0.1 Neutrino Mixing

0.1.1 Mass Generation

As we saw in Eq. **??** , the neutrino fields ν_α only couple to the associated lepton fields ℓ_α, conserving the lepton number L_α. We will now introduce two separate extensions to this part of the Standard Model.

We introduce a right-handed neutrino field, ν_R. It has the usual properties of the conventional left-handed neutrino such as hypercharge and color zero. Moreover, since the electroweak gauge group $\mathrm{SU}(2)_L$ only couple to left-handed particles and right-handed antiparticles, it transforms as a singlet under the SM symmetry group $\mathrm{SU}(3)_\mathrm{C} \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$. This neutrino is *sterile* since it doesn't participate any of the SM interactions.

We extend the SM by adding a right-handed component to the Higgs-lepton Yukawa Lagrangian from Eq. **??** with neutrino Yukawa couplings $Y'^{\nu}_{\alpha\beta}$,

$$\mathcal{L}_H = -\left(\frac{v+H}{\sqrt{2}}\right)\left[\ell'_{\alpha L} Y'^{\ell}_{\alpha\beta} \ell'_{\beta R} + \nu'_{\alpha L} Y'^{\nu}_{\alpha\beta} \nu'_{\beta R}\right] \tag{1}$$

Similar to how we diagonalized the lepton Yukawa couplings $Y'^{\ell}_{\alpha\beta}$ in Eq. **??**, we diagonalize $Y'^{\nu}_{\alpha\beta}$ as

$$V^{\nu\dagger}_{\alpha k L} Y'^{\nu}_{\alpha\beta} V^{\nu}_{\beta j R} = Y^{\nu}_{kj} \,. \tag{2}$$

Now we introduce a crucial difference between the properties of the charged lepton and the neutrino fields. While the charged lepton flavor eigenstate was uniquely determined by its mass eigenstate, the neutrino flavor is a superposition of mass eigenstates. This is because neutrinos are indirectly detected via the observation of its associated charged lepton, so there is no requirement of neutrino flavor eigenstates to have a definite mass. The flavor of a neutrino is then, by definition, the flavor of the associated charged lepton. This is commonly introduced as giving the mass eigenstates Latin numerals and letters, while the flavor eigenstates stay as Greek letters.

So, let the neutrino field with chriality X be denoted ν_X, with components having Latin numerals to distinguish them from the flavour components, i.e

$$\nu_{kX} = V^{\nu\dagger}_{k\beta X} \nu'_{\beta X} \,. \tag{3}$$

The diagonalized Lagrangian now takes the form

$$\begin{aligned}
\mathcal{L}_H &= -\left(\frac{v+H}{\sqrt{2}}\right)\left[\ell'_{\alpha L} Y'^{\ell}_{\alpha\beta} \ell'_{\beta R} + \nu'_{\alpha L} Y'^{\nu}_{\alpha\beta} \nu'_{\beta R}\right] \\
&= -\left(\frac{v+H}{\sqrt{2}}\right)\Big[\ell'_{\alpha L} V^{\ell}_{\alpha\beta L} Y^{\ell}_{\alpha\beta} V^{\ell\dagger}_{\alpha\beta R} \ell'_{\beta R} \\
&\qquad\qquad + \nu'_{\alpha L} V^{\nu}_{\alpha k L} Y^{\nu}_{kj} V^{\nu\dagger}_{\beta j R} \nu'_{\beta R}\Big] \\
&= -\left(\frac{v+H}{\sqrt{2}}\right)\left[\ell^{\dagger}_{\alpha L} Y^{\ell}_{\alpha\beta} \ell_{\beta R} + \nu^{\dagger}_{kL} Y^{\nu}_{kj} \nu_{jR}\right] \\
&= -\left(\frac{v+H}{\sqrt{2}}\right)\left[\bar{\ell}_{\alpha L} Y^{\ell}_{\alpha\beta} \ell_{\beta R} + \bar{\nu}_{kL} Y^{\nu}_{kj} \nu_{jR}\right]
\end{aligned} \tag{4}$$

By construction, $Y^{\ell}_{\alpha\beta}$ and Y^{ν}_{kj} are diagonal, so we write them as $y^{\ell}_{\alpha}\delta_{\alpha\beta}$ and $y^{\nu}_{k}\delta_{kj}$ respectively, leaving the Lagrangian as

$$\begin{aligned}
\mathcal{L}_H &= -\left(\frac{v+H}{\sqrt{2}}\right)\left[\bar{\ell}_{\alpha L} y^{\ell}_{\alpha} \delta_{\alpha\beta} \ell_{\beta R} + \bar{\nu}_{kL} y^{\nu}_{k} \delta_{kj} \nu_{jR}\right] \\
&= -\left(\frac{v+H}{\sqrt{2}}\right)\left[\bar{\ell}_{\alpha L} y^{\ell}_{\alpha} \ell_{\alpha R} + \bar{\nu}_{kL} y^{\nu}_{k} \nu_{kR}\right] \\
&= -\left(\frac{v+H}{\sqrt{2}}\right)\left[y^{\ell}_{\alpha} \bar{\ell}_{\alpha L} \ell_{\alpha R} + y^{\nu}_{k} \bar{\nu}_{kL} \nu_{kR}\right]
\end{aligned} \tag{5}$$

Now, the Dirac neutrino field is

$$\nu_k = \nu_{kL} + \nu_{kR} \,. \tag{6}$$

Multiplying ν_k with its conjugate $\bar{\nu}_k$, we get

$$\begin{aligned}\bar{\nu}_k\nu_k &= \bar{\nu}_{kL}\nu_{kL} + \bar{\nu}_{kR}\nu_{kL} + \bar{\nu}_{kL}\nu_{kR} + \bar{\nu}_{kR}\nu_{kR} \\ &= \bar{\nu}_{kL}\nu_{kR} + \bar{\nu}_{kR}\nu_{kL} \\ &= \bar{\nu}_{kL}\nu_{kR} + \text{h.c.}\end{aligned} \tag{7}$$

The same calculation for the charged lepton field yields the same result for ℓ_k. Substituting this result and expanding the Higgs VEV into the fields gives us

$$\begin{aligned}\mathcal{L}_H &= -\left(\frac{v+H}{\sqrt{2}}\right)\left[y^\ell_\alpha \bar{\ell}_\alpha \ell_\alpha + y^\nu_k \bar{\nu}_k \nu_k\right] \\ &= -\frac{y^\ell_\alpha v}{\sqrt{2}}\bar{\ell}_\alpha\ell_\alpha - \frac{y^\nu_k v}{\sqrt{2}}\bar{\nu}_k\nu_k - \frac{y^\ell_\alpha}{\sqrt{2}}\bar{\ell}_\alpha\ell_\alpha H - \frac{y^\nu_k}{\sqrt{2}}\bar{\nu}_k\nu_k H\,.\end{aligned} \tag{8}$$

Thus, this extension to the SM generates neutrino masses by the Higgs mechanism, in the same fashion as with the charged leptons and the quarks:

$$m_k = \frac{y^\nu_k v}{\sqrt{2}} \tag{9}$$

Substituting the new transformation from Eq. 3 into the weak charged current, we get

$$\begin{aligned}j^\rho_L &= 2\bar{\nu}'_{\alpha L}\gamma^\rho \ell'_{\alpha L} \\ &= 2\bar{\nu}_{kL} V'^{\nu\dagger}_{k\alpha} V'^{\ell}_{\alpha\alpha}\gamma^\rho \ell_{\alpha L}\end{aligned} \tag{10}$$

Now, the current in Eq. 10 conserves lepton number, since the neutrino field with flavor α only couples to the lepton field with flavor α. Thus, neutrino interactions still conserve lepton number. However, the Higgs-lepton Yukawa Lagrangian in Eq. 5 violates lepton number conservation since it couples the charged lepton flavor α to the neutrino mass eigenstate k, which is a superposition of flavors. There is no transformation that leaves both the interaction and kinetic Lagrangian invariant.

Call $V'^{\nu\dagger}_{k\alpha} V'^{\ell}_{\alpha\alpha} = U^\dagger_{k\alpha}$. We will refer to the matrix U built by the components $U_{k\alpha}$ as the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix. We now have

$$j^\rho_L = 2\sum_\alpha\sum_k U^\dagger_{\alpha k}\bar{\nu}_{kL}\gamma^\rho\ell_{\alpha L}\,. \tag{11}$$

What form does this new matrix U take? By construction, it provides the unitary transformation between the flavor and mass bases. Any unitary 3×3 matrix can be parametrized using three mixing angles and six phases. However, not all phases affect the charged and weak currents and are thus not observable. Moreover, both the Lagrangian and the currents are invariant under global $U(1)$ transformations, leaving only one physical phase for us to observe. Thus we are down to four degrees of freedom in the three dimensional case: three mixing angles of the form $\sin(\theta_{ij})/\cos(\theta_{ij}) = s_{ij}/c_{ij}$, and one phase of the form $e^{i\delta}$.

We construct the PMNS matrix using the rotation matrixes from $SO(3)$ as

$$\begin{aligned}U &= R_{23}R_{13,\delta_{\text{CP}}}R_{12} \\ &= \begin{bmatrix}1 & 0 & 0\\ 0 & c_{23} & s_{23}\\ 0 & -s_{23} & c_{23}\end{bmatrix}\begin{bmatrix}c_{13} & 0 & s_{13}e^{-i\delta_{\text{CP}}}\\ 0 & 1 & 0\\ -s_{13}e^{i\delta_{\text{CP}}} & 0 & c_{13}\end{bmatrix}\begin{bmatrix}c_{12} & s_{12} & 0\\ -s_{12} & c_{12} & 0\\ 0 & 0 & 1\end{bmatrix}.\end{aligned} \tag{12}$$

The term δ_{CP} determines the degree of charge-parity violation. Assuming CPT symmetry, CP violation implies T violation, thus making reverse probabilities not equal to the original. In this work, we will assume CP invariance by always setting $\delta_{CP} = 0°$. Thus,

$$\begin{aligned}U &= \begin{pmatrix}1 & 0 & 0\\ 0 & c_{23} & s_{23}\\ 0 & -s_{23} & c_{23}\end{pmatrix}\begin{pmatrix}c_{13} & 0 & s_{13}\\ 0 & 1 & 0\\ -s_{13} & 0 & c_{13}\end{pmatrix}\begin{pmatrix}c_{12} & s_{12} & 0\\ -s_{12} & c_{12} & 0\\ 0 & 0 & 1\end{pmatrix} \\ &= \begin{pmatrix}c_{12}c_{13} & s_{12}c_{13} & s_{13}\\ -s_{12}c_{23} - c_{12}s_{23}s_{13} & c_{12}c_{23} - s_{12}s_{23}s_{13} & s_{23}c_{13}\\ s_{12}s_{23} - c_{12}c_{23}s_{13} & -c_{12}s_{23} - s_{12}c_{23}s_{13} & c_{23}c_{13}\end{pmatrix}.\end{aligned} \tag{13}$$

In this work, we will be using the best-fit values from [?] with the exception of the CP-violating phase δ_{CP}. The 90% CL values are

$$\begin{aligned}&\Delta m_{21}^2 = 7.42\times 10^{-5}\,\mathrm{eV}^2,\ \ \Delta m_{31}^2 = 2.517\times 10^{-3}\,\mathrm{eV}^2,\\ &\theta_{12} = 33.44^\circ,\quad \theta_{13} = 8.57^\circ,\quad \theta_{23} = 49.2^\circ,\quad \delta_{\mathrm{CP}} = 0\,.\end{aligned} \tag{14}$$

The PMNS matrix then has the numerical values

$$U = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3}\end{pmatrix} = \begin{pmatrix} 0.825 & 0.545 & 0.149 \\ -0.455 & 0.485 & 0.746 \\ 0.334 & -0.684 & 0.649\end{pmatrix}\,. \tag{15}$$

In our study, we will only use atmospheric neutrinos, of which ν_μ are the most abundant. We see that $U_{\mu\tau}$ is the largest $U_{\mu\beta}$ term, suggesting that ν_μ primarily mixes into ν_τ, at least in vacuum. It turns out that matter effects indeed preserves this ordering, making the $\nu_\mu \to \nu_\tau$ transition the most abundant. This means we will be able to stringently constrain parameters relating to $\mu\tau$ mixing.

0.1.2 Neutrino Propagation

Since we now know how the neutrino mass and flavor eigenstates combine, and have an expression for the flavor interaction with the neutrino's charged lepton partner, we are now ready to study the flavor oscillations themselves.

Now, the Fourier expansion of the field operator $\bar{\nu}_{kL}$ in the current of Eq. 11 contains creation operators $a^\dagger_{\nu_k}$ of massive neutrinos with mass m_k. This means that the summation over the mass index k constructs a flavor neutrino, which interacts with the charged lepton field $\ell_{\alpha L}$. In other words, the charged current generates a flavor neutrino ν_α, which is a superposition of the mass eigenstates ν_k with weights $U^\dagger_{\alpha k}$. In the ket-formalism, we express this as

$$|\nu_\alpha\rangle = \sum_k U^\dagger_{\alpha k} |\nu_k\rangle\ . \tag{16}$$

It is the mass eigenstates $|\nu_k\rangle$ that are eigenstates of the Hamiltonian, with eigenvalues

$$E_k = \sqrt{\vec{p}^2 + m_k^2}\,. \tag{17}$$

The solution to the time-dependent Schrödinger equation

$$i\frac{\mathrm{d}}{\mathrm{d}t}|\nu_k(t)\rangle = H_0 |\nu_k(t)\rangle\ , \tag{18}$$

where H_0 is the Hamiltonian and the subscript signifies that we are in vacuum.

The solution to Eq. 18 gives ut the time evolution in the form of plane wave solutions:

$$|\nu_k(t)\rangle = e^{-iE_k t}|\nu_k\rangle\ . \tag{19}$$

Inserting the plane wave solution into Eq. 16, we get

$$|\nu_\alpha(t)\rangle = \sum_k U^\dagger_{\alpha k} e^{-iE_k t}|\nu_k\rangle\ . \tag{20}$$

Now we know how to evolve and combine the mass eigenstates to form a flavor eigenstate, but how about the reverse? We swap the index $k \to j$ in Eq. 16 and multiply by $U_{\alpha k}$:

$$\begin{aligned}\sum_\alpha U_{\alpha k}|\nu_\alpha\rangle &= \sum_{\alpha,j} U_{\alpha k}U^\dagger_{\alpha j}|\nu_j\rangle\\ \sum_\alpha U_{\alpha k}|\nu_\alpha\rangle &= \sum_j \delta_{kj}|\nu_j\rangle\\ \sum_\alpha U_{\alpha k}|\nu_\alpha\rangle &= |\nu_k\rangle\ ,\end{aligned} \tag{21}$$

where we have used the unitarity of the leptonic mixing matrix. Eqs. 16 and 20 yield

$$\begin{aligned}
|\nu_\alpha(t)\rangle &= \sum_k U^\dagger_{\alpha k} e^{-iE_k t} |\nu_k\rangle \\
|\nu_\alpha(t)\rangle &= \sum_k U^\dagger_{\alpha k} e^{-iE_k t} \left(\sum_\beta U_{\beta k} |\nu_\beta\rangle \right) \\
|\nu_\alpha(t)\rangle &= \sum_{k,\beta} U^\dagger_{\alpha k} U_{\beta k} e^{-iE_k t} |\nu_\beta\rangle \ .
\end{aligned} \tag{22}$$

The probability of the flavor transition $\nu_\alpha \to \nu_\beta$ at time t is $|\langle \nu_\beta | \nu_\alpha(t)\rangle|^2$:

$$P_{\nu_\alpha \to \nu_\beta}(t) = \sum_{k,j} U^\dagger_{\alpha k} U_{\beta k} U^*_{\beta j} U_{\alpha j} e^{-i(E_k - E_j)t} \, . \tag{23}$$

We assume the neutrino masses m_k to be extremely small compared to their associated energies E_k. Thus, $v \sim 1$, and $|\vec{p}| \sim E$ making the energy-dispersion relation of Eq. 17 to first order:

$$\begin{aligned}
E_k &= \sqrt{\vec{p}^2 + m_k^2} \\
&= \vec{p}^2 \sqrt{1 + \frac{m_k^2}{\vec{p}^2}} \\
&\approx E + \frac{m_k^2}{2E}
\end{aligned} \tag{24}$$

Hence, the exponential can be simplified, and simplifying the notation $P_{\nu_\alpha \to \nu_\beta}(t) \to P_{\alpha\beta}(t)$ we get

$$P_{\alpha\beta}(t) = \sum_{k,j} U^\dagger_{\alpha k} U_{\beta k} U^\dagger_{\beta j} U_{\alpha j} e^{-i(m_k^2 - m_j^2)t/2E} \, . \tag{25}$$

Now, our approximation $v \approx 1$ implies $x \approx t$, thus

$$\begin{aligned}
P_{\alpha\beta}(x) &= \sum_{k,j} U^\dagger_{\alpha k} U_{\beta k} U^*_{\beta j} U_{\alpha j} e^{-i(m_k^2 - m_j^2)x/2E} \\
&= \sum_{k,j} U^\dagger_{\alpha k} U_{\beta k} U^*_{\beta j} U_{\alpha j} \exp\left(-i \frac{\Delta m^2_{kj} x}{2E} \right) ,
\end{aligned} \tag{26}$$

where we in the last step have defined the *mass squared-difference* $\Delta m^2_{kj} = m_k^2 - m_j^2$. Since the oscillation probability depends on this quantity rather than the individual masses, it is impossible to measure the mass states m_k through neutrino oscillations. Squaring the unitarity condition $\sum_k U_{\alpha k} U^\dagger_{\beta k} = \delta_{\alpha\beta}$ yields

$$\sum_k |U_{\alpha k}|^2 |U_{\beta k}|^2 = \delta_{\alpha\beta} - 2 \sum_{k>j} \mathrm{Re}[U^\dagger_{\alpha k} U_{\beta k} U_{\alpha j} U^*_{\beta j}] \tag{27}$$

As the CP phase is going to be assumed to be zero in this work, we will always assume the mixing matrix U to be real. We thus write

$$\begin{aligned}
P_{\alpha\beta} &= \sum_{k,j} U_{\alpha k} U_{\beta k} U_{\beta j} U_{\alpha j} \exp\left(-i\frac{\Delta m^2_{kj} x}{2E}\right) \\
&= \sum_{k} |U_{\alpha k}|^2 |U_{\beta k}|^2 + \\
&\qquad + \sum_{k\neq j} U_{\alpha k} U_{\beta k} U_{\beta j} U_{\alpha j} \exp\left(-i\frac{\Delta m^2_{kj} x}{2E}\right) \\
&= \delta_{\alpha\beta} - 2\sum_{k>j} U_{\alpha k} U_{\beta k} U_{\beta j} U_{\alpha j} \\
&\qquad + \sum_{k\neq j} U_{\alpha k} U_{\beta k} U_{\beta j} U_{\alpha j} \exp\left(-i\frac{\Delta m^2_{kj} x}{2E}\right) \\
&= \delta_{\alpha\beta} - 2\sum_{k>j} U_{\alpha k} U_{\beta k} U_{\beta j} U_{\alpha j} \\
&\qquad + 2\sum_{k>j} U_{\alpha k} U_{\beta k} U_{\beta j} U_{\alpha j} \exp\left(-i\frac{\Delta m^2_{kj} x}{2E}\right) \\
&= \delta_{\alpha\beta} - 2\sum_{k>j} U_{\alpha k} U_{\beta k} U_{\beta j} U_{\alpha j} \left[1 - \exp\left(-i\frac{\Delta m^2_{kj} x}{2E}\right)\right] \\
&= \delta_{\alpha\beta} - 2\sum_{k>j} U_{\alpha k} U_{\beta k} U_{\beta j} U_{\alpha j} \left[1 - \cos\left(\frac{\Delta m^2_{kj} x}{2E}\right)\right] \\
&= \delta_{\alpha\beta} - 2\sum_{k>j} U_{\alpha k} U_{\beta k} U_{\beta j} U_{\alpha j} \sin^2\left(\frac{\Delta m^2_{kj} x}{4E}\right),
\end{aligned} \tag{28}$$

which is the probability of neutrino vacuum oscillations. The calculation for antineutrinos yield the same result if one assumes the realness of the mixing matrix.

For our purposes, the probabilities in Eq. 28 are rather uninteresting, since they are only valid in vacuum. Despite this fact, this form elucidates an important aspect of neutrino oscillations. The mixing matrix elements determine the amplitude of the oscillations, while the mass-squared differences together with the ratio L/E determine the frequency.

0.1.3 Effective Matter Potentials

In this work, we are concerned about the interactions with the neutrino and Earth-like matter, i.e. electrons, protons, and neutrons. The possible interactions are shown in Fig. 1. The left panel shows that the only flavor that can go through charged current (CC) interactions is the electron flavor. This is because the Earth doesn't consist of any muons or tau particles. The right panel shows any neutrino flavor interaction via the neutral current (NC) with Earth-like matter[1], mediated by the neutral Z boson The interaction mediated by the W boson will give rise to a effective matter potential V_{CC}, while the Z boson is responsible for V_{NC}. Our task is now to find expressions for these.

We start with the effective Hamiltonian for the CC process. The Feynman rules for the left panel give us

$$H_{CC} = \frac{G_F}{\sqrt{2}} \left[\bar{\nu_e}\gamma^\rho \left(1-\gamma^5\right) e\right] \left[\bar{e}\,\gamma_\rho \left(1-\gamma^5\right) \nu_e\right] \tag{29}$$

By using the Fierz transformation

$$\mathcal{L}^{V-A}(\psi_1, \psi_2, \psi_3, \psi_4) = \mathcal{L}^{V-A}(\psi_1, \psi_4, \psi_3, \psi_2)\,, \tag{30}$$

[1]The Earth is entirely composed of electrons, protons, and neutrons. Thus, the fundamental particles composing Earth are electrons, and up and down quarks. We refer to this as *Earth-like matter.*

figures/w-boson.pdf

(a) Charged current weak interaction between an electron neutrino and an electron, mediated by either a W^+ or W^- boson.

figures/z-boson.pdf

(b) Neutral current weak interaction between a neutrino of any flavor and a Earth-like fermion, mediated by a neutral Z boson.

Figure 1: Feyman diagrams showing the two interactions that neutrinos participate in according to the Standard Model.

we can permute the terms inside the brackets, yielding

$$H_{CC} = \frac{G_F}{\sqrt{2}} \left[\bar{\nu_e}\gamma^\rho \left(1-\gamma^5\right) \nu_e\right] \left[\bar{e}\,\gamma_\rho \left(1-\gamma^5\right) e\right] . \tag{31}$$

Now, lets consider a finite volume V with electron states defined as

$$|e(p_e, h_e)\rangle = \frac{1}{2E_e V} a_e^{(h_e)\dagger}(p_e)\,|0\rangle \ , \tag{32}$$

i.e. using the creation operator $a_e^{(h_e)\dagger}(p_e)$ to create electron states from vacuum with momenta p_e, energy E_e, and helicity h_e. The density distribution of electrons in V is $f(E_e, T)$, which we normalize to the total number of electrons as we integrate out the momenta p_e:

$$\int \mathrm{d}p_e^3 f(E_e, T) = N_e V = n_e \tag{33}$$

Here, the electron density N_e will ultimately determine the strength of the effective matter potential. To obtain the average effective Hamiltonian, project it on the electron states in Eq. 32 and integrate over the density and sum over the helicities:

$$\begin{aligned}
\bar{H}^{CC} &= \int \mathrm{d}p_e^3 \,\langle e(p_e, h_e)| \times \frac{1}{2} \sum_{h_e} H f(E_e, T)\,|e(p_e, h_e)\rangle \\
&= \frac{G_F}{\sqrt{2}} \int \mathrm{d}p_e^3 \,\langle e(p_e, h_e)| \left[\bar{\nu_e}\gamma^\rho \left(1-\gamma^5\right) \nu_e\right] f(E_e, T) \times \frac{1}{2} \sum_{h_e} \left[\bar{e}(x)\,\gamma_\rho \left(1-\gamma^5\right) e(x)\right] |e(p_e, h_e)\rangle \\
&= \frac{G_F}{\sqrt{2}} \bar{\nu_e}\gamma^\rho \left(1-\gamma^5\right) \nu_e \int \mathrm{d}p_e^3 f(E_e, T) \times \frac{1}{2} \sum_{h_e} \langle e(p_e, h_e)|\,\bar{e}(x)\,\gamma_\rho \left(1-\gamma^5\right) e(x)\,|e(p_e, h_e)\rangle \ .
\end{aligned} \tag{34}$$

First, calculate the sum using trace technology

$$\begin{aligned}\frac{1}{2}\sum_{h_e}\langle e(p_e,h_e)|\,\bar{e}(x)\,\gamma_\rho\,(1-\gamma^5)\,e(x)\,|e(p_e,h_e)\rangle &= \frac{1}{4E_eV}\sum_{h_e}\bar{u}_e^{h_e}(p_e)\,\gamma_\rho\,(1-\gamma^5)\,u_e^{h_e}(p_e)\\ &= \frac{1}{4E_eV}\,\mathrm{Tr}\left[\sum_{h_e}\bar{u}_e^{h_e}(p_e)u_e^{h_e}(p_e)\,\gamma_\rho\,(1-\gamma^5)\right]\\ &= \frac{1}{4E_eV}\,\mathrm{Tr}\left[(\not{p}_e+m_e)\,\gamma_\rho\,(1-\gamma^5)\right]\\ &= \frac{(p_e)_\rho}{E_eV}\,. \end{aligned} \tag{35}$$

Eq. 34 now becomes

$$\langle H_{CC}\rangle = \frac{G_F}{\sqrt{2}E_eV}\bar{\nu}_e\,(1-\gamma^5)\,\nu_e\int \mathrm{d}p_e^3\,\not{p}_e\,f(E_e,T)\,. \tag{36}$$

Expand the integral, and use the fact that $\vec{p}_e$ is odd:

$$\begin{aligned}\int \mathrm{d}p_e^3\,\not{p}_e\,f(E_e,T) &= \int \mathrm{d}p_e^3\,f(E_e,T)(\gamma^0E_e-\vec{p}_e\cdot\vec{\gamma})\\ &= \int \mathrm{d}p_e^3\,f(E_e,T)\gamma^0E_e\\ &= \gamma_0E_eN_eV\,. \end{aligned} \tag{37}$$

Inserting this into Eq. 36, we have

$$\begin{aligned}\langle H_{CC}\rangle &= \frac{G_FN_e}{\sqrt{2}}\bar{\nu}_e\,(1-\gamma^5)\,\nu_e\gamma_0\\ &= \sqrt{2}G_FN_e\bar{\nu}_{Le}\gamma^0\nu_{Le}\,, \end{aligned} \tag{38}$$

where the projection operator $(1-\gamma^5)$ ensures that only the left-hand compontent of the neutrino fields interact weakly. Thus,

$$V_{CC} = \sqrt{2}G_FN_e\,,\quad H_{CC}\,|\nu_k\rangle = V_{CC}\,|\nu_k\rangle\,. \tag{39}$$

Here we see a crucial difference between the eigenvectors between the vacuum Hamiltonian defined in Eq. 16 and H_{CC}, namely that the CC (and NC) interactions happen in the flavor basis rather than in the mass basis. In other words, neutrinos propagate in their mass eigenstates, but interact in their flavor eigenstate. The mixing of mass eigenstates during propagation determines if the flavor eigenstate has oscillated or not. Thus, the expressions involving the matter potential does not need to be transformed by the PMNS matrix.

For neutral current, we replace the electron field $e(x)$ in Eq. 31 by the fermion field $f(x)$, and the projection operator $(1-\gamma^5)$ with $(g_V^f-g_A^f\gamma^5)$. Again, the γ^5 will cause the spacial component of p_f to disappear after integration, and the only difference between the average effective Hamiltonian for the neutral current is then the factor g_V^f:

$$V_{NC}^f = \sqrt{2}G_FN_Ag_V^f\,. \tag{40}$$

Summing over the fermions, and assuming electrical neutrality and equal abundance of protons and neutrons, we have

$$\begin{aligned}V_{NC} &= \sum_{f\in e,p,n}V_{NC}^f\\ &= \sqrt{2}G_FN_A\sum_{f\in e,p,n}g_V^f\\ &= \sqrt{2}G_FN_A\left[-\frac{1}{2}+2\sin^2(\theta_W)+\frac{1}{2}-2\sin^2(\theta_W)-\frac{1}{2}\right]\\ &= -\frac{1}{\sqrt{2}}G_FN_e\,, \end{aligned} \tag{41}$$

where the electrical neutrality condition allows us to simply sum the vectorial couplings together, cancelling the electron and proton contributions (and hence, also the Weinberg angle dependence).

Matter oscillations

Since only ν_e undergo CC interactions in Earth-like matter, the V_{CC} potential is zero for all other flavors. However, since all flavors undergo NC interactions the total matter potential in matrix form is:

$$V = \begin{bmatrix} V_{CC} + V_{NC} & 0 & 0 \\ 0 & V_{NC} & 0 \\ 0 & 0 & V_{NC} \end{bmatrix} = V_{CC}\,\delta_{\alpha e} + V_{NC}\,. \tag{42}$$

Just as in Eq. 18, we start with a Hamiltonian that solves the time-dependent Schrödinger equation. This time, let the Hamiltonian be

$$H = H_0 + H_I\,, \tag{43}$$

where H_0 is the Hamiltonian in vacuum, and H_I is our interaction Hamiltonian associated with our matter potentials. Let the wavefunction that describes the $\nu_\alpha \to \nu_\beta$ transition be

$$\langle \nu_\beta | \nu_\alpha(t) \rangle\ , \tag{44}$$

i.e. the evolution of the state of a neutrino emitted at $t = 0$ with flavor α to flavor β at time t.

Now using Eq. 16 and Eq. 39, we are ready to see what form our Hamiltonians take. Let us start with the vaccum Hamiltonian H_0, and act on its Schrödinger equation with $\langle \nu_\beta |$:

$$i\frac{\mathrm{d}}{\mathrm{d}t} |\nu_\alpha(t)\rangle = H_0 |\nu_\alpha(t)\rangle \implies i\frac{\mathrm{d}}{\mathrm{d}t}\psi_{\alpha\beta} = \langle \nu_\beta | H_0 |\nu_\alpha(t)\rangle\ . \tag{45}$$

Reminding ourselves that the vacuum Hamiltonian H_0 has eigenstates in the mass basis, we write the following expression where use the relations 16 and 21 to switch between the flavor and mass basis with the PMNS elements:

$$\begin{aligned} \langle \nu_\beta | H_0 &= \sum_k U_{\beta k} \langle \nu_k | H_0 \\ &= \sum_k U_{\beta k} E_k \langle \nu_k | \\ &= \sum_\eta \sum_k U_{\beta k} E_k U^*_{\eta k} \langle \nu_\eta |\ . \end{aligned} \tag{46}$$

Thus,

$$\begin{aligned} \langle \nu_\beta | H_0 |\nu_\alpha(t)\rangle &= \sum_\eta \sum_k U_{\beta k} E_k U^*_{\eta k} \langle \nu_\eta | \nu_\alpha(t)\rangle \\ &= \sum_\eta \sum_k U_{\beta k} E_k U^*_{\eta k} \psi_{\alpha\eta}(t)\,. \end{aligned} \tag{47}$$

Using the ultrarelativistic approximation from Eq. 24:

$$\begin{aligned} \sum_\eta \sum_k U_{\beta k} E_k U^*_{\eta k} \psi_{\alpha\eta}(t) &= \sum_\eta \sum_k U_{\beta k} \left(p + \frac{m_k^2}{2E}\right) U^*_{\eta k} \psi_{\alpha\eta}(x) \\ &= \sum_\eta \sum_k U_{\beta k} \left(p + \frac{m_k^2}{2E}\right) U^*_{\eta k} \psi_{\alpha\eta}(x)\,. \end{aligned} \tag{48}$$

Use the fact that $\sum_k m_k^2 = \sum m_1^2 +_k m_k^2 - m_1^2 = \sum_k m_1^2 + \Delta m_{k1}^2$ to pull out common terms out of the summation:

$$\begin{aligned} \sum_\eta \sum_k U_{\beta k} \left(p + \frac{m_k^2}{2E}\right) U^*_{\eta k} \psi_{\alpha\eta}(x) &= \sum_\eta \sum_k U_{\beta k} \left(p + \frac{m_1^2}{2E} + \frac{\Delta m_{k1}^2}{2E}\right) U^*_{\eta k} \psi_{\alpha\eta}(x) \\ &= \sum_\eta \sum_k \left(p + \frac{m_1^2}{2E}\right) U_{\beta k} U^*_{\eta k} \psi_{\alpha\eta}(x) + \sum_\eta \sum_k U_{\beta k} \frac{\Delta m_{k1}^2}{2E} U^*_{\eta k} \psi_{\alpha\eta}(x)\,. \end{aligned} \tag{49}$$

Unitarity gives $\sum_k U_{\beta k} U^*_{\eta k} = \delta_{\eta\beta}$, and the first term in the last step of Eq. 49 becomes

$$\sum_\eta \left(p + \frac{m_1^2}{2E}\right) \delta_{\beta\eta} \psi_{\alpha\eta}(x) = \left(p + \frac{m_1^2}{2E}\right) \psi_{\alpha\beta}(x) \,. \tag{50}$$

Our treatment of the interaction Hamiltonian is similar except for the fact that its eigenstates lie in the flavor basis, conviniently allowing us to letting it act directly on the flavor eigenstates:

$$\begin{aligned} \langle \nu_\beta | \, H_I &= V_\beta \, \langle \nu_\beta | \\ &= \delta_{\beta\eta} V_\beta \, \langle \nu_\eta | \; . \end{aligned} \tag{51}$$

Using Eq. 42, we rewrite this as

$$\begin{aligned} \delta_{\beta\eta} V_\beta \, \langle \nu_\eta | &= \delta_{\beta\eta} (V_{CC} \delta_{\beta e} + V_{NC}) \, \langle \nu_\eta | \\ &= V_{CC} \delta_{\beta\eta} \delta_{\beta e} \, \langle \nu_\eta | + V_{NC} \, \langle \nu_\beta | \\ \implies \langle \nu_\beta | \, H_I \, | \nu_\alpha \rangle &= V_{CC} \delta_{\beta\eta} \delta_{\beta e} \, \langle \nu_\eta | \nu_\alpha \rangle + V_{NC} \, \langle \nu_\beta | \nu_\alpha \rangle \\ &= V_{CC} \delta_{\beta\eta} \delta_{\beta e} \psi_{\alpha\eta} + V_{NC} \psi_{\alpha\beta} \end{aligned} \tag{52}$$

Now, combining Eq. 49 and Eq. 52, we have for the full Hamiltonian

$$\langle \nu_\beta | \, H \, | \nu_\alpha(x) \rangle = \left(p + \frac{m_1^2}{2E} + V_{NC}\right) \psi_{\alpha\beta}(x) + \sum_\eta \sum_k \left(U_{\beta k} \frac{\Delta m_{k1}^2}{2E} U^*_{\eta k} + V_{CC} \delta_{\beta\eta} \delta_{\eta e} \right) \psi_{\alpha\eta}(x) \tag{53}$$

In this form, we see that the term $p + \frac{m_1^2}{2E} + V_{NC}$ which does not affect the probability since it is a common termt to all flavor states. It can be rotated away. Thus

$$\begin{aligned} \langle \nu_\beta | \, H \, | \nu_\alpha(x) \rangle &= \sum_\eta \sum_k \left(U_{\beta k} \frac{\Delta m_{k1}^2}{2E} U^*_{\eta k} + V_{CC} \delta_{\beta\eta} \delta_{\eta e} \right) \psi_{\alpha\eta}(x) \\ &= i \frac{\mathrm{d}}{\mathrm{d}x} \psi_{\alpha\beta}(x) \,. \end{aligned} \tag{54}$$

If we form the vector

$$\Psi_\alpha = \begin{pmatrix} \psi_{\alpha e} \\ \psi_{\alpha\mu} \\ \psi_{\alpha\tau} \end{pmatrix} \,, \tag{55}$$

we can write the Schrödinger equation on matrix form ($i \frac{\mathrm{d}}{\mathrm{d}x} \Psi_\alpha = H_F \Psi_\alpha$) and compare it with Eq. 54 to see that the flavor Hamiltonian takes the form

$$\begin{aligned} H_F &= \frac{1}{2E} (U M^2 U^\dagger + A) \\ &= \frac{1}{2E} \left[U \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Delta m_{21}^2 & 0 \\ 0 & 0 & \Delta m_{31}^2 \end{pmatrix} U^\dagger \right] + \sqrt{2} G_F N_e \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \,. \end{aligned} \tag{56}$$

This is the three-flavor neutrino oscillation Hamiltonian that we will solve numerically to obtain the evolution of Ψ_α, whose squared components are the probabilities

$$\begin{aligned} P_\alpha = |\Psi_a|^2 &= \begin{pmatrix} |\psi_{\alpha e}|^2 \\ |\psi_{\alpha\mu}|^2 \\ |\psi_{\alpha\tau}|^2 \end{pmatrix} \\ &= \begin{pmatrix} P_{\alpha e} \\ P_{\alpha\mu} \\ P_{\alpha\tau} \end{pmatrix} \end{aligned} \tag{57}$$

For $N_e = 0$, i.e. in vacuum, these probabilities are identical to the ones that we analytically derived in Eq. 28. For $N_e \neq 0$, we have closed form solutions, but they are not considered further here.

We now need to know how the electrons are distributed within the Earth. The Preliminary Earth Reference Model [?] gives us spherically symmetric piecewise polynomials for the Earth density in gcm^3

figures/potential.pdf

Figure 2: *Left panel:* Earth density asa function of distance from the core from the PREM [?]. *Right panel:* V_{CC} using the PREM density 0.5 electrons per nucleon.

shown in the left panel of Fig. 2. We note a steep discontinuity at 3480 km where the density is nearly halved. This is the core-mantle boundary, and will be visible in our oscillations.

Using a value of $Y = 0.5$ electron per nucleon, we express the matter potential as

$$\begin{aligned} V_{CC} &= \sqrt{2} G_F N_e = \sqrt{2} G_F Y N_A \, \rho \\ &= 3.8 \times 10^{-23} \, \mathrm{eV} \, \mathrm{gcm}^{-3} \, , \end{aligned} \tag{58}$$

a low number due to the smallness of G_F. This is plotted in the right panel of Fig. 2

Now, solving the Schrödinger equation with the Hamiltonian from Eq. 56 with the matter potential from the PREM, we obtain the nine combinations of probabilities through the Earth diameter. We note that due to our assumption of CP-invariance (and thus, T-invariance), the probabilities $P_{\alpha\beta}$ and $P_{\beta\alpha}$ are identically equivalent. The result for GeV neutrinos is shown in Fig. 3.

Now we need to incorporate the *zenith angle*, defined as the angle between the neutrino direction of travel and south. This way, neutrinos traveling through the entire diameter of the Earth up through the South Pole are defined as 'up-going', while neutrinos that start at the South Pole are 'down-going'. We will refer to the quantity $\cos(\theta_z)$ as the *zenith*, while θ_z is the *zenith angle*.

Since we are interested in the matter effects, we reserve our IceCube study to up-going neutrinos, i.e. neutrinos with zenith $-1 <= \cos(\theta_z) <= 0$. We now supplement our probability grid from Fig. 3 with the zenith dimension, allowing us to fully see the Earth matter effect on the oscillations in full. This is shown in Fig. 4.

The core-mantle boundary from Fig.2 is clearly displayed at $\cos(\theta_z) = -0.83$ as a sharp discontinuity for all flavors.

figures/oscillations.pdf

Figure 3

figures/oscillograms.pdf

Figure 4