TMA4140 - Homework Exercise Set 8

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1 Section 5.2

1.1 Exercise 4

Let $P(n)$ be the statement that a postage of n cents can be formed using just 4-cent stamps and 7-cent stamps. The parts of this exercise outline a strong induction proof that $P(n)$ is *true* for $n \geq 18$.

a) Show statements $P(18), P(19), P(20)$ and $P(21)$ are *true*, completing the basis step of the proof.

$$\begin{aligned} P(n) &= 4a + 7b = n \\ P(18) &= (4 \times 1) + (7 \times 2) = 18 \\ P(19) &= (4 \times 3) + (7 \times 1) = 19 \\ P(20) &= (4 \times 5) + (7 \times 0) = 20 \\ P(21) &= (4 \times 0) + (7 \times 3) = 21 \end{aligned} \tag{1}$$

b) What is the inductive hypothesis of the proof?
If for every i in $18 \leq i \leq k$, where $k \leq 21$, there is an "a" and "b" so that $i = 4a + 7b$. Then there is an "c" and "b" so that $k + 1 = 4c + 7d$ is *true*.

c) What do you need to prove in the inductive step?
In the inductive step, we assume that the inductive hypothesis holds, and use that to prove $k + 1$.

d) Complete the inductive step for $k \geq 21$.
For $k = 21$ then $P(k + 1)$ should still be *true*.

$$\begin{aligned} P(k) &= (4 \times 0) + (7 \times 3) = 21 \\ P(k + 1) = 4 + (k - 3) &= (4 \times 2) + (7 \times 2) = 22 \end{aligned} \tag{2}$$

True for base step $P(21)$ and the first inductive step $P(22)$.

e) Explain why these steps show that this statement is *true* whenever $n \geq 18$.
Since the base step and the inductive step are true, by the principle of strong induction all amount of postage where $n \geq 18$ can be obtained using only 4- and 7-cent stamps.

2 Section 5.3

2.1 Exercise 12

Prove that $f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$ when n is a positive integer.

$$\begin{aligned} f_0 &= 0 \\ f_1 &= 1 \\ \\ f_2 &= f_1 + f_0 = 1 + 0 = 1 \\ f_3 &= f_2 + f_1 = 1 + 1 = 2 \\ f_4 &= f_3 + f_1 = 2 + 1 = 3 \end{aligned} \tag{3}$$

Base step:
$P(1)$ is true because $f_1^2 = f_1 f_1$
$= 1.1$
$= 1$
$= f_1 * f_2$

Inductive step:
Assume that $P(k)$ is true.
Show that $P(k+1)$ is true:

$$\begin{aligned} f_1^2 + f_2^2 + \cdots + f_k^2 + f_{k+1}^2 &= f_k * f_{k+1} + f_{k+1}^2 \\ &= f_k * f_{k+1} + f_{k+1} * f_{k+1} \\ &= f_{k+1}(f_k + f_{k+1}) = f_{k+1} * f_{k+2} \end{aligned} \tag{4}$$

Therefore, $P(k+1)$ is true.
Hence by the principle of strong induction, we conclude that the statement is true.
And therefor $f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$ is true.

2.2 Exercise 18

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, Show that $A^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}$ when n is a positive integer.

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ Let $P(n)$ be a statement that $A^n = \begin{bmatrix} f_{n+1} & f_n \\ f_1 & f_{1-1} \end{bmatrix}$

$$A^1 = \begin{bmatrix} f_2 & f_1 \\ f_1 & f_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = A \tag{5}$$

A is true.
Inductive step... Assume that $P(k)$ is true. i.e., $A^k = \begin{bmatrix} f_{k+1} & f_k \\ f_k & f_k - 1 \end{bmatrix}$
We have to prove that $P(k+1)$ is true.

$$\begin{aligned} A^{k+1} &= A.A^k \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f_{k+1} & f_k \\ f_k & f_{k-1} \end{bmatrix} \\ &= \begin{bmatrix} 1.f_{k+1} + 1.f_k & 1.f_k + 1.f_{k-1} \\ 1.f_{k+1} + 0.f_k & 1.f_k + 0.f_{k-1} \end{bmatrix} \\ &= \begin{bmatrix} f_{k+1} + f_k & f_{k-1} + f_k \\ f_{k+1} + 0 & f_k + 0 \end{bmatrix} \\ &= \begin{bmatrix} f_{k+2} & f_{k+1} \\ f_{k+1} & f_k \end{bmatrix} \end{aligned} \tag{6}$$

Therefore $P(k+1)$ is **true**.
And from the principle of mathematical induction we can conclude that the given statement is true.

3 Section 5.4

3.1 Exercise 3

Trace Algorithm 3 when it finds $\gcd(8, 13)$. That is, show all the steps used by Algorithm 3 to find $\gcd(8, 13)$.

Input: $\gcd(8, 13)$

$$
\begin{aligned}
&since(8 < 13)\\
&\quad \gcd(8,13) = \gcd(13\textbf{mod}8, 8)\\
&\quad \gcd(8,13) = \gcd(5,8)\\
\\
&since(5 < 8)\\
&\quad \gcd(5,8) = \gcd(8\textbf{mod}5, 5)\\
&\quad \gcd(5,8) = \gcd(3,5)\\
\\
&since(3 < 5)\\
&\quad \gcd(3,5) = \gcd(5\textbf{mod}3, 3)\\
&\quad \gcd(3,5) = \gcd(2,3)\\
\\
&since(2 < 3)\\
&\quad \gcd(2,3) = \gcd(3\textbf{mod}2, 2)\\
&\quad \gcd(2,3) = \gcd(1,2)\\
\\
&since(1 < 2)\\
&\quad \gcd(1,2) = \gcd(2\textbf{mod}1, 1)\\
&\quad \gcd(1,2) = \gcd(0,1)\\
\\
&since(a = 0)\\
&\quad \gcd(0,1) = 1\\
&\quad \gcd(8,13) = \mathbf{1}
\end{aligned}
\tag{7}
$$

4 Section 9.1

4.1 Exercise 7

Determine whether the relation R on the set of all integers is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*, where $(x, y) \in R$ if and only if
a) $x \neq y$.
R is **Not reflective**. $x \neq x$ can never be true

R is **symetric**, since if x, y are integers and $x \neq y$; then $y \neq x$.
R is **not antisymetric**, as $1 \neq 5$, $5 \neq 1$ while 1 and 5 are not the same. R is **not transitive**, as $1 \neq 5$, $5 \neq 1$ while $1 = 1$

b) $xy \geq 1$.
R is **not reflexive**
R is **symetric**
R is **not antisymetric**
R is **transitive**

c) $x = y + 1$ or $x = y - 1$.
R is **not reflexive**
R is **symetric**
R is **not asymetric**
R is **not transitive**

d) $x \equiv y(mod 7)$.
R is **reflexive**
R is **symetric**
R is **not asymetric**
R is **transitive**

e) x is a multiple of y.
R is **reflective**
R is **not symmetric**
R is **not antisymetric**
R is **transitive**

f) x and y are both negative or both nonnegative.
R is **reflexitive**
R is **symmetric**
R is **not asymmetric**
R is **transitive**

g) $x = y^2$.
R is **not reflexive**
R is **not symmetric**
R is **asymmetric**

R is **not transitive**

h) $x \geq y^2$. R is **not reflexive**
R is **not symmetric**
R is **asymmetric**
R is **transitive**

4.2 Exercise 40

Let R_1 and R_2 be the "divides" and "is a multiple of" relations on the set of all positive integers, respectively.
That is, $R_1 = \{(a, b)|a \text{ divides } b\}$ and $R_2 = \{(a, b)|a \text{ is a multiple of } b\}$. Find...

4.2.1 Exercise 40.a

$R_1 \cup R_2$.
An ordered pair $(a, b) \in R_1 \cup R_2$ if and only if $(a, b) \in R_1$ or $(a, b) \in R_2$.
If a divides b for some integer k, $b = ka$.
If a is an multiple of b, then for some integer k, $b = \frac{1}{k}a$
Therefore, $R_1 \cup R_2 = \{(a, b)|b = ka$ or $b = \frac{1}{k}a$, where k is an integer.$\}$

4.2.2 Exercise 40.c

$R_1 - R_2$.
An ordered pair $(a, b) \in R_1 - R_2$ if and only if $(a, b) \in R_1$ and $(a, b) \notin R_2$.
That is, if and only if a *divides* b and a is not a multiple of b.
Therefore, $R_1 - R_2 = \{(a, b)|a \text{ is a proper divisor of } b\}$.

5 Section 9.3

5.1 Exercise 10

How many nonzero entries does the matrix representing the relation R on $A = \{1, 2, 3, ..., 1000\}$ consisting of the first 1000 positive integers have if R

is...

a) $\{(a,b)|A \leq b\}$?

$$\begin{aligned}\{(a,b)|A \leq b\} &= 1000 + 999 + 998 + \cdots + 2 + 1 \\ &= \frac{(1000)(1000+1)}{2} \\ &= (500)(1001) \\ &= 500,500\end{aligned} \tag{8}$$

b) $\{(a,b)|a = b \pm 1\}$?

$$\begin{aligned}\{(a,b)|a = b \pm 1\} &= 1 + 2(998) + 1 \\ &= 1 + 1996 + 1 \\ &= 1998\end{aligned} \tag{9}$$

c) $\{(a,b)|a + b = 1000\}$?

$$\begin{aligned}\{(a,b)|a + b = 1000\} &= 1 + 1 + 1 + \cdots + 1(999) \\ &= 999\end{aligned} \tag{10}$$

d) $\{(a,b)|a + b \leq 1001\}$?

$$\begin{aligned}\{(a,b)|a + b \leq 1001\} &= 1000 + 999 + 998 + \cdots + 2 + 1 \\ &= \frac{(1000)(1000+1)}{2} \\ &= (500)(1001) \\ &= 500,500\end{aligned} \tag{11}$$

e) $\{(a,b)|a \neq 0\}$?

$$\begin{aligned}\{(a,b)|a \neq 0\} &= 1000 + 1000 + 1000 + \cdots + 1000 + 1000(1000 times) \\ &= (1000)(1000) \\ &= 1,000,000\end{aligned} \tag{12}$$

5.2 Exercise 14

Let R_1 and R_2 be the relations represented by the matrices...

$M_{R_1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and $M_{R_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Find the matrices that represent...

5.2.1 Exercise 14.a

$R_1 \cup R_2$

$$M_{R_1 \cup R_2} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \tag{13}$$

5.2.2 Exercise 14.b

$R_1 \cap R_2$.

$$M_{R_1 \cap R_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \tag{14}$$

5.2.3 Exercise 14.c

$R_2 \circ R_1$.

$$M_{R_2 \circ R_1} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{15}$$