TMA4140 - Homework Exercise Set 7

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Contents

1 Section 8.1

1.1 Exercise 11

a) Find a recurrence relation for the number of ways to climb n stairs if the person climbing the stairs can take one stair or two stairs at a time.

$a_n = a_{n-1} + a_{n-2}$ when $n > 2$

b) What are the initial conditions?

To reach the first step there is only one solution, taking one single step $a_1 = 1$
To reach the second step, we can take either two single steps or one double step. $a_2 = 2$

c) In how many ways can this person climb a flight of eight stairs?

A lot...

$$\begin{aligned}
1+1+1+1+1+1+1+1&=8\\
2+1+1+1+1+1+1&=8\\
1+2+1+1+1+1+1&=8\\
1+1+2+1+1+1+1&=8\\
1+1+1+2+1+1+1&=8\\
1+1+1+1+2+1+1&=8\\
1+1+1+1+1+2+1&=8\\
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1+2+2+1+2&=8\\
1+2+1+2+2&=8\\
1+1+2+2+2&=8\\
2+2+2+2&=8
\end{aligned} \tag{1}$$

I missed one above, but easier to calculate it... $a_n = a_{n-1} + a_{n-2}$ when $n > 2$

$$\begin{aligned}
&a_n = a_{n-1} + a_{n-2} | n > 2 \\
&a_1 = 1, a_2 = 2 \\
&a_3 = a_{3-1} + a_{3-2} = a_2 + a_1 = 2 + 1 = 3 \\
&a_4 = a_{4-1} + a_{4-2} = a_3 + a_2 = 3 + 2 = 5 \\
&a_5 = a_{5-1} + a_{5-2} = a_4 + a_3 = 5 + 3 = 8 \\
&a_6 = a_{6-1} + a_{6-2} = a_5 + a_4 = 8 + 5 = 13 \\
&a_7 = a_{7-1} + a_{7-2} = a_6 + a_5 = 13 + 8 = 21 \\
&a_8 = a_{8-1} + a_{8-2} = a_7 + a_6 = 21 + 13 = 34
\end{aligned} \tag{2}$$

There are 34 possible ways to climb the flight of 8 stairs using only 1 or 2 steps at a time.

1.2 Exercise 20

A bus driver pays all tolls, using only nickels and dimes, by throwing one coin at a time into the mechanical toll collector.
a) Find a recurrence relation for the number of different ways the bus driver can pay a toll of n cents (where the order in which the coins are used matters).
1 Nickel = 5 cents, 1 Dime = 10 cents.
Let a_n be the number of ways the busdriver can pay a toll of n cents.
If last coin Nickel a_{n-5}, If last coin Dime a_{n-10}
Then the recurrence relation can be given as... $a_n = a_{n-5} + a_{n-10} | n \geq 10$
And since nickels and dimes are of multiple of 5 we can further write it as ...
$a_{5n} = a_{5(n-1)} + a_{5(n-2)} | n \geq 2$. Where the initial conditions are $a_0 = 1; a_5 = 1;$
b) In how many different ways can the driver pay a toll of 45 cents? We

must calculate a_{45}

$$\begin{aligned} a_{5n} &= a_{5(n-1)} + a_{5(n-2)} \\ a_0 &= 1; a_5 = 1 \\ a_{10} &= 2 \\ a_{15} &= 3 \\ a_{20} &= 5 \\ a_{25} &= 8 \\ a_{30} &= 13 \\ a_{35} &= 21 \\ a_{40} &= 34 \\ a_{45} &= 55 \end{aligned} \tag{3}$$

2 Section 8.2

2.1 TODO: Exercise 3

Solve these recurrence relations together with the initial conditions given.
c) $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2, a_0 = 1, a_1 = 0$
Comparing the given recurrence with the general relation we get $C_1 = 5, c_2 = -6$ and rest of the coefficients are 0. Our characteristic equation will be $r^2 = 5r^1 - 6r^0$, so...

$$\begin{aligned} r^2 - 5r^1 + 6r^0 &= 0 \\ r^2 - 5r + 6r &= 0 \\ r^2 - 3r - 2r + 6 &= 0 \\ (r-3)(r-2) &= 0 \end{aligned} \tag{4}$$

That is, $r = 3, 2$. Solution will be of the form $a_n = a_1 r_2^n$. Setting in the value of r and use the initial condition. $n = 0, 1...$

$$\begin{aligned} a_n &= a_1 r_1^n \\ a_n &= a_1(2)^n + a_2(3)^n \end{aligned} \tag{5}$$

When $n = 0$, then...

$$\begin{aligned} a_0 &= a_1(2)^0 + a_2(3)^0 \\ 1 &= a_1 + a_2 \\ a_1 &= 1 - a_2 \end{aligned} \tag{6}$$

When $n = 1$, then...

$$\begin{aligned} a_1 &= a_1(2)^1 + a_2(3)^1 \\ 0 &= 2a_1 + 3a_2 \end{aligned} \tag{7}$$

Put the value of $a_1 = 1 - a_2$ in the equation $2a_1 + 3a_2 = 0$...

$$\begin{aligned} 2(1 - a_2) + 3a_2 &= 0 \\ 2 - 2a_2 + 3a_2 &= 0 \\ a_2 &= -2 \end{aligned} \tag{8}$$

Put the value of $a_2 = -2$ in the equation $a_1 = 1 - a_2$ and we get $a_1 = 3$. Put the value of a_1, a_2 in the equation $a_n = a_1 r_1^n + a_2 r_2^n$ to get the final solution $a_n = 3 \times (2)^n - 2 \times (3)^n$.

d) $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2, a_0 = 6, a_1 = 8$

Comparing the given recurrence relation with the general relation we get $c_1 = 4$ $c_2 = -4$ and the rest of the coefficients are 0. The corresponding characteristic equation will be $r^2 = 4r^1 - 4r^0$. So,

$$\begin{aligned} r^2 - 4r + 4 &= 0 \\ r^2 - 2r - 2r + 4 &= 0 \\ (r - 2)(r - 2) &= 0 \end{aligned} \tag{9}$$

Since we have a repeated root, the solution will be of the form $a_n = a_1 r_1^n + a_2 n r_2^n$ **e)** $a_n = -4a_{n-1} - 4a_{n-2}$ for $n \geq 2, a_0 = 0, a_1 = 1$

g) $a_n = \frac{a_{n-2}}{4}$ for $n \geq 2, a_0 = 1, a_1 = 0$

2.2 Todo: Exercise 6

How many different messages can be transmitted in n microseconds using three different signals if one signal requires 1 microsecond for transmittal, the other two signals require 2 microseconds each for transmittal, and a signal in a message is followed immediately by the next signal?

2.3 Todo: Exercise 11

The Lucas numbers satisfy the recurrence relation

$$L_n = L_{n-1} + L_{n-2} \tag{10}$$

and the initial conditions $L_0 = 2$ and $L_1 = 1$.

a) Show that $L_n = f_{N-1} + f_{n+1}$ for $n = 2, 3, ...,$ where f_n is the nth Fibonacci number.

b) Find an explicit formula for the Lucas numbers.

2.4 Todo: Exercise 42

Show that if $a_n = a_{n-1} + a_{n-2}, a_0 = s$ and $a1 = t$, where s and t are constants, then $a_n = sf_{n-1} + tf_n$ for all positive integers n.

3 Section 5.1

3.1 Exercise 4

Let $P(n)$ be the statement that $1^3 + 2^3 + \cdots + n^3 = (n(n+1)/2)^2$ for the positive integer n.

$P(1) : 1^3 = (\frac{1(1+1)}{2})^2$ Basis step: $P(1) : 1^3 = (\frac{1(1+1)}{2})^2 = 1$
$P(1)$ is true, which completes the basis step of a proof by induction for $P(k)$

The inductive hypothesis consists of two parts:
- $P(b)$ holds true
- $P(k) \to P(k+1)$ holds true

Then $P(k), \quad \forall k > b$

In other words, if P is true for the first step, and P holds true for an arbitrary step implies P holds true for the next step, then P holds true for all steps.

You need to prove the first step $P(b)$, and then you need to prove $P(k) \to P(k+1)$

Basis step:

$$1^3 = (1(1+1)/2)^2 = 1 \qquad (11)$$

LHS = RHS

Inductive step:

$$\sum_{n=1}^{k} n^3 = \Big(\frac{k(k+1)}{2}\Big)^2 \tag{12}$$

We assume $P(k)$ is true for an arbitrary integer k. We can replace k with $k+1$. Out goal is to show that if $P(k)$ holds then $P(k+1)$ must hold

$$\sum_{n=1}^{k+1} n^3 = \Big(\frac{(k+1)((k+1)+1)}{2}\Big)^2 \tag{13}$$

$$\sum_{n=1}^{k} n^3 + (k+1)^3 = \Big(\frac{(k+1)(k+2)}{2}\Big)^2 \tag{14}$$

$$= \Big(\frac{(k^2+3k+2)}{2}\Big)^2 \tag{15}$$

$$= \Big(\frac{(k(k+1))}{2} + (k+1)\Big)^2 \tag{16}$$

$$\sum_{n=1}^{k} n^3 + (k+1)^3 = \Big(\frac{(k(k+1))}{2}\Big)^2 + k(k+1)(k+1) + (k+1)^2 \tag{17}$$

We subtract $\sum_{n=1}^{k} n^3 = \Big(\frac{k(k+1)}{2}\Big)^2$ from the equation and have

$$(k+1)^3 = k(k+1)^2 + (k+1)^2 \tag{18}$$

$$(k+1)^3 = (k+1)(k+1)^2 = (k+1)^3 \tag{19}$$

LHS = RHS. This means that if $P(k)$ holds true, then $P(k+1)$ must also be true, which by the inductive hypothesis means that $\forall k \geq 1, \quad P(k)$ holds true

$$\tag{20}$$

3.2 Exercise 6

Prove that $s1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$ whenever n is a positive integer.

Basis step:

$$1 \cdot 1! = (1+1)! - 1 \tag{21}$$
$$1 = (2)! - 1 = 2 - 1 = 1 \tag{22}$$

LHS = RHS, so $P(1)$ holds true

Inductive step:

$$\sum_{n=1}^{k} n \cdot n! = (k+1)! - 1 \tag{23}$$

We assume that $P(k)$ holds for all $k > 1$, and replace k with $k+1$

$$\sum_{n=1}^{k+1} n \cdot n! = ((k+1)+1)! - 1 \tag{24}$$
$$(k+1) \cdot (k+1)! + \sum_{n=1}^{k} n \cdot n! = (k+2)! - 1 \tag{25}$$
$$(k+1) \cdot (k+1)! + \sum_{n=1}^{k} n \cdot n! = (k+2)(k+1)! - 1 \tag{26}$$

We subtract $\sum_{n=1}^{k} n \cdot n! = (k+1)! - 1$ from the equation and get

$$(k+1) \cdot (k+1)! + = (k+2)(k+1)! - 1 - \big((k+1)! - 1\big) \tag{27}$$
$$(k+1) \cdot (k+1)! + = (k+2)(k+1)! - (k+1)! \tag{28}$$
$$(k+1) \cdot (k+1)! + = k(k+1)! + 2(k+1)! - (k+1)! \tag{29}$$
$$(k+1) \cdot (k+1)! + = k(k+1)! + (k+1)! \tag{30}$$
$$(k+1) \cdot (k+1)! + = (k+1) \cdot (k+1)! \tag{31}$$

LHS = RHS, so $P(k)$ must hold for all $k > 1$

$$\tag{32}$$

3.3 Exercise 14

Prove that for every positive integer n, $\sum_{k=1}^{n} k2^k = (n-1)2^{n+1} + 2$.

$P(k)$:

$$\sum_{n=1}^{k} n2^n = (k-1)2^{k+1} + 2 \tag{33}$$

Basis step:

$$1 \cdot 2^1 = (1-1)2^2 1 + 1 + 2 = 2 \tag{34}$$

LHS = RHS, so $P(1)$ holds true

Induction step:

Replacing n with $k+1$ in $\sum_{n=1}^{k} n2^n = (k-1)2^{k+1} + 2$ gives us

$$\sum_{n=1}^{k+1} n2^n = ((k+1)-1)2^{(k+1)+1} + 2 \tag{35}$$

$$(k+1)2^{k+1} + \sum_{n=1}^{k} n2^n = k2^{k+2} + 2 \tag{36}$$

$$(k+1)2^{k+1} + \sum_{n=1}^{k} n2^n = 2k2^{k+1} + 2 \tag{37}$$

We subtract $\sum_{n=1}^{k} n2^n = (k-1)2^{k+1} + 2$ from the equation and get

$$(k+1)2^{k+1} = 2k2^{k+1} + 2 - \big((k-1)2^{k+1} + 2\big) \tag{38}$$

$$(k+1)2^{k+1} = 2k2^{k+1} - (k-1)2^{k+1} \tag{39}$$

$$(k+1)2^{k+1} = (2k-(k-1))2^{k+1} \tag{40}$$

$$(k+1)2^{k+1} = (k+1)2^{k+1} \tag{41}$$

LHS = RHS, so $P(k)$ must hold for all $k > 1$

$$\tag{42}$$