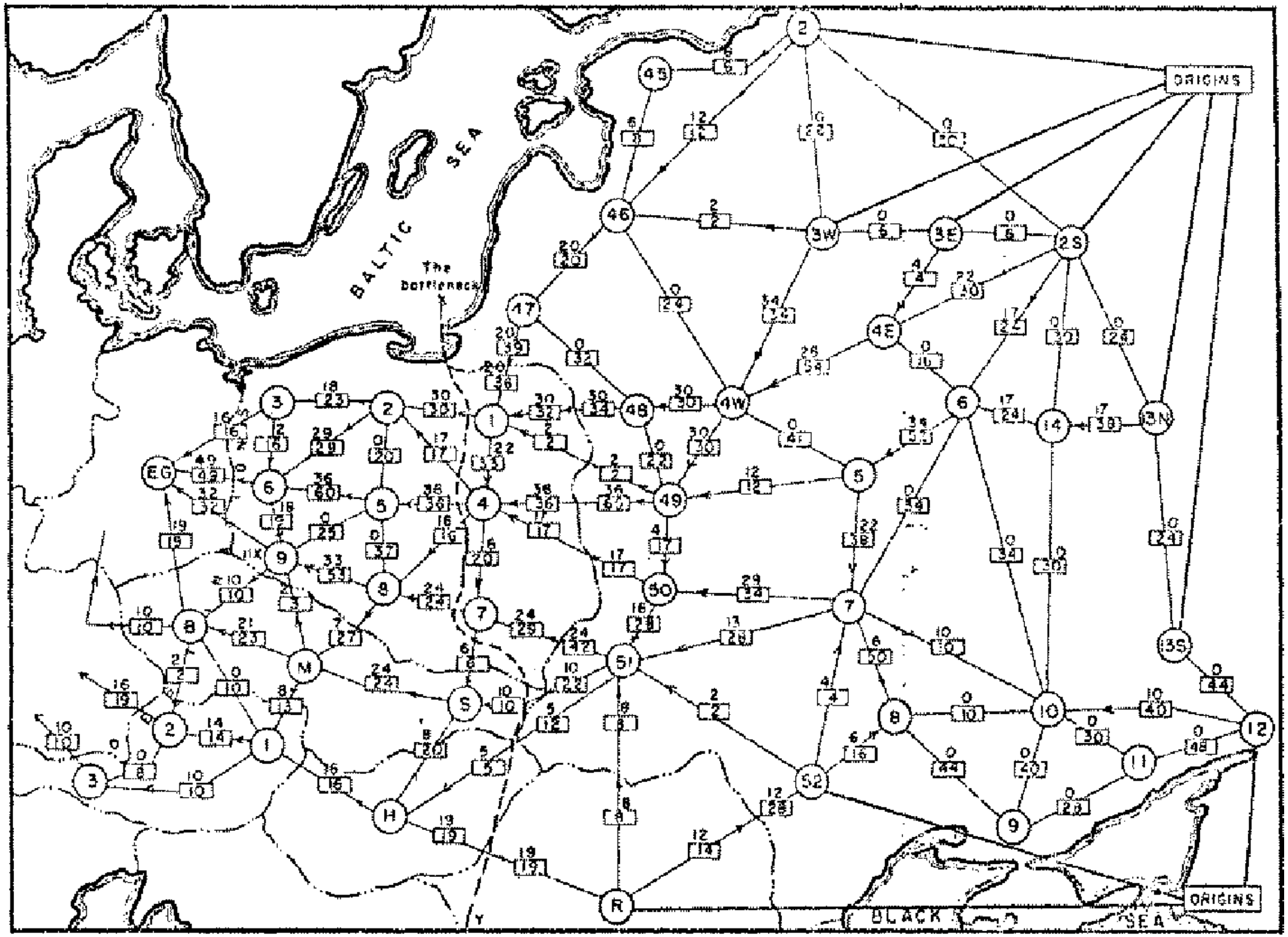
Introduction to Algorithms

Network Flows

Soviet Rail Network



Reference: *On the history of the transportation and maximum flow problems*. Alexander Schrijver in Math Programming, 91: 3, 2002.

Network Flow Applications

Max flow and min cut.

- Two very rich algorithmic problems.
- Cornerstone problems in combinatorial optimization.
- Beautiful mathematical duality.

Nontrivial applications / reductions.

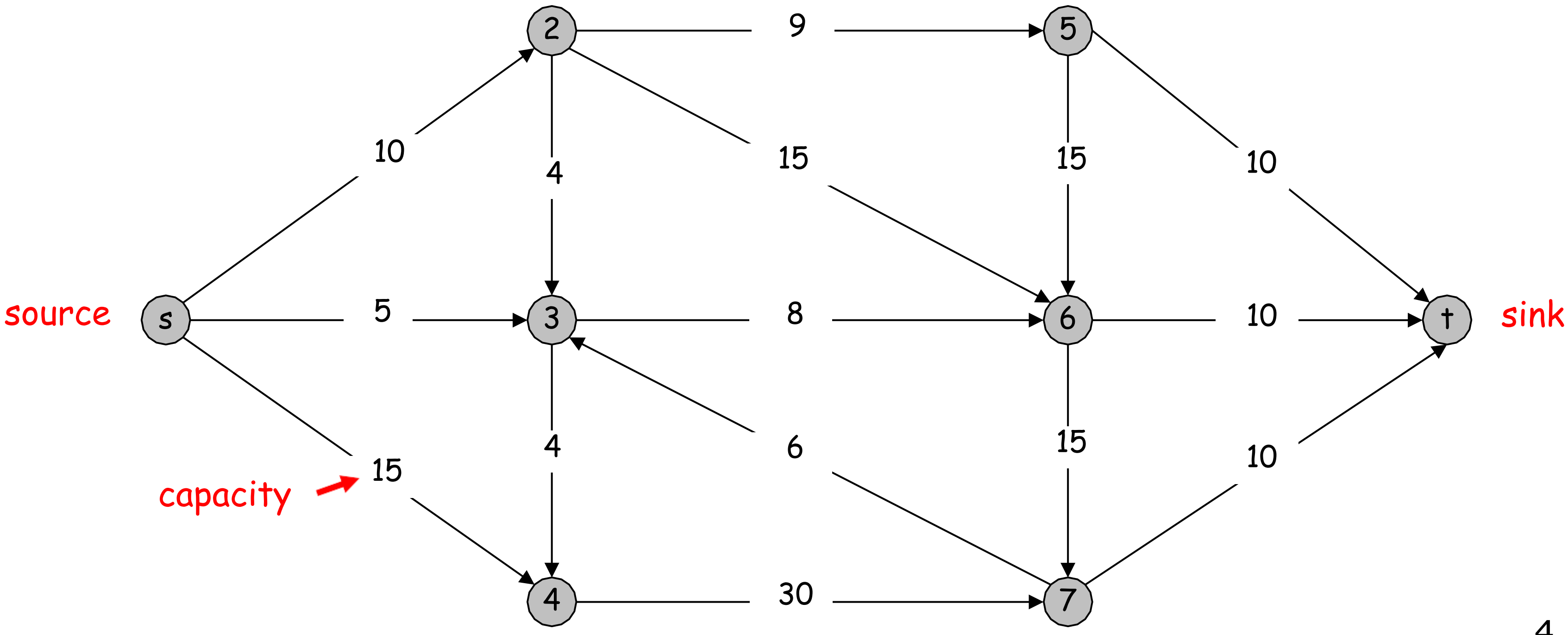
- Data mining.
- Open-pit mining.
- Project selection.
- Airline scheduling.
- Bipartite matching.
- Baseball elimination.
- Image segmentation.
- Network connectivity.

Minimum s-t Cut Problem

Given a directed graph G = (V, E) = directed graph and two distinguished nodes: s = source, t = sink.

Suppose each directed edge e has a nonnegative capacity $c(e)$

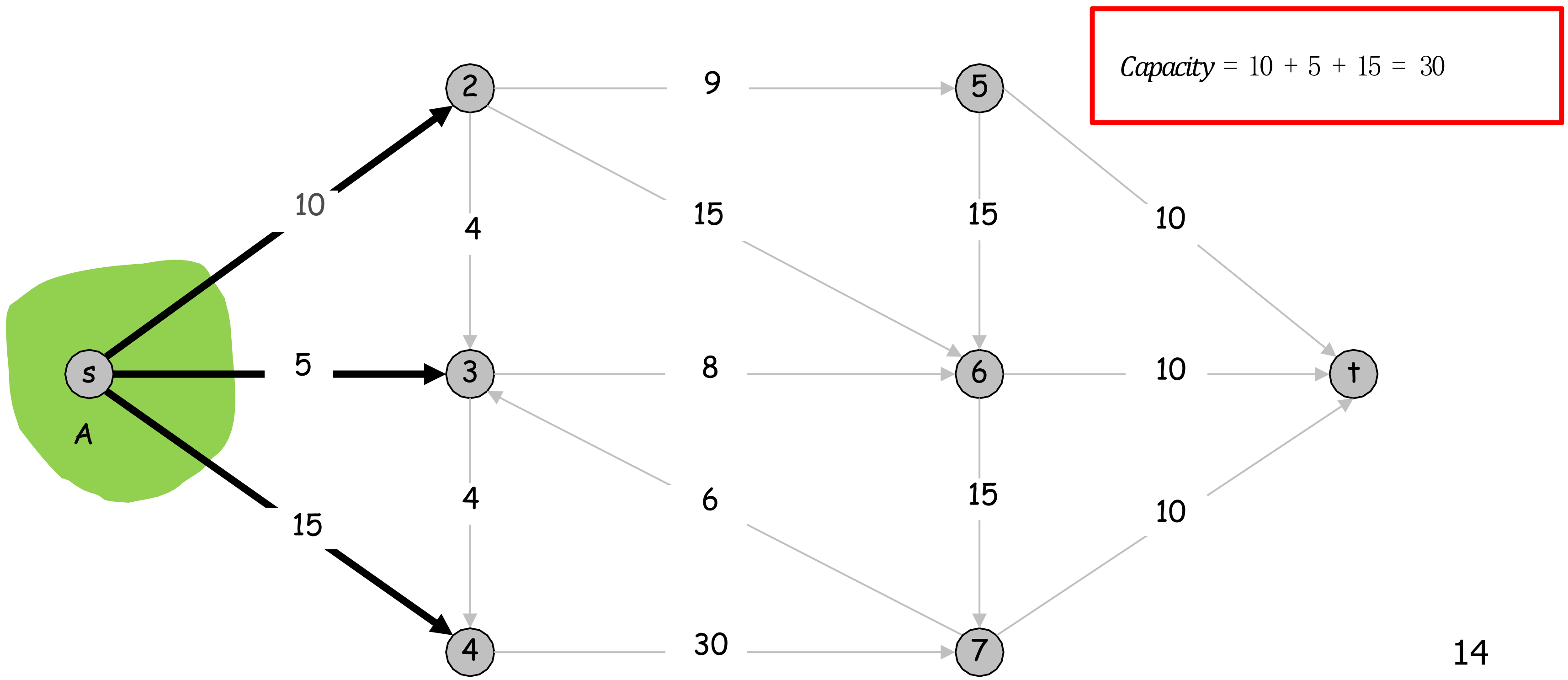
Goal: Find a cut separating $s,\ t$ that cuts the minimum capacity of edges.



s-t cuts

Def. An s-t cut is a partition (A, B) of V with s $\in$ A and t $\in$ B.

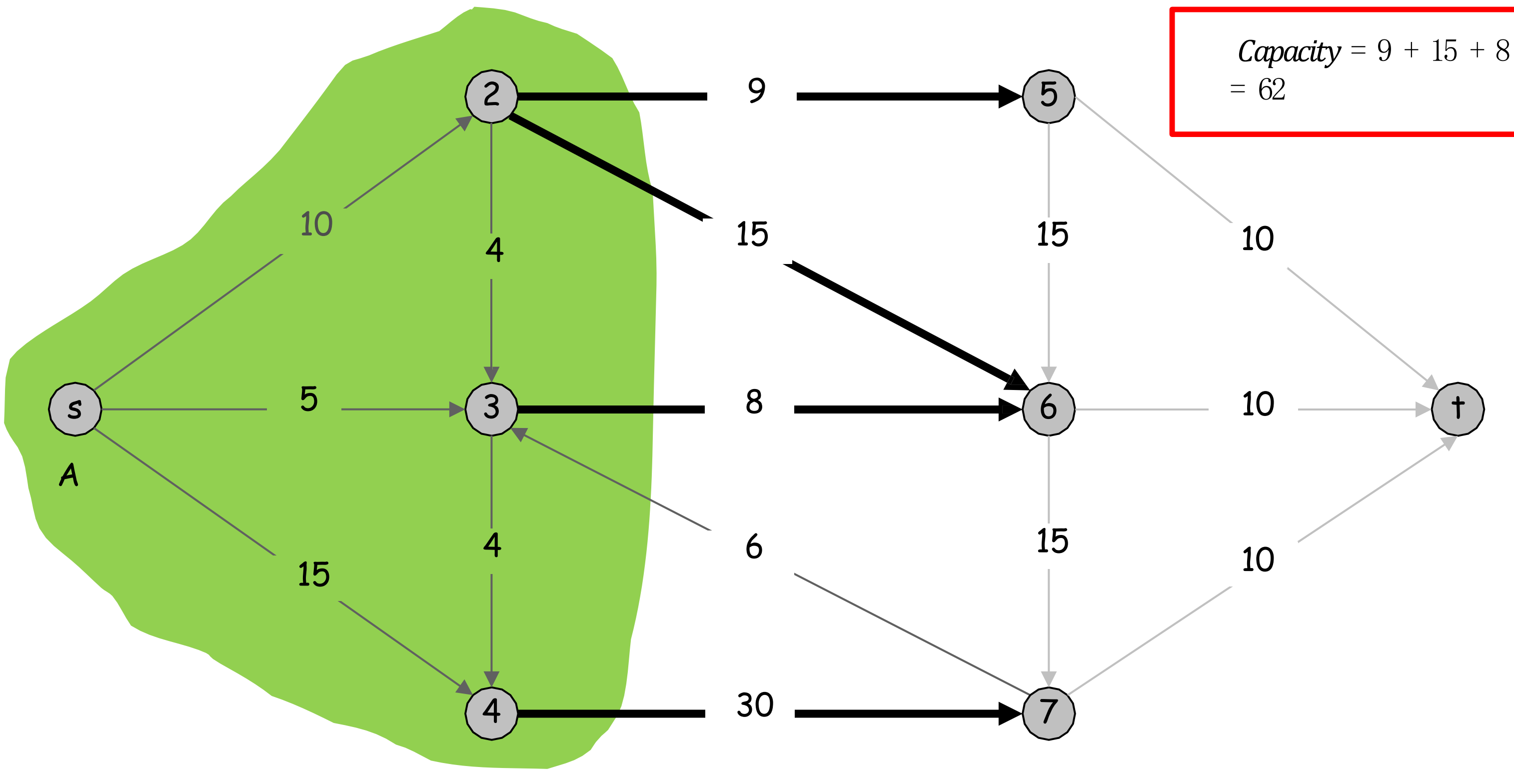
Def. The capacity of a cut (A, B): $cap(A, B) = \sum_{e \text{ out of } A} c(e)$



s-t cuts

Def. An s-t cut is a partition (A, B) of V with s $\in$ A and t $\in$ B.

Def. The capacity of a cut (A, B): $cap(A, B) = \sum_{(u,v)\,:\,u \in A,\, v \in B} c(u, v)$



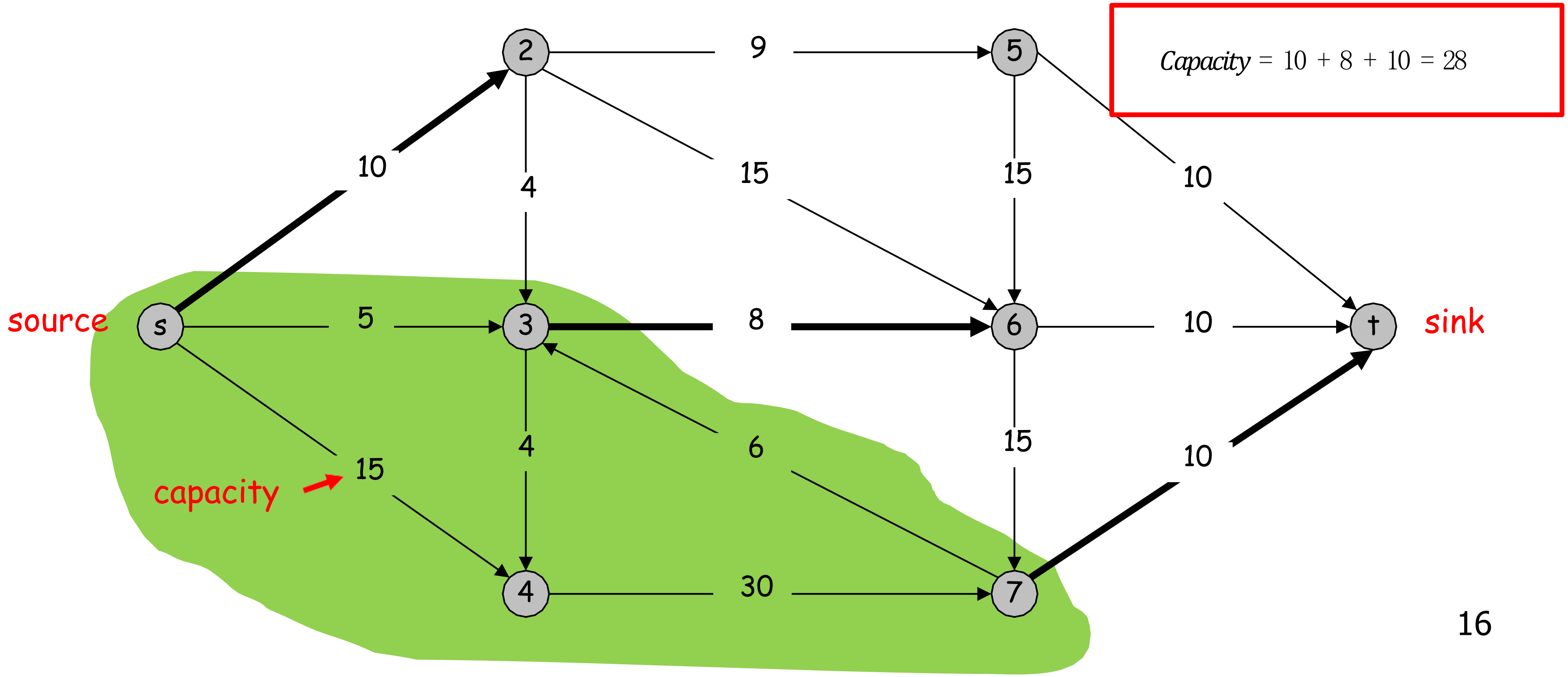
$Capacity = 9 + 15 + 8 + 30$
$= 62$

Minimum s-t Cut Problem

Given a directed graph G = (V, E) = directed graph and two distinguished nodes: s = source, t = sink.

Suppose each directed edge e has a nonnegative capacity $c(e)$

Goal: Find a s-t cut of minimum capacity

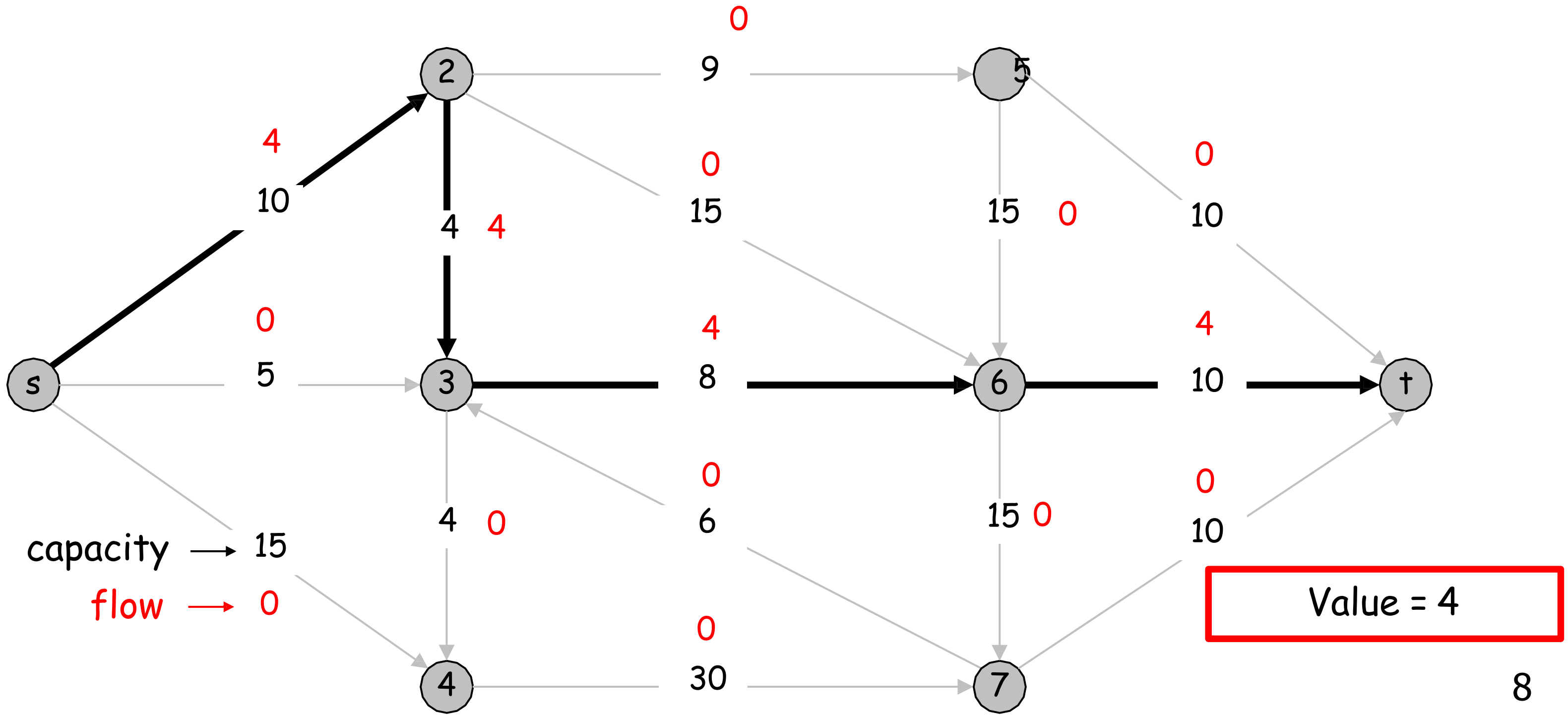


s-t Flows

Def. An s-t flow is a function that satisfies:

- For each $e \in E$: $0 \leq f(e) \leq c(e)$ (capacity)
- For each $v \in V - \{s, t\}$: $\sum_{e \text{ in to } v} f(e) = \sum_{e \text{ out of } v} f(e)$ (conservation)

Def. The value of a flow f is: $v(f) = \sum_{e \text{ out of } s} f(e)$

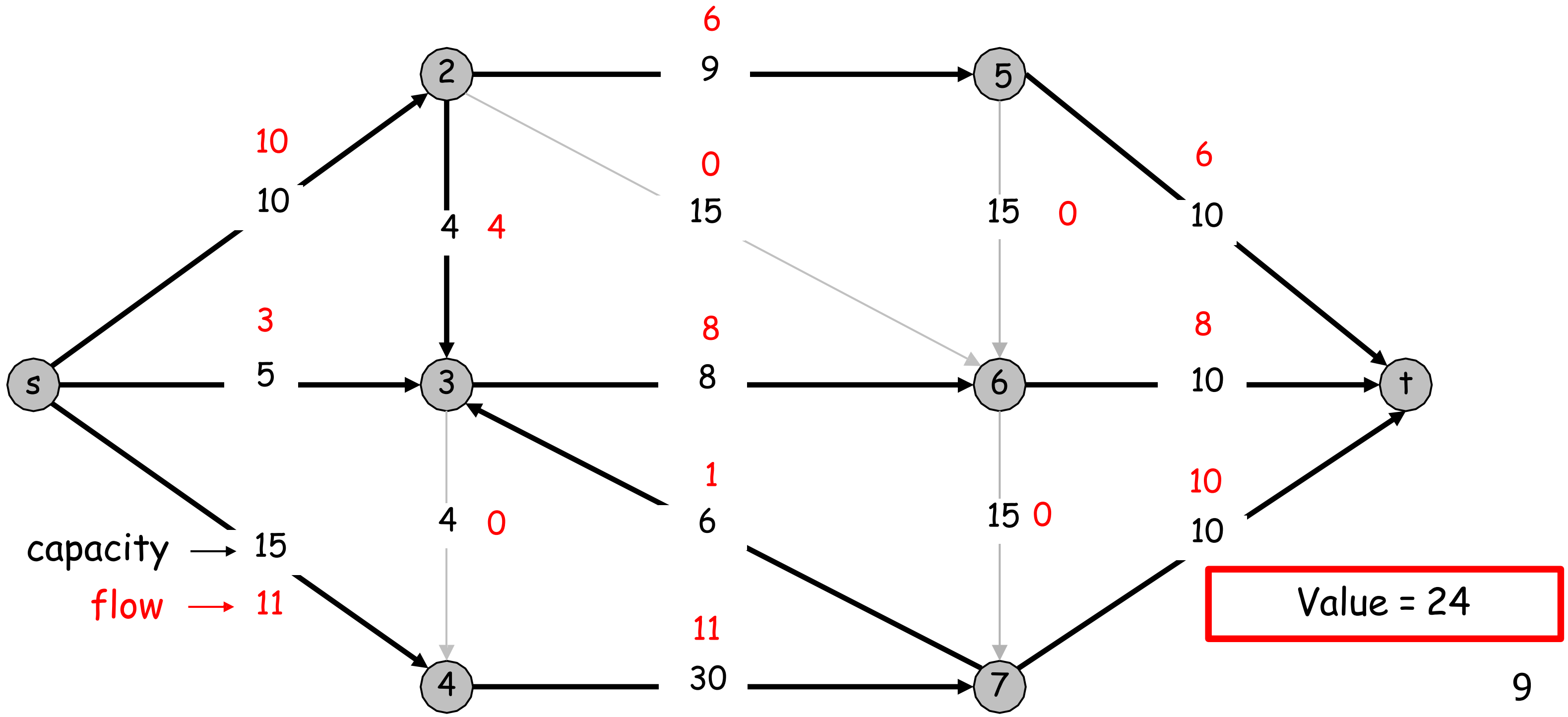


s-t Flows

Def. An s-t flow is a function that satisfies:

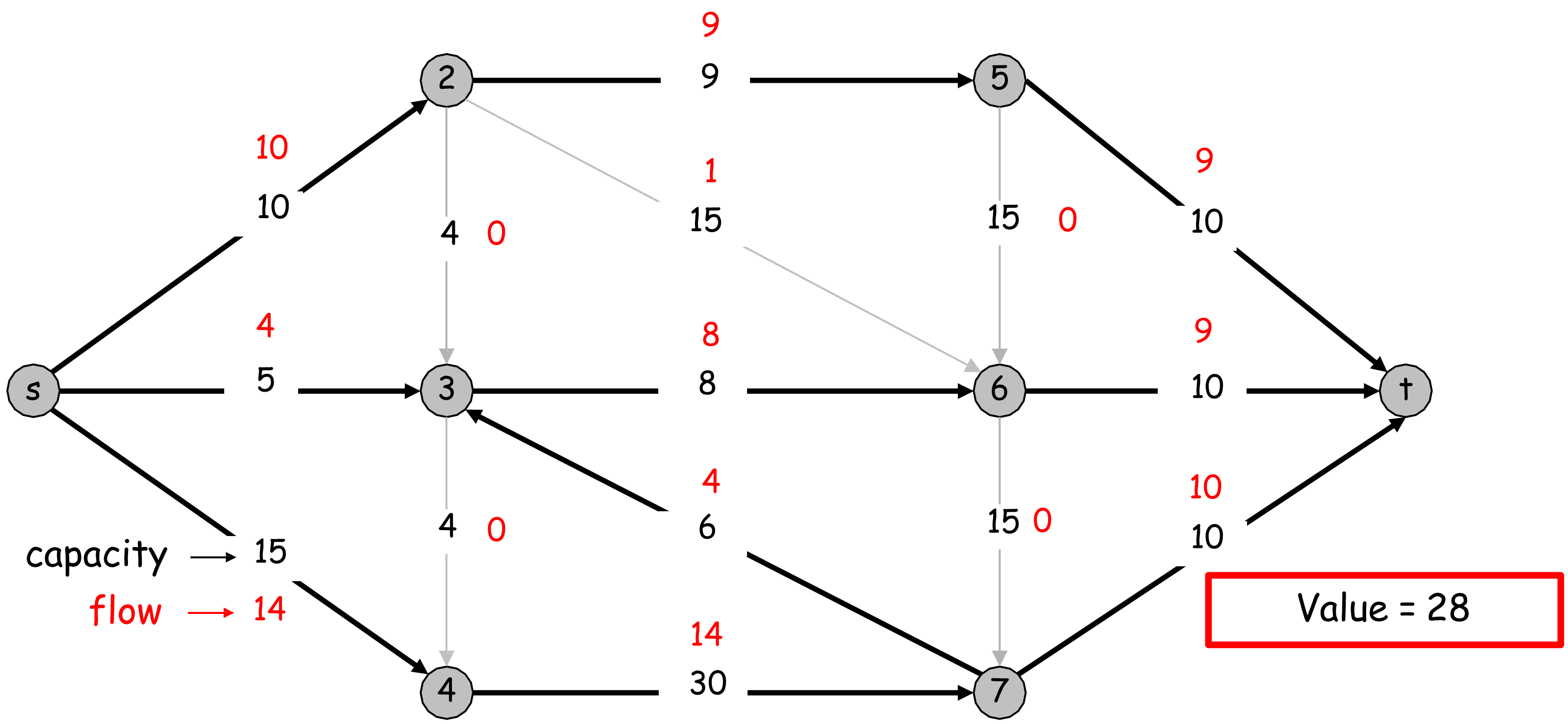
- For each $e \in E$: $0 \leq f(e) \leq c(e)$ (capacity)
- For each $v \in V - \{s, t\}$: $\sum_{e \text{ in to } v} f(e) = \sum_{e \text{ out of } v} f(e)$ (conservation)

Def. The value of a flow f is: $v(f) = \sum_{e \text{ out of } s} f(e)$



Maximum s-t Flow Problem

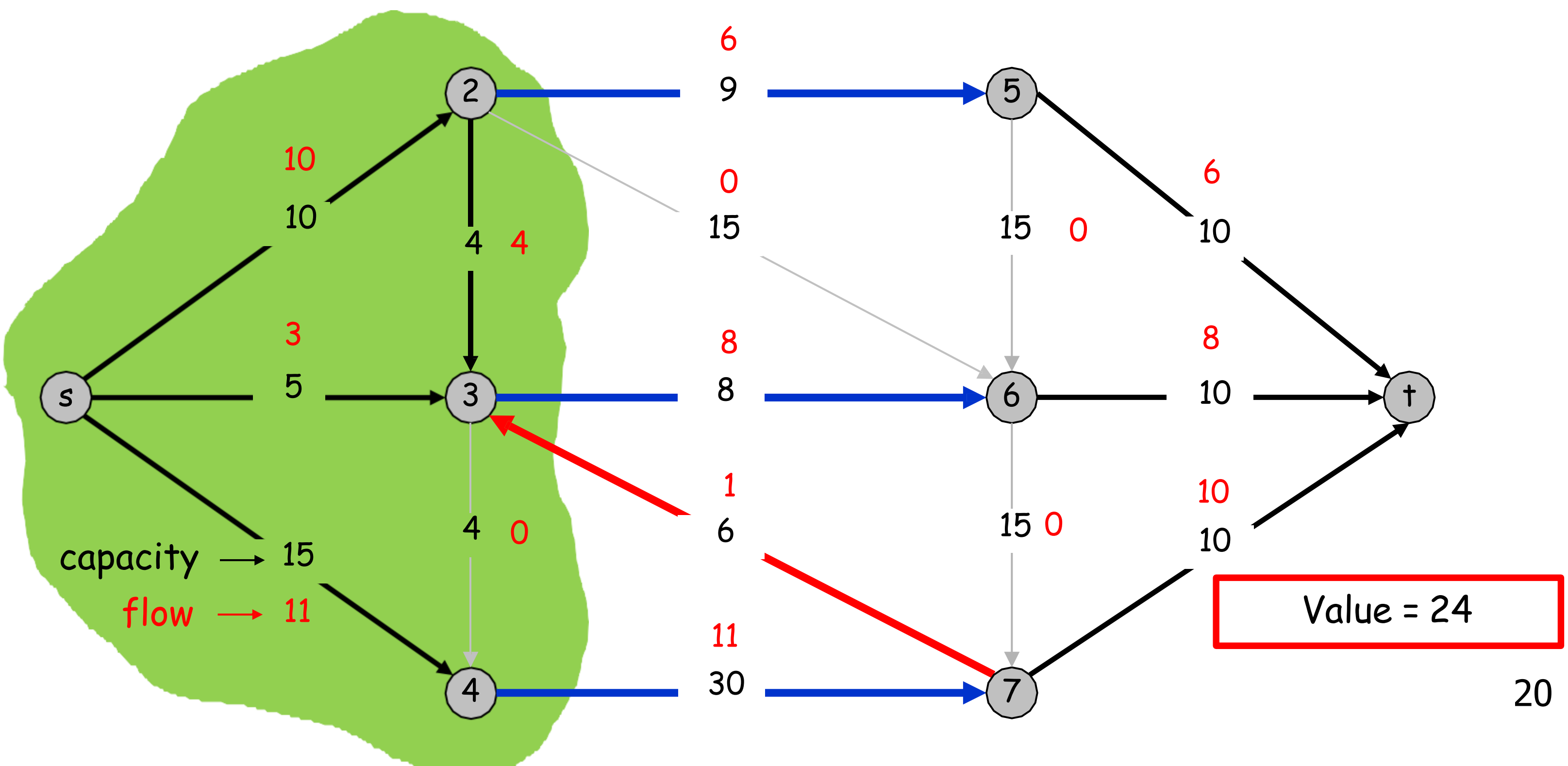
Goal: Find a s-t flow of largest value.



Flows and Cuts

Flow value lemma. Let f be any flow, and let (A, B) be any s-t cut. Then, the net flow sent across the cut is equal to the amount leaving s.

$$\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e) = v(f)$$



Pf of Flow value Lemma

Flow value lemma. Let f be any flow, and let (A, B) be any s-t cut. Then, the net flow sent across the cut is equal to the amount leaving s.

$$\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) = v(f)$$

Pf.

$$v(f) = \sum_{e \text{ out of } s} f(e)$$

By conservation of flow, all terms except v=s are 0

$$= \sum_{v \in A} \left(\sum_{e \text{ out of } v} f(e) - \sum_{e \text{ in to } v} f(e) \right)$$

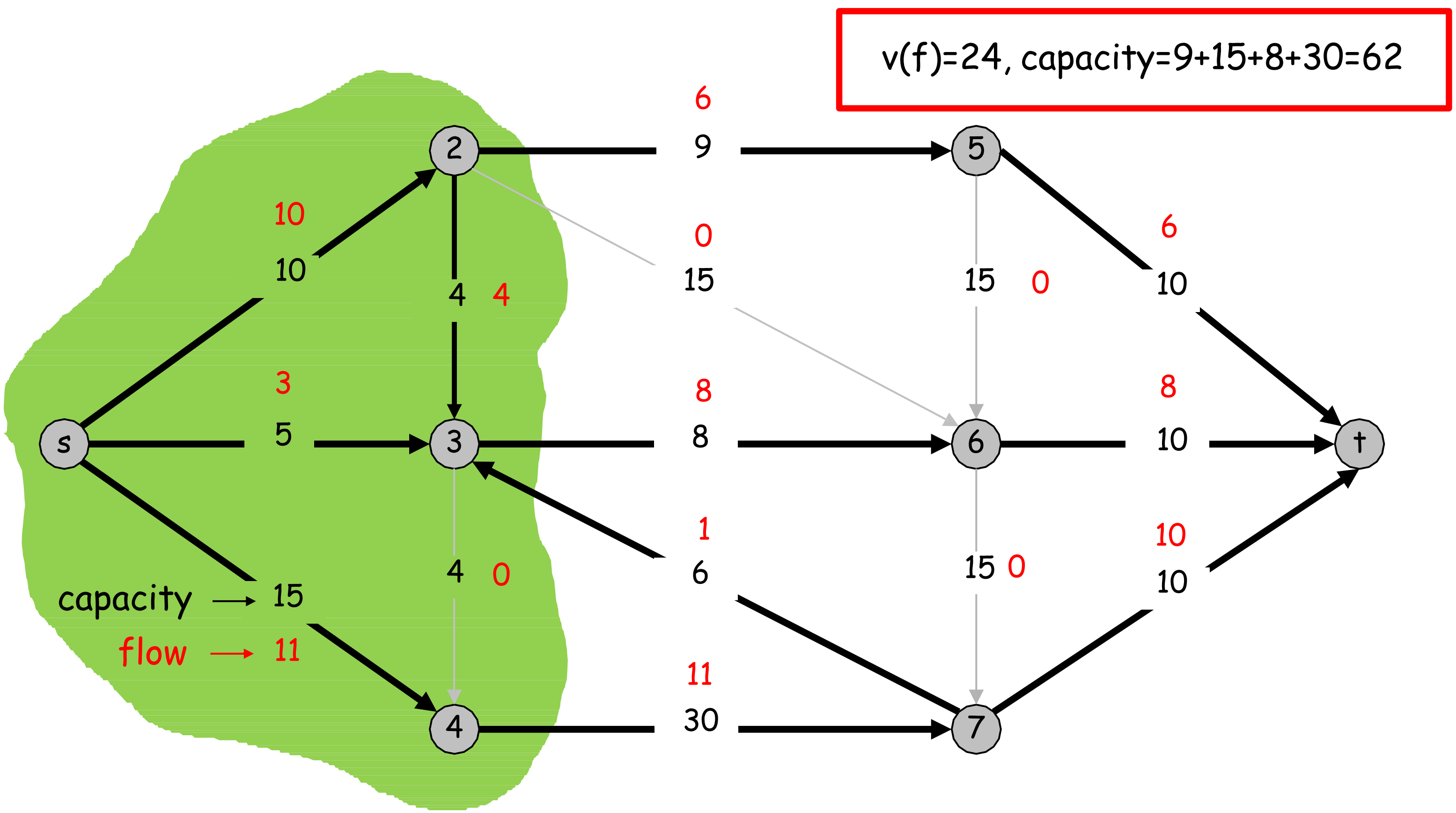
All contributions due to internal edges cancel out

$$= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e)$$

Weak Duality of Flows and Cuts

Cut Capacity lemma. Let f be any flow, and let (A, B) be any s-t cut. Then the value of the flow is at most the capacity of the cut.

$$v(f) \leqslant cap(A, B)$$



Weak Duality of Flows and Cuts

Cut capacity lemma. Let f be any flow, and let (A, B) be any s-t cut. Then the value of the flow is at most the capacity of the cut.

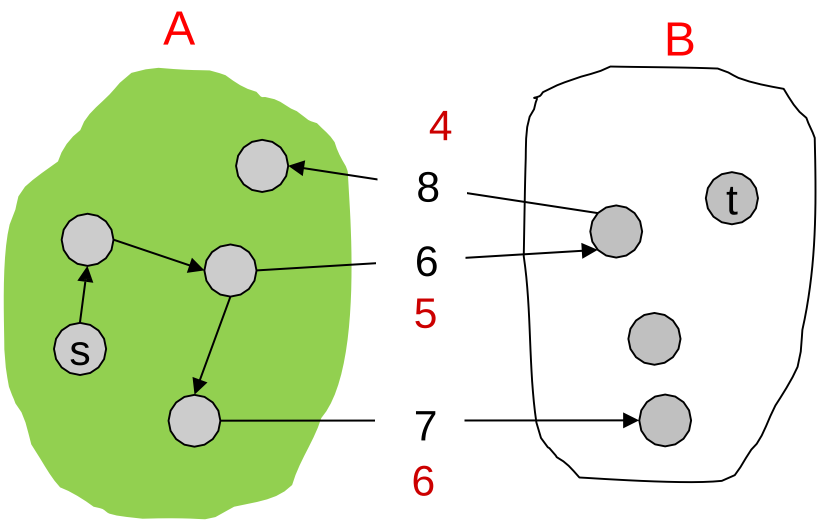
$$v(f) \leqslant cap(A, B)$$

Pf.

$$v(f) = \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e)$$

$$\leq \sum_{e \text{ out of } A} f(e)$$

$$\leq \sum_{e \text{ out of } A} c(e) = cap(A, B)$$



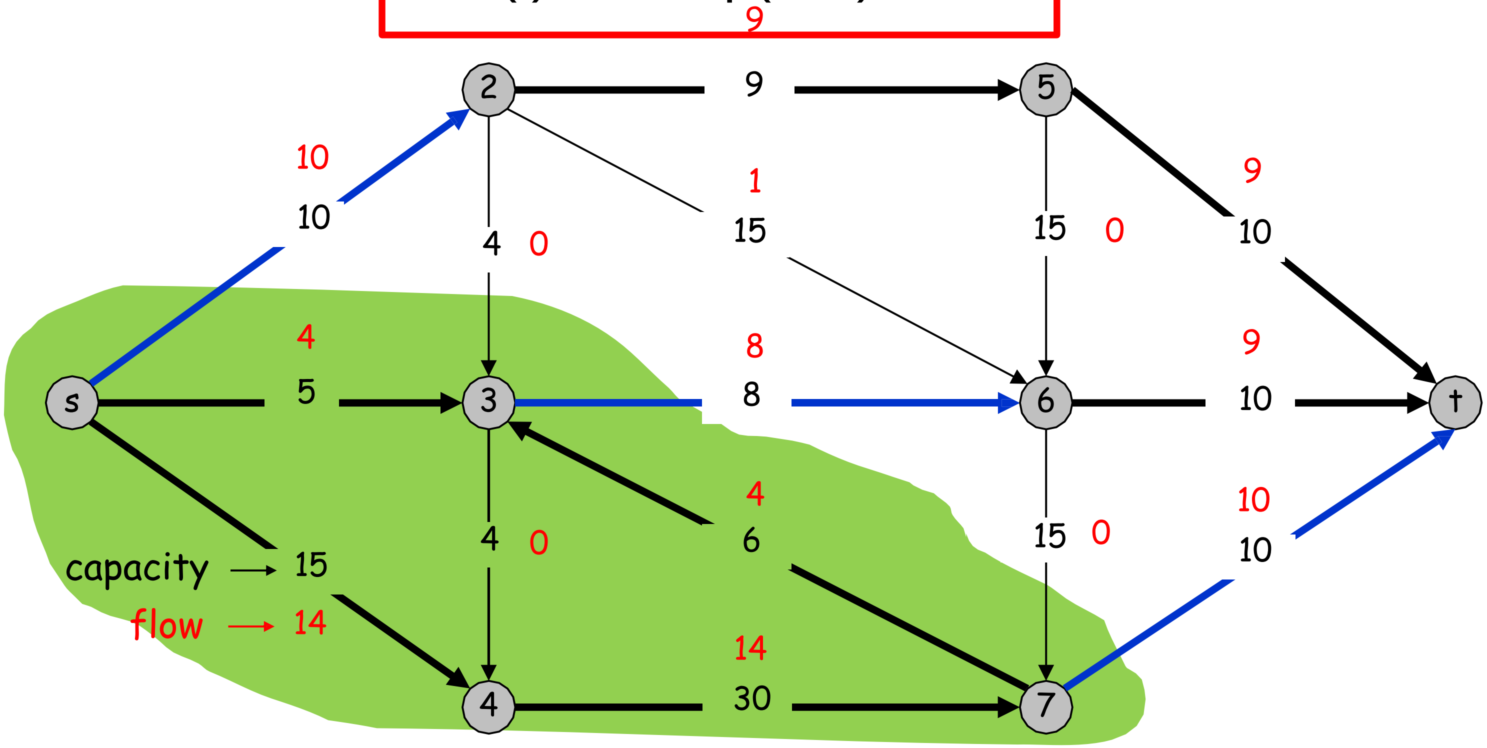
Certificate of Optimality

Corollary: Suppose there is a s-t cut (A,B) such that

$$v(f) = cap\ (A, B)$$

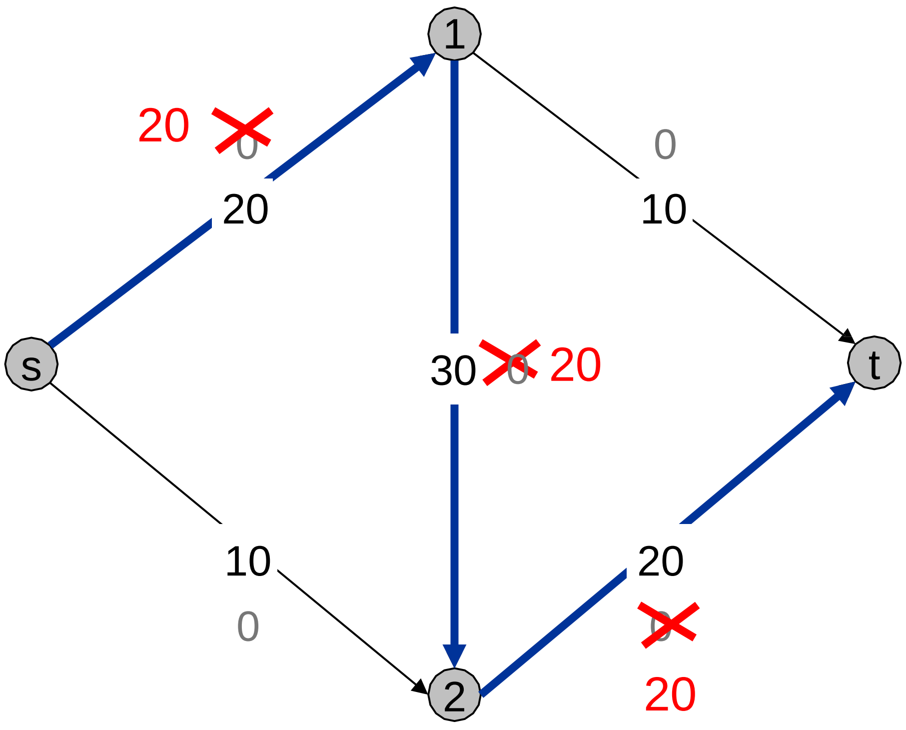
Then, f is a maximum flow and (A,B) is a minimum cut.

v(f)=28, cap(A,B)=28



A Greedy Algorithm for Max Flow

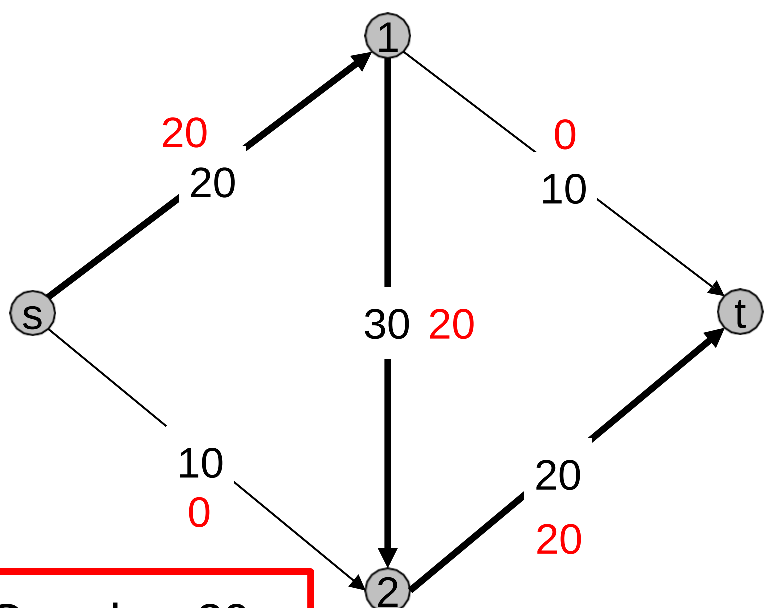
- Start with f(e) = 0 for all edge e $\in$ E.
- Find an s-t path P where each edge has f(e) < c(e).
- Augment flow along path P.
- Repeat until you get stuck.



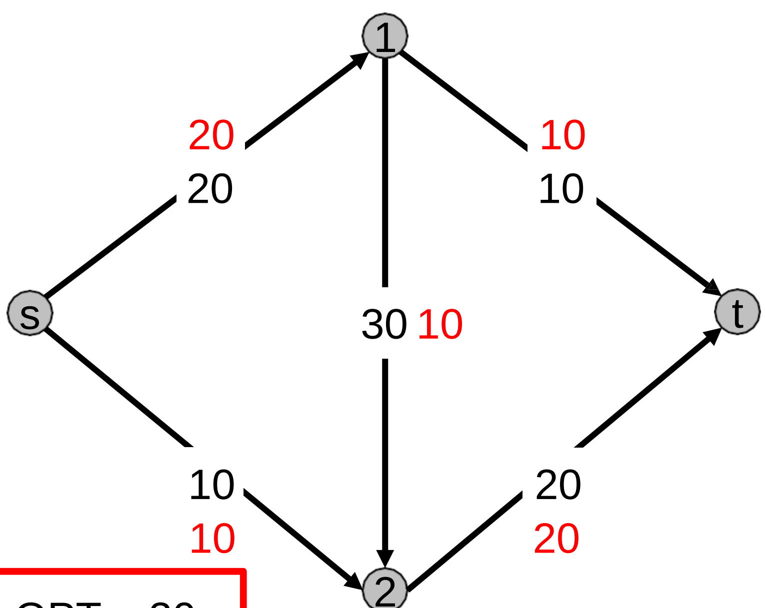
A Greedy Algorithm for Max Flow

- Start with $f(e) = 0$ for all edge $e \in E$.
- Find an s-t path P where each edge has $f(e) < c(e)$.
- Augment flow along path P.
- Repeat until you get stuck.

Local Optimum $\neq$ Global Optimum



Greedy = 20



OPT = 30