Classification of the Irreducible Representations of $\mathfrak{su}(2)$ and $SU(2)$

An elementary yet crucial result in representation theory is Schur's lemma, which is our first aim to prove. Although we will only need it for finite-dimensional representations, we will prove it for unitary group representations on Hilbert spaces. (Recall that in finite dimension, each group representation is equivalent to some unitary representation, so in that case we don't need to require unitarity.) The following result will be used in the proof of Schur's lemma.

Lemma 1. *A closed linear subspace $K \subset H$ of a Hilbert space H is invariant under a unitary representation u of a group G if and only if the corresponding projection P_K on K commutes with the representation.*

Proof. Suppose K is an invariant closed subspace of H and let $g \in G$. Then for $k \in K$ we have

$$u(g)P_K k = u(g)k$$
$$P_k u(g)k = u(g)k$$

since $u(g)k \in K$ by invariance. And for $x \in K^\perp$ we have

$$u(g)P_K x = 0$$
$$P_k u(g)x = 0$$

where the last equality relies on the fact that $u(g)x \in K^\perp$, which is the case because for all $k \in K$ we have

$$\langle u(g)x, k\rangle = \langle x, u(g)^* k\rangle = \langle x, u(g)^{-1}k\rangle = \langle x, u(g^{-1})k\rangle = 0$$

since $u(g^{-1})k \in K$ and $x \in K^\perp$. Now we can write a general vector $v \in H$ as $v = k + x$ with $k \in K$ and $x \in K^+$, and we get

$$u(g)P_k v = u(g)k = P_K u(g)v$$

so that $u(g)P_K = P_K u(g)$, proving one implication.
For the other direction, suppose that K is not invariant. Then there exist $g \in G$ and $k \in K$ such that $u(g)k \notin K$ and then

$$u(g)P_K k = u(g)k$$
$$P_K u(g)k \neq u(g)k$$

and hence P_K does not commute with the representation, proving the lemma. □

Lemma 2. *(Schur's lemma.) A (non-zero) unitary representation $u : G \to B(H)$ of a group G on a Hilbert space H is irreducible if and only if the commutant of the represention consists of scalar multiples of the identity, i.e., if $u(G)' = \mathbb{C} \cdot 1_H$.*

Proof. Suppose that the representation is not irreducible. Then there is a non-trivial invariant subspace K of H, and by the lemma, P_K commutes with the representation. But since K is a non-trivial subspace, K is not proportional to the identity, whence $u(G) \neq \mathbb{C} \cdot 1_H$, proving one implication.
On the other hand, if $u(G)' \neq \mathbb{C} \cdot 1_H$ then there exists $T \in u(G)'$ which is not a multiple of the identity.

Now note that, for all $g \in G$, $T^* u(g) = (u(g^{-1})T)^* = (Tu(g^{-1}))^* = gT^*$, so that $T^* \in u(G)'$. Since $u(G)'$ is a linear subspace of $B(H)$, we also have $T + T^* \in u(G)'$. We now use the following result: *if some $A \in B(H)$ commutes with a self-adjoint $S \in B(H)$, then A commutes with all spectral projections of S.* Thus, since each $u(g)$ commutes with $T + T^*$ (which is clearly self-adjoint), each $u(g)$ commutes with all spectral projections of $T + T^*$, i.e., all those projections lie in the commutant $u(G)'$. Since $T + T^* \notin \mathbb{C} \cdot 1_H$, there exists a nontrivial such projection, and by the lemma this implies that there exists a nontrivial closed subspace which is invariant. Hence u is irreducible, completing the proof. $\square$

Classification

The Lie algebra $\mathfrak{su}(2)$ is the 3-dimensional real vector space spanned by the basis S_1, S_2, S_3 where $S_i = -\frac{i}{2}\sigma_i$ with σ_i the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and with the Lie bracket (the commutator) given by $[S_i, S_j] = \varepsilon_{ijk} S_k$. A representation $\pi : \mathfrak{su}(2) \to M_n(\mathbb{C})$ corresponds precisely (via $\pi(S_k) = -iL_k$) to three complex $n \times n$ matrices L_i that satisfy $[L_i, L_j] = i\varepsilon_{ijk} L_k$. Note that if π is a skew-adjoint representation, i.e., $\pi(S_k)^* = \pi(-S_k)$, iff $(-iL_k)^* = iL_k$ and hence iff each L_k is self-adoint. We now define the matrices $L_\pm := L_1 \pm iL_2$, and note that $\{L_3, L_+, L_-\}$ forms a basis for $\mathfrak{su}(2)$ with Lie brackets

$$[L_3, L_\pm] = \pm L_\pm, \qquad [L_+, L_-] = 2L_3$$

Note that L_2 and L_3 are self-adjoint iff $L_\pm^* = L_\mp$. We further define $C = L_1^2 + L_2^2 + L_3^2$ (" $= \vec{L}^2$ ") and we easily check that we have the relations

$$L_+ L_- = C - L_3(L_3 - 1_H) \tag{1}$$
$$L_- L_+ = C - L_3(L_3 + 1_H) \tag{2}$$

Importantly, C commutes with the representation of the Lie algebra, as is also easily checked. If we can show that C commutes with the representation of the *group*, then Schur's lemma says that C is a scalar multiple of the identity. To this end, we invoke the following result (see Corollary 3.3 of Klaas Landsman's text):

If G is a connected and simply connected matrix Lie group with Lie algebra $\mathfrak{g}$, then the finite-dimensional skew-adjoint representations of $\mathfrak{g}$ correspond one-to-one to the finite-dimensional unitary representations u of G via

$$u(e^X) = e^{\pi(X)}$$

for all $X \in \mathfrak{g}$. And if G is compact, then this specializes to a bijective correspondence between unitary irreps of G and skew-adjoint irreps[1] of $\mathfrak{g}$.

[1] Irreducibility of a Lie algebra representation is defined (in finite dimension) in the same way as for group representations: a (non-zero) finite-dimensional Lie algebra representation is called irreducible when the only invariant subspaces are $\{0\}$ and the whole representation space.

Since $SU(2) \subset GL_n(\mathbb{C})$ is connected and simply connected, we know that our π comes from some representation u of $SU(2)$ via $u(e^X) = e^{\pi(X)}$ for all $X \in \mathfrak{su}(2)$. Thus, using the well-known result that if any complex square matrix X commutes with another such matrix Y then X also commutes with e^Y, we see that

$$u(e^X)C = e^{\pi(X)}C = Ce^{\pi(X)} = Cu(e^X)$$

for all $X \in \mathfrak{su}(2)$ since C commutes with $\pi(X)$. Thus we see that C commutes with all matrices in the representation u of $SU(2)$ and hence Schur's lemma implies that $C = c \cdot 1_H$ for some $c \in \mathbb{C}$. We will use these results to our advantage in proving the classification theorem of the $\mathfrak{su}(2)$ and $SU(2)$ irreps below.

Theorem 3. *(Classification of the $\mathfrak{su}(2)$ and $SU(2)$ irreps.) Let $\pi : \mathfrak{su}(2) \to H$ be a skew-adjoint irreducible representation of $\mathfrak{su}(2)$ on a Hilbert space H. Then, up to equivalence of representations,*

$$H = \mathbb{C}^{2j+1}$$
$$\sigma(L_3) = \{-j, -j+1, \dots, j-1, j\}$$

for some $j \in \frac{1}{2}\mathbb{N} = \{0, 1/2, 1, 3/2, 2, \dots\}$, and there exists an orthonormal basis $\{w_m : m \in \sigma(L_3)\}$ for H such that for each $m \in \sigma(L_3)$ we have

$$\begin{aligned} L_3 w_m &= m w_m \\ L_+ w_m &= \sqrt{j(j+1) - m(m+1)}\, w_{m+1} \\ L_- w_m &= \sqrt{j(j+1) - (m-1)m}\, w_{m-1} \\ C w_m &= j(j+1) w_m \end{aligned}$$

Conversely, for any such j there is a unique skew-adjoint irreducible representation of $\mathfrak{su}(2)$ on $\mathbb{C}^{2j+1}$, up to equivalence, and it satisfies the listed conditions. Furthermore, each such skew-adjoint irrep gives a unitary irrep of $SU(2)$ by exponentiation, and all unitary irreps of $SU(2)$ are obtained in this way.

Proof. By the Peter-Weyl theorem, every irreducible representation of a compact topological group is finite-dimensional, and by the theorem quoted above, for any connected and simply connected compact matrix Lie group there is a bijective correspondence between the unitary irreps of the group and the skew-adjoint irreps of its Lie algebra, and this bijection respects the dimension of the representation, so that also all skew-adjoint irreps of the Lie algebra are finite-dimensional. Since $SU(2)$ satisfies all the premises, these theorems apply and $\dim(H) < \infty$.
L_3 is self-adjoint (and $H \neq \{0\}$ for otherwise the representation would not be irreducible by definition), so L_3 has at least one eigenvector (actually, H is finite-dimensional, so any operator has an eigenvector). And if $v_\lambda \in H$ is an eigenvector of L_3, then $L_3 v_\lambda = \lambda v_\lambda$, say, and hence for $L_+ v_\lambda$ we have

$$L_3(L_+ v_\lambda) = ([L_3, L_+] + L_+ L_3) v_\lambda = (L_+ + L_+ L_3) v_\lambda = L_+ v_\lambda + \lambda L_+ v_\lambda = (1+\lambda) L_+ v_\lambda$$

so either $L_+ v_\lambda = 0$ or $L_+ v_\lambda$ is an eigenvector of L_3 with eigenvalue $\lambda + 1$. And similarly, looking at $L_- v_\lambda$, we see that

$$L_3(L_- v_\lambda) = ([L_3, L_-] + L_- L_3) v_\lambda = (-L_- + L_- L_3) v_\lambda = -L_- v_\lambda + \lambda L_- v_\lambda = (-1+\lambda) L_- v_\lambda$$

so $L_- v_\lambda$ is either zero or it is an eigenvector for L_3 with eigenvalue $\lambda - 1$. Now note that L_3 must have some smallest eigenvalue, because otherwise L_3 would have infinitely many eigenvalues, which cannot happen in a finite-dimensional Hilbert space, so we may denote this smallest eigenvalue by λ and one of the corresponding eigenvectors by v_λ. We can successively apply L_+ to this vector, thereby obtaining a new eigenvector each time, until at some point the process terminates, i.e., for some $k \in \mathbb{N}$ we will have $(L_+)^{k+1} v_\lambda = 0$, wheres for all $1 \leq l \leq k$, $(L_+)^l v_\lambda \neq 0$. (The process must terminate because otherwise, again, L_3 would have infinitely many eigenvalues, which cannot happen in a finite-dimensional Hilbert space.) We'll show that the set of eigenvectors obtained in this way spans the entire Hilbert space of the representation.
Consider the linear subspace

$$H' := \mathbb{C} \cdot v_\lambda \oplus \mathbb{C} \cdot v_{\lambda+1} \oplus \cdots \oplus \mathbb{C} \cdot v_{\lambda+k} \subset H$$

where $v_{\lambda+l} = (L_+)^l v_\lambda$ is the eigenvector with eigenvalue $\lambda + l$. This subspace is easily seen to be invariant under the generators L_3 and L_+, but it's not immediately clear that it's also invariant under L_-. However, the following argument shows that it is. Consider an L_3 eigenvector $v_{\lambda+l}$, as defined above. If $l = 0$ then $L_- v_\lambda = 0 \in H'$ (since λ was per definition the smallest eigenvalue). And if $1 \leq l \leq k$ then, using Eq. 2 and the fact that $C = c \cdot 1_H$, we see that v_{l-1} is an eigenvector of $L_- L_+$ (since it is an eigenvector of C and of L_3) and hence

$$L_- v_l = L_-(L_+ v_{l-1}) = \tilde{\lambda} v_{l-1}$$

for some $\tilde{\lambda} \in \mathbb{C}$. This shows that H' is invariant under L_- and thus under the whole Lie algebra representation. Since the representation is irreducible by assumption, we must have $H' = \{0\}$ or $H' = H$. But $H' \neq \{0\}$ since $H' \ni v_\lambda \neq 0$ and hence $H = H'$. So

$$\sigma(L_3) = \{\lambda, \lambda + 1, \ldots, \lambda + k\},$$

each eigenspace is one-dimensional, and $\dim(H) = k + 1$. But we can say more about the values that λ can take.
We let Eq. (1) act on v_λ, which yields

$$0 = (c - \lambda(\lambda - 1)) v_\lambda$$

and we let Eq. (2) act on $v_{\lambda+k} = (L_+)^k v_\lambda$, which yields

$$0 = (c - (\lambda + k)(\lambda + k + 1)) v_{\lambda+k}$$

From these two relations it follows (since $v_\lambda, v_{\lambda+k} \neq 0$) that

$$\begin{aligned}
&c = \lambda(\lambda - 1) = (\lambda + k)(\lambda + k + 1) \\
\Rightarrow\ &\lambda^2 - \lambda = \lambda^2 + 2\lambda k + \lambda + k^2 + k \\
\Rightarrow\ &0 = 2\lambda k + 2\lambda + k^2 + k = 2\lambda(k + 1) + k(k + 1) = (2\lambda + k)(k + 1)
\end{aligned}$$

and since $k + 1 > 0$ this implies that $2\lambda + k = 0$, i.e., $\boxed{\lambda = -k/2}$. Moreover, we see that $c = \lambda(\lambda - 1) = (k/2)(1 + k/2) = j(j + 1)$ with $j := k/2$.

Thus, if we denote by H_m the eigenspace of L_3 corresponding to the eigenvalue m, we have found the following so far:

$$H = \bigoplus_{m=-j}^{j} H_m \cong \mathbb{C}^{2j+1}$$

$$\sigma(L_3) = \{-j, -j+1, \dots, j-1, j\}$$

$$L_+(H_m) = \begin{cases} H_{m+1} \text{ if } m < j \\ 0 \text{ if } m = j \end{cases}$$

$$L_-(H_m) = \begin{cases} H_{m-1} \text{ if } m > -j \\ 0 \text{ if } m = j \end{cases}$$

$$C = j(j+1) \cdot 1_H$$

where $j \in \frac{1}{2}\mathbb{N}$. But we can derive more about the allowed representations. With Eq. (2) we find that

$$L_- L_+ v_m = (j(j+1) - m(m+1))v_m$$

We will use this to normalize the v_m. We can of course choose our starting vector v_{-j} such that it has norm one, and we will write $w_{-j} := v_{-j}$ for this normalized vector. And then, assuming that we have found a unit eigenvector w_m of L_3 with eigenvalue m, for some $-j \leq m < j$, and if we write $w_{m+1} = \frac{1}{N_m} L_+ w_m$ with $N_m \in \mathbb{R}$, we have

$$\langle w_{m+1}, w_{m+1} \rangle = \frac{1}{N_m^2} \langle L_+ w_m, L_+ v_m \rangle = \frac{1}{N_m^2} \langle w_m, L_- L_+ w_m \rangle = \frac{1}{N_m^2} (j(j+1) - m(m+1))$$

where in the middle step we have use that $L_\pm^* = L_\mp$ because L_1 and L_2 are self-adjoint, which in turn follows because the representation is skew-adjoint by assumption. So w_{m+1} would be properly normalized if $N_m = \sqrt{j(j+1) - m(m+1)}$. Thus if we choose $w_{-j} = v_{-j}$ with norm 1 and set

$$w_{m+1} := \frac{1}{\sqrt{j(j+1) - m(m+1)}} L_+ w_m$$

for each $-j \leq m < j$, then it follows immediately by induction the w_m furnish an orthonormal basis for H. (The w_m are orthogonal because they are eigenvectors for different eigenvalues of the self-adjoint operator L_3). We then find that

$$\begin{aligned} L_+ w_m &= \sqrt{j(j+1) - m(m+1)}\, w_{m+1}, \qquad (m < j) \\ L_- w_m &= \frac{1}{N_{m-1}} L_-(L_+ w_{m-1}) = \frac{1}{N_{m-1}} (j(j+1) - m(m-1)) v_{m-1} = \frac{1}{N_{m-1}} N_{m-1}^2 w_{m-1} \\ &= \sqrt{j(j+1) - (m-1)m}\, w_{m-1}, \qquad (m > -j) \end{aligned}$$

and we note that coincedentally these formulae are also both correct for $m = j$ and $m = -j$. This *almost* proves the first part of the theorem. The only thing we still need to show is that $H = \mathbb{C}^{2j+1}$ instead of just $H \cong \mathbb{C}^{2j+1}$, up to equivalence of representations. But this will follow from the uniqueness of the irreps combined with the explicit construction in the proof of the existence of the irreps.

Uniqueness

It is now straightforward to show that there is at most one representation for each j, up to equivalence. For suppose that we have two representations $\{L_3, L_+, L_-\}$ and $\{\widetilde{L}_3, \widetilde{L}_+, \widetilde{L}_-\}$ on $H \cong \mathbb{C}^{2j+1}$ and $\widetilde{H} \cong \mathbb{C}^{2j+1}$, respectively. Then, by the first part of the theorem, there are two orthonormal bases $\{w_m\}$ and $\{\widetilde{w}_m\}$ of H and $\widetilde{H}$, respectively such that the generators act precisely as indicated in the theorem. We define the linear map

$$U : H \to \widetilde{H}, \qquad w_m \mapsto \widetilde{w}_m$$

which is unitary, as it sends an orthonormal basis to an orthonormal basis. Then

$$\widetilde{L}_3 = U L_3 U^{-1}, \qquad \widetilde{L}_\pm = U L_\pm U^{-1}$$

which is checked easily by acting on the basis vectors $\widetilde{w}_m$, and hence the representations are equivalent.

Existence

Finally, we'll show that each allowed value for j 'does it', and that we may take $H = \mathbb{C}^{2j+1}$, by constructing the representation explicitly. Using the first part of the theorem for inspiration, we take $H = \mathbb{C}^{2j+1}$ with its standard orthonormal basis $\{e_m : m = 1, 2, \ldots, 2j+1\}$, which we however rewrite as $\{w_m : m = -j, -j+1, \ldots, j\}$, i.e., $w_m = e_{m+j+1}$, and we define operators L_3 and $L_\pm$ as

$$\begin{aligned}
L_3 w_m &= m w_m \\
L_+ w_m &= \sqrt{j(j+1) - m(m+1)}\, w_{m+1} \\
L_- w_m &= \sqrt{j(j+1) - (m-1)m}\, w_{m-1}
\end{aligned}$$

Let's check the commutation relations.

$$\begin{aligned}
[L_3, L_\pm] w_m &= L_3(\sqrt{j(j+1) - m(m\pm 1)}\, w_{m\pm 1}) - L_\pm(m w_m) \\
&= ((m \pm 1) - m)\sqrt{j(j+1) - m(m\pm 1)}\, w_{m\pm 1} \\
&= \pm L_\pm w_m \\
[L_+, L_-] w_m &= L_+(\sqrt{j(j+1) - (m-1)m}\, w_{m-1}) - L_-(\sqrt{j(j+1) - m(m+1)}\, w_{m+1}) \\
&= \sqrt{j(j+1) - (m-1)m}\sqrt{j(j+1) - m(m-1)}\, w_m - \sqrt{j(j+1) - m(m+1)}\sqrt{j(j+1) - (m+1)m}\, w_m \\
&= (j(j+1) - m(m-1) - j(j+1) + m(m+1)) w_m \\
&= 2m w_m = 2 L_3 w_m
\end{aligned}$$

The commutation relations are correct, so $L_3, L_\pm$ constitute a representation of $\mathfrak{su}(2)$. Further, from the definition of $L_3, L_\pm$ we see that

$$L_3^* = L_3, \qquad L_\pm^* = L_\mp$$

since

$$\begin{aligned}
\langle L_+ w_m, w_{m'} \rangle &= \sqrt{j(j+1) - m(m+1)}\delta_{(m+1),m'} = \sqrt{j(j+1) - m(m+1)}\delta_{m,(m'-1)} \\
&= \sqrt{j(j+1) - (m'-1)m'}\delta_{m,(m'-1)} = \langle w_m, L_- w_{m'} \rangle
\end{aligned}$$

And since $L_1 = \frac{1}{2}(L_+ + L_-) = \frac{1}{2}(L_+ + L_+^*)$ and $L_2 = \frac{1}{2i}(L_+ - L_-) = \frac{1}{2i}(L_+ - L_+^*) = -\frac{1}{2}(iL_+ + (iL_+)^*)$ this implies that L_1, L_2 and L_3 are self-adjoint. As we argued in the beginning, this means that the representation is skew-adjoint. What is left to show then is that the representation is irreducible. So suppose that $\{0\} \neq V \subset \mathbb{C}^{2j+1}$ is an invariant subspace. Let $0 \neq w \in V$. Then $w = \sum c_m w_m$ with at least one c_m nonzero. Let m_0 be the smallest value of m for which $c_m \neq 0$, and apply L_+ to w, $j - m_0$ times. Since $L_+^{j-m_0} w_m = 0$ for each $m > m_0$ this yields

$$L_+^{j-m_0} w = \sum c_m L_+^{j-m_0} w_m = c_{m_0} L_+^{j-m_0} w_{m_0} = \alpha w_{m_0+(j-m_0)} = \alpha w_j$$

for some $0 \neq \alpha \in \mathbb{C}$, and thus, by irreducibility, $w_j \in V$. Then, by applying L_- successively to w_j, we also find that $w_m \in V$ for each $-j \leq m < j$. This shows that $V = \mathbb{C}^{2j+1}$, so the representation is irreducible. This proves the last part (concerning the Lie algebra $\mathfrak{su}(2)$) of the theorem. Lastly, by the bijective correspondence between the skew-adjoint irreps of $\mathfrak{su}(2)$ and the unitary irreps of $SU(2)$, this gives a complete list of irreducible representations of $SU(2)$. $\square$