

Opt Reg Exercise 3

Problem 1

KKT - conditions

$$\nabla_x \mathcal{L}(x^*, \lambda^*, s^*) = C - A^T \lambda^* - s^* = 0$$

$$Ax^* = b$$

$$x^* \geq 0$$

$$s^* \geq 0$$

$$s_i^* x_i^* = 0, i = 1, \dots, n$$

a) The Newton direction is found by setting

$$M_k(p) = f_k + p^T \nabla f_k + \frac{1}{2} p^T \nabla^2 f_k p = 0$$

By solving for p we find that

$$p_k^N = -(\nabla^2 f_k)^{-1} \nabla f_k$$

Since the LP has no second derivative,

p_k^N is equal to zero at all points. Thus it is not defined.

b) Convex problem if

- objective function is convex (show later)
- Equality constraints are linear ($Ax=b$ is linear)
- Inequality constraints are concave. (upper half plane is both convex and concave).

objective function is convex if:

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y), \quad f(x) = C^T x$$

$$\Rightarrow C^T(\alpha x + (1-\alpha)y) \leq \alpha C^T x + (1-\alpha)C^T y$$

$$\Leftrightarrow \alpha C^T x + (1-\alpha)C^T y \leq \alpha C^T x + (1-\alpha)C^T y$$

Thus, the problem is convex.

Need to put the problem in
c) Standard form, so:

$$\min_{\lambda} -b^T \lambda \text{ s.t. } C - A^T \lambda \geq 0$$

$$L(\lambda, x) = -b^T \lambda - x(C - A^T \lambda)$$

$$\nabla_{\lambda} L(\lambda^*, x^*) = -b + A x^* = 0 \quad (i)$$

$$A x^* = b \quad (ii)$$

Using the other KKT-conditions, we get

$$x^* \geq 0 \quad (iii)$$

$$(C - A^T \lambda^*) \geq 0$$

$$x^* (C - A^T \lambda^*) \geq 0$$

by setting $s^* = (C - A^T \lambda^*)$, it is easy to see that the conditions are alike.

d) Did it last assignment. The two points are the same.

~~e) A basic feasible point is a point where both the equality constraints and the inequality constraints are fulfilled, and:
- It is a~~

e) A point is a basic feasible point if it is feasible and contained in a subset B of index set $\{1, \dots, n\}$ (columns) s.t.

- B contains exactly m indices (same as number of rows)
- $i \notin B \Rightarrow x_i = 0$

max matrix B is defined as

$$B = [A_i]_{i \in B}$$

f) If A has full row rank, then the constraints along the row vectors must fulfill LICQ

Problem 2

Formulate as a minimization problem
Rename $A = x_1, B = x_2$.

$$\min_x C^T x \quad \text{s.t.} \quad b - Ax \geq 0 \\ x \geq 0$$

Selling price given by:

$$-(-x_1 - x_2) \Rightarrow C = \begin{bmatrix} -\frac{3}{2} \\ -1 \end{bmatrix}, \quad x_1 = \frac{3}{2}x_2$$

Then, add slack variables: x_3 for R_1 and x_4 for R_2

$$b = \begin{bmatrix} 8 \\ 15 \end{bmatrix} \quad A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{bmatrix}$$

Thus we have

$$\min C^T x \quad \text{s.t.} \quad b - Ax = 0 \\ x \geq 0$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq 0$$

Problem 3

- a) The active set $\lambda(x^*)$ is a set of indices where $a_i^T x = b_i$

$$\lambda(x^*) = \{i \in I \mid a_i^T x^* = b_i\}$$

$$\begin{aligned} b) \quad \mathcal{L}(x^*, \lambda^*) &= q(x^*) - \sum_{i \in I} \lambda_i^* (a_i^T x^* - b_i) \\ &= \frac{1}{2} x^{*T} G x^* + x^{*T} c - \sum_{i \in I} \lambda_i^* (a_i^T x^* - b_i) \end{aligned}$$

$$\nabla_x \mathcal{L}(x^*, \lambda^*) = G x^* + c - \underbrace{\sum_{i \in I} \lambda_i^* a_i}_{\text{Since } \lambda_i^* = 0 \text{ for inactive constraints, we can write}}$$

Since $\lambda_i^* = 0$ for inactive constraints, we can write

$$\nabla_x \mathcal{L}(x^*, \lambda^*) = G x^* + c - \sum_{i \in \lambda(x^*)} \lambda_i^* a_i$$

We also have

$$a_i^T x^* = b_i \quad \forall i \in \lambda(x^*) \quad (\text{at constraint})$$

$$a_i^T x^* > b_i \quad \forall i \in I \setminus \lambda(x^*)$$

$$\lambda_i^* \geq 0 \quad \forall i \in I \cap \lambda(x^*)$$

Problem 2

b)

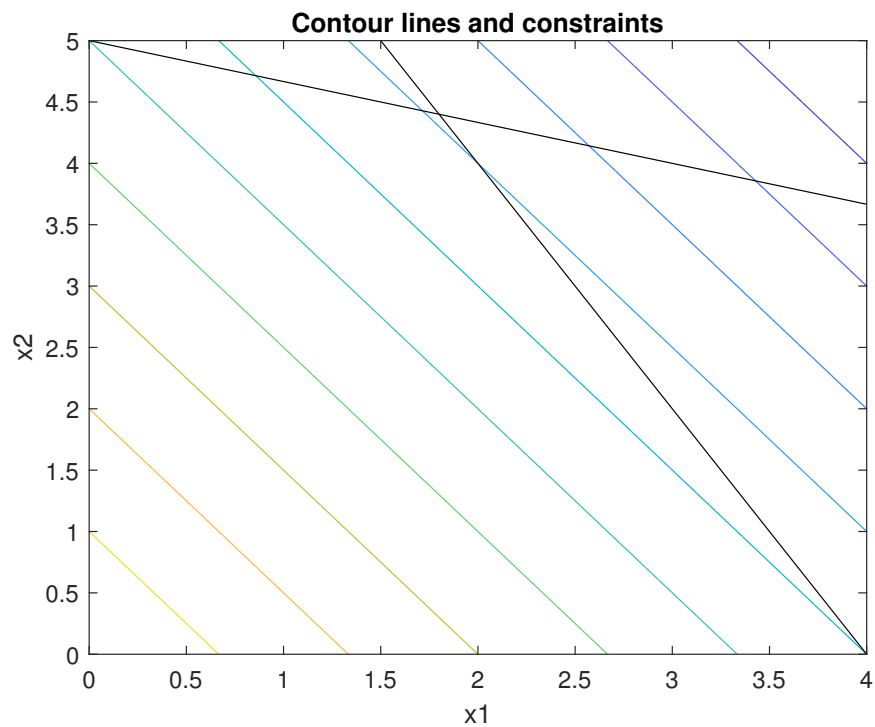


Figure 1: Contour plot with constraints

c)

Yes, the answer is at the intersection between two constraints, at the point $x = [1.8, 4.4, 0, 0]$. Since the slacking variables are at origo, all the constraints are active.

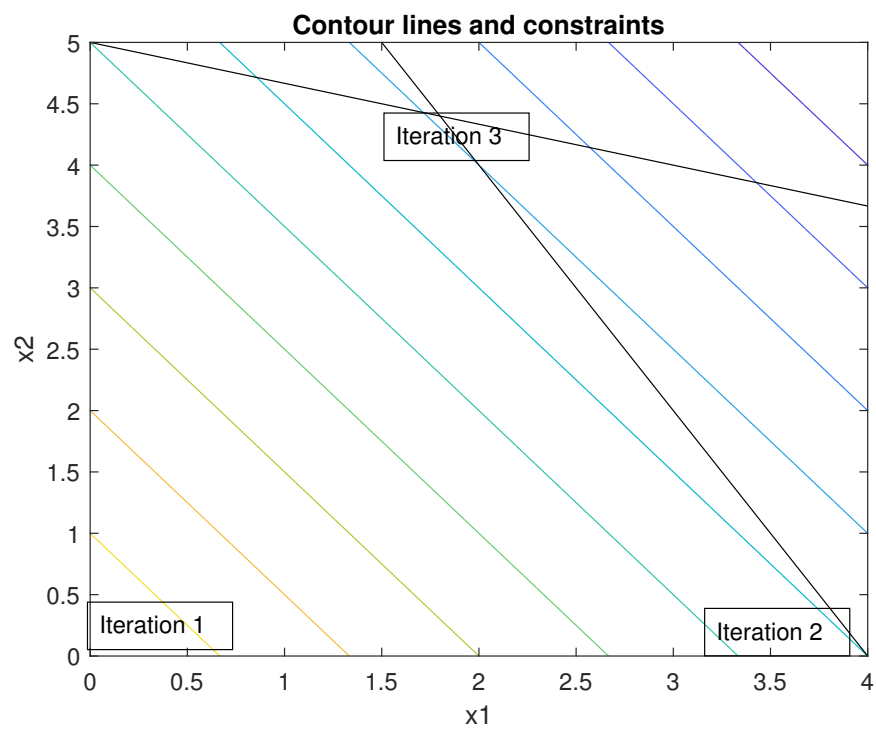


Figure 2: Marked iterations

d)

e)

Yes, everything coincides with the theory from chapter 13.3.