Opt Reg Exercise 3

Problem 1

KKT - conditions

$$\nabla_x \mathcal{L}(x^*, \lambda^*, s^*) = c - A^T\lambda^* - s^* = 0$$

$$Ax^* = b$$

$$x^* \geq 0$$

$$s^* \geq 0$$

$$s_i^* x_i^* = 0 \quad , i = 1, \dots, n$$

a) The Newton direction is found by setting

$$M_k(p) = f_k + p^T \nabla f_k + \frac{1}{2} p^T \nabla^2 f_k \, p = 0$$

By solving for p we find that

$$p_k^N = -(\nabla^2 f_k)^{-1} \nabla f_k$$

Since the LP has no second derivative, p_k^N is equal to zero at all points. Thus it is not defined.

b) convex problem if

- Objective function is convex (show later)
- Equality constraints are linear ($Ax = b$ is linear)
- Inequality constraints are concave. (upper half plane is both convex and concave).

Objective function is convex if:

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y) \quad , f(x) = c^T x$$

$$\Rightarrow \quad c^T(\alpha x + (1-\alpha)y) \leq \alpha c^T x + (1-\alpha) c^T y$$

$$\Leftrightarrow \alpha c^T x + (1-\alpha) c^T y = \alpha c^T x + (1-\alpha) c^T y .$$

Thus, the problem is convex.

c) Need to put the problem in standard form, so:

$$\min_{\lambda} -b^T\lambda \quad s.t \quad c - A^T\lambda \geq 0$$

$$\mathcal{L}(\lambda, x) = -b^T\lambda - x(c - A^T\lambda)$$

$$\nabla_\lambda \mathcal{L}(\lambda^*, x^*) = -b + Ax^* = 0 \quad (i)$$

$$Ax^* = b \quad (ii)$$

Using the other KKT-conditions, we get

$$x^* \geq 0 \quad (iii)$$

$$(c - A^T\lambda) \geq 0$$

$$x^*_i (c - A^T\lambda^*)_i \geq 0$$

by setting $s^* = (c - A^T\lambda^*)$, it is easy to see that the conditions are alike.

d) Did it last assignment. The two points are the same.

e) ~~A basic feasible point is a point where both the equality constraints and the inequality constraints are fullfilled, and: - It is a~~

e) A point is a basic feasible point if it is feasible and contained in a subset $\mathcal{B}$ of indice sets $\{1, ..., n\}$ (columns) s.t.

- $\mathcal{B}$ contains exactly m indices (same as number of rows
- $i \notin \mathcal{B} \Rightarrow x_i = 0$
- $m \times m$ matrix B is defined as

$$B = [A_i]_{i \in \mathcal{B}}$$

f) If A has full row rank, then the constraints along the row vectors must fullfill LICQ

Problem 2

Formulate as a minimization problem
Rename $A = x_1$, $B = x_2$.

$$\min_x C^T x \quad \text{s.t} \quad b - Ax \geq 0$$
$$x \geq 0$$

Selling price given by:

$$-(-x_1 - x_2) \Rightarrow C = \begin{bmatrix} -\frac{3}{2} \\ -1 \end{bmatrix}, \quad x_1 = \frac{3}{2} x_2$$

Then, add slack variables: x_3 for R_1 and x_4 for R_{11}

$$b = \begin{bmatrix} 8 \\ 15 \end{bmatrix} \quad A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{bmatrix}$$

Thus we have

$$\min C^T x \quad \text{s.t} \quad b - Ax = 0$$
$$x \geq 0$$

$$\begin{bmatrix} x \\ x_1 \end{bmatrix} \geq 0$$

Problem 3

a) The active set $\mathcal{A}(x^*)$ is a set of indices where $a_i^T x = b_i$

$$\mathcal{A}(x^*) = \{ i \in \mathcal{E} \cup \mathcal{I} \mid a_i^T x^* = b_i \}$$

b) $$\mathcal{L}(x^*, \lambda^*) = q(x^*) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i^* (a_i^T x - b_i)$$

$$= \frac{1}{2} x^{*T} G x^* + x^{*T} c - \sum_{i = \mathcal{E} \cup \mathcal{I}}^{n} \lambda_i^* (a_i^T x^* - b_i)$$

$$\nabla_x \mathcal{L}(x^*, \lambda^*) = G x^* + c - \underbrace{\sum_{i = \mathcal{E} \cup \mathcal{I}} \lambda_i^* a_i}$$

Since $\lambda_i^* = 0$ for inactive constraints, we can write

$$\nabla_x \mathcal{L}(x^*, \lambda^*) = G x^* + c - \sum_{i \in \mathcal{A}(x^*)} \lambda_i^* a_i$$

We also have

$a_i^T x^* = b_i \quad \forall \; i \in \mathcal{A}(x^*)$ (at constraint)

$a_i^T x^* > b_i \quad \forall \; i \in \mathcal{I} / \mathcal{A}(x^*)$

$\lambda_i^* \geq 0 \quad \forall \; i \in \mathcal{I} \cap \mathcal{A}(x^*)$