Unit 4: Double Integrals in Polar Coordinates

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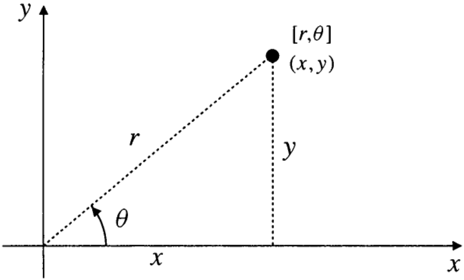
Summary

1. Polar coordinates

2. General Polar Regions of Integration

Rectangular and Polar Coordinates

Recall that the polar representation of a point P is an ordered pair (r, θ), where r is the distance from the origin to P and θ is the angle that the ray through the origin and P makes with the positive x-axis.



Relation between Rectangular and Polar Coordinates

The polar coordinates r and θ of a point (x, y) in rectangular coordinates satisfy the following relations:

- $r = \sqrt{x^2 + y^2}$ and $\tan\theta = \dfrac{y}{x}$.
- $x = r\cos\theta$ and $y = r\sin\theta$.

Polar coordinates are convenient when the domain of integration is an angular sector or a polar rectangle, as shown in the figure given below.

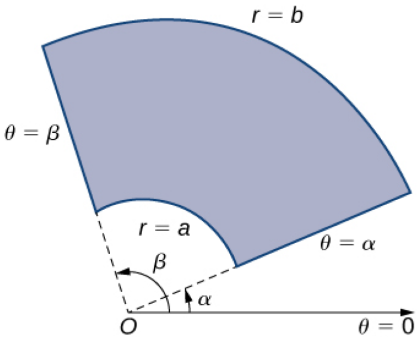


Figure 1: Polar rectangle

Moreover for many applications of double integrals, the integrand may be easier to integrate if it is in terms of polar coordinates than in terms of Cartesian coordinates.

Example

Consider the double integral

$$\iint_D e^{x^2+y^2}\, dA,$$

where D is the unit disk.

Note that we cannot directly evaluate this integral in rectangular coordinates. However, a change to polar coordinates will convert it to one we can easily evaluate. First we establish the concept of a double integral in a polar rectangular region. Then we change rectangular coordinates to polar coordinates in double integrals.

Concept of a double integral in a polar rectangle

In polar coordinates, the shape we work with is a *polar rectangle*. See the figure given below.

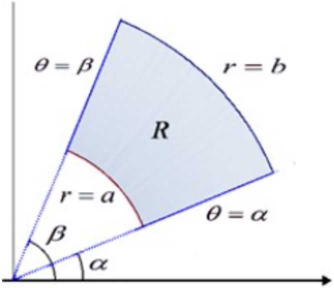


Figure 2: Polar rectangle

Polar rectangle

A polar rectangle is a region R given by

$$R = \{(r, \theta) : 0 \leq a \leq r \leq b, \alpha \leq \theta \leq \beta\},$$

where $0 \leq \beta - \alpha \leq 2\pi$.

Consider a function $f(r, \theta)$ over a polar rectangle R defined above.
Double integrals in polar coordinates are defined through a three-step process: **subdivision**, **summation**, and **passage to the limit**, as we did in the case of double integrals in rectangular coordinates.

Subdivision

We decompose R into an $n \times m$ grid of small polar subrectangles R_{ij} as follows:
We divide the interval $[a, b]$ into m subintervals $[r_{i-1}, r_i]$ of length

$$\Delta r = (b - a)/m$$

and divide the interval $[\alpha, \beta]$ into n subintervals $[\theta_{i-1}, \theta_i]$ of width

$$\Delta\theta = (\beta - \alpha)/n$$

by choosing partitions:

$$a = r_0 < r_1 < ... < r_m = b, \ \alpha = \theta_0 < \theta_1 < ... < \theta_n = \beta,$$

where m and n are positive integers.

This means that the circles of radii $r = r_i$ and rays with angles $\theta = \theta_j$ for $1 \leq i \leq m$ and $1 \leq j \leq n$ divide the polar rectangle R into smaller polar subrectangles R_{ij} as in the figure given below.

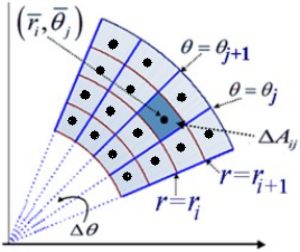


Figure 3: Polar grid

Choose the center $(\overline{r}_i, \overline{\theta}_l)$ of each polar subrectangle R_{ij} as a sample point. Then

$$\overline{r}_i = \frac{1}{2}(r_{i-1} + r_i), \quad \overline{\theta}_j = \frac{1}{2}(\theta_{j-1} + \theta_j).$$

The area of the polar subrectangle R_{ij} is given by

$$\begin{aligned}
\Delta A_i &= \frac{1}{2}r_i^2\Delta\theta - \frac{1}{2}r_{i-1}^2\Delta\theta = \frac{1}{2}(r_i^2 - r_{i-1}^2)\Delta\theta \\
&= \frac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\Delta\theta \\
&= \overline{r}_i\Delta r\Delta\theta.
\end{aligned}$$

So, the volume of each of the boxes with a base area of ΔA_i and a height of $f(\overline{r}_i, \overline{\theta}_j)$ is

$$f(\overline{r}_i, \overline{\theta}_j)\Delta A_i = f(\overline{r}_i, \overline{\theta}_j)\overline{r}_i \Delta r \Delta \theta.$$

The volume of the solid with the vase R_{ij} is now approximated as follows:

$$\iint_{R_{ij}} f(r, \theta)\ dA \approx f(\overline{r}_i, \overline{\theta}_j)\Delta A_i = f(\overline{r}_i, \overline{\theta}_j)\overline{r}_i \Delta r \Delta \theta.$$

Summation

The volume of the solid under the surface $z = f(r, \theta)$ with the vase R is now approximated as follows:

$$\begin{aligned}\iint_R f(r,\theta)\, dA &= \sum_{i=1}^{n}\sum_{j=1}^{m}\iint_{R_{ij}} f(r,\theta)\, dA \\ &\approx \sum_{i=1}^{n}\sum_{j=1}^{m} f(\overline{r}_i, \overline{\theta}_j)\ \Delta A_i \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} f(\overline{r}_i, \overline{\theta}_j)\overline{r}_i \Delta r \Delta \theta.\end{aligned}$$

Riemann sum

The expression

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(\overline{r}_i, \overline{\theta}_j)\overline{r}_i \Delta r \Delta \theta$$

is called a **Riemann sum** for the double integral of $f(r, \theta)$ over the region

$$0 \leq a \leq r \leq b, \alpha \leq \theta \leq \beta,$$

where $0 \leq \beta - \alpha \leq 2\pi$.

Passage to the limit

Double integral in polar coordinates

Let f be continuous on a polar rectangle R given by

$$0 \leq a \leq r \leq b, \ \alpha \leq \theta \leq \beta,$$

where $0 \leq \beta - \alpha \leq 2\pi$. The double integral $\iint_R f(r,\theta)\, dA$ is defined as follows:

$$\iint_R f(r,\theta)\, dA = \lim_{n,m\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(\overline{r}_i, \overline{\theta}_j)\overline{r}_i \Delta r \Delta\theta.$$

Just as in double integrals over rectangular regions, the double integral over a polar rectangular region can be expressed as an iterated integral in polar coordinates. Hence

$$\iint_R f(r,\theta)\ dA = \iint_R f(r,\theta) r\ dr\, d\theta = \int_\alpha^\beta \int_a^b f(r,\theta) r\ dr\, d\theta.$$

Notice that the expression for dA is replaced by $r\, dr\, d\theta$ when working in polar coordinates.

We have the following theorem.

Theorem

If f is continuous on a polar rectangle R given by

$$0 \leq a \leq r \leq b, \alpha \leq \theta \leq \beta,$$

where $0 \leq \beta - \alpha \leq 2\pi$, then

$$\iint\limits_D f(x, y)\ dA = \iint\limits_D f(r\cos\theta, r\sin\theta)\ r\,dr\,d\theta.$$

It is noteworthy that all the properties of the double integral in rectangular coordinates hold true for the double integral in polar coordinates as well, so we can use them without hesitation.

Change to Polar Coordinates in a Double Integral

If we are given a double integral

$$\iint\limits_D f(x, y)\ dA$$

in rectangular coordinates, we can write the corresponding iterated integral in polar coordinates by substitution.

Method for converting in polar coordinates

- Describe the domain of integration, R, and find bounds

$$a \leq r \leq b \text{ and } \alpha \leq \theta \leq \beta,$$

  where $0 \leq \beta - \alpha \leq 2\pi$.
- Convert the function $z = f(x, y)$ to a function with polar coordinates with the substitutions

$$x = r\cos\theta, y = r\sin\theta.$$

- Replace dA by $r\,dr\,d\theta$ to obtain

$$\iint\limits_R f(x, y)\, dA = \iint\limits_R f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta.$$

Example

Let $f(x,y) = e^{x^2+y^2}$ on the disk $D = \{(x,y) : \ x^2+y^2 \leq 1\}$.
Evaluate $\iint\limits_{D} f(x,y)\ dA$.

Solution.

We have the unit disk

$$D = \{(x, y) : \; x^2 + y^2 \leq 1\}.$$

We observe that

$$0 \leq r \leq 1, \; 0 \leq \theta \leq 2\pi.$$

Solution...
Using
$$x = r\cos\theta,\ y = r\sin\theta,\ dA = r\,dr\,d\theta,$$
we then have
$$\begin{aligned}\int_D e^{x^2+y^2}\ dA &= \int_0^{2\pi}\int_0^1 e^{r^2} r\ dr\,d\theta \\ &= \int_0^{2\pi} \frac{1}{2}e^{r^2}\Big|_0^1\ d\theta \\ &= \int_0^{2\pi}\frac{1}{2}(e-1)\ d\theta \\ &= \pi(e-1). \qquad \blacktriangleleft\end{aligned}$$

While there is no firm rule for when polar coordinates can or should be used, they are a natural alternative anytime the domain of integration may be expressed simply in polar form, and/or when the integrand involves expressions such as

$$\sqrt{x^2 + y^2}.$$

Example

Determine the volume of the region that lies under the sphere $x^2 + y^2 + z^2 = 9$, above the plane $z = 0$ and inside the cylinder $x^2 + y^2 = 5$.

Solution.

We know that the formula for finding the volume of a region is

$$V = \iint_D f(x, y)\, dA.$$

We have

$$f(x, y) = z = \sqrt{9 - x^2 - y^2} = \sqrt{9 - r^2}.$$

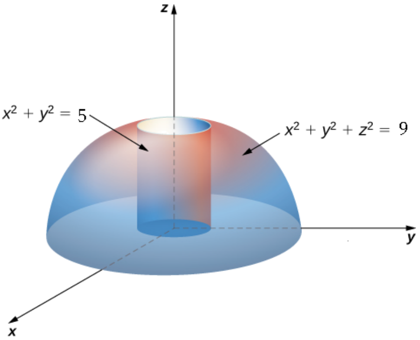
The region D is the bottom of the cylinder given by $x^2 + y^2 = 5$, that is, the disk

$$D = \{(x, y) |\ x^2 + y^2 \leq 5\}$$

in the xy-plane.

Solution...

So, the region that we want the volume for is really a cylinder with a cap that comes from the sphere.



Thus, the region D in polar coordinates is as follows:

$$D = \{(r, \theta) : \; 0 \leq r \leq \sqrt{5}, 0 \leq \theta \leq 2\pi\}$$

Solution...

Now, the volume is

$$\begin{aligned} V &= \iint\limits_D \sqrt{9 - x^2 - y^2}\, dA \\ &= \int_0^{2\pi} \int_0^{\sqrt{5}} r\sqrt{9 - r^2} dr\ d\theta \\ &= 38\pi/3. \end{aligned}$$

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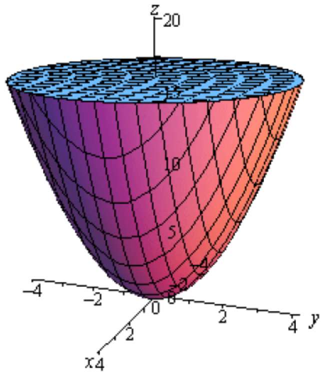
Example

Find the volume of the region that lies inside $z = x^2 + y^2$ and below the plane $z = 16$.

Solution.

Let's start this example off with a sketch of the region.



Now, we see that the top of the region, where the elliptic paraboloid intersects the plane $z = 16$, is the widest part of the region.

Solution...

So, setting $z = 16$ in the equation of the paraboloid gives

$$16 = x^2 + y^2,$$

which is the equation of a circle of radius 4 centered at the origin. Now, the domain of intgration, D, is given by

$$D = \{(r, \theta) : 0 \leq \theta \leq 2\pi,\ 0 \leq r \leq 4\}.$$

Solution...
Notice that the formula

$$\iint_D 16 \; dA.$$

will be the volume under plane $z = 16$ while the formula

$$\iint_D (x^2 + y^2) \; dA.$$

is the volume under the paraboloid $z = x^2 + y^2$, using the same D.

Solution...

Hence the required volume is

$$V = \iint\limits_D 16 \ dA - \iint\limits_D (x^2 + y^2) \ dA$$

$$= \iint\limits_D (16 - x^2 - y^2) \ dA$$

$$= \int_0^{2\pi} \int_0^4 r(16 - r^2) \ dr \ d\theta$$

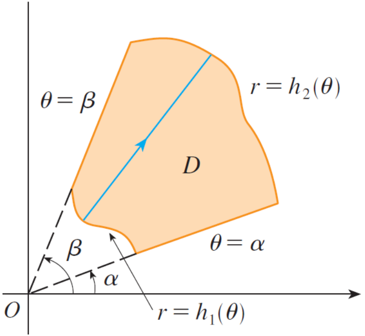
$$= 128\pi.$$

General Polar Regions of Integration

To evaluate the double integral of a continuous function by iterated integrals over general polar regions, we consider two types of regions, analogous to Type I and Type II as discussed for rectangular coordinates in Double Integrals over General Regions.

It is more common to write polar equations as $r = f(\theta)$ than $\theta = f(r)$, so we describe a general polar region as

$$D = \{(r, \theta) |\ \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$$



Example

Determine the area of the region that lies inside $r = 3 + 2\sin\theta$ and outside $r = 2$.

Solution.

Here is a sketch of the region, D, that we want to determine the shaded area (Figure (a)).

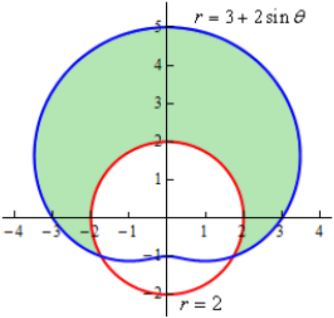


Figure 4: (a)

Solution...

To determine the range of θ, we solve the two equations. We have

$$3 + 2\sin\theta = 2 \Rightarrow \sin\theta = -\frac{1}{2} \Rightarrow \theta = \frac{7\pi}{6}, \frac{11\pi}{6}.$$

Here is a sketch of the figure with these angles added.

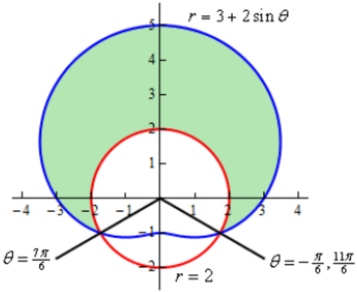


Figure 5: (b)

Solution...

Note as well that we've acknowledged that $-\frac{\pi}{6}$ is another representation for the angle

$$\frac{11\pi}{6} = 2\pi - \frac{\pi}{6}.$$

This is important since we need the range of θ to actually enclose the regions as we increase from the lower limit to the upper limit. If we'd chosen to use $\frac{11\pi}{6}$, then as we increase from $\frac{7\pi}{6}$ to $\frac{11\pi}{6}$ we would be tracing out the lower portion of the circle and that is not the region that we are after.

Solution...
To get the ranges for r the function that is closest to the origin is the lower bound and the function that is farthest from the origin is the upper bound.
So, here are the ranges that will define the region.

$$-\frac{\pi}{6} \leq \theta \leq \frac{7\pi}{6}$$

$$2 \leq r \leq 3 + 2\sin\theta.$$

Solution...

The area of the region D is then

$$
\begin{aligned}
A &= \iint\limits_D dA \\
&= \int_{-\pi/6}^{\pi/6} \int_{2}^{3+2\sin\theta} dr\, d\theta \\
&= \frac{11\sqrt{3}}{2} + \frac{14\pi}{3}.
\end{aligned}
$$

◀

Example

Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.

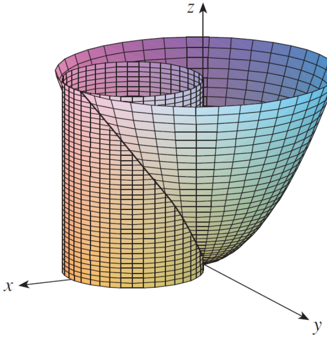
Solution.
The solid lies above the disk D whose boundary circle has equation $x^2 + y^2 = 2x$ or, after completing the square, we get

$$(x-1)^2 + y^2 = 1$$

(See the figures given below.) To find the volume of the required solid, we have to evaluate the integral:

$$V = \iint\limits_D (x^2 + y^2)\, dx\, dy.$$

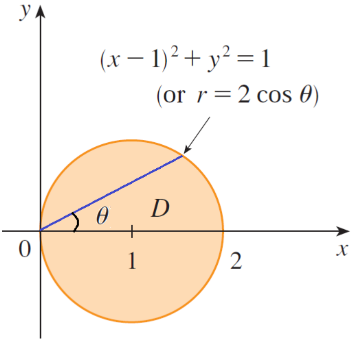
Solution...



Converting the equation of the circle in polar form, we get

$$
\begin{aligned}
x^2+y^2=2x &\Rightarrow (r\cos\theta)^2+(r\sin\theta)^2=2r\cos\theta\\
&\Rightarrow r^2=2r\cos\theta\\
&\Rightarrow r=0 \text{ or } 2\cos\theta.
\end{aligned}
$$

Solution...



Thus the disk D is given by

$$D = \{(r, \theta) : -\pi/2 \leq \theta \leq \pi/2,\ 0 \leq r \leq 2\cos\theta\}.$$

Solution.

Now, we have

$$V = \iint\limits_D (x^2 + y^2)\ dx\,dy = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^2 r\ dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left[\frac{r^4}{4}\right]_0^{2\cos\theta} d\theta = 4\int_{-\pi/2}^{\pi/2} \cos^4\theta\ d\theta$$

$$= 8\int_0^{\pi/2} \cos^4\theta\ d\theta = 8\int_0^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$= 2\int_0^{\pi/2} [1 + 2\cos 2\theta + \frac{1}{2}(1+\cos 4\theta)]\ d\theta$$

$$= 2\left[\frac{3}{2}\theta + \sin 2\theta + \frac{1}{8}\sin 4\theta\right]_0^{\pi/2} = 2(3/2)(\pi/2)$$

$$= 3\pi/2. \quad \blacktriangleleft$$