Unit 4: Change of variables in multiple integrals

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Summary

Review of the Idea of Substitution

Consider the integral

$$\int_0^2 x \cos x^2 \; dx.$$

To evaluate this integral we use the substitution $u = x^2$.
Then
As $x \to 0$, $u \to 0$ and
as $x \to 2$, $u \to 4$.
Hence

- The above substitution sends the interval $[0, 2]$ onto the interval $[0, 4]$.
- We see that there is stretching of the interval.

However,

The stretching is not uniform.

In fact,
As $x \to 0.5$, $u \to 00.25$ and
as $x \to 1$, $u \to 1$.
Hence

$$\begin{aligned} [0, 0.5] &\to [0, 0.25] && \text{contraction} \\ [0.5, 1] &\to [0.25, 1] && \text{stretch} \\ [1, 2] &\to [1, 4] && \text{stretch} \end{aligned}$$

Thus, the stretching is not uniform.

This is the reason why we need to find du.

We have

$$\frac{du}{dx} = 2x \quad \text{or} \quad \frac{dx}{du} = \frac{1}{2x}.$$

We know that

$$dx = \frac{dx}{du}\, du = \frac{1}{2x}\, du.$$

The expression $\frac{1}{2x}$ is the factor that needs to be multiplied in when we perform the substitution.

Notice for small positive values of x, this factor is greater than 1 and for large values of x, the factor is smaller than 1. This is how the stretching and contracting is accounted for.

Jacobians

We have seen that when we convert to polar coordinates, we use

$$dy\,dx = r\,dr\,d\theta.$$

With a geometrical argument, we showed why the "extra r" is included. Taking the analogy from the one variable case, the transformation to polar coordinates produces stretching and contracting. The "extra r" takes care of this stretching and contracting.

The goal for this section is to be able to find the "extra factor" for a more general transformation. We call this "extra factor" the *Jacobian of the transformation*. We can find it by taking the determinant of the 2×2 matrix of partial derivatives.

Jacobian

Let

$$x = g(u, v) \;\text{ and }\; y = h(u, v)$$

be a transformation of the plane. Then the Jacobian of this transformation is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Remark

A useful fact is that the Jacobian of the inverse transformation is the reciprocal of the Jacobian of the original transformation.

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}}.$$

This is a consequence of the fact that the determinant of the inverse of a matrix A is the reciprocal of the determinant of A.

Example

Find the Jacobian of the polar coordinates transformation

$$x(r, \theta) = r\cos\theta, \qquad y(r, \theta) = r\sin\theta.$$

Solution.

We have

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

Remark

This example confirms that

$$dydx = rdrd\theta$$

Integration and Coordinate Transformations

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$x = g(u,v), \qquad y = h(u,v)$$

be a transformation on the plane that is one to one from a region S to a region R. If g and h have continuous partial derivatives such that the Jacobian is never zero, then

$$\iint\limits_R f(x,y)\ dy\ dx = \iint\limits_S f(g(u,v), h(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \ dv\ du$$

Remark

Note that a small region $\Delta A = \Delta x \Delta y$ in the xy-plane is related to a small region in the uv-plane whose area is the product $\Delta u \Delta v$, that is,

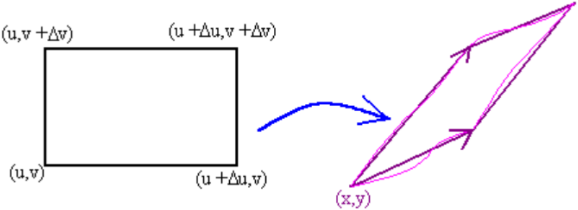
$$\Delta A \approx \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \Delta u \, \Delta v.$$

In the limiting case we have

$$dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,du\,dv.$$

The additional factor of $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$ tells us how area changes under the map T.

Idea of the Proof.
As usual, we cut S up into tiny rectangles so that the image under T of each rectangle is a parallelogram.



We need to find the area of the parallelogram.

Considering differentials, we have

$$T(u+\Delta u, v) \approx T(u,v) + (x_u \Delta u, y_u \Delta u)$$
$$T(u, v+\Delta v) \approx T(u,v) + (x_v \Delta v, y_v \Delta v)$$

Thus the two vectors that make the parallelogram are

$$P = g_u \Delta u \vec{i} + h_u \Delta u \vec{j}$$
$$Q = g_v \Delta v \vec{i} + h_v \Delta v \vec{j}$$

To find the area of this parallelogram we just cross the two vectors.

$$\begin{aligned} |P\times Q| = &\ \text{abs}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_u\Delta u & y_u\Delta u & 0 \\ x_v\Delta v & y_v\Delta v & 0 \end{vmatrix} \\ &= |(x_u y_v - x_v y_u)\Delta u\Delta v| \\ &= \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \Delta u\Delta. \end{aligned}$$

and the extra factor is revealed.

Example

Determining the image of a region under a transformation

A transformation is defined by the equations

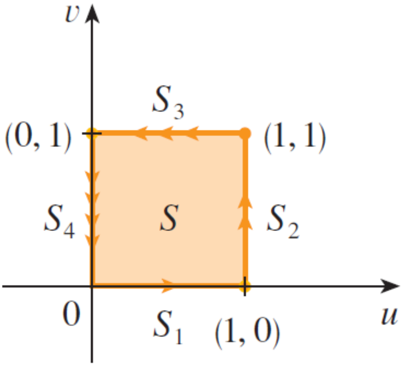
$$x = u^2 - v^2, \qquad y = 2uv.$$

Find the image of the square
$S = \{(u, v) :\ 0 \le u \le 1,\ 0 \le v \le 1\}$.

Solution. The transformation maps the boundary of S into the boundary of the image. So we begin by finding the images of the sides of S. The first side is given by

$$S_1 = \{(u, 0) : \ 0 \leq u \leq 1\}.$$

(See Figure 2.)



From the given equations we have

$$x = u^2, \qquad y = 0$$

and so $0 \leq x \leq 1$. Thus, S_1 is mapped into the line segment from $(0, 0)$ to (1,0) in the xy-plane.
The second side is

$$S_2 = \{(1, v) : \ 0 \leq v \leq 1\}$$

and, putting $u = 1$ in the given equations, we get

$$x = 1 - v^2, \qquad y = 2v$$

Eliminating v, we obtain

$$x = 1 - \frac{y^2}{4} \qquad 0 \leq x \leq 1, \tag{1}$$

which is part of a parabola. Similarly, S_3 is given by

$$S_3 = \{(u, 1) : \ 0 \leq u \leq 1\},$$

whose image is the parabolic arc

$$x = \frac{y^2}{4} - 1 \qquad -1 \leq x \leq 0, \tag{2}$$

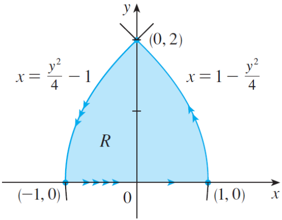
Finally, S_4 is given by

$$S_3 = \{(0, v) :\ 0 \leq v \leq 1\},$$

whose image is

$$x = -v^2, y = 0,$$

that is, $-1 \leq x \leq 0$.

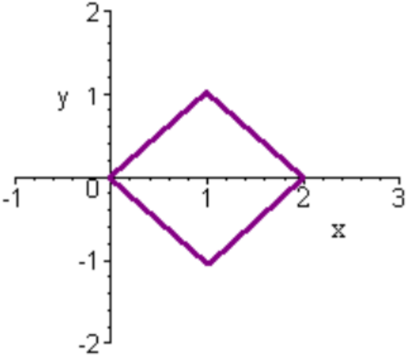


(Notice that as we move around the square in the counterclockwise direction, we also move around the parabolic region in the counterclockwise direction.) The image of S is the region (shown in Figure 2) bounded by the x-axis and the parabolas .

Use an appropriate change of variables to find the volume of the region below

$$z = (x - y)^2$$

above the x-axis, over the parallelogram with vertices $(0, 0), (1, 1), (2, 0)$, and $(1, -1)$.



Solution.
We find the equations of the four lines that make the parallelogram to be

$$y = x, \quad y = x - 2, \quad y = -x, \quad y = -x + 2,$$

that is,

$$x - y = 0, \quad x - y = 2, \quad x + y = 0, \quad x + y = 2$$

The region is given by

$$0 < x - y < 2 \text{ and } 0 < x + y < 2$$

This leads us to the inverse transformation

$$u(x,y) = x - y, \quad v(x,y) = x + y$$

The Jacobian of the inverse transformation is

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2.$$

Since the Jacobian is the reciprocal of the inverse Jacobian we get

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{2}.$$

The region is given by

$$0 < u < 2 \text{ and } 0 < v < 2$$

and the function is given by

$$z = u^2$$

Putting this all together, we get the double integral

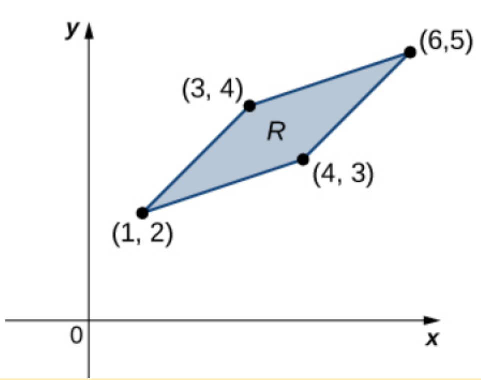
$$\int_0^2 \int_0^2 u^2 \frac{1}{2}\ du\ dv = \int_0^2 \left[\frac{u^3}{6}\right]_0^2\ dv$$
$$= \int_0^2 \frac{4}{3}\ dv = \frac{8}{3}.$$

Changing Variables

Consider the integral

$$\iint_R (x-y)dy\,dx,$$

where R is the parallelogram joining the points $(1,2)$, $(3,4)$, $(4,3)$, and $(6,5)$ (See the figure given below).

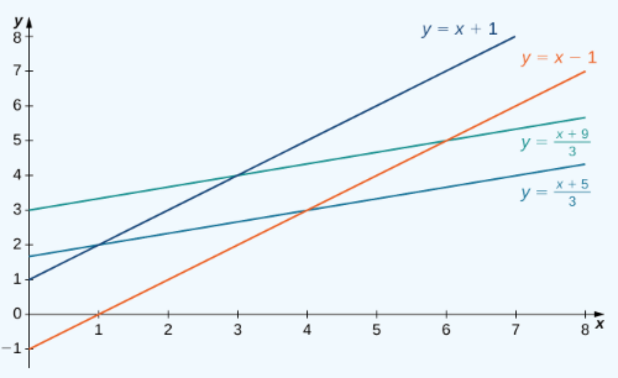


Make appropriate changes of variables, and write the resulting integral.

Solution First, we need to understand the region over which we are to integrate. The sides of the parallelogram are

$$x - y + 1,\ x - y - 1 = 0,\ x - 3y + 5 = 0 \text{ and } x - 3y + 9 = 0$$

See the figure.



Another way to look at them is

$$x - y = -1,\ x - y = 1,\ x - 3y = -5,\ \text{and } x - 3y = -9.$$

Clearly the parallelogram is bounded by the lines

$$y = x + 1,\ y = x - 1,\ y = \frac{1}{3}(x + 5),\ y = \frac{1}{3}(x + 9).$$

Notice that if we were to make $u = x - y$ and $v = x - 3y$, then the limits on the integral would be

$$-1 \leq u \leq 1 \text{ and } -9 \leq v \leq -5.$$

To solve for x and y, we multiply the first equation by 3 and subtract the second equation,

$$3u - v = (3x - 3y) - (x - 3y) = 2x.$$

Then we have

$$x = \frac{3u - v}{2}.$$

Moreover, if we simply subtract the second equation from the first, we get

$$u - v = (x - y) - (x - 3y) = 2y \text{ and } y = \frac{u - v}{2}.$$

Thus, we can choose the transformation

$$T(u, v) = \left(\frac{3u - v}{2}, \frac{u - v}{2}\right)$$

and compute the Jacobian J(u,v).

We have

$$J(u,v)=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{vmatrix}$$

$$=\begin{vmatrix}3/2 & -1/2\\ 1/2 & -1/2\end{vmatrix}=-\frac{3}{4}+\frac{1}{4}=-\frac{1}{2}$$

Therefore, $|J(u,v)|=\frac{1}{2}$.

Also, the original integrand becomes

$$x - y = \frac{1}{2}[3u - v - u + v] = \frac{1}{2}[3u - u] = \frac{1}{2}[2u] = u.$$

Therefore, by the use of the transformation T, the integral changes to

$$\begin{aligned}\iint_R (x-y)dy\,dx &= \int_{-9}^{-5}\int_{-1}^{1} J(u,v)u\,du\,dv\\ &= \int_{-9}^{-5}\int_{-1}^{1}\left(\frac{1}{2}\right)u\,du\,dv,\end{aligned}$$

which is much simpler to compute.

Jacobians and Triple Integrals

For transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$, we define the Jacobian in a similar way

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}.$$

Evaluating a Triple Integral with a Change of Variables
Evaluate the triple integral

$$\int_0^3 \int_0^4 \int_{y/2}^{(y/2)+1} \left(x + \frac{z}{3}\right) dx\, dy\, dz$$

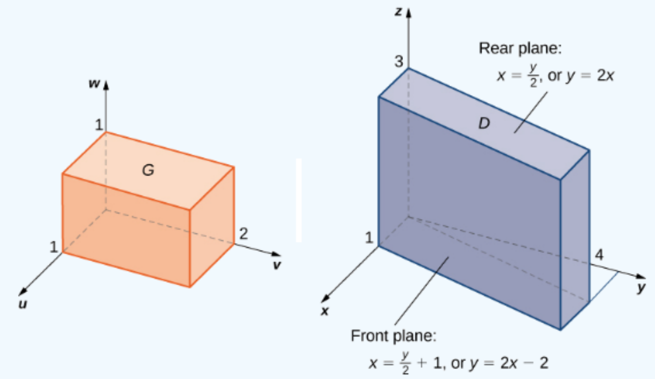
In xyz-space by using the transformation

$$u = (2x - y)/2,\ v = y/2, and w = z/3.$$

Then integrate over an appropriate region in uvw-space.

Solution.

As before, some kind of sketch of the region G in xyz-space over which we have to perform the integration can help identify the region D in uvw-space (see the figure PageIndex13).



Clearly G in xyz-space is bounded by the planes

$$x = y/2,\ x = (y/2) + 1,\ y = 0,\ y = 4,\ z = 0,\ \text{and } z = 4.$$

We also know that we have to use

$$u = (2x - y)/2,\ v = y/2,\ \text{and } w = z/3$$

for the transformations. We need to solve for x, y and z. Here we find that

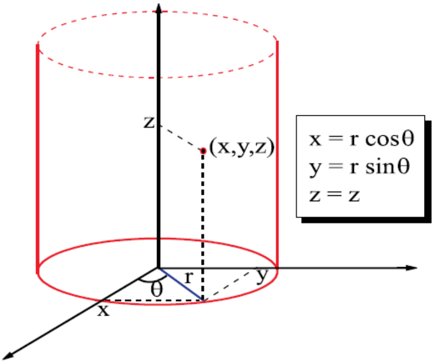
$$x = u + v,\ y = 2v,\ \text{and } z = 3w.$$

Cylindrical Coordinates

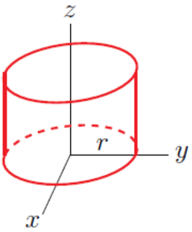
When we were working with double integrals, we saw that it was often easier to convert to polar coordinates. For triple integrals we have been introduced to three coordinate systems. The rectangular coordinate system (x, y, z) is the system that we are used to. The other two systems are cylindrical coordinates (r, θ, z) and spherical coordinates (r, θ, ϕ).

Cylindrical coordinates are denoted by r, θ and z, and are defined by

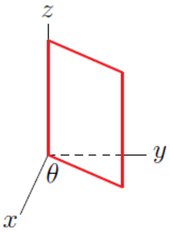
$$\begin{aligned} r = & \text{ the distance from } (x, y, 0) \text{ to } (0,0,0) \\ = & \text{ the distance from } (x, y, z) \text{ to the } z\text{-axis} \\ \theta = & \text{ the angle between the positive } x\text{-axis and} \\ & \quad \text{the line joining } (x, y, 0) \text{ to } (0,0,0) \\ z = & \text{ the signed distance from } (x, y, z) \text{ to the } xy\text{-plane} \end{aligned}$$



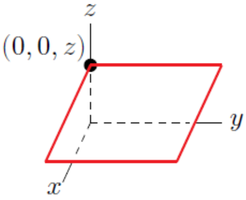
Here are sketches of surfaces of constant r, constant θ, and constant z.



surface of constant r
(a cylindrical shell)



surface of constant θ
(a plane)



surface of constant z
(a plane)

The Cartesian and cylindrical coordinates are related by Recall that cylindrical coordinates are most appropriate when the expression

$$x^2 + y^2$$

occurs. The construction is just an extension of polar coordinates.

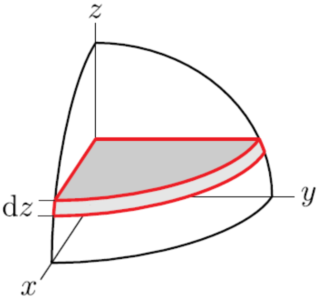
$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

We now establish that

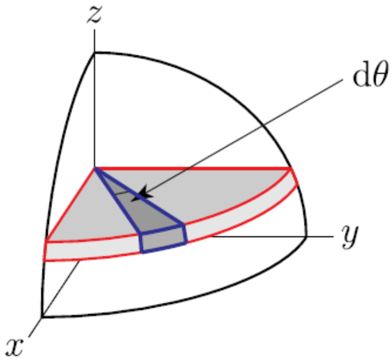
$$dV = r dr\, d\theta\, dz.$$

If we cut up a solid by

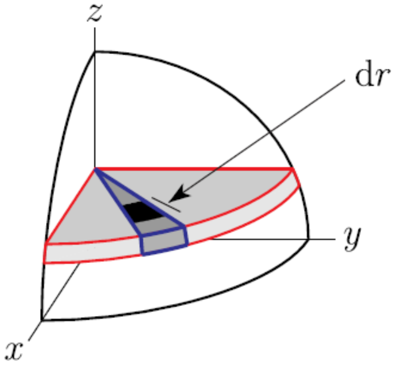
- first slicing it into horizontal plates of thickness dz by using planes of constant z,



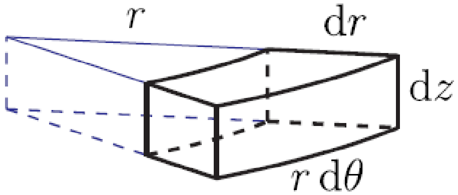
- and then subdividing the plates into wedges using surfaces of constant θ, say with the difference between successive θ's being $d\theta$,



- and then subdividing the wedges into approximate cubes using surfaces of constant r, say with the difference between successive r's being dr,



we end up with approximate cubes that look like



When we introduced slices using surfaces of

- constant r, the difference between the successive r's was dr, so the indicated edge of the cube has length dr.
- constant z, the difference between the successive z's was dz, so the vertical edges of the cube have length dz.
- constant θ the difference between the successive θ's was $d\theta$ so the remaining edges of the cube are circular arcs of radius essentially r that subtend an angle θ and so have length $rd\theta$.

Find the Jacobian for the cylindrical coordinate transformation

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z$$

Solution.
We compute the Jacobian

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} x_r & x_\theta & 0 \\ y_r & y_\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r.$$

Spherical Coordinates

This example indicates that in the case of the spherical coordinate transformation we have

$$dx\,dy\,dz = r\,dr\,d\theta\,dz.$$

This leads us to the following theorem:

Theorem (Integration With Cylindrical Coordinates)

Let $f(x, y, z)$ be a continuous function on a solid B. Then

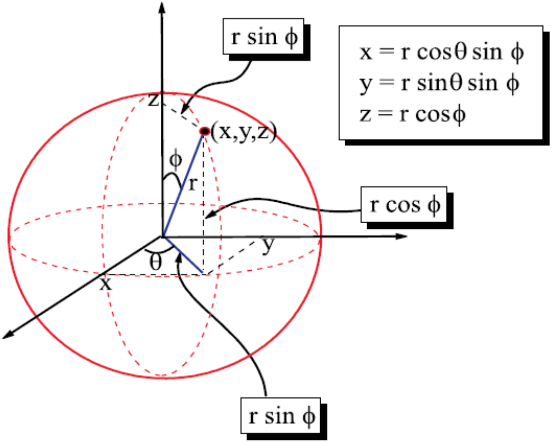
$$\iiint_B f(x, y, z)\ dz\,dy\,dx$$

$$= \iiint_B f(r\cos\theta, r\sin\theta, z)r\ dz\,dr\,d\theta.$$

Another coordinate system that often comes into use is the spherical coordinate system.
Spherical coordinates are denoted by r, θ and ϕ, and are defined by

$$\begin{aligned} r =\ & \text{the distance from } (0,0,0) \text{ to } (x,y,z) \\ \theta =\ & \text{the angle between the positive } x\text{-axis and} \\ & \quad \text{the line joining } (x,y,0) \text{ to } (0,0,0) \\ \phi =\ & \text{the angle between the positive } z\text{-axis and} \\ & \quad \text{the line joining } (x,y,z) \text{ to } (0,0,0) \end{aligned}$$

where $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$.


r sin ϕ
x = r cosθ sin ϕ
y = r sinθ sin ϕ
z = r cosϕ
z
(x,y,z)
ϕ
r
r cos ϕ
y
θ
x
r sin ϕ

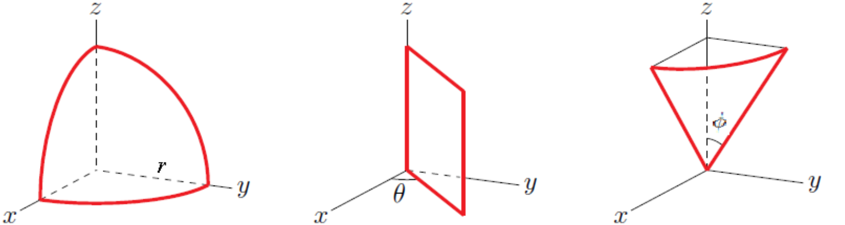
The spherical coordinate θ is the same as the cylindrical coordinate θ. The spherical coordinate ϕ is new. It runs from 0 (on the positive z-axis) to π (on the negative z-axis). The Cartesian and spherical coordinates are related by

$$x = r\cos\theta\sin\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\phi,$$

$$r^2 = x^2 + y^2 + z^2, \qquad \theta = \arctan\frac{y}{x}, \qquad \phi = \arctan\frac{\sqrt{x^2+y^2}}{z}.$$

arctan= tan^-1()

Here are sketches of surfaces of constant r, constant θ, and constant ϕ.



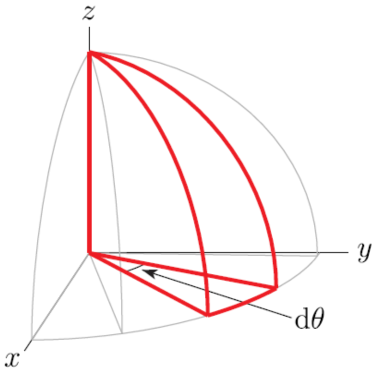
surface of constant r (a sphere)

surface of constant θ (a plane)

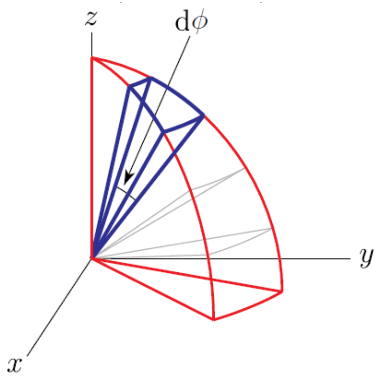
surface of constant ϕ (a cone)

If we cut up a solid by

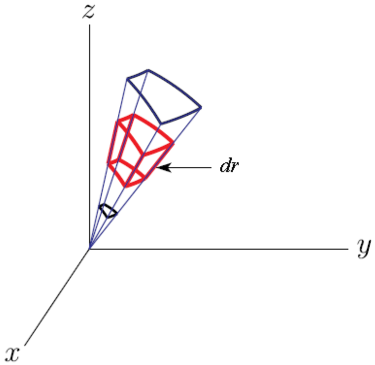
- first slicing it into segments (like segments of an orange) by using planes of constant θ, say with the difference between successive θ's being $d\theta$,



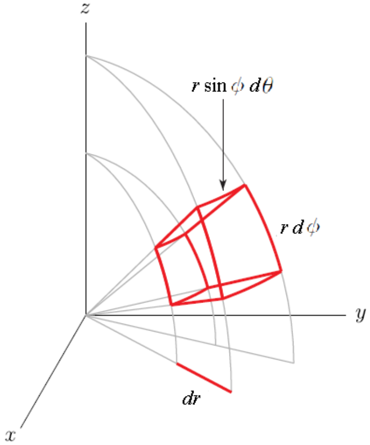
- and then subdividing the segments into "searchlights" (like the searchlight outlined in blue in the figure below) using surfaces of constant ϕ, say with the difference between successive ϕ's being$d\phi$,



- and then subdividing the searchlights into approximate cubes using surfaces of constant r, say with the difference between successive r's beingdr,



We end up with approximate cubes that look like the red one in the figure given above.



Example

Find the Jacobian for the spherical coordinate transformation

$$x = r\cos\theta\sin\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\phi,$$

where $0 \le \theta \le 2$ and $0 \le \phi \le \pi$ and $r^2 = x^2 + y^2 + z^2$.

Solution.

We compute the Jacobian

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \cos\theta\sin\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\phi & 0 & -r\sin\phi \end{vmatrix}$$

$$= r^2\sin\phi.$$