TMA4140 - Homework Exercise Set 7

Henry S. Sjøen & Toralf Tokheim

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1 Section 8.1

1.1 Exercise 11

a) Find a recurrence relation for the number of ways to climb n stairs if the person climbing the stairs can take one stair or two stairs at a time.
$a_n = a_{n-1} + a_{n-2}$ when $n > 2$
b) What are the initial conditions?
To reach the first step there is only one solution, taking one single step $a_1 = 1$
To reach the second step, we can take either two single steps or one double step. $a_2 = 2$
c) In how many ways can this person climb a flight of eight stairs?

A lot...

$$
\begin{aligned}
1+1+1+1+1+1+1+1&=8\\
2+1+1+1+1+1+1&=8\\
1+2+1+1+1+1+1&=8\\
1+1+2+1+1+1+1&=8\\
1+1+1+2+1+1+1&=8\\
1+1+1+1+2+1+1&=8\\
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1+2+2+1+2&=8\\
1+2+1+2+2&=8\\
1+1+2+2+2&=8\\
2+2+2+2&=8
\end{aligned}
\tag{1}
$$

I missed one above, but easier to calculate it... $a_n = a_{n-1} + a_{n-2}$ when $n > 2$

$$\begin{aligned}
&a_n = a_{n-1} + a_{n-2} | n > 2 \\
&a_1 = 1, a_2 = 2 \\
&a_3 = a_{3-1} + a_{3-2} = a_2 + a_1 = 2 + 1 = 3 \\
&a_4 = a_{4-1} + a_{4-2} = a_3 + a_2 = 3 + 2 = 5 \\
&a_5 = a_{5-1} + a_{5-2} = a_4 + a_3 = 5 + 3 = 8 \\
&a_6 = a_{6-1} + a_{6-2} = a_5 + a_4 = 8 + 5 = 13 \\
&a_7 = a_{7-1} + a_{7-2} = a_6 + a_5 = 13 + 8 = 21 \\
&a_8 = a_{8-1} + a_{8-2} = a_7 + a_6 = 21 + 13 = 34
\end{aligned} \tag{2}$$

There are 34 possible ways to climb the flight of 8 stairs using only 1 or 2 steps at a time.

1.2 Exercise 20

A bus driver pays all tolls, using only nickels and dimes, by throwing one coin at a time into the mechanical toll collector.
a) Find a recurrence relation for the number of different ways the bus driver can pay a toll of n cents (where the order in which the coins are used matters).
1 Nickel = 5 cents, 1 Dime = 10 cents.
Let a_n be the number of ways the busdriver can pay a toll of n cents.
If last coin Nickel a_{n-5}, If last coin Dime a_{n-10}
Then the recurrence relation can be given as... $a_n = a_{n-5} + a_{n-10} | n \geq 10$
And since nickels and dimes are of multiple of 5 we can further write it as ...
$a_{5n} = a_{5(n-1)} + a_{5(n-2)} | n \geq 2$. Where the initial conditions are $a_0 = 1; a_5 = 1;$
b) In how many different ways can the driver pay a toll of 45 cents? We

must calculate a_{45}

$$\begin{aligned} a_{5n} &= a_{5(n-1)} + a_{5(n-2)} \\ a_0 &= 1; a_5 = 1 \\ a_{10} &= 2 \\ a_{15} &= 3 \\ a_{20} &= 5 \\ a_{25} &= 8 \\ a_{30} &= 13 \\ a_{35} &= 21 \\ a_{40} &= 34 \\ a_{45} &= 55 \end{aligned} \tag{3}$$

2 Section 8.2

2.1 TODO: Exercise 3

Solve these recurrence relations together with the initial conditions given.
c) $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2, a_0 = 1, a_1 = 0$
Comparing the given recurrence with the general relation we get $C_1 = 5, c_2 = -6$ and rest of the coefficients are 0. Our characteristic equation will be $r^2 = 5r^1 - 6r^0$, so...

$$\begin{aligned} r^2 - 5r^1 + 6r^0 &= 0 \\ r^2 - 5r + 6r &= 0 \\ r^2 - 3r - 2r + 6 &= 0 \\ (r-3)(r-2) &= 0 \end{aligned} \tag{4}$$

That is, $r = 3, 2$. Solution will be of the form $a_n = a_1 r_2^n$. Setting in the value of r and use the initial condition. $n = 0, 1...$

$$\begin{aligned} a_n &= a_1 r_1^n \\ a_n &= a_1(2)^n + a_2(3)^n \end{aligned} \tag{5}$$

When $n = 0$, then...

$$\begin{aligned} a_0 &= a_1(2)^0 + a_2(3)^0 \\ 1 &= a_1 + a_2 \\ a_1 &= 1 - a_2 \end{aligned} \tag{6}$$

When $n = 1$, then...

$$\begin{aligned} a_1 &= a_1(2)^1 + a_2(3)^1 \\ 0 &= 2a_1 + 3a_2 \end{aligned} \tag{7}$$

Put the value of $a_1 = 1 - a_2$ in the equation $2a_1 + 3a_2 = 0$...

$$\begin{aligned} 2(1 - a_2) + 3a_2 &= 0 \\ 2 - 2a_2 + 3a_2 &= 0 \\ a_2 &= -2 \end{aligned} \tag{8}$$

Put the value of $a_2 = -2$ in the equation $a_1 = 1 - a_2$ and we get $a_1 = 3$. Put the value of a_1, a_2 in the equation $a_n = a_1 r_1^n + a_2 r_2^n$ to get the final solution $a_n = 3 \times (2)^n - 2 \times (3)^n$.

d) $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2, a_0 = 6, a_1 = 8$

Comparing the given recurrence relation with the general relation we get $c_1 = 4$ $c_2 = -4$ and the rest of the coefficients are 0. The corresponding characteristic equation will be $r^2 = 4r^1 - 4r^0$. So,

$$\begin{aligned} r^2 - 4r + 4 &= 0 \\ r^2 - 2r - 2r + 4 &= 0 \\ (r - 2)(r - 2) &= 0 \end{aligned} \tag{9}$$

Since we have a repeated root, the solution will be of the form $a_n = a_1 r_1^n + a_2 n r_2^n$ **e)** $a_n = -4a_{n-1} - 4a_{n-2}$ for $n \geq 2, a_0 = 0, a_1 = 1$

g) $a_n = \frac{a_{n-2}}{4}$ for $n \geq 2, a_0 = 1, a_1 = 0$

2.2 Todo: Exercise 6

How many different messages can be transmitted in n microseconds using three different signals if one signal requires 1 microsecond for transmittal, the other two signals require 2 microseconds each for transmittal, and a signal in a message is followed immediately by the next signal?

2.3 Todo: Exercise 11

The Lucas numbers satisfy the recurrence relation

$$L_n = L_{n-1} + L_{n-2} \tag{10}$$

and the initial conditions $L_0 = 2$ and $L_1 = 1$.

a) Show that $L_n = f_{N-1} + f_{n+1}$ for $n = 2, 3, ...,$ where f_n is the nth Fibonacci number.

b) Find an explicit formula for the Lucas numbers.

2.4 Todo: Exercise 42

Show that if $a_n = a_{n-1} + a_{n-2}, a_0 = s$ and $a1 = t$, where s and t are constants, then $a_n = sf_{n-1} + tf_n$ for all positive integers n.

3 Section 5.1

3.1 Todo: Exercise 4

Let $P(n)$ be the statement that $1^3 + 2^3 + \cdots + n^3 = (n(n+1)/2)^2$ for the positive integer n.

a) What is the statement $P(1)$?

b) Show that $P(1)$ is $true$, completing the basis step of the proof.

c) What is the inductive hypothesis?

d) What do you need to prove in the inductive step?

e) Complete the inductive step, identifying where you use the inductive hypothesis.

f) Explain why these steps show that this formula is $true$ whenever n is a positive integer. $P(1) : 1^3 = (\frac{1(1+1)}{2})^2$ Basis step: $P(1) : 1^3 = (\frac{1(1+1)}{2})^2 = 1$

$P(1)$ is true, which completes the basis step of a proof by induction for $P(k)$

The inductive hypothesis consists of two parts:

- $P(b)$ holds true
- $P(k) \to P(k+1)$ holds true

Then $P(k), \quad \forall k > b$

In other words, if P is true for the first step, and P holds true for an arbitrary step implies P holds true for the next step, then P holds true for all steps.

You need to prove the first step $P(b)$, and then you need to prove $P(k) \to P(k+1)$

Basis step:

$$1^3 = (1(1+1)/2)^2 = 1 \tag{11}$$

LHS = RHS

Inductive step:

$$\sum_{n=1}^{k} n^3 = \Big(\frac{k(k+1)}{2}\Big)^2 \tag{12}$$

We assume $P(k)$ is true for an arbitrary integer k. We can replace k with $k+1$. Out goal is to show that if $P(k)$ holds then $P(k+1)$ must hold

$$\sum_{n=1}^{k+1} n^3 = \Big(\frac{(k+1)((k+1)+1)}{2}\Big)^2 \tag{13}$$

$$\sum_{n=1}^{k} n^3 + (k+1)^3 = \Big(\frac{(k+1)(k+2)}{2}\Big)^2 \tag{14}$$

$$= \Big(\frac{(k^2+3k+2)}{2}\Big)^2 \tag{15}$$

$$= \Big(\frac{(k(k+1))}{2} + (k+1)\Big)^2 \tag{16}$$

$$\sum_{n=1}^{k} n^3 + (k+1)^3 = \Big(\frac{(k(k+1))}{2}\Big)^2 + k(k+1)(k+1) + (k+1)^2 \tag{17}$$

We subtract $\sum_{n=1}^{k} n^3 = \Big(\frac{k(k+1)}{2}\Big)^2$ from the equation and have

$$(k+1)^3 = k(k+1)^2 + (k+1)^2 \tag{18}$$

$$(k+1)^3 = (k+1)(k+1)^2 = (k+1)^3 \tag{19}$$

LHS = RHS. This means that if $P(k)$ holds true, then $P(k+1)$ must also be true, which by the inductive hypothesis means that $\forall k \geq 1, \quad P(k)$ holds true

$$\tag{20}$$

3.2 Exercise 6

Prove that $s1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$ whenever n is a positive integer.

Basis step:

$$1 \cdot 1! = (1+1)! - 1 \tag{21}$$
$$1 = (2)! - 1 = 2 - 1 = 1 \tag{22}$$

LHS = RHS, so $P(1)$ holds true

Inductive step:

$$\sum_{n=1}^{k} n \cdot n! = (k+1)! - 1 \tag{23}$$

We assume that $P(k)$ holds for all $k > 1$, and replace k with $k+1$

$$\sum_{n=1}^{k+1} n \cdot n! = ((k+1)+1)! - 1 \tag{24}$$
$$(k+1) \cdot (k+1)! + \sum_{n=1}^{k} n \cdot n! = (k+2)! - 1 \tag{25}$$
$$(k+1) \cdot (k+1)! + \sum_{n=1}^{k} n \cdot n! = (k+2)(k+1)! - 1 \tag{26}$$

We subtract $\sum_{n=1}^{k} n \cdot n! = (k+1)! - 1$ from the equation and get

$$(k+1) \cdot (k+1)! + = (k+2)(k+1)! - 1 - \big((k+1)! - 1\big) \tag{27}$$
$$(k+1) \cdot (k+1)! + = (k+2)(k+1)! - (k+1)! \tag{28}$$
$$(k+1) \cdot (k+1)! + = k(k+1)! + 2(k+1)! - (k+1)! \tag{29}$$
$$(k+1) \cdot (k+1)! + = k(k+1)! + (k+1)! \tag{30}$$
$$(k+1) \cdot (k+1)! + = (k+1) \cdot (k+1)! \tag{31}$$

LHS = RHS, so $P(k)$ must hold for all $k > 1$

$$\tag{32}$$

3.3 DONE?: Exercise 14

Prove that for every positive integer n, $\sum_{k=1}^{n} k2^k = (n-1)2^{n+1} + 2$.

$P(k)$:

$$\sum_{n=1}^{k} n2^n = (k-1)2^{k+1} + 2 \tag{33}$$

Basis step:

$$1 \cdot 2^1 = (1-1)2^2 1 + 1 + 2 = 2 \tag{34}$$

LHS = RHS, so $P(1)$ holds true

Induction step:

Replacing n with $k+1$ in $\sum_{n=1}^{k} n2^n = (k-1)2^{k+1} + 2$ gives us

$$\sum_{n=1}^{k+1} n2^n = ((k+1)-1)2^{(k+1)+1} + 2 \tag{35}$$

$$(k+1)2^{k+1} + \sum_{n=1}^{k} n2^n = k2^{k+2} + 2 \tag{36}$$

$$(k+1)2^{k+1} + \sum_{n=1}^{k} n2^n = 2k2^{k+1} + 2 \tag{37}$$

We subtract $\sum_{n=1}^{k} n2^n = (k-1)2^{k+1} + 2$ from the equation and get

$$(k+1)2^{k+1} = 2k2^{k+1} + 2 - \big((k-1)2^{k+1} + 2\big) \tag{38}$$

$$(k+1)2^{k+1} = 2k2^{k+1} - (k-1)2^{k+1} \tag{39}$$

$$(k+1)2^{k+1} = (2k - (k-1))2^{k+1} \tag{40}$$

$$(k+1)2^{k+1} = (k+1)2^{k+1} \tag{41}$$

LHS = RHS, so $P(k)$ must hold for all $k > 1$

$$\tag{42}$$