Partial derivatives of order 3 or higher

Example 25. Calculate $f_{xx\,yz}$ if $f(x, y, z) = \sin(3x + yz)$.

5 Partial Differential Equations

A **partial differential equation** (PDE) is a differential equation involving functions of several variables and their partial derivatives. Many important and interesting phenomena are modelled by functions of several variables that satisfy certain partial differential equations. We mention three particular partial differential equations that arise frequently in mathematics and the physical sciences.

- Laplace's equation
- Wave equation
- Heat equation.

We will also introduce the Cobb-Douglas Production model and the associated partial differential equation.

5.1 Laplace's equation

Laplace's equation

The partial differential equation of the form:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is called **Laplace's equation**. Solutions of this equation are called **harmonic functions**.

Laplace's equations play a role in problems of heat conduction, fluid flow, and electric potential.

Example 26. Show that the function $u(x, y) = e^x \sin y$ is a solution of Laplace's equation.

Solution. ...

5.2 Wave equation

Wave equation

The partial differential equation of the form:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

is called a **wave equation**.

This equation describes the motion of a waveform, which could be an ocean wave, a sound wave, a light wave, or a wave traveling along a vibrating string.

If t measures time, then $f(x-ct)$ represents a waveform travelling to the right along the x-axis with speed c depending on the density of the string and on the tension in the string.. (See the figure given below.)

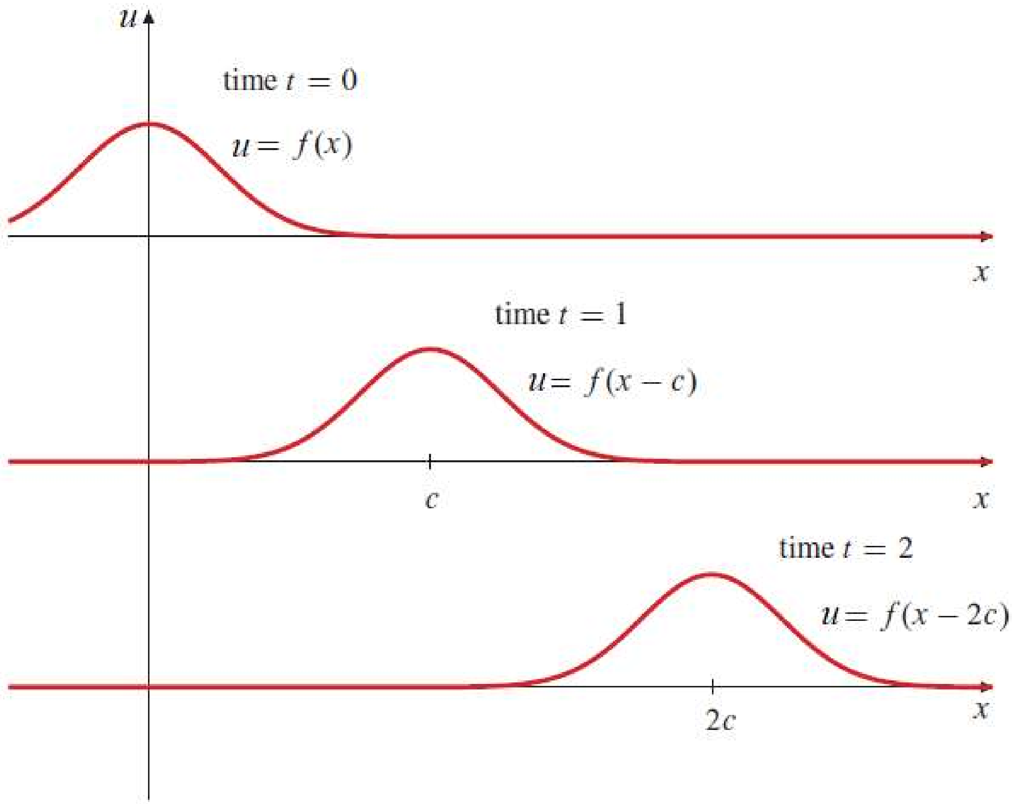

Figure 20: $u = f(x - ct)$ represents a waveform moving to the right with speed c.

Similarly, $g(x + ct)$ represents a waveform travelling to the left with speed c. Unlike the solutions of Laplace's equation that must be infinitely differentiable, solutions of the wave equation need only have enough derivatives to satisfy the differential equation. The functions f and g are otherwise arbitrary.

Example 27. Verify that the function $u(x, y) = \sin(x-ct)$ satisfies the wave equation.

Solution. ...

5.2.1 Heat (diffusion) equation

Wave equation

The partial differential equation of the form:

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

is called a **heat equation** or **diffusion equation**.

Example 28. Verify that the function $u(x, y) = e^{-a^2 t} \sin(ax)$ satisfies the heat equation with $c = 1$.

Solution. ...

5.3 Cobb-Douglas Production Function

Let

P : total production of an economic system

L : the amount of labor required to produce P

K : the capital investment required to produce P.

Then the total production P can be described as a function of L and K.

Let the production function be denoted by $P = P(L, K)$. Then

- The partial derivative $\partial P/\partial L$ is the rate at which production changes with respect to the amount of labor. Economists call it the **marginal production with respect to labor** or the **marginal productivity of labor**.
- The partial derivative $\partial P/\partial K$ is the rate of change of production with respect to capital and is called the **marginal productivity of capital**.

Problem 12. The assumptions made by Cobb and Douglas can be stated as follows.

(i) If either labor or capital vanishes, then so will production.

(ii) The marginal productivity of labor is proportional to the amount of production per unit of labor.

(iii) The marginal productivity of capital is proportional to the amount of production per unit of capital.

Construct the Cobb–Douglas production model.

Solution. Because the production per unit of labor is P/L, assumption (ii) says that

$$\frac{\partial P}{\partial L} = \alpha \frac{P}{L}$$

for some constant α. Assumption (i) shows that $\alpha > 0$. If we keep $K = K_0$ constant , then this partial differential equation becomes an ordinary differential equation:

$$\frac{dP}{dL} = \alpha \frac{P}{L}.$$

If we solve this separable differential equation, we get

$$P(L, K_0) = C_1(K_0)L^{\alpha}. \tag{1}$$

Notice that we have written the constant C_1 as a function of K_0 because it could depend on the value of K_0.

Similarly, assumption (iii) says that

$$\frac{\partial P}{\partial K} = \beta \frac{P}{K},$$

for some constant β. Assumption (i) shows that $\beta > 0$. Keeping L constant($L = L_0$), and we can solve this differential equation to get

$$P(L_0, K) = C_2(L_0)K^{\beta}. \tag{2}$$

Comparing Equations (1) and (2), we have

$$P(L, K) = cL^{\alpha}K^{\beta}, \tag{3}$$

where c is a constant that is independent of both L and K.

Thus,

Cobb-Douglas production model

Let

P: total production of an economic system

L: amount of labor required to produce P

K: capital investment required to produce P.

Also, let c be a constant independent of both L and K, and let $\alpha, \beta > 0$. The function P given by the equation

$$P(L, K) = cL^{\alpha}K^{\beta},$$

is called the **Cobb-Douglas production model**.

Example 29. Consider the Cobb-Douglas production model given by the formula $P = 1.01L^{0.75}K^{0.25}$. Its level curves are shown below.
In the figure level curves are labeled with the value of the production P. For instance, the level curve labeled 140 shows all values of the labor L and capital investment K that result in a production of $P = 140$. We see that, for a fixed value of P, as L increases K decreases, and vice versa.

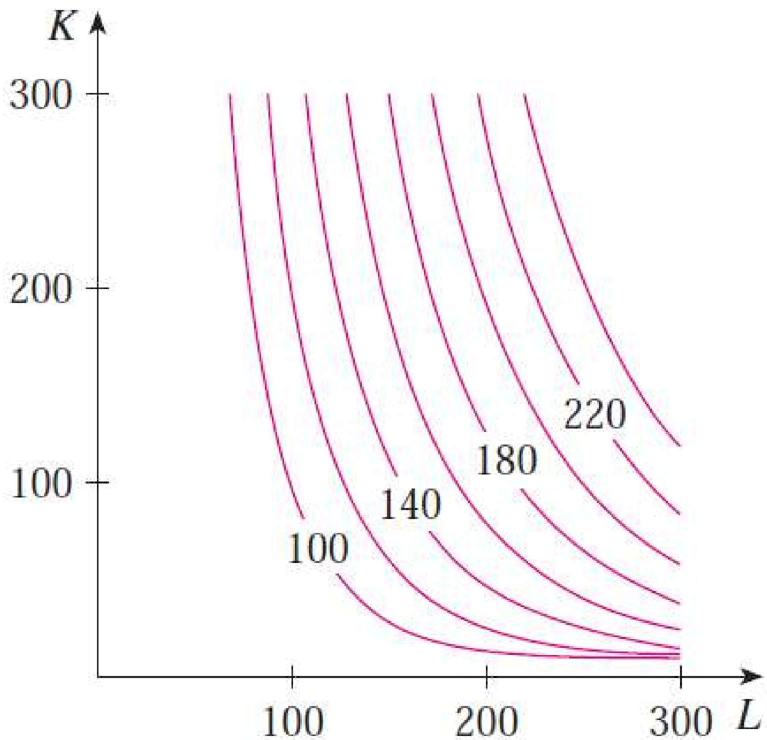

Figure 21: Level curves for $P = 1.01L^{0.75}K^{0.25}$.

Example 30. Show that the Cobb-Douglas production function $P = cL^{\alpha}K^{\beta}$ satisfies the equation

$$L\frac{\partial P}{\partial L} + K\frac{\partial P}{\partial K} = (\alpha + \beta)P.$$

Notice from Equation (3) that if labor and capital are both increased by a factor m, then

$$P(mL, mK) = b(mL)^{\alpha}(mK)^{\beta} = m^{\alpha+\beta}P(L, K)$$

If $\alpha + \beta = 1$, then $P(mL, mK) = mP(L, K)$, which means that production is also increased by a factor of m. Hence Cobb and Douglas assumed that $\alpha + \beta = 1$ and therefore

$$P(L, K) = cL^{\alpha}K^{1-\alpha}.$$

Note that if $\alpha+\beta = 1$, then the production function P satisfies the differential equation:

$$L\frac{\partial P}{\partial L} + K\frac{\partial P}{\partial K} = P.$$

Cobb-Douglas production function

Let

P: total production of an economic system

L: amount of labor required to produce P

K: capital investment required to produce P.

Also, let c be a constant independent of both L and K, and let $0 < \alpha < 1$. The **Cobb-Douglas production function** P is given by the equation

$$P(L, K) = cL^{\alpha}K^{1-\alpha}.$$

6 Tangent planes and linear approximation

In this section we will extend the notion of differentiability to functions of two variables. Our definition of differentiability will be based on the idea that a function is differentiable at a point provided it can be very closely approximated by a linear function near that point. In the process, we will expand the concept of a "differential" to functions of more than one variable and define the "local linear approximation" of a function. Thus, we

- Determine the equation of a plane tangent to a given surface at a point.

- Use the tangent plane to approximate a function of two variables at a point.
- Explain when a function of two variables is differentiable.
- Use the total differential to approximate the change in a function of two variables.

Tangent Planes

A function of one variable: $y = f(x)$.

The slope of the tangent line at the point $x = a$: $m = f'(a)$.

The equation of the tangent line at the point $x = a$:

$$y = f(a) + f'(a)(x - a).$$

What is the slope of a tangent plane?

Tangent plane

Let $P_0 = (x_0, y_0, z_0)$ be a point on a surface S, and let C be any curve passing through P_0 and lying entirely in S. If the tangent lines to all such curves C at P_0 lie in the same plane, then this plane is called the **tangent plane** to S at P_0.

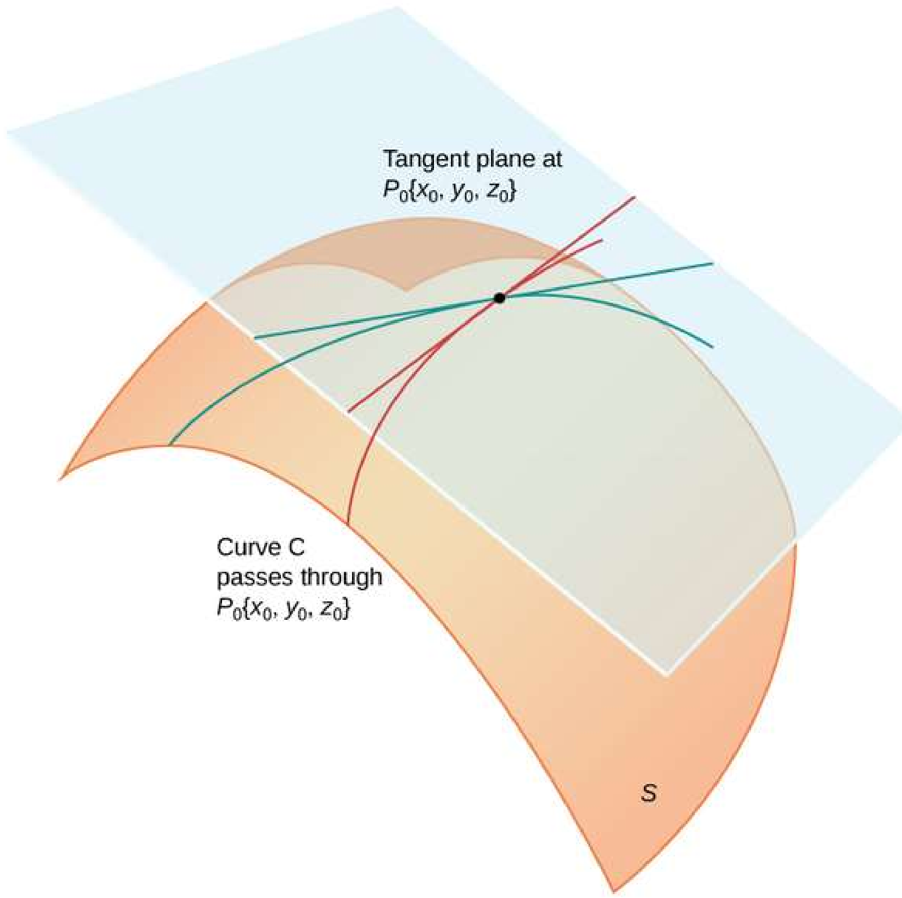

Figure 22: The tangent plane to a surface at a point contains all the tangent lines to curves in that pass through.

Equation of a tangent plane

We know that any plane passing through the point (x_0, y_0, z_0) has an equation of the form

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0. \tag{4}$$

Dividing this equation by C, we can write it in the form

$$z - z_0 = a(x - x_0) + b(y - y_0). \tag{5}$$

Equation of a tangent plane

Let S be a surface defined by a function $z = f(x, y)$ and $P_0 = (x_0, y_0)$ a point in the domain of f. Suppose that f has continuous partial derivatives. Then the **tangent plane** to S at P_0 is given by the equation

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0). \quad (6)$$

Example 31. Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.

Solution. ...

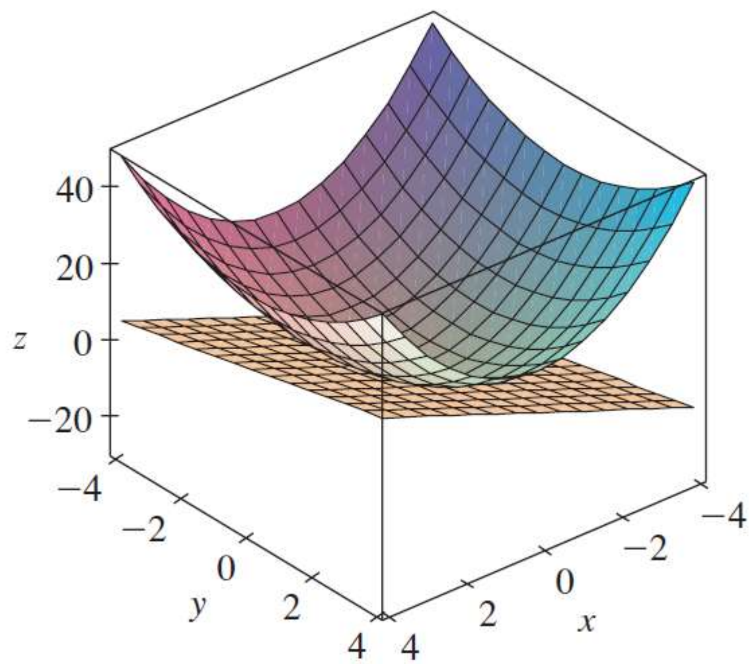

Figure 23: The tangent plane to $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.

Linear Approximations

One nice use of tangent planes is they give us a way to approximate a surface near a point. As long as we are near to the point (x_0, y_0) then the tangent plane should nearly approximate the function at that point.

In Example 31, we found that an equation of the tangent plane to the graph of the function $f(x, y) = 2x^2 + y^2$ at the point $(1, 1, 3)$ is

$$z = 4x + 2y - 3.$$

Set

$$L(x, y) = 4x + 2y - 3.$$

This is a linear function in two variables

For instance, at the point $(1.1, 0.95)$ we have

$$\begin{aligned} f(1.1, 0.95) &= 2(1.1)^2 + (0.95)^2 = 3.3225, \\ L(1.1, 0.95) &= 4(1.1) + 2(0.95) - 3 = 3.3 \end{aligned}$$

Clearly, $f(x, y) \approx L(x, y)$. But if we take a point farther away from $(1, 1)$, such as $(2, 3)$, we no longer get a good approximation. In fact,

$$\begin{aligned} f(2, 3) &= 2(2)^2 + (3)^2 = 17, \\ L(2, 3) &= 4(2) + 2(3) - 3 = 11. \end{aligned}$$

Thus, $L(x, y)$ is a good approximation to $f(x, y)$ when (x, y) is near (1, 1). The function L is called the **linearization** of f at $(1, 1)$ and the approximation

$$f(x, y) \approx 4x + 2y - 3$$

is called the **linear approximation** or **tangent plane approximation** of f at $(1,1)$.

Because of this we define the linear approximation to be,

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Linear approximation

In general, if f has continuous partial derivatives, then the equation of a tangent plane to the graph of a function $f(x,y)$ at the point $(a,b,f(a,b))$ is

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

Put

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

The expression $L(x,y)$ is called the **linear approximation** or the **tangent plane approximation** of f at (a,b). Indeed,

$$f(x,y) \approx L(x,y) \text{ near } (a,b).$$

We have defined tangent planes for surfaces $z = f(x,y)$, where f has continuous first partial derivatives. What happens if f_x and f_y are not continuous?

Example 32. Consider the following function:

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

We can verify that its partial derivatives exist at the origin and, in fact, $f_x(0,0) = f_y(0,0) = 0$, but f_x and f_y are not continuous. The linear approximation would be $f(x,y) \approx 0$, but $f(x,y) = \frac{1}{2}$ at all points on the line $y = x$. So a function of two variables can behave badly even though both of its partial derivatives exist.

To rule out such behavior as in the above example, we formulate the idea of a differentiable function of two variables.

Differentiable functions of two variables

Let us return to the one-dimensional case. If Δx is an increment in x, then the increment in y, Δy, is defined as

$$\Delta y = f(x + \Delta x) - f(x).$$

By definition, the derivative of $f : \mathbb{R} \to \mathbb{R}$ is given by

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

Put

$$\frac{\Delta y}{\Delta x} - f'(x) = \varepsilon. \tag{7}$$

Clearly,

$$\varepsilon \to 0 \text{ as } \Delta x \to 0.$$

From Equation (7), we get

$$\Delta y = f'(x)\Delta x + \varepsilon \Delta x, \quad \text{where } \varepsilon \to 0 \text{ as } \Delta x \to 0. \tag{8}$$

Thus, a function $y = f(x)$ has a derivative (or is differentiable) if (8) holds true.

Now consider a function of two variables, $z = f(x, y)$, and suppose that Δx is an increment in x and Δy an increment in y. Then the corresponding increment of z is given by

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$$

By analogy with (8) we define the differentiability of a function of two variables as follows.

Differentiable functions

Let $z = f(x, y)$. We say that f is differentiable at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

Example 33. Show that the function $f(x, y) = x^2 + 3y$ is differentiable at every point in the plane.

Solution. ...

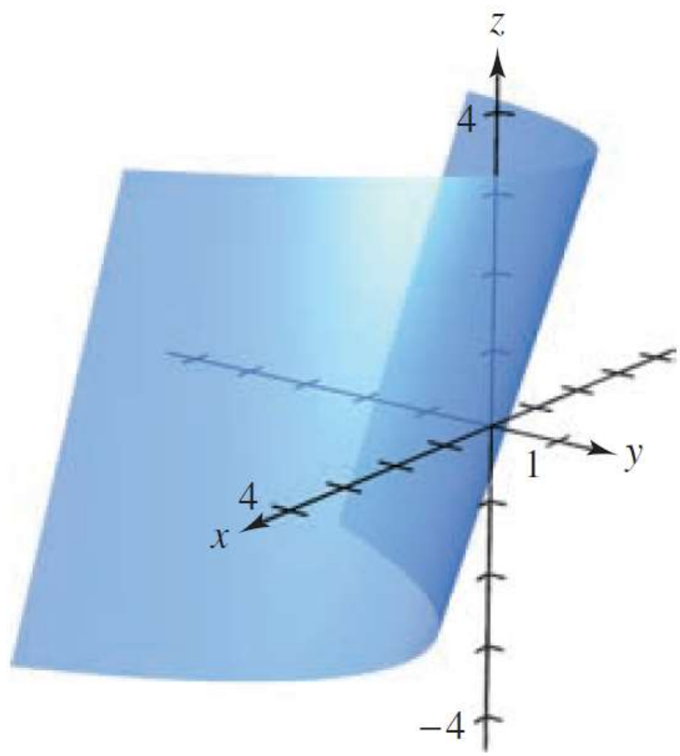

Figure 24: The surface $f(x, y) = x^2 + 3y$.

The above definition says that a differentiable function f is one for which the linear approximation

$$\begin{aligned} f(x, y) &\approx L(x, y) \\ &= f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \end{aligned}$$

is a good approximation when (x, y) is near (a, b). In other words, the tangent plane approximates the graph of f well near the point of tangency.

It's sometimes hard to use above definition directly to check the differentiability of a function, but the next theorem provides a convenient sufficient condition for differentiability.

Theorem 6.1. *If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).*

Example 34 (Using a linearization to estimate a function value)**.** Show that $f(x, y) = xe^{xy}$ is differentiable at $(1, 0)$ and find its linearization there. Then use it to approximate $f(1.1, -0.1)$.

Solution. The partial derivatives are

$$f_x(x, y) = e^{xy} + xye^{xy}, \qquad f_y(x, y) = x^2e^{xy}.$$

This implies that

$$f_x(1, 0) = 1, \qquad f_y(1, 0) = 1.$$

Both f_x and f_y are continuous functions, so f is differentiable. The linearization is

$$\begin{aligned} L(x, y) &= f(1, 0) + f_x(1, 0)(x - 1) + f_y(1, 0)(y - 0) \\ &= 1 + 1(1 - 0) + 1 \cdot y = x + y. \end{aligned}$$

The corresponding linear approximation is

$$xe^{xy} \approx x + y$$

and so

$$f(1.1, -0.1) \approx 1.1 - 0.1 = 1.$$

Compare this with the actual value of

$$f(1.1, -0.1) = 1.1e^{-0.11} \approx 0.98542.$$

Theorem 6.2 (Differentiability Implies Continuity)**.** *If a function $z = f(x, y)$ is differentiable at a point, then it is continuous at the point.*

Proof.

Assume that f is differentiable at a point (a, b). To prove that f is continuous at (a, b) we must show that

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b).$$

Let

$$\Delta x = x - a, \quad \Delta y = y - b,$$

Then

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$. However, by definition, we know that

$$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$$

Setting $x = a + \Delta x$ and $y = b + \Delta y$ produces

$$\begin{aligned} & f(x, y) - f(a, b) \\ =& f_x(a, b)\Delta x + f_y(a, b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y \\ =& f_x(a, b)(x - a) + f_y(a, b)(y - b) + \varepsilon_1(x - a) + \varepsilon_2(y - b) \end{aligned}$$

Taking the limit as $(x, y) \to (a, b)$, we have

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b).$$

which means that f is continuous at (a, b). ◀

Differentials

Recall: one dimensional case.

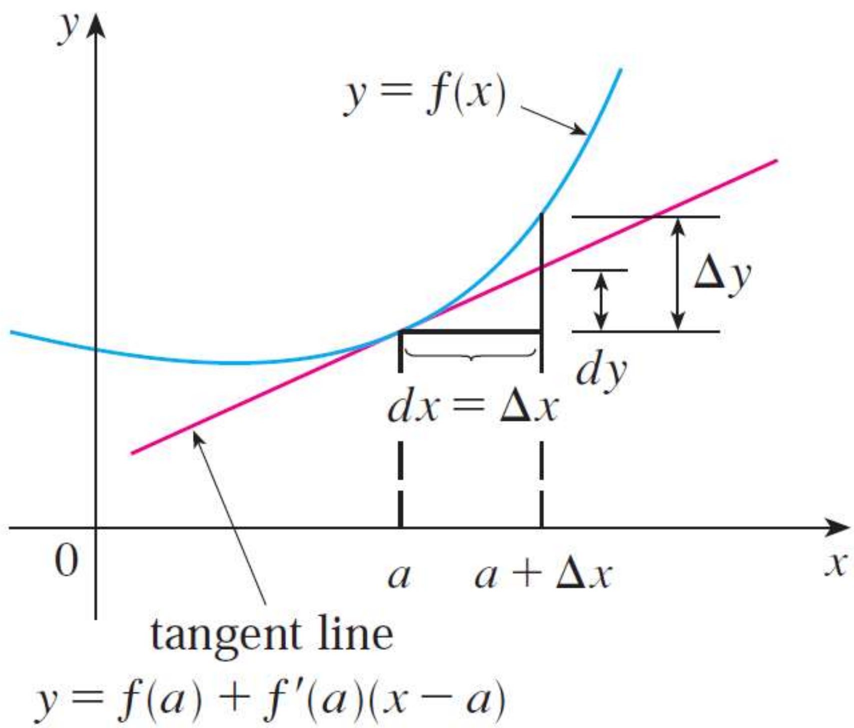

Figure 25: The tangent line

For a differentiable function of two variables, $z = f(x, y)$, we define the differentials dx and dy to be independent variables; that is, they can be given any values.

Total differential

Let $z = f(x, y)$. If Δx and Δy are increments in x and y, the differentials of the independent variables and are

$$dx = \Delta x,\ dy = \Delta y.$$

The **total differential**, dz, of the dependent variable z is defined by

$$dz = f_x(x, y)dx + f_y(x, y)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

We know that

$$\begin{aligned} f(x,y) &\approx L(x,y) \\ &= f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b). \end{aligned}$$

If we take

$$dx = \Delta x = x - a, \quad dy = \Delta y = y - b,$$

then the differential dz is

$$dz = f_x(x,y)(x-a) + f_y(x,y)(y-b).$$

So, the linear approximation can be written as

$$\boxed{f(x,y) \approx f(a,b) + dz = L(x,y).}$$

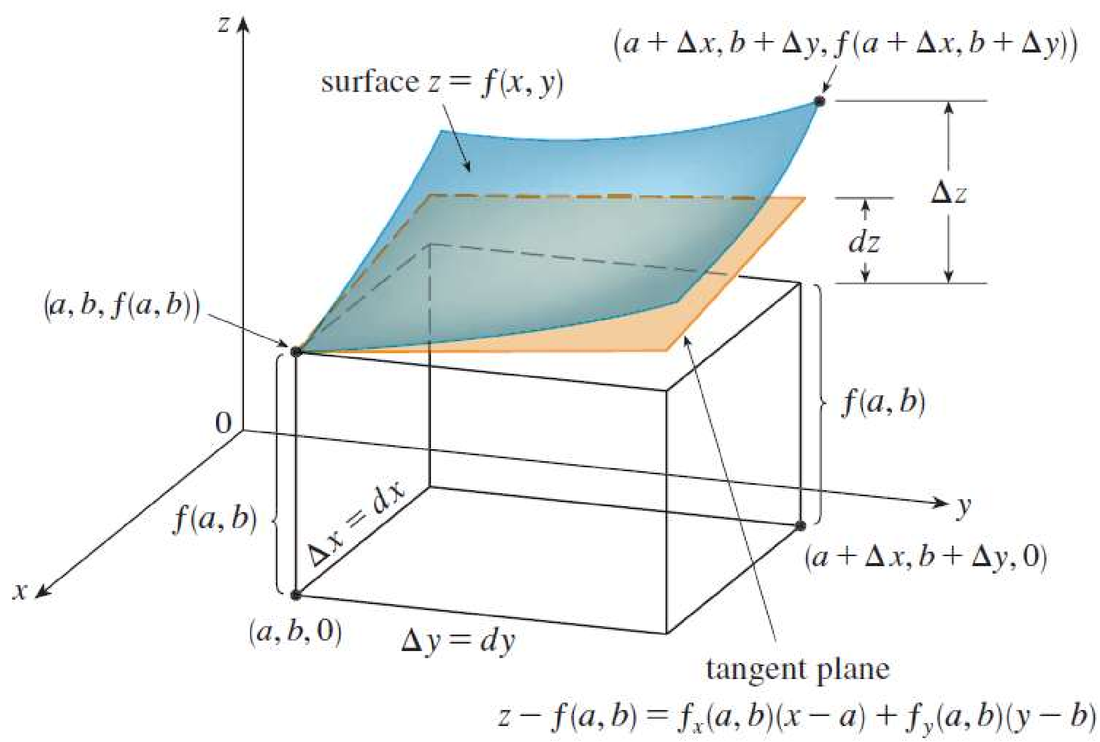

Figure 26: The tangent plane

Figure 26 is the three-dimensional counterpart of Figure 25 and shows the geometric interpretation of the differential and the increment : represents the change in height of the tangent plane, whereas represents the change in height of the surface when changes from (a, b) to $(a + \Delta x, b + \Delta y)$.

Example 35 (Differentials versus increments)**.**

(a) If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential dz.

(b) If x changes from 2 to 2.05 and y changes from 3 to 2.96, compare the values of Δz and dz.

Solution. (a)

(b) Putting

$$x = 2, dx = \Delta x = 0.05, \quad y = 3, \quad dy = \Delta y = -0.04,$$

we get

$$dz = 2(2) + 3(3)0.05[3(2) - 2(3)](-0.04) = 0.65$$

The increment of z is

$$\begin{aligned}\Delta z &= f(2.05, 2.96) - f(2, 3)\\ &= [2.05^2 + 3(2.05)(2.96) - 2.96^2] - [2^2 + 3(2)(3) - 3^2]\\ &= 0.6449.\end{aligned}$$

Therefore, $\Delta z \approx dz$.

Example 36 (Using differentials to estimate an error:)**.** The base radius and height of a right circular cone are measured as 10 cm and 25 cm, respectively, with a possible error in measurement of as much as 0.1 cm in each. Use differentials to estimate the maximum error in the calculated volume of the cone.

Solution. We know that the volume of a cone with base radius r and height h is

$$V = \frac{1}{3}\pi r^2 h.$$

So

$$dV = \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial h}dh = \frac{2\pi rh}{3}dr + \frac{\pi r^2}{3}dh.$$

Since each error is at most 0.1 cm, we have

$$|\Delta r| \leq 0.1, \quad |\Delta h| \leq 0.1$$

To find the largest error in the volume we take the largest error in the measurement of r and of h. Therefore we take $dr = 0.1$ and $dh = 0.1$ along with $r = 10$, $h = 25$. This gives

$$dV = \frac{500\pi}{3}(0.1) + \frac{100\pi}{3}(0.1) = 20\pi.$$

Thus the maximum error in the calculated volume is about

$$20\pi \text{ cm}^3 = 63 \text{ cm}^3.$$

Functions of Three or More Variables

Linear approximation:

$$
\begin{aligned}
f(x, y, z) &\approx L(x, y, z) \\
&= f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) \\
&\quad + f_x(a, b, c)(z - c).
\end{aligned}
$$

Differentiability

Let $u = f(x, y, z)$. We say that f is differentiable at (a, b, c) if Δu can be expressed in the form

$$
\begin{aligned}
\Delta u = {} & f_x(a, b, c)\Delta x + f_y(a, b, c)\Delta y + f_z(a, b, c)\Delta z + \varepsilon_1 \Delta x \\
& + \varepsilon_2 \Delta y + \varepsilon_3 \Delta z,
\end{aligned}
$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3 \to 0$ as $(\Delta x, \Delta y, \Delta z) \to (0, 0, 0)$.

Total differentials

The differential du, also called the **total differential**, is defined by

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial yz} dz.$$

Example 37. The dimensions of a rectangular box are measured to be 75 cm, 60 cm, and 40 cm, and each measurement is correct to within 0.2 cm. Use differentials to estimate the largest possible error when the volume of the box is calculated from these measurements.

Solution. If the dimensions of the box are x, y, and z, its volume is

$$v = xyz$$

and so

$$\begin{aligned} du &= \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial yz}dz \\ &= yzdx + xzdy + xydz. \end{aligned}$$

We are given that

$$\Delta x, \Delta y, \Delta z \leq 0.2.$$

To find the largest error in the volume, we therefore use

$$dx = dy = dz = 0.2$$

together with

$$x = 75, y = 60, z = 40.$$

We have

$$\Delta V = dV = (60)(40)(0.2)+(75)(40)(0.2)+(75)(60)(0.2) = 1980$$

Thus an error of only 0.2 cm in measuring each dimension could lead to an error of as much as 1980 cm in the calculated volume! This may seem like a large error, but it's only about 1% of the volume of the box.