Unit 4: Applications of Double Integrals

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December 1, 2023

Summary

1. **Applications of double integrals**
   - Mass
   - Moments and Center of Mass
   - Moments of Inertia
   - Surface area

We saw before that the double integral over a region of the constant function 1 measures the area of the region. If the region has uniform density 1, then

$$\begin{aligned}\text{Mass} &= \text{Density} \times \text{Area}\\ &= 1 \times \text{Area}\\ &= \text{Area}\,.\end{aligned}$$

What if the density is not constant. Suppose that the density is given by the continuous function

$$\text{Density } = \rho(x, y)$$

In this case we can cut the region into tiny rectangles where the density is approximately constant. The area of mass rectangle is given by

$$\text{Mass } = \text{ Density } \times \text{ Area } = \rho(x, y)\Delta x\ \Delta y$$

You probably know where this is going. If we add all to masses together and take the limit as the rectangle size goes to zero, we get a double integral.

Mass

Let $\rho(x, y)$ be the density of a lamina (flat sheet) D at the point (x, y). Then the total mass of the lamina is the double integral

$$\text{Mass } = \iint_D \rho(x, y)\, dy\, dx.$$

Finding the mass of a lamina with constant density

Find the mass of a square lamina, with side length 1, with a density of $\rho = 3 \text{ gm/cm}^2$.

Solution.

We represent the lamina with a square region in the plane as shown in the figure given below.

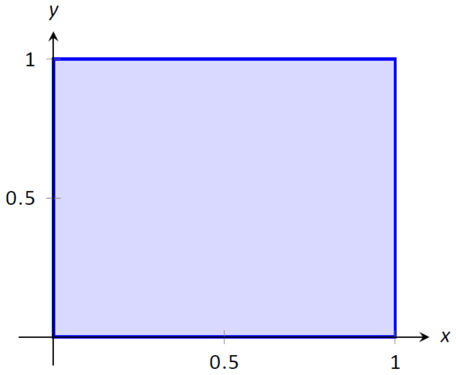


Figure 1: A region R representing a lamina.

As the density is constant, it does not matter where we place the square. Now, the mass M of the lamina is

$$M = \iint_R 3 \; dA = \int_0^1 \int_0^1 3 \; dx \, dy$$
$$= 3 \text{ gm.} \qquad \blacktriangleleft$$

This is all very straightforward.

Finding the mass of a lamina with variable density

Find the mass of a square lamina, represented by the unit square with lower lefthand corner at the origin (see the Figure), with variable density $\rho(x, y) = (x + y + 2)$ gm/cm^2.

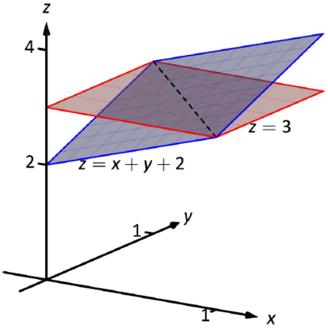


Figure 2: Graphing the density functions: $z = 3$ and $z = x + y + 2$

Solution.
The variable density ρ, in this example, is very uniform, giving a density of 3 in the center of the square and changing linearly. A graph of $\rho(x, y)$ can be seen in Figure 2; notice how "same amount" of density is above $z = 3$ as below.
The mass M is found by integrating $\rho(x, y)$ over R. The order of integration is not important; we choose $dx\, dy$ arbitrarily.

Solution...

Thus

$$\begin{aligned}
\iint_R (x+y+2)\,dA &= \int_0^1 \int_0^1 (x+y+2)\,dx\,dy \\
&= \int_0^1 \left((1/2)x^2 + x(y+2)\right)\Big|_0^1 dy \\
&= \int_0^1 \left(\frac{5}{2} + y\right) dy \\
&= \left(\frac{5}{2}y + \frac{1}{2}y^2\right)\Big|_0^1 \\
&= 3 \text{ gm.}
\end{aligned}$$

◀

It turns out that since the density of the lamina is so uniformly distributed "above and below" $z = 3$ that the mass of the lamina is the same as if it had a constant density of 3. The density functions in the last two Examples are graphed in Figure 2, which illustrates this concept.

Moments

We know that the moments about an axis are defined by the product of the mass times the distance from the axis.

$$M_x = (\text{ Mass })(y), \qquad M_y = (\text{ Mass })(x).$$

If we have a region D with density function $\rho(x, y)$, then we do the usual thing. We cut the region into small rectangles for which the density is constant and add up the moments of each of these rectangles. Then take the limit as the rectangle size approaches zero. This will give us the total moment.

Moments of Mass and Center of Gravity

Suppose that $\rho(x, y)$ is a continuous density function on a lamina D. Then the **moments of mass** are

$$M_x = \iint\limits_D \rho(x, y)y \; dy \; dx, \qquad M_y = \iint\limits_D \rho(x, y)x \; dy \; dx.$$

and if m is the mass of the lamina, then the **center of mass** or **center of gravity** is

$$(\overline{x}, \overline{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m} \right)$$

Finding the center of mass of a lamina

Find the center mass of a square lamina, with side length 1, with a density of $\rho = 3$ gm/cm^2.

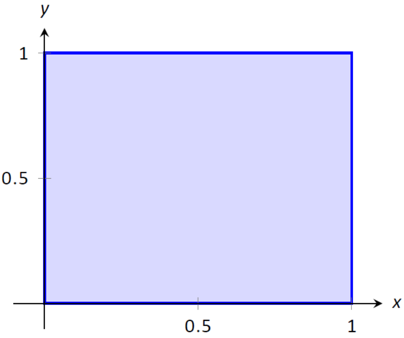


Figure 3: A region R representing a lamina.

Solution.

We represent the lamina with a square region in the plane as shown in the Figure. We have

$$M = \iint\limits_R \rho(x,y)\ dA = \int_0^1 \int_0^1 3\ dx\,dy = 3 \text{ gm}.$$

$$M_x = \iint\limits_R \rho(x,y)y\ dA = \int_0^1 \int_0^1 3y\ dx\,dy = 3/2 = 1.5.$$

$$M_y = \iint\limits_R \rho(x,y)x\ dA = \int_0^1 \int_0^1 3x\ dx\,dy = 3/2 = 1.5.$$

Thus the center of mass is

$$(\overline{x}, \overline{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right) = (1.5/3, 1.5/3) = (0.5, 0.5).$$

Finding the mass of a lamina with variable density

Find the center of the mass of a square lamina, represented by the unit square with lower lefthand corner at the origin (see the Figure), with variable density $\rho(x, y) = (x + y + 2)$ gm/cm^2.

Solution.

We represent the lamina with a square region in the plane as before. We have

$$M = \iint\limits_R \rho(x,y)\;dA = \int_0^1\int_0^1 (x+y+2)\;dx\,dy = 3\text{ gm.}$$

$$M_x = \iint\limits_R \rho(x,y)y\;dA = \int_0^1\int_0^1 (x+y+2)y\;dx\,dy = 19/12.$$

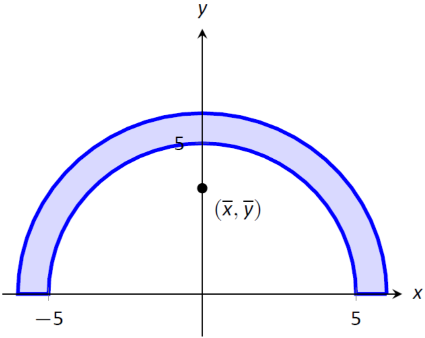
$$M_y = \iint\limits_R \rho(x,y)x\;dA = \int_0^1\int_0^1 (x+y+2)x\;dx\,dy = 19/12.$$

Thus the center of mass is

$$(\overline{x},\overline{y}) = \left(\frac{M_y}{m},\frac{M_x}{m}\right) = (19/36, 19/36) = (0.528, 0.528).$$

Example

Find the center of mass of the lamina represented by the region R which is half an annulus with outer radius 6 and inner radius 5, with constant density 2 lb/ft^2.



Solution. Here, it is useful to represent R in polar coordinates. Using the description of R, we see that

$$R = \{(r, \theta) :\ 5 < r < 6,\ 0 < \theta < \pi\}.$$

As the lamina is symmetric about the y-axis, we should expect $M_y = 0$. We compute M, M_x and M_y.

Solution...

We have

$$M = \int_0^{\pi} \int_5^{6} 2r \; dr \, d\theta = 11\pi \text{ lb.}$$

$$M_x = \int_0^{\pi} \int_5^{6} (r \sin\theta) 2r \; dr \, d\theta = \frac{364}{3} = 121.33.$$

$$M_y = \int_0^{\pi} \int_5^{6} (r \cos\theta) 2r \; dr \, d\theta = 0.$$

Thus the center of mass is

$$(\overline{x}, \overline{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m} \right) = (0, 364/(33\pi)) = (0, 3.51).$$

Example

Set up the integrals that give the center of mass of the rectangle with vertices $(0,0),(1,0),(1,1)$, and $(0,1)$ and density function proportional to the square of the distance from the origin.

Solution.
Since the density function $\rho(x, y)$ is proportional to the square of the distance from the origin, $x^2 + y^2$, the mass is given by

$$m = \int_0^1 \int_0^1 k(x^2 + y^2)\ dy\ dx = \frac{2k}{3}.$$

The moments are given by

$$M_x = \int_0^1 \int_0^1 k(x^2 + y^2)y\ dy\ dx = 5k/12$$
$$M_y = \int_0^1 \int_0^1 k(x^2 + y^2)x\ dy\ dx = 5k/12$$

Solution...

It should not be a surprise that the moments are equal since there is complete symmetry with respect to x and y. Finally, we divide to get

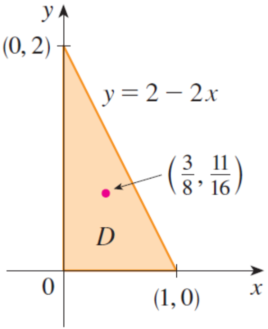
$$(\overline{x}, \overline{y}) = (5/8, 5/8)$$

This tells us that the metal plate will balance perfectly if we place a pin at (5/8 , 5/8).

Example

Find the mass and center of mass of a triangular lamina with vertices $(0,0), (1,0)$, and $(0,2)$ if the density function is

$$\rho(x,y) = 1 + 3x + y.$$



Moments of Inertia

We often call M_x and M_y the first moments. They have first powers of y and x in their definitions and help find the center of mass. We define the moments of inertia (or second moments) by introducing squares of y and x in their definitions. The moments of inertia help us find the kinetic energy in rotational motion. Below is the definition.

Moments of Inertia

Suppose that $\rho(x, y)$ is a continuous density function on a lamina D. Then the moments of inertia about the x-axis and the y-axis are

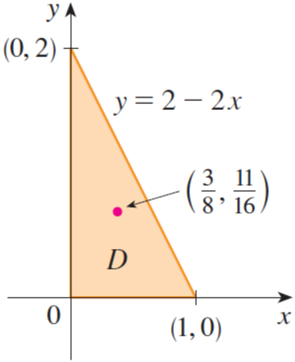
$$I_x = \iint\limits_D \rho(x, y)y^2 \; dy \; dx, \qquad I_y = \iint\limits_D \rho(x, y)x^2 \; dy \; dx.$$

It is also of interest to consider the moment of inertia about the origin, also called the **polar moment of inertia**:

$$I_x = \iint\limits_D \rho(x, y)(x^2 + y^2) \; dy \; dx.$$

Example

Find the moments of inertia for the square metal plate with vertices $(0, 0), (1, 0), (1, 1)$, and $(0, 1)$.



Problem

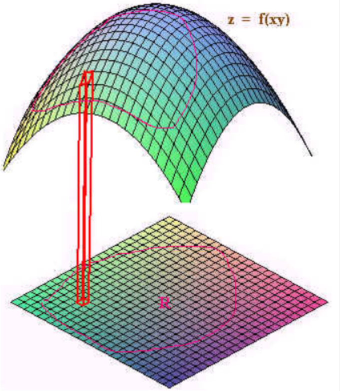
Find the moments of inertia I_x, I_y, and I_o of a homogeneous disk with density $\rho(x, y) = \rho$, center the origin, and radius a.

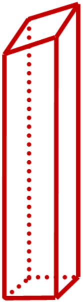
Solution.

The boundary of D is the circle $x^2 + y^2 = a$ and in polar coordinates D is described by $0 \leq \theta \leq 2\pi, 0 \leq r \leq a$. First compute I_o. Then use $I_x + I_y = I_o$ and $I_x = I_y$ (due to the symmetry of the problem) to find I_x and I_y. ◀

Surface area

Let $z = f(x, y)$ be a surface in $\mathbb{R}^3$ defined over a region D in the xy-plane. cut the xy-plane into rectangles. Each rectangle will project vertically to a piece of the surface as shown in the figure below.





Although the area of the rectangle in D is

$$\text{Area } = \Delta y \Delta x,$$

the area of the corresponding piece of the surface will not be $\Delta y \Delta x$ since it is not a rectangle. Even if we cut finely, we will still not produce a rectangle, but rather will approximately produce a parallelogram. With a little geometry we can see that the two adjacent sides of the parallelogram are (in vector form)

$$u = \Delta x \vec{i} + f_x(x, y) \Delta x \vec{k}$$

and

$$v = f_y(x, y) \Delta y \vec{i} + \Delta y \vec{k}$$

We can see this by realizing that the partial derivatives are the slopes in each direction. If we run Δx in the $\vec{i}$ direction, then we will rise $f_x(x,y)\Delta x$ in the $\vec{k}$ direction so that

$$\frac{\text{rise}}{\text{run}} = f_x(x,y),$$

which agrees with the slope idea of the partial derivative. A similar argument will confirm the equation for the vector v. Now that we know the adjacent vectors we recall that the area of a parallelogram is the magnitude of the cross product of the two adjacent vectors.

We have

$$
\begin{aligned}
|v \times w| &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \Delta x & 0 & f_x(x,y)\Delta x \\ 0 & \Delta y & f_y(x,y)\Delta y \end{vmatrix} \\
&= |-(f_y(x,y)\Delta y\Delta x)\vec{i} - (f_x(x,y)\Delta y\Delta x)\vec{j} + (\Delta y\Delta x)\vec{k}| \\
&= \sqrt{f_y^2(x,y)(\Delta y\Delta x)^2 + f_x^2(x,y)(\Delta y\Delta x)^2 + (\Delta y\Delta x)^2} \\
&= \sqrt{f_y^2(x,y) + f_x^2(x,y) + 1}\ \Delta y\Delta x.
\end{aligned}
$$

This is the area of one of the patches of the quilt. To find the total area of the surface, we add up all the areas and take the limit as the rectangle size approaches zero. This results in a double Riemann sum, that is a double integral. We state the definition below.

Surface Area

Let $z = f(x, y)$ be a differentiable surface defined over a region D. Then its surface area is given by

$$\text{Surface Area } = \iint\limits_{D} \sqrt{1 + f_x^2(x, y) + f_y^2(x, y)}\; dy\; dx.$$

Problem

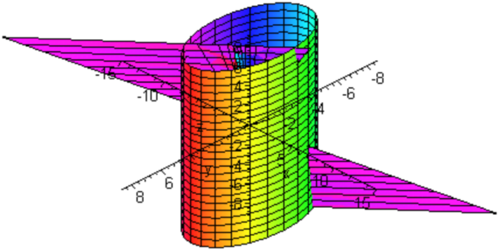
Find the surface area of the part of the plane

$$z = 8x + 4y$$

that lies inside the cylinder

$$x^2 + y^2 = 16.$$

Solution



We calculate partial derivatives

$$f_x(x, y) = 8, \qquad f_y(x, y) = 4$$

so that

$$1 + f_x^2(x, y) + f_y^2(x, y) = 1 + 64 + 16 = 81$$

Taking a square root and integrating, we get

$$\iint\limits_D 9\ dy\ dx.$$

We could work this integral out, but there is a much easier way. The integral of a constant is just the constant times the area of the region. Since the region is a circle, we get

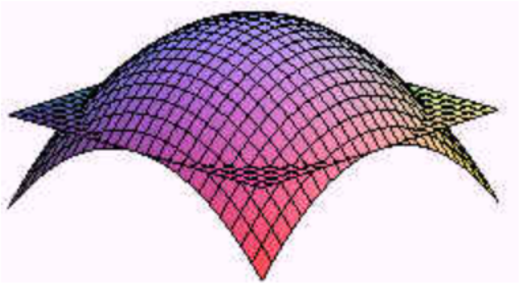
$$\text{Surface Area } = 9(16\pi) = 144\pi.$$

Example

Find the surface area of the part of the paraboloid

$$z = 25 - x^2 - y^2$$

that lies above the xy-plane.



Solution.
We calculate partial derivatives

$$f_x(x,y) = -2x \qquad f_y(x,y) = -2y$$

so that

$$1 + f_x^2(x,y) + f_y^2(x,y) = 1 + 4x^2 + 4y^2.$$

Using "Polar Coordinates", we realize that the region is just the circle

$$r = 5$$

Now convert the integrand to polar coordinates to get

$$\int_0^{2\pi} \int_0^5 \sqrt{(}1 + 4r^2)\ rdr\ d\theta$$

Solution...

Now let

$$u = 1 + 4r^2, \qquad du = 8rdr$$

and substitute

$$\frac{1}{8}\int_0^{2\pi}\int_1^{101} u^{1/2}\ du\ d\theta = \frac{1}{12}\int_0^{2\pi}\left[u^{3/2}\right]_1^{101}\ d\theta \approx 169.3\pi.$$