Unit 4: Double Integrals in Rectangles

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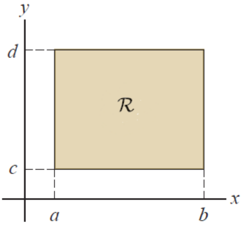
Summary

1. **Double integrals**
   - Subdivision
   - Geometric interpretation
   - The Midpoint Rule
   - Properties of Double Integrals
2. **Iterated integration**
   - Partial integration and iterated integrals
3. **Double integrals and iterated integrals**

Domain of integration: Rectangle

We consider a rectangle in $\mathbb{R}^2$ given by

$$\begin{aligned} R &= [a, b] \times [c, d] \\ &= \{(x, y) \in \mathbb{R}^2 : \ a \le x \le b, \ c \le y \le d\}. \end{aligned}$$



Let

$$A = \text{Area of } R.$$

Solid under a surface above a rectangle

Also, we will initially assume that

$$z = f(x, y) \geq 0.$$

The graph of f is a surface with equation $z = f(x, y)$. Let S be the solid that lies above R and under the graph of f, that is,

$$S = \{(x, y, z) \in \mathbb{R}^3 : \ 0 \leq z \leq f(x, y), \ (x, y) \in R\}.$$

Here is a sketch of this set up.

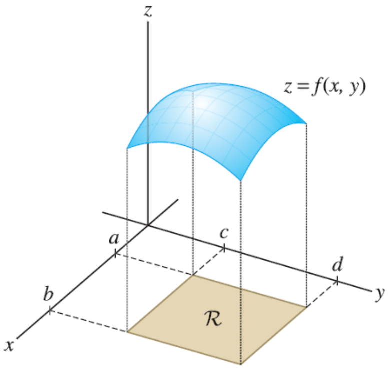


Figure 1: The solid above R and under f.

We now ask: What is the volume of the solid S?
Ans.: It is expressed by a *double integral*.

Three-step process of definition

Like integrals in one variable, double integrals are defined through a three-step process: **subdivision**, **summation**, and **passage to the limit**.

We first divide up $[a, b]$ into m subintervals of equal width $\Delta x = (b - a)/m$ and $[c, d]$ into n subintervals of equal width $\Delta y = (d - c)/n$ by choosing partitions:

$$a = x_0 < x_1 < ... < x_m = b, c = y_0 < y_1 < ... < y_n = d,$$

where m and n are positive integers to create an $n \times m$ grid of subrectangles R_{ij}.

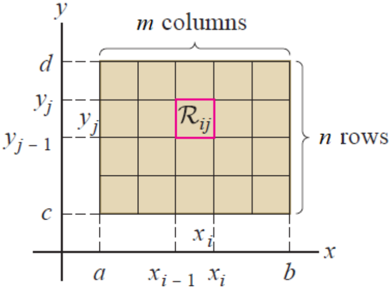


Figure 2: An $n \times m$ grid of R

The area of each subrectangle R_{ij} is given by

$$\Delta A = \Delta x \Delta y.$$

From each of these subrectangles we will choose a point (x_i^*, y_j^*), as shown in the figure given below.

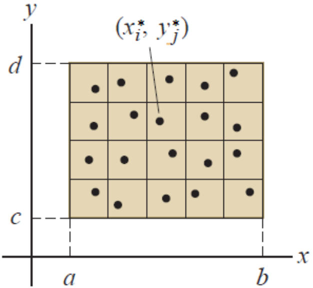


Figure 3: An $n \times m$ grid of R with sample points (x_i^*, y_j^*)

Now, over each of these smaller rectangles we will construct a box whose height is given by $f(x_i^*, y_j^*)$. See the figure given below.

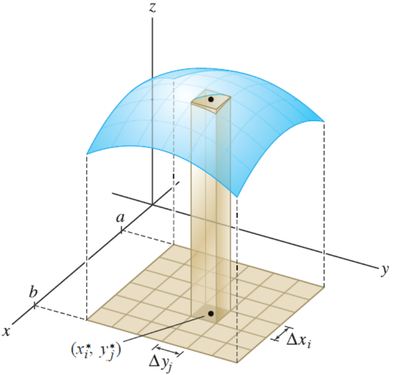


Figure 4: A box with volume $f(x_i^*, y_j^*)\Delta A$

Each of the rectangles has a base area of ΔA and a height of $f(x_i^*, y_j^*)$ so the volume of each of these boxes is

$$f(x_i^*, y_j^*)\Delta A.$$

Summation

The volume of the solid S is now approximated as follows:

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*) \Delta A.$$

Double Riemann sum

A **double Riemann sum** is defined as

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*) \Delta A.$$

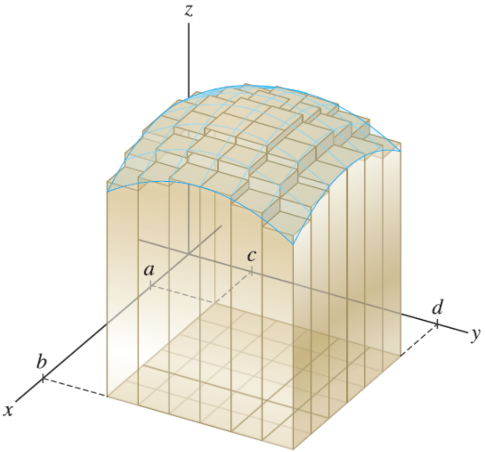


Figure 5: The solid S is approximated by the sum of the volumes of all boxes.

Passage to the limit

- We have a double sum since we will need to add up volumes in both the x and y directions.
- To get a better estimation of the volume we will take n and m larger and larger.
- And to get the exact volume we will need to take the limit as both n and m go to infinity.

In other words,

$$V = \lim_{n,m\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta A.$$

This looks a lot like the definition of the integral of a function of single variable. In fact, this is also the definition of an integral of a function of two variables over a rectangle.

Here is the formal definition of a double integral of a function of two variables over a rectangle R as well as the notation that we'll use for it.

Double integral over a rectangle

$$\iint_R f(x,y)\,dA = \lim_{n,m\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta A.$$

The sample point (x_i^*, y_j^*) in the definition can be chosen to be any point (x_i, y_j) in the subrectangle.

If f happens to be a positive function, then the double Riemann sum represents the sum of volumes of columns, as in Figure 5, and is an approximation to the volume of the solid under the graph of f and above the rectangle R. Thus, we have the following difinition:

Geometric interpretation of a double integral

If $f(x, y) \geq 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x, y)$ is

$$\text{Volume } = \iint_R f(x, y)\, dA.$$

Example

Estimate the volume of the solid that lies above the square

$$R = [0, 2] \times [0, 2]$$

and below the elliptic paraboloid

$$z = 16 - x^2 - 2y^2.$$

Divide R into four equal squares and choose the sample point to be the upper right corner of each square . Sketch the solid and the approximating rectangular boxes.

Solution

The paraboloid is the graph of $f(x, y) = 16 - x^2 - 2y^2$ and the area of each square is $\Delta A = 1$. The squares are shown in Figure 6.

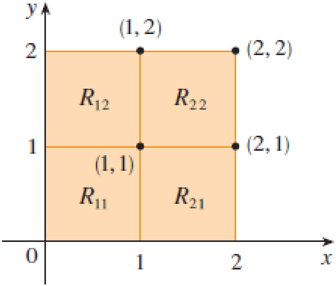


Figure 6:

Solution ...

Approximating the volume by the Riemann sum with $m = n = 2$, we have

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_i)\Delta A$$
$$= f(1,1)\Delta A + f(1,1)\Delta A + f(1,2)\Delta A + f(2,2)\Delta A$$
$$= 13(1) + 7(1) + 10(1) + 4(1) = 34.$$

Solution ...

This is the volume of the approximating rectangular boxes shown in Figure 7.

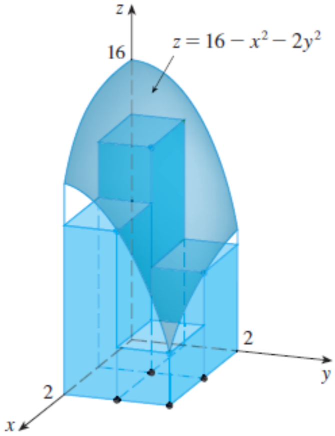


Figure 7:

Let $\overline{x}_i$ be the midpoint of $[x_{i-1}, x_i]$ and $\overline{y}_j$ be the midpoint of $[y_{j-1}, y_j]$. Then we can choose the center $(\overline{x}_i, \overline{y}_j)$ of R_{ij} as the sample point (x_i^*, y_j^*).

Midpoint Rule for Double Integrals

$$\iint_A f(x,y)\, dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(\overline{x}_i, \overline{y}_j) \Delta A,$$

where $\overline{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\overline{y}_j$ is the midpoint of $[y_{j-1}, y_j]$.

Example

Use the Midpoint Rule with $m = n = 2$ to estimate the value of the integral $\iint_R (x-3y)^2 \, dA$, where

$$R = \{(x,y) |\ 0 \le x \le 2, 1 \le y \le 2\}.$$

Solution

In using the Midpoint Rule with $m = n = 2$, we evaluate $f(x, y) = (x - 3y^2)$ at the centers of the four subrectangles shown in Figure 8.

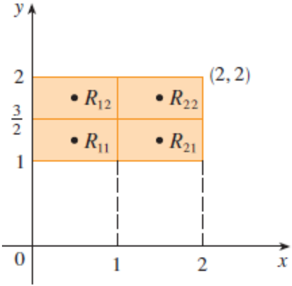


Figure 8:

Solution...

Since $m = n = 2$, we have

$$\Delta x = 1 - 0 = 2 - 1 = 1,$$
$$\Delta y = \tfrac{3}{2} - 1 = 2 - \tfrac{3}{2} = \tfrac{1}{2}.$$

The area of each subrectangle is

$$\Delta A = \Delta x \Delta y = 1 \times \tfrac{1}{2} = \tfrac{1}{2}.$$

We also have

$$\overline{x}_1 = \tfrac{1}{2}, \overline{x}_2 = \tfrac{3}{2}, \overline{y}_1 = \tfrac{5}{4}, \text{ and } \overline{y}_2 = \tfrac{7}{4}.$$

Thus, the centers of the rectangles $R_{11}, R_{12}, R_{21}, R_{22}$ are respectively

$$(\tfrac{1}{2}, \tfrac{5}{4}),\ (\tfrac{1}{2}, \tfrac{7}{4}),\ (\tfrac{3}{2}, \tfrac{5}{4}),\ (\tfrac{3}{2}, \tfrac{7}{4}).$$

Solution...
Thus

$$
\begin{aligned}
\iint_R (x - 3y^2)\, dA &\approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\overline{x}_i, \overline{y}_j)\Delta A \\
&= f(\overline{x}_1, \overline{y}_1)\Delta A + f(\overline{x}_1, \overline{y}_2)\Delta A + f(\overline{x}_2, \overline{y}_1)\Delta A \\
&\quad + f(\overline{x}_2, \overline{y}_2)\Delta A \\
&= f(\tfrac{1}{2}, \tfrac{5}{4})\Delta A + f(\tfrac{1}{2}, \tfrac{7}{4})\Delta A + f(\tfrac{3}{2}, \tfrac{5}{4})\Delta A \\
&\quad + f(\tfrac{3}{2}, \tfrac{7}{4})\Delta A \\
&= (-\tfrac{67}{16})\tfrac{1}{2} + (-\tfrac{130}{16})\tfrac{1}{2} + (-\tfrac{51}{16})\tfrac{1}{2} + (-\tfrac{123}{16})\tfrac{1}{2} \\
&= -\tfrac{95}{8} = -11.875.
\end{aligned}
$$

Average Value

Let f be a function of two variables defined on a rectangle R. We define the average value of f to be

$$f_{ave} = \frac{1}{A(R)} \iint_R f(x,y)\, dA,$$

where $A(R)$ is the area of R.
If $f(x, y \geq 0$, the equation

$$A(R) \times f_{ave} = \iint_R f(x,y)\, dA$$

says that the box with base R and height f_{ave} has the same volume as the solid that lies under the graph of f.

Here are some properties of the double integral. Note that all three of these properties are really just extensions of properties of single integrals that have been extended to double integrals.

Some properties of the double integral

1. $$\iint_R [f(x,y)+g(x,y)]\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA$$

2. If c is a constant, then
$$\iint_R cf(x,y)\,dA = c\iint_R f(x,y)\,dA$$

3. If $f(x,y) \geq g(x,y)$ for all in $(x,y) \in R$, then
$$\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA$$

Iterated Integrals

Just like with the definition of a single integral, it is usually difficult to evaluate double integrals from first principles. So we need to start looking into how we actually compute double integrals.

In the previous unit we found that it was useful to differentiate functions of several variables with respect to one variable, while treating all the other variables as constants or coefficients. We can integrate functions of several variables in a similar way.

For instance, suppose that $f_x(x, y) = 2xy$. We can treat y as staying constant and integrate to obtain $f(x, y)$:

$$\begin{aligned} f(x,y) &= \int f_x(x,y)\ dx \\ &= \int 2xydx \\ &= \int x^2y + C. \end{aligned}$$

Make a careful note about the constant of integration, C. This "constant" is something with a derivative of 0 with respect to x, so it could be any expression that contains only constants and functions of y.

For instance, if

$$f(x,y) = x^2y + \sin y + y^3 + 17,$$

then $f_x(x,y) = 2xy$. To signify that C is actually a function of y, we write:

$$f(x,y) = \int f_x(x,y)\, dx = x^2y + C(y).$$

Using this process we can evaluate definite integrals.

Example

Evaluate the integral $\int_1^4 2xy \ dx$.

Solution. We consider y as a constant and integrate with respect to x:

$$\begin{aligned}\int_1^4 2xy\ dx &= x^2y\Big|_1^4\\ &= 4^2y - 1^2y\\ &= 15y.\end{aligned}$$

We have considered y to be a constant. So, the limits of the integral may be functions of y as in the above example.

Example

Evaluate the integral $\int_1^{2y} 2xy \, dx$.

Solution. We consider y as a constant and integrate with respect to x:

$$\begin{aligned}\int_1^{2y} 2xy\, dx &= x^2y\Big|_1^{2y}\\ &= (2y)^2y - 1^2y\\ &= 4y^3 - y.\end{aligned}$$

Remark

Note how the limits of the integral are from $x = 1$ to $x = 2y$ and that the final answer is a function of y.

Example

Evaluate the integral $\int_1^x (5x^3y^{-3} + 6y^2)\ dy$.

Solution. Here, we consider x to be a constant and integrate with respect to y:

$$\int_1^x (5x^3y^{-3} + 6y^2)\ dy = \left(\frac{5x^3y^{-2}}{-2} + \frac{6y^3}{3}\right)\Bigg|_1^x$$

$$= \left(-\frac{5}{2}x^3x^{-2} + 2x^3\right) - \left(-\frac{5}{2}x^3\right.$$

$$= \frac{9}{2}x^3 - \frac{5}{2}x - 2.$$

Remark

Note how the limits of the integral are from $y = 1$ to $y = x$ and that the final answer is a function of x.

We can integrate the result obtained in the previous example with respect to x as well. This process is known as **iterated integration**, or **multiple integration**.

Example

Evaluate the integral

$$\int_1^2 \left(\int_1^x (5x^3y^{-3} + 6y^2)\ dy \right) dx.$$

Solution. We follow a standard "order of operations" and perform the operations inside parentheses first (which is the integral evaluated in the previous example.

$$
\begin{aligned}
& \int_1^2 \left(\int_1^x (5x^3y^{-3} + 6y^2)\ dy \right) dx \\
= \quad & \int_1^2 \left(\frac{5x^3y^{-2}}{-2} + \frac{6y^3}{3} \right) \Bigg|_1^x dx \\
= \quad & \int_1^2 \left(\frac{9}{2}x^3 - \frac{5}{2}x - 2 \right)\ dx \\
= \quad & \left(\frac{9}{8}x^4 - \frac{5}{4}x^2 - 2x \right) \Bigg|_1^2 \\
= \quad & \frac{89}{8}.
\end{aligned}
$$

Remark

Note how the limits of the integral are from $x = 1$ to $x = 2$ and that the final result was a number.

The previous example showed how we could perform something called an *iterated integral*; we do not yet know why we would be interested in doing so nor what the result, such as the number $89/8$, means. Before we investigate these questions, we offer some definitions.

We will continue to assume that we are integrating $f(x, y)$ over the rectangle

$$R = [a, b] \times [c, d].$$

If $x = x_0$ is kept fixed, we obtain a cross-section bounded by vertical lines $y = c$ and $y = d$, the horizontal line $z = 0$, and by the curve $z = f(x_0, y)$. The area of the cross-section is therefore given by

$$A(x_0) = \int_c^d f(x_0, y)\, dy.$$

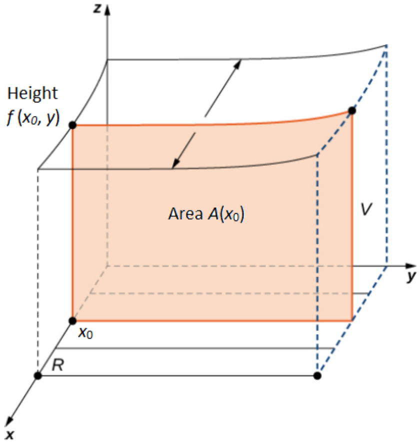


Figure 9: The cross-section $A(x_0)$.

Partial integration

We use the notation

$$\int_c^d f(x,y)\,dy$$

to mean that x is held fixed and $f(x,y)$ is integrated with respect to y from c to d. This procedure is called **partial integration** with respect to y.

We see that the cross-section area $\int_c^d f(x,y)\,dy$ is a number that depends on the value of x, so it defines a function of x:

$$\boxed{A(x) = \int_c^d f(x,y)\,dy.}$$

We now integrate the function A with respect to x from a to b. We then get

$$\int_a^b A(x)\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx.$$

The integral on the right side is called an **iterated integral**. Usually the brackets are omitted.

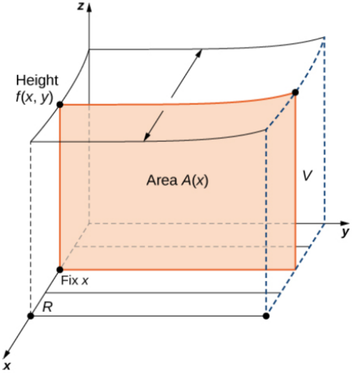
Thus

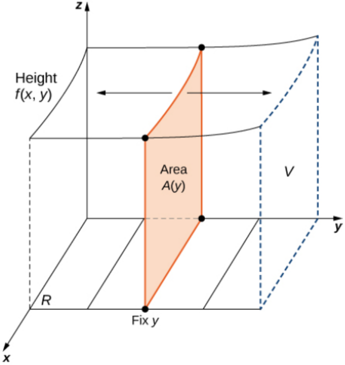
$$\int_a^b \int_c^d f(x,y)\ dydx = \int_a^b \left[\int_c^d f(x,y)\ dy\right] dx.$$

This means that we first integrate with respect to y from c to d and then with respect to x from a to b.

Similarly, we define the iterated integral:

$$\int_c^d \int_a^b f(x,y)\ dxdy = \int_c^d \left[\int_a^b f(x,y)\ dx\right] dy.$$





Integrating first w.r.t. y and then w.r.t. x to find the area $A(x)$ and then the volume V.

Integrating first w.r.t. x and then w.r.t. y to find the area $A(y)$ and then the volume V.

Thus,

Iterated integrals

- $$\int_a^b \int_c^d f(x,y)\ dydx = \int_a^b \left[\int_c^d f(x,y)\ dy\right] dx.$$
- $$\int_c^d \int_a^b f(x,y)\ dxdy = \int_c^d \left[\int_a^b f(x,y)\ dx\right] dy.$$

Example

Evaluate the iterated integrals:
(a) $\int_0^1 \int_1^2 x^2 y\ dydx$ (b) $\int_1^2 \int_0^1 x^2 y\ dxdy$.

Double integrals
and
Iterated integrals

The previous example illustrates a general fact: The value of an iterated integral does not depend on the order in which the integration is performed. This is a part of Fubini's Theorem. Even more important, Fubini's Theorem gives a practical method for evaluating a double integral by expressing it as an iterated integral.

Fubini's theorem

Suppose that $f(x, y)$ is is continuous over a rectangle

$$R = \{(x, y) \in \mathbb{R}^2 | a \le x \le b,\ c \le y \le d\}.$$

Then

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dydx = \int_c^d \int_a^b f(x, y)\, dxdy.$$

More generally, this is true if f is bounded on R and f is discontinuous only on a finite number of continuous curves.

What Fubini's Theorem says is that

- The value of an iterated integral does not depend on the order in which the integration is performed.

$$\int_a^b \int_c^d f(x,y)\, dydx = \int_c^d \int_a^b f(x,y)\, dxdy.$$

- A double integral can be calculated as an iterated integral.

$$\iint_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dydx$$

- The volume V can be calculated as the integral of the cross section perpendicular to the x or y-axis.

$$\iint_R f(x,y)\, dA = \int_a^b A(x)\, dx = \int_c^d A(y)\, dy.$$

Example

Compute the following double integral over the indicated rectangle.

$$\iint_R x\ dA,\ \ R = [0,2] \times [0,1].$$

Fubini's theorem offers an easier way to evaluate the double integral by the use of an iterated integral. Note how the boundary values of the region R become the upper and lower limits of integration.

Example

Compute the following double integral over the indicated rectangle.

$$\iint_R (2x - 4y^3)\ dA,\ \ R = [-5, 4] \times [0, 3].$$

Fubini's theorem offers an easier way to evaluate the double integral by the use of an iterated integral. Note how the boundary values of the region R become the upper and lower limits of integration.

Find the volume under the surface $z = \sqrt{1 - x^2}$ and above the triangle formed by $y = x, x = 1$, and the x-axis.

Solution

Let's consider the two possible ways to set this up:

$$\int_0^1 \int_0^x \sqrt{1-x^2}\, dy\, dx \quad \text{or} \quad \int_0^1 \int_y^1 \sqrt{1-x^2}\, dx\, dy.$$

Which appears easier? In the first, the inner integral is easy, because we need an anti-derivative with respect to y, and the entire integrand $\sqrt{1-x^2}$ is constant with respect to y.

Of course, the outer integral may be more difficult. In the second, the inner integral is mildly unpleasant – a trigonometric substitution.

Solution...

So let's try the first one, since the first step is easy, and see where that leaves us.

$$\int_0^1 \int_0^x \sqrt{1-x^2}\, dy\, dx = \int_0^1 y\sqrt{1-x^2}\Big|_0^x\, dx$$
$$= \int_0^1 x\sqrt{1-x^2}\, dx.$$

This is quite easy, since the substitution $u = 1 - x^2$ works:

$$\int x\sqrt{1-x^2}\, dx = -\frac{1}{2}\int \sqrt{u}\, du$$
$$= -\frac{1}{3}u^{2/3}$$
$$= -\frac{1}{3}(1-x^2)^{2/3}.$$

Therefore,

$$\int_0^1 x\sqrt{1-x^2}\,dx = -\frac{1}{3}(1-x^2)^{2/3}\Big|_0^1 = \frac{1}{3}.$$

This is a good example of how the order of integration can affect the complexity of the problem. In this case it is possible to do the other order, but it is a bit messier. In some cases one order may lead to a very difficult or impossible integral; it's usually worth considering both possibilities before going very far.

Compute the following double integral over the indicated rectangle.

$$\iint_R y\sin(xy)\ dA,\ \ R = [1,2]\times[0,\pi].$$

It is easier to integrate first with respect to x and then with respect to y.

Find the volume of the solid S enclosed by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$, and three coordinate planes.

Solution

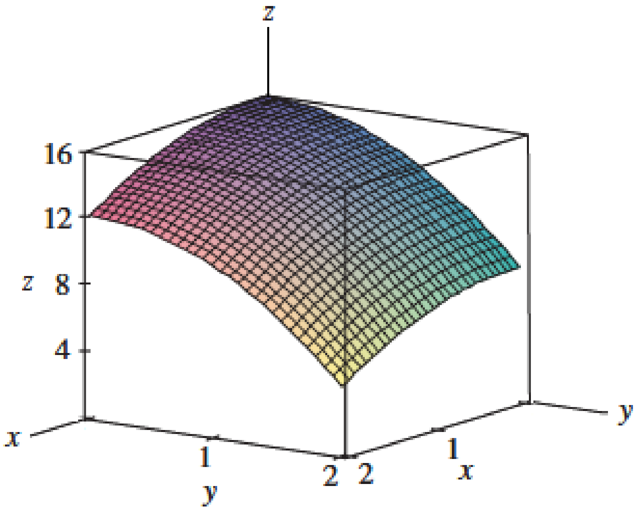


Figure 10:

Find the volume of the solid enclosed by the planes $4x + 2y + z = 10, y = 3x, z = 0, x = 0.$

Solution

Notice that the planes $4x + 2y + z = 10$ is the top of the volume and the planes $z = 0$ and $x = 0$ indicate that the plane $4x + 2y + z = 10$ does not go past the xy-plane and the yz-plane. So we are really looking for the volume under the plane

$$z = 10 - 4x - 2y$$

and above the region R in the xy-plane.

Solution...

The second plane, $y = 3x$, gives one of the sides of the volume as shown below.

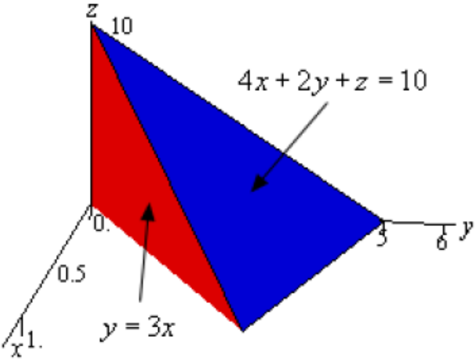


Figure 11:

The region R will be the region in the xy-plane (i.e. $z = 0$) that is bounded by $y = 3x, x = 0$, and the line where $4x + 2y + z = 10$ intersects the xy-plane. We can determine where $4x + 2y + z = 10$ intersects the xy-plane by plugging $z = 0$ into it.

$$\begin{aligned} & 4x + 2y + 0 = 10 \\ \Rightarrow \quad & 2x + y = 5 \\ \Rightarrow \quad & y = -2x + 5. \end{aligned}$$

Solution...

So, here is a sketch the region R.

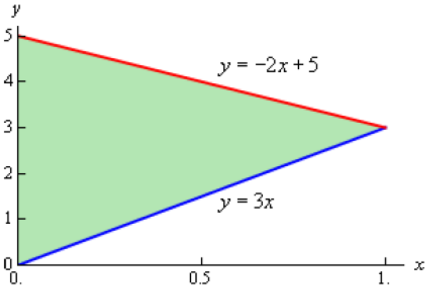


Figure 12:

Solution...

The region R is really where this solid will sit on the xy-plane and here are the inequalities that define the region.

$$0 \leq x \leq 1$$
$$3x \leq y \leq -2x + 5.$$

A special case:

$$f(x, y) = f(x)h(y) \text{ on } R = [a, b] \times [c, d].$$

In this case,

$$\begin{aligned}\iint_R f(x, y)\ dA &= \int_a^b \int_c^d g(x)h(y)\ dy\, dx\\ &= \int_a^b g(x)\ dx \int_c^d h(y)\ dy.\end{aligned}$$

Example

Compute the double integral of

$$f(x, y) = \frac{1 + x^2}{1 + y^2},$$

in the rectangular region $R = [0, 2] \times [0, 1]$.