Bits, Bytes and Words

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Based on slides by Randal E. Bryant and David R. O'Hallaron

Agenda

Representing information as bits

Bit-level manipulation

Integers
- Representation: unsigned and signed
- Conversion, casting
- Expanding, truncating

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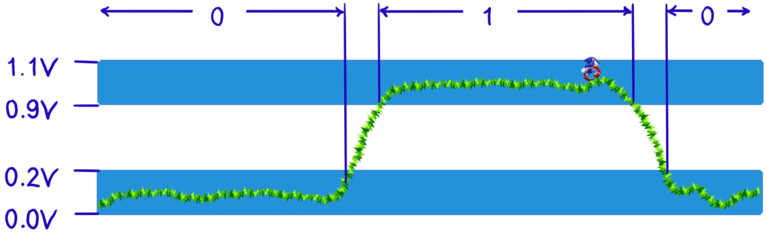
Everything is bits

- Each bit is 0 or 1
- By interpreting sets of bits in various ways...
  - ...computers determine what to do.
  - ...represent and manipulate numbers, sets, strings—*data*.

Why bits? Why not decimals? Could it have been some some other way?

Everything is bits

- **Why bits? Electronic implementation.**
  - Easy to store with bistable elements.
  - Reliably transmitted on noisy and inaccurate wires (error correction).



- **... But there exist models that do not use bits.**
  - The Soviet Setun computer used ternary *trits*.
  - Quantum computers use *qubits* that are in a superposition of the two states.
    - ...error correction is the main challenge here.

Binary numbers

- **Base 2 number representation.**
  - Represent 15213_{10} as 11101101101101_2
  - Represent 1.20_{10} as $1.0011001100110011[0011]\ldots_2$
  - Represent 1.5213×10^4 as $1.1101101101101_2 \times 2^{13}$
- **Machine numbers are of some finite size.**
  - If we use k bits to represent a number, only 2^k distinct values are possible.
  - How we interpret those bits can vary.
  - **Why do we use finite-sized numbers?**

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  - If we use k bits to represent a number, only 2^k distinct values are possible.
  - How we interpret those bits can vary.
  - **Why do we use finite-sized numbers?**
  - A "k-bit machine" handles numbers of up to k bits "natively" (meaning fast).

Encoding byte values

Byte = 8 bits

- (Machine-specific, but is true for all mainstream machines.)
- 256 different values.
- Binary 00000000_2 to 11111111_2.
- Decimal 0_{10} to 255_{10}.
- Hexadecimal 00_{16} to FF_{16}.
  - Base 16 number representation.
  - Uses characters 0—9 and A—F.
  - In C we write $FA1D37B_{16}$ as
    - `0xFA1D37B`
    - `0xfa1d37b` (case does not matter)

Hex	Dec	Bin
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

Let’s play a game

http://topps.diku.dk/compsys/integers.html

Example data representations

C data type	Typical 16-bit	Typical 32-bit	Typical 64-bit	x86-64
`char`	1	1	1	1
`short`	1	2	2	2
`int`	2	4	4	4
`long`	4	4	8	8
`int32_t`	4	4	4	4
`int64_t`	8	8	8	8
`float`	4	4	4	4
`double`	8	8	8	8
`long double`	-	-	-	10
`pointer`	2	4	8	8

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Boolean algebra

Developed by George Boole in 19th century

- Algebraic representation of logic ("truth values").
- Encode *true* as 1 and *false* as 0.

And

&	0	1
0	0	0
1	0	1

Or

<table>
<tr><th>|</th><th>0</th><th>1</th></tr>
<tr><td>0</td><td>0</td><td>1</td></tr>
<tr><td>1</td><td>1</td><td>1</td></tr>
</table>

Not

~	
0	1
1	0

Exclusive-or

^	0	1
0	0	1
1	1	0

- These operations can be implemented with tiny electronic *gates*.

General boolean algebras

- The truth tables generalise to operate on *bit vectors*, applied elementwise.

```
  01101001      01101001      01101001
& 01010101    | 01010101    ^ 01010101    ~ 01101001
----------    ----------    ----------    ----------
  01000001      01111101      00111100      10010110
```

- This is the form they take when available in programming languages such as C.

Bit-level operations in C

Operations &, |, ~, ^ available in C.

- Apply to any integral type.
  - E.g. `long`, `int`, `short`, `char`…
- Interpret operands as bit vectors.
- Applied bit-wise.

Examples

- `~0x41 = 0xBE`
  - ~01000001_2 = 10111110_2
- `~0x00 = 0xFF`
  - ~00000000_2 = 11111111_2
- `0x69 & 0x55 = 0x41`
  - 01101001_2 & 01010101_2 = 01000001_2
- `0x69 & 0x55 = 0x7D`
  - 01101001_2 & 01010101_2 = 01111101_2

Contrast: logical operators in C

The logical operators interpret numbers as *single boolean values*, not as bit vectors!

- **&&, ||, !**
  - View 0 as false.
  - Anything nonzero as true.
  - Always produce 0 or 1.
  - **Early termination:** `1 || (0/0)` is safe.
- **Examples**
  - `!0x41 = 0x00`
  - `!0x00 = 0x01`
  - `!!0x41 = 0x01`
  - `0x69 && 0x55 = 0x01`
  - `0x69 || 0x55 = 0x01`
- **Do not confuse the logical and bitwise operators!**

Shift operations

- **Left shift x << y**
  - Shift bit-vector x left by y positions.
    - Throws away excess bits on the left.
    - Fills with zeroes on right.
- **Right shift x >> y**
  - Shift bit-vector x right by y positions.
    - Throws away excess bits on the left.
  - Logical shift: Fill with 0s on left.
  - Arithmetic shift: Replicate most significant bit on left.
- **Undefined behaviour**
  - Shifting a negative amount or by the vector size or more.

```
x                   01100010
----------------------------
x << 3              00010000
x >> 2 (log)        00011000
x >> 2 (arith)      00011000
```

```
x                   10100010
----------------------------
x << 3              00010000
x >> 2 (log)        00101000
x >> 2 (arith)      11101000
```

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Encoding integers

Suppose x_i is the ith bit of a w-bit word (with x_0 being the least significant bit).

Unsigned

$$B2U(X) = \sum_{i=0}^{w-1} x_i \cdot 2^i$$

Two's complement

$$B2T(X) = -x_{w-1} \cdot 2^{w-1} + \sum_{i=0}^{w-2} x_i \cdot 2^i$$

```
int16_t x =  15213;
int16_t y = -15213;
```

	Decimal	Hex	Binary
x	15213	3 B 5 D	0011 1011 0110 1101
y	-15213	C 4 9 3	1100 0100 1001 0011

Sign bit

- For 2's complement, most significant bit (x_{w-1}) indicates sign.
  - 0 for non-negative.
  - 1 for negative.

Two's complement encoding example

```
int16_t x =  15213; // 0011 1011 0110 1101
int16_t y = -15213; // 1100 0100 1001 0011
```

Weight	15213		-15213	
1	1	1	1	1
2	0	0	1	2
4	1	4	0	0
8	1	8	0	0
16	0	0	1	16
32	1	32	0	0
64	1	64	0	0
128	0	0	1	128
256	1	256	0	0
512	1	512	0	0
1024	0	0	1	1024
2048	1	2047	0	0
4096	1	4096	0	0
8192	1	8192	0	0
16384	0	0	1	16384
-32768	0	0	1	-32768
Sum		**15213**		**-15213**

Numeric ranges

Unsigned

$$\mathrm{UMin} = 0 = 0\ldots0_2$$
$$\mathrm{UMax} = 2^w - 1 = 1\ldots1_2$$

Two's complement

$$\mathrm{TMin} = -2^{w-1} = 10\ldots0_2$$
$$\mathrm{TMax} = 2^{w-1} - 1 = 01\ldots1_2$$
$$-1 = 1\ldots1_2$$

Values for $w = 16$:

	Decimal	Hex	Binary
UMax	65535	F F F F	1111 1111 1111 1111
TMax	32767	7 F F F	0111 1111 1111 1111
TMin	-32768	8 0 0 0	1000 0000 0000 0000
-1	-1	F F F F	1111 1111 1111 1111
0	0	0 0 0 0	0000 0000 0000 0000

Values for different word sizes

	W			
	8	**16**	**32**	**64**
UMax	255	65,535	4,294,967,295	18,446,744,073,709,551,615
TMax	127	32,767	2,147,483,647	9,223,372,036,854,775,807
TMin	-128	-32,768	-2,147,483,648	-9,223,372,036,854,775,808

Observations

$$|\text{TMin}| = \text{TMax} + 1$$
$$|\text{UMax}| = 2 \cdot \text{TMax} + 1$$

Note the assymetric range.

C Programming

- #include <limits.h>
- Declares constants, e.g:
  - ULONG_MAX
  - LONG_MAX
  - LONG_MIN
- Values are platform-specific.

Unsigned and signed numeric values (here $w = 4$)

x	$B2U(x)$	$B2T(x)$
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	8	-8
1001	9	-7
1010	10	-6
1011	11	-5
1100	12	-4
1101	13	-3
1110	14	-2
1111	15	-1

- **Equivalence**
  - Same encoding for non-negative values.
- **Uniqueness**
  - Every bit pattern represents distinct integer value.
  - Each representable integer has unique bit encoding.
  - The representation is **bijective**.
- **Can invert mappings**
  - $U2B(x) = B2U^{-1}(x)$
    - Bit pattern for unsigned integer.
  - $T2B(x) = B2T^{-1}(x)$
    - Bit pattern for two's complement integer.

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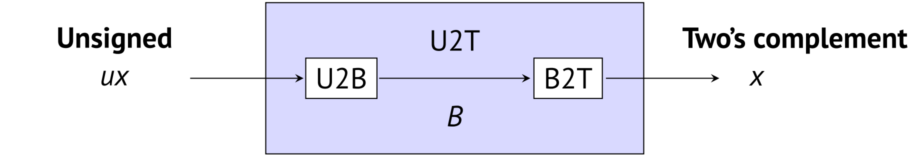
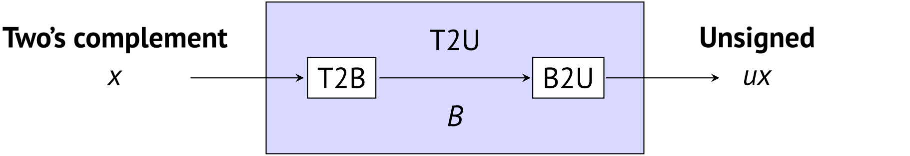
Mapping between signed and unsigned



Maintain same bit pattern



Mapping between unsigned and two's complement numbers:
Keep bit representations and reinterpret.

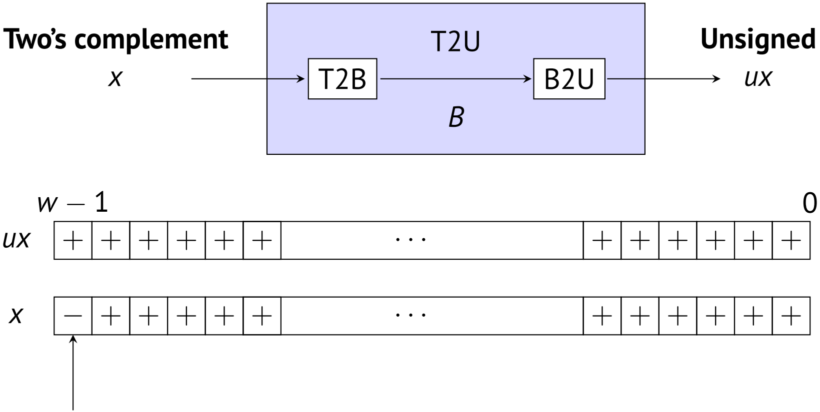
Mapping signed ⇔ unsigned

Bits	Signed	Unsigned
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	-8	8
1001	-7	9
1010	-6	10
1011	-5	11
1100	-4	12
1101	-3	13
1110	-2	14
1111	-1	15

$\longrightarrow$ T2U $\longrightarrow$

$\longleftarrow$ U2T $\longleftarrow$

Relation between signed and unsigned

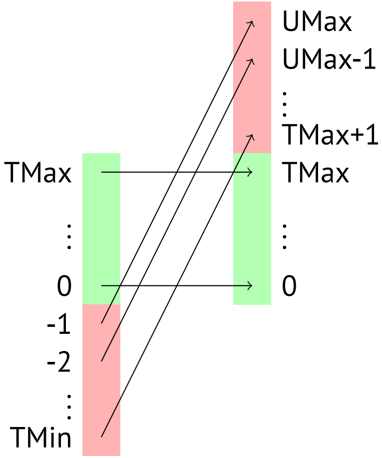


Large negative weight becomes **large positive weight.**

Conversion (that is, *reinterpretation*) visualized

Two's complement to unsigned

- Ordering inversion.
- Negative numbers become large positive numbers.



Signed versus unsigned in C

C makes working with this more error-prone than it should be.

Types
- Signedness part of type: `unsigned int`, `int32_t`, `uint32_t`.

Constants
- By default are considered signed integers.
- Unsigned with U suffix: `0U, 4294967259U`

Casting
- Explicit casting between signed and unsigned:

```
int tx, ty;
unsigned int ux, uy;
tx = (int) ux;
uy = (unsigned int) ty;
```

- Implicit casting due to assignments and other expressions:

```
tx = ux;
uy = ty;
```

Casting surprises

Evaluation

- If there is a mix of unsigned and signed in single expression, *signed values implicitly cast to unsigned*.
- Including comparison operations <, >, ==, <=, >=.
- Examples for $w = 32$: $TMIN = -2,147,483,648$, $TMAX = 2,147,483,647$:

Const LHS	Relation	Const RHS	Evaluation
0	==	0U	

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-1	<	0	

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-1	>	0U	unsigned
2147483647	>	-2147483647-1	

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2147483647U	<	-2147483647-1	

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2147483647U	<	-2147483647-1	unsigned
-1	>	-2	

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(unsigned int)-1	>	-2	

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2147483647	<	2147483648U	

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Casting between signed and unsigned: basic rules

- Bit pattern is maintained.
- ...but reinterpreted.
- Can have unexpected effects: adding or subtracting 2^w.

- Expression containing signed and unsigned int:
  - `int` is cast to `unsigned int`!
  - **When can this go bad?**

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Advice: Never do arithmetic on unsigned types—only use them for bit operations.

But: Some C operators (`sizeof`) and many functions return unsigned types (e.g. `size_t`).

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Truncation

Task

- Given $k + w$-bit signed integer x.
- Convert it to w-bit integer x' with same value i possible.

Approach

- Remove the k most significant bits.
- Equivalent to computing $x' = x \bmod 2^w$.
- Can cause numerical change if number has no representation in w bits.
- Otherwise safe.

w	Bits	Two's complement
8	11111111_2	-1_{10}
4	1111_2	-1_{10}
8	10000000_2	-128_{10}
4	0000_2	0_{10}

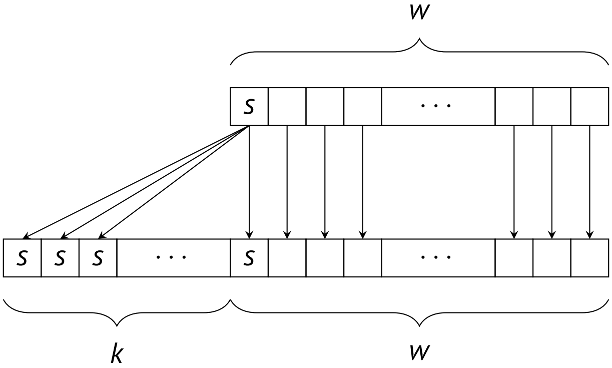
Sign extension

Task

- Given w-bit signed integer x.
- Convert it to $w+k$-bit integer x' with same value.

Approach

- Make k copies of sign bit (most significant bit):
- $x' = \underbrace{x_{w-1}, \ldots, x_{w-1}}_{k \text{ copies of sign bit.}}, x_{w-1}, \ldots, x_0$



Sign extension example

```
short int x = 15213;
int       ix = (int) x;
short int y = -15213;
int       iy = (int) y;
```

	Decimal	Hex	Binary
x	15213	3B 6D	0011 1011 0110 1101
ix	15213	00 00 3B 6D	0000 0000 0000 0000 0011 1011 0110 1101
y	-15213	C4 93	1100 0100 1001 0011
iy	-15213	FF FF C4 93	1111 1111 1111 1111 1100 0100 1001 0011

Summary: basic rules for expanding and truncating

Expanding (e.g. `short` to `int`)

- Unsigned: zeros added.
- Signed: sign extension.
- Both yield expected result.

Truncating (e.g. `unsigned int` to `unsigned short`)

- Bits are truncated.
- Result reinterpreted.
- Unsigned: modulo operation.
- Signed: similar to a modulo operation.
- For small numbers yield expected behaviour.