Floating point numbers

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Based on slides by Randal E. Bryant and David R. O’Hallaron.
Some material by Michael Kirkedal Tomsen.

Agenda

Preliminaries: biased numbers

Floating point arithmetic

- Background: Fractional binary numbers
- IEEE floating point standard
- Examples and properties
- Rounding, addition, and multiplication
- Floating point in C

Summary

Biased number representation

For *biased numbers*, the raw bits are interpreted as unsigned, and then a constant *bias* is subtracted.

Unsigned

$$B2U(X) = \sum_{i=0}^{w-1} x_i \cdot 2^i$$

Two's complement

$$B2T(X) = -x_{w-1} \cdot 2^{w-1} + \sum_{i=0}^{w-2} x_i \cdot 2^i$$

Biased

$$B2I(X) = (\sum_{i=0}^{w-1} x_i \cdot 2^i) - \text{Bias}$$

- Typically

$$Bias = 2^{w-1} - 1$$

- **Examples for** $w - 8$, **Bias** $= 127$

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Integral binary numbers

We have seen that

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is basically interpreted like

$$149_{10}$$

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Can we do the same thing for fractional numbers?

$$1011.101_2$$

Fractional numbers

$$123.456 = 1 \cdot 10^2 + 2 \cdot 10^1 + 3 \cdot 10^0 + 4 \cdot 10^{-1} + 5 \cdot 10^{-2} + 6 \cdot 10^{-3}$$

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Generally

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Even more generally, for radix r

$$a_{m-1} \cdots a_0 . a_{-1} \cdots a_{-n} = \sum_{i=-n}^{m-1} a_i \cdot r^i$$

Fractional binary numbers

Weight	2^{m-1}	2^{m-2}	$\cdots$	4	2	1	$1/2$	$1/4$	$1/8$	$\cdots$	2^{-n}
Digits	b_{m-1}	b_{m-2}	$\cdots$	b_2	b_1	b_0	b_{-1}	b_{-2}	b_{-3}	$\cdots$	b_{-n}

Representation

- Bits to the right of "binary point" represents fractional powers of 2.
- Represents rational number

$$b_{m-1} \cdots b_0 . b_{-1} \cdots b_{-n} = \sum_{i=-n}^{m-1} b_i \cdot 2^i$$

Examples of fractional binary numbers

Value	Representation
$5\frac{3}{4}$	101.11_2
$2\frac{7}{8}$	10.111_2
$1\frac{7}{16}$	1.0111_2

Observations

- Divide by 2 by logical shifting right.
- Multiply by 2 by shifting left.
- Numbers of form $0.111\ldots$ are just below 1.0.
  - $1/2 + 1/4 + 1/8 + \cdots 1/2^n + \cdots \sim 1.0$.
  - Use notation $1.0 - \epsilon$.

Representable numbers

Limitation #1

- Can only represent fractional part of form $x/2^k$
- Other rational numbers have repeating bit representation

Value	**Representation**
$\frac{1}{3}$	$0.0101010101[01]\cdots_2$
$\frac{1}{5}$	$0.001100110011[0011]\cdots_2$
$\frac{1}{10}$	$0.0001100110011[0011]\cdots_2$

Limitation #2

- Just one setting of binary point within the w bits.
  - Limited range of numbers—very small values? Very large?

The fixed-point dilemma

Consider $w = 8$

1 **bit for fraction**

- Largest number: $1111111.1_2 = 127.5_{10}$
- Increment: $0000000.1_2 = 0.5_{10}$

7 **bits for fraction**

- Largest number: $1.1111111_2 = 1.9921875_{10}$
- Increment: $0.0000001_2 = 0.0078125_{10}$

4 **bits for fraction**

- Largest number: $1111.1111_2 = 15.9375_{10}$
- Increment: $0000.0001_2 = 0.0625_{10}$

Fixed-point has same absolute precision everywhere, but this means relative precision is worse for numbers close to 0!

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IEEE Floating Point

IEEE Standard 754

- Established in 1985 as uniform standard for floating point.
  - Many idiosyncratic formats before then.
- Supported by all major CPUs.

Driven by numerical concerns

- Nice standards for rounding, overflow, underflow.
- Hard to make fast in hardware.
  - Numerical analysts predominated over hardware designers in defining standard.
  - ... but (later) Turing Award winner William Kahan secretly knew that Intel had figured out how.
  - **Beware the wrath of Kahan!**
    - `http://people.eecs.berkeley.edu/~wkahan/`

Floating Point Representation

Numerical form

$$(-1)^S \cdot M \cdot 2^E$$

- **Sign bit** S determines whether number negative or positive.
- **Significand** M normally a fractional value in range $[1, 2)$.
- **Exponent** E weights value by power of two.

Encoding

- Most significant bit is sign bit.
- Exp field encodes E (but is not equal to E).
- Frac field encodes M (but is not equal to M).

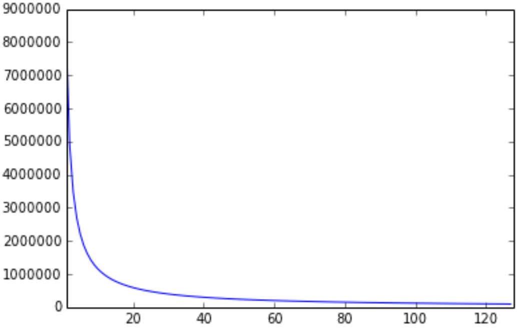
S	Exp	Frac

Why such a weird format?

The point is floating

- No fixed number of bits allocated to “fraction”.
- More bits close to 0, fewer bits for numbers with large magnitude.
- Symmetric around 0.

Density of floats



https://stackoverflow.com/a/24179424/6131552

Precision options

32-bit single precision: `float`

S	Exp	Frac
1 bit	8 bits	23 bits

64-bit double precision: `double`

S	Exp	Frac
1 bit	11 bits	52 bits

80-bit Extended precision (Intel only, never use): `long double`

S	Exp	Frac
1 bit	15 bits	64 bits

Normalised values when Exp $\neq 0\cdots 0$ and Exp $\neq 1\cdots 1$

$v = (-1)^S \cdot M \cdot 2^E$

S	Exp	Frac

- **Exponent encoded as *biased* value**

$$E = \text{Exp} - \text{Bias}$$

  - Exp: unsigned value of Exp field.
  - Bias $= 2^{k-1} - 1$, where k is number of Exp bits.
    - Single precision: 127 (Exp $\in [1, 254]$, $E \in [-126, 127]$).
    - Double precision: 1023 (Exp $\in [1, 2046]$, $E \in [-1022, 1023]$).
- **Significand coded with implied leading 1:**

$$M = 1.xxx\cdots x_2$$

  - $xxx\cdots x$: bits of Frac field.
  - Minimum when Frac $= 0000\cdots 0$ ($M = 1$).
  - Maximum when Frac $= 1111\cdots 1$ ($M = 2 - \epsilon$).
  - Get extra leading bit for free.

Normalised encoding example

$$\boxed{v = (-1)^S \cdot M \cdot 2^E} \quad \boxed{E = \text{Exp} - \text{Bias}}$$

Value: `float F = 15213.0`

$$15213_{10} = 11101101101101_2$$
$$= 1.1101101101101_2 \cdot 2^{13}$$

Significand

$$M = 1.1101101101101_2$$
$$\text{Frac} = \quad 11011011011010000000000_2$$

Exponent

$$E = 13_{10}$$
$$\text{Bias} = 127_{10}$$
$$\text{Exp} = 140_{10} = 10001100_2$$

Result	0	10001100	11011011011010000000000

Denormal values

$$\boxed{v = (-1)^S \cdot M \cdot 2^E} \qquad \boxed{E = 1 - \text{Bias}}$$

Occur when Exp $= 000\cdots 0_2$.

- **Exponent encoded as**

$$E = 1 - \text{Bias}$$

- **Significand coded with implied leading 0:**

$$M = 0.xxx\cdots x_2$$

- **Cases**
  - $\text{Exp} = 000\cdots 0_2, \text{Frac} = 000\cdots 0_2$
    - Represents zero value.
    - Note distinct values $-0, +0$ – why do you think that is?
  - $\text{Exp} = 000\cdots 0_2, \text{Frac} \neq 000\cdots 0_2$
    - Numbers closest to 0.0.
    - Called **Subnormal numbers.**
    - Ensure that $x \neq y \Rightarrow x - y \neq 0$, i.e. avoid overflow.

Special values

Occur when Exp $= 111\cdots1_2$.

When Exp $= 111\cdots1_2$, Frac $= 000\cdots0_2$

- Represents $\pm\infty$.
- Typically the result of *overflow*.
  - Overflow can be negative!
  - *Underflow* is when the result becomes zero due to rounding.
- Both positive and negative.
- Examples:

$$\frac{1}{0} = \frac{-1}{-0} = \infty \qquad \frac{1}{-0} = -\infty$$

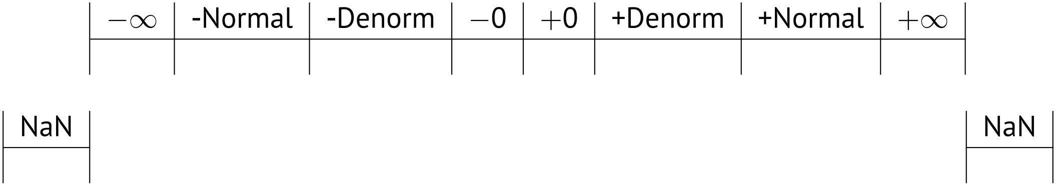
When Exp $= 111\cdots1_2$, Frac $\neq 000\cdots0_2$

- Not A Number (NaN).
- Represents case when no numeric value can be determined.
- Examples:

$$\texttt{sqrt}(-1) \qquad \infty - \infty \qquad \infty \cdot 0$$

The floating point number line

$\leftarrow$ very negative E very positive E $\rightarrow$



Note that NaNs are unordered:

- NaN is different from everything *even other NaNs*!
  - `NaN == NaN` is false.
  - Floating-point equality is not reflexive!
- `NaN > x` and `NaN < x` is false for all x.

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Play the game

`https://topps.diku.dk/compsys/floating-point.html`

Tiny 8-bit floating point example

s	Exp	Frac
1b	4b	3b

8-bit floating point representation

- Sign bit is the most significant bit (leftmost).
- The next four bits are Exp with a bias of 7.
- The last three bits are Frac.

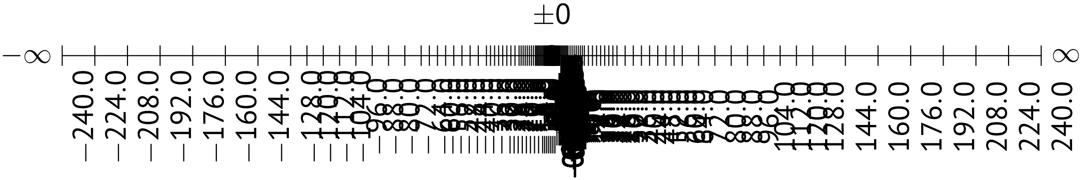
Same general form as IEEE Format

- Normalised, denormalised.
- Representation of 0, NaN, both infinities.

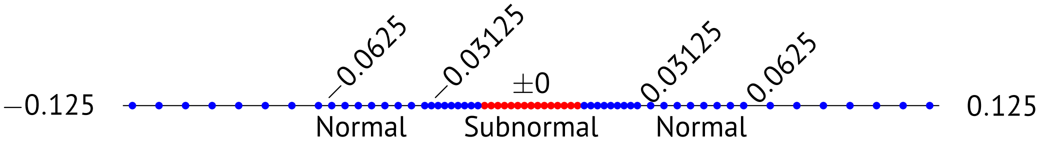
Dynamic range of positive numbers

	Sign	Exp	Frac	Exp	Value	
Denormalised	0	0000	000	-6	0	
	0	0000	001	-6	$1/8 \cdot 1/64 = 1/512$	closest to zero
	0	0000	010	-6	$2/8 \cdot 1/64 = 2/512$	
	...					
	0	0000	111	-6	$7/8 \cdot 1/64 = 7/512$	largest denorm
Normalised	0	0001	000	-6	$8/8 \cdot 1/64 = 8/512$	smallest norm
	0	0001	001	-6	$9/8 \cdot 1/64 = 9/512$	
	...					
	0	0110	110	-1	$14/8 \cdot 1/2 = 14/16$	
	0	0110	111	-1	$15/8 \cdot 1/2 = 15/16$	Closest to 1
	0	0111	000	0	$8/8 \cdot 1 = 1$	
	0	0111	001	0	$9/8 \cdot 9/8 = 1$	Closest to 1
	0	0111	010	0	$10/8 \cdot 10/8 = 1$	
	...					
	0	1110	110	7	$14/8 \cdot 128 = 224$	
	0	1110	111	7	$15/8 \cdot 128 = 240$	
	0	1111	000	N/A	∞	

Distribution of values



Distribution of values (zooming in)



- Note how the distribution gets denser towards zero.
- Note the big gap there would be around 0 if we did not have subnormals.
- Each of the spans with same distance between neighbors corresponds to numbers with same Exp.

s	Exp	Frac
1b	4b	3b

Useful properties of the IEEE encoding

S	Exp	Frac

- **Floating-point zero same as integer zero**
  - All bits 0.
  - ...but negative zero is different.
- **Can almost compare floats with unsigned integer comparisons**
  - Must first compare sign bit.
  - Must consider $-0 = 0$.
  - NaNs problematic:
    - Greater than any other value (because Exp $= 111 \cdots 1_2$).
    - What should comparison yield?
  - Otherwise OK:
    - Normalised and denormalised compare as expected.
    - Infinities ordered properly relative to finities.

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Basic idea behind floating point operations

$$x +_f y = \text{Round}(x + y)$$

$$x \times_f y = \text{Round}(x \times y)$$

- **Basic idea**
  - First *compute exact result*!
  - Then round it to fit into desired precision.
    - Overflow if exponent too large.
    - *Round to fit* into Frac.

Rounding and rounding modes

- There's more than one way to round a number, here to an integer.

	1.40	1.60	1.50	2.50	-1.50
Towards zero					

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Towards ∞	2	2	2	3	-1
Nearest even					

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Towards ∞	2	2	2	3	-1
Nearest even ∞	1	2	2	3	-2

- "Round to nearest, ties to even" is the default rounding mode.

Closer look at *nearest even*

- **Default rounding mode**
  - But can be changed dynamically.
    - `https://www.gnu.org/software/libc/manual/html_node/Rounding.html`
    - Never do this.
  - All others are statistically biased.
    - Biased: Sum of set of positive numbers will consistently be over- or under-estimated.
- **Applying to other decimal places / bit positions**
  - When exactly halfway between two possible values:
    - Round so that least significant digit is even.
  - E.g. rounding to nearest hundredth:
    - 7.8949999:

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    - 7.8990001:

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    - 7.8850000: 7.88

Rounding binary numbers

- **Binary fractional numbers**
  - "Even" when least significant bit is 0.
  - "Half way" when bits to right of rounding position are $100\cdots_2$.
- **Examples**
  - Round to nearest $1/4$ (2 bits right of binary point).

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Rounding binary numbers

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Floating point multiplication (assuming operands are numbers)

$$((-1)^{S_3} \cdot M_3 \cdot 2^{E_3}) = ((-1)^{S_1} \cdot M_1 \cdot 2^{E_1}) \cdot ((-1)^{S_2} \cdot M_2 \cdot 2^{E_2})$$

- **Exact result**

$$\begin{aligned} S_3 &= S_1 \oplus S_2 \\ M_3 &= M_1 \cdot M_2 \\ E_3 &= E_1 + E_2 \end{aligned}$$

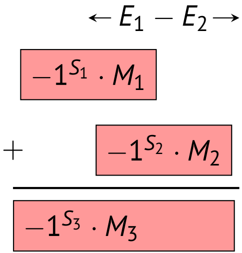
  where $\oplus$ is exclusive-or.
- **Fixing**
  - If $M_3 \geq 2$, shift M_3 right and increment E_e.
  - If E_3 out of range, overflow to ∞.
  - Round M_3 to fit Frac precision.
- **Implementation**
  - Biggest chore is multiplying significands.

Floating point addition (assuming operands are numbers)

$$((-1)^{S_3} \cdot M_3 \cdot 2^{E_3}) = ((-1)^{S_1} \cdot M_1 \cdot 2^{E_1}) + ((-1)^{S_2} \cdot M_2 \cdot 2^{E_2})$$

- **Approach**
  - Assume without loss of generality that $E_1 \geq E_2$.
  - Rewrite smaller number such that its exponent matches E_1:

$$((-1)^{S_3} \cdot M_3 \cdot 2^{E_3}) = ((-1)^{S_1} \cdot M_1 \cdot 2^{E_1}) + ((-1)^{S_2} \cdot M_2' \cdot 2^{E_1})$$



- **Exact result**
  - Sign S_3, significant M_3:
    - Result of signed addition.
- **Fixing**
  - If $M_3 \geq 2$, shift M_3 right and increment E_3.
  - If $M_3 < 1$, shift M left k positions and decrement E_3 by k.
  - If E_3 out of range, overflow to ∞.
  - Round M to fit Frac precision.

Example of floating-point addition with a 2-bit significand

$$
\begin{aligned}
& (-1.01 \cdot 2^2) + (1.1 \cdot 2^4) & \\
= \quad & (-1.01 \cdot 2^2) + (110.0 \cdot 2^2) & \text{Align exponents} \\
= \quad & (-1.01 + 110.0) \cdot 2^2 & \text{Distributivity} \\
= \quad & 100.11 \cdot 2^2 & \text{Add significands} \\
= \quad & 1.0011 \cdot 2^4 & \text{Normalise} \\
= \quad & 1.01 \cdot 2^4 & \text{Perform rounding}
\end{aligned}
$$

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Preliminaries: biased numbers

Floating point arithmetic

- Background: Fractional binary numbers
- IEEE floating point standard
- Examples and properties
- Rounding, addition, and multiplication
- Floating point in C

Summary

Floating point in C

- **C guarantees two types**
  - `float`: 32-bit single precision.
  - `double`: 64-bit single precision.
- **Conversions/casting**
  - Casting between `int`, `float`, and `double` changes bit represensation.
  - `double`/`float` to `int`
    - Truncates fractional part.
    - Like rounding toward zero.
    - Not defined when out of range or NaN: generally sets to TMin.
  - `int` to `double`
    - Exact conversion as long as `int` fits in 53 bits.
  - `int` to `float`
    - Will round according to rounding mode.

Floating point is exciting!

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- A `double` storing horizontal velocity of the rocket was converted to a 16-bit signed integer.
- The number was larger than 32767 so the conversion failed, causing an exception, crashing the guidance module.

Floating point puzzles

For each of the following C expressions, either

- Argue that it is true for all argument values.
- Explain why it's not.

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- **IEEE floating point has clear properties.**
  - But they may not match your intuition.
- **Represents numbers of the form $M \cdot 2^E$.**
- One can reason about operations independent of implementation.
  - Computed with perfect precision and then rounded.
  - But rounded after *every* “primitive” operation (e.g. addition, multiplication).
- **Not the same as $\mathbb{Q}/\mathbb{R}$ arithmetic.**
  - Violates associativity and distributivity, mostly due to rounding.
  - Sometimes makes life difficult for heavy-duty numerical programming.
  - But carefully designed such that “naive” use mostly does what one expects.