex being considered by the while loop. Since the while loop considers $c * n$ vertices the second for loop runs in total at most $c * n * e$ times. The loop itself performs set operations on the set of configurations which can be done in $O(c)$. This gives a total time complexity for the attractor set of $O(n * c^2 * e)$.

For the symbolic representation set operations can be done in $O(c^2)$ so we get a time complexity of $O(n * c^3 * e)$.

This gives the following time complexities

	Base	Attractor set
Set-wise	$O(c * n)$	$O(c * e)$
Function-wise explicit	$O(n)$	$O(n * c^2 * e)$
Function-wise symbolic	$O(n)$	$O(n * c^3 * e)$

Recursion. The three algorithms behave the same way with regards to their recursion, so we analyse the running time of the set-wise variant and can derive the time complexity of the others using the result.

The algorithm has two recursions, the first recursion lowers the number of distinct priorities by 1. The second recursion removes at least one edge, however the game is comprised of disjoint projections. We can use this fact use in the analyses. Consider unified PG G and A as specified by the algorithm. Now consider the projection of G to an arbitrary configuration q , $G|_q$. If $(G \setminus A)|_q$ contains a vertex that is won by player $\bar{\alpha}$ then this vertex is removed in the second recursion step. If there is no vertex won by player $\bar{\alpha}$ then the game is won in its entirety and the only vertices won by player $\bar{\alpha}$ are in different projections. We can conclude that for every configuration q the second recursion either removes a vertex or $(G \setminus A)|_q$ is entirely won by player α . Let \bar{n} denote be the maximum number of vertices that are won by player $\bar{\alpha}$ in game $(G \setminus A)|_q$. Since

every projection has at most n vertices the value for \bar{n} can be at most n . Furthermore since \bar{n} depends on A , which depends on the maximum priority, the value \bar{n} gets reset when the top priority is removed in the first recursion. We can now write down the recursion of the algorithm:

$$T(d, \bar{n}) \leq T(d-1, n) + T(d, \bar{n}-1) + O(c * e)$$

When $\bar{n} = 0$ we will get $W_{\bar{\alpha}} = \emptyset$ as a result of the first recursion. In such a case there will be only 1 recursion.

$$T(d, 0) \leq T(d-1, n) + O(c * e)$$

Finally we have the base case where there is 1 priority:

$$T(1, \bar{n}) \leq O(c * n)$$

Expanding the second recursion gives

$$\begin{aligned} T(d) &\leq (n+1)T(d-1) + (n+1)O(c * e) \\ T(1) &\leq O(c * n) \end{aligned}$$

We will now prove that $T(d) \leq (n+d)^d O(c * e)$ by induction on d .

Base $d = 1$: $T(1) \leq O(c * n) \leq O(c * e) \leq (n+1)^1 O(c * e)$

Step $d > 1$:

$$\begin{aligned} T(d) &\leq (n+1)T(d-1) + (n+1)O(c * e) \\ &\leq (n+1)(n+d-1)^{d-1} O(c * e) + (n+1)O(c * e) \end{aligned}$$

Since $n+1 \leq n+d-1$ we get:

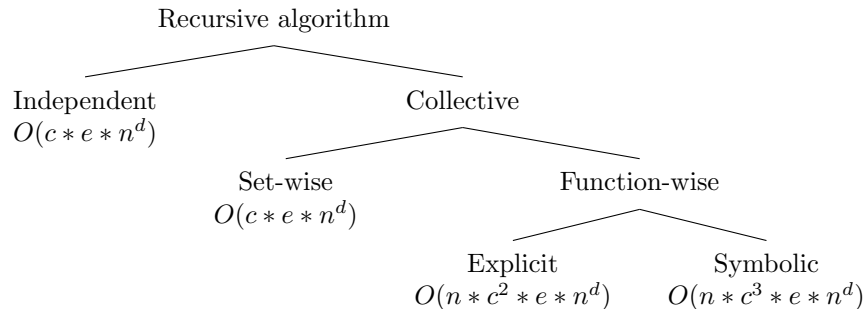
$$\begin{aligned} T(d) &\leq (n+d-1)(n+d-1)^{d-1} O(c * e) + (n+1)O(c * e) \\ &\leq (n+d-1)^d O(c * e) + (n+1)O(c * e) \\ &\leq ((n+d-1)^d + n+1)O(c * e) \end{aligned}$$

Using lemma A.1 we get

$$T(d) \leq (n+d)^d O(c * e)$$

This gives a time complexity of $O(c * e * (n+d)^d) = O(c * e * n^d)$. Note that the base time complexity is subsumed in the recursion by the time complexity of the attractor set. Since the time complexity of the attractor set is higher than the time complexity of the base cases for all three variants of algorithms we can simply fill in the attractor time complexity to get $O(n * c^2 * e * n^d)$ for the function-wise explicit algorithm and $O(n * c^3 * e * n^d)$ for the function-wise symbolic algorithm.

The different algorithms, including their time complexities, are repeated in the diagram below:



The function wise time complexities consist of three parts:

- the number of edges in the queue during the attractor calculation,
- the time complexity of set operations on subsets of \mathfrak{C} and
- the number of recursions.

The number of vertices in the queue during attracting is at most $c * n$, however this number will only be large if we attract a very small number of configurations at a per time we evaluate an edge. Most likely we will be able to attract many configurations at the same time, especially when the VPG originates from an FTS there is a good change that many edges admit most or all configurations in \mathfrak{C} . So when there are many similarities in behaviour between the different configurations in the FTS we will have a low number of vertices in the queue.

The time complexity of set operations is $O(c)$ when using an explicit representation and $O(c^2)$ when using a symbolic one. However, as shown in [9], a breadth-depth first implementation of BDDs keeps a table of already computed results. This allows us to get already calculated results in sublinear time. In total there are 2^c possible sets and therefore 2^{2c} possible set combinations and $O(2^c)$ possible set operations that can be computed. However when solving a VPG originating from an FTS there will most likely be a relatively small number of different edge guards, in which case the number of unique sets considered in the algorithm will be small and we can often retrieve a set calculation from the computed table.

We can see that even though the running time of the collective symbolic algorithm is the worse, its actual running time might be good when we are able to attract multiple configurations at the same time and have a small number of different edge guards.

10 Incremental pre-solve algorithm

Next we explore an algorithm where we try to solve the game for all configurations as much as possible, then split the configurations in two sets, create subgames using those two configuration sets and recursively repeat the process. Specifically we try to find vertices that are won by the same player for all configurations in \mathfrak{C} , if we have a vertex that is won by the same player for all configurations we say such a vertex is pre-solved. The algorithm tries to recursively the set of pre-solved vertices until all vertices are either pre-solved or only 1 configuration remains. Pseudo code is presented in algorithm 6. The algorithm is based around finding sets P_0 and P_1 , furthermore we want to find these sets in a clever such that the algorithm doesn't spend time finding vertices that are already pre-solved. Furthermore when there is only 1 configuration left we want an algorithm that solves the parity game $G|_c$ in an efficient manner by using the vertices that are pre-solved.

Algorithm 6 INCPRESOLVE($G = (V, V_0, V_1, E, \Omega, \mathfrak{C}, \theta), P_0, P_1$)

```

1: if  $|\mathfrak{C}| = 1$  then
2:    $\{c\} \leftarrow \mathfrak{C}$ 
3:    $(W'_0, W'_1) \leftarrow \text{solve } G|_c \text{ using } P_0 \text{ and } P_1$ 
4:   return  $(\mathfrak{C} \times W'_0, \mathfrak{C} \times W'_1)$ 
5: end if
6:  $P'_0 \leftarrow \text{find vertices won by player 0 for all configuration in } \mathfrak{C}$ 
7:  $P'_1 \leftarrow \text{find vertices won by player 1 for all configuration in } \mathfrak{C}$ 
8: if  $P'_0 \cup P'_1 = V$  then
9:   return  $(\mathfrak{C} \times P'_0, \mathfrak{C} \times P'_1)$ 
10: end if
11:  $\mathfrak{C}^a, \mathfrak{C}^b \leftarrow \text{partition } \mathfrak{C} \text{ in non-empty parts}$ 
12:  $(W_0^a, W_1^a) \leftarrow \text{INCPRESOLVE}(G \cap \mathfrak{C}^a, P'_0, P'_1)$ 
13:  $(W_0^b, W_1^b) \leftarrow \text{INCPRESOLVE}(G \cap \mathfrak{C}^b, P'_0, P'_1)$ 
14:  $W_0 \leftarrow W_0^a \cup W_0^b$ 
15:  $W_1 \leftarrow W_1^a \cup W_1^b$ 
16: return  $(W_0, W_1)$ 

```

The subgames created are based on creating a subgame based on a set of configurations, we define the subgame operator as follows:

Definition 10.1. Given VPG $G = (V, V_0, V_1, E, \Omega, \mathfrak{C}, \theta)$ and non-empty set $\mathfrak{X} \subseteq \mathfrak{C}$ we define the subgame $G \cap \mathfrak{X} = (V, V_0, V_1, E', \Omega, \mathfrak{C}', \theta')$ such that

- $\mathfrak{C}' = \mathfrak{C} \cap \mathfrak{X}$,
- $\theta'(e) = \theta(e) \cap \mathfrak{C}'$ and
- $E' = \{e \in E \mid \theta'(e) \neq \emptyset\}$.

VPGs are total, meaning that for every configuration and every vertex there is an outgoing edge from that vertex admitting that configuration. In subgames the set of configurations is restricted and only edge guards and edges are removed for configurations that fall outside the restricted set, therefore we still have totality.

Furthermore it is trivial to see that every projection $G|_c$ is equal to $(G \cap \mathfrak{X})|_c$ for any $c \in \mathfrak{C} \cap \mathfrak{X}$.

Finally the subgame operator is associative, meaning $(G \cap \mathfrak{X}) \cap \mathfrak{X}' = G \cap (\mathfrak{X} \cap \mathfrak{X}') = G \cap \mathfrak{X} \cap \mathfrak{X}'$.

10.1 Finding P_0 and P_1

We can find P_0 and P_1 by using *pessimistic* parity games; a pessimistic PG is a parity game created from a VPG for a player $\alpha \in \{0, 1\}$ such that the PG allows all edges that player $\bar{\alpha}$ might take but only allows edges for α when that edge admits all the configurations in \mathfrak{C} .

Definition 10.2. Given VPG $G = (V, V_0, V_1, E, \Omega, \mathfrak{C}, \theta)$, we can create pessimistic PG $G_{\triangleright\alpha}$ for player $\alpha \in \{0, 1\}$. We have

$$G_{\triangleright\alpha} = \{V, V_0, V_1, E', \Omega\}$$

such that

$$E' = \{(v, w) \in E \mid v \in V_{\bar{\alpha}} \vee \theta(v, w) = \mathfrak{C}\}$$

Note that pessimistic parity games are not necessarily total. A play in a PG that is not total might result in a finite path, in such a case the player that can't make a move loses the play.

When solving a pessimistic PG $G_{\triangleright\alpha}$ we get winning sets W_0, W_1 , every vertex in W_α is winning for player α in G played for any configuration, as shown in the following theorem.

Theorem 10.1. Given:

- VPG $G = (V, V_0, V_1, E, \Omega, \mathfrak{C}, \theta)$,
- configuration $c \in \mathfrak{C}$,
- winning sets W_0^c, W_1^c for game G ,
- player $\alpha \in \{0, 1\}$ and
- pessimistic PG $G_{\triangleright\alpha}$ with winning sets P_0 and P_1

we have $P_\alpha \subseteq W_\alpha^c$.

Proof. Player α has a strategy in game $G_{\triangleright\alpha}$ such that vertices in P_α are won. We will show that this strategy can also be applied to game $G|_c$ to win the same or more vertices.

First we observe that any edge that is taken by player α in game $G_{\triangleright\alpha}$ can also be taken in game $G|_c$ so player α can play the same strategy in game $G|_c$.

For player $\bar{\alpha}$ there are possibly edges that can be taken in $G_{\triangleright\alpha}$ but can't be taken in $G|_c$, in such a case player $\bar{\alpha}$'s choices are limited in game $G|_c$ compared to $G_{\triangleright\alpha}$ so if player $\bar{\alpha}$ can't win a vertex in $G_{\triangleright\alpha}$ then he/she can't win that vertex in $G|_c$.

We can conclude that applying the strategy from game $G_{\triangleright\alpha}$ in game $G|_c$ for player α wins the same or more vertices. \square

Figure 13 shows an example VPG with corresponding pessimistic games, after solving the pessimistic games we find $P_0 = \{v_2\}$ and $P_1 = \{v_0\}$.

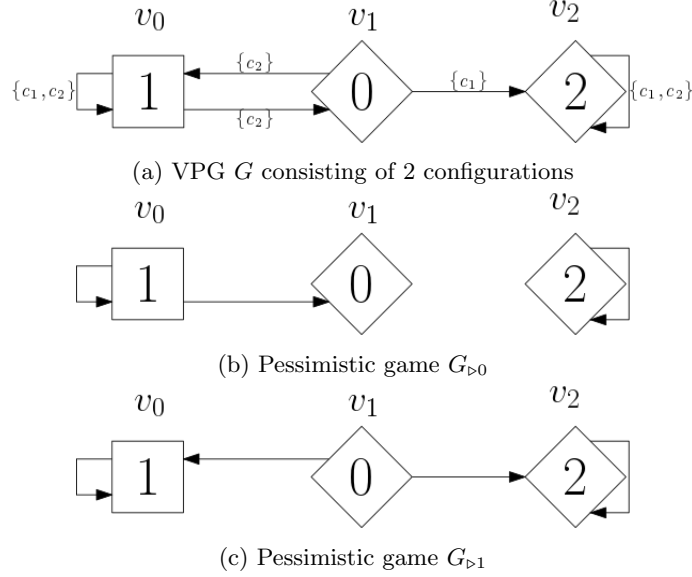


Figure 13: A VPG with its corresponding pessimistic games

10.1.1 Pessimistic subgames

Vertices in winning set P_α for $G_{>\alpha}$ are also winning for player α in pessimistic subgames of G , as shown in the following lemma.

Lemma 10.2. *Given:*

- VPG $G = (V, V_0, V_1, E, \Omega, \mathfrak{C}, \theta)$,
- P_0 being the winning set of game $G_{>0}$ for player 0,
- P_1 being the winning set of game $G_{>1}$ for player 1,
- non-empty set $\mathfrak{X} \subseteq \mathfrak{C}$,
- player $\alpha \in \{0, 1\}$ and
- winning sets Q_0, Q_1 for game $(G \cap \mathfrak{X})_{>\alpha}$

we have

$$P_0 \subseteq Q_0$$

$$P_1 \subseteq Q_1$$

Proof. Let edge (v, w) be an edge in game $G_{>\alpha}$ with $v \in V_\alpha$. Edge (v, w) admits all configuration in \mathfrak{C} so it also admits all configuration in $\mathfrak{C} \cap \mathfrak{X}$, therefore we can conclude that edge (v, w) is also an edge of game $(G \cap \mathfrak{X})_{>\alpha}$.

Let edge (v, w) be an edge in game $(G \cap \mathfrak{X})_{>\alpha}$ with $v \in V_\alpha$. The edge admits some configuration in $\mathfrak{C} \cap \mathfrak{X}$, this configuration is also in \mathfrak{C} so we can conclude that edge (v, w) is also an edge of game $G_{>\alpha}$.

We have concluded that game $(G \cap \mathfrak{X})_{>\alpha}$ has the same or more edges for player α as game $G_{>\alpha}$ and the same or less edges for player $\bar{\alpha}$. Therefore we can conclude that any vertex won by player α in $G_{>\alpha}$ is also won by α in game $(G \cap \mathfrak{X})_{>\alpha}$, ie. $P_\alpha \subseteq Q_\alpha$.

Let $v \in P_{\bar{\alpha}}$, using theorem 10.1 we find that v is winning for player $\bar{\alpha}$ in $G|_c$ for any $c \in \mathfrak{C}$. Because projections of subgames are the same as projections of the original game we can conclude that v is winning for player $\bar{\alpha}$ in $(G \cap \mathfrak{X})|_c$ for any $c \in \mathfrak{C} \cap \mathfrak{X}$.

Assume $v \notin Q_{\bar{\alpha}}$ then $v \in Q_\alpha$ and using theorem 10.1 we find that v is winning for player α in $(G \cap \mathfrak{X})|_c$ for any $c \in \mathfrak{C} \cap \mathfrak{X}$. This is a contradiction so we can conclude $v \in Q_{\bar{\alpha}}$ and therefore $P_{\bar{\alpha}} \subseteq Q_{\bar{\alpha}}$. \square

10.2 Algorithm

In order to find P_0 and P_1 we need to solve a pessimistic parity game, specifically we want to solve a parity game that uses the vertices that are already pre-solved to efficiently solve the parity game. Note that when there is 1 configuration left we need a similar algorithm. In algorithm 8 we present the INCPRESOLVE algorithm using pessimistic parity games. The algorithm uses a SOLVE algorithm for solving parity games using the pre-solved vertices. First we will show the correctness of the algorithm, later we will explore an appropriate SOLVE algorithm.

Algorithm 7 INCPRESOLVE($G = (V, V_0, V_1, E, \Omega, \mathfrak{C}, \theta), P_0, P_1$)

```

1: if  $|\mathfrak{C}| = 1$  then
2:    $\{c\} \leftarrow \mathfrak{C}$ 
3:    $(W'_0, W'_1) \leftarrow \text{SOLVE}(G|_c, P_0, P_1)$ 
4:   return  $(\mathfrak{C} \times W'_0, \mathfrak{C} \times W'_1)$ 
5: end if
6:  $(P'_0, -) \leftarrow \text{SOLVE}(G_{\triangleright 0}, P_0, P_1)$ 
7:  $(-, P'_1) \leftarrow \text{SOLVE}(G_{\triangleright 1}, P_0, P_1)$ 
8: if  $P'_0 \cup P'_1 = V$  then
9:   return  $(\mathfrak{C} \times P'_0, \mathfrak{C} \times P'_1)$ 
10: end if
11:  $\mathfrak{C}^a, \mathfrak{C}^b \leftarrow$  partition  $\mathfrak{C}$  in non-empty parts
12:  $(W_0^a, W_1^a) \leftarrow \text{INCPRESOLVE}(G \cap \mathfrak{C}^a, P'_0, P'_1)$ 
13:  $(W_0^b, W_1^b) \leftarrow \text{INCPRESOLVE}(G \cap \mathfrak{C}^b, P'_0, P'_1)$ 
14:  $W_0 \leftarrow W_0^a \cup W_0^b$ 
15:  $W_1 \leftarrow W_1^a \cup W_1^b$ 
16: return  $(W_0, W_1)$ 

```

A SOLVE algorithm must correctly solve a game as long as the sets P_0 and P_1 are in fact vertices that are won by player 0 and 1 respectively. We prove that this is the case in the INCPRESOLVE algorithm.

Theorem 10.3. *Given VPG \hat{G} . For every $\text{SOLVE}(G, P_0, P_1)$ that is invoked during $\text{INCPRESOLVE}(\hat{G}, \emptyset, \emptyset)$ we have winning sets W_0, W_1 for game G for which the following holds:*

$$P_0 \subseteq W_0$$

$$P_1 \subseteq W_1$$

Proof. When $P_0 = \emptyset$ and $P_1 = \emptyset$ the theorem holds trivially. So we will start the analyses after the first recursion.

After the first recursion the game is $\hat{G} \cap \mathfrak{X}$ with \mathfrak{X} being either \mathfrak{C}^a or \mathfrak{C}^b . The set P_0 is the winning set for player 0 for game $\hat{G}_{\triangleright 0}$ and the set P_1 is the winning set for player 1 for game $\hat{G}_{\triangleright 1}$. In the next recursion the game is $\hat{G} \cap \mathfrak{X} \cap \mathfrak{X}'$ with P_0 being the winning set for player 0 in game $(\hat{G} \cap \mathfrak{X})_{\triangleright 0}$ and P_1 being the winning set for player 1 in game $(\hat{G} \cap \mathfrak{X})_{\triangleright 1}$. In general, after the k th recursion, with $k > 0$, the game is of the form $(\hat{G} \cap \mathfrak{X}^0 \cap \dots \cap \mathfrak{X}^{k-1}) \cap \mathfrak{X}^k$. Furthermore P_0 is the winning set for player 0 for game $(\hat{G} \cap \mathfrak{X}^0 \cap \dots \cap \mathfrak{X}^{k-1})_{\triangleright 0}$ and P_1 is the winning set for player 1 for game $(\hat{G} \cap \mathfrak{X}^0 \cap \dots \cap \mathfrak{X}^{k-1})_{\triangleright 1}$.

Next we inspect the three places SOLVE is invoked:

1. Consider the case where there is only one configuration in \mathfrak{C} (line 1-5). Because P_0 is the winning set for player 0 for game $(\hat{G} \cap \mathfrak{X}^0 \cap \dots \cap \mathfrak{X}^{k-1})_{\triangleright 0}$ the vertices in P_0 are won by player 0 in game $G|_c$ for all $c \in \mathfrak{X}^0 \cap \dots \cap \mathfrak{X}^{k-1}$ (using theorem 10.1). This includes the one element in \mathfrak{C} . So we can conclude $P_0 \subseteq W_0$ where W_0 is the winning set for player 0 in game $G|_c$ where $\{c\} = \mathfrak{C}$.

Similarly for player 1 we can conclude $P_1 \subseteq W_1$ and the theorem holds in this case.

2. On line 6 the game $G_{\triangleright 0}$ is solved with P_0 and P_1 . Because $G = \hat{G} \cap \mathfrak{X}^0 \cap \dots \cap \mathfrak{X}^{k-1} \cap \mathfrak{X}^k$ and P_0 is the winning set for player 0 for game $(\hat{G} \cap \mathfrak{X}^0 \cap \dots \cap \mathfrak{X}^{k-1})_{\triangleright 0}$ and P_1 is the winning set for player 1 for game $(\hat{G} \cap \mathfrak{X}^0 \cap \dots \cap \mathfrak{X}^{k-1})_{\triangleright 1}$ we can apply lemma 10.2 to conclude that the theorem holds in this case.

3. On line 7 we apply the same reasoning and lemma to conclude that the theorem holds in this case. \square

Next we prove the correctness of the algorithm, assuming the correctness of the SOLVE algorithm.

Theorem 10.4. *Given VPG $\hat{G} = (\hat{V}, \hat{V}_0, \hat{V}_1, \hat{E}, \hat{\Omega}, \mathfrak{C}, \theta)$ and $(W_0, W_1) = \text{INCPRESOLVE}(\hat{G}, \emptyset, \emptyset)$. For every configuration $c \in \mathfrak{C}$ and winning sets \hat{W}_0^c, \hat{W}_1^c for game \hat{G} player for c it holds that:*

$$(c, v) \in W_0 \iff v \in \hat{W}_0^c$$

$$(c, v) \in W_1 \iff v \in \hat{W}_1^c$$

Proof. We will prove the theorem by applying induction on \mathfrak{C} .

Base $|\mathfrak{C}| = 1$, when there is only one configuration, being c , then the algorithm solves game $G|_c$. The product of the winning sets and $\{c\}$ is returned, so the theorem holds.

Step Consider P'_0 and P'_1 as calculated in the algorithm (line 6-7). By theorem 10.1 all vertices in P'_0 are won by player 0 in game $G|_c$ for any $c \in \mathfrak{C}$, similarly for P'_1 and player 1.

If $P'_0 \cup P'_1 = V$ then the algorithm returns $(\mathfrak{C} \times P'_0, \mathfrak{C} \times P'_1)$. In which case the theorem holds because there are no configuration vertex combinations that are not in either winning set and theorem 10.1 proves the correctness.

If $P'_0 \cup P'_1 \neq V$ then we have winning sets (W_0^a, W_1^a) for which the theorem holds (by induction) for game $G \cap \mathfrak{C}^a$ and (W_0^b, W_1^b) for which the theorem holds (by induction) for game $G \cap \mathfrak{C}^b$. The algorithm returns $(W_0^a \cup W_0^b, W_1^a \cup W_1^b)$. Since $\mathfrak{C}^a \cup \mathfrak{C}^b = \mathfrak{C}$ all vertex configuration combinations are in the winning sets and the correctness follows from induction. \square

10.3 A parity game algorithm using pre-solved vertices

We can use the fixed-point iteration algorithm to solve parity games using pre-solved vertices. First recall the fixed-point formula to calculate W_0 :

$$S(G = (V, V_0, V_1, E, \Omega)) = \nu Z_{d-1}. \mu Z_{d-2}. \dots. \nu Z_0. F_0(Z_{d-1}, \dots, Z_0)$$

Let G be a PG and let sets P_0 and P_1 be such that vertices in P_0 are won by player 0 and vertices in P_1 are won by player 1. We can fixed-point approximate $S(G)$ to calculate W_0 , we know that W_0 is bounded by P_0 and P_1 , specifically we have

$$P_0 \subseteq W_0 \subseteq V \setminus P_1$$

We can use this restriction to efficiently approximate $S(G)$. If no bounds are know we would approximate fixed-point formula $\nu Z_{d-1} \dots$ by starting at $Z_{d-1}^0 = V$ which is the largest value possible, however given the bounds we can start our approximations of greatest fixed-point variables at $V \setminus P_1$ and start our approximations of least fixed-point variables at P_0 . The following lemma's and theorems prove this.

Lemma 10.5. *Given*

- *A complete lattice $\langle 2^A, \subseteq \rangle$,*
- *monotonic function $f : 2^A \rightarrow 2^A$ and*
- *$R^\perp \subseteq A$ and $R^\top \subseteq A$ such that $R^\perp \subseteq \nu X.f(X) \subseteq R^\top$*

we approximate X by starting with $X^0 = R^\top$. For any $i \geq 0$ it holds that

$$R^\perp \subseteq f(X^i) \subseteq R^\top$$

Proof. Assume $R^\perp \supset f(X^i)$. By fixed-point approximation we have $\nu X.f(X) = \bigcap_{j \geq 0} X^j$, so we find $R^\perp \supset \nu X.f(x)$ which is a contradiction so $R^\perp \subseteq f(X^i)$.

Assume $f(X^i) \supset R^\top$. Because of monotonicity we find $X^i \subseteq R^\top$ and therefore $f(X^i) \supset R^\top \supseteq X^i$. Using the Knaster-Tarski theorem (7.1) we can conclude that the greatest fixed-point of $f(X)$ is larger than $f(X^i)$, so we find $\nu X.f(X) \supset R^\top$ which is a contradiction so $f(X^i) \subseteq R^\top$. \square

Lemma 10.6. *Given*

- *A complete lattice $\langle 2^A, \subseteq \rangle$,*
- *monotonic function $f : 2^A \rightarrow 2^A$ and*
- *$R^\perp \subseteq A$ and $R^\top \subseteq A$ such that $R^\perp \subseteq \mu X.f(X) \subseteq R^\top$*

we approximate X by starting with $X^0 = R^\perp$. For any $i \geq 0$ it holds that

$$R^\perp \subseteq f(X^i) \subseteq R^\top$$

Theorem 10.7. *Given PG $G = (V, V_0, V_1, E, \Omega)$ with P_0 and P_1 such that vertices in P_0 are won by player 0 in game G and vertices in P_1 are won by player 1 in game G we can approximate the fixed-point variables by starting at P_0 for least fixed-points and starting at $V \setminus P_1$ for greatest fixed-points.*

Proof. Let $f(Z_{d-1}) = \mu Z_{d-2} \dots \nu Z_0.F_0(Z_{d-1}, \dots, Z_0)$. Because $\nu Z_{d-1}.f(Z_{d-1})$ calculates W_0 we know $P_0 \subseteq \nu Z_{d-1}.f(Z_{d-1}) \subseteq V \setminus P_1$ so we can start the fixed-point approximation at $Z_{d-1}^0 = V \setminus P_1$. Using lemma 10.5 we find for any $i \geq 0$ we have $P_0 \subseteq f(Z_{d-1}^i) \subseteq V \setminus P_1$.

Let $g(Z_{d-2}) = \nu Z_{d-3} \dots \nu Z_0.F_0(Z_{d-1}^i, Z_{d-2}, \dots, Z_0)$. We found $P_0 \subseteq f(Z_{d-1}^i) \subseteq V \setminus P_1$, therefore we have $P_0 \subseteq \mu Z_{d-2}.g(Z_{d-2}) \subseteq V \setminus P_1$ so we can start the fixed-point approximation at $Z_{d-2}^0 = P_0$. Using lemma 10.6 we find that for any $j \geq 0$ we have $P_0 \subseteq g(Z_{d-2}^j) \subseteq V \setminus P_1$.

We can repeat this logic up until Z_0 to conclude that the theorem holds. \square

We take the fixed-point algorithm presented in [15] and modify it by starting at P_0 and $V \setminus P_1$. The pseudo code is presented in algorithm 8, its correctness follows from [15] and theorem 10.7.

Algorithm 8 Fixed-point iteration with P_0 and P_1

<pre> 1: function FPITER($G = (V, V_0, V_1, E, \Omega), P_0, P_1$) 2: for $i \leftarrow d-1, \dots, 0$ do 3: INIT(i) 4: end for 5: repeat 6: $Z'_0 \leftarrow Z_0$ 7: $Z_0 \leftarrow \text{DIAMOND}() \cup \text{BOX}()$ 8: $i \leftarrow 0$ 9: while $Z_i = Z'_i \wedge i < d-1$ do 10: $i \leftarrow i+1$ 11: $Z'_i \leftarrow Z_i$ 12: $Z_i \leftarrow Z_{i-1}$ 13: INIT($i-1$) 14: end while 15: until $i = d-1 \wedge Z_{d-1} = Z'_{d-1}$ 16: return ($Z_{d-1}, V \setminus Z_{d-1}$) 17: end function </pre>	<pre> 1: function INIT(i) 2: $Z_i \leftarrow P_0$ if i is odd, $V \setminus P_1$ otherwise 3: end function 1: function DIAMOND 2: return $\{v \in V_0 \mid \exists w \in V (v, w) \in E \wedge w \in Z_{\Omega(w)}\}$ 3: end function 1: function BOX 2: return $\{v \in V_1 \mid \forall w \in V (v, w) \in E \implies w \in Z_{\Omega(w)}\}$ 3: end function </pre>
---	---

This algorithm is appropriate to use as a SOLVE algorithm in INCPRESOLVE since it solves parity games that are not necessarily total and uses P_0 and P_1 .

10.4 Running time

We will consider the running time for solving VPG $G = (V, V_0, V_1, E, \Omega, \mathfrak{C}, \theta)$ independently and collectively. We will use n to denote the number of vertices, e the number of edges, c the number of configurations and d the number of distinct priorities.

The fixed-point iteration algorithm without P_0 and P_1 runs in $O(e * n^d)$ ([15]). We can use this algorithm to solve G independently, ie. solve all the projections of G . This gives a time complexity of $O(c * e * n^d)$.

Next we consider the INCPRESOLVE algorithm for a collective approach, observe that in the worst case we have to split the set of configurations all the way down to individual configurations. We can consider the recursion as a tree where the leafs are individual configurations and at every internal node the set of configurations is split in two. Since in the worst case there are c leaves, there are at most $c - 1$ internal nodes. At every internal node the algorithm solves two games and at every leaf the algorithm solves 1 game, so we get $c + 2c - 2 = O(c)$ games that are being solved by INCPRESOLVE. In the worst case there are no similarities between the configuration in G and at every iteration P_0 and P_1 are empty. In this case the FPITE algorithm behaves the same as the original algorithm and has a time complexity of $O(e * n^d)$, this gives an overall time complexity of $O(c * e * n^d)$ which is equal to an independent solving approach.

10.5 Local variant

When solving a VPG for model checking purposes we don't actually need to solve the entire game, due to the way the VPGs are build there is a certain vertex, say v_0 , such that iff player 0 wins it then the μ -calculus formula holds for the FTS. So finding out for which configurations v_0 is won by player 0 is sufficient, this is called *locally* solving a VPG. Solving the entire VPG is called *globally* solving the VPG. Similarly we can solve a PG locally, ie. only determining the winner for v_0 , or solve it globally, ie. solve all vertices.

The incremental pre-solve algorithm is particularly appropriate for local solving because if we find v_0 in either P_0 or P_1 we are done, this can potentially reduce the depth of the recursion of the algorithm and therefore reduce the number of (pessimistic) games we solve.

Furthermore when there is only one configuration the incremental pre-solve algorithm solves the corresponding parity game. When taking a local approach it is sufficient to solve this PG locally. Note that the pessimistic games still should be solved globally to find as much assistance as possible for further recursions.

We can modify the fixed-point approximation algorithm to solve a game locally by distinguishing two cases:

1. If $d - 1$ is even then the outermost fixed-point variable is a greatest fixed-point variable. When at some point $v_0 \notin Z_{d-1}$ then we know 0 is never won by player 1 and we are done.
2. If $d - 1$ is odd then the outermost fixed-point variable is a least fixed-point variable. When at some point $v_0 \in Z_{d-1}$ then we know 0 is won by player 0 and we are done.

We can also use the local fixed-point approximation algorithm for an independent solving approach.

Part III

Experimental evaluation

11 Introduction

The algorithms proposed to solve VPGs don't have a better worst case running time complexity, generally speaking this is the case because the projections of the VPG might have very little in common. However the aim of the algorithms is to solve VPGs effectively when there are commonalities. In order to evaluate the performance of the algorithms when there are commonalities the algorithms are implemented and ran on a number of VPG test sets. In this part we discuss the implementation and look at the results.

12 Implementation

The algorithms are implemented in C++ version 14 and use BuDDy¹ as a BDD library. The complete source is hosted on github².

The implementation is split in three phases: parsing, solving and solution printing. The solving part contains the implementation of the algorithm presented. The parsing and solution printing parts are implemented trivially and hardly optimized, their running times are not considered in the experimental evaluation.

The parsing phase of the algorithm creates BDDs from the input file, in doing so parts of the BDD cache gets filled. After parsing the BDD cache is cleared to make sure that the work done in the solving phase corresponds with the algorithms presented and no work to assist it has been done prior to this phase.

12.1 Game representation

The graph is represented by using adjacency lists for incoming and outgoing edges, furthermore every edge maps to a set of configurations representing the θ value for the edge (either explicitly or implicit). For product based algorithms the edges are not mapped to sets of configurations. Finally sets of vertices are represented using bit-vectors.

Note that only the representation of the games used during the algorithm is relevant, since we don't evaluate the parsing phase it is not relevant how we store the games in files.

12.2 Product based algorithms

Three product based algorithms are implemented, ie. algorithms that solve parity games:

- Zielonka's recursive algorithm.
- Fixed-point iteration algorithm, global variant.
- Fixed-point iteration algorithm, local variant.

A few optimizations are applied to the fixed-point iteration algorithm. The following three are described in [15]:

- For fixed-point variable Z_i its value is only ever used to check if a vertex with priority i is in Z_i . So instead of storing all vertices in Z_i we only have to store the vertices that have priority i . We can store all fixed-point variables in a single bit-vector, named Z , of size n .
- The algorithm only reinitializes a certain range of fixed-point variables. So the diamond and box operations can use the previous result and only reconsider vertices that have an edge to a vertex that has a priority for which its fixed-point variable is reset.

¹<https://sourceforge.net/projects/buddy/>

²<https://github.com/SjefvanLoo/VariabilityParityGames/tree/master/implementation/VPGSolver>

- The algorithm updates variables Z_0 to Z_m and reinitializes Z_0 to Z_{m-1} , however if Z_m is a least fixed-point variable then Z_m has just increased and due to monotonicity the other least fixed-point formula's, ie. Z_{m-2}, Z_{m-4}, \dots , will also increase so there is no need to reset them. Similarly for greatest fixed-point variables. So we only to reset half of the variables instead of all of them.

Finally the vertices in the game are reordered such that they are sorted by parity first and secondly by priority. Using the above optimizations the algorithm needs to reset variables Z_m, Z_{m-2}, \dots , these variables are stored in a single bit-vector Z . By reordering the variables to be sorted by parity and priority these vertices that need to be reset are always consecutively stored in Z , so resetting this sequence can be done by a memory copy instead of iterating all the different vertices. Note that when the algorithm is used by the pre-solve algorithm the variables are not reset to simply \emptyset and V but are reset to two specific bit-vectors that are given by the pre-solve algorithm. These bit-vectors have the same order and resetting can be done by copying a part of them into Z .

12.3 Family based algorithms

Four family based algorithms are implemented:

- Zielonka's recursive family based algorithm with explicit configuration set representation.
- Zielonka's recursive family based algorithm with symbolic configuration set representation.
- Incremental pre-solve algorithm, global variant.
- Incremental pre-solve algorithm, local variant.

The incremental pre-solve algorithms use the fixed-point iteration algorithm as described above to solve the (pessimistic) PGs.

12.4 Random verification

In order to prevent implementation mistakes 200 VPGs are randomly created, every VPG is projected to all its configurations to get a set of PGs. These PGs are solved by the PGSolver tool ([17]). All algorithms implemented are used to solve the 200 VPGs and the solutions are compared to the solutions created by the PGSolver.

13 Test cases

The algorithms are run on VPGs created from different test cases. We have two model checking problems as well as random VPGs. The model checking VPGs are created as described in part I, with the exception that only vertices are added when they are reachable from the initial vertex. So these games are never disjointed, random games can be disjointed.

13.1 Model checking games

We use two software product line examples, first the minepump example as described in [18] and implemented in the mCRL2 toolset as described in [2]. The implementation creates an LTS with parametrized actions, the parameters describe the boolean formula's guarding the transitions, effectively making it an FTS. This FTS is interpreted in combination with 9 different μ -calculus formula's to create 9 VPGs.

Next we have the elevator example, described in [19]. Again an mCRL2 implementation³ (created by T.A.C. Willemse) is used to create 7 VPGs.

³<https://github.com/SjefvanLoo/VariabilityParityGames/blob/master/implementation/Elevator.tar.gz>

13.2 Random games

We can create a random variability parity game by creating a random parity game and creating sets of configurations that guard the edges. For these sets we need to consider two factors: how large are the sets guarding the edges and how are they created.

The guard sets in the minepump and elevator games have a very specific distribution where of the sets admit either 100% or 50% of the configurations. An edge requiring the presence or absence of one specific feature results in a set admitting 50%, this explains the distribution. The average edge in the examples admits 92% of the configurations.

We use λ to denote the average relative size of guard sets in a VPG. We create a random game with a specific λ by using a probabilistic distribution ranging from 0 to 1 to determine the size of an individual guard set. Such a distribution must have a mean equal to λ . We consider two distributions:

- A modified Bernoulli distribution; in a Bernoulli distribution there is a probability of p to get an outcome of 1 and a probability of $1 - p$ to get an outcome of 0. We modify this such that there is a probability of p to get 1 and a probability of $1 - p$ to get 0.5. This gives a mean of $1p + 0.5(1 - p) = 0.5p + 0.5$. So to get a mean of λ we choose $p = 2\lambda - 1$. Note that we can't use this distribution when $\lambda < 0.5$ because p becomes less than 0.
- A beta distribution; a beta distribution ranges from 0 to 1 and is curved such that it has a specific mean. The beta distribution has two parameters: α and β and a mean of $\frac{\alpha}{\alpha + \beta}$. We pick $\beta = 1$ and $\alpha = \frac{\lambda\beta}{1-\lambda}$ to get a mean of λ .

Figures 14, 15 and 16 show the shapes of the distribution for different values for λ .

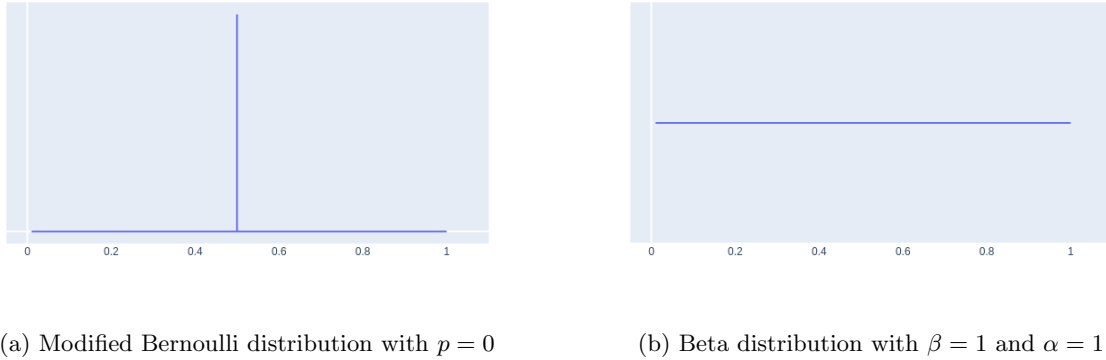
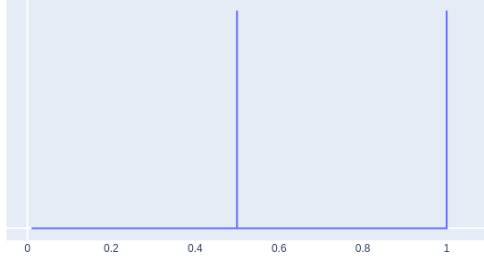
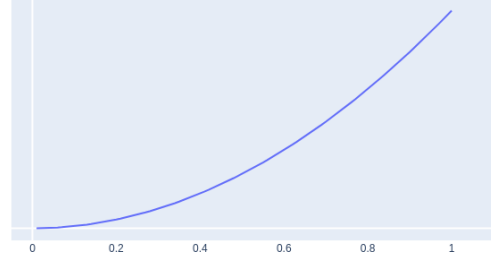


Figure 14: Edge guard size distribution for $\lambda = 0.5$

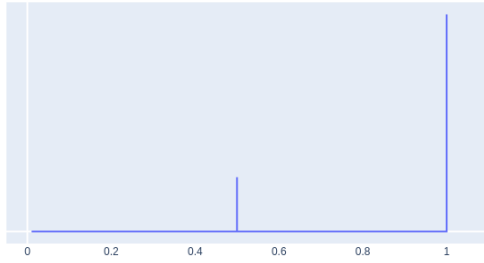


(a) Modified Bernoulli distribution with $p = 0.5$

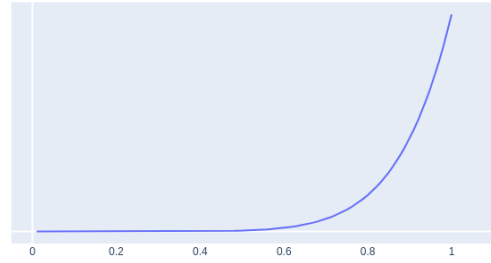


(b) Beta distribution with $\beta = 1$ and $\alpha = 3$

Figure 15: Edge guard size distribution for $\lambda = 0.75$



(a) Modified Bernoulli distribution with $p = 0.8$



(b) Beta distribution with $\beta = 1$ and $\alpha = 9$

Figure 16: Edge guard size distribution for $\lambda = 0.9$

Once we have determined the size for a configuration set we need to consider how the set is created. We can simply create a random set of configurations without any notion of features, we call this a *configuration based* approach. Alternatively we can use a *feature based* approach where we create sets by looking at features. Consider features f_0, \dots, f_m , we can create a boolean function that is the conjunction of k features where every feature in the conjunction has probability 0.5 of being negated. For example when using $k = 3$ and $m = 5$ we might get boolean formula $f_1 \wedge \neg f_2 \wedge \neg f_4$. Such a boolean formula corresponds to a set of configurations of size 2^{m-k} and a relative size $\frac{2^{m-k}}{2^m} = 2^{-k}$, so when creating a set with relative size r we choose $k = \min(m, \lfloor -\log_2 r \rfloor)$. When using a feature based approach we can only create sets that have a relative size of $\frac{1}{2^i}$ for some $i \in \mathbb{N}$.

We can create 4 types of games:

1. Bernoulli distributed and feature based. These games are most similar to the model verification games.
2. Bernoulli distributed and configuration based. These games do have the characteristics of a model verification game in terms of set size but have unstructured sets guarding the edges. Furthermore with a configuration based approach less guard sets will be identical than with a feature based approach.
3. Beta distributed and configuration based. These games are most different from the model verification games.

4. Beta distributed and feature based. Games created in this way don't have the average relative set size of λ because the feature based approach can only create sets of size $\frac{1}{2^i}$ for any $\lambda \geq \frac{1}{2}$ almost all the sets have either relative size $\frac{1}{2}$ or 1, so effectively this creates the same games as using the Bernoulli distribution only with an incorrect average relative set size. Therefore we will not consider this category of games.

For random games of type 1,2 and 3 we create 50 games: game 50 to game 99, where game i has $\lambda = \frac{i}{100}$ and a random number of features, nodes, edges and maximum priority.

Furthermore we create 48 games to evaluate how the algorithm scales when the number of features becomes larger. For every $i \in [2, 12]$ we create random games ia , ib , ic and id of type 1 with $\lambda = 0.92$, i features and a random number of nodes, edges and maximum priority.

14 Results

In this section the experimental results are presented. The following algorithms are used to solve the 6 categories of games:

1. Zielonka's recursive algorithm, product based
2. Fixed-point iteration, product based
3. Fixed-point iteration, local product based
4. Zielonka's recursive algorithm, family based with explicit configuration representation
5. Zielonka's recursive algorithm, family based with symbolic configuration representation
6. Incremental pre-solve algorithm
7. Incremental pre-solve algorithm, local

The numbers presented are the times it took to solve the games; times for parsing, projecting (for product based approaches) and printing solutions are excluded. So the product based results are the sum of the solve times of the projections, parsing and projecting are not included in the result.

The exact times can be found in appendix B, in this section the results are visualized and presented to be easily interpreted. In some cases the results in a graph are normalized meaning that the running times are divided by the running times of the first algorithm in the graph. Specifically for the random games the running times vary a lot so normalizing is required to properly visualize the results. The times are presented on a logarithmic scale, games that were unable to be completed (due to memory) are marked with a grey bar.

All the experiments are ran on a Linux x64 operating system with an Intel i5-4570 processor and 8GB of RAM.

14.1 Zielonka's family based

First we compare the running times of the Zielonka's family based approaches with the Zielonka's product based approach. First we look at the model verification games.

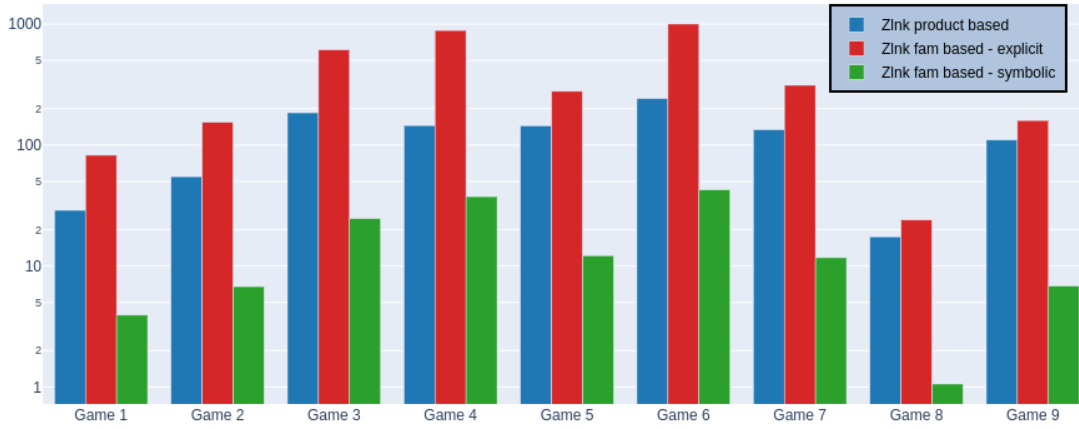


Figure 17: Running time of Zielonka's algorithms on the minepump problem

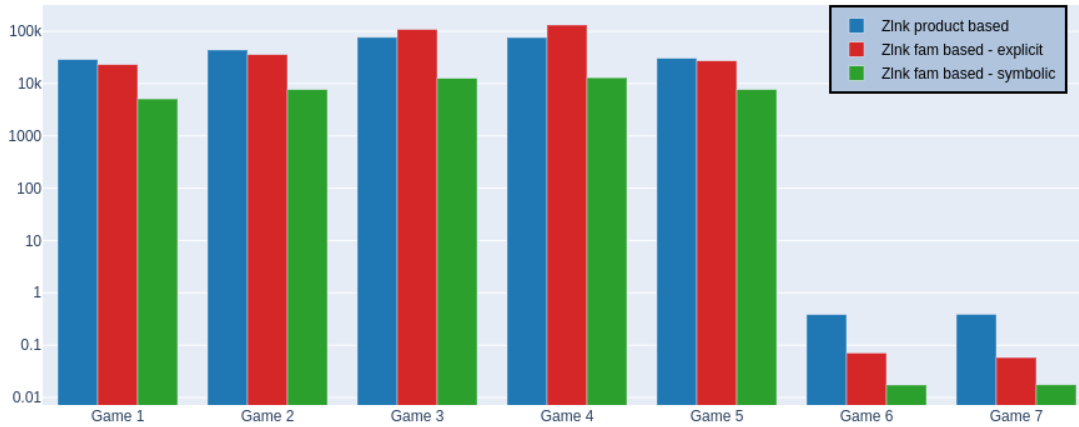


Figure 18: Running time of Zielonka's algorithms on the elevator problem

We can see that the performance of the explicit variant varies a lot between games. The symbolic variant greatly outperforms the product based approach for every problem.

Next we inspect the random games, first we look at the games with a variable λ and a random number of features. The graphs are normalized.

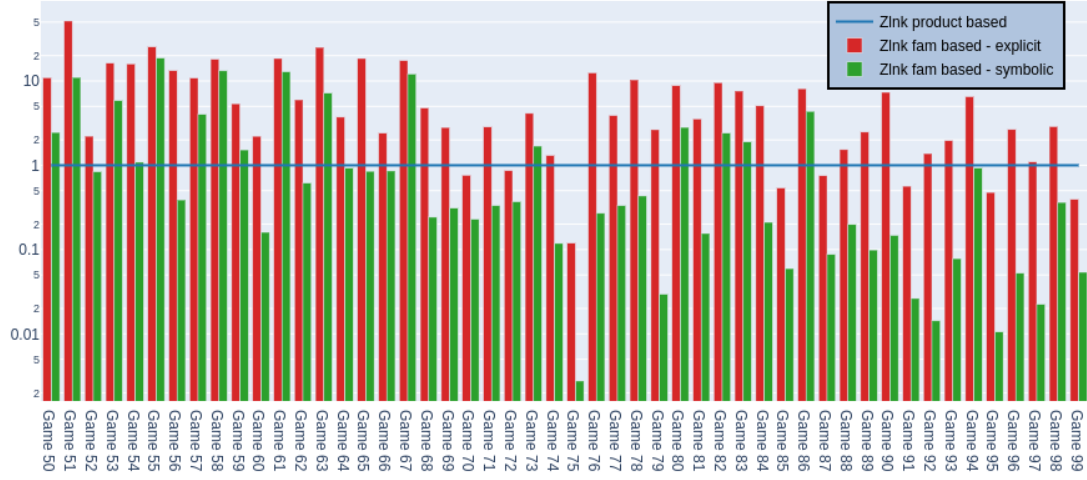


Figure 19: Running time of Zielonka's algorithms on randomgames of type 1 with $\lambda = \frac{\text{game nr}}{100}$, times are normalized

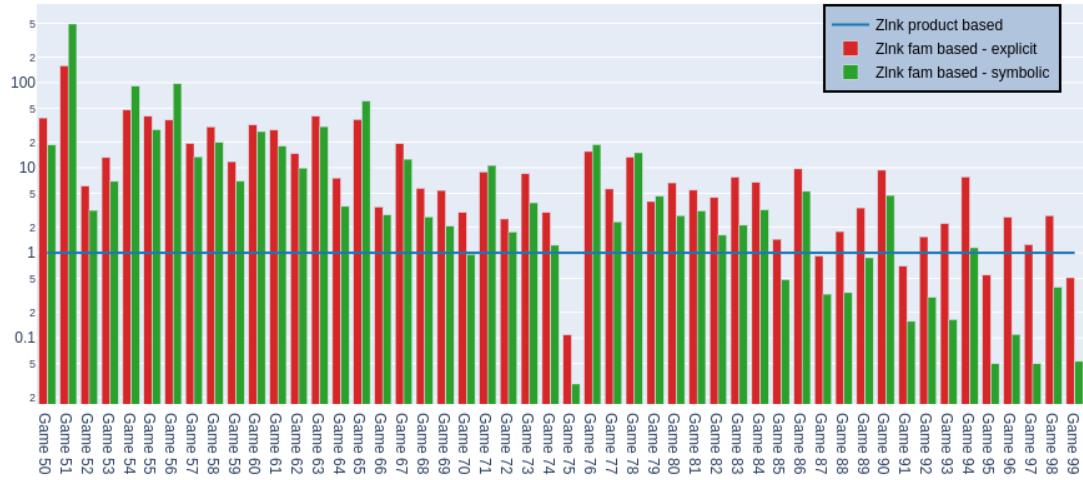


Figure 20: Running time of Zielonka's algorithms on randomgames of type 2 with $\lambda = \frac{\text{game nr}}{100}$, times are normalized



Figure 21: Running time of Zielonka’s algorithms on randomgames of type 3 with $\lambda = \frac{\text{game nr}}{100}$, times are normalized

For type 1 games we see that when λ gets bigger the family based symbolic approach starts winning from the product based approach. There are a few exceptions to this, games 80, 82 and 86, all three of these games have only 4 features. As we will see later the less features there are in a game the worse the family based approaches perform.

For type 2 the explicit variant performs very similar to the type 1 games, however the symbolic approach performs much worse. This is due to the unstructured nature of the configuration sets which negatively influences bdd performance but has no effect on the explicit set operations. We also see the explicit algorithm outperforming the symbolic algorithm in some cases.

For type 3 games the product based approach performs generally better than the family based approaches unless λ becomes very high.

Next we inspect how the algorithms scales in terms of number of features



Figure 22: Running time of Zielonka’s algorithms on randomgames of type 1 with $\lambda = 0.92$ and the number of features equal to the *game nr*, times are normalized

We can clearly conclude that as the number of features increases the family based symbolic approach performs better compared to the product based approach.

Overall we can conclude that the explicit algorithm performs somewhat arbitrary, however the symbolic algorithm performs really well for model checking problems and random games that have similar properties. Also we can conclude that the algorithms scales well in terms of number of features.

14.2 Incremental pre-solve algorithm

We compare the running times of the incremental pre-solve approaches with the fixed-point iteration product based algorithm. First we look at the model verification games.

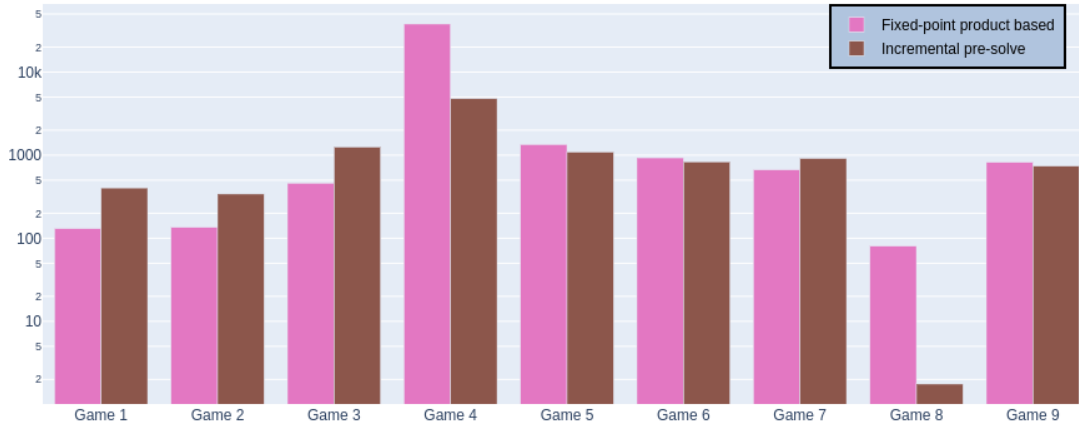


Figure 23: Running time of the incremental pre-solve algorithms on the minepump problem

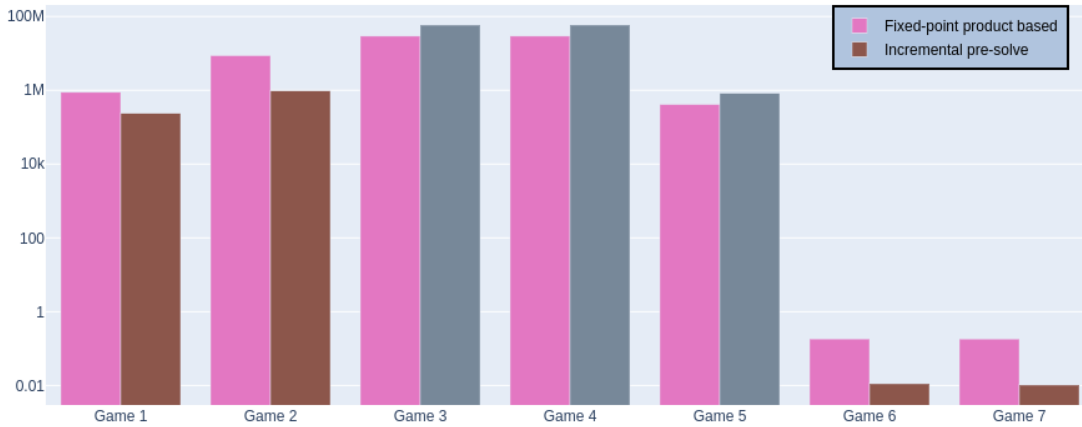


Figure 24: Running time of the incremental pre-solve algorithms on the elevator problem. Game 3, 4 and 5 were not able to be finished

The incremental pre-solve algorithms doesn't show any significant improvements over the product based fixed-point algorithm.

Next we inspect the random games, first we look at the games with a variable λ and a random number of features. The graphs are normalized.

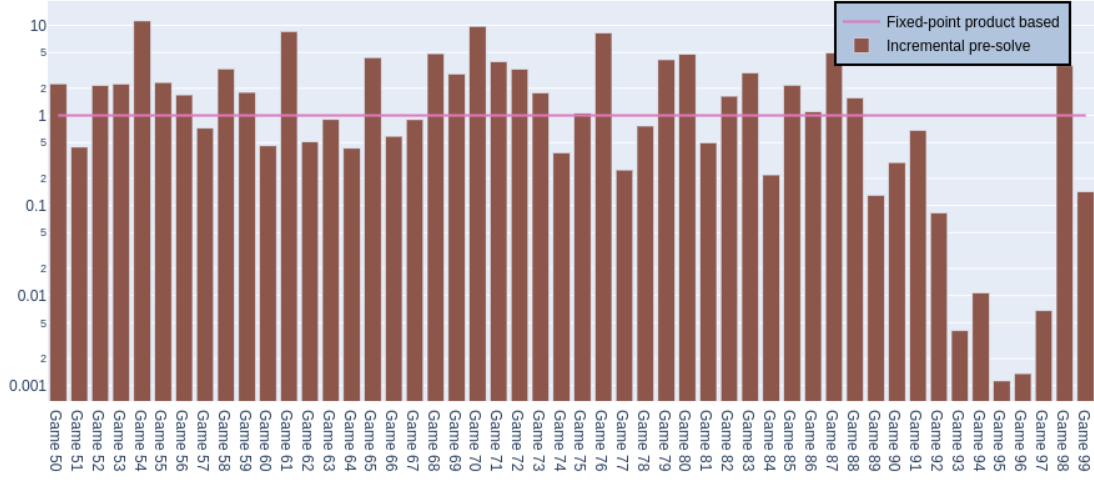


Figure 25: Running time of the incremental pre-solve algorithms on randomgames of type 1 with $\lambda = \frac{\text{game nr}}{100}$, times are normalized

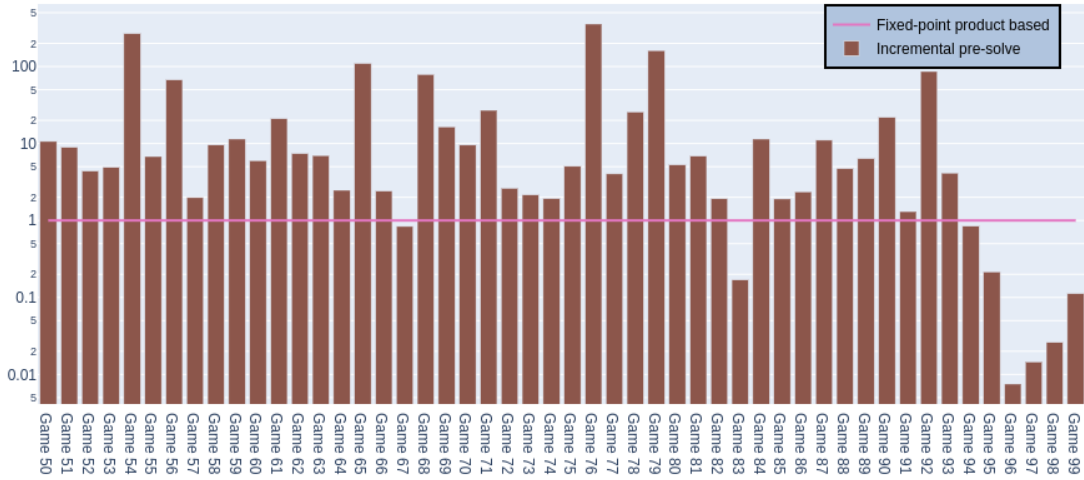


Figure 26: Running time of incremental pre-solve algorithms on randomgames of type 2 with $\lambda = \frac{\text{game nr}}{100}$, times are normalized

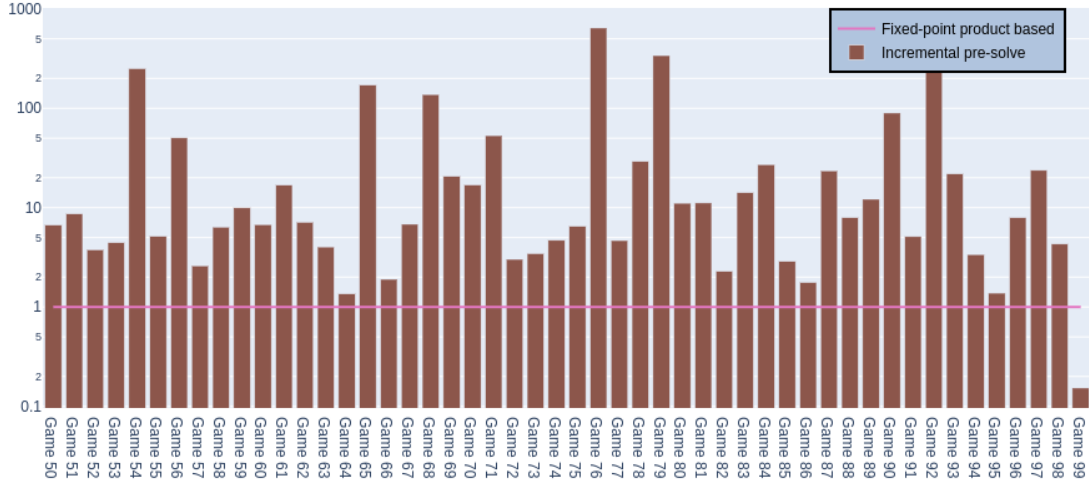


Figure 27: Running time of incremental pre-solve algorithms on randomgames of type 3 with $\lambda = \frac{\text{game nr}}{100}$, times are normalized

For type 1 and type 2 games we see that when λ gets bigger the incremental pre-solve start performing better, however it this trends only appears when λ get larger than about 0.95.

Again we see that for type 3 games the family based algorithm performs worse.

Next we inspect how the algorithms scales in terms of number of features

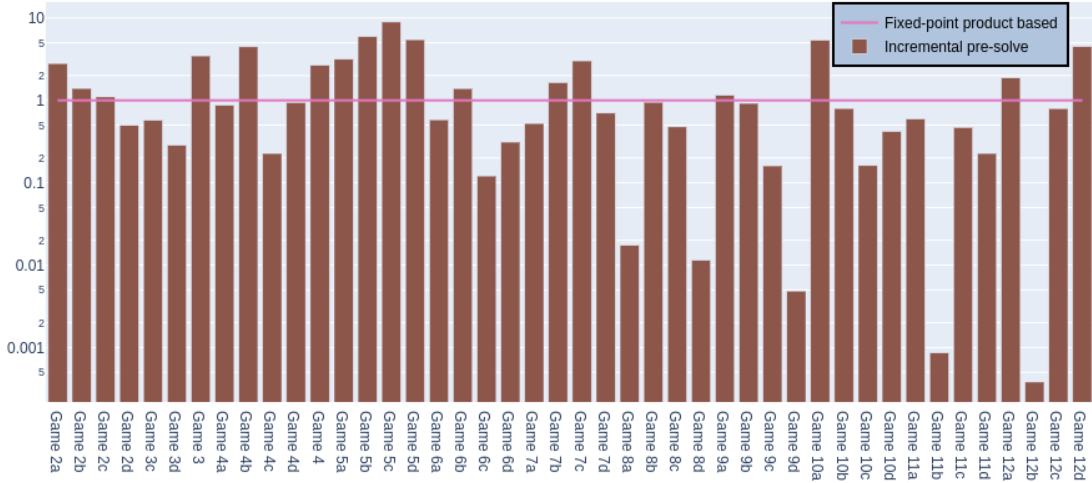


Figure 28: Running time of incremental pre-solve algorithms on randomgames of type 1 with $\lambda = 0.92$ and the number of features equal to the *game nr*, times are normalized

There is a downwards trend when it comes to the number of features, however it is quite inconsistent. Note that these games have $\lambda = 0.92$. Which is lower than the threshold observed in figure 25.

Overall the performance is not a significant improvement on the product based approach.

14.3 Incremental pre-solve local algorithm

Next we again compare the incremental pre-solver with the fixed-point iteration product based approach, however this time we compare the local variants. We start with the model-checking games.

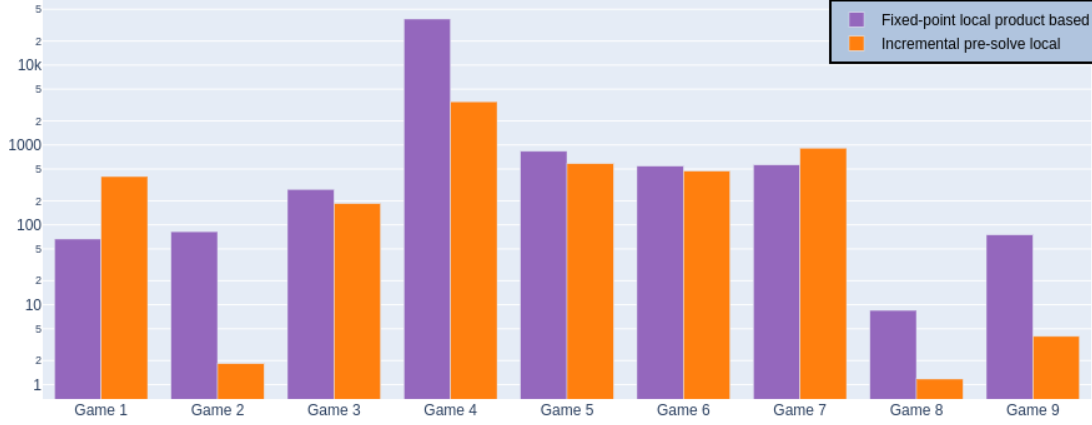


Figure 29: Running time of the incremental pre-solve local algorithms on the minepump problem

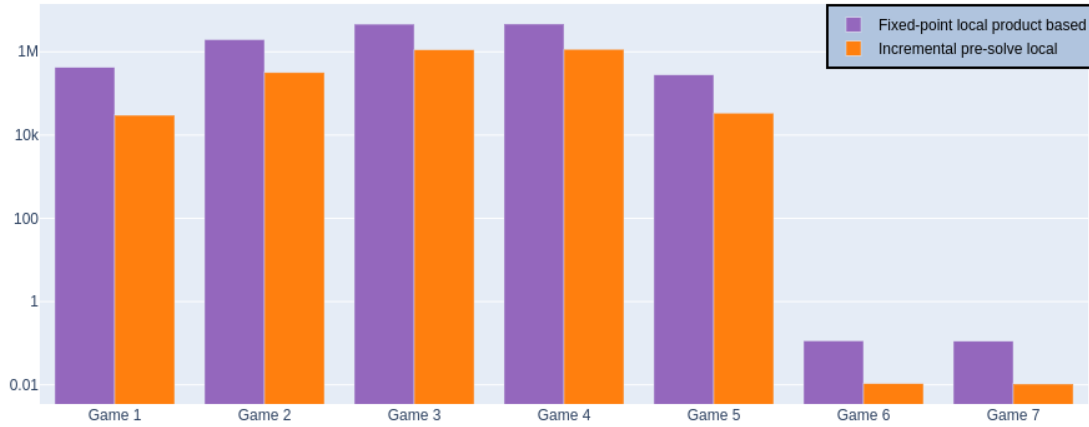


Figure 30: Running time of the incremental pre-solve local algorithms on the elevator problem

Generally the performance of the incremental pre-solver local algorithm is better, also games 3, 4 and 5 for the elevator problem were solved which was not the case for the global algorithm.

Next we inspect the random games, first we look at the games with a variable λ and a random number of features. The graphs are normalized.

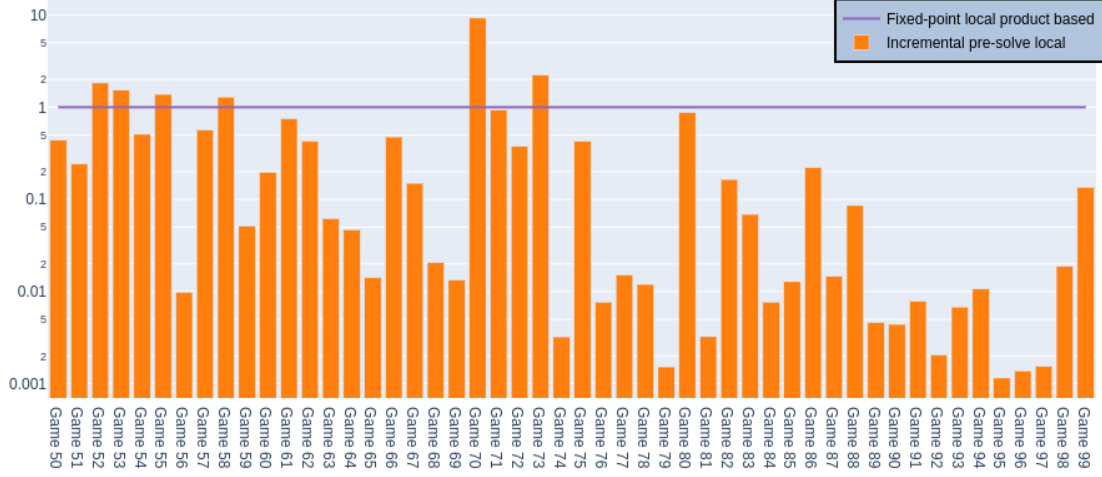


Figure 31: Running time of the incremental pre-solve local algorithms on randomgames of type 1 with $\lambda = \frac{\text{game nr}}{100}$, times are normalized

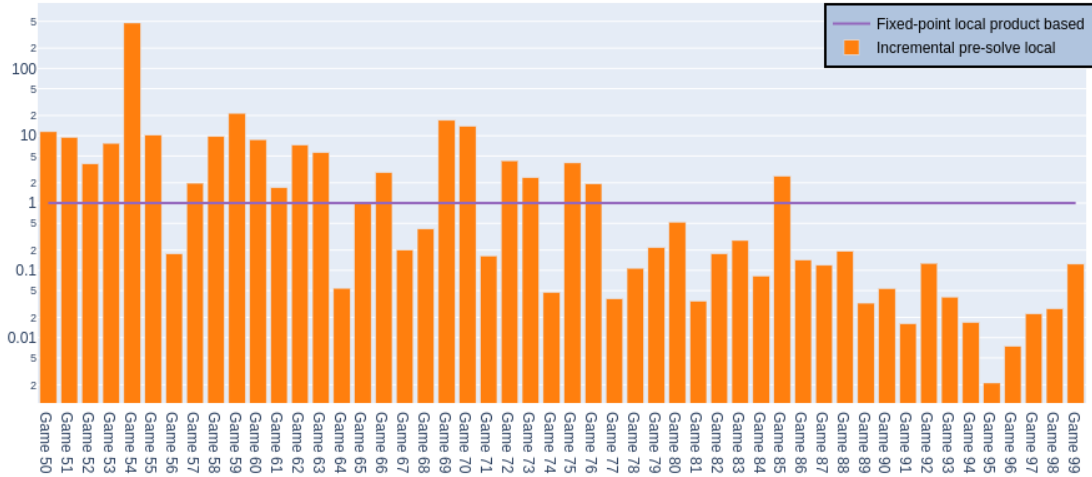


Figure 32: Running time of incremental pre-solve local algorithms on randomgames of type 2 with $\lambda = \frac{\text{game nr}}{100}$, times are normalized

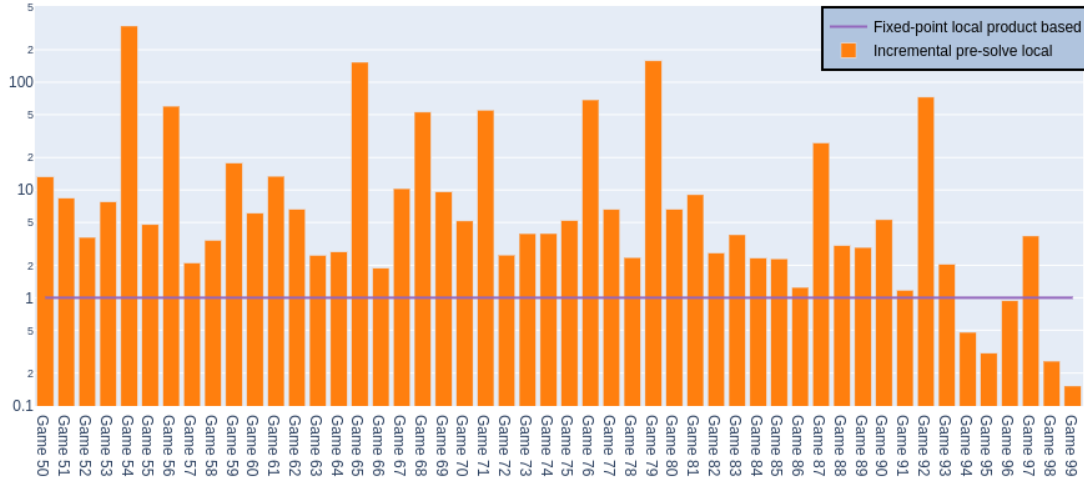


Figure 33: Running time of incremental pre-solve local algorithms on randomgames of type 3 with $\lambda = \frac{\text{game } nr}{100}$, times are normalized

For type 1 games the local incremental pre-solve algorithm greatly outperforms the product based approach, for type 2 games this is still the case but we can see differences becoming a bit smaller. Finally for type 3 games it only outperforms the product based approach when λ gets sufficiently high.

Next we inspect how the algorithms scales in terms of number of features

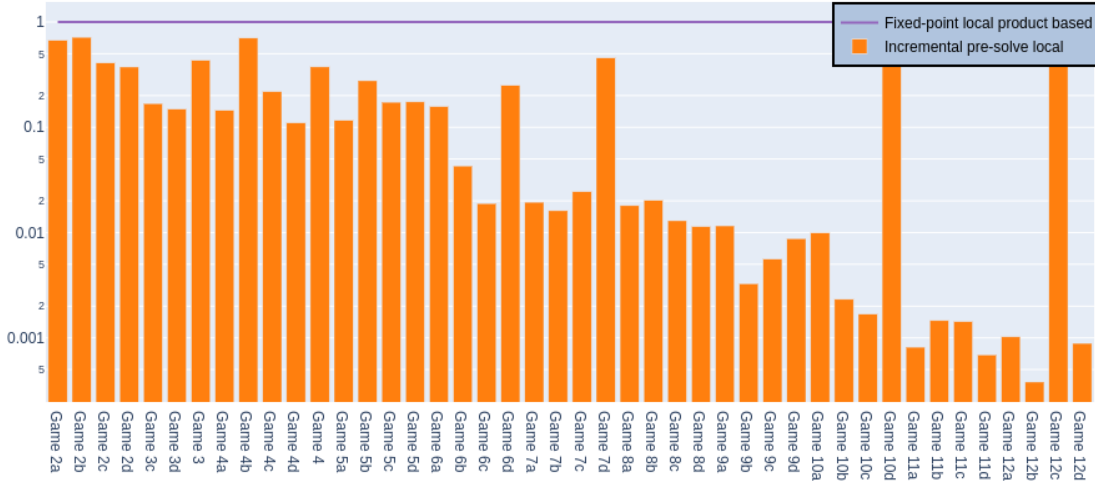


Figure 34: Running time of incremental pre-solve local algorithms on randomgames of type 1 with $\lambda = 0.92$ and the number of features equal to the *game nr*, times are normalized

Clearly the performance improves when the number of features becomes larger.

We can conclude that for model-checking games the algorithm performs decent, however the biggest

improvements are seen on random games.

14.4 Comparison

After comparing family based and product based variants of the same algorithm we compare the algorithms overall. We do so by comparing the mean solve time of the for every set of games. The incremental pre-solve global algorithm didn't complete 3 games, for these games we use the fixed-point product based global times to calculate the mean.

First we compare the product based algorithms:

	Zlnk product based	Fixed-point product based global	Fixed-point product based local
Minepump	118 ms	4719 ms	4469 ms
Elevator	36540 ms	9842835 ms	1713073 ms
Random games type 1	55 ms	413 ms	324 ms
Random games type 2	54 ms	413 ms	381 ms
Random games type 3	55 ms	416 ms	351 ms
Random scale games	108 ms	408 ms	381 ms

Next we compare the family based algorithms:

	Zlnk fam based explicit	Zlnk fam based symbolic	Incremental pre-solve global	Incremental pre-solve local
Minepump	391 ms	16 ms	1154 ms	672 ms
Elevator	46775 ms	6592 ms	8612914 ms	381041 ms
Random games type 1	513 ms	59 ms	274 ms	40 ms
Random games type 2	1149 ms	2569 ms	8283 ms	1507 ms
Random games type 3	1117 ms	2114 ms	12733 ms	4365 ms
Random scale games	397 ms	9 ms	205 ms	24 ms

Appendices

A Auxiliary theorems and lemma's

Lemma A.1. For $d, n \in \mathbb{N}$ with $d > 1$ and $n \geq 0$ the following inequality holds:

$$(n + d - 1)^d + n + 1 \leq (n + d)^d$$

Proof. We expand the inequality.

$$\begin{aligned} (n + d - 1)^d + n + 1 &\leq (n + d)^d \\ (n + d - 1)(n + d - 1)^{d-1} + n + 1 &\leq (n + d)(n + d)^{d-1} \\ n(n + d - 1)^{d-1} + d(n + d - 1)^{d-1} - (n + d - 1)^{d-1} + n + 1 &\leq n(n + d)^{d-1} + d(n + d)^{d-1} \end{aligned}$$

Since $d > 1$ and $n \geq 0$ we can see that the left hand term $n(n + d - 1)^{d-1}$ is less or equal to the right hand term $n(n + d)^{d-1}$, similarly the left hand term $d(n + d - 1)^{d-1}$ is less or equal to the right hand term $d(n + d)^{d-1}$. Finally the term $(n + d - 1)^{d-1} \geq (n + 1)^{d-1} \geq n + 1$ and therefore $-(n + d - 1)^{d-1} + n + 1 \leq 0$. This proves the lemma. \square

B Running time results

B.1 minepump

	Zlnk product based		Fixed-point product based		Fixed-point local product based
Game 1	28.882998 ms	Game 1	130.587291 ms	Game 1	66.477775 ms
Game 2	54.796994 ms	Game 2	135.382598 ms	Game 2	82.189287 ms
Game 3	184.780354 ms	Game 3	460.184224 ms	Game 3	278.245894 ms
Game 4	145.049639 ms	Game 4	37898.67001 ms	Game 4	37764.888965 ms
Game 5	144.539523 ms	Game 5	1346.308334 ms	Game 5	833.712816 ms
Game 6	242.921996 ms	Game 6	928.083238 ms	Game 6	544.744721 ms
Game 7	134.345216 ms	Game 7	669.829938 ms	Game 7	562.618467 ms
Game 8	17.444557 ms	Game 8	80.328725 ms	Game 8	8.503434 ms
Game 9	110.081099 ms	Game 9	823.229266 ms	Game 9	75.415805 ms

	Zlnk fam based - explicit		Zlnk fam based - symbolic		Incremental pre-solve
Game 1	82.447263 ms	Game 1	3.92863 ms	Game 1	402.898316 ms
Game 2	154.849897 ms	Game 2	6.760552 ms	Game 2	341.198461 ms
Game 3	615.161096 ms	Game 3	24.704502 ms	Game 3	1255.949764 ms
Game 4	886.476284 ms	Game 4	37.467636 ms	Game 4	4803.836362 ms
Game 5	278.163607 ms	Game 5	12.192468 ms	Game 5	1092.277622 ms
Game 6	1001.927374 ms	Game 6	42.792651 ms	Game 6	831.198424 ms
Game 7	312.183059 ms	Game 7	11.718343 ms	Game 7	917.083359 ms
Game 8	24.019659 ms	Game 8	1.058332 ms	Game 8	1.761178 ms
Game 9	159.633291 ms	Game 9	6.853061 ms	Game 9	742.183235 ms

	Incremental pre-solve local
Game 1	402.728056 ms
Game 2	1.827667 ms
Game 3	184.726048 ms
Game 4	3480.033761 ms
Game 5	583.474132 ms
Game 6	473.593019 ms
Game 7	914.134888 ms
Game 8	1.174847 ms
Game 9	4.019017 ms

B.2 elevator

	Zlnk product based		Fixed-point product based		Fixed-point local product based
Game 1	29073.014986 ms	Game 1	880895.014563 ms	Game 1	423068.957211 ms
Game 2	43881.36677 ms	Game 2	8790000.10349 ms	Game 2	1983814.2006 ms
Game 3	76465.44404 ms	Game 3	29387446.6626 ms	Game 3	4635988.08945 ms
Game 4	75902.19748 ms	Game 4	29423372.371 ms	Game 4	4666038.77282 ms
Game 5	30456.61109 ms	Game 5	418128.687347 ms	Game 5	282602.487182 ms
Game 6	0.386628 ms	Game 6	0.182554 ms	Game 6	0.11491 ms
Game 7	0.390919 ms	Game 7	0.182125 ms	Game 7	0.111502 ms

	Zlnk fam based - explicit		Zlnk fam based - symbolic		Incremental pre-solve
Game 1	23043.749564 ms	Game 1	5105.955006 ms	Game 1	236171.44533 ms
Game 2	35840.520344 ms	Game 2	7716.33222 ms	Game 2	960813.557869 ms
Game 3	109270.750667 ms	Game 3	12591.04199 ms	Game 3	†
Game 4	131809.825388 ms	Game 4	12980.005372 ms	Game 4	†
Game 5	27461.829276 ms	Game 5	7751.122504 ms	Game 5	†
Game 6	0.071153 ms	Game 6	0.017326 ms	Game 6	0.011157 ms
Game 7	0.058017 ms	Game 7	0.017462 ms	Game 7	0.0103 ms

	Incremental pre-solve local
Game 1	29941.840644 ms
Game 2	317217.608246 ms
Game 3	1133904.4743 ms
Game 4	1152374.57835 ms
Game 5	33845.973375 ms
Game 6	0.010736 ms
Game 7	0.010461 ms

B.3 Randomgames type 1

	Zlnk product based		Fixed-point product based		Fixed-point local product based
Game 50	12.228259 ms	Game 50	20.566329 ms	Game 50	20.595845 ms
Game 51	181.379715 ms	Game 51	6744.284861 ms	Game 51	6292.631423 ms
Game 52	2.14455 ms	Game 52	1.47327 ms	Game 52	1.398527 ms
Game 53	1.439006 ms	Game 53	4.295792 ms	Game 53	4.302088 ms
Game 54	37.311392 ms	Game 54	29.14741 ms	Game 54	29.24289 ms
Game 55	2.712425 ms	Game 55	14.425234 ms	Game 55	7.193027 ms
Game 56	195.442821 ms	Game 56	912.128577 ms	Game 56	472.727188 ms
Game 57	7.462226 ms	Game 57	199.442715 ms	Game 57	169.646163 ms
Game 58	2.260288 ms	Game 58	6.650932 ms	Game 58	6.664684 ms
Game 59	9.033436 ms	Game 59	30.331524 ms	Game 59	10.358405 ms
Game 60	39.138796 ms	Game 60	266.762206 ms	Game 60	267.307192 ms
Game 61	2.380189 ms	Game 61	2.756214 ms	Game 61	1.508336 ms
Game 62	46.281275 ms	Game 62	215.868882 ms	Game 62	205.976637 ms
Game 63	17.259779 ms	Game 63	226.614402 ms	Game 63	130.156094 ms
Game 64	6.796232 ms	Game 64	83.090986 ms	Game 64	29.198676 ms
Game 65	140.116512 ms	Game 65	266.050094 ms	Game 65	197.182221 ms
Game 66	12.213586 ms	Game 66	121.173001 ms	Game 66	120.811351 ms
Game 67	2.650634 ms	Game 67	30.477553 ms	Game 67	20.417376 ms
Game 68	33.353115 ms	Game 68	43.78565 ms	Game 68	18.646781 ms
Game 69	15.443378 ms	Game 69	28.032054 ms	Game 69	28.136298 ms
Game 70	0.909655 ms	Game 70	0.396067 ms	Game 70	0.381035 ms
Game 71	23.199077 ms	Game 71	28.951018 ms	Game 71	27.067918 ms
Game 72	0.44719 ms	Game 72	0.200488 ms	Game 72	0.204019 ms
Game 73	2.605452 ms	Game 73	10.418019 ms	Game 73	3.462291 ms
Game 74	21.374703 ms	Game 74	143.527764 ms	Game 74	143.903704 ms
Game 75	82.953938 ms	Game 75	46.377756 ms	Game 75	36.805055 ms
Game 76	185.460826 ms	Game 76	173.248219 ms	Game 76	167.128798 ms
Game 77	36.707668 ms	Game 77	1023.572304 ms	Game 77	383.250688 ms
Game 78	142.722988 ms	Game 78	1500.871069 ms	Game 78	564.751313 ms
Game 79	313.657386 ms	Game 79	473.520308 ms	Game 79	473.792807 ms
Game 80	1.210273 ms	Game 80	1.31479 ms	Game 80	1.086947 ms
Game 81	74.671357 ms	Game 81	1054.141901 ms	Game 81	1053.657412 ms
Game 82	0.706624 ms	Game 82	1.317754 ms	Game 82	1.324338 ms
Game 83	8.991237 ms	Game 83	24.022014 ms	Game 83	14.837088 ms
Game 84	73.511991 ms	Game 84	947.266091 ms	Game 84	357.204857 ms
Game 85	7.586758 ms	Game 85	9.411088 ms	Game 85	9.476771 ms
Game 86	2.602492 ms	Game 86	17.27655 ms	Game 86	7.262061 ms
Game 87	7.513792 ms	Game 87	5.299216 ms	Game 87	5.352538 ms
Game 88	3.337442 ms	Game 88	5.561753 ms	Game 88	2.610671 ms
Game 89	101.685549 ms	Game 89	1103.733905 ms	Game 89	543.43552 ms
Game 90	259.567841 ms	Game 90	642.737385 ms	Game 90	367.617943 ms
Game 91	51.463351 ms	Game 91	210.113964 ms	Game 91	87.154413 ms
Game 92	187.563902 ms	Game 92	140.735555 ms	Game 92	142.924215 ms
Game 93	49.047239 ms	Game 93	124.311004 ms	Game 93	74.032315 ms
Game 94	38.267368 ms	Game 94	465.467471 ms	Game 94	465.642971 ms
Game 95	92.163047 ms	Game 95	1940.174842 ms	Game 95	1939.324955 ms
Game 96	148.701171 ms	Game 96	1066.142203 ms	Game 96	1068.923695 ms
Game 97	57.192324 ms	Game 97	202.031576 ms	Game 97	202.097029 ms
Game 98	14.124868 ms	Game 98	24.381076 ms	Game 98	24.263109 ms
Game 99	0.593981 ms	Game 99	0.153665 ms	Game 99	0.154271 ms

	Zlnk fam based - explicit		Zlnk fam based - symbolic		Incremental pre-solve
Game 50	134.115221 ms	Game 50	30.008469 ms	Game 50	46.080898 ms
Game 51	9343.895693 ms	Game 51	1999.800971 ms	Game 51	2992.042268 ms
Game 52	4.751699 ms	Game 52	1.808211 ms	Game 52	3.151228 ms
Game 53	23.414928 ms	Game 53	8.461786 ms	Game 53	9.583558 ms
Game 54	596.248278 ms	Game 54	40.608858 ms	Game 54	325.711561 ms
Game 55	69.013627 ms	Game 55	50.971201 ms	Game 55	33.458447 ms
Game 56	2601.566566 ms	Game 56	75.923961 ms	Game 56	1539.005235 ms
Game 57	81.407788 ms	Game 57	29.997034 ms	Game 57	143.532033 ms
Game 58	41.16784 ms	Game 58	29.874202 ms	Game 58	21.824657 ms
Game 59	48.588163 ms	Game 59	13.7632 ms	Game 59	54.624111 ms
Game 60	86.673604 ms	Game 60	6.277646 ms	Game 60	122.370786 ms
Game 61	44.042146 ms	Game 61	30.812104 ms	Game 61	23.554931 ms
Game 62	276.921593 ms	Game 62	28.648873 ms	Game 62	109.501093 ms
Game 63	432.521275 ms	Game 63	124.170511 ms	Game 63	203.673008 ms
Game 64	25.353502 ms	Game 64	6.285064 ms	Game 64	35.808988 ms
Game 65	2592.790921 ms	Game 65	119.129832 ms	Game 65	1165.840461 ms
Game 66	29.396552 ms	Game 66	10.465418 ms	Game 66	71.074647 ms
Game 67	46.417311 ms	Game 67	32.170605 ms	Game 67	27.291058 ms
Game 68	160.569425 ms	Game 68	8.085773 ms	Game 68	212.068661 ms
Game 69	43.433305 ms	Game 69	4.788345 ms	Game 69	80.825415 ms
Game 70	0.694694 ms	Game 70	0.209629 ms	Game 70	3.859968 ms
Game 71	66.114718 ms	Game 71	7.7442 ms	Game 71	114.442028 ms
Game 72	0.387431 ms	Game 72	0.16619 ms	Game 72	0.649724 ms
Game 73	10.769351 ms	Game 73	4.405239 ms	Game 73	18.543163 ms
Game 74	28.102874 ms	Game 74	2.530264 ms	Game 74	55.220584 ms
Game 75	9.883799 ms	Game 75	0.229024 ms	Game 75	48.855116 ms
Game 76	2326.323546 ms	Game 76	50.011567 ms	Game 76	1419.192729 ms
Game 77	143.74654 ms	Game 77	12.211849 ms	Game 77	253.275833 ms
Game 78	1473.697901 ms	Game 78	61.78517 ms	Game 78	1142.785118 ms
Game 79	829.334874 ms	Game 79	9.310871 ms	Game 79	1962.348341 ms
Game 80	10.683794 ms	Game 80	3.41224 ms	Game 80	6.247378 ms
Game 81	265.53113 ms	Game 81	11.595323 ms	Game 81	524.071394 ms
Game 82	6.763731 ms	Game 82	1.704003 ms	Game 82	2.152315 ms
Game 83	68.374198 ms	Game 83	17.046606 ms	Game 83	71.00827 ms
Game 84	373.221211 ms	Game 84	15.492898 ms	Game 84	206.907479 ms
Game 85	4.104353 ms	Game 85	0.451486 ms	Game 85	20.351748 ms
Game 86	21.095341 ms	Game 86	11.369413 ms	Game 86	18.925971 ms
Game 87	5.654448 ms	Game 87	0.657659 ms	Game 87	26.118424 ms
Game 88	5.123875 ms	Game 88	0.662482 ms	Game 88	8.694553 ms
Game 89	252.185215 ms	Game 89	10.069447 ms	Game 89	143.203366 ms
Game 90	1906.86293 ms	Game 90	38.409335 ms	Game 90	192.609233 ms
Game 91	29.10855 ms	Game 91	1.355012 ms	Game 91	143.36905 ms
Game 92	258.287784 ms	Game 92	2.680355 ms	Game 92	11.604601 ms
Game 93	96.508631 ms	Game 93	3.834215 ms	Game 93	0.505494 ms
Game 94	250.738963 ms	Game 94	35.697677 ms	Game 94	4.989098 ms
Game 95	43.95263 ms	Game 95	0.972757 ms	Game 95	2.18487 ms
Game 96	395.296555 ms	Game 96	7.797964 ms	Game 96	1.447915 ms
Game 97	62.937372 ms	Game 97	1.282742 ms	Game 97	1.369827 ms
Game 98	40.836075 ms	Game 98	5.126716 ms	Game 98	86.552119 ms
Game 99	0.235165 ms	Game 99	0.031976 ms	Game 99	0.02177 ms

	Incremental pre-solve local
Game 50	9.018887 ms
Game 51	1523.362457 ms
Game 52	2.56276 ms
Game 53	6.587165 ms
Game 54	14.87686 ms
Game 55	9.903866 ms
Game 56	4.660675 ms
Game 57	96.08021 ms
Game 58	8.569419 ms
Game 59	0.534212 ms
Game 60	52.722595 ms
Game 61	1.132779 ms
Game 62	88.117407 ms
Game 63	8.078743 ms
Game 64	1.370403 ms
Game 65	2.78775 ms
Game 66	57.445895 ms
Game 67	3.053301 ms
Game 68	0.384612 ms
Game 69	0.376439 ms
Game 70	3.533979 ms
Game 71	25.211715 ms
Game 72	0.076533 ms
Game 73	7.696058 ms
Game 74	0.464487 ms
Game 75	15.709783 ms
Game 76	1.285309 ms
Game 77	5.801185 ms
Game 78	6.765665 ms
Game 79	0.722754 ms
Game 80	0.953798 ms
Game 81	3.425409 ms
Game 82	0.217681 ms
Game 83	1.021352 ms
Game 84	2.749428 ms
Game 85	0.121502 ms
Game 86	1.616307 ms
Game 87	0.078561 ms
Game 88	0.224175 ms
Game 89	2.50544 ms
Game 90	1.621967 ms
Game 91	0.683825 ms
Game 92	0.294685 ms
Game 93	0.503414 ms
Game 94	4.977672 ms
Game 95	2.243018 ms
Game 96	1.475905 ms
Game 97	0.312385 ms
Game 98	0.45833 ms
Game 99	0.020862 ms

B.4 Random games type 2

	Zlnk product based		Fixed-point product based		Fixed-point local product based
Game 50	10.918245 ms	Game 50	25.788145 ms	Game 50	23.491823 ms
Game 51	173.983561 ms	Game 51	6841.202002 ms	Game 51	6358.738676 ms
Game 52	1.38404 ms	Game 52	1.142889 ms	Game 52	1.120596 ms
Game 53	1.563996 ms	Game 53	3.264741 ms	Game 53	1.841316 ms
Game 54	37.693475 ms	Game 54	30.670339 ms	Game 54	16.842579 ms
Game 55	2.884703 ms	Game 55	10.21778 ms	Game 55	5.598839 ms
Game 56	186.982175 ms	Game 56	1312.727673 ms	Game 56	1316.381878 ms
Game 57	7.849308 ms	Game 57	336.312221 ms	Game 57	323.38378 ms
Game 58	2.402196 ms	Game 58	3.993694 ms	Game 58	2.154177 ms
Game 59	9.357524 ms	Game 59	19.763114 ms	Game 59	8.593163 ms
Game 60	40.283558 ms	Game 60	419.528973 ms	Game 60	266.971185 ms
Game 61	2.453017 ms	Game 61	1.847166 ms	Game 61	0.974743 ms
Game 62	37.682566 ms	Game 62	157.99114 ms	Game 62	153.270569 ms
Game 63	18.900662 ms	Game 63	123.032105 ms	Game 63	123.573552 ms
Game 64	7.005823 ms	Game 64	50.739421 ms	Game 64	50.76468 ms
Game 65	151.08974 ms	Game 65	339.604908 ms	Game 65	151.33958 ms
Game 66	10.608501 ms	Game 66	55.454246 ms	Game 66	43.175666 ms
Game 67	2.668589 ms	Game 67	56.577501 ms	Game 67	56.524643 ms
Game 68	33.591869 ms	Game 68	35.266073 ms	Game 68	35.68747 ms
Game 69	16.203081 ms	Game 69	31.136878 ms	Game 69	27.232735 ms
Game 70	0.951235 ms	Game 70	0.623003 ms	Game 70	0.42265 ms
Game 71	23.205072 ms	Game 71	28.043222 ms	Game 71	28.190491 ms
Game 72	0.578534 ms	Game 72	0.720565 ms	Game 72	0.384138 ms
Game 73	2.070027 ms	Game 73	14.39249 ms	Game 73	12.369724 ms
Game 74	22.614131 ms	Game 74	233.078601 ms	Game 74	86.760695 ms
Game 75	47.03979 ms	Game 75	20.684427 ms	Game 75	20.558411 ms
Game 76	181.067279 ms	Game 76	156.408315 ms	Game 76	71.018835 ms
Game 77	39.142096 ms	Game 77	411.702241 ms	Game 77	412.161454 ms
Game 78	155.973436 ms	Game 78	798.928246 ms	Game 78	800.264776 ms
Game 79	296.588672 ms	Game 79	463.166073 ms	Game 79	462.523326 ms
Game 80	1.168711 ms	Game 80	1.714026 ms	Game 80	0.923882 ms
Game 81	73.937408 ms	Game 81	735.675399 ms	Game 81	735.152819 ms
Game 82	0.708429 ms	Game 82	1.853504 ms	Game 82	1.870616 ms
Game 83	8.857286 ms	Game 83	14.961404 ms	Game 83	9.113067 ms
Game 84	82.055301 ms	Game 84	462.353075 ms	Game 84	295.219217 ms
Game 85	9.470796 ms	Game 85	18.109145 ms	Game 85	11.739621 ms
Game 86	2.697405 ms	Game 86	10.22887 ms	Game 86	10.344082 ms
Game 87	7.927669 ms	Game 87	4.766618 ms	Game 87	4.326818 ms
Game 88	3.402374 ms	Game 88	4.285175 ms	Game 88	2.003342 ms
Game 89	93.767298 ms	Game 89	569.014841 ms	Game 89	569.378285 ms
Game 90	247.483212 ms	Game 90	1137.612069 ms	Game 90	1133.058396 ms
Game 91	51.416592 ms	Game 91	380.726518 ms	Game 91	225.493779 ms
Game 92	194.94486 ms	Game 92	159.622577 ms	Game 92	160.648632 ms
Game 93	50.602247 ms	Game 93	190.273961 ms	Game 93	107.219647 ms
Game 94	40.277615 ms	Game 94	595.608117 ms	Game 94	596.556856 ms
Game 95	93.780179 ms	Game 95	3054.323181 ms	Game 95	3053.310575 ms
Game 96	148.719081 ms	Game 96	1172.194676 ms	Game 96	1175.237651 ms
Game 97	61.45583 ms	Game 97	120.681324 ms	Game 97	76.270106 ms
Game 98	14.088651 ms	Game 98	27.819839 ms	Game 98	27.717006 ms
Game 99	0.614478 ms	Game 99	0.184036 ms	Game 99	0.168606 ms

	Zlnk fam based - explicit		Zlnk fam based - symbolic		Incremental pre-solve
Game 50	420.892527 ms	Game 50	202.77158 ms	Game 50	274.513552 ms
Game 51	27572.326974 ms	Game 51	85292.257449 ms	Game 51	61404.869851 ms
Game 52	8.493117 ms	Game 52	4.367472 ms	Game 52	5.041807 ms
Game 53	20.603836 ms	Game 53	10.823653 ms	Game 53	16.068485 ms
Game 54	1820.06217 ms	Game 54	3445.363545 ms	Game 54	8252.836326 ms
Game 55	116.424002 ms	Game 55	81.243074 ms	Game 55	69.251691 ms
Game 56	6818.05413 ms	Game 56	18201.386708 ms	Game 56	88278.874709 ms
Game 57	152.332163 ms	Game 57	105.554749 ms	Game 57	666.755859 ms
Game 58	72.735493 ms	Game 58	47.737789 ms	Game 58	38.229257 ms
Game 59	110.421616 ms	Game 59	65.431177 ms	Game 59	226.141247 ms
Game 60	1293.173118 ms	Game 60	1073.714993 ms	Game 60	2514.769282 ms
Game 61	68.153455 ms	Game 61	44.252954 ms	Game 61	38.780713 ms
Game 62	554.753817 ms	Game 62	373.83758 ms	Game 62	1176.272233 ms
Game 63	762.954768 ms	Game 63	575.551737 ms	Game 63	858.516357 ms
Game 64	53.042922 ms	Game 64	24.641367 ms	Game 64	125.34107 ms
Game 65	5594.930519 ms	Game 65	9258.326986 ms	Game 65	37422.726912 ms
Game 66	36.767411 ms	Game 66	29.829226 ms	Game 66	133.928952 ms
Game 67	51.269817 ms	Game 67	33.649774 ms	Game 67	47.395902 ms
Game 68	192.865339 ms	Game 68	88.905642 ms	Game 68	2787.521288 ms
Game 69	87.789411 ms	Game 69	33.595287 ms	Game 69	512.772417 ms
Game 70	2.835724 ms	Game 70	0.912352 ms	Game 70	5.953089 ms
Game 71	207.624583 ms	Game 71	247.337206 ms	Game 71	757.297336 ms
Game 72	1.446822 ms	Game 72	1.017398 ms	Game 72	1.893045 ms
Game 73	17.618806 ms	Game 73	8.016097 ms	Game 73	30.999393 ms
Game 74	67.377396 ms	Game 74	27.786453 ms	Game 74	452.229702 ms
Game 75	5.127006 ms	Game 75	1.348222 ms	Game 75	104.657565 ms
Game 76	2816.136694 ms	Game 76	3402.734031 ms	Game 76	55820.453259 ms
Game 77	220.696102 ms	Game 77	90.493295 ms	Game 77	1661.170935 ms
Game 78	2085.753805 ms	Game 78	2341.770188 ms	Game 78	20559.171143 ms
Game 79	1189.430678 ms	Game 79	1382.17279 ms	Game 79	74687.630715 ms
Game 80	7.765884 ms	Game 80	3.201824 ms	Game 80	9.092573 ms
Game 81	403.953604 ms	Game 81	230.685978 ms	Game 81	5052.554785 ms
Game 82	3.16458 ms	Game 82	1.145025 ms	Game 82	3.555746 ms
Game 83	68.881758 ms	Game 83	18.733033 ms	Game 83	2.529063 ms
Game 84	552.718294 ms	Game 84	263.509541 ms	Game 84	5286.51502 ms
Game 85	13.66698 ms	Game 85	4.604088 ms	Game 85	34.527788 ms
Game 86	26.188807 ms	Game 86	14.244522 ms	Game 86	24.111287 ms
Game 87	7.311722 ms	Game 87	2.588723 ms	Game 87	52.726666 ms
Game 88	6.040884 ms	Game 88	1.156411 ms	Game 88	20.382232 ms
Game 89	316.303413 ms	Game 89	82.005214 ms	Game 89	3639.537318 ms
Game 90	2319.834539 ms	Game 90	1172.033005 ms	Game 90	24874.107362 ms
Game 91	35.78069 ms	Game 91	8.036287 ms	Game 91	496.947157 ms
Game 92	300.939324 ms	Game 92	58.713405 ms	Game 92	13755.969387 ms
Game 93	112.089537 ms	Game 93	8.327577 ms	Game 93	785.924095 ms
Game 94	314.73919 ms	Game 94	46.196091 ms	Game 94	502.19285 ms
Game 95	51.41534 ms	Game 95	4.695648 ms	Game 95	656.998381 ms
Game 96	392.095848 ms	Game 96	16.346571 ms	Game 96	8.802605 ms
Game 97	77.036153 ms	Game 97	3.059106 ms	Game 97	1.757358 ms
Game 98	38.558349 ms	Game 98	5.570903 ms	Game 98	0.72427 ms
Game 99	0.312622 ms	Game 99	0.032517 ms	Game 99	0.020685 ms

	Incremental pre-solve local
Game 50	269.718821 ms
Game 51	60355.20469 ms
Game 52	4.278918 ms
Game 53	14.222552 ms
Game 54	7967.297797 ms
Game 55	57.062669 ms
Game 56	231.673885 ms
Game 57	638.551196 ms
Game 58	20.990244 ms
Game 59	185.289698 ms
Game 60	2338.294662 ms
Game 61	1.645522 ms
Game 62	1120.42437 ms
Game 63	697.99699 ms
Game 64	2.75111 ms
Game 65	149.314436 ms
Game 66	122.997515 ms
Game 67	11.312594 ms
Game 68	14.694391 ms
Game 69	461.680203 ms
Game 70	5.849632 ms
Game 71	4.622708 ms
Game 72	1.631972 ms
Game 73	29.434576 ms
Game 74	4.10057 ms
Game 75	81.1602 ms
Game 76	137.661231 ms
Game 77	15.629775 ms
Game 78	85.947227 ms
Game 79	101.015369 ms
Game 80	0.48114 ms
Game 81	25.734733 ms
Game 82	0.330751 ms
Game 83	2.53277 ms
Game 84	24.373237 ms
Game 85	29.590797 ms
Game 86	1.480927 ms
Game 87	0.514867 ms
Game 88	0.38692 ms
Game 89	18.561496 ms
Game 90	60.996587 ms
Game 91	3.65302 ms
Game 92	20.437045 ms
Game 93	4.298427 ms
Game 94	10.100405 ms
Game 95	6.47525 ms
Game 96	8.752235 ms
Game 97	1.736223 ms
Game 98	0.749523 ms
Game 99	0.020884 ms

B.5 Random games type 3

	Zlnk product based		Fixed-point product based		Fixed-point local product based
Game 50	9.976095 ms	Game 50	33.39416 ms	Game 50	13.808627 ms
Game 51	168.106822 ms	Game 51	5534.871778 ms	Game 51	5326.973496 ms
Game 52	2.037131 ms	Game 52	1.26391 ms	Game 52	1.276611 ms
Game 53	1.455536 ms	Game 53	3.170443 ms	Game 53	1.64824 ms
Game 54	38.714198 ms	Game 54	26.331521 ms	Game 54	12.136227 ms
Game 55	2.791138 ms	Game 55	11.795348 ms	Game 55	6.576087 ms
Game 56	187.030532 ms	Game 56	1504.589399 ms	Game 56	944.411779 ms
Game 57	7.542981 ms	Game 57	191.552096 ms	Game 57	191.884138 ms
Game 58	2.237842 ms	Game 58	5.875631 ms	Game 58	5.840379 ms
Game 59	9.146537 ms	Game 59	22.604068 ms	Game 59	10.144003 ms
Game 60	45.055167 ms	Game 60	420.111746 ms	Game 60	410.714964 ms
Game 61	2.406704 ms	Game 61	2.346122 ms	Game 61	1.14639 ms
Game 62	35.729423 ms	Game 62	210.076795 ms	Game 62	205.434014 ms
Game 63	17.61155 ms	Game 63	268.083218 ms	Game 63	267.867202 ms
Game 64	7.421523 ms	Game 64	188.800481 ms	Game 64	77.285434 ms
Game 65	146.089908 ms	Game 65	246.453054 ms	Game 65	223.400633 ms
Game 66	16.845018 ms	Game 66	109.679181 ms	Game 66	106.379365 ms
Game 67	2.781907 ms	Game 67	24.159304 ms	Game 67	13.235959 ms
Game 68	35.396676 ms	Game 68	32.304285 ms	Game 68	32.293533 ms
Game 69	16.607279 ms	Game 69	35.825679 ms	Game 69	35.856809 ms
Game 70	0.967567 ms	Game 70	0.344868 ms	Game 70	0.342402 ms
Game 71	23.912201 ms	Game 71	21.781779 ms	Game 71	20.189885 ms
Game 72	0.51349 ms	Game 72	0.602684 ms	Game 72	0.587294 ms
Game 73	2.033038 ms	Game 73	7.749658 ms	Game 73	6.687538 ms
Game 74	22.84673 ms	Game 74	212.354638 ms	Game 74	209.573048 ms
Game 75	62.986938 ms	Game 75	25.740094 ms	Game 75	24.833401 ms
Game 76	187.89891 ms	Game 76	154.448659 ms	Game 76	154.408502 ms
Game 77	38.757606 ms	Game 77	756.884077 ms	Game 77	301.337573 ms
Game 78	137.862444 ms	Game 78	1174.479264 ms	Game 78	1173.799697 ms
Game 79	297.547201 ms	Game 79	422.807031 ms	Game 79	220.204762 ms
Game 80	1.183585 ms	Game 80	1.094506 ms	Game 80	1.11143 ms
Game 81	73.797893 ms	Game 81	934.68797 ms	Game 81	414.742317 ms
Game 82	0.79472 ms	Game 82	2.723153 ms	Game 82	2.324309 ms
Game 83	8.998882 ms	Game 83	18.059568 ms	Game 83	10.075222 ms
Game 84	77.547555 ms	Game 84	414.709718 ms	Game 84	415.978428 ms
Game 85	7.718849 ms	Game 85	20.150961 ms	Game 85	15.497647 ms
Game 86	2.819101 ms	Game 86	20.67089 ms	Game 86	8.184581 ms
Game 87	7.591827 ms	Game 87	5.526844 ms	Game 87	4.216409 ms
Game 88	3.20876 ms	Game 88	4.486084 ms	Game 88	4.532044 ms
Game 89	98.268788 ms	Game 89	792.944337 ms	Game 89	793.009891 ms
Game 90	250.712267 ms	Game 90	755.171693 ms	Game 90	448.992638 ms
Game 91	49.413216 ms	Game 91	380.795967 ms	Game 91	381.016753 ms
Game 92	189.968957 ms	Game 92	162.856618 ms	Game 92	77.706188 ms
Game 93	50.349972 ms	Game 93	131.000885 ms	Game 93	131.040994 ms
Game 94	38.956918 ms	Game 94	402.117363 ms	Game 94	402.395126 ms
Game 95	103.698611 ms	Game 95	3293.720234 ms	Game 95	3293.137131 ms
Game 96	159.487239 ms	Game 96	1700.935701 ms	Game 96	1060.421164 ms
Game 97	59.231515 ms	Game 97	80.308982 ms	Game 97	80.576536 ms
Game 98	14.638905 ms	Game 98	40.934591 ms	Game 98	23.666594 ms
Game 99	0.576372 ms	Game 99	0.13178 ms	Game 99	0.130765 ms

	Zlnk fam based - explicit		Zlnk fam based - symbolic		Incremental pre-solve
Game 50	273.792074 ms	Game 50	136.556033 ms	Game 50	223.466021 ms
Game 51	13646.451841 ms	Game 51	36507.220998 ms	Game 51	47937.178942 ms
Game 52	7.25798 ms	Game 52	3.313925 ms	Game 52	4.769585 ms
Game 53	11.243911 ms	Game 53	6.821058 ms	Game 53	14.069063 ms
Game 54	1055.519768 ms	Game 54	1742.951911 ms	Game 54	6622.38747 ms
Game 55	106.227548 ms	Game 55	75.711339 ms	Game 55	60.795483 ms
Game 56	6484.58168 ms	Game 56	15763.37959 ms	Game 56	75834.067772 ms
Game 57	81.66421 ms	Game 57	57.252336 ms	Game 57	497.534014 ms
Game 58	63.750863 ms	Game 58	44.878245 ms	Game 58	37.426839 ms
Game 59	109.994729 ms	Game 59	69.754642 ms	Game 59	225.083612 ms
Game 60	1603.233246 ms	Game 60	1792.61042 ms	Game 60	2836.00766 ms
Game 61	62.352276 ms	Game 61	43.232763 ms	Game 61	39.508389 ms
Game 62	898.041632 ms	Game 62	889.723099 ms	Game 62	1499.171081 ms
Game 63	798.875525 ms	Game 63	691.620014 ms	Game 63	1077.431545 ms
Game 64	110.70652 ms	Game 64	59.568202 ms	Game 64	256.577525 ms
Game 65	6624.857714 ms	Game 65	12383.166294 ms	Game 65	42208.617471 ms
Game 66	128.211484 ms	Game 66	96.249389 ms	Game 66	208.42148 ms
Game 67	76.392448 ms	Game 67	53.606178 ms	Game 67	164.479026 ms
Game 68	381.266139 ms	Game 68	386.686147 ms	Game 68	4443.06214 ms
Game 69	149.083001 ms	Game 69	86.865845 ms	Game 69	742.156512 ms
Game 70	1.31729 ms	Game 70	0.329505 ms	Game 70	5.855942 ms
Game 71	200.448827 ms	Game 71	277.223349 ms	Game 71	1155.989937 ms
Game 72	1.191567 ms	Game 72	0.992797 ms	Game 72	1.819665 ms
Game 73	23.675776 ms	Game 73	14.837991 ms	Game 73	26.794277 ms
Game 74	133.06584 ms	Game 74	76.712306 ms	Game 74	996.17373 ms
Game 75	8.497955 ms	Game 75	9.734143 ms	Game 75	167.774753 ms
Game 76	5812.819166 ms	Game 76	11627.779415 ms	Game 76	99258.828158 ms
Game 77	418.636531 ms	Game 77	414.721386 ms	Game 77	3530.63729 ms
Game 78	3857.542835 ms	Game 78	6413.826884 ms	Game 78	34238.537422 ms
Game 79	2842.369859 ms	Game 79	5931.527685 ms	Game 79	143423.447357 ms
Game 80	13.008812 ms	Game 80	5.962704 ms	Game 80	12.132405 ms
Game 81	823.914959 ms	Game 81	1046.228509 ms	Game 81	10427.95421 ms
Game 82	3.90781 ms	Game 82	3.105238 ms	Game 82	6.241549 ms
Game 83	114.703845 ms	Game 83	47.268526 ms	Game 83	255.292567 ms
Game 84	1032.435205 ms	Game 84	1192.05896 ms	Game 84	11249.112673 ms
Game 85	7.073889 ms	Game 85	3.327481 ms	Game 85	57.947425 ms
Game 86	35.204818 ms	Game 86	24.712157 ms	Game 86	36.634892 ms
Game 87	33.873518 ms	Game 87	18.135895 ms	Game 87	129.742853 ms
Game 88	7.400586 ms	Game 88	4.466311 ms	Game 88	35.592823 ms
Game 89	606.091825 ms	Game 89	579.281233 ms	Game 89	9632.501725 ms
Game 90	4348.115576 ms	Game 90	5432.643684 ms	Game 90	67689.612809 ms
Game 91	113.76588 ms	Game 91	78.93278 ms	Game 91	1946.877179 ms
Game 92	783.015408 ms	Game 92	744.988497 ms	Game 92	43094.739497 ms
Game 93	226.238564 ms	Game 93	73.713132 ms	Game 93	2865.112811 ms
Game 94	529.478261 ms	Game 94	232.290929 ms	Game 94	1356.038341 ms
Game 95	178.521841 ms	Game 95	98.273438 ms	Game 95	4543.837972 ms
Game 96	841.505137 ms	Game 96	429.448153 ms	Game 96	13465.080253 ms
Game 97	123.269255 ms	Game 97	14.449629 ms	Game 97	1906.935486 ms
Game 98	66.753929 ms	Game 98	15.714123 ms	Game 98	176.16669 ms
Game 99	0.234607 ms	Game 99	0.031156 ms	Game 99	0.020168 ms

	Incremental pre-solve local
Game 50	182.975723 ms
Game 51	44671.815935 ms
Game 52	4.637662 ms
Game 53	12.782434 ms
Game 54	4032.451405 ms
Game 55	31.408177 ms
Game 56	56605.457611 ms
Game 57	404.737995 ms
Game 58	19.957296 ms
Game 59	180.302945 ms
Game 60	2498.227687 ms
Game 61	15.375766 ms
Game 62	1364.469331 ms
Game 63	664.211163 ms
Game 64	205.724963 ms
Game 65	34218.480539 ms
Game 66	199.78587 ms
Game 67	136.205044 ms
Game 68	1704.763075 ms
Game 69	345.406447 ms
Game 70	1.766606 ms
Game 71	1111.678948 ms
Game 72	1.462348 ms
Game 73	26.320828 ms
Game 74	826.088687 ms
Game 75	129.308793 ms
Game 76	10589.753997 ms
Game 77	1986.122688 ms
Game 78	2772.260825 ms
Game 79	34775.330723 ms
Game 80	7.359347 ms
Game 81	3749.058872 ms
Game 82	6.064545 ms
Game 83	38.694408 ms
Game 84	974.61078 ms
Game 85	35.633124 ms
Game 86	10.211757 ms
Game 87	115.535395 ms
Game 88	13.844376 ms
Game 89	2323.685826 ms
Game 90	2379.568429 ms
Game 91	447.783042 ms
Game 92	5649.308915 ms
Game 93	268.755977 ms
Game 94	192.249596 ms
Game 95	1006.087455 ms
Game 96	993.745517 ms
Game 97	301.626611 ms
Game 98	6.109814 ms
Game 99	0.019839 ms

B.6 Random scale games

	Zlnk product based		Fixed-point product based		Fixed-point local product based
Game 2a	0.218619 ms	Game 2a	0.289869 ms	Game 2a	0.275598 ms
Game 2b	0.254285 ms	Game 2b	0.823527 ms	Game 2b	0.493619 ms
Game 2c	0.685084 ms	Game 2c	3.852203 ms	Game 2c	3.795691 ms
Game 2d	0.202601 ms	Game 2d	4.197048 ms	Game 2d	4.231017 ms
Game 3c	2.169628 ms	Game 3c	37.01689 ms	Game 3c	36.927433 ms
Game 3d	0.641398 ms	Game 3d	16.07463 ms	Game 3d	16.312832 ms
Game 3	1.642876 ms	Game 3	2.356911 ms	Game 3	2.346755 ms
Game 4a	1.382486 ms	Game 4a	2.746074 ms	Game 4a	2.793239 ms
Game 4b	0.761629 ms	Game 4b	0.547347 ms	Game 4b	0.243132 ms
Game 4c	0.387448 ms	Game 4c	0.126752 ms	Game 4c	0.125161 ms
Game 4d	1.558909 ms	Game 4d	5.695754 ms	Game 4d	5.710197 ms
Game 4	0.471874 ms	Game 4	0.462942 ms	Game 4	0.453259 ms
Game 5a	2.155584 ms	Game 5a	1.702403 ms	Game 5a	1.713909 ms
Game 5b	7.940156 ms	Game 5b	7.33214 ms	Game 5b	3.857951 ms
Game 5c	1.077075 ms	Game 5c	0.51776 ms	Game 5c	0.52551 ms
Game 5d	12.452995 ms	Game 5d	20.973557 ms	Game 5d	11.633533 ms
Game 6a	4.941249 ms	Game 6a	29.071679 ms	Game 6a	5.333565 ms
Game 6b	7.448886 ms	Game 6b	12.778665 ms	Game 6b	12.767724 ms
Game 6c	8.854734 ms	Game 6c	196.926396 ms	Game 6c	198.298965 ms
Game 6d	3.303204 ms	Game 6d	40.164514 ms	Game 6d	37.896252 ms
Game 7a	15.773086 ms	Game 7a	103.270116 ms	Game 7a	66.951652 ms
Game 7b	8.192256 ms	Game 7b	18.017859 ms	Game 7b	18.030418 ms
Game 7c	4.651644 ms	Game 7c	3.16192 ms	Game 7c	3.183493 ms
Game 7d	4.684234 ms	Game 7d	12.278696 ms	Game 7d	12.293851 ms
Game 8a	42.609582 ms	Game 8a	45.395534 ms	Game 8a	44.225411 ms
Game 8b	17.150868 ms	Game 8b	67.554086 ms	Game 8b	22.873975 ms
Game 8c	13.403176 ms	Game 8c	96.495009 ms	Game 8c	49.403749 ms
Game 8d	34.365665 ms	Game 8d	54.088157 ms	Game 8d	54.40273 ms
Game 9a	170.513824 ms	Game 9a	215.733053 ms	Game 9a	130.934114 ms
Game 9b	24.262356 ms	Game 9b	92.432614 ms	Game 9b	92.897594 ms
Game 9c	18.42389 ms	Game 9c	26.96885 ms	Game 9c	18.349991 ms
Game 9d	66.014055 ms	Game 9d	115.984155 ms	Game 9d	64.62795 ms
Game 10a	85.933823 ms	Game 10a	54.51221 ms	Game 10a	28.647424 ms
Game 10b	46.613012 ms	Game 10b	61.192662 ms	Game 10b	61.268989 ms
Game 10c	248.446715 ms	Game 10c	1047.345171 ms	Game 10c	1049.365977 ms
Game 10d	115.753201 ms	Game 10d	2488.498826 ms	Game 10d	2165.865324 ms
Game 11a	151.191006 ms	Game 11a	899.060691 ms	Game 11a	898.492652 ms
Game 11b	337.831038 ms	Game 11b	1336.42121 ms	Game 11b	797.898309 ms
Game 11c	74.969157 ms	Game 11c	104.520117 ms	Game 11c	85.833675 ms
Game 11d	96.280452 ms	Game 11d	615.371781 ms	Game 11d	614.757797 ms
Game 12a	937.437731 ms	Game 12a	922.246974 ms	Game 12a	927.441418 ms
Game 12b	1191.637453 ms	Game 12b	8159.747723 ms	Game 12b	8156.109629 ms
Game 12c	110.383665 ms	Game 12c	132.80073 ms	Game 12c	132.008086 ms
Game 12d	870.642396 ms	Game 12d	906.988802 ms	Game 12d	913.086655 ms

	Zlnk fam based - explicit		Zlnk fam based - symbolic		Incremental pre-solve
Game 2a	1.573788 ms	Game 2a	1.041346 ms	Game 2a	0.811764 ms
Game 2b	2.325468 ms	Game 2b	1.543646 ms	Game 2b	1.148929 ms
Game 2c	8.002301 ms	Game 2c	6.373157 ms	Game 2c	4.239638 ms
Game 2d	0.493537 ms	Game 2d	0.325945 ms	Game 2d	2.081608 ms
Game 3c	36.545318 ms	Game 3c	35.437264 ms	Game 3c	21.286546 ms
Game 3d	1.472726 ms	Game 3d	1.125151 ms	Game 3d	4.600061 ms
Game 3	29.143848 ms	Game 3	27.267114 ms	Game 3	8.164722 ms
Game 4a	6.431603 ms	Game 4a	2.663953 ms	Game 4a	2.394831 ms
Game 4b	2.338128 ms	Game 4b	0.688344 ms	Game 4b	2.44522 ms
Game 4c	0.225128 ms	Game 4c	0.046223 ms	Game 4c	0.028754 ms
Game 4d	6.151446 ms	Game 4d	2.337379 ms	Game 4d	5.364946 ms
Game 4	1.695825 ms	Game 4	1.430764 ms	Game 4	1.237886 ms
Game 5a	7.05817 ms	Game 5a	1.450956 ms	Game 5a	5.368428 ms
Game 5b	84.048738 ms	Game 5b	28.215689 ms	Game 5b	43.541702 ms
Game 5c	1.787185 ms	Game 5c	0.576556 ms	Game 5c	4.628053 ms
Game 5d	251.057677 ms	Game 5d	83.525374 ms	Game 5d	114.214773 ms
Game 6a	7.081844 ms	Game 6a	1.562035 ms	Game 6a	16.892639 ms
Game 6b	32.570335 ms	Game 6b	7.174414 ms	Game 6b	17.772473 ms
Game 6c	7.962399 ms	Game 6c	1.408088 ms	Game 6c	23.784703 ms
Game 6d	3.237656 ms	Game 6d	0.621669 ms	Game 6d	12.506237 ms
Game 7a	19.884909 ms	Game 7a	2.678508 ms	Game 7a	54.140063 ms
Game 7b	11.873086 ms	Game 7b	1.082131 ms	Game 7b	29.67486 ms
Game 7c	2.195627 ms	Game 7c	0.223138 ms	Game 7c	9.529289 ms
Game 7d	6.091123 ms	Game 7d	0.60704 ms	Game 7d	8.606841 ms
Game 8a	243.813714 ms	Game 8a	17.830424 ms	Game 8a	0.786969 ms
Game 8b	17.348234 ms	Game 8b	0.987863 ms	Game 8b	64.062096 ms
Game 8c	8.437889 ms	Game 8c	0.63709 ms	Game 8c	46.006536 ms
Game 8d	133.539403 ms	Game 8d	9.918959 ms	Game 8d	0.619124 ms
Game 9a	1344.320405 ms	Game 9a	51.749732 ms	Game 9a	248.430049 ms
Game 9b	19.089173 ms	Game 9b	0.79506 ms	Game 9b	85.026739 ms
Game 9c	7.582653 ms	Game 9c	0.286808 ms	Game 9c	4.321948 ms
Game 9d	207.958891 ms	Game 9d	8.256069 ms	Game 9d	0.561722 ms
Game 10a	204.418578 ms	Game 10a	4.034914 ms	Game 10a	292.491686 ms
Game 10b	10.469885 ms	Game 10b	0.248462 ms	Game 10b	48.732983 ms
Game 10c	1253.036232 ms	Game 10c	24.639604 ms	Game 10c	169.264165 ms
Game 10d	78.559265 ms	Game 10d	8.788762 ms	Game 10d	1042.297044 ms
Game 11a	76.917888 ms	Game 11a	0.962753 ms	Game 11a	536.153076 ms
Game 11b	1055.452837 ms	Game 11b	10.355926 ms	Game 11b	1.147536 ms
Game 11c	57.274867 ms	Game 11c	0.654151 ms	Game 11c	49.02516 ms
Game 11d	87.216235 ms	Game 11d	1.07811 ms	Game 11d	139.684456 ms
Game 12a	3368.892219 ms	Game 12a	17.111062 ms	Game 12a	1728.508003 ms
Game 12b	6525.556889 ms	Game 12b	32.849372 ms	Game 12b	3.109087 ms
Game 12c	32.864634 ms	Game 12c	0.346623 ms	Game 12c	105.602598 ms
Game 12d	2224.366375 ms	Game 12d	11.343972 ms	Game 12d	4076.344935 ms

	Incremental pre-solve local
Game 2a	0.184919 ms
Game 2b	0.353704 ms
Game 2c	1.550435 ms
Game 2d	1.58245 ms
Game 3c	6.207901 ms
Game 3d	2.427378 ms
Game 3	1.015665 ms
Game 4a	0.404769 ms
Game 4b	0.171154 ms
Game 4c	0.027441 ms
Game 4d	0.629948 ms
Game 4	0.169857 ms
Game 5a	0.199643 ms
Game 5b	1.068162 ms
Game 5c	0.090949 ms
Game 5d	2.029073 ms
Game 6a	0.843247 ms
Game 6b	0.547847 ms
Game 6c	3.744065 ms
Game 6d	9.462903 ms
Game 7a	1.300875 ms
Game 7b	0.291092 ms
Game 7c	0.077919 ms
Game 7d	5.607273 ms
Game 8a	0.797932 ms
Game 8b	0.463523 ms
Game 8c	0.640107 ms
Game 8d	0.62179 ms
Game 9a	1.517523 ms
Game 9b	0.304071 ms
Game 9c	0.103037 ms
Game 9d	0.566604 ms
Game 10a	0.284593 ms
Game 10b	0.142399 ms
Game 10c	1.76313 ms
Game 10d	918.460802 ms
Game 11a	0.733337 ms
Game 11b	1.165118 ms
Game 11c	0.122856 ms
Game 11d	0.424249 ms
Game 12a	0.948749 ms
Game 12b	3.106324 ms
Game 12c	104.64924 ms
Game 12d	0.809707 ms

References

- [1] J. F. Groote and M. R. Mousavi, *Modeling and Analysis of Communicating Systems*. The MIT Press, 2014.
- [2] M. ter Beek, E. de Vink, and T. Willemse, “Family-based model checking with mcr12,” in *Fundamental Approaches to Software Engineering* (M. Huisman and J. Rubin, eds.), Lecture Notes in Computer

- Science (including subseries Lecture Notes in Artificial Intelligence and Lecture Notes in Bioinformatics), (Germany), pp. 387–405, Springer, 2017.
- [3] A. Classen, M. Cordy, P.-Y. Schobbens, P. Heymans, A. Legay, and J.-F. Raskin, “Featured transition systems: Foundations for verifying variability-intensive systems and their application to ltl model checking,” *IEEE Transactions on Software Engineering*, vol. 39, pp. 1069–1089, 2013.
 - [4] J. Bradfield and I. Walukiewicz, *The mu-calculus and Model Checking*, pp. 871–919. Cham: Springer International Publishing, 2018.
 - [5] R. S. Streett and E. A. Emerson, “An automata theoretic decision procedure for the propositional mu-calculus,” *Information and Computation*, vol. 81, no. 3, pp. 249 – 264, 1989.
 - [6] M. J. Fischer and R. E. Ladner, “Propositional dynamic logic of regular programs,” *Journal of Computer and System Sciences*, vol. 18, no. 2, pp. 194 – 211, 1979.
 - [7] I. Wegener, *Branching Programs and Binary Decision Diagrams*. Society for Industrial and Applied Mathematics, 2000.
 - [8] R. E. Bryant, *Binary Decision Diagrams*, pp. 191–217. Cham: Springer International Publishing, 2018.
 - [9] *3. Ordered Binary Decision Diagrams (OBDDs)*, pp. 45–67.
 - [10] G. Birkhoff, *Lattice Theory*. No. v. 25, dl. 2 in American Mathematical Society colloquium publications, American Mathematical Society, 1940.
 - [11] A. Tarski, “A lattice-theoretical fixpoint theorem and its applications,” *Pacific J. Math.*, vol. 5, no. 2, pp. 285–309, 1955.
 - [12] W. Zielonka, “Infinite games on finitely coloured graphs with applications to automata on infinite trees,” *Theoretical Computer Science*, vol. 200, no. 1, pp. 135 – 183, 1998.
 - [13] I. Walukiewicz, “Monadic second-order logic on tree-like structures,” *Theoretical Computer Science*, vol. 275, no. 1, pp. 311 – 346, 2002.
 - [14] O. Friedmann and M. Lange, “Solving parity games in practice,” in *Automated Technology for Verification and Analysis* (Z. Liu and A. P. Ravn, eds.), (Berlin, Heidelberg), pp. 182–196, Springer Berlin Heidelberg, 2009.
 - [15] F. Bruse, M. Falk, and M. Lange, “The fixpoint-iteration algorithm for parity games,” *Electronic Proceedings in Theoretical Computer Science*, vol. 161, 08 2014.
 - [16] E. A. Emerson and C. Lei, “Model checking in the propositional mu-calculus,” tech. rep., Austin, TX, USA, 1986.
 - [17] O. Friedmann and M. Lange, “The pgsolver collection of parity game solvers version 3,” 2010.
 - [18] J. Kramer, J. Magee, M. Sloman, and A. Lister, “Conic: an integrated approach to distributed computer control systems,” 1983.
 - [19] M. Plath and M. Ryan, “Feature integration using a feature construct,” *Science of Computer Programming*, vol. 41, no. 1, pp. 53 – 84, 2001.