TMA4140 - Homework Set 2

Basic structures: Sets, Functions, Sequences and Sums

RETTES

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Contents

1 Chapter 2.1 - Sets

1.1 Exercise 5

Determine whether each of these pairs are equal.
a) $\{1, 3, 3, 3, 5, 5, 5, 5\}, \{5, 3, 1\} \Rightarrow True.$
b) $\{\{1\}\}, \{1, \{1\}\} \Rightarrow False.$
c) $\emptyset, \{\emptyset\} \Rightarrow False$

1.2 Exercise 24

Determine wether each of these sets is the power set of a set, where a and b are distinct elements.
a) $\emptyset$, is False. $P(\emptyset) = \{\emptyset\{\emptyset\}\}$
b) $\{\emptyset, \{a\}\}$, is True. $P(\{a\}) = \{\emptyset, \{a\}\}$
c) $\{\emptyset, \{a\}, \{\emptyset, a\}\}$, is False.
d) $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$, is True.

The powerset of any set S is the set of all subsets of S, including the empty set $\emptyset$ and S itself. [1] If a and b are distinct elements of a set A. $A = \{a, b\}$ Then the powerset of A is $p(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$.

[1] Wikipedia, 2018-09-14, 13:54, `https://en.wikipedia.org/wiki/Power_set`

2 Chapter 2.2 - Set Operations

2.1 Exercise 18c

Let A, B and C be sets. Show that: $(A \cap B) \subseteq (A \cup B \cup C)$

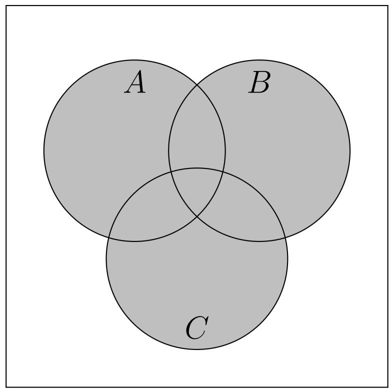
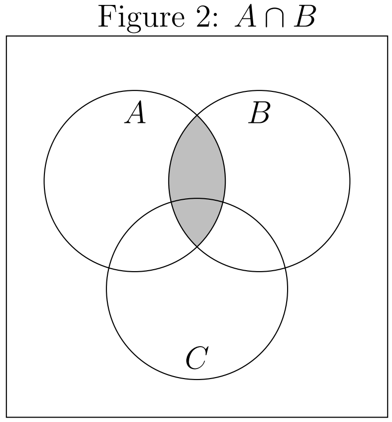


Figure 1: $(A \cup B \cup C)$

As illustrated in figure 1 we can se the venndiagram for $(A \cup B \cup C)$. Coloring out $(A \cap B)$, as shown in figure 2, we can see that $(A \cap B)$ is a subset of $(A \cup B \cup C)$.

Figure 2: $A \cap B$



2.2 Exercise 18d

Let A, B and C be sets. Show that: $(A - B) - C \subseteq A - C$

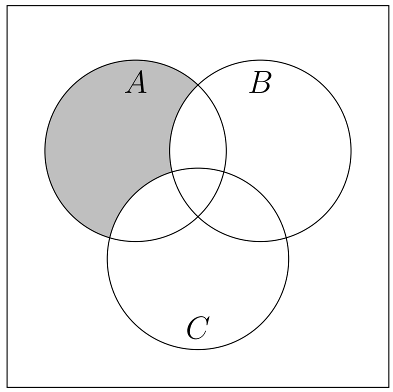


Figure 3: $(A - B) - C$

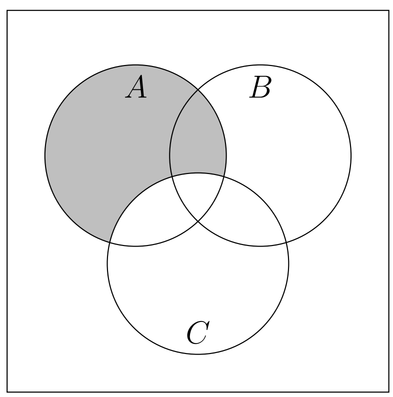


Figure 4: $A - C$

We can see that $(A - B) - C$ is a subset of $A - C$, see figure 3 and 4.

2.3 Exercise 46

Show that if A, B, and C are sets, then:[2]

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \quad (1)$$

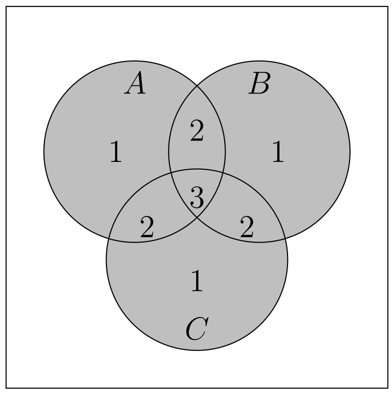


Figure 5: $|A| + |B| + |C|$

Here we have counted some elements more than once (figure 5), lets correct that.

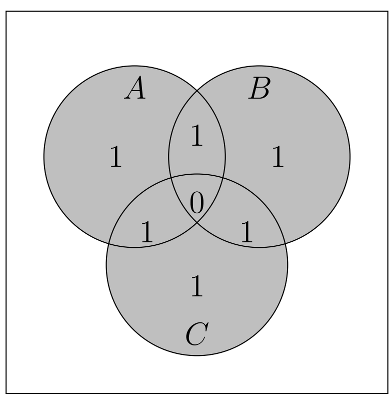


Figure 6: $|A| + |B| + |C| - (|A \cap B| - |A \cap C| - |B \cap C|)$

[2]This is a special case of the inclusion-exclusion principle, which will be studied in Chapter 8. Also... Wikipedia `https://en.wikipedia.org/wiki/Inclusion-exclusion_principle`

But now, the intersection of A,B and C is not counted. (figure 6 on the preceding page). Lets count the intersection once $|A \cap B \cap C|$ as shown in figure 7.

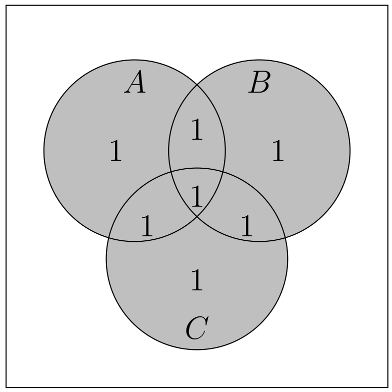


Figure 7: $|A| + |B| + |C| - (|A \cap B| - |A \cap C| - |B \cap C|) + |A \cap B \cap C|$

We have now shown that and we can intuitively see that it's the same as counting all elements inside the union of the three sets: $|A \cup B \cup C|$. (figure 8)

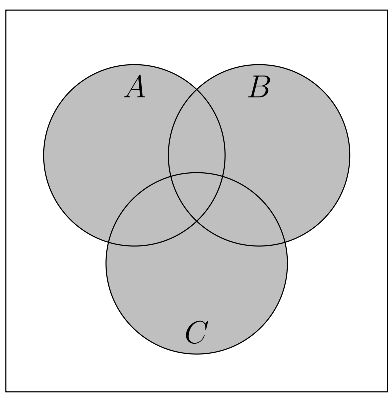


Figure 8: Union of the sets A,B and C

3 Chapter 2.3 - Functions

3.1 Exercise 12c

Determine whether each of these functions from Z to Z is one-to-one.

True. $f(n) = n^3$, is One-to-One, because it passes both the vertical and horizontal line test.[3]

3.2 Exercise 38

Let $f(x) = ax + b$ and $g(x) = cx + d$, where a,b,c, and d are constants. Determine necessary and sufficient conditions on the constants a,b,c, and d so that $f \cdot g = g \cdot f$.

$$\begin{aligned} f \cdot g = g \cdot f = a(cx + d) + b &= acx + ad + b \\ g \cdot f = g(f(x)) = c(ax + b) + d &= acx + cb + d \\ f \cdot g = g \cdot f \Leftrightarrow ad + b &= cb + d \end{aligned} \tag{2}$$

3.3 Exercise 42

Let f be the function from R to R defined by $f(x) = x^2$. Find:
a) $f^{-1}(\{1\}) = \pm 1$
b) $f^{-1}(\{x|0 < x < 1\}) = \pm\{x| -1 < x < 1 \wedge x \neq 0\}$
c) $f^{-1}(\{x|x > 4\}) = \{x| -2 > x \wedge x > 2\}$

[3] http://www.mathwords.com/o/one_to_one_function.htm

4 Chapter 2.4 - Sequences and Summations

4.1 Exercise 12c

Show that the sequence a_n is a solution of the recurrence relation $a_n = -3a_{n-1} + 4a_{n-2}$ if $a_n = (-4)^n$

$$\begin{aligned} a_n =& -3a_{n-1} + 4a_{n-2} = (-4)^n \\ =& -3a_{n-1} + 4a_{n-2} \\ =& (-4)^{n-1} + 4(-4)^{n-2} \\ =& (-4)^{n-2}[(-3)(-4) + 4] \\ =& (-4)^{n-2}(16) \\ =& (-4)^{n-2}(-4)^2 \\ =& (-4)^n \end{aligned} \tag{3}$$

4.2 Exercise 33d

Compute the double sum

$$\begin{aligned} \sum_{i=0}^{2}\sum_{j=1}^{3} ij &= (0*1+0*2+0*3)+(1*1+1*2+1*3)+(2*1+2*2+2*3) \\ &= (0+0+0)+(1+2+3)+(2+4+6) \\ &= 0+6+12 \\ &= 18 \end{aligned} \tag{4}$$

5 Chapter 2.5 - Cardinality of Sets

5.1 TODO: Exercise 16

Exercise: Show that a subset of a countable set is also countable.
Answer: A set is countable if it is finite or is the same size as $\mathbb{N}$. To show that A is countable, it is sufficient to show that there is an injection from A to $\mathbb{N}$.