Introduction to Algorithms

Stable Matching

Summary

Stable matching problem: Given n men and n women, and their preferences, find a stable matching if one exists.

- Gale-Shapley algorithm: Guarantees to find a stable matching for any problem instance.

- Q: How to implement GS algorithm efficiently?
- Q: If there are multiple stable matchings, which one does GS find?
- Q: How many stable matchings are there?

Propose-And-Reject Algorithm [Gale-Shapley'62]

```
Initialize each person to be free.
while (some man is free and hasn't proposed to every woman) {
    Choose such a man m
    w = 1st woman on m's list to whom m has not yet proposed
    if (w is free)
        assign m and w to be engaged
    else if (w prefers m to her fiancé m')
        assign m and w to be engaged, and m' to be free
    else
        w rejects m
}
```

Implementation of GS Algorithm

Problem size

$N=2n^2$ words

- $2n$ people each with a preference list of length n

Brute force algorithm

Try all $n!$ possible matchings

Do any of them work?

Gale-Shapley Algorithm

n^2 iterations, each costing constant time as follows:

Efficient Implementation

We describe $O(n^2)$ time implementation.

Representing men and women:

Assume men are named 1, …, n.
Assume women are named n+1, …, 2n.

Engagements.

Maintain a list of free men, e.g., in a queue.
Maintain two arrays **wife**[**m**], and **husband**[**w**].

- set entry to **0** if unmatched
- if **m** matched to **w** then **wife**[**m**]=**w** and **husband**[**w**]=**m**

Men proposing:

For each man, maintain a list of women, ordered by preference.
Maintain an array **count**[**m**] that counts the number of proposals made by man **m**.

Efficient Implementation

Women rejecting/accepting.

Does woman **w** prefer man **m** to man **m'**?

For each woman, create inverse of preference list of men.

Constant time access for each query after **O(n)** preprocessing per woman.

O(n^2) total reprocessing cost.

w_i	1st	2nd	3rd	4th	5th	6th	7th	8th
Pref	8	3	7	1	4	5	6	2

w_i	1	2	3	4	5	6	7	8
Inverse	4th	8th	2nd	5th	6th	7th	3rd	1st

```
for i = 1 to n
   for j = 1 to n
      inverse[i][pref[i][j]] = j
```

w_i prefers man **3** to **6**

since `inverse[i][3]`=**2** < **7**=`inverse[i][6]`

Summary

- Stable matching problem: Given **n** men and **n** women, and their preferences, find a stable matching if one exists.

- Gale-Shapley algorithm guarantees to find a stable matching for any problem instance.

- GS algorithm finds a stable matching in **O**$(\boldsymbol{n}^2)$ time.

- Q: If there are multiple stable matchings, which one does GS find?

- Q: How many stable matchings are there?

Understanding the Solution

Q. For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

An instance with two stable matchings:

- $(m_1, w_1), (m_2, w_2)$.
- $(m_1, w_2), (m_2, w_1)$.

	1st	2nd
m_1	w_1	w_2
m_2	w_2	w_1

	1st	2nd
w_1	m_2	m_1
w_2	m_1	m_2

Man Optimal Assignments

Definition: Man m is a valid partner of woman w if there exists some stable matching in which they are matched.

Man-optimal matching: Each man receives the best valid partner (according to his preferences).

- Not that each man receives his most favorite woman.

Example

Here
Valid-partner(m_1) = $\{w_1, w_2\}$
Valid-partner(m_2) = $\{w_1, w_2\}$
Valid-partner(m_3) = $\{w_3\}$.

Man-optimal matching $\{m_1, w_1\}, \{m_2, w_2\}, \{m_3, w_3\}$

	favorite ↓ 1st	2nd	least favorite ↓ 3rd
m_1	w_1	w_2	w_3
m_2	w_2	w_1	w_3
m_3	w_1	w_2	w_3

	favorite ↓ 1st	2nd	least favorite ↓ 3rd
w_1	m_2	m_1	m_3
w_2	m_1	m_2	m_3
w_3	m_1	m_2	m_3

Man Optimal Assignments

Definition: Man m is a valid partner of woman w if there exists some stable matching in which they are matched.

Man-optimal matching: Each man receives the best valid partner (according to his preferences).

- Not that each man receives his most favorite woman.

Claim: All executions of GS yield a man-optimal matching, which is a stable matching!

- So, output of GS is unique!!

Man Optimality

S
(m, w)
(m', w')
...

Claim: GS matching **S*** is man-optimal.
Proof: (by contradiction)

Suppose some man is paired with someone other than his best partner. Men propose in decreasing order of preference $\Rightarrow$ some man is rejected by a valid partner.

Let m be the man who is the first such rejection, and let w be the women who is first valid partner that rejects him.

Let **S** be a stable matching where m and w are matched.
In building **S*,** when m is rejected, w forms (or reaffirms) engagement with a man, say m' whom she prefers to m.

Let w' be m' partner in **S**.
In building **S*,** m' is not rejected by any valid partner at the point when m is rejected by w. Thus, m' *prefers* w *to* w'.

since this is the first rejection by a valid partner

But w prefers m' to m.
Thus (m', w) is unstable in **S**. ■

Man Optimality Summary

Man-optimality: In version of GS where men propose, each man receives the best valid partner.

> w is a valid partner of m if there exist some stable matching where m and w are paired

Q: Does man-optimality come at the expense of the women?

Woman Pessimality

Woman-pessimal assignment: Each woman receives the worst valid partner.

Claim. GS finds woman-pessimal stable matching **S***.

Proof.

Suppose (m, w) matched in **S***, but m is not worst valid partner for w.

There exists stable matching **S** in which w is paired with a man, say m', whom she likes less than m.

- Let w' be m partner in **S**.
- m prefers w to w'. ← man-optimality of **S***
- Thus, (m, w) is an unstable in **S**.

■

Summary

- Stable matching problem: Given **n** men and **n** women, and their preferences, find a stable matching if one exists.

- Gale-Shapley algorithm guarantees to find a stable matching for any problem instance.

- GS algorithm finds a stable matching in **O**$(\boldsymbol{n}^2)$ time.
- GS algorithm finds man-optimal woman pessimal matching

Extensions: Matching Residents to Hospitals

Men ≈ hospitals, Women ≈ med school residents.

- Variant 1: Some participants declare others as unacceptable.
- Variant 2: Unequal number of men and women.

e.g. A resident not interested in Cleveland

- Variant 3: Limited polygamy.

e.g. A hospital wants to hire **3** residents

Def: Matching **S** is unstable if there is hospital **h** and resident **r** s.t.

- **h** and **r** are acceptable to each other; and
- either **r** is unmatched, or **r** prefers **h** to her assigned hospital; and
- either **h** does not have all its places filled, or **h** prefers **r** to at least one of its assigned residents.

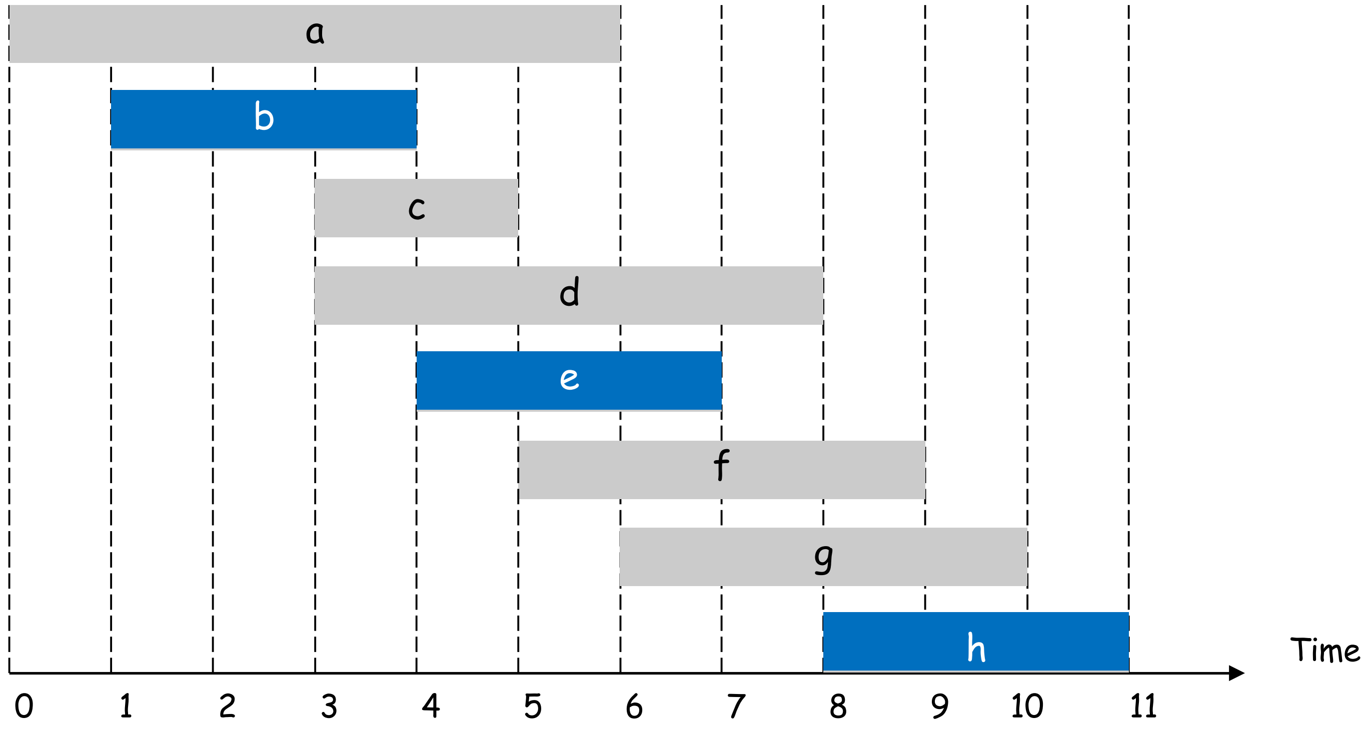
Four Representative Problems

1. Interval Scheduling
2. Weighted Interval Scheduling
3. Bipartite Matching
4. Independent Set Problem

Interval Scheduling

Input: Given a set of jobs with start/finish times

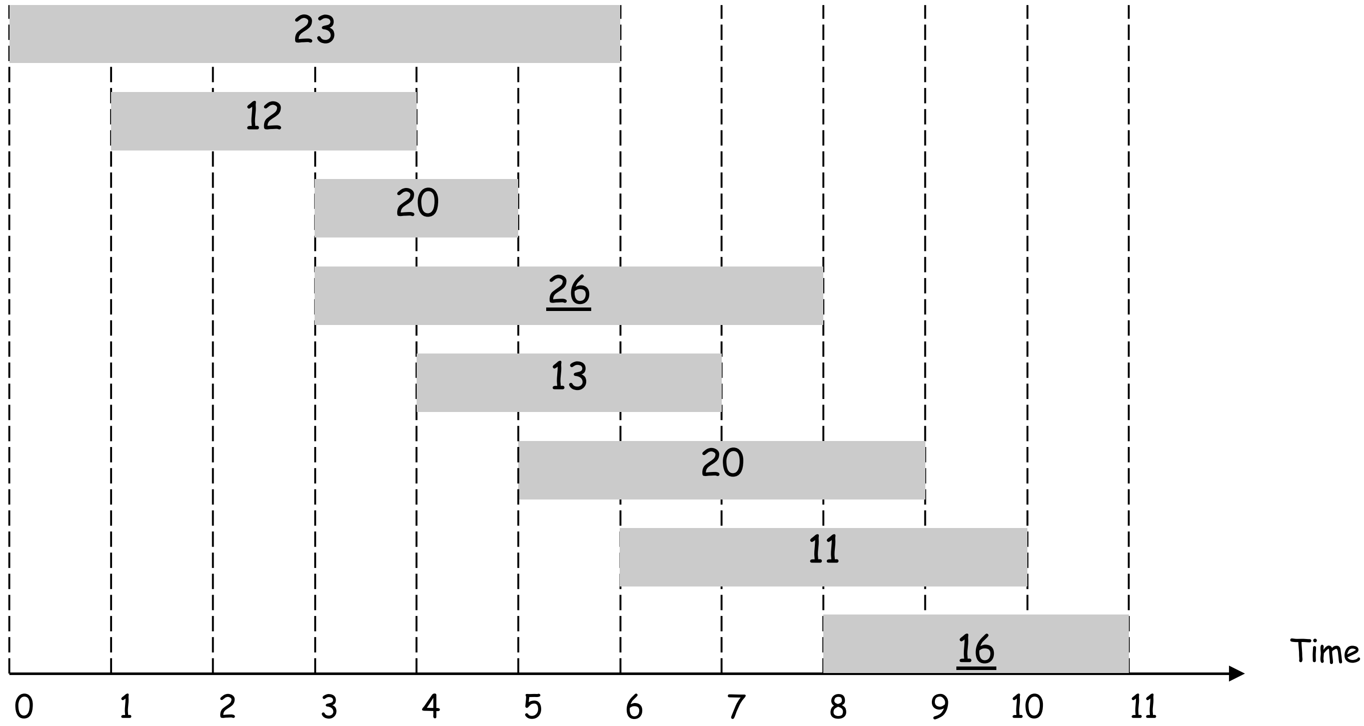
Goal: Find the maximum cardinality subset of jobs that can be run on a single machine.



Interval Scheduling

Input: Given a set of jobs with start/finish times

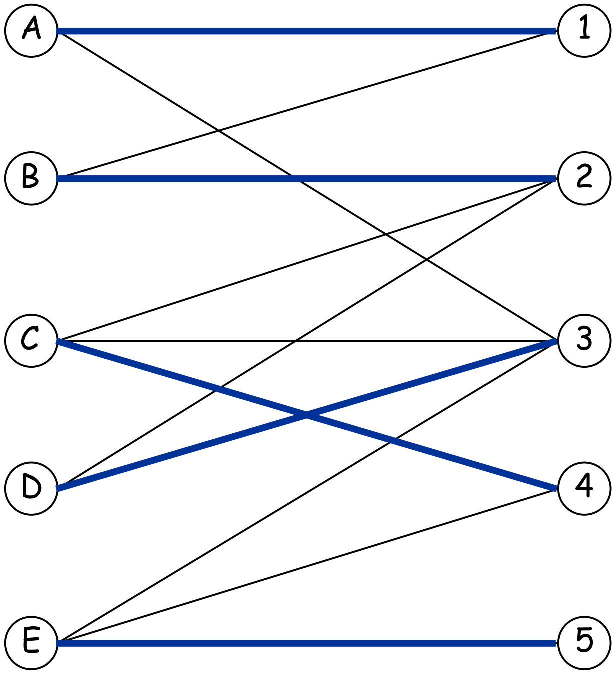
Goal: Find the maximum weight subset of jobs that can be run on a single machine.



Bipartite Matching

Input: Given a bipartite graph

Goal: Find the maximum cardinality matching

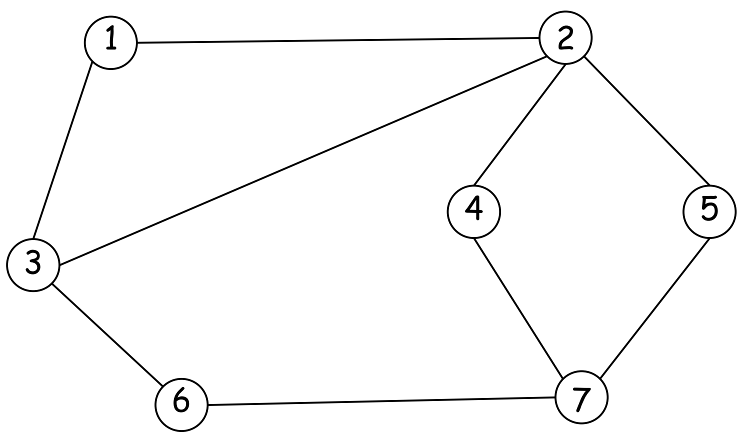


Independent Set

Input: A graph

Goal: Find the maximum independent set
(https://zhuanlan.zhihu.com/p/55932619)

Subset of nodes that no two joined by an edge



Four Representative Problems

Variation of a theme: Independent set Problem

1. Interval Scheduling
   $n \log n$ greedy algorithm

2. Weighted Interval Scheduling
   $n \log n$ dynamic programming algorithm

3. Bipartite Matching
   n^k maximum flow based algorithm

4. Independent Set Problem: NP-complete