Tribhuvan University
School of Mathematical Sciences, Kirtipur,Kathmandu, Nepal
Problem Set For Master in Data Sciences I Year (II Sem)-2080
Course: Multivariable Calculus For Data Science -II (MDS 554)
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Unit II: Vector Functions

Unit 2: **Vector Functions**
Vector functions and space curves
Derivatives and integrals of vector functions
Arc length and curvature
Motion in space

Basic Formulae on Differentiation

If $\vec{r}$, $\vec{r}_1$ and $\vec{r}_2$ be three differentiable vector functions of scalar variable t and ϕ is a differentiable scalar function of t, then we have

1. $\frac{d}{dt}(\vec{r}_1 \pm \vec{r}_2) = \frac{d\vec{r}_1}{dt} \pm \frac{d\vec{r}_2}{dt}$.

2. If $\vec{r} = \vec{a}$, be a constant vector, then $\frac{d\vec{r}}{dt} = \vec{0}$.

3. $\frac{d}{dt}(\phi\, \vec{r}) = \frac{d\phi}{dt}\vec{r} + \phi\frac{d\vec{r}}{dt}$. In particular, if k is a constant, then $\frac{d}{dt}(k\vec{r}) = k\frac{d\vec{r}}{dt}$

4. $\frac{d}{dt}(\vec{r}_1 . \vec{r}_2) = \vec{r}_1 . \frac{d\vec{r}_2}{dt} + \frac{d\vec{r}_1}{dt} . \vec{r}_2$

5. $\frac{d}{dt}(\vec{r}_1 \times \vec{r}_2) = \vec{r}_1 \times \frac{d\vec{r}_2}{dt} + \frac{d\vec{r}_1}{dt} \times \vec{r}_2$.

6. **The Derivative of Scalar Triple Product:**

   The derivative of the scalar triple product $[\vec{r}_1\ \vec{r}_2\ \vec{r}_3]$ of three vectors $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$ is

$$\frac{d}{dt}[\vec{r}_1\ \vec{r}_2\ \vec{r}_3] = \left[\frac{d\vec{r}_1}{dt}\ \vec{r}_2\ \vec{r}_3\right] + \left[\vec{r}_1\ \frac{d\vec{r}_2}{dt}\ \vec{r}_3\right] + \left[\vec{r}_1\ \vec{r}_2\ \frac{d\vec{r}_3}{dt}\right].$$

7. **The Derivative of Vector Triple Product:**

   The derivative of the vector triple product $\vec{r}_1 \times (\vec{r}_2 \times \vec{r}_3)$ of three vectors $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$ is

$$\frac{d}{dt}[\vec{r}_1 \times (\vec{r}_2 \times \vec{r}_3)] = \frac{d\vec{r}_1}{dt} \times (\vec{r}_2 \times \vec{r}_3) + \vec{r}_1 \times \left(\frac{d\vec{r}_2}{dt} \times \vec{r}_3\right) + \vec{r}_1 \times \left(\vec{r}_2 \times \frac{d\vec{r}_3}{dt}\right).$$

Geometrical Interpretation of Derivative

Theorem:1. If $\vec{r} = \vec{f}(t)$ be a vector function of scalar variable t, then geometrically, the derivative $\frac{d\vec{r}}{dt}$ at a point P represents a vector along the tangent in the sense of t increasing.

Theorem: 2 The derivative $\frac{d\vec{r}}{dt}$ also represents the velocity of the particle at point P along the tangent PT and the second derivative $\frac{d^2\vec{r}}{dt^2}$ represents the acceleration of the particle at P along the tangent PT.

Examples and Exercises

<u>Example-1:</u> If $\vec{r} = \vec{a}\, e^{nt} + \vec{b}\, e^{nt}$, where $\vec{a}$ and $\vec{b}$ are constant vectors, show that: $\frac{d^2\vec{r}}{dt^2} - (m + n)\frac{d\vec{r}}{dt} + mn\vec{r} = \vec{0}$.

Proof:

Given that $\vec{r} = \vec{a}\, e^{mt} + \vec{b}\, e^{nt}$ .... (1)

$\therefore \quad \dfrac{d\vec{r}}{dt} = m\vec{a}\, e^{mt} + n\vec{b}\, e^{nt}$ .... (2), and $\dfrac{d^2\vec{r}}{dt^2} = m^2\vec{a}\, e^{mt} + n^2\, \vec{b}\, e^{nt}$ .... (3)

Using (1), (2) and (3), we get

$$\frac{d^2\vec{r}}{dt^2} - (m+n)\frac{d\vec{r}}{dt} + mn\vec{r} = (m^2\vec{a}\, e^{mt} + n^2\vec{b}\, e^{nt}) - (m+n)\,(m\vec{a}\, e^{mt} + n\vec{b}\, e^{nt}) + mn\,(\vec{a}\, e^{mt} + \vec{b}\, e^{nt}) = \vec{0}.$$

Example-2: **If $\vec{r} = \vec{a}\cos\omega t + \vec{b}\sin\omega t$,**

show that : $\vec{r} \times \dfrac{d\vec{r}}{dt} = \omega\,(\vec{a} \times \vec{b})$ and $\dfrac{d^2\vec{r}}{dt^2} = -\,\omega^2\vec{r}$, where $\vec{a}$ and $\vec{b}$ are constant vectors and ω is a constant.

Solution:

Here, $\vec{a}\cos\omega t + \vec{b}\sin\omega t$ .... (1)

$\therefore \quad \dfrac{d\vec{r}}{dt} = -\,\vec{a}\omega\sin\omega t + \vec{b}\omega\cos\omega t$

and $\dfrac{d^2\vec{r}}{dt^2} = -\,\vec{a}\,\omega^2\cos\omega t - b\omega^2\sin\omega t = -\,\omega^2\,(\vec{a}\cos\omega t + \vec{b}\sin\omega t) = -\,\omega^2\vec{r}$. [$\because$ By using (1)]

To prove the first part:

Now, $\vec{r} \times \dfrac{d\vec{r}}{dt} = (\vec{a}\cos\omega t + \vec{b}\sin\omega t) \times (-\,\vec{a}\omega\sin\omega t + \vec{b}\omega\cos\omega t) = (\vec{a} \times \vec{b})\omega$.

$\therefore \quad \vec{r} \times \dfrac{d\vec{r}}{dt} = (\vec{a} \times \vec{b})\omega$.

Problem: **If $\vec{r} = \vec{a}\, e^{nt} + \vec{b}\, e^{-nt}$, where $\vec{a}$ and $\vec{b}$ are constant vectors. Show that : $\dfrac{d^2\vec{r}}{dt^2} - n^2\vec{r} = 0$.**

Example-3: **If $\vec{r}$ be a unit vector, prove that : $\left|\vec{r} \times \dfrac{d\vec{r}}{dt}\right| = \left|\dfrac{d\vec{r}}{dt}\right|$**

Solution:

Since $\vec{r}$ is a unit vector, so that $|\vec{r}| = 1$ i.e., $r = 1$ and $r^2 = 1$. $\therefore\ \vec{r}.\vec{r} = 1$ . ... (1)

Differentiating both sides of (1) w.r.t t, we get

$\dfrac{d\vec{r}}{dt}.\vec{r} + \vec{r}.\dfrac{d\vec{r}}{dt} = 0$ or, $2\vec{r}.\dfrac{d\vec{r}}{dt} = 0$ or, $\vec{r}.\dfrac{d\vec{r}}{dt} = 0$.

This shows that $\vec{r}$ and $\dfrac{d\vec{r}}{dt}$ are perpendicular to each other and therefore, the angle between $\vec{r}$ and $\dfrac{d\vec{r}}{dt}$ is 90°.

Hence, $\left|\vec{r} \times \dfrac{d\vec{r}}{dt}\right| = |\vec{r}|\left|\dfrac{d\vec{r}}{dt}\right| \sin 90° = \left|\dfrac{d\vec{r}}{dt}\right|$.

Example-4: **If $\vec{r} = t^2\vec{i} - t\vec{j} + (2t+1)\,\vec{k}$. Find (i) $\dfrac{d\vec{r}}{dt}.\dfrac{d^2\vec{r}}{dt^2}$ (ii) $\left|\dfrac{d\vec{r}}{dt}\right|$ (iii) $\left|\dfrac{d^2\vec{r}^2}{dt^2}\right|$ at $t = 0$.**

Solution:

Now, $\vec{r} = t^2\vec{i} - t\vec{j} + (2t+1)\vec{k}$ $\therefore \quad \dfrac{d\vec{r}}{dt} = 2t\,\vec{i} - \vec{j} + 2\,\vec{k}$ and $\dfrac{d^2\vec{r}}{dt^2} = 2\,\vec{i}$.

(i) $\dfrac{d\vec{r}}{dt}.\dfrac{d^2\vec{r}}{dt^2} = (2t\vec{i} - \vec{j} + 2\vec{k}).2\,\vec{i} = 4t\,(\vec{i}\,.\,\vec{i}) - 2(\vec{j}\,.\,\vec{i}) + 4\,(\vec{k}\,.\,\vec{i}) = 4t.$

$\therefore$ **At $t = 0$,** $\dfrac{d\vec{r}}{dt}.\dfrac{d^2\vec{r}}{dt^2} = 4.0 = 0$.

(ii) $\left|\dfrac{d\vec{r}}{dt}\right| = |2t\vec{i} - \vec{j} + 2\vec{k}| = \sqrt{(2t)^2 + (-1)^2 + (2)^2} = \sqrt{4t^2 + 5}$. $\therefore$ **At $t = 0$,** $\left|\dfrac{d\vec{r}}{dt}\right| = \sqrt{5}$.

(iii) $\left|\dfrac{d^2\vec{r}}{dt^2}\right| = |2\vec{i}\,| = \sqrt{2^2} = 2.$ $\qquad\therefore$ **At $t = 0$,** $\left|\dfrac{d^2\vec{r}}{dt^2}\right| = 2.$

Example-5: If $\vec{r} = a\cos t\,\vec{i} + b\sin t\,\vec{j} + ct\vec{k}$, find $\dot{\vec{r}}$, $\ddot{\vec{r}}$, $|\dot{\vec{r}}|$ and $|\ddot{\vec{r}}|$. Also, find their values at $t = 0$.
Solution:

Given that $\vec{r} = a\cos t\,\vec{i} + b\sin t\,\vec{j} + ct\vec{k}$. .... (1)
Differentiating bothsides of (1) two times w.r.t. 't', we get

$\dot{\vec{r}} = -a\sin t\,\vec{i} + b\cos t\,\vec{j} + c\vec{k}$ .... (2)

and $\ddot{\vec{r}} = -a\cos t\,\vec{i} - b\sin t\,\vec{j}$. .... (3)
From (2) and (3), we get

$|\dot{\vec{r}}| = \sqrt{a^2\sin^2 t + b^2\cos^2 t + c^2}$ and $|\ddot{\vec{r}}| = \sqrt{a^2\cos^2 t + b^2\sin^2 t}$.
At $t = 0$

$\dot{\vec{r}} = b\vec{j} + c\vec{k}$ and $\ddot{\vec{r}} = -a\vec{i}$ $\quad\therefore\ |\dot{\vec{r}}| = \sqrt{b^2 + c^2}$ and $|\ddot{\vec{r}}| = a$.

Problem: **If $\vec{r} = a\cos t\,\vec{i} + a\sin t\,\vec{j} + t\vec{k}$, find $\dfrac{d\vec{r}}{dt}$, $\dfrac{d^2\vec{r}}{dt^2}$ and $\left|\dfrac{d^2\vec{r}}{dt^2}\right|$.**

Ans: $-a\sin t\,\vec{i} + a\cos t\,\vec{j} + \vec{k}$; $-a\cos t\,\vec{i} - a\sin t\,\vec{j}$; a.

Example-6: If $\vec{r_1} = t^3\vec{i} + t^2\vec{j} + t\vec{k}$ and $\vec{r_2} = (t+1)\vec{i} + (t+2)\vec{j} - 3t\,\vec{k}$. Find (a) $\dfrac{d}{dt}(\vec{r_1}.\vec{r_2})$ (b) $\dfrac{d}{dt}(\vec{r_1}\times\vec{r_2})$ at $t = 2$.

Solution:

Now, $\vec{r_1} = t^3\vec{i} + t^2\vec{j} + t\vec{k}$ $\qquad\therefore \dfrac{d\vec{r_1}}{dt} = 3t^2\vec{i} + 2t\vec{j} + \vec{k}$

and $\vec{r_2} = (t+1)\vec{i} + (t+2)\vec{j} - 3t\,\vec{k}$ $\qquad\therefore \dfrac{d\vec{r_2}}{dt} = \vec{i} + \vec{j} - 3\vec{k}$.

$$\textbf{(a)}\ \frac{d}{dt}(\vec{r_1}.\vec{r_2}) = \vec{r_1}.\frac{d\vec{r_2}}{dt} + \frac{d\vec{r_1}}{dt}.\vec{r_2}$$
$$= (t^3\vec{i} + t^2\vec{j} + t\vec{k}).(\vec{i} + \vec{j} - 3\vec{k}) + (3t^2\vec{i} + 2t\vec{j} + \vec{k})\ .[t+1)\vec{i} + (t+2)\vec{j} - 3t\vec{k}]$$
$$= 4t^3 + 6t^2 - 2t.$$

$\therefore$ **At t = 2,**

$$\frac{d}{dt}(\vec{r_1}.\vec{r_2}) = 4(2)^3 + 6(2)^2 - 2.2 = 32 + 24 - 4 = 52.$$

$$\textbf{(b)}\ \frac{d}{dt}(\vec{r_1}\times\vec{r_2}) = \vec{r_1}\times\frac{d\vec{r_2}}{dt} + \frac{d\vec{r_1}}{dt}\times\vec{r_2}$$
$$= [(t^3\vec{i} + t^2\vec{j} + t\vec{k})\times(\vec{i} + \vec{j} - 3\vec{k})] + (3t^2\vec{i} + 2t\vec{j} + k)\times[(t+1)\vec{i} + (t+2)\vec{j} - 3t\vec{k}]$$
$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ t^3 & t^2 & t \\ 1 & 1 & -3 \end{vmatrix} + \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3t^2 & 2t & 1 \\ (t+1) & (t+2) & (-3t) \end{vmatrix}$$

$\therefore$ **At t = 2,**

$$\frac{d}{dt}(\vec{r_1}\times\vec{r_2}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 8 & 4 & 2 \\ 1 & 1 & -3 \end{vmatrix} + \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 12 & 4 & 1 \\ 3 & 4 & -6 \end{vmatrix}$$
$$= \vec{i}\,(-12-2) + \vec{j}\,(2+24) + \vec{k}\,(8-4) + \vec{i}\,(-2\,4-4) + \vec{j}(3+72) + \vec{k}\,(48-12)$$
$$= -42\vec{i} + 101\vec{j} + 40\vec{k}.$$

Problem: 1. If $\vec{u} = t^2\vec{i} - t\vec{j} + (2t+1)\vec{k}$ and $\vec{v} = (2t-3)\,\vec{i} + \vec{j} - t\vec{k}$,

find (a) $\frac{d}{dt}(\vec{u}.\vec{v})$ (b) $\frac{d}{dt}(\vec{u} \times \vec{v})$,at t = 1.

Ans: At $t = 1$, $\frac{d}{dt}(\vec{u}.\vec{v}) = -6$ and $\frac{d}{dt}(\vec{u} \times \vec{v}) = 7\vec{j} + 3\vec{k}$.

<u>Problem: 2.</u> If $\vec{r} = (a \cos t)\vec{i} + (a \sin t)\vec{j} + (at \tan\alpha)\vec{k}$, find $\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right|$ and $\left[\frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\ \frac{d^3\vec{r}}{dt^3}\right]$.

Ans: $\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right| = a^2 \sec\alpha$ and $\left[\frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\ \frac{d^3\vec{r}}{dt^3}\right] = \frac{d^3\vec{r}}{dt^3}.\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2} = a^3 \tan\alpha.$

<u>Example-8:</u> If $\vec{r_1} = (a \cos t, b \sin t, 0)$, $\vec{r_2} = (-a \sin t, b \cos t, t)$ and $\vec{r_3} = (1, 2, 3)$, find $\frac{d}{dt}[\vec{r_1}.(\vec{r_2} \times \vec{r_3})]$.

Solution:

Here, $\vec{r_1} = (a \cos t, b \sin t, 0)$, $\vec{r_2} = (-a \sin t, b \cos t, t)$ and $\vec{r_3} = (1, 2, 3)$.

Now, $\vec{r_1}.\vec{r_2} \times \vec{r_3} = (a \cos t, b \sin t, 0).[(-a \sin t, b \cos t, t) \times (1, 2, 3)]$

$$= \begin{vmatrix} a\cos t & b \sin t & 0 \\ -a \sin t & b \cos t & t \\ 1 & 2 & 3 \end{vmatrix} = 3ab - 2at\cos t + bt \sin t .$$

$$\therefore \frac{d}{dt}[\vec{r_1}.\vec{r_2} \times \vec{r_3}] = -2a(\cos t - t \sin t) + b(\sin t + t \cos t).$$

$$\therefore \textbf{At } t = 0, \frac{d}{dt}[\vec{r_1}.\vec{r_2} \times \vec{r_3}] = -2a(1 - 0) + b(0 + 0) = -2a.$$

<u>Problem: 1.</u>For the curve $x = 3t$, $y = 3t^2$, $z = 2t^3$, show that $[\dot{\vec{r}}\ \ddot{\vec{r}}\ \dddot{\vec{r}}] = 216$.

2. If $\vec{r_1} = (2t + 1)\vec{i} - t^2\vec{j} + 3t^3\vec{k}$ and $\vec{r_2} = t^2\vec{i} + t\vec{j} - (t - 1)\vec{k}$, verify that:

(a) $\frac{d}{dt}(\vec{r_1}.\vec{r_2}) = \vec{r_1}.\frac{d\vec{r_2}}{dt} + \frac{d\vec{r_1}}{dt}.\vec{r_2}$ (b) $\frac{d}{dt}(\vec{r_1} \times \vec{r_2}) = \vec{r_1} \times \frac{d\vec{r_2}}{dt} + \frac{d\vec{r_1}}{dt} \times \vec{r_2}$.

<u>Example-9:</u> Evaluate the derivatives of the following w.r.t t.

(a) $\frac{\vec{r}}{r}$ (b) $\vec{r}^2 + \frac{1}{\vec{r}^2}$ (c) $\frac{\vec{r} \times \vec{a}}{\vec{r}.\vec{a}}$ (d) $\frac{\vec{r} + \vec{a}}{\vec{r} \times \vec{a}}$, where $\vec{a}$ is a constant vector.

(e) $\vec{r} \times \left(\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right)$

Solutions:

(a) **Let $\vec{u} = \frac{\vec{r}}{r}$.**

Differentiating both sides w.r.t t yields

$$\frac{d\vec{u}}{dt} = \frac{d}{dt}\left(\frac{\vec{r}}{r}\right) = \left[\mathrm{r}\frac{d\vec{r}}{dt} - \frac{dr}{dt}\vec{r}\right]\frac{1}{r^2} = \frac{1}{r}\frac{d\vec{r}}{dt} - \frac{\vec{r}}{r^2}\frac{dr}{dt}.$$

(b) Let $u = \vec{r}^2 + \frac{1}{\vec{r}^2} = r^2 + \frac{1}{r^2}$ $(\because \vec{r}^2 = r^2$ etc)

Differentiating w.r.t t, we get

$$\frac{du}{dt} = \frac{d}{dt}\left(r^2 + \frac{1}{r^2}\right) = 2r\frac{dr}{dt} + (-2r^{-3})\frac{dr}{dt} = 2\left(r - \frac{1}{r^3}\right)\frac{dr}{dt}.$$

(c) Let $\vec{u} = \frac{\vec{r} \times \vec{a}}{\vec{r}.\vec{a}}$.

Differentiating bothsides w.r.t t, we get

$$\frac{d\vec{u}}{dt} = \frac{(\vec{r}.\vec{a})\frac{d}{dt}(\vec{r} \times \vec{a}) - (\vec{r} \times \vec{a})\frac{d}{dt}(\vec{r}.\vec{a})}{(\vec{r}.\vec{a})^2} = \frac{1}{\vec{r}.\vec{a}}\left(\frac{d\vec{r}}{dt} \times \vec{a} + \vec{r} \times \frac{d\vec{a}}{dt}\right) - \frac{\vec{r} \times \vec{a}}{(\vec{r}.\vec{a})^2}\left(\vec{r}.\frac{d\vec{a}}{dt} + \frac{d\vec{r}}{dt}.\vec{a}\right)$$

Since $\vec{a}$ is a constant vector, therefore $\frac{d\vec{a}}{dt} = \vec{0}$.

$$\therefore \quad \frac{d\vec{u}}{dt} = \frac{1}{\vec{r}.\vec{a}}\left(\frac{d\vec{r}}{dt} \times \vec{a}\right) - \frac{(\vec{r} \times \vec{a})}{(\vec{r}.\vec{a})^2}\left(\frac{d\vec{r}}{dt}.\vec{a}\right).$$

(d) Let $\vec{u} = \dfrac{\vec{r} + \vec{a}}{\vec{r} \times \vec{a}}$.

Differentiating both sides w.r.t t, we get

$$\frac{d\vec{u}}{dt} = \frac{(\vec{r} \times \vec{a})\frac{d}{dt}(\vec{r} + \vec{a}) - (\vec{r} + \vec{a})\frac{d}{dt}(\vec{r} \times \vec{a})}{(\vec{r} \times \vec{a})^2} = \frac{1}{\vec{r} \times \vec{a}}\left(\frac{d\vec{r}}{dt} + \frac{d\vec{a}}{dt}\right) - \frac{\vec{r} + \vec{a}}{(\vec{r} + \vec{a})^2}\left[\vec{r} \times \frac{d\vec{a}}{dt} + \frac{d\vec{r}}{dt} \times \vec{a}\right]$$

Since $\vec{a}$ is a constant vector, therefore $\dfrac{d\vec{a}}{dt} = \vec{0}$.

$$\therefore \quad \frac{d\vec{u}}{dt} = \frac{1}{\vec{r} \times \vec{a}}\frac{d\vec{r}}{dt} - \frac{\vec{r} + \vec{a}}{(\vec{r} \times \vec{a})^2}\left(\frac{d\vec{r}}{dt} \times \vec{a}\right).$$

(e) Let $\vec{u} = \vec{r} \times \left(\dfrac{d\vec{r}}{dt} \times \dfrac{d^2\vec{r}}{dt^2}\right)$

Differentiating both sides w.r.t t, we get

$$\frac{d\vec{u}}{dt} = \frac{d}{dt}\left[\vec{r} \times \left(\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right)\right] = \vec{r} \times \frac{d}{dt}\left(\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right) + \frac{d\vec{r}}{dt} \times \left(\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right)$$

$$= \vec{r} \times \left[\frac{d^2\vec{r}}{dt^2} \times \frac{d^2\vec{r}}{dt^2} + \frac{d\vec{r}}{dt} \times \frac{d^3\vec{r}}{dt^3}\right] + \frac{d\vec{r}}{dt} \times \left(\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right)$$

Since $\dfrac{d^2\vec{r}}{dt^2} \times \dfrac{d^2\vec{r}}{dt^2} = \vec{0}$. So we have

$$\frac{d\vec{u}}{dt} = \vec{r} \times \left(\frac{d\vec{r}}{dt} \times \frac{d^3\vec{r}}{dt^3}\right) + \frac{d\vec{r}}{dt} \times \left(\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right).$$

Problems: **Find the derivative of the following:**

(a) $\dfrac{\vec{r} + \vec{a}}{r^2 + a^2}$ **(b)** $\dfrac{\vec{r} + \vec{a}}{\vec{r}.\vec{a}}$ **(c)** $\vec{r}.\dfrac{d\vec{r}}{dt} \times \dfrac{d^2\vec{r}}{dt^2}$

Ans: (a) $\dfrac{1}{r^2 + a^2}\dfrac{d}{dt}(\vec{r} + \vec{a}) - \dfrac{\vec{r} + \vec{a}}{(r^2 + a^2)^2}2\vec{r}.\dfrac{d\vec{r}}{dt}$ **(b)** $\dfrac{(\vec{r}.\vec{a})\dfrac{d\vec{r}}{dt} - (\vec{r} + \vec{a})\left(\dfrac{d\vec{r}}{dt}.\vec{a}\right)}{(\vec{r}.\vec{a})^2}$ **(c)** $\left[\vec{r}\ \dfrac{d\vec{r}}{dt}\ \dfrac{d^3\vec{r}}{dt^3}\right]$

<u>Example-11:</u> If $\dfrac{d\vec{a}}{dt} = \vec{c} \times \vec{a}$ and $\dfrac{d\vec{b}}{dt} = \vec{c} \times \vec{b}$, show that: $\dfrac{d}{dt}(\vec{a} \times \vec{b}) = \vec{c} \times (\vec{a} \times \vec{b})$.

Solution:

Now,
$$\begin{aligned}
\frac{d}{dt}(\vec{a} \times \vec{b}) &= \frac{d\vec{a}}{dt} \times \vec{b} + \vec{a} \times \frac{d\vec{b}}{dt} \\
&= (\vec{c} \times \vec{a}) \times \vec{b} + \vec{a} \times (\vec{c} \times \vec{b}) \\
&= -[\vec{b} \times (\vec{c} \times \vec{a})] + \vec{a} \times (\vec{c} \times \vec{b}) \\
&= -[(\vec{b}.\vec{a})\,\vec{c} - (\vec{b}.\vec{c})\vec{a}] + (\vec{a}.\vec{b})\vec{c} - (\vec{a}.\vec{c})\vec{b} \\
&= -(\vec{a}.\vec{b})\,\vec{c} + (\vec{c}.\vec{b})\vec{a} + (\vec{a}.\vec{b})\vec{c} - (\vec{c}.\vec{a})\vec{b} \\
&= (\vec{c}.\vec{b})\,\vec{a} - (\vec{c}.\vec{a})\vec{b} \\
&= \vec{c} \times (\vec{a} \times \vec{b})
\end{aligned}$$

Geometrical Problems:

Example-12:. Find the velocity and acceleration of a particle which moves along the curve $x = 2\sin 3t$, $y = 2\cos 3t$, $z = 8t$ at time $t = \frac{\pi}{3}$. Also, find also their magnitudes.

Solution:

Let $\vec{r}$ be the position vector of a particle at any time t. So that

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ $\quad\therefore\quad \vec{r} = 2\sin 3t\,\vec{i} + 2\cos 3t\,\vec{j} + 8t\vec{k}$.

Since $\frac{d\vec{r}}{dt}$ and $\frac{d^2\vec{r}}{dt^2}$ represent the velocity and acceleration of the moving particle at any time t.

Therefore, velocity $(\vec{v}) = \frac{d\vec{r}}{dt} = 6\cos 3t\,\vec{i} - 6\sin 3t\,\vec{j} + 8\vec{k}$

and acceleration $(\vec{a}) = \frac{d^2\vec{r}}{dt^2} = -18\sin 3t\,\vec{i} - 18\cos 3t\,\vec{j}$.

At $t = \frac{\pi}{3}$, *the velocity and acceleration are*

$\vec{v} = 6\cos\pi\,\vec{i} - 6\sin\pi\,\vec{j} + 8\vec{k} = -6\vec{i} + 8\vec{k}$ and $\vec{a} = -18\sin\pi\,\vec{i} - 18\cos\pi\,\vec{j} = 18\vec{j}$.

The magnitude of velocity and acceleration are

$|\vec{v}| = \sqrt{(-6)^2 + (8)^2} = 10$ and $|\vec{a}| = \sqrt{(18)^2} = 18$.

Example-13: A particle P is moving on a circle of radius r with constant angular velocity $\omega = \frac{d\theta}{dt}$. Show that its acceleration is $-\omega^2\vec{r}$.

Hint:

Since the particle P is moving on a circle of radius r with constant angular velocity ω. So that $x = r\cos\omega t$ and $y = r\sin\omega t$.

Therefore, $\vec{r} = x\vec{i} + y\vec{j} = r\cos\omega t\,\vec{i} + r\sin\omega t\,\vec{j}$.

Now, $\frac{d\vec{r}}{dt} = -r\omega\sin\omega t\,\vec{i} + r\omega\cos\omega t\,\vec{j}$

and $\frac{d^2\vec{r}}{dt^2} = -r\omega^2\cos\omega t\,\vec{i} - r\omega^2\sin\omega t\,\vec{j} = -\omega^2(r\cos\omega t\,\vec{i} + r\sin\omega t\,\vec{j}) = -\omega^2\vec{r}$.

Therefore, $\frac{d^2\vec{r}}{dt^2} = -\omega^2\vec{r}$.

Example-14: A particle moves such that it's position vector is given by $\vec{r} = \cos\omega t\,\vec{i} + \sin\omega t\,\vec{j}$, where ω is constant. Then show that (a) Velocity $\vec{v}$ is perpendicular to $\vec{r}$. (b) $\vec{r} \times \vec{v}$ is a constant vector.

Hint:

(a) Since $\vec{r} = \cos\omega t\,\vec{i} + \sin\omega t\,\vec{j}$ $\quad\therefore\quad \vec{v} = \frac{d\vec{r}}{dt} = (-\sin\omega t)\,\omega\vec{i} + (\cos\omega t)\omega\vec{j}$

Now, $\vec{v}.\vec{r} = [(-\omega\sin\omega t)\vec{i} + (\omega\cos\omega t)\vec{j}].(\cos\omega t\,\vec{i} + \sin\omega t\,\vec{j}) = 0$.

Therefore, the velocity $\vec{v}$ is perpendicular to $\vec{r}$.

(b) Now, $\vec{r} \times \vec{v} = (\cos\omega t\,\vec{i} + \sin\omega t\,\vec{j}) \times (-\omega\sin\omega t\,\vec{i} + \omega\cos\omega t\,\vec{j}) = \omega\vec{k}$. (Prove it)

Therefore, $\vec{r} \times \vec{v} = \omega\vec{k}$, which is constant vector as ω is constant.

EXERCISE

1. (a) A particle moves along the curve $x = 2\sin 3t$, $y = 2\cos 3t$, $z = 8t$. Find the magnitude of velocity and acceleration at time $t = \frac{\pi}{3}$. **Ans:** Magnitude of velocity = 10 and acceleration = 18.

(b) A particle moves along the curve $x = 4\cos t$, $y = 4\sin t$, $z = 6t$. Find the magnitude of velocity and acceleration at time $t = 0$ and $t = \pi$. **Ans:** $4\vec{i} + 6\vec{k}, -4\vec{j} + 6\vec{k}, -4\vec{i}, 4\vec{i}$

Integration of Vector Functions

Examples/ Exercises

Example-1: Evaluate: $\int \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) dt$.

Solution:

Since $\frac{d}{dt}\left(\vec{r} \times \frac{d\vec{r}}{dt}\right) = \vec{r} \times \frac{d^2\vec{r}}{dt^2} + \frac{d\vec{r}}{dt} \times \frac{d\vec{r}}{dt} = \vec{r} \times \frac{d^2\vec{r}}{dt^2}$.

Therefore, we have

$$\int \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) dt = \vec{r} \times \frac{d\vec{r}}{dt} + \vec{c}.$$

Example-2: Let $\vec{r}_1 = 3t^2\,\vec{i} + 4(t-2)\,\vec{j} + 5t^2\vec{k}$ and $\vec{r}_2 = 2(t-2)\,\vec{i} + t^2\vec{j} + (t-3)\vec{k}$. Find the value of $\int_0^1 (\vec{r}_1 \cdot \vec{r}_2)\, dt$.

Solution:

Now, $\vec{r}_1.\vec{r}_2 = [3t^2\vec{i} + 4(t-2)\vec{j} + 5t^2\vec{k}].[2(t-2)\,\vec{i} + t^2\vec{j} + (t-3)\vec{k}] = 15t^3 - 35t^2$.

$$\therefore \int_0^1 (\vec{r}_1.\vec{r}_2)\, dt = \int_0^1 (15t^3 - 35t^2)\, dt = \left[15\frac{t^4}{4} - 35\frac{t^3}{3}\right]_0^1 = \frac{15}{4} - \frac{35}{3} = -\frac{95}{12}.$$

Example-2: If $\vec{r} = t^2\vec{i} + t\vec{j} + t^2\vec{k}$, then evaluate: $\int_0^2 \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) dt$.

Solution:

Here, $\vec{r} = t^2\vec{i} + t\vec{j} + t^2\vec{k}$. $\quad\therefore \frac{d\vec{r}}{dt^2} = 2t\vec{i} + \vec{j} + 2t\vec{k}$ and $\frac{d^2\vec{r}}{dt^2} = 2\vec{i} + 2\vec{k}$.

Now, $\vec{r} \times \frac{d^2\vec{r}}{dt^2} = (t^2\vec{i} + t\vec{j} + t^2\vec{k}) \times (2\vec{i} + 2\vec{k}) = 2t\vec{i} - 2t\vec{k}$.

Now, $\int_0^2 \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) dt = \int_0^2 (2t\vec{i} - 2t\vec{k})\, dt = [t^2\vec{i} - t^2\vec{k}]_0^2 = 4\vec{i} - 4\vec{k}$.

Example-3: Solve the equation: $\frac{d^2\vec{r}}{dt^2} = \vec{a}t$, for any constant vector $\vec{a}$.

Solution:

Here, $\frac{d^2\vec{r}}{dt^2} = \vec{a}t$. Integrating twice w.r.t. t, we get

$$\frac{d\vec{r}}{dt} = \frac{\vec{a}t^2}{2} + \vec{b}.$$

or, $\vec{r} = \vec{a}\frac{t^3}{6} + \vec{b}t + \vec{c}$, where $\vec{b}$ and $\vec{c}$ are the constants of integration.

Example-4: If the acceleration of a particle at any time t is given by the function $\vec{a}(t) = e^{2t}\,\vec{i} + e^t\vec{j} + 2\vec{k}$, find the velocity $\vec{v}$ and displacement $\vec{r}$ at time t, given that $\vec{v} = \vec{i} + \vec{j}$ and $\vec{r} = \vec{0}$ at $t = 0$.

Solution:

Here, acceleration function $\vec{a}(t) = e^{2t}\,\vec{i} + e^t\vec{j} + 2\vec{k}$. Integrating w.r.t. t, the velocity function is

$$\vec{v}(t) = \int a(t)dt = \int (e^{2t}\,\vec{i} + e^t\,\vec{j} + 2\vec{k})\, dt$$

$\therefore \quad \vec{v}(t) = \frac{e^{2t}}{2}\ \vec{i} + e^t\vec{j} + 2t\vec{k} + \vec{c}$. ………(1)

At $t = 0$, $\vec{v}(0) = \vec{i} + \vec{j}$, $\vec{v}(0) = \frac{1}{2}\vec{i} + \vec{j} + \vec{c}_1$ or, $\vec{i} + \vec{j} = \frac{1}{2}\vec{i} + \vec{j} + \vec{c}_1$ $\quad \therefore \quad \vec{c}_1 = \frac{1}{2}\vec{i}$.

Hence (1) becomes

$$\vec{v}(t) = \frac{e^{2t}}{2}\ \vec{i} + e^t\vec{j} + 2t\vec{k} + \frac{1}{2}\ \vec{i}$$

Again integrating, we get

$$\vec{r}(t) = \int \vec{v}(t)\, dt = \int \left[\frac{e^{2t}}{2}\vec{i} + e^t\vec{j} + 2t\vec{k} + \frac{1}{2}\ \vec{i}\right] dt = \frac{e^{2t}}{4}\ \vec{i} + e^t\vec{j} + t^2\vec{k} + \frac{1}{2}\ \vec{i} + \vec{c}_2$$

But $\vec{r}(t) = \vec{0}$, when $t = 0$.

$\therefore \quad \vec{0} = \frac{\vec{i}}{4} + \vec{j} + \frac{1}{2}\vec{i} + \vec{c}_2 \qquad \therefore \qquad \vec{c}_2 = -\frac{3\vec{i}}{4} - \vec{j}$.

Hence, $\vec{r}(t) = \frac{e^{2t}}{4}\vec{i} + e^t\vec{j} + t^2\vec{k} + \frac{1}{2}\vec{i} - \frac{3}{4}\vec{i} - \vec{j} \;\; = \frac{e^{2t}}{4}\vec{i} + e^t\vec{j} + t^2\vec{k} - \frac{1}{4}\vec{i} - \vec{j}$.

Example-5: If $\vec{r}.d\vec{r} = 0$, show that $|\vec{r}|$ is constant.

Solution:

Here, $\vec{r}.d\vec{r} = 0$ or, $2\vec{r}.\,d\vec{r} = 0 \quad \therefore \quad d(\vec{r}.\vec{r}) = 0$.

Integrating, we get

$\vec{r}.\vec{r} = c$ or, $\vec{r}^{\,2} = c$ i.e. $|\vec{r}|^2 = c$. i.e., $|\vec{r}|$ is constant.

Exercise (Vector Integration)

1. If $\vec{r} = 3t^2\vec{i} - 2\vec{j} + 3(t^2 - 1)\,\vec{k}$, find a) $\int \vec{r}\, dt$ b) $\int_0^1 \vec{r}\, dt$. ***Ans:*** a) $t^3\vec{i} - 2t\vec{j} + (t^3 - 3t)\,\vec{k}$, b) $\vec{i} - 2\vec{j} - 2\vec{k}$.

2. Given $\vec{r} = (t - t^2)\,\vec{i} + 2t^3\vec{j} - 3\vec{k}$, evaluate: $\int_1^2 \vec{r}\, dt$. ***Ans:*** $\frac{1}{6}(-5\vec{i} + 45\vec{j} - 18\vec{k})$

3. If $\vec{r}_1 = 2\vec{i} + t\vec{j} - \vec{k}$, $\vec{r}_2 = t\vec{i} + 2\vec{j} + 3\vec{k}$ and $\vec{r}_3 = 2\vec{i} - 3\vec{j} + 4\vec{k}$, find: a) $\int_0^2 (\vec{r}_1 \times \vec{r}_2)\, dt$ b) $\int_0^2 [\vec{r}_1\ \vec{r}_2\ \vec{r}_3]\, dt$.
Ans: a) $10\vec{i} - 14\vec{j} + \frac{16}{3}\vec{k}$ b) $\frac{253}{6}$

4. Evaluate: $\int_1^2 \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) dt$ for $\vec{r} = 2t^2\vec{i} + t\vec{j} - 3t^3\vec{k}$. ***Ans:*** $-42\vec{i} + 90\vec{j} - 6\vec{k}$

5. Integrate: $\frac{d^2\vec{r}}{dt^2} = -n^2\vec{r}$. ***Ans:*** $\left(\frac{d\vec{r}}{dt}\right)^2 = -n^2\vec{r}^{\,2} + c$

6. Solve: $\frac{d^2\vec{r}}{dt^2} = t\vec{a} + \vec{b}$, where $\vec{a}$ and $\vec{b}$ are constant vectors, given that $\vec{r} = \vec{0}$ and $\frac{d\vec{r}}{dt} = \vec{0}$, when $t = 0$.
Ans: $\vec{r}(t) = \frac{t^3}{6}\ \vec{a} + \frac{t^2}{2}\vec{b}$

7. a) The acceleration of a particle is given by $\frac{d^2\vec{r}}{dt^2} = \frac{d\vec{v}}{dt} = e^{-t}\,\vec{i} - 6(t + 1)\vec{j} + 3sint\,\vec{k}$.
Find the velocity $\vec{v}$ and displacement $\vec{r}$, given that $\vec{v} = \vec{0}$, $\vec{r} = \vec{0}$ when $t = 0$.
Ans: $\vec{v} = (1 - e^{-t})\,\vec{i} - (3t^2 + 6t)\vec{j} + (3 - 3\cos t)\,\vec{k}$ and $\vec{r} = (t - 1 + e^{-t})\,\vec{i} - (t^3 + 3t^2)\vec{j} + (3t - 3\,sin\,t)\,\vec{k}$.

b) The acceleration of a moving particle at any time t is $\vec{a}\,(t) = \frac{d\vec{v}}{dt} = 12\,cos2t\,\vec{i} - 8\,sin2t\,\vec{j} + 16t\,\vec{k}$.
Find the velocity $\vec{v}$ and the displacement $\vec{r}$ at any time t, given that if $t = 0$, $\vec{v} = \vec{0}$ and $\vec{r} = \vec{0}$.
Ans: $\vec{v} = 6sin\,2t\,\vec{i} + 4(cos2t - 1)\,\vec{j} + 8t^2\vec{k}$; and $\vec{r} = 3(1 - cos\,2t)\,\vec{i} + 2(sin2t - 2t)\,\vec{j} + \frac{8}{3}\,t^3\,\vec{k}$.

□□