OptReg 5

Problem 1

a) Stable if $|eig(A)| \leq 1$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & 0 & 0 \\ 0 & \lambda & -1 \\ -0.1 & 0.79 & \lambda - 1.78 \end{vmatrix} = \lambda\left(\lambda(\lambda - 1.78) + 0.79\right)$$

$$= \lambda^3 - 1.78\lambda^2 + 0.79\lambda = \lambda(\lambda^2 - 1.78\lambda + 0.79) = 0$$

$$\underline{\underline{\lambda_1 = 0}}$$

$$\lambda_{2,3} = \frac{1.78 \pm \sqrt{1.78^2 - 4 \cdot 0.79}}{2} \approx \frac{1.78 \pm 0.09}{2} = \underline{\underline{0.845 \wedge 0.935}}$$

Since $|\lambda_i| \leq 1$ for all λ, the system is stable.

b) $\dim(\underline{x}_t) = 3$, $\dim(u) = 1$

$$y_{t+1} = C\underline{x}_{t+1} = x_{3,t+1}$$

want (2) on the form

$$f(z) = \frac{1}{2} \sum_{t=0}^{N-1} 2x_{3,t+1}^2 + 2u_t$$

$$\Rightarrow f(z) = \frac{1}{2} \sum_{t=0}^{N-1} [2\underline{x}_{t+1}^T C^T C \underline{x}_{t+1} + u_t \cdot 2 \cdot u_t]$$

$$Q = 2C^T C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad R = 2$$

c) The minimization problem is convex since:

$Q \geq 0$ (positive semidefinite)

$R > 0$ (positive definite).

Thus the objective function is convex, and since there are only linear constraints, the feasible set is also convex. Thus the problem is convex.

Convexity only depends on Q and R.

d) Firstly, we have

$$z = [\underline{x}_1^T, \underline{x}_2^T, \dots, \underline{x}_N^T, u_0, u_1, \dots, u_{N-1}]^T$$

where

$$\underline{x}_{t+1} = A\underline{x}_t + Bu_t$$

which gives

$$\left.\begin{aligned} \underline{x}_1 &= A\underline{x}_0 + Bu_0 \\ \underline{x}_2 &= A\underline{x}_1 + Bu_1 \\ &\vdots \\ x_N &= A\underline{x}_{N-1} + Bu_{N-1} \end{aligned}\right\} \quad \begin{aligned} \underline{x}_1 - Bu_0 &= A\underline{x}_0 \\ \underline{x}_2 - A\underline{x}_1 - Bu_1 &= 0 \\ &\vdots \\ x_N - A\underline{x}_{N-1} - Bu_{N-1} &= 0 \end{aligned}$$

This gives us the matrix

$$\begin{array}{cc} N \cdot \dim(x) + N \cdot \dim(u) & \\ A_{eq} = \begin{bmatrix} I & 0 & \cdots & 0 & -B & 0 & \cdots & 0 \\ -A & I & \cdots & 0 & 0 & -B & \cdots & 0 \\ 0 & -A & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & I & 0 & 0 & \cdots & -B \end{bmatrix} & N \cdot \dim(x) \end{array} \quad b_{eq} = \begin{bmatrix} A x_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} N \cdot \dim(x)$$

d) Since we have

$$f(z) = \frac{1}{2} \sum_{t=0}^{N-1} \left(x_{t+1}^T Q \, x_{t+1} + u_t^T R u_t \right)$$

where

$$z = [\underline{x}_1^T, \underline{x}_2^T, ---, \underline{x}_N^T, u_0, ---, u_{N-1}]^T$$

we want a matrix G on the form

$$G = \begin{bmatrix} Q & 0 & 0 \cdots & 0 & 0 & 0 \cdots & 0 \\ 0 & Q & 0 \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & 0 & Q & 0 & \cdots & 0 \\ \vdots & \vdots & & & & & \\ 0 & 0 & \cdots & 0 & R & \ddots & \\ \vdots & \vdots & & & & \ddots & \\ 0 & 0 & 0 \cdots & 0 & 0 & \cdots 0 & R \end{bmatrix}$$

Want to minimize a function on the form

$$\min q(x) = \frac{1}{2} x^T G x + x^T c$$
$$\text{s.t. } Ax = b$$

$$\begin{bmatrix} G & -A_{eq}^T \\ A_{eq} & 0 \end{bmatrix} \begin{bmatrix} \underline{x}^* \\ \underline{\lambda}^* \end{bmatrix} = \begin{bmatrix} -\underline{c} \\ \underline{b} \end{bmatrix}$$

In our case

$$\begin{bmatrix} G & -A_{eq}^T \\ A_{eq} & 0 \end{bmatrix} \begin{bmatrix} z^* \\ \lambda^* \end{bmatrix} = \begin{bmatrix} \underline{0} \\ \underline{b}_{eq} \end{bmatrix}$$