Useful Identities

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Contents

1 Vector Calculus

1.1 Products in $\mathbb{R}^3$

$$A \cdot (B \times C) = B(C \times A) = C \cdot (A \times B) \tag{1}$$

$$A \times (B \times C) = (A \cdot C)B - (A \cdot B)C \tag{2}$$

$$(A \times B) \cdot (C \times D) = (A \cdot C)(B \cdot D) - (A \cdot D)(B \cdot C) \tag{3}$$

1.2 The Levi-Civita symbol

Provided that $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ is a *righthanded* basis for $\mathbb{R}^2$, we can *define* $\varepsilon_{ijk} = \boldsymbol{e}_i \cdot (\boldsymbol{e}_j \times \boldsymbol{e}_k)$. Then it follows that indeed

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ is an even permutation of } 123 \\ -1 & \text{if } ijk \text{ is an odd permutation of } 123 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

and we have the following properties.

$$\boldsymbol{e}_i \times \boldsymbol{e}_j = \varepsilon_{ijk} \boldsymbol{e}_k \tag{5}$$

$$(\boldsymbol{a} \times \boldsymbol{b})_i = \varepsilon_{ijk} a_j b_k \tag{6}$$

$$\tag{7}$$

$$\varepsilon_{ijk} = \begin{vmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{vmatrix}, \qquad \varepsilon_{ijk}\varepsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} \tag{8}$$

$$\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \tag{9}$$

$$\varepsilon_{imn}\varepsilon_{jmn} = 2\delta_{ij} \tag{10}$$

$$\varepsilon_{ijk}\varepsilon_{ijk} = 6 \tag{11}$$

For a 3×3 matrix A, we have

$$\det A = \varepsilon_{ijk} A_{1i} A_{2j} A_{3k} \tag{12}$$

$$\varepsilon_{ijk} \det A = \varepsilon_{lmn} A_{il} A_{jm} A_{kn} \tag{13}$$

1.3 Rotations in $\mathbb{R}^2$

Under a rotation about a counterclockwise angle φ, a vector $(x, y) \in \mathbb{R}^2$ transforms as

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{14}$$

1.4 Spherical Coordinates

The transformation from spherical coordinates (r, θ, φ) to Cartesian coordinates (x, y, z), is given by the a function $F : (0, \infty) \times (0, \pi) \times (0, 2\pi) \to \mathbb{R}^3$, $(r, \theta, \varphi) \mapsto (x, y, z)$ where

$$x = r \sin\theta \cos\varphi \tag{15}$$
$$y = r \sin\theta \sin\varphi \tag{16}$$
$$z = r \cos\theta \tag{17}$$

with the inverse relations

$$r = \sqrt{x^2 + y^2 + z^2} \tag{18}$$
$$\theta = \arccos \frac{z}{\sqrt{x^2 + y^2 + z^2}} = \arccos \frac{z}{r} \tag{19}$$
$$\varphi = \text{angle}(y, x) \tag{20}$$

where the function *angle* is the restriction to the image of F of the angle function that is defined in (27). The Jacobian matrix of the coordinate transformation F is given by

$$J_{\mathbf{F}}(r, \theta, \varphi) = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{bmatrix} \tag{21}$$

with determinant $\det J_{\mathbf{F}}(r, \theta, \varphi) = r^2 \sin\theta$ and hence with volume element $\mathrm{d}V = r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\varphi$. On the specified domain F is a C^∞-diffeomorphism. Note that if the origin $(r = 0)$ or a pole $(\theta = 0, \pi)$ is included in the domain then F is not even bijective!

Unit vectors are related by

$$\hat{\mathbf{r}} = \sin\theta\cos\varphi\,\hat{\mathbf{x}} + \sin\theta\sin\varphi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \tag{22}$$

$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\varphi\,\hat{\mathbf{x}} + \cos\theta\sin\varphi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \tag{23}$$

$$\hat{\boldsymbol{\varphi}} = -\sin\varphi\,\hat{\mathbf{x}} + \cos\varphi\,\hat{\mathbf{y}} \tag{24}$$

1.5 The function atan2 and the Angle Function

It is often convenient, especially in programming, to have a function that for a point $(x, y) \in \mathbb{R}^2$ returns the angle that the vector (x, y) makes with the positive x-axis. On the first quadrant of $\mathbb{R}^2$, i.e., for $x, y > 0$, the arctangent function does exactly this, but when we allow all four quadrants, this does not work anymore. A commonly used function that does work is atan2, which is how this function is usually denoted. For any $(x, y) \neq 0$, $\text{atan2}(y, x)$ is the angle between the vector and the positive x-axis, in the following sense. The angle is positive in the upper half-plane ($y \geq 0$), and negative in the lower half-plane ($y < 0$). In terms of the standard arctangent, arctan: $\mathbb{R} \to (-\pi/2, \pi/2)$, it is given by

$$\operatorname{atan2}(y,x) = \begin{cases} \arctan(\frac{y}{x}) & \text{if } x > 0 \\ \arctan(\frac{y}{x}) + \pi & \text{if } x < 0 \text{ and } y \geq 0 \\ \arctan(\frac{y}{x}) - \pi & \text{if } x < 0 \text{ and } y < 0 \\ +\frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0 \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0 \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0 \end{cases} \tag{25}$$

Sometimes it is more convenient to use a different version, that I call the *angle function*, that is defined as

$$\text{angle} : \mathbb{R}^2 \backslash \{0\} \to [0, 2\pi) \tag{26}$$

$$\text{angle}(y,x) = \begin{cases} \arctan(\frac{y}{x}) & \text{if } x > 0 \text{ and } y \geq 0 \\ \arctan(\frac{y}{x}) + 2\pi & \text{if } x > 0 \text{ and } y < 0 \\ \arctan(\frac{y}{x}) + \pi & \text{if } x < 0 \\ +\frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0 \\ +\frac{3\pi}{2} & \text{if } x = 0 \text{ and } y < 0 \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0 \end{cases} \tag{27}$$

For each $(x, y) \neq 0$, angle(y, x) is the *positive* angle of the vector (x, y) with the positive x-axis, measured counter-clockwise. So its range is $(0, 2\pi)$.

2 Maxwell Equations

In SI units:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \tag{28}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{29}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{30}$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \tag{31}$$

together with $F = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. In cgs-Gauss units:

$$\nabla \cdot \mathbf{E} = 4\pi\rho \tag{32}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \tag{33}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{34}$$

$$\nabla \times \mathbf{B} = \frac{1}{c} \left(4\pi\mathbf{J} + \frac{\partial \mathbf{E}}{\partial t} \right) \tag{35}$$

together with $F = q(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B})$. These equations (except for the force equation) can be converted into each other by making the substitutions

$$\mathbf{E}_{\mathrm{SI}} \to \frac{\mathbf{E}_{\text{cgs-Gauss}}}{\sqrt{4\pi\epsilon_0}}, \quad \mathbf{B}_{\mathrm{SI}} \to \sqrt{\frac{\mu_0}{4\pi}}\mathbf{B}_{\text{cgs-Gauss}}, \quad q_{\mathrm{SI}} \to \sqrt{4\pi\epsilon_0}\, q_{\text{cgs-Gauss}} \tag{36}$$

where q is charge. Also

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \tag{37}$$

3 Lorentz Transformations

If I and I' are two intertial frames such that:

- at $t = 0$ the two frames coincide;
- I' moves with velocity v in the positive x-direction relative to I, i.e., an observer in I sees (e.g. the origin of) I' move with velocity v in the positive x-direction,

then the following Lorentz boost relates the coordinates of the two frames.

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \tag{38}$$
$$x' = \gamma\,(x - vt) \tag{39}$$
$$y' = y \tag{40}$$
$$z' = z \tag{41}$$

where $v \in (-c, c)$ and $\gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}}$. Equivalently,

$$ct' = ct\cosh\eta - x\sinh\eta \tag{42}$$
$$x' = x\cosh\eta - ct\sinh\eta \tag{43}$$
$$y' = y \tag{44}$$
$$z' = z \tag{45}$$

Here $\eta \in (-\infty, \infty)$, where $\eta > 0$ and $\eta < 0$ correspond to motion in the positive and negative x-direction, respectively, and $\eta = 0$ corresponds to no motion. We have the relations

$$\frac{v}{c} = \tanh\eta \qquad \gamma = \cosh\eta \tag{46}$$

4 The Poincaré Group

(metric 1,-1,-1,-1) The Poincaré group (sometimes called the inhomogeneous Lorentz group) is the group of isometries of Minkowski spacetime, i.e., the group of transformations that leave Minkowski inner products invariant. The Lorentz group (sometimes called the homogeneous Lorentz group) is the subgroup of the Poincaré group consisting of all linear transformations, or equivalently, all transformations that leave the origin fixed. Schematically:

$$\overbrace{\textbf{Translations}\ (4)\qquad \underbrace{\textbf{Rotations}\ (3)\qquad \textbf{Boosts}\ (3)}_{\text{Lorentz group}}}^{\text{Poincaré group}} \tag{47}$$

The characteristic property of elements Λ of the Lorentz group is that they satisfy $\eta_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$. For the Lorentz transformations connected continuously to the identity (elements of the the proper orthochronous Lorentz group), we can write an infinitesimal transformation as $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$ with $\omega^\mu{}_\nu$ infinitesimal. Using the characteristic property of Lorentz transformations, it is easily shown that $\omega_{\mu\nu} = -\omega_{\nu\mu}$. If $U(\Lambda)$ forms a representation of the (proper orthochronous) Lorentz group, then we should have, infinitesimally, $U(\Lambda) = \mathbb{I} - \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}$. The Lie algebra therefore consists of elements $\omega_{\mu\nu}J^{\mu\nu}$. Since $\omega_{\mu\nu}$ is antisymmetric, we must take the $J^{\mu\nu}$ to be **antisymmetric**, because otherwise $J^{\mu\nu}$ would itself not be part of the algebra (because $J^{\mu\nu} = \delta^\mu_\rho\delta^\nu_\sigma J^{\rho\sigma}$ and the coefficient $\delta^\mu_\rho\delta^\nu_\sigma$ is not antisymmetric under $\rho \leftrightarrow \sigma$). The generators $J^{\mu\nu}$ have the Lie bracket (commutator)

$$-i[J^{\mu\nu}, J^{\rho\sigma}] = g^{\nu\rho}J^{\mu\sigma} - g^{\nu\sigma}J^{\mu\rho} - g^{\mu\rho}J^{\nu\sigma} + g^{\mu\sigma}J^{\nu\rho} \tag{48}$$

We can define $L^i = \frac{1}{2}\epsilon^{ijk}J^{jk}$ and $K^i = J^{0i}$ for $i = 1, 2, 3$, which yields

$$[L^i, L^j] = i\epsilon^{ijk}L^k \tag{49}$$

$$[K^i, K^j] = -i\epsilon^{ijk}L^k \tag{50}$$

$$[L^i, K^j] = i\epsilon^{ijk}K^k \tag{51}$$

(When deriving this, note that $\epsilon^{ijk}g^{lk} = -\epsilon^{ijk}\delta^{lk} = -\epsilon^{ijl}$ and use antisymmetry of $J^{\mu\nu}$.) Moreover we can define $\vec{J}_\pm = \frac{1}{2}(\vec{L} \pm i\vec{K})$, and this yields

$$[J^i_\pm, J^j_\pm] = i\epsilon^{ijk}K^k_\pm \tag{52}$$

$$[\vec{J}_+, \vec{J}_-] = 0 \tag{53}$$

This shows that the complexified lie algebra of the Lorentz group is isomorphic to the direct sum of two $\mathfrak{sl}(2, \mathbb{C})$ algebras.

4.1 Realizations

If γ^μ are Dirac matrices (54), then the matrices $S^{\mu\nu} := \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ satisfy the algebra (48).

The operators $J^{\mu\nu} := i(x^\mu\partial^\nu - x^\nu\partial^\mu)$ satisfy the algebra (48).

The 4×4 matrices $J^{\mu\nu}$ with $(J^{\mu\nu})^\alpha{}_\beta = i(g^{\mu\alpha}g^\nu{}_\beta - g^\mu{}_\beta g^{\nu\alpha})$ satisfy the algebra (48).

5 Dirac Matrices, the Clifford Algebra, and Spinors

5.1 Dirac Matrices and the Clifford Algebra

(metric (1,-1,-1,-1) The four Dirac matrices γ^μ are defined by the anti-commutation relations

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \tag{54}$$

Note that γ^μ is a matrix *for each value of* $\mu = 0, 1, 2, 3$. Moreover, we define

$$\gamma_\mu := g_{\mu\nu}\gamma^\nu \tag{55}$$
$$\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{56}$$
$$\sigma^{\mu\nu} := \tfrac{i}{2}\left[\gamma^\mu, \gamma^\nu\right] \tag{57}$$
$$\not{a} := a_\mu\gamma^\mu \tag{58}$$
$$\omega_\pm := \frac{1}{2}(1 \pm \gamma^5) \tag{59}$$

Then we have the following identities, where Greek indices $\mu, \nu \ldots$ run over $0, 1, 2, 3$ while latin indices $k, l \ldots$ run over $1, 2, 3$.

$$(\gamma^0)^2 = 1 \tag{60}$$
$$(\gamma^k)^2 = -1 \tag{61}$$
$$(\gamma^5)^2 = 1 \tag{62}$$
$$\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu, \quad (\nu \neq \mu) \tag{63}$$
$$\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5 \tag{64}$$
$$\gamma_\mu\gamma^\mu = 4 \tag{65}$$
$$(\gamma^0)^\dagger = \gamma^0 \tag{66}$$
$$(\gamma^k)^\dagger = -\gamma^k \tag{67}$$
$$(\gamma^5)^\dagger = \gamma^5 \tag{68}$$
$$\tag{69}$$
$$\not{a}\not{b} + \not{b}\not{a} = 2(a \cdot b) \tag{70}$$
$$\not{a}^2 = a^2 \tag{71}$$
$$\gamma^\mu\not{a}\gamma_\mu = -2\not{a} \tag{72}$$
$$\gamma^\mu\not{a}\not{b}\gamma_\mu = 4(a \cdot b) \tag{73}$$
$$\tag{74}$$
$$\omega{\pm}^2 = \omega_\pm \tag{75}$$
$$\omega_\pm\omega_\mp = 0 \tag{76}$$

The Clifford algebra consists of all elements of the form

$$\Gamma = S\,1 + V_\mu\gamma^\mu + T_{\mu\nu}\sigma^{\mu\nu} + A_\mu\gamma^5\gamma^\mu + P\gamma^5 \tag{77}$$

where $S, V_\mu, T_{\mu\nu}, A_\mu, P$ are coefficients. In the following, N the dimension of (the representation of) the Clifford algebra.

$$\begin{aligned}
\mathrm{Tr}(1) &= N && (78)\\
\mathrm{Tr}(\gamma^\mu) &= 0 && (79)\\
\mathrm{Tr}(\gamma^\mu\gamma^\nu) &= Ng^{\mu\nu} && (80)\\
\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= N(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) && (81)\\
\mathrm{Tr}(\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_{2n+1}}) &= 0 && (82)\\
\mathrm{Tr}(\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_n}) &= \mathrm{Tr}(\gamma^{\mu_n}\gamma^{\mu_{n-1}}\cdots\gamma^{\mu_1}) && (83)\\
& && (84)\\
\mathrm{Tr}(\gamma^5) &= 0 && (85)\\
\mathrm{Tr}(\gamma^5\gamma^\mu) &= 0 && (86)\\
\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu) &= 0 && (87)\\
\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= iN\epsilon^{\mu\nu\rho\sigma} && (88)\\
& && (89)\\
\mathrm{Tr}(\sigma^{\mu\nu}) &= 0 && (90)
\end{aligned}$$

If Γ is a Clifford algebra element as in (76), its components can be obtained via

$$\begin{aligned}
NS &= \mathrm{Tr}(\Gamma) && (91)\\
NV^\mu &= \mathrm{Tr}(\Gamma\,\gamma^\mu) && (92)\\
2NT^{\mu\nu} &= \mathrm{Tr}(\Gamma\,\sigma^{\mu\nu}) && (93)\\
-NA^\mu &= \mathrm{Tr}(\Gamma\,\gamma^5\gamma^\mu) && (94)\\
NP &= \mathrm{Tr}(\Gamma\,\gamma^5) && (95)
\end{aligned}$$

The Dirac conjugates of spinors ξ and algebra elements Γ, and the *reverse* of string of gamma matrices:

$$\begin{aligned}
\overline{\xi} &:= \xi^\dagger\gamma^0\\
\overline{\Gamma} &:= \gamma^0\Gamma^\dagger\gamma^0\\
(\gamma^{\mu_1}\ldots\gamma^{\mu_n})^R &:= \gamma^{\mu_n}\ldots\gamma^{\mu_1}
\end{aligned}$$

Then

$$
\begin{aligned}
\overline{\overline{\xi}} &= \xi \\
(\overline{\eta}\Gamma\xi)^* &= \overline{\xi}\,\overline{\Gamma}\,\eta \\
\overline{\sigma_{\mu\nu}} &= \sigma_{\mu\nu} \\
\overline{\gamma^5\gamma^\mu} &= \gamma^5\gamma^\mu \\
\overline{\gamma^\mu} &= \gamma^\mu \\
\overline{\gamma^5} &= -\gamma^5 \\
\overline{\omega_\pm} &= \omega_\mp \\
\overline{\eta}\Gamma\xi &= \mathrm{Tr}(\xi\Gamma\overline{\eta}) \quad \text{(Casimir's trick)} \\
\overline{\not{a}} &= \not{a}, \quad (a \in \mathbb{R})
\end{aligned}
$$

$$
\gamma_\mu \mathrm{Tr}(\Gamma\gamma^\mu) = \frac{N}{2}(\Gamma + \Gamma^R) \quad \text{(Chisholm's identity)}
$$

5.2 Dirac Spinors

From here on we set $N = 4$ (the dimension of the Clifford algebra). We define (ambiguously) the two Dirac spinors $u(p,s), v(p,s)$ for momentum p and spin s (for p on-shell and $p^0 \geq 0$) by requiring

$$
\begin{aligned}
u(p,s)\overline{u}(p,s) &= \frac{1}{2}(\not{p} + m)(1 + \gamma^5\not{s}) \\
v(p,s)\overline{v}(p,s) &= \frac{1}{2}(\not{p} - m)(1 + \gamma^5\not{s})
\end{aligned}
$$

Then, if we require $p \cdot p = m^2$, $s \cdot s = -1$, $p \cdot s = 0$, we have

$$\begin{aligned}
(\not{p} \pm m)^2 &= \pm 2m(\not{p} \pm m) \\
(1 \pm \gamma^5 \not{s})^2 &= 2(1 \pm \gamma^5 \not{s}) \\
(\not{p} + m)(\not{p} - m) &= 0 \\
(1 + \gamma^5 \not{s})(1 - \gamma^5 \not{s}) &= 0 \\
[\not{p} \pm m, 1 + \gamma^5 \not{s}] &= 0
\end{aligned}$$

$$\begin{aligned}
\overline{u}(p,s)u(p,s) &= 2m \\
\overline{u}(p,s)u(p,-s) &= 0 \\
\overline{v}(p,s)v(p,s) &= -2m \\
\overline{u}(p,s)v(p,s') &= 0
\end{aligned}$$

$$\begin{aligned}
\overline{u}(p,s)\gamma^\mu u(p,s) &= 2p^\mu \\
\overline{u}(p,s)\gamma^5\gamma^\mu u(p,s) &= -2ms^\mu \\
\overline{v}(p,s)\gamma^\mu v(p,s) &= 2p^\mu \\
\overline{v}(p,s)\gamma^5\gamma^\mu v(p,s) &= 2ms^\mu
\end{aligned}$$

Is s_1, s_2 are two orthogonal spins (that hence span the space of all spins) then

$$\begin{aligned}
\sum_s \overline{u}(p,s)u(p,s) &= \not{p} + m \\
\sum_s \overline{v}(p,s)v(p,s) &= \not{p} - m
\end{aligned}$$

5.2.1 Massless Particles and their Standard Form

Denote by $u_\pm(p) \equiv u(p, s_\pm)$ the '$\pm$ helicity states' where s_+ means that the spin is in the direction of motion (of $\vec{p}$), and s_- means that it is opposite to that direction. Then **in the limit $m \to 0$** we have

$$\begin{aligned}
u_\pm(p)\overline{u_\pm}(p) &= \omega_\pm \not{p} \\
v_\pm(p)\overline{v_\pm}(p) &= \omega_\pm \not{p}
\end{aligned}$$

In the massless case we can use the *standard form* for massless particles. Choose two basis vectors $k_0^m u, k_1^m u$ that satisfy $k_0 \cdot k_0 = k_0 \cdot k_1 = 0$ and $k_1 \cdot k_1 = -1$ **and assume that $k_0 \cdot p \neq 0$ for any massless momentum p^μ encountered in the calculation**. Define

$u_0 \equiv u_-(k_0)$ and define all other massless spinors by

$$\begin{aligned} u_+(p) &= \frac{1}{\sqrt{2p\cdot k_0}} \not{p} u_0 \\ u_-(p) &= \frac{1}{\sqrt{2p\cdot k_0}} \not{p}\not{k}_1 u_0 \end{aligned}$$

It follows indeed that $u_\pm(p)\overline{u_\pm}(p) = \omega_\pm \not{p}$.
Now define

$$s_\pm(p,q) := \overline{u}_\pm(p) u_\mp(q)$$

Then[1]

$$\begin{aligned} s_\pm(p,q) &= -s_\pm(q,p) \\ s_\pm(p,q)^* &= -s_\mp(p,q) \\ s_\pm(p,q)s_\mp(p,q) &= 2(p\cdot q) \end{aligned}$$

<u>Some more very useful identities</u> (where $\lambda = \pm 1$):

$$\begin{aligned} \not{p} u_\lambda(p) &= 0 \\ \omega_{-\lambda} u_\lambda(p) &= 0 \\ \omega_\lambda u_\lambda(p) &= u_\lambda(p) \end{aligned}$$

$$\begin{aligned} \overline{u}_\lambda(p) u_\lambda(q) &= 0 \quad (\forall p,q) \\ |s_\lambda(p,q)|^2 &= 2(p\cdot q) \end{aligned}$$

$$\gamma_\mu u_\lambda(p)\overline{u}_\lambda(q)\gamma^\mu = -2u_{-\lambda}(q)\overline{u}_{-\lambda}(p) \quad \text{(Elimination of repeated indices)}$$

$$\left[\overline{u}_\lambda(p)\gamma_\mu u_\lambda(q)\right]\gamma^\mu = 2\left[u_\lambda(q)\overline{u}_\lambda(p) + u_{-\lambda}(p)\overline{u}_{-\lambda}(q)\right] \quad \text{(Chisholm Identity)}$$

$$\overline{u}_{\lambda_1}(p)\Gamma u_{\lambda_2}(q) = \lambda_1\lambda_2 \overline{u}_{-\lambda_2}(q)\Gamma^R u_{-\lambda_1}(p) \quad \text{(Reversal)}$$

$$s_\lambda(q,p_1)s_\lambda(p_2,p_3) + s_\lambda(q,p_2)s_\lambda(p_3,p_1) + s_\lambda(q,p_3)s_\lambda(p_1,p_2) = 0 \quad \text{(Schouten identity)}$$

[1] If e.g. $k_0^\mu = (1,1,0,0)$ and $k_1^\mu = (0,0,1,1)$ then we have the following explicit form of s_+:

$$s_+(p,q) = (p^2 + ip^3)\sqrt{\frac{q^0 - q^1}{p^0 - p^1}} - (q^2 + iq^3)\sqrt{\frac{p^0 - p^1}{q^0 - q^1}}$$

5.2.2 The Standard Form for Massive Particles

Let p^μ be the momentum of a particle with generally nonzero mass m. Define

$$u_\pm(p) := \frac{1}{\sqrt{2p \cdot k_0}}(\not{p} + m)u_\mp(k_0)$$
$$v_\pm(p) := \frac{1}{\sqrt{2p \cdot k_0}}(\not{p} - m)u_\mp(k_0)$$

Then $u_\pm(p) = u(p, \pm s_0)$ with

$$s_0^\mu = \frac{1}{m}p^\mu - \frac{m}{(p \cdot k_0)}k_0^\mu$$

and indeed $s_0 \cdot s_0 = -1$ and $p \cdot s_0 = 0$.

5.3 Vector Particles

The (complex) polarization vector ε^μ of a particle with momentum p^μ satisfies $\varepsilon \cdot p = 0$. There are (**for massive particles**) three orthonormal polarizations, denoted ϵ_λ, where $\lambda = -1, 0, 1$, satisfying

$$(\varepsilon_\lambda)_\mu \overline{(\varepsilon_{\lambda'})^\mu} = -\delta_{\lambda\lambda'}, \qquad \sum_{\lambda=-1}^{1} (\varepsilon_\lambda)^\mu \overline{(\varepsilon_\lambda)^\nu} = -g^{\mu\nu} + \frac{1}{m^2}p^\mu p^\nu \tag{96}$$

(Note the minus sign in front of the δ.)

For **massless vector particles** like photons there are only two polarization states, namely two maximal helicity states. The third (longitudinal) polarization decouples completely. (Otherwise it would necessarily violate unitarity at some point when boosted enough.) Any polarization vector must now be a superposition of

$$\frac{1}{\sqrt{2}}(x + iy)^\mu \text{ and } \frac{1}{\sqrt{2}}(x - iy)^\mu \tag{97}$$

given that the particle is traveling in the z-direction. We must have

$$k \cdot \varepsilon = 0, \quad \varepsilon^0 = 0, \quad \vec{k} \cdot \vec{\varepsilon} = 0$$

We have

$$\sum_{\lambda=\pm} (\varepsilon_\lambda)^\mu \overline{(\varepsilon_\lambda)^\nu} = -g^{\mu\nu} + \frac{1}{k \cdot r}(k^\mu r^\nu + r^\mu k^\nu) \tag{98}$$

where r^μ is the *gauge vector*: it is an arbitrarily chosen massless vector not parallel to k^μ.

6 Pauli Matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{99}$$

$$[\sigma_i, \sigma_j] = 2\epsilon_{ijk}\sigma_k \tag{100}$$

7 Integrals

$$\int_0^\infty \frac{\sin(x)}{x}\,\mathrm{d}x = \frac{\pi}{2} \tag{101}$$

$$\int_0^\infty e^{-x^2/2}\,\mathrm{d}x = \sqrt{\frac{\pi}{2}} \tag{102}$$

8 Constants and Units

8.1 Constants

$$\mu_0 = 4\pi \cdot 10^{-7}\,\mathrm{N/A^2} = 4\pi \cdot 10^{-7}\,\mathrm{kg\,m\,s^2/A^2} \tag{103}$$

8.2 Units

statcoulomb

$$\mathrm{statC} = \mathrm{g}^{1/2}\mathrm{cm}^{3/2}\mathrm{s}^{-1} \tag{104}$$

9 Math Definitions

Definition 1. *A **monoid** is a set S with an operation $\circ : S \times S \to S$, such that*

- *(associativity):* $(x \circ y) \circ z = x \circ (y \circ z)$ *for all* $x, y, z \in S$
- *(identity element): There exists* $e \in S$ *such that for all* $x \in S$, $ex = xe = x$

Definition 2. *A **group** is a set G with an operation $\circ : G \times G \to G$, such that*

- *(associativity):* $(a \circ b) \circ c = a \circ (b \circ c)$ *for all* $a, b, c \in G$
- *(identity element): There exists* $e \in S$ *such that for all* $a \in G$, $e \circ a = a \circ e = a$
- *(inverse element): For all* $a \in G$ *there exists* $a^{-1} \in G$ *such that* $a \circ a^{-1} = a^{-1} \circ a = e$

Equivalently, a group is a monoid in which every element has an inverse.

Definition 3. *A **ring** is a set R with two operations $+, \circ : R \times R \to R$, such that $(R, +)$ is an abelian group, and such that $(R, \circ)$ is a monoid, and moreover such that*

- *(distributivity):* $(x + y) \circ z = x \circ z + y \circ z$ *and* $x \circ (yz) = x \circ y + x \circ z$

for all $x, y, z \in R$.

Definition 4. *A **field** is a set K with two operations $+, \circ : K \times K \to K$, such that $(K, +)$ is an abelian group with identity 0, say, and such that $(K \setminus \{0\}, \circ)$ is a group, and moreover such that*

- *(distributivity):* $(x + y) \circ z = x \circ z + y \circ z$ *and* $x \circ (yz) = x \circ y + x \circ z$

for all $x, y, z \in K$.

Definition 5. *A **vector space** over a field K is an abelian group $(V, +)$, together with a 'scalar multiplication' operation $K \times V \to V$, $(\lambda, x) \mapsto \lambda x$ such that*

- *I (identity):* $1x = x$
- *D (distributivity of scalar multiplication):* $\lambda(x + y) = \lambda x + \lambda y$
- *C (compatibility)* $\lambda(\mu x) = (\lambda\mu)x$
- *D (distributivity of vector addition):* $(\lambda + \mu)x = \lambda x + \mu x$

for all $x, y \in V$ *and* $\lambda, \mu \in K$, *where* 1 *is the multiplicative identity of the field* K.

Definition 6. *A (left) **module** over a ring R is an abelian group $(M, +)$, together with a 'scalar multiplication' operation $R \times M \to M$, $(\lambda, x) \mapsto \lambda x$ such that*

- *I (identity):* $1x = x$
- *D (distributivity of scalar multiplication):* $\lambda(x + y) = \lambda x + \lambda y$
- *C (compatibility)* $\lambda(\mu x) = (\lambda\mu)x$

- *D (distributivity of vector addition):* $(\lambda + \mu)x = \lambda x + \mu x$

for all $x, y \in M$ *and* $\lambda, \mu \in R$*, where* 1 *is the multiplicative identity of the ring* R*.*

Definition 7. *An **algebra** over a field* K *is a vector space* $(V, +)$ *over* K *together with a ('multiplication') operation* $V \times V \to V$*,* $(x, y) \mapsto xy$*, that 'has ADHD', i.e., that satisfies*

- *A (associativity):* $(xy)z = x(yz)$
- *D (distributivity):* $(x + y)z = xz + yz$
- *H (homogeneity):* $\lambda(xy) = (\lambda x)y = x(\lambda y)$
- *D (distributivity from the other side):* $x(y + z) = xy + xz$

for all $x, y, z \in V$ *and* $\lambda \in K$*.*