Curvature of a curve at a point

1. Formula for Curvature and radius of Curvature of Cartesian Curves

Formula-1. For the Cartesian equation $y = f(x)$:

Radius of curvature at any point (x, y), $\rho = \dfrac{(1 + y_1^2)^{3/2}}{y_2}$, where $y_1 = \dfrac{dy}{dx}$ and $y_2 = \dfrac{d^2y}{dx^2} \neq 0$.

But if the equation of the curve be $x = f(y)$, then

Radius of curvature at any point (x, y) , $\rho = \dfrac{(1 + x_1^2)^{3/2}}{x_2}$, where $x_1 = \dfrac{dx}{dy}$ and $x_2 = \dfrac{d^2x}{dy^2} \neq 0$.

Formula-2. The curvature(κ) = $\dfrac{1}{\textbf{radius of curvature}} = \dfrac{1}{\rho}$.

Examples

1. Show that the circle $x^2 + y^2 = a^2$ is a curve of uniform curvature κ and its radius of curvature ρ at every point is constant.

Solution:

Consider a circle $x^2 + y^2 = a^2$ ...(i)

Differentiating w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} = 0 \qquad \therefore \frac{dy}{dx} = -\frac{x}{y}.$$

Again, differentiating w.r.t.x, we get

$$\frac{d^2y}{dx^2} = \frac{x\dfrac{dy}{dx} - y}{y^2} = \frac{x.\left(-\dfrac{x}{y}\right) - y}{y^2} = \frac{-(x^2 + y^2)}{y^3} = \frac{-a^2}{y^3}.$$

If (x, y) be any point on the circle, then,

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2} = \frac{\left(1 + \dfrac{x^2}{y^2}\right)^{3/2}}{\dfrac{-a^2}{y^3}} = -\frac{(x^2 + y^2)^{3/2}}{y^3} \times \frac{y^3}{a^2} = -\frac{(a^2)^{3/2}}{a^2} = -a. \qquad \therefore \rho = a, \text{ numerically}.$$

Hence, curvature (κ) = $\dfrac{1}{\rho} = \dfrac{1}{a}$, which is constant.

2. Find the radius of curvature ρ at any point (x, y) on the parabola $y^2 = 4ax$

Solution:

Here, $y^2 = 4ax$ ....(i)

Differentiating both sides w.r.t. x, we get

$$2yy_1 = 4a \qquad \therefore y_1 = \frac{2a}{y} \qquad ......(ii)$$

Again differentiating w.r.t.x,

$$y_2 = \frac{-2a}{y^2}.y_1 = \frac{-2a}{y^2}.\frac{2a}{y} = -\frac{4a^2}{y^3} \quad [\because \text{ From (ii)}]$$

Applying the formula

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2} = \frac{\left[1 + \left(\dfrac{2a}{y}\right)^2\right]^{3/2}}{\dfrac{-4a^2}{y^3}} = -\frac{(y^2 + 4a^2)^{3/2}}{4a^2}$$

But $y^2 = 4ax$, then

$$\therefore \rho = -\frac{(4ax + 4a^2)^{3/2}}{4a^2} = -(4a)^{3/2}\frac{(x + a)^{3/2}}{4a^2} = \frac{-8a^{3/2}(x + a)^{3/2}}{4a^2} = -\frac{2(x + a)^{3/2}}{\sqrt{a}} = \frac{2(x + a)^{3/2}}{\sqrt{a}} \text{ (numerically)}.$$

3. Find the radius of curvature ρ and curvature (κ) at any point (x, y) on the catenary $y = c \cosh (x/c)$

Solution:

Here, $y = c \cosh (x/c)$ ....(i)

Differentiating (i) w.r.t. x, we get

$\frac{dy}{dx} = c. \sinh \frac{x}{c} . \frac{1}{c}$ $\therefore$ $y_1 = \sinh \frac{x}{c}$

Again differentiating, we get $y_2 = \frac{1}{c} \cosh \frac{x}{c}$.

$$\text{Now, } \rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{\left(1 + \sinh^2 \frac{x}{c}\right)^{3/2}}{\frac{1}{c}\cosh \frac{x}{c}} = \frac{c\left(\cosh^2 \frac{x}{c}\right)^{3/2}}{\cosh \frac{x}{c}} = c\,\frac{\cosh^3 \frac{x}{c}}{\cosh \frac{x}{c}} = c \cosh^2 \frac{x}{c} = c . \frac{y^2}{c^2} = \frac{y^2}{c^2} \quad [\text{ By (i)}]$$

Hence, curvature (κ) $= \frac{1}{\rho} = \frac{c^2}{y^2}$, which is constant

4. Find the radius of curvature ρ and curvature (κ) at any point (x, y) on the rectangular hyperbola $xy = c^2$

Solution:

Here, $y = \frac{c^2}{x}$. ....(i)

Differentiating (i) w.r.t. x, we get

$y_1 = \frac{-c^2}{x^2}$ and $y_2 = -c^2 . (-2x^{-3}) = \frac{2c^2}{x^3}$.

Applying the formula for radius of curvature in Cartesian form i.e.

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{\left(1 + \frac{c^4}{x^4}\right)^{3/2}}{\frac{2c^2}{x^3}} = \frac{(x^4 + c^4)^{3/2}}{2c^2x^3}$$

But $c^2 = xy$ and therefore, $c^4 = x^2y^2$

$$\therefore \quad \rho = \frac{(x^4 + x^2y^2)^{3/2}}{2c^2x^3} = \frac{x^3(x^2+y^2)^{3/2}}{2c^2x^3} = \frac{(x^2+y^2)^{3/2}}{2c^2}.$$

Hence, curvature (κ) $= \frac{1}{\rho} = \frac{2c^2}{(x^2+y^2)^{3/2}}$

3. Show that the radius of curvature ρ at any point (x, y) on the curve $y = a \log \sec \frac{x}{a}$ is of constant length.

Solution:

Here, $y = a \log \sec \frac{x}{a}$ ....(i)

Differentiating successively (i) w.r.t x, we get

$y_1 = a. \frac{1}{\sec \frac{x}{a}} .\sec \frac{x}{a} .\tan \frac{x}{a} . \frac{1}{a} = \tan \frac{x}{a}$ and $y_2 = \frac{1}{a} \sec^2 \frac{x}{a}$.

Applying the formula for radius of curvature in Cartesian form i.e.

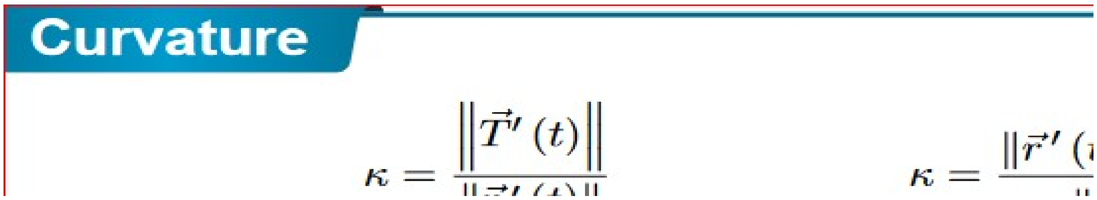
$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{\left(1 + \tan^2 \frac{x}{a}\right)^{3/2}}{\frac{1}{a}\sec^2 \frac{x}{a}} = \frac{\left(\sec^2 \frac{x}{a}\right)^{3/2}}{\frac{1}{a}\sec^2 \frac{x}{a}} = a \sec \frac{x}{a}.$$

4. Find the curvature of the curve $y = 2x^4$ at point $x = 2$.

5. Find the curvature of the curve $y = 3x^4$ at point $x = 1$.

Curvature in Parametric Curve for Vector Function:

1. Find the curvature of the vector function $\vec{r}(t) = p\cos t\,\vec{i} + p\sin t\,\vec{j}$, where p is constant.
Hint:



2. Determine the curvature for $\vec{r}(t) = (t, 3\sin(t), 3\cos(t))$.

3. Find the normal vector and binormal vector of the space curve $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ where $x = t^2, y = t^2, z = t^3$ at point (1, 1, 1).

4. Find the normal vector of the space curve

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ where $x = t^2, y = t^2, z = t^3$ at point (1, 0, 1).

2.Formula for Curvature and radius of Curvature of Parametric Curves

For the parametric equations $x = \phi(t), y = \Psi(t)$

The radius of curvature ρ is given by $\rho = \dfrac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''}$, where $x'y'' - y'x'' \neq 0$ and

Where, $x' = \dfrac{dx}{dt}$ and $y' = \dfrac{dy}{dt}$ etc.

Examples

1. Find the radius of curvature at any point θ for the parametric curve (Circle) : $x = a\cos\theta, y = a\sin\theta$

Solution:

Here, $x = a\cos\theta$ and $y = a\sin\theta$

Differentiating both sides w.r.t. θ, we get

$x' = -a\sin\theta$ and $y' = a\cos\theta$

Again differentiating, we get

$x'' = -a\cos\theta$ and $y'' = -a\sin\theta$.

Applying the formula of radius of curvature in parametric form i.e.

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''} = \frac{(a^2\sin^2\theta + a^2\cos^2\theta)^{3/2}}{(-a\sin\theta)\,.\,(-a\sin\theta) - (a\cos\theta)(-a\cos\theta)} = \frac{(a^2)^{3/2}\,.\,(1)^{3/2}}{a^2(\sin^2\theta + \cos^2\theta)} = \frac{a^3}{a^2} = a.$$

$\therefore \quad \rho = a$ Ans.

2. Find the radius of curvature at any point ϕ for the parametric curve (Ellipse): $x = a\cos\phi, y = b\sin\phi$.

Solution:

Here, $x = a\cos\phi$ and $y = b\sin\phi$

Differentiating both sides w.r.t. ϕ, we get

$x' = -a\sin\phi$ and $y' = b\cos\phi$

Again differentiating, we get

$x'' = -a\cos\phi$ and $y'' = -b\sin\phi$

Applying the formula for radius of curvature in parametric form

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - x''y'} = \frac{(a^2 \sin^2 \phi + b^2 \cos^2 \phi)^{3/2}}{(-a \sin \phi)(-b \sin \phi) - (-a \cos \phi) \cdot b \cos \phi}$$

$$= \frac{(a^2 \sin^2 \phi + b^2 \cos^2 \phi)^{3/2}}{ab(\sin^2 \phi + \cos^2 \phi)} = \frac{(a^2 \sin^2 \phi + b^2 \cos^2 \phi)^{3/2}}{ab}.$$

3. Prove that the radius of curvature at any point $\theta = 0$ for the parametric curve (Cycloid): $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ is $\rho = 4a$.

Solution:

Here, $x = a(\theta + \sin\theta)$ and $y = a(1 - \cos\theta)$

Differentiating w.r.t. θ, we get

$x' = a(1 + \cos\theta)$ and $y' = a \sin\theta$

Again, differentiating w.r.t. θ,

$x'' = -a \sin\theta$ and $y'' = a \cos\theta$

At $\theta = 0$: $x' = a(1 + 1) = 2a$ and $y' = a \sin 0 = 0$

$x'' = -a \sin 0 = 0$ and $y'' = a \cos 0 = a$.

Applying the formula

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - x''y'} = \frac{[(2a)^2 + 0^2]^{3/2}}{2a \cdot a - 0 \cdot 0} = \frac{(4a^2)^{3/2}}{2a^2} = \frac{(2a)^3}{2a^2} = 4a.$$ Hence proved.

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