TMA4140 - Homework Exercise Set 6

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1 Chapter 6.3

1.1 Exercise 13

A group contains n men and n women. How many ways are there to arrange these people in a row if the men and women alternate? Total number of people in the group is n-men and n-women therefore $2n$ in total. For men we have: $n! \times n! = (n!)^2$ and it's the same for women. Therefore when combined we get the total number of arrangements, which are $(n!)^2 + (n!)^2 = 2(n!)^2$ if the row alternates between men and women.

1.2 Exercise 34

Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have more women than men?

3 possible type of ways to form a committee:
Possibility number 1: 6 women and 0 men gives us 5005 ways.

$$C(15,6) \times C(10,0) = 5005 \tag{1}$$

Possibility number 2: 5 women and 1 man gives us 30030 ways.

$$C(15,5) \times C(10,1) = 30030 \tag{2}$$

Possibility number 3: 4 women and 2 men gives us 61425 ways.

$$C(15,4) \times C(10,2) = 61425 \tag{3}$$

So to combine all the possible ways to form a committee we get:

$$\begin{aligned} C(15,6) \times C(10,0) \times C(15,5) \times C(10,1) \times C(15,4) \times C(10,2) \\ = 5005 + 30030 + 61425 = 96460 \end{aligned} \tag{4}$$

2 Chapter 6.4

2.1 Exercise 4

Find the coefficient of x^5y^8 in $(x+y)^{13}$.

From the binomial theorem it follows that this coefficient is

$$\binom{13}{8} = \frac{13!}{8!(13-8)!} = \frac{13!}{8!5!} = 1\ 287 \tag{6}$$

$$
\begin{aligned}
(x+y)^n &= \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j \\
&= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + ... + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n
\end{aligned} \tag{5}
$$

2.2 Exercise 9

What is the coefficient of $x^{101}y^{99}$ in the expansion of $(2x-3y)^{200}$?

$$
\begin{aligned}
\binom{200}{99} * 2^{101} * (-3)^{99} &= -\left(2^{101} * 3^{99} * \binom{200}{99}\right) \\
&= -\left(2^{101} * 2^{99} \frac{200!}{99!(200-99)!}\right) \\
&= -\left(2^{101} * 2^{99} \frac{200!}{99!101!}\right)
\end{aligned} \tag{7}
$$

2.3 Exercise 12

The row of Pascal's triangle containing the binomial coefficients $\binom{10}{k}, 0 \leq k \leq 10$, is: $1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1$.

Use Pascal's identity to produce the row immediately following this row in Pascal's triangle.

$$
\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k} \tag{8}
$$

Setting $= 10$ and $k = 1$.

$$
\begin{aligned}
\binom{10+1}{1} &= \binom{10}{1-1} + \binom{10}{1} \\
\frac{11!}{1!(10)!} &= \frac{10!}{0!(10)!} + \frac{10!}{1!(9)!} \\
11 &= 1 + 10 \\
11 &= 11
\end{aligned} \tag{9}
$$

Pascal's identity is verified.
Using the binomial coefficients $\binom{n+1}{k}$ to find the immediate following row in Pascal's triangle.

Substituting $n = 10$ and $k = 0$ in $\binom{n+1}{k}$

$$\binom{10+1}{0} = \binom{11}{0} = 1 \tag{10}$$

Substituting $n = 10$ and $k = 1$ in $\binom{n+1}{k}$ to find the immediately following number in the row... and so on.

$$\binom{10+1}{1} = \binom{11}{1} = 11 \tag{11}$$

Substituting $n = 10$ and $k = 2$

$$\binom{10+1}{2} = \binom{11}{2} = 55 \tag{12}$$

Substituting $n = 10$ and $k = 3$

$$\binom{10+1}{3} = \binom{11}{3} = 165 \tag{13}$$

Substituting $n = 10$ and $k = 4$

$$\binom{10+1}{4} = \binom{11}{4} = 330 \tag{14}$$

Substituting $n = 10$ and $k = 5$

$$\binom{10+1}{5} = \binom{11}{5} = 462 \tag{15}$$

Substituting $n = 10$ and $k = 6$

$$\binom{10+1}{6} = \binom{11}{6} = 462 \tag{16}$$

Substituting $n = 10$ and $k = 7$

$$\binom{10+1}{7} = \binom{11}{7} = 330 \tag{17}$$

Substituting $n = 10$ and $k = 8$

$$\binom{10+1}{8} = \binom{11}{8} = 165 \tag{18}$$

Substituting $n = 10$ and $k = 9$

$$\binom{10+1}{9} = \binom{11}{9} = 55 \tag{19}$$

Substituting $n = 10$ and $k = 10$

$$\binom{10+1}{10} = \binom{11}{10} = 11 \tag{20}$$

Substituting $n = 10$ and $k = 11$

$$\binom{10+1}{11} = \binom{11}{11} = 1 \tag{21}$$

Which gives us the row

$$\begin{aligned}[\binom{11}{0}, \binom{11}{1}, \binom{11}{2}, \binom{11}{3}, \binom{11}{4}, \binom{11}{5}, \binom{11}{6}, \binom{11}{7}, \binom{11}{8}, \binom{11}{9}, \binom{11}{10}, \binom{11}{11}] \\ = [1, 11, 55, 165, 330, 462, 462, 330, 165, 55, 11, 1]\end{aligned} \tag{22}$$

3 Chapter 6.5

3.1 Exercise 6

How many ways are there to select five unordered elements from a set with three elements when repetition is allowed?

There are $C(n+r-1, r) = C(n+r-1, n-1)$ r-combinations from a set with n elements when repetition is allowed. Setting $n = 3$ and $r = 5$.

$$C(3 + 5 - 1, 5) = C(7, 5) \tag{23}$$

Since $C(n, r) = C(n, n - r)$, we get...

$$C(7, 2) = \frac{7 \times 6}{2} = \frac{42}{2} = 21 \tag{24}$$

There are therefore 21 different ways to select 5 unordered elements from a set of 3 elements when repetition is allowed.

3.2 Exercise 14

How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 17$, where x_1, x_2, x_3, and x_4 are nonnegative integers?

There are $C(n+r-1, r) = C(n+r-1, n-1) r$-combinations from a set with n elements when repetition is allowed.

Select 17 items from a set with four elements... Set $n = 4$ and $r = 17$.

$$\begin{aligned} C(4+17-1, 17) &= C(4+17-1, 3) \\ &= C(20, 3) \\ &= 1140 \end{aligned} \tag{25}$$

There are 1140 Solutions.

3.3 Exercise 30

How many different strings can be made from the letters in *MISSISSIPPI*, using all the letters?

There are 11 number of letters, so... $n = 11$. There are 4 unique letters, therefore...$k = 4$.

$$\frac{n!}{n_1! n_2! ... n_k!} = \frac{11!}{1!4!4!2!} = 34650 \tag{26}$$

We can form 34650 different strings from the letters.

3.4 Exercise 54

How many ways are there to distribute five indistinguishable objects into three indistinguishable boxes?

Assuming we can have empty boxes, Then we can have the following distinct arrangements:

$$\begin{gathered} [0][0][5] \\ [0][1][4] \\ [0][2][3] \\ [1][1][3] \\ [1][2][2] \end{gathered} \tag{27}$$

So there are 5 distinct orderings / total number of partitions.

4 Chapter 6.6

4.1 Exercise 5a

Find the next larger permutation in lexicographic order after the permutation of 1432. Next order is 2134.

4.2 Exercise 5c

Find the next larger permutation in lexicographic order after the permutation of 12453. Next order is 12534.