Opt Reg 6

Problem 1

$$F = ma = m\ddot{x} \qquad (1)$$

a) By rewriting (1) to state space form, we get

$$\left.\begin{array}{l} x_1 = x \\ x_2 = \dot{x} \end{array}\right\} \Rightarrow \begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = \frac{u}{m} \end{array}$$

$$A_c = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad b_c = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

which gives

$$\underline{\dot{x}} = A_c \underline{x} + \underline{b_c}\,\underline{u}, \quad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

b) Simple fourier series for $e^{A_c T}$

$$e^{AT} = I + A_c T + \frac{(A_c T)^2}{2!} + \frac{(A_c T)^3}{3!} + \dots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0.5 \\ 0 & 0 \end{bmatrix} = \underline{\begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}}$$

observe that all orders 2 or higher are matrices of 0.

$$b = \int_0^{0.5} \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix} d\tau \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 & \frac{1}{2}\cdot 0.5^2 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 0.125 \\ 0.5 \end{bmatrix}}}$$

c)

The Ricatti equation is given by

$$P_t = Q_t + A^T P_{t+1} (I + B_t R^{-1} b_t P_{t+1})^{-1} A_t \quad , t = 0, \dots, N-1 \quad (2)$$

$$P_N = Q_N = Q \qquad (LTI)$$

Since there are only equality constraints, the solution to the LQ problem can be found explicitly. Thus we can calculate the optimal gain matrix, K_t, independently of the states.

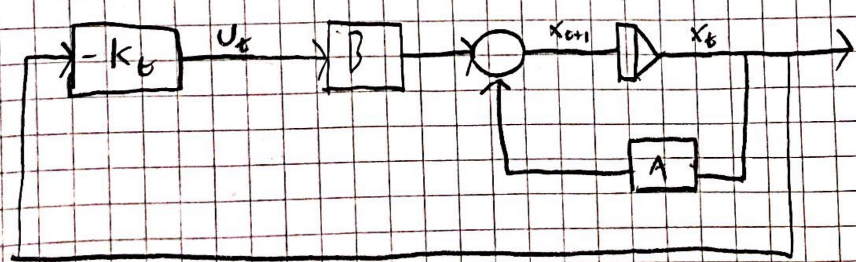
By iterating (2) backwards in time, all matrices $P_1, P_2, \dots, P_{N-1}$ can be found independent of the states. By inserting these into

$$K_t = R^{-1} B^T P_{t+1} (I + B R^{-1} b P_{t+1})^{-1} A$$

Implementing

$$U = -K_t x_t$$

will give an optimal control of the system which minimizes $f(z)$



Problem1

d)

The code is as shown below:

```
%% Constants and system matrices
m = 1;
A = [1  0.5;
     0  1];
B = [0.125/m 0.5/m]';

%Control matrices - dlqr
Q = [2  0;
     0  2];
R = 2;

%% Solve Ricatti equation
[K, P, E] = dlqr(A, B, 1/2*Q, 1/2*R, []);
```

This code gives the system's eigenvalues to be $\lambda_1 = 0.6307{+}0.1628i$ and $\lambda_2 = 0.6307{-}0.1628i$ which are inside the unit circle. Thus the system is stable.

e)

The controller does not always necessarily provide a stable system. In order for this controller to provide a stable system, it must be both stabilizable and detectable.

Problem 2

$$x_{t+1} = \underset{a}{3}x_t + \underset{b}{2}u_t$$

$$f^{\infty}(z) = \frac{1}{2}\sum_{t=0}^{\infty} q x_{t+1}^2 + u_t^2$$

a) Riccati equation given by

$$P_t = Q + A^T P_{t+1} (I + BR^{-1}BP_{t+1})^{-1} A \quad , t = 0, \dots, N-1$$

$$P_N = Q$$

In our case

$$p_t = q + a p_{t+1}\left(1 + \frac{b^2 p_{t+1}}{r}\right)^{-1} a \quad , t=0, \dots, N-1$$

$$p_N = Q_N$$

Stationary $\Rightarrow$ $p_t = p_{t+1} = p$

$$p = q + ap \cdot \frac{r}{r + b^2 p} a = \underline{\underline{q + \frac{a^2 p r}{r + b^2 p}}}$$

$q=2$, $a=3$, $b=2$, $r=1$

$$p = 2 + \frac{9p}{1+4p}$$

$$p(1+4p) = 2(1+4p) + 9p$$

$$4p^2 + p - 8p - 9p - 2 = 0$$

$$p^2 - 4p - \frac{1}{2} = 0$$

$$p = \frac{4 \pm \sqrt{16+2}}{2} = 2 \pm \frac{3}{\sqrt{2}}$$

p must be positive definite, so the solution is

$$p = \underline{\underline{2 + \frac{3}{\sqrt{2}}}}$$

b) $$k = \frac{bPa}{r\left(1 + \frac{b^2P}{r}\right)} = \frac{bPa}{r + b^2P} = \frac{2 \cdot 3 \cdot \left(2 + \frac{3}{\sqrt{2}}\right)}{1 + 4\left(2 + \frac{3}{\sqrt{2}}\right)} = \underline{\underline{\sqrt{2}}}$$

Optimal feedback is therefore

$$u_t = -\sqrt{2}\, x_t$$

c) The closed loop system is asymptotically stable if the system is stabilizable and detectable.

Problem 3

b)

The code gives the plot as shown in Figure 1. As opposed to Figure 2, we have less freedom in our control input, making the output slightly worse.

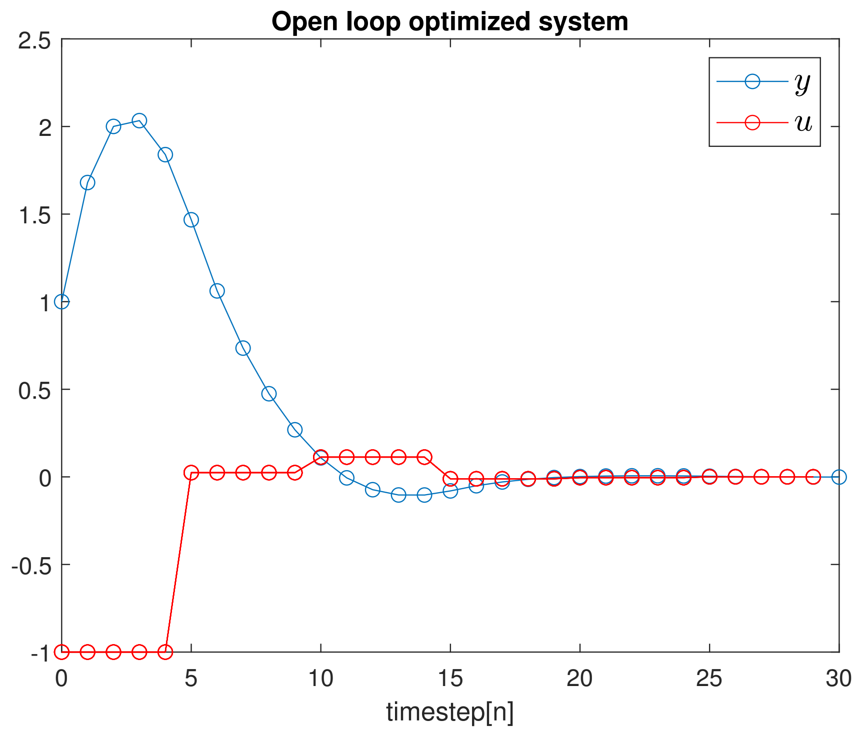


Figure 1: Optimal solution with input blocks

Saw in the solution that quadprog is supposed to use 2 iterations, but mine still uses 5. Is the solution old, because it uses active-set method, which is not supported by quadprog anymore.

c)

The plot is shown in Figure 3. Notice that the result is slightly worse than the previous compared to Figure 1(It does not land on the 0-line).
Quadprog still uses 5 iterations, which is no change from earlier. I would believe it should use fewer, since there are fewer inputs in z. NOTE: i have compared it to the solution given on BB, i get the same curve.

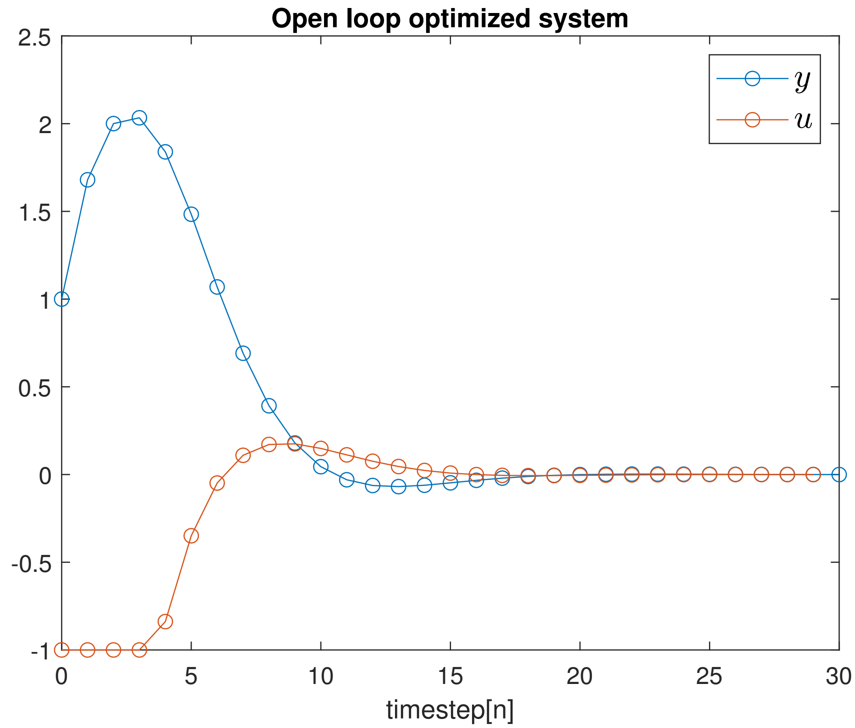


Figure 2: Optimal solution from last exercise

e)

The plots are shown in Figure 4, and Figure 5. The two plots are very similar, and this is because the MPC controller generates new block inputs on each iteration.

f)

The effect of blocked inputs on MPC is to reduce computation time. Computation becomes simpler, thus faster, when there are fewer states to account for. We use steps of increasing length because it both allows the controller to correct alot when the transients are at their worst. When they are starting to "die out", we don't need the same freedom in the controller.

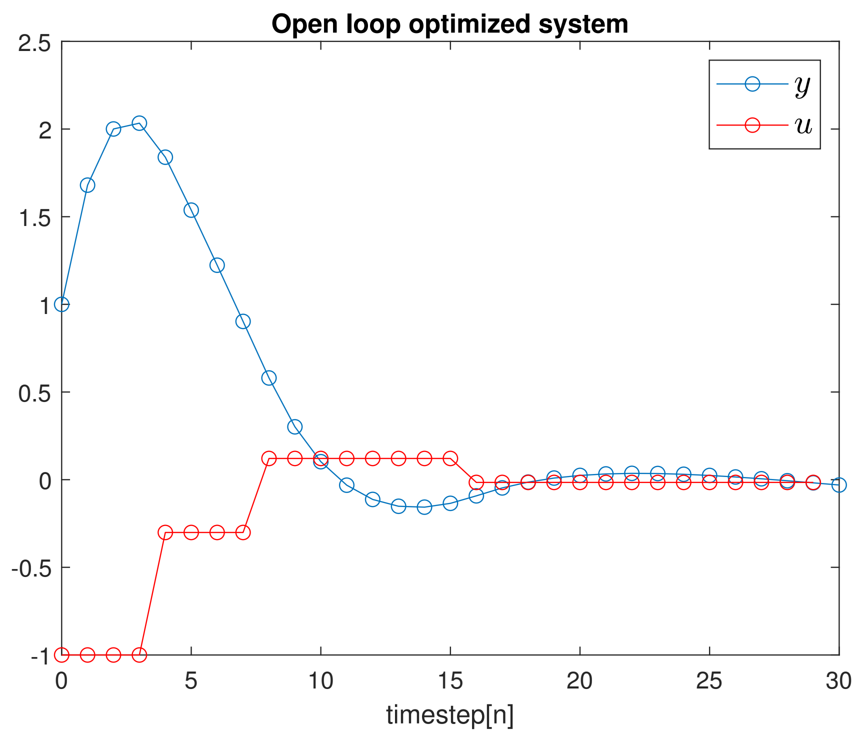


Figure 3: Increasing length input blocks

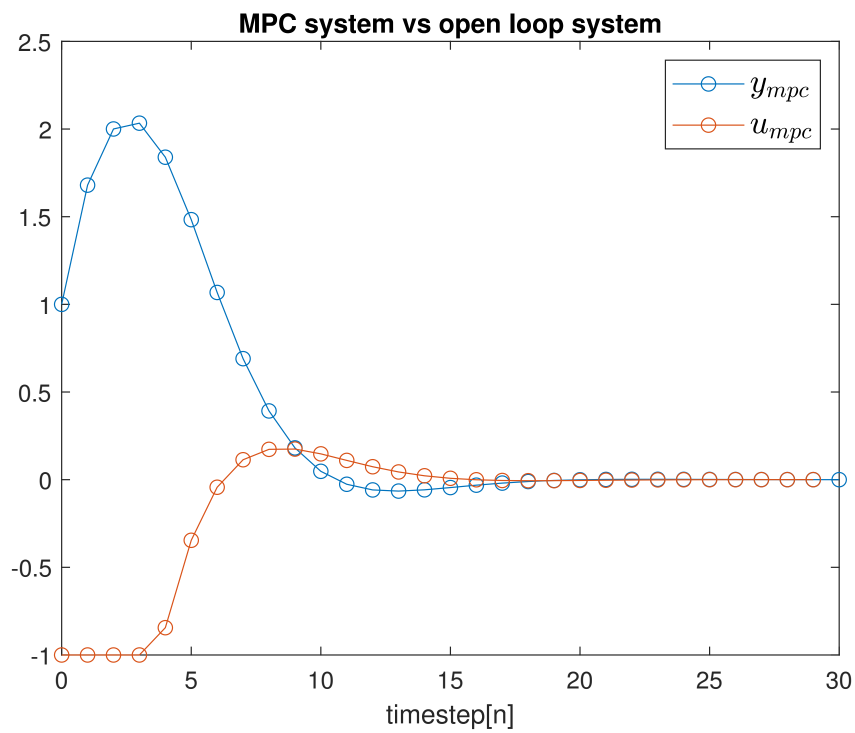


Figure 4: MPC with input blocks

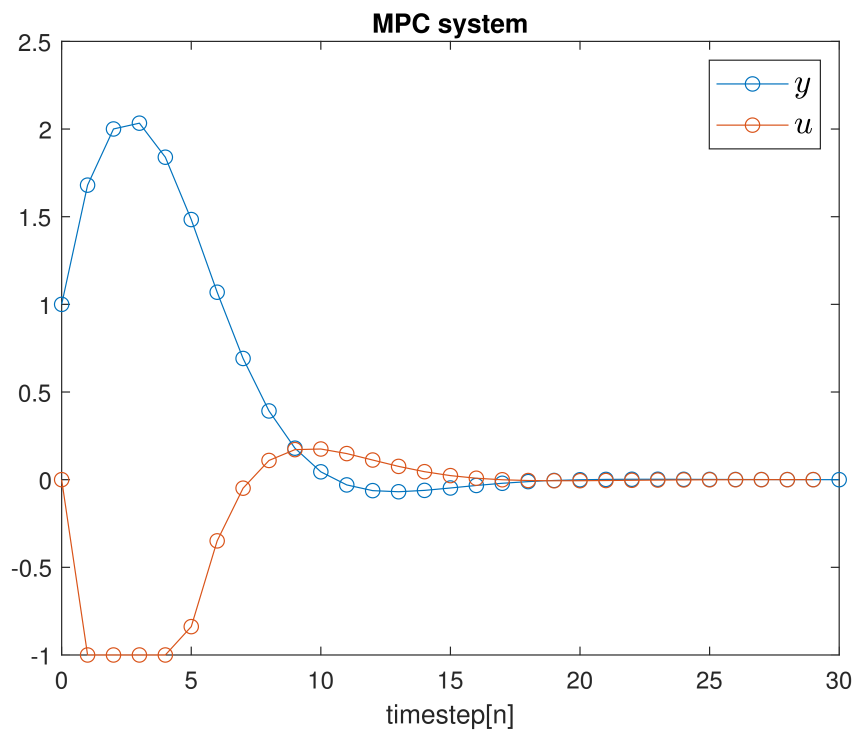


Figure 5: Optimal solution from last exercise with MPC