The Crepant Resolution Conjecture for Donaldson-Thomas invariants via wallcrossing

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Calabi-Yau threefolds & DT-invariants

Convention. *Throughout, Y denotes a smooth projective complex variety. By a curve in Y we mean an at most one-dimensional closed subscheme of Y.*

Enumerative geometry studies deformation invariants of moduli spaces of curves on varieties. Depending on Y, there are many types of invariants: Gromow-Witten, **Donaldson-Thomas**, Pandharipande-Thomas, and Gopakumar-Vafa, BPS, and many (conjectural) relations. All invariants exist if Y is a Calabi-Yau threefold!

Definition 1. *A (projective)* ***Calabi-Yau threefold*** *Y is a three-dimensional complex projective variety such that (i) $\omega_Y = \mathcal{O}_Y$, and (ii) $H^1(Y, \mathcal{O}_Y) = 0$.*

The first condition is a flatness statement and gives Serre duality a crucial symmetry. The second condition excludes abelian threefolds.

Example 2. *Any quintic hypersurface in $\mathbb{P}^4$. (Adjunction, calculation.) As a concrete smooth example, there is the Fermat quintic $x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 = 0$.*

Take a curve class $\beta \in \mathrm{N}_1(Y)$, $n \in \mathbb{Z}$, and let $\mathrm{Hilb}_Y(\beta, n)$ be the Hilbert scheme parametrising curves $C \subset Y$ of class $[C] = \beta$ and $\chi(\mathcal{O}_C) = n$.

Definition 3. *The* ***Donaldson-Thomas invariant*** *of Y of class (β, n) is*

$$\mathrm{DT}_Y(\beta, n) = \chi(\mathrm{Hilb}_Y(\beta, n), \nu) = \sum_{k \in \mathbb{Z}} k\chi(\nu^{-1}(k)) \in \mathbb{Z}.$$

Here $\nu \equiv \nu_{(\beta,n)} \colon \mathrm{Hilb}_Y(\beta, n) \to \mathbb{Z}$ is Behrend's constructible function. It encodes $\mathrm{Hilb}_Y(\beta, n)$'s singularities. If M is smooth, $\nu_M = (-1)^{\dim M}$.

Crepant Resolution Conjecture (CRC) I

Curves on a Calabi-Yau threefold Y are trajectories of strings moving through Y. There are many conjectures in the string theory literature concerning curve-counting. The **Crepant Resolution Conjecture** is due to Ruan for GW-invariants, and to Bryan-Graber for DT-invariants .

Conjecture 4 (CRC). *Let $\pi \colon \mathcal{X} \to X$ be a smooth projective three-dimensional Calabi-Yau orbifold with coarse moduli space X, and let $f \colon Y \to X$ be its natural* ***crepant*** *($f^*\omega_X = \omega_Y$) resolution [1] such that $\dim(f) \leq 1$. We have*

$$\frac{\mathrm{DT}(Y)}{\mathrm{DT}_{exc}(Y)} = \frac{\mathrm{DT}_{mr}(\mathcal{X})}{\mathrm{DT}_0(\mathcal{X})}, \tag{1}$$

where exc *denotes curve classes contracted by f, the subscript 0 denotes zero-dimensional curve classes, and* mr *denotes multi-regular curve classes.*

We have gathered DT-invariants in suitable generating functions, using a formal variable q and writing $(\beta, n) \in \mathrm{N}_1(Y) \oplus \mathbb{Z}$. For example

$$\mathrm{DT}_{\mathrm{exc}}(Y) = \sum_{(\beta,n)\,:\, f_*\beta = 0} \mathrm{DT}_Y(\beta, n) q^{(\beta,n)}. \tag{2}$$

Crepant Resolution Conjecture (CRC) II

Remark 5. *Etale-locally on X, the morphisms π and f are of the form [1]*

$$\pi \colon [\mathbb{C}^3/G] \longrightarrow \mathbb{C}^3/G \longleftarrow G\text{-}\mathrm{Hilb}(\mathbb{C}^3) \colon f$$

where $G \leq \mathrm{SL}_2(\mathbb{C}), \mathrm{SO}(3)$ in $\mathrm{SL}_3(\mathbb{C})$ is a finite subgroup, as $\dim(f) \leq 1$. The G-Hilbert scheme encodes length $|G|$ subschemes $Z \subset \mathbb{C}^3 : H^0(\mathcal{O}_Z) \cong \mathbb{C}[G]$.

John Calabrese [2] has proven the analogous formula on point classes.

Theorem 6. *As in Conjecture 4, there is an equality of generating series*

$$\mathrm{DT}_0(\mathcal{X}) = \frac{\mathrm{DT}^\vee_{exc}(Y)}{\mathrm{DT}_0(Y)} \mathrm{DT}_{exc}(Y) \equiv \overline{\mathrm{DT}}^\vee_{exc}(Y)\, \mathrm{DT}_{exc}(Y).$$

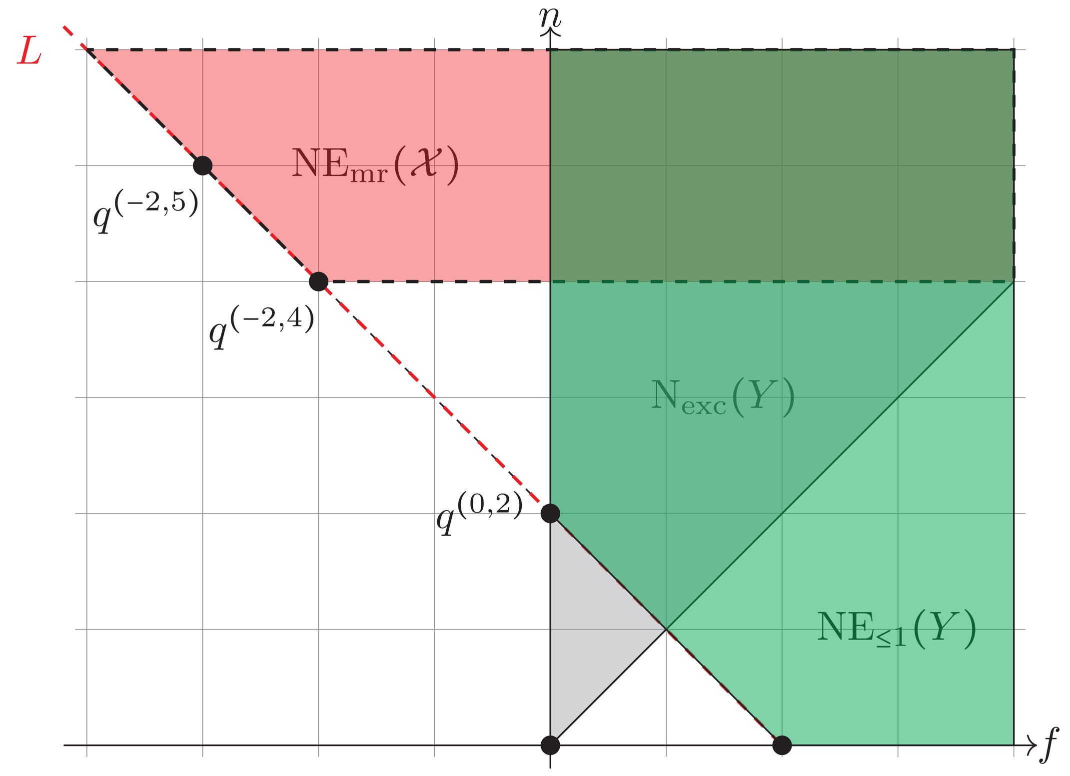
Here $\mathrm{DT}^\vee_{exc}(Y) = \sum_{(\beta,n)\,:\, f_*\beta = 0} \mathrm{DT}_Y(\beta, n) q^{(-\beta,n)}$ similar to $\mathrm{DT}_{exc}(Y)$.

The equality holds in a completed power series ring, formal variables are identified through (3) and **analytic continuation**. A reformulation:

Goal. *Complete the proof of the CRC: $\mathrm{DT}_{mr}(\mathcal{X}) = \overline{\mathrm{DT}}^\vee_{exc}(Y)\, \mathrm{DT}(Y)$.*

Example: we need analytic continuation

Let $f \colon (E \subset Y) \to (C \subset X)$ be a crepant resolution as above, where $C = \mathbb{P}^1$ embeds as the trivial family of A_1-singularities. The exceptional locus is $E = \mathbb{P}^1 \times \mathbb{P}^1$ with normal bundle $\mathcal{N}_{E/Y} = \mathcal{O}_E(-2,-2)$. A curve class on $Y = \mathrm{Tot}_E(\mathcal{N}_{E/Y})$ is determined by a triple $(s, f, n) \in \mathbb{Z}^3$. The diagram shows the numerically effective cones of $\mathcal{X}, Y$ for $s = 2$.



The DT-generating functions restricted to $L \subset \{s = 2\}$ with $x = q^{(1,-1)}$

$$\left.\frac{\mathrm{DT}(Y)}{\mathrm{DT}_{\mathrm{exc}}(Y)}\right|_{s=2} \equiv_L 3q^{(0,2)} \left(\frac{1}{1+x}\right)^2 \stackrel{!}{=} 3q^{(-2,4)} \left(\frac{1}{1+x^{-1}}\right)^2 \equiv_L \left.\frac{\mathrm{DT}_{\mathrm{mr}}(\mathcal{X})}{\mathrm{DT}_0(\mathcal{X})}\right|_{s=2}$$

Key: the CRC only holds after **analytic continuation** $q^{(1,-1)} = x \leftrightarrow x^{-1}$.

Approach: derived categories & Hall algebra

Following [2], the derived McKay correspondence of [1] and [2] yield

$$\Phi \colon \mathrm{D}^b(Y) \xrightarrow{\sim} \mathrm{D}^b(\mathcal{X}), \qquad \text{sending} \quad \Phi(\mathcal{O}_Y) = \mathcal{O}_{\mathcal{X}}, \tag{3}$$

identifying $\mathrm{Per}(Y/X) \cong \mathrm{Coh}(\mathcal{X})$: a curve on $\mathcal{X}$ **is** a perverse curve on Y. A **perverse coherent sheaf** is a complex $E = (E^{-1} \to E^0)$ in $\mathrm{D}^b(Y)$ s.t.

$$0 \to F[1] \to E \to T \to 0,$$

where F, T are sheaves on Y such that $f_*F = 0$ and $\mathbf{R}^1 f_* T = 0$. But a **sheaf** G on Y is an extension $T \hookrightarrow G \twoheadrightarrow F$. We need a **wall-crossing**, a procedure to relate curves on Y to perverse curves on Y (curves on $\mathcal{X}$). By work of Joyce, Bridgeland, and later Toda for any heart $\mathcal{A} \subset \mathrm{D}^b(Y)$:

Theorem 7. *There exists an associative unital* ***motivic Hall algebra*** *$\mathrm{H}(\mathcal{A})$ whose product encodes extensions in the abelian category $\mathcal{A}$. There exists a unique* Poisson algebra morphism called* ***integration map*** *on a subquotient of $\mathrm{H}(\mathcal{A})$ called* ***semi-classical Hall algebra***, *i.e., $\mathrm{I} \colon \mathrm{H}_{sc}(\mathcal{A}) \to \mathbb{C}[\![q]\!]$.*

Example 8. *Take $\mathcal{A} = \mathrm{Coh}(Y)$. Then we have the identity $\mathcal{A} = \mathcal{T} * \mathcal{F}$ in $\mathrm{H}(\mathcal{A})$. Why? Any $E \in \mathcal{A}$ is a* ***unique*** *extension $0 \to T \to E \to F \to 0$ in $\mathcal{A}$.*

Result: wallcrossing formula for the CRC

In [2], $\mathrm{Hilb}(Y)$ parametrises structure sheaves with section. For us, it parametrises **ideal sheaves**. These moduli spaces are isomorphic on Y. Ideal sheaves live in the abelian category $\mathcal{C}(\mathcal{X}) = \langle \mathcal{O}_{\mathcal{X}}, \mathrm{Coh}_{\leq 1}(\mathcal{X})[-1] \rangle_{\mathrm{ex}}$.

Theorem 9. *There is an identity in the Hall algebra $\mathrm{H} = \mathrm{H}(\mathcal{C}(\mathcal{X}))$*

$$\mathrm{Hilb}_{\leq 1}(\mathcal{X}) * \mathcal{F} = \mathcal{F} * \mathrm{E}_f \otimes \mathrm{Hilb}_{\leq 1}(Y). \tag{4}$$

Here E_f = {line bundles associated to divisors on the exceptional locus of f}.

Lemma 10. *The torsion filtration induces $\mathcal{F} = \mathcal{F}_1 * \mathcal{F}_2$ in H, where $\dim(F) = i$ for $F \in \mathcal{F}_i$. We obtain $\mathcal{F}_1^{-1} * \mathrm{Hilb}_{\leq 1}(\mathcal{X}) * \mathcal{F}_1 = \mathcal{F}_2 * \mathrm{E}_f \otimes \mathrm{Hilb}_{\leq 1}(Y) * \mathcal{F}_2^{-1}$.*

To apply the integration map I we need a **stability condition** to apply Joyce's work and show $[\mathbb{C}^\times] \log(\mathcal{F}_i) \in \mathrm{H}_{sc}(\mathcal{C}(\mathcal{X}))$. For $\mathcal{F}_1$, this is in [2].

Proposition 11. *Integrating the term with $\mathcal{F}_1$ yields $\overline{\mathrm{DT}}^\vee_{exc}(Y)^{-1}\, \mathrm{DT}_{mr}(\mathcal{X})$.*

Goal. *To prove CRC, we must show $I(\mathcal{F}_2 * \mathrm{E}_f \otimes \mathrm{Hilb}_{\leq 1}(Y) * \mathcal{F}_2^{-1}) = \mathrm{DT}(Y)$.*

References

[1] T. Bridgeland, A. King, M. Reid (2001), *The McKay correspondence as an equivalence of derived categories*, J. Amer. Math. Soc., Vol. 14, 535–554.

[2] J. Calabrese (2016), *On the crepant resolution conjecture for Donaldson-Thomas invariants* J. Algebraic Geom., Vol. 25, 1–18.