Opt Reg 6

Problem 1

$$F = ma = m\ddot{x} \qquad (1)$$

a) By rewriting (1) to state space form, we get

$$\left.\begin{array}{l} x_1 = x \\ x_2 = \dot{x} \end{array}\right\} \Rightarrow \begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = \frac{u}{m} \end{array}$$

$$A_c = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad b_c = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

which gives

$$\underline{\dot{x}} = A_c \underline{x} + \underline{b_c}\, \underline{u}, \quad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

b) Simple fourier series for e^{A_cT}

$$e^{AT} = I + A_cT + \frac{(A_cT)^2}{2!} + \frac{(A_cT)^3}{3!} + \dots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0.5 \\ 0 & 0 \end{bmatrix} = \underline{\begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}}$$

observe that all orders 2 or higher are matrices of 0.

$$b = \int_0^{0.5} \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix} d\tau \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 & \frac{1}{2}\cdot 0.5^2 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 0.125 \\ 0.5 \end{bmatrix}}}$$

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c)

The Ricatti equation is given by

$$P_t = Q_t + A_t^T P_{t+1} (I + B_t R_t^{-1} b_t P_{t+1})^{-1} A_t \quad , t = 0, \dots, N-1 \quad (2)$$

$$P_N = Q_N = Q \qquad (LTI)$$

Since there are only equality constraints, the solution to the LQ problem can be found explicitly. Thus we can calculate the optimal gain matrix, K_t, independently of the states.

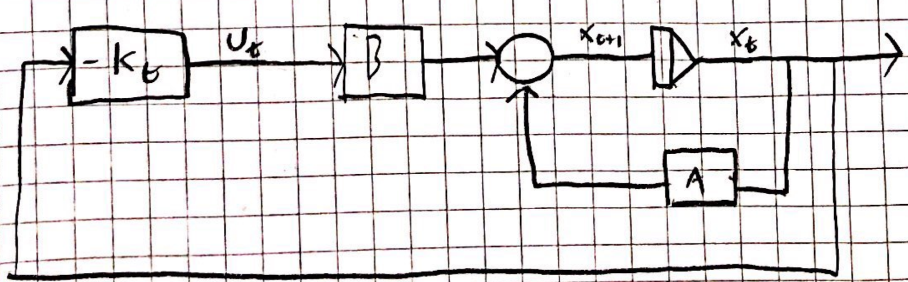
By iterating (2) backwards in time, all matrices $P_1, P_2, \dots, P_{N-1}$ can be found independent of the states. By inserting these into

$$K_t = R^{-1} B^T P_{t+1} (I + BR^{-1} b P_{t+1})^{-1} A$$

Implementing

$$U = -K_t x_t$$

will give an optimal control of the system which minimizes $f(z)$



Problem 2

$$x_{t+1} = \overset{a}{3}x_t + \overset{b}{2}u_t$$

$$f^{\infty}(z) = \frac{1}{2}\sum_{t=0}^{\infty} q x_{t+1}^2 + u_t^2$$

a) Riccati equation given by

$$P_t = Q + A^T P_{t+1}(I + BR^{-1}BP_{t+1})^{-1}A \quad , \quad t=0,\dots,N-1$$

$$P_N = Q$$

In our case

$$p_t = q + a p_{t+1}\left(1 + \frac{b^2 p_{t+1}}{r}\right)^{-1} a \quad , \quad t=0,\dots,N-1$$

$$p_N = Q_N$$

Stationary $\Rightarrow$ $p_t = p_{t+1} = p$

$$p_t = q + ap\cdot\frac{r}{r+b^2p}a = \underline{\underline{q + \frac{a^2pr}{r+b^2p}}}$$

$q=2,\ a=3,\ b=2,\ r=1$

$$p = 2 + \frac{9p}{1+4p}$$

$$p(1+4p) = 2(1+4p) + 9p$$

$$4p^2 + p - 8p - 9p - 2 = 0$$

$$p^2 - 4p - \frac{1}{2} = 0$$

$$p = \frac{4 \pm \sqrt{16+2}}{2} = 2 \pm \frac{3}{\sqrt{2}}$$

p must be positive definite, so the solution is

$$p = \underline{\underline{2 + \frac{3}{\sqrt{2}}}}$$

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b) $k = \frac{b p a}{r\left(1 + \frac{b^2 p}{r}\right)} = \frac{b p a}{r + b^2 p} = \frac{2 \cdot 3 \cdot \left(2 + \frac{3}{\sqrt{2}}\right)}{1 + 4\left(2 + \frac{3}{\sqrt{2}}\right)} = \underline{\underline{\sqrt{2}}}$

Optimal feedback is therefore

$u_t = -\sqrt{2}\, x_t$

c) The closed loop system is asymptotically stable if the system is stabilizable and detectable.