INDEX

NAME: Arpan Sapkota STD.: I-II SEC.: MDS ROLL NO.: 07 SUB.: Multivariable Calculus for Data Science.

S. No.	Date	Title	Page No.	Teacher's Sign / Remarks
		Multivariable Calculus for Data Science.	MDS 554	
	1.	Vectors and the Geometry of Space.	6 hrs	NPP
	2.	Vector Functions	8 hrs	NPP
	3.	Partial Derivatives	12 hrs	PMB
	4.	Multiple Integrals	10 hrs.	PMB
	5.	Vector Calculus	12 hrs.	NPP

Unit 1

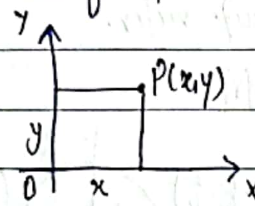
Vector and the Geometry of Space.

$f(x,y,z) = x^2+y^2=1$, $z=0$
what ? object.

Analytical Geometry

1. Plane Geometry (2D) → Vector Method
2. Space Geometry (3D) → Vector Method.

Position of a point in plane.

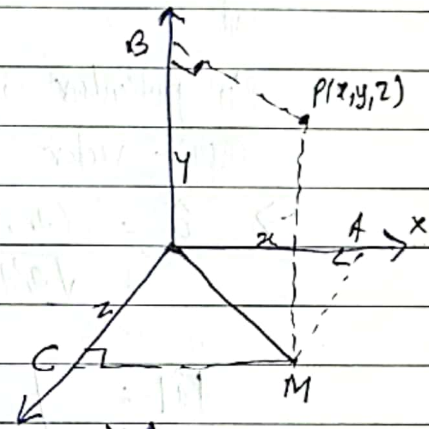


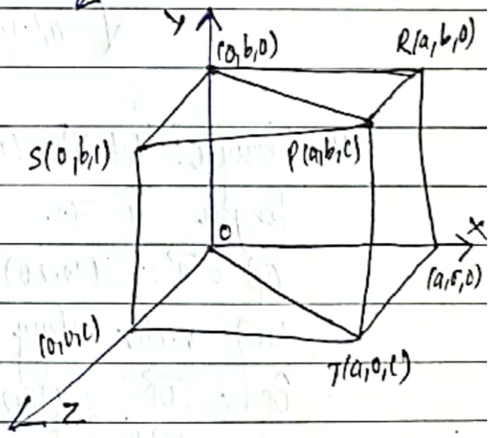
R^2 ordinate
$\vec{OP} = (x, y)$
abscissa

$R^2 : \{(x,y) : (x,y) \in R\}$
So, the locus (x,y) generates R^2.

Position of a point in Space.

Take point P on space. Draw ⊥r PM from P on XZ plane. Draw ⊥r MA & MC from M on OX & OZ
Draw PB ⊥r from P on OY. Measure OA, OB, OC.
If OA = x , OB = y & OC = z.
Then the coordinate of P is (x,y,z).
Draw a parallelopiped with edges a, b & c.





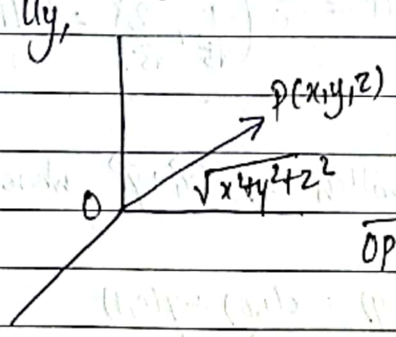
Distance Formula

If $P(x_1,y_1,z_1)$, $Q(x_2,y_2,z_2)$ then,
$$PQ = |\vec{PQ}| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

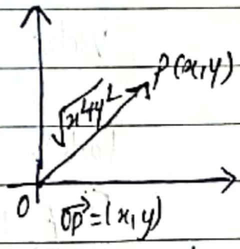
Position Vector

A vector which is defined w.r.t some references (say origin) is called position vector.

Hy,



$\vec{OP} = (x,y,z)$, Space vector.



Plane vector.

Modulus of a Vector

Modulus of a vector $\overrightarrow{OP}$ is the distance from O to P. It is denoted by $|\overrightarrow{OP}|$. Thus,

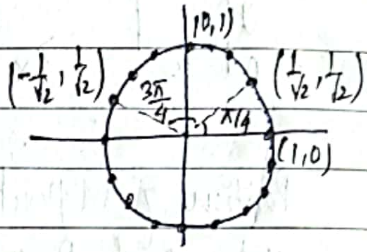
If $\overrightarrow{OP} = (x,y)$ then, $|\overrightarrow{OP}| = \sqrt{x^2+y^2}$

& if $\overrightarrow{OP} = (x,y,z)$ then, $|\overrightarrow{OP}| = \sqrt{x^2+y^2+z^2}$

If $|\overrightarrow{OP}| = \sqrt{x^2+y^2} = 1$, then $\overrightarrow{OP}$ is called unit vector.

Example: Let $\overrightarrow{OP} = (\cos t, \sin t)$ $\quad 0 \le t \le 2\pi$

Then the locus of $\overrightarrow{OP}$ for all given value of t represents a unit circle. So each point on the circle are unit vector.



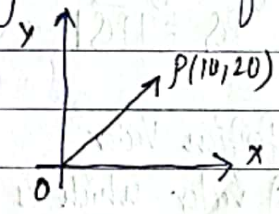
In particular, if $\vec{a} = (a_1, a_2)$ be a vector. Then $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$ is always unit vector. why?

$$\Rightarrow \quad \hat{a} = \frac{(a_1, a_2)}{\sqrt{a_1^2+a_2^2}} = \left(\frac{a_1}{\sqrt{a_1^2+a_2^2}}, \frac{a_2}{\sqrt{a_1^2+a_2^2}}\right)$$

$$|\hat{a}| = \sqrt{\frac{a_1^2}{a_1^2+a_2^2} + \frac{a_2^2}{a_1^2+a_2^2}} = 1.$$

Example: let $\vec{a} = (10,20)$, then find the vector along the direction of $\vec{a}$ whose length is 7.

$\overrightarrow{OP} = \vec{a} = (10,20)$



Unit vector along $\overrightarrow{OP}$

$$\widehat{OP} = \frac{\overrightarrow{OP}}{|\overrightarrow{OP}|} = \frac{(10,20)}{\sqrt{10^2+20^2}} = \frac{(10,20)}{10\sqrt5} = \left(\frac{1}{\sqrt5}, \frac{2}{\sqrt5}\right)$$

Vector along $\vec{a}$ with length $7 = 7\,\widehat{OP} = 7\left(\frac{1}{\sqrt5}, \frac{2}{\sqrt5}\right) = (7/\sqrt5, 14/\sqrt5)$

Facts:

Every plane vector $\overrightarrow{OP} = (x,y)$ can be written as $x\vec{i} + y\vec{j}$ where $\vec{i} = (1,0)$ & $\vec{j} = (0,1)$ and conversely.

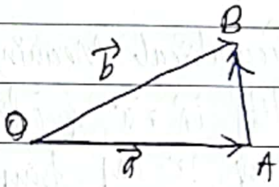
proof: $(x,y) = (x+0, 0+y) = (x,0) + (0,y) = x(1,0) + y(0,1)$
$= x\vec{i} + y\vec{j}$

Similarly, Every space vector $\vec{OP} = (x, y, z)$ can be written as
$\vec{OP} = x\vec{i} + y\vec{j} + z\vec{k}$, where, $\vec{i} = (1,0,0)$
$\vec{j} = (0,1,0)$
$\vec{k} = (0,0,1)$

Proof: $(x,y,z) = (x+0+0, y+0+0, z+0+0) = (x,0,0) + (0,y,0) + (0,0,z)$
$= x(1,0,0) + y(0,1,0) + z(0,0,1)$
$= x\vec{i} + y\vec{j} + z\vec{k}$

Like and Unlike Vector.
Two vectors $\vec{a}$ & $\vec{b}$ are said to be
(a). Like if $\vec{a} = k_1\vec{b}$, for same $k_1 > 0$ $\quad k =$ constant.
(b). Unlike if $\vec{a} = k_2\vec{b}$, for same $k_2 < 0$

Facts:
Let $\vec{OA} = \vec{a}$ & $\vec{OB} = \vec{b}$



Then, $\vec{OB} = \vec{OA} + \vec{AB}$
$\therefore \vec{AB} = \vec{OB} - \vec{OA}$
$\therefore \vec{BA} = \vec{OA} - \vec{OB}$

O Last - O First

Find $\vec{AB}$ if $\vec{OA} = (1,2)$, $\vec{OB} = (6,7)$
$\vec{AB} = \vec{OB} - \vec{OA} = (6,7) - (1,2) = (5,5)$

Triangle Inequality
If $\vec{a}$ & $\vec{b}$ be two vectors, then $|\vec{a}+\vec{b}| \leq |\vec{a}| + |\vec{b}|$

Prove Triangle Inequality (I) By vector Method (II). By Cauchy Schwarz Inequality.
$|\vec{a}\cdot\vec{b}| \leq |\vec{a}|.|\vec{b}|$

Dot Product / Scalar Product of Two Vectors

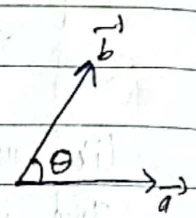
If $\vec{a} = (a_1, a_2)$ & $\vec{b} = (b_1, b_2)$ be two vectors, then

$\boxed{\vec{a}\cdot\vec{b} = a_1 b_1 + a_2 b_2}$, which is scalar

Similarly, in space vectors;

$\vec{a}\cdot\vec{b} = (a_1, a_2, a_3)\cdot(b_1, b_2, b_3) = a_1b_1 + a_2b_2 + a_3b_3$

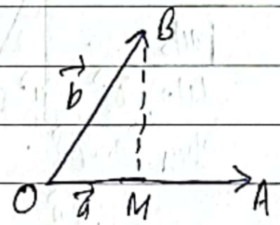
Cosine of the Angle



If θ is the angle between two vectors $\vec{a}$ & $\vec{b}$, then, cosine of the angle between $\vec{a}$ & $\vec{b}$ is

$\boxed{\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}}$ It can also written as, $\boxed{\cos\theta = \dfrac{\vec{a}}{|\vec{a}|}\cdot\dfrac{\vec{b}}{|\vec{b}|} = \hat{a}\cdot\hat{b}}$

Geometrical Meaning of Dot Product



Let $\vec{OA} = \vec{a}$ & $\vec{OB} = \vec{b}$ be given two vectors

Draw $\perp^r$ BM from B on OA. If θ is the angle between $\vec{a}$ & $\vec{b}$.

In ΔOBM, $\cos\theta = \dfrac{OM}{OB}$

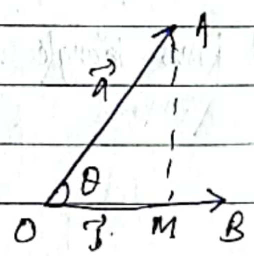
$\therefore OM = OB\cos\theta$

or, Projection of $\vec{b}$ on $\vec{a}$ $= |\vec{b}|\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right) = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

Thus, $\vec{a}\cdot\vec{b} = |\vec{a}|$ (projection of $\vec{b}$ on $\vec{a}$)

$\therefore \vec{a}\cdot\vec{b}$ = (Magnitude of $\vec{a}$)(Projection of $\vec{b}$ on $\vec{a}$)

\# Find the projection of $\vec{a}$ on $\vec{b}$



Let $\vec{OA} = \vec{a}$ & $\vec{OB} = \vec{b}$ be given two vectors.

Draw $\perp$r AM from A on OB. θ is the $\angle$ between $\vec{a}$ & $\vec{b}$.

In ΔOAM,

$\cos\theta = \dfrac{OM}{OA}$ $\Rightarrow OM = OA\cos\theta$

or, Projection of $\vec{a}$ on $\vec{b}$ $= |\vec{a}|\cdot\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right) = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

$\vec{a}\cdot\vec{b}$ = (magnitude of $\vec{b}$)(projection of $\vec{a}$ on $\vec{b}$)

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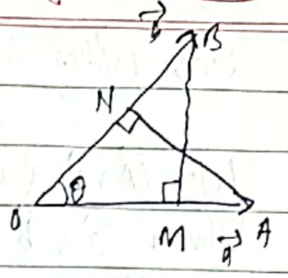
$\vec{ON}$

Vector Projection of $\vec{a}$ on $\vec{b}$ = $\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\right)\hat{b}$

Vector projection of $\vec{b}$ on $\vec{a}$ = $\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right)\hat{a}$

$\vec{OM}$



Note: The two scalar projection are equal if $\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

or, $|\vec{b}| = |\vec{b}|$

The two vectors have equal length.

HW. If $\vec{a} = (1,2,3)$, $\vec{b} = (1,0,2)$. find ⓘ SP ⓘⓘ VP: $\vec{a}$ on $\vec{b}$ and $\vec{b}$ on $\vec{a}$.

Properties of dot product:

If $\vec{a}, \vec{b}, \vec{c}$ be three vectors, then

(i). The dot product of the two vectors is commutative. $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$.

(ii). $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$. (Distributive law)

Vector Product of Two Vectors / Cross Product

let $\vec{a} = (a_1, a_2, a_3)$ & $\vec{b} = (b_1, b_2, b_3)$ be two vectors, then their VP or CP is defined by,

$$\vec{a}\times\vec{b} = (a_1, a_2, a_3) \times (b_1, b_2, b_3)$$

$$= \begin{matrix} a_1 & a_2 & a_3 & a_1 & a_2 \\ b_1 & b_2 & b_3 & b_1 & b_2 \end{matrix}$$

$$= (a_2b_3 - b_2a_3,\ a_3b_1 - b_3a_1,\ a_1b_2 - b_1a_2)$$

which is ordered triple, a vector. Hence the name "vector product" justified.

Vector Product in terms of Determinant

If $\vec{a} = (a_1, a_2, a_3)$ & $\vec{b} = (b_1, b_2, b_3)$ then $\vec{a}\times\vec{b}$ can be written in terms of determinant, given by

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - b_2a_3)\vec{i} + (a_3b_1 - b_3a_1)\vec{j} + (a_1b_2 - b_1a_2)\vec{k}$$

$$= (a_2b_3 - b_2a_3,\ a_3b_1 - b_3a_1,\ a_1b_2 - b_1a_2)$$

Example: If $\vec{i} = (1,0,0)$, $\vec{j} = (0,1,0)$, $\vec{k} = (0,0,1)$, find $\vec{i} \times \vec{j}$ & $\vec{j} \times \vec{k}$.

$$\vec{i} \times \vec{j} = (1,0,0) \times (0,1,0) = \begin{array}{cccccc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{array}$$

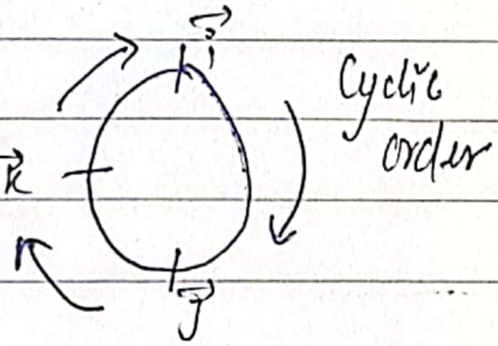
$$\therefore \vec{i} \times \vec{j} = (0-0,\ 0-0,\ 1-0) = (0,0,1) = \vec{k}.$$

& $\vec{j} \times \vec{k} = (1,0,0) = \vec{i}$.

Similarly,

$\vec{j} \times \vec{k} = \vec{i}$

$\vec{k} \times \vec{i} = \vec{j}$



Facts: $\boxed{\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}}$

IIy,

$\vec{j} \times \vec{i} = -(\vec{i} \times \vec{j}) = -\vec{k}$

$\vec{i} \times \vec{k} = -(\vec{k} \times \vec{i}) = -\vec{j}$

$\vec{k} \times \vec{j} = -(\vec{j} \times \vec{k}) = -\vec{i}$.

fact: $\vec{a} \times \vec{b}$ is always $\perp$r to $\vec{a}$ & $\vec{b}$ i.e. $(\vec{a} \times \vec{b}) \cdot \vec{a} = 0$

$(\vec{a} \times \vec{b}) \cdot \vec{b} = 0$.

Example: let $\vec{a} = (1,2,3)$, $\vec{b} = (2,0,1)$. find vector $\vec{r}$ such that $\vec{r} \cdot \vec{a} = 0$ & $\vec{r} \cdot \vec{b} = 0$.

Soln: Here $\vec{r}$ is $\perp$r to both $\vec{a}$ & $\vec{b}$, so by above fact.

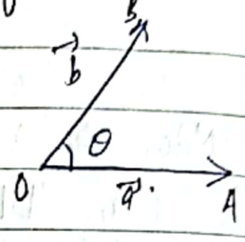
$$\vec{r} = \vec{a} \times \vec{b}$$

$$= \begin{array}{ccccc} 1 & 2 & 3 & 1 & 2 \\ 2 & 0 & 1 & 2 & 0 \end{array} = (2-0,\ 6-1,\ 0-4)$$

$$\therefore \vec{r} = (2,5,-4) \text{ //}$$

Sine of the angle between two vectors.

Let $\vec{OA} = \vec{a}$ & $\vec{OB} = \vec{b}$ be two vectors then the sine of the angle θ between $\vec{a}$ & $\vec{b}$ is

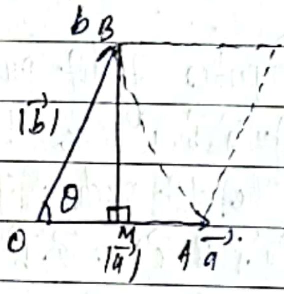


$$\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}|\,|\vec{b}|} \quad \text{— (i)}$$

Here, $0 \leq \sin\theta \leq 1$

$$\therefore |\vec{a}\times\vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta \quad \text{— (ii)}$$

Geometrical Meaning of Cross Product.



To show that $|\vec{a}\times\vec{b}|$ always represents the area of the parallelogram with edges (sides) $\vec{a}$ & $\vec{b}$.

proof:

Let $\vec{OA} = \vec{a}$ & $\vec{OB} = \vec{b}$ be two vectors represented as shown in the figure.

Draw $\perp$r BM from B on OA.

In ΔOBM, $\sin\theta = \frac{BM}{OB} = \frac{BM}{|\vec{b}|}$

$\therefore BM = |\vec{b}| \sin\theta$

Now, Area of $\Delta OBA = \frac{1}{2} \times OA \times BM = \frac{1}{2} \times |\vec{a}|\,|\vec{b}| \sin\theta$

$$= \frac{1}{2} |\vec{a}|\,|\vec{b}| \cdot \frac{|\vec{a}\times\vec{b}|}{|\vec{a}|\,|\vec{b}|}$$

$$= \frac{1}{2} |\vec{a}\times\vec{b}|$$

Also, Area of parallelogram $OACB = 2 \times$ Area of ΔOBA

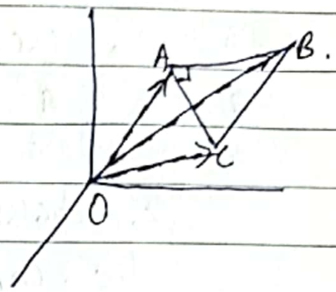
$$= 2 \times \frac{1}{2} |\vec{a}\times\vec{b}| = |\vec{a}\times\vec{b}|$$

$\therefore |\vec{a}\times\vec{b}|$ always represents the area of parallelogram with side $\vec{a}$ & $\vec{b}$.

Example:

Let the three points A, B, C with position vectors $2\vec{i}-\vec{j}+\vec{k}$, $\vec{i}-3\vec{j}-5\vec{k}$ & $3\vec{i}-4\vec{j}-4\vec{k}$. Is the triangle ABC, a right angled? what are the remaining angles?

Let O be the origin, Then,



$\vec{OA} = 2\vec{i}-\vec{j}+\vec{k}$

$\vec{OB} = \vec{i}-3\vec{j}-5\vec{k}$

$\vec{OC} = 3\vec{i}-4\vec{j}-4\vec{k}$

Then,

$\vec{AB} = \vec{OB}-\vec{OA} = -\vec{i}-2\vec{j}-6\vec{k} = (-1,-2,-6)$

$\vec{AC} = \vec{OC}-\vec{OA} = \vec{i}-3\vec{j}-5\vec{k} = (1,-3,-5)$

$\vec{BC} = \vec{OC}-\vec{OB} = 2\vec{i}-\vec{j}+\vec{k} = (2,-1,1)$

Now,

$$\cos A = \frac{\vec{AB}\cdot\vec{AC}}{|\vec{AB}||\vec{AC}|} = \frac{(-1)\times 1 + (-2)\times(-3) + (-6)\times(-5)}{\sqrt{(-1)^2+(-2)^2+(-6)^2}\sqrt{1^2+(-3)^2+(-5)^2}} = \frac{35}{\sqrt{41}\sqrt{35}}$$

$$\cos A = \sqrt{\frac{35}{41}} \qquad \therefore A = \cos^{-1}\left(\sqrt{\frac{35}{41}}\right)$$

Also,

$$\cos B = \frac{\vec{BA}\cdot\vec{BC}}{|\vec{BA}||\vec{BC}|}$$

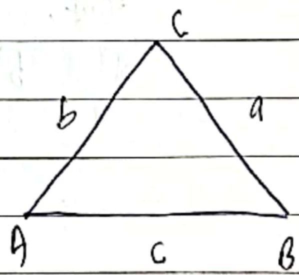
....

Other angles are: $90°$ & $\cos^{-1}\sqrt{\frac{6}{41}}$

Example: (Cosine formula)

In any ΔABC, prove by vector method

$$\cos A = \frac{b^2+c^2-a^2}{2bc}, \qquad \cos B = \frac{c^2+a^2-b^2}{2ca}$$



Let A, B, C be three points with

$\therefore AB = c$, $BC = a$ & $CA = b$.

Then by Triangle law of vector addition,

$$\vec{AB}+\vec{BC}+\vec{CA} = \vec{0}$$

Now to get $\cos A$,

$\vec{BC} = -\vec{AB} - \vec{CA}$

Squaring on both sides,

$(\vec{BC})^2 = (\vec{AB}+\vec{CA})^2$

or, $(\vec{BC})^2 = (AB)^2 + 2\vec{AB}\cdot\vec{CA} + (CA)^2$

or, $a^2 = c^2 + 2(AB)(CA)\cos(\pi - A) + b^2$

or, $a^2 = c^2 + 2cb(-\cos A) + b^2$

$\therefore 2bc\cos A = b^2+c^2-a^2$

$\therefore \cos A = \dfrac{b^2+c^2-a^2}{2bc}$

Scalar Triple Product (STP)

Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors Then their scalar Triple product is denoted by $[\vec{a}\ \vec{b}\ \vec{c}]$ or $(\vec{a}\ \vec{b}\ \vec{c})$ and is defined by,

$[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a}\cdot(\vec{b}\times\vec{c})$

Since $\vec{b}\times\vec{c}$ is vector & $\vec{a}$ is also vector, so their dot product $\vec{a}\cdot(\vec{b}\times\vec{c})$ is a scalar.

Hence the ~~hip~~ title "Scalar Triple product" is justified.

Properties of Scalar Triple Product

If $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$ & $\vec{c} = (c_1, c_2, c_3)$ Then,

Property I:

The Scalar Triple Product $[\vec{a}\ \vec{b}\ \vec{c}]$ can be expressed in terms of determinant.

i.e. $[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a}\cdot\vec{b}\times\vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Hint:

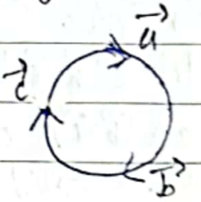
$\vec{b}\times\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (b_2c_3 - c_2b_3)\vec{i} + (b_3c_1 - c_3b_1)\vec{j} + (b_1c_2 - c_1b_2)\vec{k}$

$= (b_2c_3 - c_2b_3,\ b_3c_1 - c_3b_1,\ b_1c_2 - c_1b_2)$

$$\therefore \vec{a}\cdot\vec{b}\times\vec{c} = (a_1\ a_2\ a_3)\cdot(b_2c_3 - c_2b_3,\ b_3c_1 - c_3b_1,\ b_1c_2 - c_1b_2)$$
$$= a_1(b_2c_3 - c_2b_3) + a_2(b_3c_1 - c_3b_1) + a_3(b_1c_2 - c_1b_2)$$
$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Property II: The value of STP remain unchanged if the cyclic orders of the two vectors are maintained.

i.e.



$$[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$$

$$[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a}\cdot\vec{b}\times\vec{c}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (-)\begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = (-)(-)\begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = \vec{b}\cdot\vec{c}\times\vec{a}$$

Property III: If two vectors are parallel, then Scalar Triple Product is zero.

i.e. if $\vec{a}$ & $\vec{b}$ are parallel. Then,

$$\vec{a} = k\vec{b} = k(b_1, b_2, b_3)$$

Then,

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} kb_1 & kb_2 & kb_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = k\begin{vmatrix} b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= k\cdot 0 \quad \because (R_1 = R_2)$$

Geometrical Meaning of STP:

Property IV: If $\vec{a}, \vec{b}, \vec{c}$ be three vectors, Then STP i.e. $[\vec{a}\ \vec{b}\ \vec{c}]$ always represents the volume of the parallelopiped with edge determined by $\vec{a}, \vec{b}, \vec{c}$

Example: Find the volume of the parallelopiped with edges,

$\vec{OA} = \vec{a} = (1, 2, 3)$

$\vec{OB} = \vec{b} = (2, 3, 0)$

$\vec{OC} = \vec{c} = (4, 0, 1)$

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Soln,

Required volume $= [\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 0 \\ 4 & 0 & 1 \end{vmatrix} = -37$

$\therefore$ volume = 37 cu unit (Taking +ve sign)

Vector Triple Product

Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors. Then the product of the form $\vec{a}\times(\vec{b}\times\vec{c})$ or $\vec{b}\times(\vec{c}\times\vec{a})$ or $\vec{c}\times(\vec{a}\times\vec{b})$ are called vector Triple product of $\vec{a}, \vec{b}$ & $\vec{c}$.

Expression of $\vec{a}\times(\vec{b}\times\vec{c})$

The value of $\vec{a}\times(\vec{b}\times\vec{c})$ is

$$\boxed{\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}} \qquad (f\cdot l)m - (f\cdot m)l$$

Here, it is of the form

$$\vec{a}\times(\vec{b}\times\vec{c}) = l\vec{b} + m\vec{c}$$

This shows that $\vec{a}\times(\vec{b}\times\vec{c})$ represents the vectors coplaner with $\vec{b}$ & $\vec{c}$.

(How??)

HW If $\vec{a} = (1,2,5)$ & $\vec{b} = (0,1,0)$, $\vec{c} = (0,0,3)$ Then verify the formula.

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$$

Hint:

$$\vec{b}\times\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (\ ,\ ,\ )$$

$$\vec{a}\times(\vec{b}\times\vec{c}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ & & \\ & & \end{vmatrix} = (\ ,\ ,\ ) \quad \text{— (I)}$$

Find. $\vec{a}\cdot\vec{c} = 9$

$\vec{a}\cdot\vec{b} = 2.$

RHS $= (a\cdot c)\vec{b} - (a\cdot b)\vec{c} = (\ ,\ ,\ )$ — (II)

From (I) & (II), LHS = RHS

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Direction cosine & Direction Ratios

If α, β, γ be the angle made by a line with positive direction of x, y & z axis, then $\cos\alpha, \cos\beta$ & $\cos\gamma$ are the direction cosines of the line op. They are denoted by l, m, n.

i.e. $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$

Facts:

$l^2+m^2+n^2 = 1.$

Note:

→ Direction Cosines of x-axis are $\cos 0, \cos 90, \cos 90$ i.e. $1, 0, 0$

→ Direction cosines of y-axis are $\cos 90, \cos 0, \cos 90$ i.e. $0, 1, 0$.

→ Direction cosines of z-axis are $\cos 90, \cos 90, \cos 0$ i.e. $0, 0, 1$

Direction Ratios

Any three numbers a, b, c that are proportional to the direction cosines l, m, n are called direction ratios

i.e. $\frac{a}{l} = \frac{b}{m} = \frac{c}{n} = \frac{\sqrt{a^2+b^2+c^2}}{\sqrt{l^2+m^2+n^2}} = \sqrt{a^2+b^2+c^2}$.

$\therefore l = \frac{a}{\sqrt{a^2+b^2+c^2}}$

$\therefore m = \frac{b}{\sqrt{a^2+b^2+c^2}}$

$\therefore n = \frac{c}{\sqrt{a^2+b^2+c^2}}$

Equation of straight line.

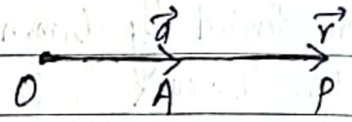
The equation of straight line which passes through a point (x_1, y_1, z_1) with direction cosines l, m, n is.

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$$

Vector Equation of Straight Line.

To find the vector equation of st. line passing through origin and with direction of $\vec{a}$ is of the form $\vec{r} = t\vec{a}$.

Proof:



Let $\vec{OA} = \vec{a}$ passing through origin

Let P be any point on the line such that

$\vec{OP} = \vec{r}$

Then $\vec{OA}$ & $\vec{OP}$ are collinear,

i.e. $\vec{OP} = t\,\vec{OA}$, where t is scalar.

or, $\vec{r} = t\vec{a}$, which is required equation of st. line.

Verification:

Let $\vec{r} = (x, y, z)$ & $\vec{a} = (a_1, a_2, a_3)$

Then, $\vec{r} = t\vec{a}$

or $(x, y, z) = t(a_1, a_2, a_3)$

$= (ta_1, ta_2, ta_3)$

$\therefore x = ta_1,\ y = ta_2,\ z = ta_3$

$\therefore \dfrac{x}{a_1} = t,\ \dfrac{y}{a_2} = t,\ \dfrac{z}{a_3} = t.$

$\therefore \dfrac{x}{a_1} = \dfrac{y}{a_2} = \dfrac{z}{a_3}$ which is cartesian form of st. line passing through origin.

Example:

Find the equation of st. line (By vector method) passing through origin and in the direction of $\vec{a} = (1, 2, 3)$.

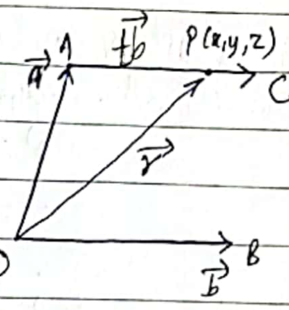
i.e. $\vec{a} = (1, 2, 3)$

$\vec{r} = (x, y, z)$.

Use $\vec{r} = t\vec{a}$

$\dfrac{x}{1} = \dfrac{y}{2} = \dfrac{z}{3}$

Vector Equation of St. line passing through $\vec{a}$ & parallel to $\vec{b}$



let $\vec{OA} = \vec{a}$ be given, $\vec{OB} = \vec{b}$,
we have to find vector equation of AC, which is parallel to $\vec{OB} = \vec{b}$
let P(x,y,z) be any point on AC such that $\vec{OP} = \vec{r}$
Then by Triangle law of Vector addition,

$$\vec{r} = \vec{OA} + \vec{AP} \quad \text{--- (1)}$$

But, $\vec{AP}$ is parallel to $\vec{OB} = \vec{b}$

$\therefore \vec{AP} = t\vec{b}$, where t is some scalar

Hence equation (1) becomes.

$\boxed{\vec{r} = \vec{a} + t\vec{b}}$ which is required equation.

Verification:
Using Cartesian co-ordinates,
let $\vec{r} = (x,y,z)$
$\vec{a} = (a_1, a_2, a_3)$
$\vec{b} = (b_1, b_2, b_3)$

Then eqn $\vec{r} = \vec{a} + t\vec{b}$ becomes,

or, $(x,y,z) = (a_1,a_2,a_3) + t(b_1,b_2,b_3)$

or, $(x,y,z) = (a_1+tb_1, a_2+tb_2, a_3+tb_3)$

Equating on both sides,

$x = a_1+tb_1$, $y = a_2+tb_2$, $z = a_3+tb_3$. —(*) (Parametric Eqn)

$\therefore \dfrac{x-a_1}{b_1} = t$, $\dfrac{y-a_2}{b_2} = t$, $\dfrac{z-a_3}{b_3} = t$.

$\therefore \dfrac{x-a_1}{b_1} = \dfrac{y-a_2}{b_2} = \dfrac{z-a_3}{b_3}$, which is the required eqn in Cartesian form.

Note: The Equation (*) is called parametric eqn of st. line of (1).

Find the equation of the st. line which is parallel to $3\vec{i} - 6\vec{j} + 8\vec{k}$ & passes through (1,2,0).
(i). Parametric Equation
(ii). Cartesian Equation

Soln, Here, $\vec{b} = (3,-6,8)$

$\vec{a} = (1,2,0)$

The required equation of st. line is

$\vec{r} = \vec{a} + t\vec{b}$

$= (1,2,0) + t(3,-6,8)$

$= (1,2,0) + (3t,-6t,8t)$

$\therefore \vec{r} = (1+3t, 2-6t, 8t)$

$x = 1+3t, \quad y = -2-6t, \quad z = 8t$ (parametric Eqn)

$\frac{x-1}{3} = \frac{y-2}{6} = \frac{z}{8}$ (Cartesian Eqn)

Find the vector equation of st. line, whose cartesian eqn is

$\frac{x-3}{2} = \frac{y-4}{3} = \frac{z-1}{6}$

Ans: $\vec{r} = \vec{a} + t\vec{b}$

where $\vec{r} = (x,y,z)$

$\vec{a} = (3,4,1)$

$\vec{b} = (2,3,6)$

let $\frac{x-3}{2} = \frac{y-4}{3} = \frac{z-1}{6} = t$ (say)

$x = 3+2t, \; y = 3t+4, \; z = 6t+1.$

$(x,y,z) = (2t+3, 3t+4, 6t+1)$

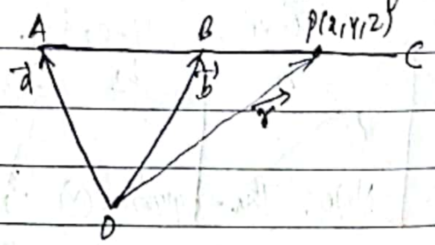
$= (3,4,1) + (2t,3t,6t).$ where $\vec{a} = (3,4,1)$

$= (3,4,1) + t(2,3,6)$ $\vec{b} = (2,3,6)$

which is of the form, $\vec{r} = \vec{a} + t\vec{b}$,

Theorem: Vector equation of st. line passes through $\vec{a}$ & $\vec{b}$.

To find the vector equation of st. line which passes through two points A & B with position vector. $\vec{OA} = \vec{a}$ & $\vec{OB} = \vec{b}$ is of the form: $\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$



Proof:

Let $\vec{OA} = \vec{a}$ & $\vec{OB} = \vec{b}$ be given,

let $P(x,y,z)$ be any point on the AC, whose equation is to be determined.

let $\vec{OP} = \vec{r}$

Now, In Δ OAB, by vector addition

$\vec{AB} = \vec{OB} - \vec{OA} = \vec{b} - \vec{a}$

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Again, In ΔOAP, $\vec{OP} = \vec{OA} + \vec{AP}$ — (i)

Since $\vec{AP}$ and $\vec{AB}$ are collinear,

$\therefore \vec{AP} = t\vec{AB}$ — (ii)

Hence, eqn (i) becomes,

$\vec{OP} = \vec{OA} + t\vec{AB}$

or $\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$

$\therefore \boxed{\vec{r} = \vec{a} + t(\vec{b} - \vec{a})}$ which is the required vector equation of st. line passing through $\vec{a}$ & $\vec{b}$.

Verification: (Change into Parametric & Cartesian form)

let $\vec{r} = (x, y, z)$

$\vec{a} = (a_1, a_2, a_3)$

$\vec{b} = (b_1, b_2, b_3)$

Then eqn $\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$ becomes,

or $(x, y, z) = (a_1, a_2, a_3) + t(b_1 - a_1, b_2 - a_2, b_3 - a_3)$

or $(x, y, z) = (a_1 + t(b_1 - a_1), a_2 + t(b_2 - a_2), a_3 + t(b_3 - a_3))$

Equating the corresponding components

$x = a_1 + t(b_1 - a_1)$, $y = a_2 + t(b_2 - a_2)$, $z = a_3 + t(b_3 - a_3)$ — (*).

or $\dfrac{x - a_1}{b_1 - a_1} = \dfrac{y - a_2}{b_2 - a_2} = \dfrac{z - a_3}{b_3 - a_3} = t$

$\therefore$ Cartesian eqn is

$$\frac{x - a_1}{b_1 - a_1} = \frac{y - a_2}{b_2 - a_2} = \frac{z - a_3}{b_3 - a_3}.$$

The eqn (*) is the parametric Eqn.

\# Find the Cartesian eqn of st. line which passes through $\vec{a} = (1, 2, 3)$ & $\vec{b} = (4, 5, 0)$

Hint: Reqd eqn is: $\vec{r} = \vec{a} + t(b - a)$

Ans: $\dfrac{x-1}{4-1} = \dfrac{y-2}{5-2} = \dfrac{z-3}{0-3}$

\# Example:

Find the equation of st. line in the form $\dfrac{x}{p} + \dfrac{y}{q} = 1$ by vector method.

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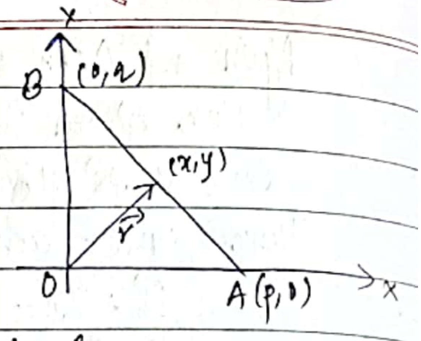
Soln,

Let the St. line makes intercepts p & q on x-axis and y-axis at A & B.



$\therefore \vec{OA} = \vec{a} = (p,0)$

$\vec{OB} = \vec{b} = (0,q)$

Then the equation (vector) of st. line in two point form is

$\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$

$= (p,0) + t\left((0,q) - (p,0)\right)$

$= (p,0) + t(-p,q)$

$= (p,0) + (-tp, tq)$

$\therefore (x,y) = (p - tp, tq)$

$\therefore x = p - tp \quad , \quad y = tq$

$x = p(1-t)$

$\frac{x}{p} = 1 - t \quad \text{--- (i)} \quad \& \quad \frac{y}{q} = t \quad \text{--- (ii)}$

Eliminating t between (i) & (ii).

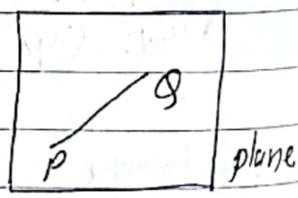
$\frac{x}{p} = 1 - \frac{y}{q}$.

$\frac{x}{p} + \frac{y}{q} = 1$

Equation of a plane

plane:

A locus (or surface) is said to be plane if given any two points P & Q on the locus, the st. line joining P & Q i.e. wholly lies on the locus.



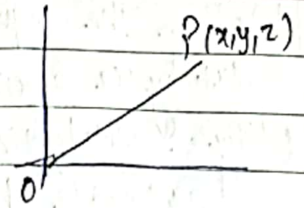
General equation of plane passes through origin O(0,0,0) is,

$ax + by + cz = 0$

This can be written as,

$(a,b,c) \cdot (x,y,z) = 0 \quad \text{--- } (*)$

Let $\vec{r} = \vec{OP} = (x, y, z)$
& $\vec{n} = \vec{ON} = (a, b, c)$, which is Normal vector
Then (*) gives,
$$\vec{n} \cdot \vec{r} = 0$$



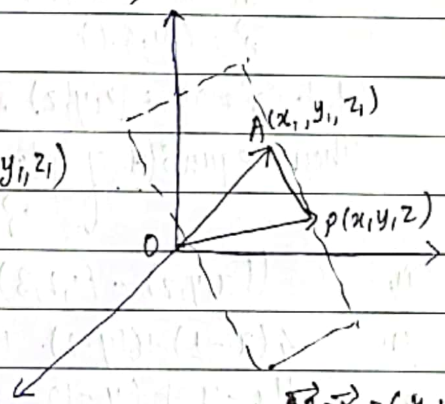
This shows that a plane is completely determined by a point $\vec{r} = P(x,y,z)$ if normal vector $\vec{n} = N(a,b,c)$ is given.

Note: In the equation of plane $ax+by+cz=0$, The numbers a, b, c are called direction ratio of the normal to the plane.

Determination of Equation of plane

To find the equation of the plane which passes through the point $A(x_1, y_1, z_1)$ and having normal vector $\vec{n} = (a, b, c)$

Let, $P(x, y, z)$ be any point on the plane
Such that $\vec{OP} = \vec{r} = (x, y, z)$
Also, the given point on the plane $\vec{OA} = \vec{a} = (x_1, y_1, z_1)$
$\vec{AP} = \vec{OP} - \vec{OA} = \vec{r} - \vec{a}$
Here, $\vec{AP}$ lies completely on the plane.
Here, $\vec{ON} = \vec{n} = (a, b, c)$ is the normal vector
$\therefore$ Normal vector $\vec{n}$ is perpendicular to $\vec{AP}$
$\therefore$ Applying condition of perpendicular, we get
$$\therefore \vec{AP} \cdot \vec{n} = 0$$
or, $$(\vec{r} - \vec{a}) \cdot \vec{n} = 0$$
Which is the required equation of plane which passes through $\vec{OA} = \vec{a} = (x_1, y_1, z_1)$ with normal vector $\vec{n} = (a, b, c)$.



$\vec{OP} = \vec{r} = (x, y, z)$
$\vec{OA} = \vec{a} = (x_1, y_1, z_1)$
$\vec{AP} = \vec{OP} - \vec{OA} = \vec{r} - \vec{a}$

Verification : Into Cartesian Coordinate
If $\vec{r} = (x, y, z)$, $\vec{a} = (x_1, y_1, z_1)$ & $\vec{n} = (a, b, c)$, Then equation of plane $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$ becomes,
$$((x, y, z) - (x_1, y_1, z_1)) \cdot (a, b, c) = 0.$$
or, $$(x - x_1, y - y_1, z - z_1) \cdot (a, b, c) = 0.$$
or, $$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0 \quad \text{—(*)}$$ which is eqn of plane passes through (x_1, y_1, z_1) & direction ratio of normal to the plane a, b, c.

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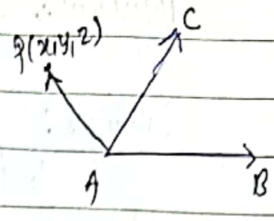
Note:- The equation (*) becomes
$ax+by+cz = ax_1+by_1+cz_1$ — (**)
Letting $ax_1+by_1+cz_1 = d$ (say), Then the eqn of plane becomes
$ax+by+cz=d$.

Fact: If plane passes through origin, then, $(x_1, y_1, z_1) = (0,0,0)$
So that $d=0$. Eqn of plane is,
$\therefore$ $ax+by+cz=0$

Find the equation of plane which passes through $A(1,2,3)$ with normal vector $\vec{n} = (4,6,0)$.
Soln,
Let $\vec{OA} = \vec{a} = (1,2,3)$
$\vec{n} = (4,6,0)$
Let $\vec{r} = \vec{OP} = (x,y,z)$ be any point on the plane.
Then equation of plane is.
$$(\vec{r}-\vec{a}).\vec{n} = 0.$$
or $((x,y,z)-(1,2,3)).(4,6,0) = 0$
or $4(x-1)+6(y-2)+(z-3).0=0$
or $4x-4+6y-12=0$
or $4x+6y-16=0$
or, $2x+3y-8=0$

Plane through three points
To find the equation of the plane which passes through the three points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$
let, $\vec{OA} = (x_1, y_1, z_1)$
$\vec{OB} = (x_2, y_2, z_2)$
$\vec{OC} = (x_3, y_3, z_3)$ be given vectors.
$\vec{AB} = \vec{OB}-\vec{OA} = (x_2-x_1, y_2-y_1, z_2-z_1)$
$\vec{AC} = \vec{OC}-\vec{OA} = (x_3-x_1, y_3-y_1, z_3-z_1)$
Here, $\vec{AB}\times\vec{AC}$ always represent the vector normal to the

plane containing $\vec{AB}$ & $\vec{AC}$.



Let $P(x,y,z)$ be any point on the plane,
such that $\vec{OP} = (x,y,z)$

$\therefore \vec{AP} = \vec{OP} - \vec{OA} = (x-x_1, y-y_1, z-z_1)$.
which lie on the plane.

Here. $\vec{AP}$ is perpendicular to $\vec{AB} \times \vec{AC}$, so their dot product is zero.

i.e. $\vec{AP} \cdot \vec{AB} \times \vec{AC} = 0$

i.e. Scalar Triple Product is zero.

i.e. $$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

which is required equation of plane passes through 3 points.

Example: Find the eqn of plane passes through three points $(1,2,3)$, $(2,0,8)$ & $(6,1,0)$

Let, $\vec{OA} = (1,2,3)$, $\vec{OB} = (2,0,8)$ and $\vec{OC} = (6,1,0)$.

Let $P(x,y,z)$ be any point on the plane.

$\therefore \vec{OP} = (x,y,z)$

$\vec{AP} = \vec{OP} - \vec{OA}$

$\vec{AB} = \vec{OB} - \vec{OA}$

$\vec{AC} = \vec{OC} - \vec{OA}$

Then, $\vec{AP}$ is $\perp r$ to $AB \times AC$.

$$\vec{AP} \cdot \vec{AB} \times \vec{AC} = 0$$

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 2-1 & 0-2 & 8-3 \\ 6-1 & 1-2 & 0-3 \end{vmatrix} = 0$$

$$(x-1)\begin{vmatrix} -2 & 5 \\ -1 & -3 \end{vmatrix} - (y-2)\begin{vmatrix} 1 & 5 \\ 5 & -3 \end{vmatrix} + (z-3)\begin{vmatrix} 1 & -2 \\ 5 & -1 \end{vmatrix} = 0.$$

or $(x-1)\{6+5\} - (y-2)\{-3-25\} + (z-3).\{-1+10\} = 0$.

or $(x-1)\,11 + (y-2)(+28) + (z-3)\,9 = 0$

or $11x - 11 + 28y - 56 + 9z - 27 = 0$

$\therefore 11x + 28y + 9y = 94$ //

Intersection of a line with a plane

Example: find the point of intersection of line $\frac{x-3}{2} = \frac{y-2}{1} = \frac{z-1}{3}$ to the plane. $2x+5y+6z=1$

Soln,

let $\frac{x-3}{2} = \frac{y-2}{1} = \frac{z-1}{3} = r$ (say)

$\therefore x = 2r+3, \quad y = r+2, \quad z = 3r+1$.

To find the point of intersection of line & plane,

The point $(2r+3, r+2, 3r+1)$ lie on plane $2x+5y+6z=1$ for suitable value of r.

$$2(2r+3) + 5(r+2) + 6(3r+1) = 1$$

$$\therefore r = -\frac{21}{27}.$$

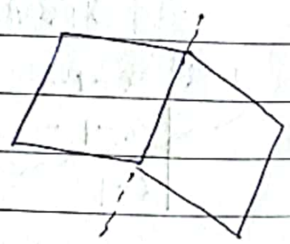
The required points of intersection is,

$$(2r+3, r+2, 3r+1) = \left(2\left(\frac{-21}{27}\right)+3, \ -\frac{21}{27}+2, \ 3\left(\frac{-21}{27}\right)+1\right)$$

$$= \left(\frac{13}{9}, \frac{11}{9}, \frac{-12}{9}\right)$$

Angle between two Planes

The angle between two planes is defined as the angle between their normal vectors.



The angle between two planes

$P_1: a_1x + b_1y + c_1z = d_1$

$P_2: a_2x + b_2y + c_2z = d_2$

are determined in the following steps.

1. The normal vector of both planes P_1 & P_2 are,
$\vec{n_1} = (a_1, b_1, c_1)$ & $\vec{n_2} = (a_2, b_2, c_2)$

2. If θ is the angle between these two normal vectors i.e. angle between two planes.

$$\cos\theta = \frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}|\,|\vec{n_2}|} = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

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Case-I : If two planes are parallel, then their normals are parallel.
In this case, $\theta = 0$.

$$1 = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

This gives after simplification. (Squaring & Adding)

$$(a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (c_1a_2 - a_1c_2) = 0.$$

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

i.e. a_1, b_1, c_1 and a_2, b_2, c_2 are proportional.

Case-II : If two planes are perpendicular, then their normal are also ⊥r.
In this case, $\theta = 90°$

$$0 = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

$$\therefore a_1a_2 + b_1b_2 + c_1c_2 = 0.$$

Example: Find the angle between two planes,
$2x + 3y + z = 3$, $4x + y + 2z = 1$

Soln,

$2x + 3y + z = 3$ — (I)

$4x + y + 2z = 1$ — (II).

The normal vector of both planes (I) & (II) are,

$\vec{n_1} = (2, 3, 1)$

$\vec{n_2} = (4, 1, 2)$.

If θ is the angle between these plane then,

$$\cos\theta = \frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}|.|\vec{n_2}|} = \frac{2\times4 + 3\times1 + 1\times2}{\sqrt{2^2+3^2+1^2}\sqrt{4^2+1^2+2^2}} = \frac{13}{7\sqrt{6}}.$$

$$\therefore \theta = \cos^{-1}\left(\frac{13}{7\sqrt{6}}\right) = 40.69°$$

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Equation of st. lines through the intersection of two planes.

let $P_1 \Rightarrow a_1x + b_1y + c_1z + d_1 = 0$ — (I)

and $P_2 \Rightarrow a_2x + b_2y + c_2z + d_2 = 0$ — (II)

be two planes.

To find equation of st. lines through the intersection of planes (I) & (II), we need

(a). a point (x_1, y_1, z_1) from which the line passes.

let $z_1 = 0$ be a point from which (I) & (II) intersect.

Then, $a_1x + b_1y + d_1 = 0$

$a_2x + b_2y + d_2 = 0$

Solving by cross multiplication

$$\begin{array}{cccccc} & x & & y & & 1 \\ a_1 & b_1 & d_1 & a_1 & b_1 \\ a_2 & b_2 & d_2 & a_2 & b_2 \end{array}$$

$$\frac{x}{b_1d_2 - b_2d_1} = \frac{y}{d_1a_2 - d_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

$$\therefore x = \frac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1} \Rightarrow x_1 \text{ (say)}$$

$$\therefore y = \frac{d_1a_2 - d_2a_1}{a_1b_2 - a_2b_1} \Rightarrow y_1 \text{ (say)}$$

(b) The direction cosine (ratio) of the lines.

Since the required line lie on the both planes P_1 & P_2.

So the required line L is perpendicular to both normal vectors of the planes P_1 and P_2.

Where normal vectors are:

$\vec{n_1} = (a_1, b_1, c_1)$ and $\vec{n_2} = (a_2, b_2, c_2)$.

$\therefore$ Required direction ratio of the line $= \vec{n_1} \times \vec{n_2}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$\Rightarrow \underbrace{(b_1c_2 - b_2c_1)}_{a}\vec{i} + \underbrace{(c_1a_2 - a_1c_2)}_{b}\vec{j} + \underbrace{(a_1b_2 - a_2b_1)}_{c}\vec{k}$$

$= a\vec{i} + b\vec{j} + c\vec{k}$

$= (a, b, c)$

$\therefore$ Required equation of st. line is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}, \text{ (where } z_1 = 0 \text{ in this case)}$$

Example: Find the eqn of st. line through the intersection of two planes. $3x + y + z = 7$, $x + y + 2z = 5$.

Soln,

$P_1 : 3x + y + z = 7$ — (i)

$P_2 : x + y + 2z = 5$ — (ii)

Step I: We need a point which the line passes.

let $z = 0$ be a point on required line,

$3x + y = 7$ — (*) Then, (**) gives,

$-\ x + y = 5$ — (**) $y = 4$.

$2x = 2$

$\therefore x = 1$

$\therefore$ Point on the line $(x_1, y_1, z_1) = (1, 4, 0)$

Step II: Required direction ratio, $= \vec{n_1} \times \vec{n_2}$

$= (3, 1, 1) \times (1, 1, 2)$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix}$$

$= \vec{i} - 5\vec{j} + 2\vec{k}$

$= (1, -5, 2) = (a, b, c)$

$\therefore$ Required line is,

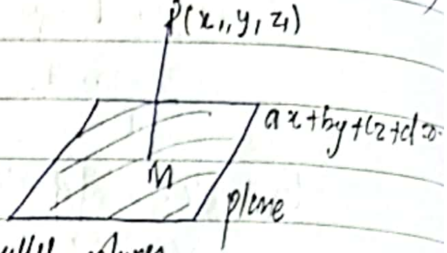
$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}.$$

$$\Rightarrow \frac{x - 1}{1} = \frac{y - 4}{-5} = \frac{z - 0}{2}$$

Length of ⊥r from a point $P(x_1, y_1, z_1)$ to a plane $ax+by+cz+d=0$
(Shortest Distance | ⊥r Distance)

The formula for perpendicular distance

$$p = \left| \frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}} \right|$$



Also, to find the distance between two parallel planes.

$$ax+by+cz+d_1=0 \quad \text{—} \; ①$$
$$ax+by+cz+d_2=0 \quad \text{—} \; ⑪$$

In this case, take a point $z=0$, $y=0$ in ①,

$$x = -\frac{d_1}{a}$$

$\therefore$ Point on ① is $P\left(-\frac{d_1}{a}, 0, 0\right)$

Then find ⊥r distance from P on plane P_2.

Example: Find the distance between two parallel planes,

$P_1: 2x+3y+z=5$ —①

$P_2: 6x+9y+3z=10$ —⑪

Soln,

Let $y=0$ & $z=0$ be a point on ①, Then,

$2x+3.0+0=5.$

$\therefore x = 5/2$

$\therefore$ Required point on the plane ① i.e. P_1 is $(5/2, 0, 0)$

$\therefore$ length of ⊥r from $(5/2, 0, 0)$ on the plane P_2 is

$$p = \left| \frac{6 \times 5/2 + 9\times 0 + 3\times 0 - 10}{\sqrt{6^2+9^2+3^2}} \right| = \frac{5}{\sqrt{126}}$$

Skewed Lines

The two lines which are neither parallel nor intersect are called Skewed Lines

Example:

Consider two lines $\vec{r} = (1,1,0) + t(1,-1,2)$ — (i) & $\vec{r} = (2,0,2) + s(-1,+1,0)$ — (ii)

(i) Are they parallel

(ii) Determine skewness. Find the point of intersection.

(iii) Find the eqn of plane in which they lie.

Soln,

(i) From (i),

$(x,y,z) = (1,1,0) + t(1,-1,2)$

or $(x-1, y-1, z) = t, -t, 2t$

$x-1 = t \quad , \quad y-1 = -t \quad , \quad z = 2t$

$\frac{x-1}{1} = \frac{y-1}{-1} = \frac{z}{2} = t \qquad \therefore l_1 = 1, \; m_1 = -1, \; n_1 = 2$

Similarly from (ii)

$\frac{x-2}{-1} = \frac{y}{1} = \frac{z-2}{0} = s. \qquad \therefore l_2 = -1, \; m_2 = 1, \; n_2 = 0$

The condition of parallelism is,

$\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$ i.e. $\frac{1}{-1} = \frac{-1}{1} = \frac{2}{0}$ is not parallel.

$\therefore$ The two lines are not parallel.

(ii) From (i) & (ii)

$x = t+1 \;,\; y = 1-t \;,\; z = 2t$ — (*)

$x = 2-s \;,\; y = s \quad z = 2$ — (**)

If they intersect, then for some appropriate value of t & s, above two points are equal.

$\therefore t+1 = 2-s$ — (***) $, \quad 1-t = s, \qquad 2t = 2.$

or $1-t = s. \qquad \therefore t = 1$

$\Rightarrow 1-1 = s$

$\therefore s = 0$

Substituting value of s & t in (***) : $t+1 = 2-s$

$\Rightarrow 1+1 = 2-0$

i.e. $2 = 2$ (True).

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So the above set (x) and (xx) consistent. So the two lines intersect.

To find point of intersection
putting value of t (or s) in (x) and (xx)

From (x)

$x = 1+1 = 2$ Not Skewed

$y = 1-1 = 0$

$z = 2\times1 = 2$

$\therefore$ Required point of intersection $= (2,0,2)$

(iii) To find equation of plane in which they lie.

The equation of plane passes through $(2,0,2)$ is

$a(x-2)+b(y-0)+c(z-2) = 0$ —(iii)

where $\vec{r} = (a,b,c)$ are normal vector

Direction ratio of normal to the plane.

Where, $(a,b,c) = (1,-1,2) \times (-1,1,0)$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 2 \\ -1 & 1 & 0 \end{vmatrix} = -2\vec{i} - 2\vec{j} + 0\vec{k}$$

$= (-2,-2,0)$.

$\therefore a = -2,\ b = -2,\ c = 0$

Hence, eqn (iii) becomes,

$-2(x-2) + (-2)(y-0) + 0(z-2) = 0$

or, $-2x+y-2y = 0$

or, $x+y = 2$

Unit 2

Vector functions

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Vector function of Scalar Variable.

Let t be a scalar variable defined on some interval (a,b). Let $\vec{r}$ be a vector function which depends on scalar variable t. Then we say that $\vec{r}$ is a vector function t of we write $\vec{r} = F(\vec{t})$.

Example: Vector Equation of parabola.

Let $\vec{r} = t^2\vec{i} + 2t\vec{j} = \vec{F}(t)$ —①

Here comparing ① with $\vec{r} = x\vec{i} + y\vec{j}$

we get,

$$x = t^2 \quad \& \quad y = 2t$$

Eliminating t, we get.

$t^2 = x$ & $t^2 = \frac{y^2}{4}$

$$x = \frac{y^2}{4}$$

$\therefore y^2 = 4t$.

which is parabola in cartesian form.

So, the equation

$$\vec{r} = at^2\vec{i} + 2at\vec{j}, \; a = 1$$

always represents vector equation of parabola.

Example The vector $\vec{r} = a\cos t\,\vec{i} + a\sin t\,\vec{j}$

$x = a\cos t$, $y = a\sin t$.

$x^2 + y^2 = a^2 \rightarrow$ circle. (circle).

Example: The vector $\vec{r} = a\cos t\,\vec{i} + b\sin t\,\vec{j}$ represent vector equation of ellipse.

$x = a\cos t$ $\qquad$ $y = b\sin t$

$\cos t = \frac{x}{a}$ $\qquad$ $\sin t = \frac{y}{b}$

$\therefore \frac{x^2}{a} + \frac{y^2}{b} = 1$. (Ellipse)

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Example: The vector $\vec{r} = a\sec t\,\vec{i} + b\tan t\,\vec{j}$ represent hyperbola.

$x = a\sec t \qquad y = b\tan t$

$\sec t = \frac{x}{a} \qquad \tan t = \frac{y}{b}$

Now $\sec^2 t - \tan^2 t = 1$

$\therefore \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (Hyperbola)

Limit of a vector function.

We had: A function $f(x)$ was said to have limit L as $x \to a$ if.

$$\lim_{x\to a} f(x) = L.$$

Here, L is called limiting value of the function $f(x)$ at point $x = a$

A vector function $\vec{r} = \vec{F}(t)$ is said to have limit $\vec{L}$ as $t \to t_0$ if

$$\lim_{t\to t_0} \vec{F}(t) = \vec{L}.$$

Evaluate: find $\lim_{t\to t_0} \vec{F}(t)$ if $t_0 = 0$ & $\vec{F}(t) = e^{-2t}\vec{i} + \frac{t^3}{\sin^3 t}\vec{j} + \cos 3t\,\vec{k}$

Here.

$\lim_{t\to t_0} \vec{F}(t)$

$= \lim_{t\to 0} \left(e^{-2t}\vec{i} + \frac{t^3}{\sin^3 t}\vec{j} + \cos 3t\,\vec{k}\right)$

$= \vec{i}\lim_{t\to 0} e^{-2t} + \vec{j}\lim_{t\to 0} \frac{t^3}{\sin^3 t} + \vec{k}\lim_{t\to 0} \cos 3t$

$= \vec{i}.1 + j\lim_{t\to 0}\left(\frac{t}{\sin t}\right)^3 + k\cos 0$

$= \vec{i} + \vec{j}(1)^3 + \vec{k}.1.$

$= \vec{i} + \vec{j} + \vec{k}$

$= (1,1,1).$

$$\boxed{\begin{array}{l} e^0 = 1 \\ \lim_{x\to 0} \frac{\sin x}{x} = 1 \\ \lim_{x\to 0} \frac{x}{\sin x} = 1 \end{array}}$$

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Continuity of a Vector Function.

We had, A real valued function $f(x)$ was said to be continuous at point $x=a$ if

$\lim_{x\to a} f(x) = f(a)$ i.e. limiting value = functional value of $f(x)$ at point $x=a$.

limiting value functional value.

A vector function $\vec{F}(t)$ is said to be continuous at point $t=t_0$ if

$\lim_{t\to t_0} \vec{F}(t) = \vec{F}(t_0)$

i.e. limiting value = functional value of $\vec{F}(t)$ as $t=t_0$

Derivative of Vector function.

let $\vec{r} = \vec{F}(t)$ be a vector function of scalar variable t. let Δt be the small change in t. then the derivative of $\vec{F}(t)$ wrt t is denoted by, $\frac{d\vec{F}(t)}{dt}$ or $\vec{F}'(t)$ and is defined by

$$\frac{d\vec{F}(t)}{dx} = \lim_{\Delta t\to 0} \frac{\vec{F}(t+\Delta t)-\vec{F}(t)}{\Delta t}$$, provided the limit on right hand side exists.

Formula on Derivative

let $\vec{r_1}, \vec{r_2}, \vec{r_3}$ be three vector function & ϕ be a scalar function,

Then

(i) $\frac{d}{dt}(\vec{r_1}+\vec{r_2}) = \frac{d\vec{r_1}}{dt} + \frac{d\vec{r_2}}{dt}$

(ii) $\frac{d}{dt}(\phi\vec{r}) = \phi\frac{d\vec{r}}{dt} + \vec{r}\frac{d\phi}{dt}$

In particular, if $\vec{r}$ is a constant vector, then $\frac{d\vec{r}}{dt} = \vec{0}$

(iii) $\frac{d}{dt}(\vec{r_1}\cdot\vec{r_2}) = \vec{r_1}\cdot\frac{d\vec{r_2}}{dt} + \vec{r_2}\cdot\frac{d\vec{r_1}}{dt}$ or $\frac{d\vec{r_1}}{dt}\cdot\vec{r_2} + \vec{r_1}\cdot\frac{d\vec{r_2}}{dt}$ (Commutative)

(iv) $\frac{d}{dt}(\vec{r_1}\times\vec{r_2}) = \vec{r_1}\times\frac{d\vec{r_2}}{dt} + \frac{d\vec{r_1}}{dt}\times\vec{r_2}$ (Not Commutative).

or, $\frac{d\vec{r_1}}{dt}\times\vec{r_2} + \vec{r_1}\times\frac{d\vec{r_2}}{dt}$

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(v) Derivative of Scalar Triple Product.

let $\vec{r_1}, \vec{r_2}, \vec{r_3}$ be three vector functions then the derivative of Scalar Triple Product $[\vec{r_1}\,\vec{r_2}\,\vec{r_3}]$ is obtained as follows.

$$\frac{d}{dt}[\vec{r_1}\,\vec{r_2}\,\vec{r_3}] = \frac{d}{dt}[\vec{r_1}.(\vec{r_2}\times\vec{r_3})]$$

$$= \frac{d\vec{r_1}}{dt}.(\vec{r_2}\times\vec{r_3}) + \vec{r_1}.\frac{d}{dt}(\vec{r_2}\times\vec{r_3})$$

$$= \frac{d}{dt}\vec{r_1}.(\vec{r_2}\times\vec{r_3}) + \vec{r_1}.\left[\frac{d\vec{r_2}}{dt}\times\vec{r_3} + \vec{r_2}\times\frac{d\vec{r_3}}{dt}\right]$$

$$= \frac{d\vec{r_1}}{dt}.(\vec{r_2}\times\vec{r_3}) + \vec{r_1}\left(\frac{d}{dt}\vec{r_2}\times\vec{r_3}\right) + \vec{r_1}.\vec{r_2}\times\frac{d\vec{r_3}}{dt}$$

$$= \left[\frac{d\vec{r_1}}{dt}\,\vec{r_2}\,\vec{r_3}\right] + \left[\vec{r_1}.\frac{d\vec{r_2}}{dt}\,\vec{r_3}\right] + \left[r_1\,r_2\,\frac{d\vec{r_3}}{dt}\right]$$

(vi) Derivative of Vector Triple Product

let $\vec{r_1}, \vec{r_2}, \vec{r_3}$ be three vector functions then the derivative of Vector Triple product $\vec{r_1}\times(\vec{r_2}\times\vec{r_3})$ is obtained as follows.

$$\frac{d}{dt}(\vec{r_1}\times(\vec{r_2}\times\vec{r_3})) = \frac{d}{dt}[\vec{r_1}\times(\vec{r_2}\times\vec{r_3})]$$

$$= \frac{d}{dt}\vec{r_1}\times(\vec{r_2}\times\vec{r_3}) + \vec{r_1}\times\frac{d}{dt}(\vec{r_2}\times\vec{r_3})$$

$$= \frac{d}{dt}\vec{r_1}\times(\vec{r_2}\times\vec{r_3}) + \vec{r_1}\times\left[\frac{d\vec{r_2}}{dt}\times\vec{r_3} + \vec{r_2}\times\frac{d\vec{r_3}}{dt}\right]$$

$$= \frac{d}{dt}\vec{r_1}\times(\vec{r_2}\times\vec{r_3}) + r_1\times\left(\frac{d}{dt}\vec{r_2}\times r_3\right) + \vec{r_1}\times\left(\vec{r_2}\times\frac{d\vec{r_3}}{dt}\right)$$

Fact:

If $\vec{F}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ Then,

$$\frac{d}{dt}\vec{F}(t) = \frac{d}{dt}(f(t))\vec{i} + \frac{d}{dt}(g(t))\vec{j} + \frac{d}{dt}(h(t))\vec{k}$$

$$= \left(\frac{d}{dt}f(t), \frac{d}{dt}(g(t)), \frac{d}{dt}(h(t))\right)$$

$$= (\dot{f}(t), \dot{g}(t), \dot{h}(t))$$, where dot(.) denotes derivatives w.r.t t.

$= b\cdot c\times a$
$= c\cdot a\times b$

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Example:
Let $\vec{r} = a\cos t\,\vec{i} + a\sin t\,\vec{j} + at\tan\alpha\,\vec{k}$. find:

(i) $\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right|$ (ii). $\left[\frac{d\vec{r}}{dt}\cdot\frac{d^2\vec{r}}{dt^2}\cdot\frac{d^3\vec{r}}{dt^3}\right]$

Soln,

$\vec{r} = a\cos t\,\vec{i} + a\sin t\,\vec{j} + at\tan\alpha\,\vec{k}$.

$\therefore \frac{d\vec{r}}{dt} = -a\sin t\,\vec{i} + a\cos t\,\vec{j} + a\tan\alpha\,\vec{k}$.

$\frac{d^2\vec{r}}{dt^2} = -a\cos t\,\vec{i} - a\sin t\,\vec{j} + 0.\vec{k}$

$= -a\cos t\,\vec{i} - a\sin t\,\vec{j} + 0\vec{k}$

$\frac{d^3\vec{r}}{dt^3} = a\sin t\,\vec{i} - a\cos t\,\vec{j} + 0\vec{k}$.

Now,

(i) $\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -a\sin t & a\cos t & a\tan\alpha \\ -a\cos t & -a\sin t & 0 \end{vmatrix} = (0 + a^2\sin t\tan\alpha)\vec{i} + (-a^2\tan\alpha\cos t - 0)\vec{j} + (a^2\sin^2 t + a^2\cos^2 t)\vec{k}$.

$= (a^2\sin t\tan\alpha)\vec{i} - a^2\tan\alpha\cos t\,\vec{j} + a^2\vec{k}$.

$\therefore \left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right| = \sqrt{(a^2\sin t\tan\alpha)^2 + (-a^2\tan\alpha\cos t)^2 + (a^2)^2}$

$= \sqrt{a^4\tan^2\alpha(\sin^2 t + \cos^2 t) + a^4}$

$= \sqrt{a^4\tan^2\alpha + a^4}$

$= a^2\sqrt{\tan^2\alpha + 1}$

$= a^2\sqrt{\sec^2\alpha}$

$= a^2\sec\alpha$.

(ii) $\left[\frac{d\vec{r}}{dt}\cdot\frac{d^2\vec{r}}{dt^2}\cdot\frac{d^3\vec{r}}{dt^3}\right] = \left(\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right)\cdot\frac{d^3\vec{r}}{dt^3}$

$\because [a\,b\,c] = \vec{a}\cdot\vec{b}\times\vec{c}$
$= \vec{b}\cdot\vec{c}\times\vec{a}$
$= \vec{c}\cdot\vec{a}\times\vec{b}$
$= (\vec{a}\times\vec{b})\cdot\vec{c}$.

$\hookrightarrow [abc] = \vec{a}\cdot\vec{b}\times\vec{c}$
$= \vec{a}\times\vec{b}\cdot\vec{c}$

$= ((a^2\sin t\tan\alpha)\vec{i} - a^2\tan\alpha\cos t\,\vec{j} + a^2\vec{k}).$
$(a\sin t\,\vec{i} - a\cos t\,\vec{j} + 0\vec{k})$

$= a^3 \sin^2 t \tan\alpha + a^3 \cos^2 t \tan\alpha + 0$

$= a^3 \tan\alpha (\sin^2 t + \cos^2 t)$

$= a^3 \tan\alpha.$

Example:

If $\vec{r}$ is a unit vector, then prove that $\left|\vec{r} \times \frac{d\vec{r}}{dt}\right| = \left|\frac{d\vec{r}}{dt}\right|$

proof:

Since $\vec{r}$ is unit vector so,

$\vec{r}\cdot\vec{r} = 1 \quad —(1)$

Diff (1). w.r.t t

$\frac{d\vec{r}}{dt}\cdot\vec{r} + \vec{r}\cdot\frac{d\vec{r}}{dt} = 0$

$\therefore 2\vec{r}\,\frac{d\vec{r}}{dt} = 0.$

or, $\vec{r}\cdot\frac{d\vec{r}}{dt} = 0.$

This shows that $\vec{r}$ is orthogonal to $\frac{d\vec{r}}{dt}$.

$\therefore$ Angle between them $\theta = 90$

Now, $\left|\vec{r} \times \frac{d\vec{r}}{dt}\right| = |\vec{r}| \left|\frac{dr}{dt}\right| \sin 90.$

$= (1) \left(\frac{d\vec{r}}{dt}\right) (1) \qquad (\because |\vec{r}| = 1)$

$\therefore \left|\vec{r} \times \frac{d\vec{r}}{dt}\right| = \left|\frac{d\vec{r}}{dt}\right|$

$\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$

$|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta.$

Example:

Show that $\frac{d}{dt}(\vec{a}\times\vec{b}) = \vec{c}\times(\vec{a}\times\vec{b})$ where $\frac{d\vec{a}}{dt} = \vec{c}\times\vec{a}$ & $\frac{d\vec{b}}{dt} = \vec{c}\times\vec{b}$

Soln Here,

$\frac{d}{dt}(\vec{a}\times\vec{b}) = \frac{d\vec{a}}{dt}\times\vec{b} + \vec{a}\times\frac{d\vec{b}}{dt}$

$= (\vec{c}\times\vec{a})\times\vec{b} + \vec{a}\times(\vec{c}\times\vec{b})$

$= -[\vec{b}\times(\vec{c}\times\vec{a})] + \vec{a}\times(\vec{c}\times\vec{b})$

$= -[(\vec{b}\cdot\vec{a})\vec{c} - (\vec{b}\cdot\vec{c})\vec{a}] + (\vec{a}\cdot\vec{b})\vec{c} - (\vec{a}\cdot\vec{c})\vec{b}$

$= (\vec{b}\cdot\vec{c})\vec{a} - (\vec{a}\cdot\vec{c})\vec{b}$

$= (\vec{c}\cdot\vec{b})\vec{a} - (\vec{c}\cdot\vec{a})\vec{b}$

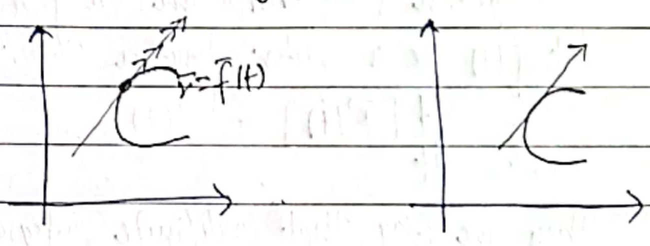
$= \vec{c}\times(\vec{a}\times\vec{b})$

$=$ RHS //

Geometrical Meaning of Derivative.

If $\vec{r} = f(t)$ be a vector function of scalar variable t, then the

(i) Derivative $\frac{d\vec{r}}{dt}$ always represents the vector along the tangent in the sense of t increasing.



(ii) The second derivative $\frac{d^2\vec{r}}{dt^2}$ always represents acceleration.

Example:

A particle moves along the curve $x = 2\sin 3t$, $y = 2\cos 3t$, $z = 8t$. Find the magnitude of velocity and acceleration at time $t = \pi/3$.

Soln,

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$= (2\sin 3t)\vec{i} + (2\cos 3t)\vec{j} + 8t\vec{k}$

$\therefore \frac{d\vec{r}}{dt} = 6\cos 3t\,\vec{i} - 6\sin 3t\,\vec{j} + 8\vec{k}$

At $t = \pi/3$, $\frac{d\vec{r}}{dt} = 6\cos\pi\,\vec{i} - 6\cdot 0 + 8\vec{k} = -6\vec{i} + 8\vec{k}$.

$\therefore$ velocity vector $\frac{d\vec{r}}{dt} = -6\vec{i} + 8\vec{k}$.

$\therefore$ Magnitude of velocity $= \left|\frac{d\vec{r}}{dt}\right| = \sqrt{(-6)^2 + 8^2} = 10$.

Similarly,

$\frac{d^2\vec{r}}{dt^2}$ = acceleration vector $= -18\sin 3t\,\vec{i} - 18\cos 3t\,\vec{j} + 0\vec{k}$.

$\therefore$ At $t = \pi/3$

$$\frac{d^2\vec{r}}{dt^2} = 0\vec{i} + 18\vec{j} + 0\vec{k}.$$

Magnitude of acceleration: $\left|\frac{d^2\vec{r}}{dt^2}\right| = \sqrt{18^2} = 18.$

Vector Integration.

Integration & differentiation are just reverse process.

Let $\vec{f}(t)$ be a vector function scalar variable t and

$$\frac{d}{dt}[\vec{F}(t)] = \vec{F}(t)$$

Then we say that indefinite integral of $\vec{F}(t)$ w.r.t t is $\vec{f}(t)$ and we write.

$$\int \vec{F}(t)dt = \vec{f}(t) + \vec{c}$$

where c is constant vector.

Definite Integral.

Let $\frac{d}{dx}[\vec{f}(t)] = \vec{f}(t)$

Then by first fundamental theorem of integral calculus,

$$\int_a^b \vec{F}(t)\,d(t) = \left[\vec{f}(t)\right]_a^b$$

$$= \vec{f}(b) - \vec{f}(a)$$

Example: Let $\vec{r_1} = 2t\vec{i} + t\vec{j} - \vec{k}$, $\vec{r_2} = t\vec{i} + 2\vec{j} + 3\vec{k}$, $\vec{r_3} = 2\vec{i} - 3\vec{j} + 4\vec{k}$.

Find: (a). $\int_0^2 (\vec{r_1} \times \vec{r_2})\,dt$

(b). $\int_0^2 [\vec{r_1}\ \vec{r_2}\ \vec{r_3}]\,dt$.

$$\int\left(\frac{d\vec{r}}{dt}\right)dt = \int d\vec{r} = \vec{r}+\vec{c}$$

Soln,

$$\vec{r_1}\times\vec{r_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & t & -1 \\ t & 2 & 3 \end{vmatrix} = (3t+2)\vec{i}+(-t-6)\vec{j}+(4-t^2)\vec{k}.$$

Also,

$$[\vec{r_1}\ \vec{r_2}\ \vec{r_3}] = \begin{vmatrix} 2 & t & -1 \\ t & 2 & 3 \\ 2 & -3 & 4 \end{vmatrix} = 2\begin{vmatrix} 2 & 3 \\ -3 & 4 \end{vmatrix} - t\begin{vmatrix} t & 3 \\ 2 & 4 \end{vmatrix} - 1\begin{vmatrix} t & 2 \\ 2 & -3 \end{vmatrix}$$

$$= 34 - 4t^2 + 6t + 3t + 4$$

$$= 38 - 4t^2 + 9t.$$

(a). $$\int_0^2 (\vec{r_1}\times\vec{r_2})dt = \int_0^2 \left[(3t+2)\vec{i}+(-t-6)\vec{j}+(4-t^2)\vec{k}\right]dt.$$

$$= \left[\left(\frac{3t^2}{2}+2t\right)\vec{i}+\left(-\frac{t^2}{2}-6t\right)\vec{j}+\left(4t-\frac{t^3}{3}\right)\vec{k}\right]_0^2$$

$$= \left[\left(\frac{3\times2^2}{2}+2\times2\right)\vec{i}+\left(-\frac{2^2}{2}-6\times2\right)\vec{j}+\left(4\times2-\frac{2^3}{3}\right)\vec{i}\right]-\vec{0}.$$

$$= 10\vec{i}-14\vec{j}+\frac{16}{3}\vec{k}.$$

(b). $$\int_0^2 [r_1\ r_2\ r_3]dt = \int_0^2 (-4t^2+9t+38)dt = \left[-\frac{4t^3}{3}+\frac{9t^2}{2}+38t\right]_0^2$$

$$= 83.33.$$

Formula

If $\vec{r} = \vec{f}(t)$ be a vector function of scalar variable t, then we know,

$\vec{v}$ = velocity vector $= \frac{d\vec{r}}{dt}$

$$\therefore \int\left(\frac{d\vec{r}}{dt}\right)dt = \int\vec{v}dt.$$

$$\vec{r} = \int\vec{v}dt.$$

Similarly,

$\vec{a}$ = acceleration vector $= \frac{d^2\vec{r}}{dt^2}$, After integration.

$$\int\frac{d^2\vec{r}}{dt^2}dt = \int\vec{a}dt$$

$\int dx = x$

or, $\int \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) dt = \int \vec{a}\, dt$

or, $\int d\left(\frac{d\vec{r}}{dt}\right) = \int \vec{a}\, dt$

or, $\frac{d\vec{r}}{dt} = \int \vec{a}\, dt$

Again integration,

$\vec{r} = \int\left[\int \vec{a}\, dt\right] dt$

Example: Solve $\frac{d^2\vec{r}}{dt^2} = t\vec{a} + \vec{b}$ where $\vec{a}$ & $\vec{b}$ are constant vector

Given that if $t=0$, then $\vec{r} = \vec{a}$ & $\frac{d\vec{r}}{dt} = \vec{a}$.

Here,

$\frac{d^2\vec{r}}{dt^2} = t\vec{a} + \vec{b}$

Integrating, we get,

$\int \frac{d^2\vec{r}}{dt^2} dt = \int (t\vec{a} + \vec{b}) dt$

or, $\frac{d\vec{r}}{dt} = \frac{t^2}{2}\vec{a} + \vec{b}t + \vec{c_1}$ — (i)

Again, Integration.

$\vec{r} = \frac{t^3}{6}\vec{a} + \vec{b}\frac{t^2}{2} + \vec{c_1}t + \vec{c_2}$ — (ii)

From (i).

$\vec{0} = \vec{0} + \vec{0} + \vec{c_1}$

$\therefore \vec{c_1} = 0$.

From (ii).

$\vec{0} = \vec{0} + \vec{0} + \vec{0} + \vec{c_2}$.

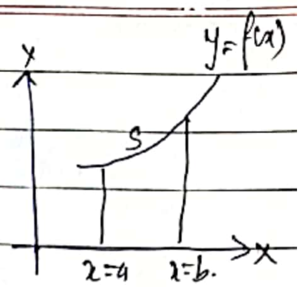
$\therefore \vec{c_2} = 0$.

Substituting the value of $\vec{c_1}$ & $\vec{c_2}$ on (i).

$\vec{r} = \frac{t^3}{6}\vec{a} + \vec{b}\frac{t^2}{2}$.

Arc length of a Vector function.

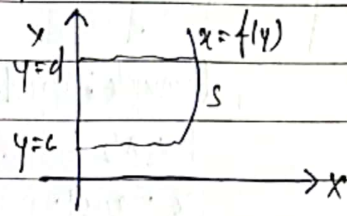
The arc length of the curve $y = f(x)$ (cartesian curve) from $x = a$ to $x = b$ is,



$$S = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

where, $x = a$ to $x = b$.

Moreover, If the curve is expressed in terms of y i.e. $x = f(y)$, then the arc length of the curve $y = c$ to $y = d$ is,



$$s = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

In Case of Parametric curve,
i.e. $x = f(t)$ & $y = g(t)$

We use $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ we also use, $\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

Hence, $\frac{ds}{dt} = \frac{ds}{dx} \cdot \frac{dx}{dt} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \times \frac{dx}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

$\therefore$ Arc length 's' of the curve $x = f(t)$ & $y = g(t)$ from $t = t_1$ to $t = t_2$ is

$$S = \int_{t=t_1}^{t=t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

If we extend this concept to vector function, $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$
where, $x = f(t)$, $y = g(t)$, $z = h(t)$.

$$\boxed{S = \int_{t=t_1}^{t=t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt} \quad \text{—}(*)$$

is the arc length of the curve. $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ from $t = t_1$ to $t = t_2$

Example: Find the arc length of the vector function: $\vec{r} = \cos t\,\vec{i} + \sin t\,\vec{j} + t\,\vec{k}$ from $t = 0$ to $t = 2\pi$

Soln: Here, $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ $\quad\therefore x = \cos t$, $y = \sin t$, $z = t$.

$\frac{dx}{dt} = -\sin t$, $\frac{dy}{dt} = \cos t$, $\frac{dz}{dt} = 1$.

The required arc length, $s = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$

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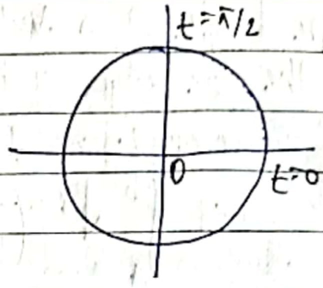
$$s = \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t + 1}\, dt$$
$$= \int_0^{2\pi} \sqrt{2}\, dt$$
$$= \sqrt{2}\,[t]_0^{2\pi}$$
$$= \sqrt{2}(2\pi - 0)$$
$$= 2\pi\sqrt{2}$$



Example: find the arc length of the curve $\vec{r} = a\cos t\,\vec{i} + a\sin t\,\vec{j}$ from $t=0$ to $t=\pi/2$

$x = a\cos t$

$y = a\sin t$

$\frac{2\pi r}{4} = \frac{\pi a}{2}$

* Note:

If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, where x, y, z are function of t

$\therefore \frac{d\vec{r}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$

$\therefore \left|\frac{d\vec{r}}{dt}\right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$

Hence, Above formula (*) of arc length, $s = \int_{t=t_1}^{t=t_2} \left|\frac{dr}{dt}\right| dt$.

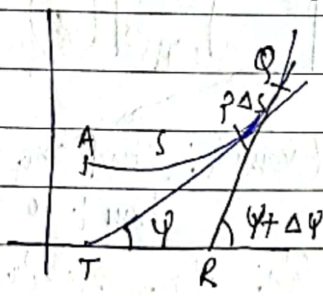
Curvature.

The curvature is simply means as the rate at which the curve curves.

Let P & Q be two points at the curve $y = f(x)$ such that,

$AP = s$ & $AQ = s + \Delta s$

so that $PQ = \Delta s$.



Draw tangents PT & QR from P & Q such that:

$\angle PTX = \psi$ & $\angle QRX = \psi + \Delta\psi$

$\therefore \Delta\psi$ is the change of angle of tangent when P moves to Q along the curve.

$\frac{\Delta s}{\Delta \psi}$ is called average curvature. Also $\frac{ds}{d\psi}$ is called curvature at point P.

Some Definition:

We had,

We know if $\vec{r} = \vec{f}(t)$ be a vector function of t. Then $\frac{d\vec{r}}{dt}$ always represents the vector along the tangent. Also unit vector along the tangent is denoted by T(t) & given by,

$$T(t) = \frac{d\vec{r}/dt}{|d\vec{r}/dt|} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

Curvature of Vector function (K)

The curvature of a vector function $\vec{r} = \vec{r}(t)$ is defined as the rate of change of unit vector T(t) (along the tangent) w.r.t. the arc length s. It is denoted by K (kappa). Thus,

$\boxed{K = \left|\frac{dT}{ds}\right|}$. Also it can be written as $\boxed{K = \left|\frac{dT/dt}{ds/dt}\right|}$

Example: find the curvature of circle of radius a.

OR

fact: Show that a circle is the curve of constant curvature.

Soln:

We know the vector function of circle of radius a is $\vec{r} = a\cos t\,\vec{i} + a\sin t\,\vec{j}$

Here $\frac{d\vec{r}}{dt} = -a\sin t\,\vec{i} + a\cos t\,\vec{j}$.

$$T(t) = \frac{d\vec{r}/dt}{|d\vec{r}/dt|} = \frac{-a\sin t\,\vec{i} + a\cos t\,\vec{j}}{\sqrt{a^2\sin^2 t + a^2\cos^2 t}} = -\sin t\,\vec{i} + \cos t\,\vec{j}.$$

$\therefore \frac{dT}{dt} = -\cos t\,\vec{i} - \sin t\,\vec{j}$.

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

* Note $(\vec{r} = x\vec{i} + y\vec{j})$

$$\boxed{\begin{aligned} r'(t) &= \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} \\ |r'(t)| &= \sqrt{(dx/dt)^2 + (dy/dt)^2} = \frac{ds}{dt} \end{aligned}}$$

Also, $\frac{ds}{dt} = |\vec{r}'(t)| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = a$.

$\therefore k = \left|\frac{dT/dt}{ds/dt}\right| = \left|\frac{-\cos t\,\vec{i} - \sin t\,\vec{j}}{a}\right| = \frac{1}{a}\sqrt{(\cos t)^2 + (-\sin t)^2} = \frac{1}{a}$

which is constant.

Alternative formula to find curvature.

If $\vec{r} = \vec{r}(t)$ be the curve, then the curvature is.

$$k(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$

Example: Find the curvature of the curve, $\vec{r}(t) = t^2\vec{i} + \ln t\,\vec{j} + t\ln t\,\vec{k}$ at $t=1$

Soln, $r'(t) = 2t\vec{i} + \frac{1}{t}\vec{j} + (\ln t + 1)\vec{k}$

$r''(t) = 2\vec{i} - \frac{1}{t^2}\vec{j} + \frac{1}{t}\vec{k}$.

$\therefore \vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t & 1/t & (\ln t + 1) \\ 2 & -1/t^2 & 1/t \end{vmatrix} = i\left(\frac{1}{t^2} + \frac{\ln t}{t^2} + \frac{1}{t^2}\right) + j\left[2(\ln t + 1) - 2\right] + \vec{k}\left(-\frac{2}{t} - \frac{2}{t}\right)$

$= \vec{i}\left(\frac{2}{t^2} + \frac{\ln t}{t^2}\right) + \vec{j}\,2\ln t - \vec{k}\,\frac{4}{t}$.

At $t=1$, $\vec{r}'(t) \times \vec{r}''(t) = 2\vec{i} + 0\vec{j} - 4\vec{k}$.

$\therefore |\vec{r}'(t) \times \vec{r}''(t)| = \sqrt{2^2 + 0^2 + (-4)^2} = \sqrt{20}$.

$r'(t) = 2\vec{i} + \vec{j} + \vec{k}$.

$\therefore r'(t) = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$.

$\therefore k(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|r'(t)|^3} = \frac{\sqrt{20}}{(\sqrt{6})^3}$

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Curvature of a Cartesian Curve $y = f(x)$

We know,

$$\vec{r} = x\vec{i} + y\vec{j}$$
$$= x\vec{i} + f(x)\vec{j}$$

$$K(x) = \frac{|r'(x) \times r''(x)|}{|r'(x)|^3}$$

$$\therefore \vec{r}'(x) = 1\vec{i} + f'(x)\vec{j}$$
& $$r''(x) = 0\vec{i} + f''(x)\vec{j}.$$

$$\therefore r'(x) \times r''(x) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & f'(x) & 0 \\ 0 & f''(x) & 0 \end{vmatrix} = 0\vec{i} + 0\vec{j} + f''(x)\vec{k}$$

$$\therefore |r'(x) \times r''(x)| = \sqrt{f''(x)^2} = |f''(x)|$$
& $$|r'(x)| = \sqrt{1^2 + [f'(x)]^2} = (1 + (f'(x))^2)^{1/2}$$

$$\therefore K(x) = \frac{|f''(x)|}{[1 + f'(x))^2]^{3/2}} = \frac{|y_2|}{(1 + y_1^2)^{3/2}}$$

Similarly, if the curve is expressed as $x = f(y)$, then

$$K(y) = \frac{|f''(y)|}{[(1 + (f'(y))^2]^{3/2}} = \frac{|x_2|}{(1 + x_1^2)^{3/2}}$$

Example: find the curvature of the curve $y^2 = 4ax$ at point $x = 1$.

Soln, $y^2 = 4ax$ $\quad \therefore y = \sqrt{4ax}.$

Diff. w.r.t. x.

$$y_1 = \frac{dy}{dx} = \sqrt{4a} \cdot \frac{1}{2} x^{\frac{1}{2}-1} = \frac{\sqrt{a}}{\sqrt{x}} \quad — (*)$$

$$\therefore \text{At } x = 1, \quad (y_1)_{x=1} = \frac{\sqrt{a}}{\sqrt{1}} = \sqrt{a}$$

Again Diff. (*)

$$y_2 = \sqrt{a} \frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) = \sqrt{a} \cdot \frac{-1}{2} x^{-\frac{1}{2}-1} = \sqrt{a}\left(-\frac{1}{2x^{3/2}}\right) = \frac{-\sqrt{a}}{2x\sqrt{x}}$$

$$\text{At } x = 1, \quad (y_2)_{x=1} = -\frac{\sqrt{a}}{2 \times 1 \times \sqrt{1}} = \frac{-\sqrt{a}}{2}$$

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$\therefore$ Required Curvature at point $x=1$ is,

$$k(1) = \frac{|y_2|}{(1+y_1^2)^{3/2}} = \frac{\sqrt{a}/2}{(1+a)^{3/2}}$$

Fact : The reciprocal of curvature is called radius of curvature and is denoted by ρ.

Thus, $\boxed{\rho = \frac{1}{k}}$

In above example, the radius of curvature of the parabola $y^2 = 4ax$ at $x=1$ is,

$$\rho = \frac{1}{k} = \frac{2(1+a)^{3/2}}{\sqrt{a}}$$

Formula: If $x = f(t)$ & $y = g(t)$ be the parametric curve, then

$$\vec{r} = x\vec{i} + y\vec{j}$$
$$= f(t)\vec{i} + g(t)\vec{j}$$

In this case,

$$y_1 = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\dot{y}}{\dot{x}} \quad (\text{function of } t)$$

Again, $y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx} = \frac{d}{dt}\left(\frac{\dot{y}}{\dot{x}}\right) \times \frac{1}{\dot{x}}$

$$= \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x})^2} \times \frac{1}{\dot{x}}$$ Quotient Rule

$$y_2 = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x})^3}$$

$$\therefore \rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{\left[1 + (\dot{y}/\dot{x})^2\right]^{3/2}}{\frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x})^2}}$$

$$\rho(t) = \frac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}$$ Also, $k(t) = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}$

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Q. Find the curvature & radius of curvature of the ellipse, $x = a\cos\theta$, $y = b\sin\theta$ at point $\theta = \frac{\pi}{3}$ OR $\vec{r} = a\cos\theta\,\vec{i} + b\sin\theta\,\vec{j}$.

Soln;

$\Rightarrow x = a\cos\theta,\ y = b\sin\theta$.

$$\dot{x} = \frac{dx}{d\theta} = a(-\sin\theta) = -a\sin\theta.$$

$$\ddot{x} = -a\cos\theta$$

$$\dot{y} = b\cos\theta$$

$$\ddot{y} = -b\sin\theta$$

$$\therefore\ k(t) = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{3/2}} = \frac{-a\sin\theta(-b\sin\theta) - b\cos\theta \times (-a\cos\theta)}{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}$$

$$\therefore\ k(\theta) = \frac{ab}{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}.$$

At $\theta = \pi/3$

$$k(\pi/3) = \frac{ab}{\left(a^2\left(\frac{\sqrt{3}}{2}\right)^2 + b^2\left(\frac{1}{2}\right)^2\right)^{3/3}} = \frac{8ab}{(3a^2 + b^2)^{3/2}}$$

Normal & Binormal Vector.

Fact:

If $\vec{r} = \vec{r}(t)$ is a constant vector, then $\vec{r}\cdot\vec{r}'(t) = 0$.

Example: let $\vec{r} = a\cos t\,\vec{i} + a\sin t\,\vec{j}$ be a circle so that.

$$|\vec{r}| = \sqrt{a^2\cos^2 t + a^2\sin^2 t} = \sqrt{a^2} = a \qquad \therefore\ \vec{r} \text{ is constant}$$

Then,

$$\vec{r}' = \frac{d\vec{r}}{dt} = -a\sin t\,\vec{i} + a\cos t\,\vec{j} = 0\vec{i} + a\vec{j} = (0, a).$$

$$\therefore\ \vec{r}\cdot\vec{r}' = -a^2\sin t\cos t + a^2\sin t\cos t = 0.$$

This shows that $\vec{r}$ is perpendicular to its derivative $\vec{r}'(t)$.

Recall:

For a vector function $\vec{r} = \vec{r}(t)$, we have the unit tangent vector

$$T(t) = \frac{\vec{r}\,'(t)}{|r'(t)|} \quad — (x)$$

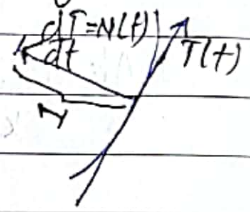
There are infinitely many vectors produced by the vector (x), But they are unit length 1.

Dfn: The above fact can be written by replacing $\vec{r}\,'(t)$ by $T(t)$, then we have

$$T(t) \cdot T'(t) = 0$$

This shows that $T(t)$ is $\perp r$ to $\frac{d\vec{T}}{dt}$ or $\vec{T}\,'(t)$.

The principal normal vector is denoted by $N(t)$ and is defined by



$$N(t) = \frac{T'(t)}{|T'(t)|}$$

Also, $|\vec{N}(t)| = 1$