Unit 4: Triple Integrals

Prof.Dr.P.M.Bajracharya

School of Mathematical Sciences
T.U., Kirtipur

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Summary

Triple integrals

We have seen that the geometry of a double integral involves

- cutting the two dimensional region into tiny rectangles,
- multiplying the areas of the rectangles by the value of the function there,
- adding the areas up, and
- taking a limit as the size of the rectangles approaches zero.

We have also seen that this is equivalent to finding the double iterated integral.

We will now take this idea to the next dimension.

- Instead of a region in the xy-plane, we will consider a solid in xyz-space.
- Instead of cutting up the region into rectangles, we will cut up the solid into rectangular solids.
- And instead of multiplying the function value by the area of the rectangle, we will multiply the function value by the volume of the rectangular solid.

- We define the triple integral as the limit of the sum of the product of the function times the volume of the rectangular solids.
- Instead of the double integral being equivalent to the double iterated integral, the triple integral is equivalent to the triple iterated integral.

We can define a rectangular box B in $\mathbb{R}^3$ as

$$B = \{(x, y, z) |\ a \leq x \leq b, c \leq y \leq d, p \leq z \leq q\}.$$

We follow a similar procedure to what we did in Double Integrals over Rectangular Regions.

We divide

- the interval $[a, b]$ into l subintervals $[x_{i-1}, x_i]$ of equal length
$$\Delta x = \frac{b-a}{l},$$
- the interval $[c, d]$ into m subintervals $[y_{i-1}, y_i]$ of equal length
$$\Delta y = \frac{d-c}{m},$$
- the interval $[p, q]$ into n subintervals $[z_{i-1}, z_i]$ of equal length
$$\Delta z = \frac{q-p}{n}.$$

Then the rectangular box B is subdivided into lmn subboxes

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{i-1}, y_i] \times [z_{i-1}, z_i],$$

as shown in the figure given below.

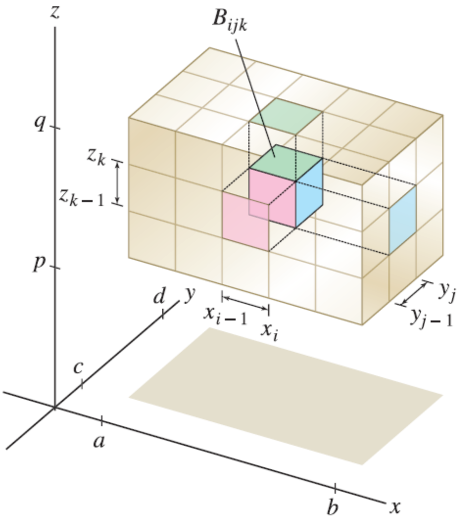


Figure 1: The box $B = [a, b] \times [c, d] \times [p, q]$ decomposed into smaller boxes B_{ijk}.

For each i, j, and k, consider a sample point $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ in each sub-box B_{ijk}. We see that its volume is

$$\Delta V = \Delta x \Delta y \Delta z.$$

We define the triple integral in terms of the limit of a triple Riemann sum, as we did for the double integral in terms of a double Riemann sum.

Triple Integral

Let $f(x, y, z)$ be a continuous function of three variables defined over a solid B. Then the triple integral over B is defined as

$$\iiint\limits_B f(x, y, z)\, dx\, dy\, dz$$

$$= \lim_{i,j,k\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \Delta x \Delta y \Delta z.$$

Just as for double integrals, the practical method for evaluating triple integrals is to express them as iterated integrals as follows.

Fubini's Theorem for Triple Integrals

If f is continuous on a rectangular box

$$B = [a, b] \times [c, d] \times [p, q],$$

then

$$\iiint\limits_B f(x, y, z)\; dx\, dy\, dz = \int_a^b \int_c^d \int_p^q f(x, y, z)\; dx\, dy\, dz.$$

Remark

We compute triple integrals using Fubini's Theorem rather than using the Riemann sum definition. We follow the order of integration in the same way as we did for double integrals (that is, from inside to outside).

Example

Evaluate the triple integral $\iiint_B x^2yz\ dV$, over the box

$$B = \{(x,y,z) :\ -2 \le x \le 1, 0 \le y \le 3, 1 \le z \le 5\}.$$

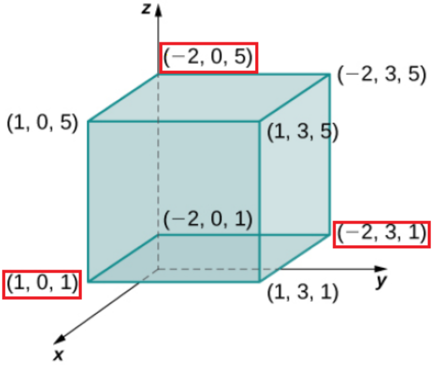


Figure 2: The domain of integration.

Triple integral over a general bounded region

We now expand the definition of the triple integral to compute a triple integral over a more general bounded region E in $\mathbb{R}^3$. We consider three types of region E.

Type I of region E

Type I of region E is a region between the graphs of two continuous functions $u_1(x, y)$ and $u_2(x, y)$ on $D \subseteq \mathbb{R}^2$. That is,

$$E = \{(x, y, z) : \ (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}.$$

Here, D is the projection of E onto the xy-plane.

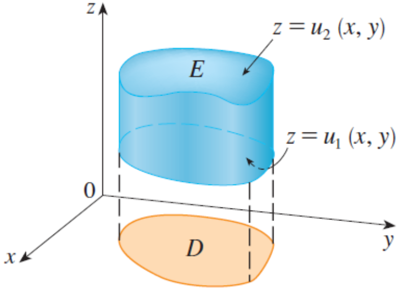


Figure 3: A type I solid region E

Type II of region E

Type II of region E is a region between the graphs of two continuous functions $u_1(y,z)$ and $u_2(y,z)$ on $D \subseteq \mathbb{R}^2$. That is,

$$E = \{(x,y,z) : \ (y,z) \in D, u_1(y,z) \leq x \leq u_2(y,z)\}.$$

Here, D is the projection of E onto the yz-plane.

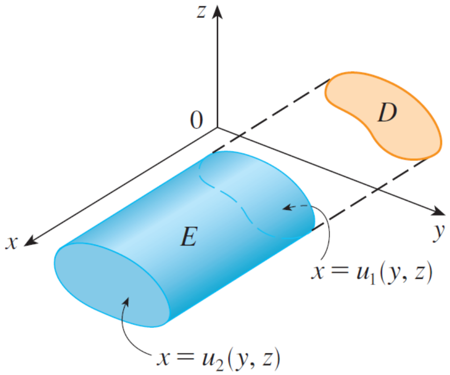


Figure 4: A type II solid region E

Type III of region E

Type III of region E is a region between the graphs of two continuous functions $u_1(x,z)$ and $u_2(x,z)$ on $D \subseteq \mathbb{R}^2$. That is,

$$E = \{(x,y,z) : \ (x,z) \in D, u_1(x,z) \leq y \leq u_2(x,z)\}.$$

Here, D is the projection of E onto the xz-plane.

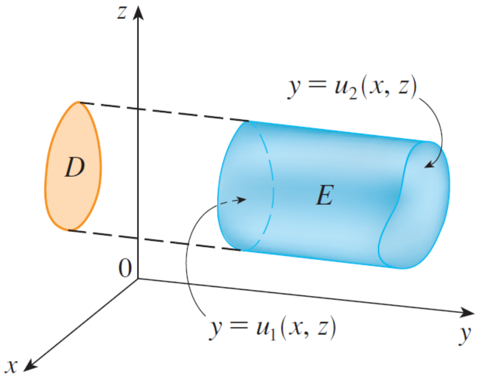


Figure 5: A type III solid region E

Note that the plane region D in any of the three types of region E may be of Type I or Type II as described in **Double Integrals over General Regions**.
If D in the xy-plane is of Type I (Figure 15.4.5), then

$$E = \{(x, y, z)|\ a \leq x \leq b,\\ g_1(x) \leq y \leq g_2(x), u_1(x, y) \leq z \leq u_2(x, y)\}.$$

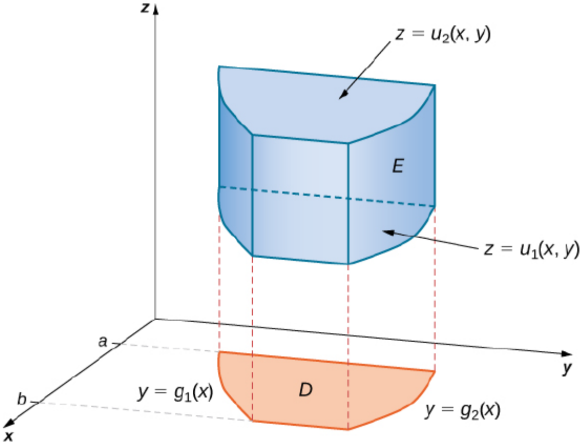


Figure 6: A box E with the projection D in the xy-plane is of Type I

Then the triple integral is equal to the triple iterated integral.

$$\iiint_E f(x, y, z)\ dxdydz$$

$$= \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\ dz\, dy\, dx.$$

Example

Evaluate

$$\iiint\limits_E f(x,y,z)\ dzdydx,$$

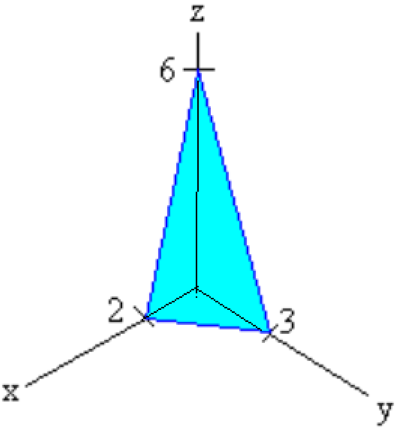
where

$$f(x,y,z) = 1 - x$$

and E is the solid that lies in the first octant and below the plain

$$3x + 2y + z = 6.$$

Solution The picture of the region E is



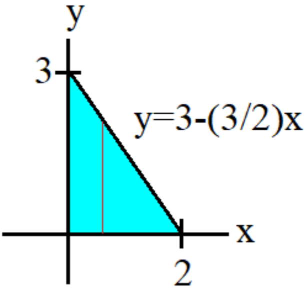
The challenge here is to find the limits. We work on the innermost limit first which corresponds with the variable "z". Think of standing vertically. Your feet will rest on the lower limit and your head will touch the higher limit. The lower limit is the xy-plane or

$$z = 0.$$

The upper limit is the given plane. Solving for z, we get

$$z = 6 - 3x - 2y.$$

Now we work on the middle limits that correspond to the variable "y". We look at the projection of the surface in the xy-plane. It is shown below.



Now we find the limits just as we found the limits of double integrals. The lower limit is just

$$y = 0.$$

If we set $z = 0$ and solve for y, we get for the upper limit

$$y = 3 - (3/2)x.$$

Next we find the outer limits, corresponding to the variable "x". The lowest x gets is 0 and highest x gets is 2. Hence we can describe the solid region tetrahedron E as

$$\begin{aligned} E = \{(x, y, z)|\ & 0 \leq x \leq 2, \\ & 0 \leq y \leq 3 - (3/2)x, 0 \leq z \leq 6 - 3x - 2y\}. \end{aligned}$$

The integral is thus

$$\int_0^2 \int_0^{3-3x/2} \int_0^{6-3x-2y} (1-x)\ dzdydx$$

$$= \int_0^2 \int_0^{3-3x/2} \left[z - xz\right]_{z=0}^{6-3x-2y}\ dydx$$

$$= \int_0^2 \int_0^{3-3x/2} [(6-3x-2y)-(6x-3x^2-2xy)]\ dydx$$

$$= \int_0^2 \int_0^{3-3x/2} (6-9x-2y+3x^2+2xy)\ dydx$$

$$= \int_0^2 \left[6y-9xy-y^2+3x^2y+xy^2\right]_{y=0}^{3-3x/2}\ dx$$

$$= \int_0^2 (9-18x+(45/4)x^2-(9/4)x^3)\ dx$$

$$= \left[9x-9x^2+(15/4)x^3-(9/16)x^4\right]_0^2 = 3.$$

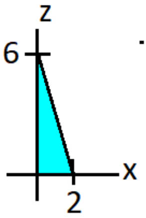
Example

Switch the order of integration from the previous example so that $dy\,dx\,dz$ appears.

Solution. This time we work on the "y" variable first. The lower limit for the y-variable is 0. For the upper limit, we solve for y in the plane to get

$$y = 3 - 3/2x - 1/2z$$

To find the "x" limits, we project onto the xz-plane as shown below



The lower limit for x is 0. To find the upper limit we set $y = 0$ and solve for x to get

$$x = 2 - (1/3)z$$

Finally, to get the limits for z, we see that the smallest z will get is 0 and the largest z will get is 6. We get

$$0 < z < 6$$

We can write

$$\int_0^6 \int_0^{2-z/3} \int_0^{6-3x/2-z=/3} (1-x)\; dy\, dx\, dz.$$

Example

Evaluate

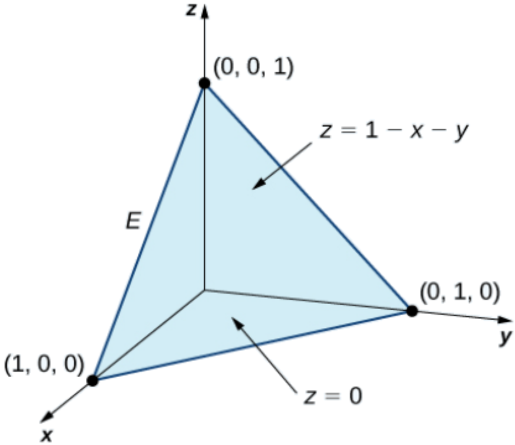
$$\iiint\limits_E (5x - 3y)\ dzdydx,$$

where E is the solid tetrahedron bounded by the planes

$$x = 0,\ y = 0,\ z = 0,\ x + y + z = 1.$$

Solution.

The picture of the region E is



The challenge here is to find the limits.

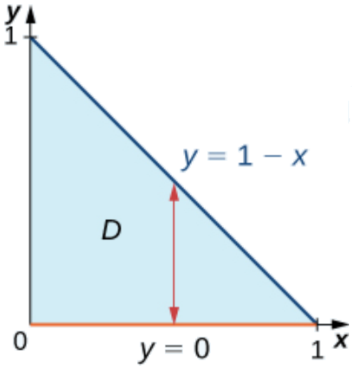
Solution.
We work on the innermost limit first which corresponds with the variable "z". Think of standing vertically. Your feet will rest on the lower limit and your head will touch the higher limit. The lower limit is the xy-plane or

$$z = 0.$$

The upper limit is the given plane. Solving for z, we get

$$z = 1 - x - y.$$

Now we work on the middle limits that correspond to the variable "y". We look at the projection of the surface in the xy-plane. It is shown below.



Now we find the limits just as we found the limits of double integrals. The lower limit is just

$$y = 0.$$

If we set $z = 0$ and solve for y, we get for the upper limit

$$y = 1 - x.$$

Next we find the outer limits, corresponding to the variable "x". The lowest x gets is 0 and highest x gets is 2. Hence we can describe the solid region tetrahedron E as

$$\begin{aligned} E = \{(x, y, z)|\ 0 \leq x \leq 1, \\ 0 \leq y \leq 1 - x, 0 \leq z \leq 1 - x - y\}. \end{aligned}$$

The integral is thus

$$\begin{aligned}
&\int_0^1 \int_0^{1-x} \int_0^{1-x-y} (5x-3y)\ dzdydx \\
&= \int_0^1 \int_0^{1-x} \left[5xz - 3yz\right]_{z=0}^{1-x-y}\ dydx \\
&= \int_0^1 \int_0^{1-x} \left[(5x-5x^2-5xy)-(3y-3xy-3y^2)\right]\ dydx \\
&= \int_0^1 \int_0^{1-x} (5x-3y-5x^2+3y^2-2xy)\ dydx
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 \left[5xy - (3/2)y^2 - 5x^2y + y^3 - xy^2\right]_{y=0}^{1-x} \, dx \\
&= \int_0^1 [5x(1-x) - (3/2)(1-x)^2 - 5x^2(1-x) \\
&\qquad + (1-x)^3 - x(1-x)^2] \, dx \\
&= \int_0^1 (2 + 7x - 6x^2 + 3x^3) \, dx \\
&= \left[2x + (7/2)x^2 - 2x^3 + (3/4)x^4\right]_0^1 \\
&= 2 + (7/2) - 2 + (3/4) \\
&= 17/4.
\end{aligned}
$$

Applications of Triple Integrals

Volume

Let's begin with the special case where $f(x, y, z) = 1$ for all points in E. Then the triple integral does represent the volume of E :

The triple integral as the volume

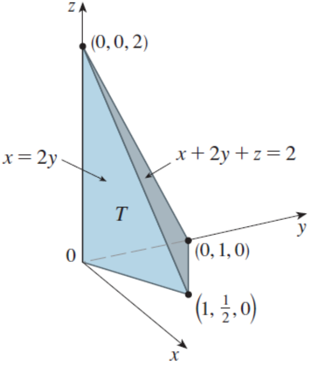
$$V(E) = \iiint\limits_{E} dV.$$

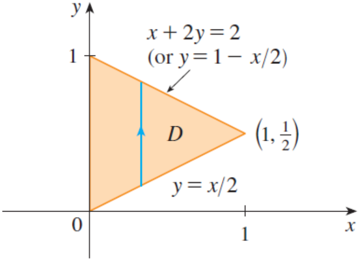
Example

Use a triple integral to find the volume of the tetrahedron T bounded by the planes

$$x + 2y + z = 2, \; x = 2y, \; z = 0.$$

Solution.
The tetrahedron T and its projection D onto the XY-plane are shown in the figure.





Solution...

The lower boundary of T is the plane $z = 0$ and the upper boundary is the plane $x + 2y + z = 2$, that is, $z = 2 - x - 2y$. Therefore we have

$$V(E) = \iiint\limits_E dV = \int_0^1 \int_{x/2}^{1-x/2} \int_0^{2-x-2y} dz\, dy\, dx$$

$$= \int_0^1 \int_{x/2}^{1-x/2} (2 - x - 2y)\, dy\, dx = \frac{1}{3}.$$

Mass

- For a three dimensional solid E with constant density, the mass is the density times the volume.
- If the density is not constant but rather a continuous function of x, y, and z, then we can cut the solid into very small rectangular solids so that on each rectangular solid the density is approximately constant.
- The volume of the small rectangular solid is then given by

$$\Delta\text{Mass} \;=\; (\text{Density})(\Delta\text{Volume}) \;= f(x,y,z)\Delta x\Delta y\Delta z$$

Now do the usual thing. We add up all the small masses and take the limit as the rectangular solids get small. This will give us the triple integral

The triple integral as the mass

$$\text{Mass } = \iiint_E f(x, y, z)\, dz\, dy\, dx.$$

We find the center of mass of a solid just as we found the center of mass of a lamina. Since we are in three dimensions, instead of the moments about the axes, we find the moments about the coordinate planes. We state the definitions from physics below.

Moments

Let $\rho(x, y, z)$ be the density of a solid E. Then the first moments about the coordinate planes are

$$M_{yz} = \iiint\limits_E x\rho(x, y, z)\ dz\, dy\, dx$$

$$M_{xz} = \iiint\limits_E y\rho(x, y, z)\ dz\, dy\, dx$$

$$M_{xy} = \iiint\limits_E z\rho(x, y, z)\ dz\, dy\, dx.$$

Center of Mass

The center of mass is given by

$$(x, y, z) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m}\right).$$

Notice that letting the density function being identically equal to 1 gives the volume

$$\text{Volume } = \iiint_E \, dz\, dy\, dx.$$

Just as with lamina, there are formulas for moments of inertial about the three axes. They involve multiplying the density function by the square of the distance from the axes.

We have

Moments of inertia

Let $\rho(x, y, z)$ be the density of a solid E. Then the first moments of inertia about the coordinate axes are

$$I_x = \iiint_E (y^2 + z^2)\rho(x, y, z)\ dz\,dy\,dx$$

$$I_y = \iiint_E (x^2 + z^2)\rho(x, y, z)\ dz\,dy\,dx$$

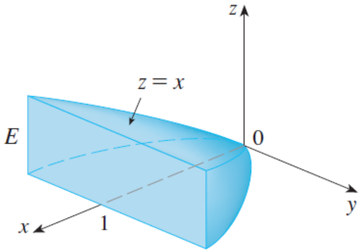
$$I_z = \iiint_E (x^2 + y^2)\rho(x, y, z)\ dz\,dy\,dx$$

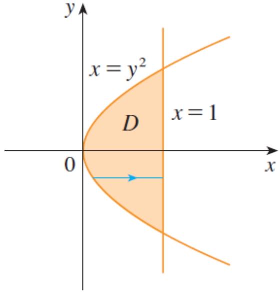
Example

Find the center of mass of a solid E of constant density that is bounded by the parabolic cylinder $x = y^2$ and the planes $x = z, z = 0$, and $x = 1$.

Solution.

The solid E and its projection D onto the xy-plane are shown in the figure given below.





The lower and upper surfaces of E are the planes $z = 0$ and $z = x$, so we describe as a type 1 region:

$$E = \{(x, y, z) : \ -1 \le y \le 1, y^2 \le x \le 1, \ 0 \le z \le x\}.$$

Then, if the density is $\rho(x, y, z) = \rho$, the mass is

$$m = \iiint\limits_E \rho \, dV = \int_{-1}^{1} \int_{y^2}^{1} \int_0^x \rho \, dz \, dy \, dx = \frac{4\rho}{5}.$$

Because of the symmetry of E and ρ about the xz-plane, we can immediately say that M_{xz} and therefore $\overline{y} = 0$. The other moments are

$$M_{yz} = \iiint\limits_E x\rho\, dV = \int_{-1}^{1} \int_{y^2}^{1} \int_0^{x} x\rho\, dz\, dy\, dx = \frac{4\rho}{7}$$

$$M_{xy} = \iiint\limits_E z\rho\, dV = \int_{-1}^{1} \int_{y^2}^{1} \int_0^{x} z\rho\, dz\, dy\, dx = \frac{2\rho}{7}$$