Date
Page

(20) Prove that the space of all 3×2 matrices over the field of real number $\mathbb{R}$, has dimension 3×2.

Solution:-

Let V be set of all 3×2 matrices with real entries. Then V is the vector space where addition and scalar multiplication are defined by

(i) Let $A = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}_{3\times 2}$ & $B = (b_{ij}) = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}$

Then,

$$A+B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \\ a_{31}+b_{31} & a_{32}+b_{32} \end{pmatrix}_{3\times 2}$$

(ii) For $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$ and scalar $c \in F$

We have,

$$cA = c\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

$$= \begin{pmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \\ ca_{31} & ca_{32} \end{pmatrix}_{3\times 2}$$

Date
Page

Matrices

(1) Prove the right distributive law:-
$(B+C)A = BA+CA$.

Solution:-

Let $A=(a_{ij})$, $B(b_{ij})$ and $C=(c_{ij})$ be three matrices of order $n\times p$, $m\times n$ and $m\times n$ respectively. Then $(B+C)A$ is an $m\times p$ matrix. Also, since both BA and CA are $m\times p$ matrices, $(BA+CA)$ is also of an $m\times p$ matrix.

Thus order of $(B+C)A$ = order of $(BA+CA)$

Again, we have $B+C=(b_{ij}+c_{ij})$.

Now, $(i,j)^{th}$ element of $(B+C)A = \sum_{k=1}^{n}(b_{ik}+c_{ik})a_{kj}$

$$= \sum_{k=1}^{n} b_{ik}a_{kj} + \sum_{k=1}^{n} c_{ik}a_{kj}$$

$= (i,j)^{th}$ element of BA + $(i,j)^{th}$ element of CA

$= (i,j)^{th}$ element of $BA+CA$.

$\therefore (B+C)A = BA+CA$.

Date
Page

If A and B are matrices of appropriate size suitable for multiplication, then prove that $AB \neq BA$, in general.

Solution:-

Let $A = (a_{ij})$, $B = (b_{ij})$ be square matrix of same order then we can compute AB and BA.

Let the order be say $m \times m$.

Then,

$AB = D = (d_{ij})$ of $m \times m$ matrix, where

$$d_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj} \quad \text{----} \quad (I)$$

Again,

$BA = C = (c_{ij})$ of $m \times m$ matrix, where,

$$c_{ij} = \sum_{k=1}^{m} b_{ik} a_{kj} \quad \text{----} \quad (II)$$

In above, $a_{ik} b_{kj}$ may not be equal to $b_{ik} a_{kj}$ as b_{kj} and b_{ik}, a_{ik} and a_{kj} are element of different position.

Hence, AB may not be equal to BA in every case. #

For example;-

let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then

$$AB = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Also

$$BA = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Hence, $AB \neq BA$ #

Date
Page

If A and B are matrices of appropriate size suitable for multiplication, then $AB=0$ may or may not imply $A=0$ or $B=0$.

Solution:-

let $A=\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $B=\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$ Then,

$$AB=\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$$

$$=\begin{pmatrix} 1-1 & 0-0 \\ 1-1 & 0-0 \end{pmatrix}$$

$$\therefore AB=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O$$

Here, neither A nor B is a zero matrix. So $AB=0$ may or may not imply $A=0$ or $B=0$. #

Date _____
Page _____

(8) For any positive integer n, show that
$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}.$$

Solution :-
Let $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda^1 & 1\lambda^{1-1} \\ 0 & \lambda \end{pmatrix}$

Now,

$A^2 = A \times A$
$$= \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$
$$= \begin{pmatrix} \lambda^2 & \lambda+\lambda \\ 0 & 0+\lambda^2 \end{pmatrix}$$
$$= \begin{pmatrix} \lambda^2 & 2\lambda^{2-1} \\ 0 & \lambda^2 \end{pmatrix}$$

Again,

$A^3 = A^2 \times A$
$$= \begin{pmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{pmatrix}\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$
$$= \begin{pmatrix} \lambda^3 & \lambda^2+2\lambda^2 \\ 0 & \lambda^3 \end{pmatrix}$$
$$= \begin{pmatrix} \lambda^3 & 3\lambda^{3-1} \\ 0 & \lambda^3 \end{pmatrix}$$

Similarly, continuing same process we can conclude that $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$

where n is positive integer.

Date
Page

(9) If $a_1, a_2, \ldots a_n$ are numbers, and $A_i = \begin{pmatrix} 1 & a_i \\ 0 & 1 \end{pmatrix}$, show that $A_1 A_2 \cdots A_n = \begin{pmatrix} 1 & \sum_{i=1}^{n} a_i \\ 0 & 1 \end{pmatrix}$

Solution,

$$A_i = \begin{pmatrix} 1 & a_i \\ 0 & 1 \end{pmatrix}$$

Now,

$$A_1 = \begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 1 & a_3 \\ 0 & 1 \end{pmatrix}$$

$\vdots$

$$A_n = \begin{pmatrix} 1 & a_n \\ 0 & 1 \end{pmatrix}$$

Then

$$A_1 \cdot A_2 = \begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & a_2 + a_1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & a_1 + a_2 \\ 0 & 1 \end{pmatrix}$$

Again,

$$A_1 \cdot A_2 \cdot A_3 = \begin{pmatrix} 1 & a_1 + a_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_3 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & a_3 + a_1 + a_2 \\ 0 & 1 \end{pmatrix}$$

CamScanner

$$= \begin{pmatrix} 1 & a_1+a_2+a_3 \\ 0 & 1 \end{pmatrix}$$

Then,

$$A_1 \cdot A_2 \cdot A_3 \cdot A_4 = \begin{pmatrix} 1 & a_1+a_2+a_3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_4 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & a_4+a_1+a_2+a_3 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & a_1+a_2+a_3+a_4 \\ 0 & 1 \end{pmatrix}$$

Continuing similar process we can conclude that

$$A_1 \cdot A_2 \cdot A_3 \cdots A_n = \begin{pmatrix} 1 & a_1+a_2+a_3+\cdots+a_n \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \sum_{j=1}^{n} a_j \\ 0 & 1 \end{pmatrix}$$

#

Date
Page

(10) If $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \end{pmatrix}$, show that A^TA and AA^T are symmetric.

Solution:-

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \end{pmatrix}_{2\times3}$$

$$A^T = \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 0 & -1 \end{pmatrix}_{3\times2}$$

Now,

$$AA^T = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \end{pmatrix}_{2\times3} \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 0 & -1 \end{pmatrix}_{3\times2}$$

$$= \begin{pmatrix} 1+4+0 & 2+6+0 \\ 2+6+0 & 4+9+1 \end{pmatrix}_{2\times2}$$

$$= \begin{pmatrix} 5 & 8 \\ 8 & 14 \end{pmatrix}_{2\times2}$$

Hence, AA^T is symmetric of 2×2.

Again,

$$A^TA = \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 0 & -1 \end{pmatrix}_{3\times2} \begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \end{pmatrix}_{2\times3}$$

$$= \begin{pmatrix} 1+4 & 2+6 & 0-2 \\ 2+6 & 4+9 & 0-3 \\ 0-2 & 0-3 & 0+1 \end{pmatrix}$$

CamScanner

Date
Page

$$= \begin{pmatrix} 5 & 8 & -2 \\ 8 & 13 & -3 \\ -2 & -3 & 1 \end{pmatrix}_{3\times 3}$$

Hence, A^TA is symmetric of 3×3.

(11) If $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 5 \\ 1 & 3 & 4 \end{pmatrix}$, verify that $\frac{1}{2}(A+A^T)$ is symmetric and $\frac{1}{2}(A-A^T)$ is skew-symmetric and $A = \frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T)$.

Solution:-

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 5 \\ 1 & 3 & 4 \end{pmatrix}_{3\times 3}$$

$$A^T = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 3 \\ 3 & 5 & 4 \end{pmatrix}_{3\times 3}$$

Now,

$$\frac{1}{2}(A+A^T) = \frac{1}{2}\left\{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 5 \\ 1 & 3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 3 \\ 3 & 5 & 4 \end{pmatrix}\right\}$$

$$= \frac{1}{2}\begin{pmatrix} 2 & 4 & 4 \\ 4 & 0 & 8 \\ 4 & 8 & 8 \end{pmatrix}$$

Date
Page

$$= \begin{pmatrix} 1 & 2 & 2 \\ 2 & 0 & 4 \\ 2 & 4 & 4 \end{pmatrix}$$

Above we can see $\frac{1}{2}(A+A^T)$ is symmetric.

Then,

$$\frac{1}{2}(A-A^T) = \frac{1}{2}\left\{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 5 \\ 1 & 3 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 3 \\ 3 & 5 & 4 \end{pmatrix}\right\}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 2 \\ -2 & -2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}$$

Hence, we can say $\frac{1}{2}(A-A^T)$ is skew-symmetric

Finally, we have to show $A = \frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T)$

So,

$$\frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T) = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 0 & 4 \\ 2 & 4 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 5 \\ 1 & 3 & 4 \end{pmatrix} = A.$$

Date
Page

$$\therefore A = \frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T)$$ #

- Simple properties of transpose
- Adjoint of a square matrix

(1) Verify the property $A\cdot(\text{Adj}\,A) = \text{Adj}\,A = |A|\,I$ and find the inverse for each of the following non-singular.

(a) $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Solution,

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $|A| = (ad - bc)$

$\text{Adj}\,A = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Now,

$$A\cdot(\text{Adj}\,A) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \begin{pmatrix} ad - bc & -ab + ab \\ cd - cd & -bc + ad \end{pmatrix}$$

$$= \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix}$$

$$= (ad - bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{2\times 2}$$

$$= (ad - bc)\, I_2$$

$$A\cdot(\text{Adj}\,A) = |A|\, I_2$$

Date
Page

Again,

$$(\text{Adj}\cdot A)\cdot A = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \begin{pmatrix} ad-bc & bd-bd \\ -ac+ac & -bc+ad \end{pmatrix}$$

$$= \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

$$= (ad-bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{2\times 2}$$

$$(\text{Adj}\cdot A)\cdot A = |A|\cdot I_2$$

Hence, $(\text{Adj}\cdot A)\cdot A = A\cdot(\text{Adj}\cdot A) = |A|\, I_2$

Finally,

$$A^{-1} = \frac{1}{|A|}\cdot(\text{Adj}\cdot A)$$

$$= \frac{1}{(ad-bc)}\cdot\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

#

(2) Which of the following matrices have reciprocal? If they exist, find their reciprocal

(a) $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 5 & 7 \end{pmatrix}$

Solution,

$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 5 & 7 \end{pmatrix}$

$|A| = 1(21-20) + 2(4-14) + 3(10-3)$

$= 1 - 20 + 21$

$= 2$

$\therefore A^{-1}$ exist.

Now, To find Adjoint of A.

first we have to find cofactors:-

$A_{11} = (-1)^2 \begin{vmatrix} 3 & 4 \\ 5 & 7 \end{vmatrix} = 21 - 20 = 1$

$A_{12} = (-1)^3 \begin{vmatrix} 2 & 4 \\ 1 & 7 \end{vmatrix} = -(14-4) = -10$

$$A_{13} = (-1)^4 \begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} = 10 - 3 = 7$$

$$A_{21} = (-1)^3 \begin{vmatrix} 2 & 3 \\ 5 & 7 \end{vmatrix} = -(14-15) = 1$$

$$A_{22} = (-1)^4 \begin{vmatrix} 1 & 3 \\ 1 & 7 \end{vmatrix} = 7 - 3 = 4$$

$$A_{23} = (-1)^5 \begin{vmatrix} 1 & 2 \\ 1 & 5 \end{vmatrix} = -(5-2) = -3$$

$$A_{31} = (-1)^4 \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = (8-9) = -1$$

$$A_{32} = (-1)^5 \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = -(4-6) = +2$$

$$A_{33} = (-1)^6 \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = (3-4) = -1$$

$$\therefore \text{Adj}: A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & -10 & 7 \\ 1 & 4 & -3 \\ -1 & +2 & -1 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & 1 & -1 \\ -10 & 4 & +2 \\ 7 & -3 & -1 \end{bmatrix}$$

So, we have,

$$A^{-1} = \frac{1}{|A|} \cdot Adj\,A$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 \\ -10 & 4 & -2 \\ 7 & -3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -5 & 2 & +1 \\ 7/2 & -3/2 & -1/2 \end{bmatrix} \#$$

Date
Page

(B) Show that the inverse of the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \text{ is } \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}.$$

Solution:-

Here, $\omega = \frac{-1+\sqrt{3}i}{2}$, $\omega^2 = \frac{-1-\sqrt{3}i}{2}$

Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}$

$|A| = 1(\omega^2-\omega) - 1(\omega-\omega^2) + (\omega^2-\omega)$

$= \omega^2-\omega-\omega+\omega^2+\omega^2-\omega$

$= 3\omega^2-3\omega$

$= 3(\omega^2-\omega) \neq 0$

So, A^{-1} exists.

Now,

$A_{11} = \omega^2-\omega$

$A_{12} = -(\omega-\omega^2) = \omega^2-\omega$

$A_{13} = (\omega^2-\omega)$

$A_{21} = -(\omega-\omega^2) = \omega^2-\omega$

$A_{22} = (\omega-1)$

$A_{23} = -(\omega^2-1) = 1-\omega^2$

$A_{31} = \omega^2-\omega$

Date
Page

$A_{32} = -(\omega^2 - 1) = 1 - \omega^2$

$A_{33} = (\omega - 1)$

Now,

$$\text{Adj } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T = \begin{bmatrix} \omega^2-\omega & \omega^2-\omega & \omega^2-\omega \\ \omega^2-\omega & \omega-1 & 1-\omega^2 \\ \omega^2-\omega & 1-\omega^2 & \omega-1 \end{bmatrix}$$

$$= \begin{bmatrix} \omega^2-\omega & \omega^2-\omega & \omega^2-\omega \\ \omega^2-\omega & \omega-1 & 1-\omega^2 \\ \omega^2-\omega & 1-\omega^2 & \omega-1 \end{bmatrix}$$

Then,

$$A^{-1} = \frac{1}{|A|} \text{Adj } A$$

$$= \frac{1}{3(\omega^2-\omega)} \begin{bmatrix} \omega^2-\omega & \omega^2-\omega & \omega^2-\omega \\ \omega^2-\omega & \omega-1 & 1-\omega^2 \\ \omega^2-\omega & 1-\omega^2 & \omega-1 \end{bmatrix}$$

$$= \frac{(\omega-1)}{3\omega(\omega^2-1)} \begin{bmatrix} \omega & \omega & \omega \\ \omega & 1 & -1-\omega \\ \omega & -1-\omega & 1 \end{bmatrix}$$

$$= \frac{\cancel{\omega(\omega-1)}}{3\cancel{\omega(\omega-1)}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{1}{\omega} & -\frac{1}{\omega}-1 \\ 1 & -\frac{1}{\omega}-1 & \frac{1}{\omega} \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & -\omega^2-1 \\ 1 & -\omega^2-1 & \omega^2 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix} \quad [\because \omega = -\omega^2 - 1] \qquad \therefore A = A^{-1}$$

proved

Date
Page

(4) Verify the property $(A^T)^{-1} = (A^{-1})^T$ for the matrix: $A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$

Solution:-

$A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$

$|A| = 3-2 = 1$

Now,

$\text{Adj}\,A = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}$

$A^{-1} = \frac{1}{|A|}(\text{Adj}\,A)$

$= \frac{1}{1}\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$

$= \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$

$A^T = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$

$|A^T| = 3-2 = 1$

Now,

$\text{Adj}\,A^T = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$

$(A^T)^{-1} = \frac{1}{|A^T|}(\text{Adj}\,A^T)$

$= \frac{1}{1}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$

$[A^{-1}]^T = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}^T$

$= \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$

$\therefore [A^{-1}]^T = [A^T]^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$ #

CamScanner

Date
Page

(5) Verify the property $(AB)^{-1} = B^{-1}A^{-1}$ for the following matrices:

(a) $A = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 0 \\ 1 & 2 \end{pmatrix}$

Solution:-

$$AB = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 15+1 & 0+2 \\ 20+2 & 0+4 \end{pmatrix}$$

$$= \begin{pmatrix} 16 & 2 \\ 22 & 4 \end{pmatrix}$$

Now, For $(AB)^{-1}$

$$|AB| = \begin{vmatrix} 16 & 2 \\ 22 & 4 \end{vmatrix}$$

$$= 64 - 44$$

$$= 20$$

$$\text{Adj}: AB = \begin{bmatrix} 4 & -2 \\ -22 & 16 \end{bmatrix}$$

So, $(AB)^{-1} = \dfrac{1}{|AB|} \text{Adj}: AB$

$$= \frac{1}{20}\begin{bmatrix} 4 & -2 \\ -22 & 16 \end{bmatrix}$$

$$= \begin{bmatrix} 1/5 & -\frac{1}{10} \\ -\frac{11}{10} & \frac{4}{5} \end{bmatrix}$$

Date
Page

For A^{-1}

$$|A| = \begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix}$$
$$= 6-4$$
$$= 2$$

$$\text{Adj } A = \begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{Adj } A$$

$$= \frac{1}{2}\begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} \\ -2 & \frac{3}{2} \end{bmatrix}$$

For B^{-1}

$$|B| = \begin{vmatrix} 5 & 0 \\ 1 & 2 \end{vmatrix}$$
$$= 10-0$$
$$= 10$$

$$\text{Adj } B = \begin{bmatrix} 2 & 0 \\ -1 & 5 \end{bmatrix}$$

$$B^{-1} = \frac{1}{|B|} \text{Adj } B$$

$$= \frac{1}{10}\begin{bmatrix} 2 & 0 \\ -1 & 5 \end{bmatrix}$$
$$= \begin{bmatrix} 1/5 & 0 \\ -\frac{1}{10} & 1/2 \end{bmatrix}$$

Now,

$$B^{-1}A^{-1} = \begin{bmatrix} 1/5 & 0 \\ -\frac{1}{10} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & -\frac{1}{2} \\ -2 & \frac{3}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{5}+0 & -\frac{1}{10}+0 \\ -\frac{1}{10}-\frac{1}{2}\times 2 & +\frac{1}{20}+\frac{3}{4} \end{bmatrix}$$

$$= \begin{bmatrix} 1/5 & -1/10 \\ -11/10 & \frac{1+15}{20} \end{bmatrix}$$

$$= \begin{bmatrix} 1/5 & -1/10 \\ -11/10 & \frac{16}{20} \end{bmatrix}$$

$$= \begin{bmatrix} 1/5 & -1/10 \\ -11/10 & 4/5 \end{bmatrix}$$

CamScanner

Linear Transformations

Date
Page

(1) Are the following mapping linear?

(i) $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x,y) = (x+y, x-y)$

solution:- $v_1 = (x_1, y_1)$, $v_2 = (x_2, y_2)$

let $v_1, v_2 \in \mathbb{R}^2$ such that

$$T(v_1 + v_2) = T((x_1, y_1) + (x_2, y_2))$$
$$= T(x_1 + x_2, y_1 + y_2)$$
$$= (x_1 + x_2 + y_1 + y_2, x_1 + x_2 - y_1 - y_2)$$

Again,

$$T(v_1) + T(v_2) = (x_1 + y_1, x_1 - y_1) + (x_2 + y_2, x_2 - y_2)$$
$$= (x_1 + y_1 + x_2 + y_2, x_1 - y_1 + x_2 - y_2)$$
$$= (x_1 + x_2 + y_1 + y_2, x_1 + x_2 - y_1 - y_2)$$

Here, we get, $T(v_1 + v_2) = T(v_1) + T(v_2)$

let $v = (x, y)$

let $v \in V$ and $c \in F$ such that

$$T(cv) = T(cx, cy)$$
$$= (cx + cy, cx - cy)$$
$$= (c(x+y); c(x-y))$$
$$= c(x+y, x-y)$$
$$= cT(v)$$

$\therefore T(cv) = cT(v)$

$\therefore$ T is linear. #

[i.e. $T: \mathbb{R}^2 \to \mathbb{R}^2$ is linear.

(ii) $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x,y) = (x, y, x+y)$

Solution:-

let $v_1 = (x_1, y_1)$ & $v_2 = (x_2, y_2) \in \mathbb{R}^2$ such that

$$T(v_1 + v_2) = T((x_1, y_1) + (x_2, y_2))$$
$$= T(x_1 + x_2, y_1 + y_2)$$
$$= (x_1 + x_2, y_1 + y_2, x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2)$$

Again, $T(v_1) + T(v_2) = (x_1, y_1, x_1y_1) + (x_2, y_2, x_2y_2)$
$$= (x_1 + x_2, y_1 + y_2, x_1y_1 + x_2y_2)$$

Here, we get,

$$T(v_1 + v_2) \neq T(v_1) + T(v_2)$$

Hence, $T: \mathbb{R}^2 \to \mathbb{R}^3$ is non-linear.

(iii) $T: \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x,y,z) = (x+y+z, 0)$

Solution:-

let $v_1 = (x_1, y_1, z_1)$ and $v_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$ such that

$$T(v_1 + v_2) = T((x_1, y_1, z_1) + (x_2, y_2, z_2))$$
$$= T(x_1 + x_2, y_1 + y_2, z_1 + z_2)$$
$$= (x_1 + x_2 + y_1 + y_2 + z_1 + z_2, 0)$$

Again, $T(v_1) + T(v_2) = (x_1 + y_1 + z_1, 0) + (x_2 + y_2 + z_2, 0)$
$$= (x_1 + y_1 + z_1 + x_2 + y_2 + z_2, 0)$$
$$= (x_1 + x_2 + y_1 + y_2 + z_1 + z_2, 0)$$

So, from above, we get,

$$T(v_1 + v_2) = T(v_1) + T(v_2)$$

let $v = (x, y, z) \in \mathbb{R}^3$ and $c \in F$ such that

$$T(cv) = T(cx, cy, cz)$$
$$= (cx + cy + cz, 0)$$
$$= c(x + y + z, 0)$$
$$= cT(v)$$

$\therefore T(cv) = cT(v)$

$\therefore T$ is linear. [$T: \mathbb{R}^3 \to \mathbb{R}^2$ is linear.]

(iv) $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x+1, y)$

Solution:-

let $v_1 = (x_1, y_1)$ and $v_2 = (x_2, y_2) \in \mathbb{R}^2$ such that

$$T(v_1 + v_2) = T((x_1, y_1) + (x_2, y_2))$$
$$= T(x_1 + x_2, y_1 + y_2)$$
$$= (x_1 + x_2 + 1, y_1 + y_2)$$

Again,

$$T(v_1) + T(v_2) = (x_1 + 1, y_1) + (x_2 + 1, y_2)$$
$$= (x_1 + x_2 + 2, y_1 + y_2)$$

Here, we get,

$$T(v_1 + v_2) \neq T(v_1) + T(v_2)$$

Hence, $T: \mathbb{R}^2 \to \mathbb{R}^2$ is non-linear.

(v) $T: \mathbb{R} \to \mathbb{R}$ defined by $T(x) = x+2$

Solution:-

Let $x_1, x_2 \in \mathbb{R}$ such that

$T(x_1 + x_2) = x_1 + x_2 + 2$

$T(x_1) + T(x_2) = x_1 + 2 + x_2 + 2$

$= x_1 + x_2 + 4$

Here, $T(x_1 + x_2) \neq T(x_1) + T(x_2)$

Hence, $T: \mathbb{R} \to \mathbb{R}$ is not linear.

Date
Page

Exercises :-

(1) Let $T: \mathbb{R}^2 \to \mathbb{R}$ be the linear transformation for which $T(1,1) = 3$ and $T(0,1) = -2$, Prove that

(a) $T(1,0) = 5$

Solution,

We express $(1,0)$ as the linear combination of $(1,1)$ and $(0,1)$.

So, let c_1 & $c_2 \in F$ such that

$(1,0) = c_1(1,1) + c_2(0,1)$ --- (I)

$(1,0) = (c_1, c_1) + (0, c_2)$

$(1,0) = (c_1, c_1 + c_2)$

Here, $c_1 = 1$

$c_1 + c_2 = 0$

$\therefore c_2 = -1$

Substituting c_1 & c_2 in (I) we get,

$(1,0) = 1(1,1) - 1(0,1)$ ---- (II)

Applying T on both side of (II) we get,

$T(1,0) = T\{1(1,1) - 1(0,1)\}$

$= T(1,1) - T(0,1)$

$= 3 - (-2)$

$= 3 + 2$

$= 5$

$\therefore T(1,0) = 5$ #

(b) $T(1, 1/2) = 4$

Solution:-

We express $(1, 1/2)$ as the linear combination ~~[illegible]~~ of $(1,1)$ and $(0,1)$.

So, let c_1 and $c_2 \in F$ such that

$(1, 1/2) = c_1(1,1) + c_2(0,1)$ --- (1)

$(1, 1/2) = (c_1, c_1) + (0, c_2)$

$(1, 1/2) = (c_1, c_1 + c_2)$

$c_1 = 1$

$c_1 + c_2 = 1/2$

Solving we get $c_1 = 1$ and $c_2 = -1/2$

Substituting c_1 and c_2 in (1) we get,

$(1, 1/2) = 1(1,1) - \frac{1}{2}(0,1)$ --- (1)

Applying T on both side we get,

$$T(1, 1/2) = T\{1(1,1) - \tfrac{1}{2}(0,1)\}$$

$$= T(1,1) - \frac{1}{2}(T(0,1))$$

$$= 3 - \frac{1}{2} \times (-2)$$

$$= 3 + 1 \quad = 4$$

$\therefore T(1, 1/2) = 4$ #

(c) $T(2,5) = 0$

Solution,

We express $(2,5)$ as the linear combination of $(1,1)$ and $(0,1)$.

So, let $c_1, c_2 \in F$ such that

$(2,5) = c_1(1,1) + c_2(0,1)$ ——— (1)

$(2,5) = (c_1, c_1 + c_2)$

Here,

$c_1 = 2$ and $c_2 = 3$

Substituting c_1 & c_2 in (1) we get,

$(2,5) = 2(1,1) + 3(0,1)$

Applying T on both side we get,

$$\begin{aligned} T(2,5) &= T\{2(1,1) + 3(0,1)\} \\ &= 2T(1,1) + 3T(0,1) \\ &= 2 \times 3 + 3 \times (-2) \\ &= 6 - 6 \\ &= 0 \end{aligned}$$

$\therefore T(2,5) = 0$ #

Date
Page

(2) Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(0,1) = (2,1)$, $T(1,4) = (0,-2)$. If T is linear, prove that $T(a,b) = (2b-8a, b-6a)$.

→ Solution:-

We express (a,b) as the linear combination of $(0,1)$ and $(1,4)$.

So, let $c_1, c_2 \in F$ such that,

$(a,b) = c_1(0,1) + c_2(1,4)$ ——— (I)

$(a,b) = (0, c_1) + (c_2, 4c_2)$

$(a,b) = (c_2, c_1 + 4c_2)$

Here,

$c_2 = a$, $c_1 = b-4a$

Substituting c_1 & c_2 in (I) we get,

$(a,b) = (b-4a)(0,1) + a(1,4)$ ——— (II)

Applying T on both side we get,

$$T(a,b) = T\{(b-4a)(0,1) + a(1,4)\}$$
$$= (b-4a)T(0,1) + aT(1,4)$$
$$= (b-4a)(2,1) + a(0,-2)$$
$$= (2b-8a, b-6a)$$

$\therefore T(a,b) = (2b-8a, b-6a)$ #

Date
Page

(3) Let $T: \mathbb{R}^3 \to \mathbb{R}$ be the linear transformation for which $T(1,1,0)=1$, $T(1,1,1)=2$, $T(0,1,0)=-1$. Prove that

(a) $T(a,b,c) = 2a-b+c$.

Solution,

We express (a,b,c) as the linear combination of $(1,1,0)$, $(1,1,1)$ and $(0,1,0)$.

So, let $x_1, x_2, x_3 \in F$ such that,

$(a,b,c) = x_1(1,1,0) + x_2(1,1,1) + x_3(0,1,0)$

$(a,b,c) = (x_1, x_1, 0) + (x_2, x_2, x_2) + (0, x_3, 0)$

$(a,b,c) = (x_1+x_2, x_1+x_2+x_3, x_2)$

Here,

$x_1+x_2 = a$, $x_1+x_2+x_3 = b$, $x_2 = c$.

Solving, we get,

$x_1 = a-c$, $x_2 = c$, $x_3 = b-a$

Substituting x_1, x_2, x_3 in (1) we get,

$(a,b,c) = (a-c)(1,1,0) + c(1,1,1) + (b-a)(0,1,0)$

Applying T on both side we get,

$$\begin{aligned} T(a,b,c) &= T\{(a-c)(1,1,0) + c(1,1,1) + (b-a)(0,1,0)\} \\ &= (a-c)T(1,1,0) + cT(1,1,1) + (b-a)T(0,1,0) \\ &= (a-c)1 + c\cdot 2 + (b-a)(-1) \\ &= a-c+2c+a-b \\ &= 2a-b+c \end{aligned}$$

$\therefore T(a,b,c) = 2a-b+c$ #

(b) $T(2,1,0) = 3$.

Solution,

We have,

$T(a,b,c) = 2a - b + c$

let $a = 2$, $b = 1$ and $c = 0$; we get,

$T(2,1,0) = 2\times 2 - 1 + 0$
$= 3$

$\therefore T(2,1,0) = 3$

(4) Show that $T(a,b) = (2b-a, b-3a, 7a-b)$, where $T: \mathbb{R}^2 \to \mathbb{R}^3$ is linear and is defined by $T(1,2) = (3,-1,5)$ and $T(0,1) = (2,1,-1)$

Solution:-

We express (a,b) as the linear combination of $(1,2)$ and $(0,1)$.

So, let c_1 and $c_2 \in F$ such that,

$(a,b) = c_1(1,2) + c_2(0,1)$ ----- (1)

$(a,b) = (c_1, 2c_1) + (0, c_2)$

$(a,b) = (c_1, 2c_1 + c_2)$

Here,

$c_1 = a$

$2c_1 + c_2 = b$

$2a + c_2 = b$

$c_2 = b - 2a$

$\therefore c_1 = a$, $c_2 = b - 2a$

Date
Page

substituting α_1 and α_2 in (1) we get,

$(a, b) = a(1, 2) + (b-2a)(0, 1)$

Applying T on both side we get,

$T(a, b) = T\{a(1, 2) + (b-2a)(0, 1)\}$

$= aT(1, 2) + (b-a)T(0, 1)$

$= a.(3, -1, 5) + (b-2a)(2, 1, -1)$

$= (3a, -a, 5a) + (2b-4a, b-2a, 2a-b)$

$= (3a+2b-4a, -a+b-2a, 5a+2a-b)$

$= (2b-a, b-3a, 7a-b)$

$\therefore T(a, b) = (2b-a, b-3a, 7a-b)$ #

Kernel and Image of linear Transformation

(1) let $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ and $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T_1(x,y) = (x+y, y)$ and $T_2(x,y) = (x,0)$. Find formula defining the transformations.

(a) $T_1 - T_2$

Solution,

let $v = (x,y) \in \mathbb{R}^2$

$$(T_1 - T_2)v = (x+y, y) - (x, 0)$$
$$= (\cancel{x}+y-\cancel{x}, y)$$
$$= (y, y)$$

$\therefore (T_1 - T_2)v = (x, y)$ #

(b) $2T_1 + 3T_2$

Solution,

let $v = (x,y) \in \mathbb{R}^2$

$$(2T_1 + 3T_2)v = 2(x+y, y) + 3(x, 0)$$
$$= (2x+2y, 2y) + (3x, 0)$$
$$= (5x+2y, 2y)$$

$\therefore (2T_1 + 3T_2)v = (5x+2y, 2y)$ #

(c) $T_1 \circ T_2$

Solution,

let $v = (x,y) \in \mathbb{R}^2$

$$(T_1 \circ T_2)v = T_1(T_2(v))$$
$$= T_1(T_2(x,y))$$
$$= T_1(x, 0)$$
$$= (x+0, 0)$$
$$= (x, 0)$$

$\therefore (T_1 \circ T_2)v = (x, 0)$ #

(d) $T_2 \circ T_1$

Solution,

let $v = (x, y) \in \mathbb{R}^2$

$$T_2 \circ T_1 = T_2(T_1(v))$$
$$= T_2(T_1(x,y))$$
$$= T_2(x+y, y)$$
$$= (x+y, 0)$$

$\therefore (T_2 \circ T_1) v = (x+y, 0)$

(e) $T_2 \circ T_2$

Solution

let $v = (x, y) \in \mathbb{R}^2$

$$T_2 \circ T_2 = T_2(T_2(v))$$
$$= T_2(T_2(x,y))$$
$$= T_2(x, 0)$$
$$= (x, 0)$$

$\therefore (T_2 \circ T_2)(v) = (x, 0)$

Date
Page

(2) Let T_1, T_2 and T_3 be the linear transformations on $\mathbb{R}^3$ defined by

$T_1(x, y, z) = (0, y, z)$

$T_2(x, y, z) = (x, 0, z)$

$T_3(x, y, z) = (x, y, 0)$, Then show that

$$T_1(T_2 + T_3) = T_1T_2 + T_1T_3$$

Solution,

Let $v = (x, y, z) \in \mathbb{R}^3$

$T_1(v) = (0, y, z)$

$T_2(v) = (x, 0, z)$

$T_3(v) = (x, y, 0)$

Now,

$$\begin{aligned} T_1(T_2(v) + T_3(v)) &= T_1((x, 0, z) + (x, y, 0)) \\ &= T_1(2x, y, z) \\ &= (0, y, z) \end{aligned}$$

Again,

$$\begin{aligned} T_1T_2(v) &= T_1(T_2(v)) \\ &= T_1(x, 0, z) \\ &= (0, 0, z) \end{aligned}$$

$$\begin{aligned} T_1T_3(v) &= T_1(T_3(v)) \\ &= T_1(x, y, 0) \\ &= (0, y, 0) \end{aligned}$$

So,

$$\begin{aligned} T_1T_2(v) + T_1T_3(v) &= (0, 0, z) + (0, y, 0) \\ &= (0, y, z) \end{aligned}$$

$\therefore T_1(T_2 + T_3) = T_1T_2 + T_1T_3$ #

Inverse of a linear Transformation

(4) (a) Let T be a linear transformation on $\mathbb{R}^2$ defined by $T(x,y) = (x, y+x)$. Find a formula for T^{-1}.

Solution,

Suppose, $T(x,y) = (s,t)$ ——— (I) and

$T^{-1}(s,t) = (x,y)$.

We have,

$T(x,y) = (x, x+y)$ ——— (II)

From (I) & (II), we get,

$s = x$

$t = x+y$

Solving above, we get,

$x = s$ and $y = t-s$

Thus, we have,

$T^{-1}(s,t) = (x,y)$

$= (s, t-s)$

$\therefore T^{-1}(s,t) = (s, t-s)$

i.e. $T^{-1}(x,y) = (x, y-x)$ #

Date
Page

(5) Let T be the linear transformation defined on $\mathbb{R}^2$ such that $T(x,y) = (x+2y, 3y)$. Find T^{-1}.

Solution:-

Let $T(x,y) = (s,t)$ --- (I) and

$T^{-1}(s,t) = (x,y)$

We have,

$T(x,y) = (x+2y, 3y)$ --- (II)

From (I) & (II), we get,

$x + 2y = s$

$3y = t$

Solving we get,

$x = s - \frac{2t}{3}$ $\quad y = t/3$

So, we have,

$T^{-1}(s,t) = \left(s - \frac{2t}{3}, \frac{t}{3}\right)$

$\therefore T^{-1}(s,t) = \left(s - \frac{2t}{3}, \frac{t}{3}\right)$

i.e. $T^{-1}(x,y) = \left(x - \frac{2y}{3}, \frac{y}{3}\right)$ #

Date
Page

(2) (a) Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x,y,z) = (x-y, y-z, z)$. Find a formula for T^{-1}.

Solution:-

Let $T(x,y,z) = (a, b, c)$ ---- (i)

$T^{-1}(a,b,c) = (x,y,z)$

We have,

$T(x,y,z) = (x-y, y-z, z)$ ---- (ii)

From (i) & (ii), we have,

$x-y = a$

$y-z = b$

$z = c$,

Solving, we get,

$x = a+b+c \qquad y = b+c \qquad z = c$

Thus, we get,

$T^{-1}(a,b,c) = (x,y,z)$

$= (a+b+c, b+c, c)$

$\therefore T^{-1}(a,b,c) = (a+b+c, b+c, c)$

i.e. $T^{-1}(x,y,z) = (x+y+z, y+z, z)$ #

Date
Page

(b) Show that the following operator on $\mathbb{R}^3$ is invertible and find a formula for T^{-1}.
where $T(x, y, z) = (x+z, x-z, y)$.

Solution :-

(i) <u>T is one-one</u> :-

let $T(u) = T(v)$ where $u, v \in \mathbb{R}^3$

also, let $u = (x_1, y_1, z_1)$ & $v = (x_2, y_2, z_2)$

So, we get,

$(x_1+z_1, x_1-z_1, y_1) = (x_2+z_2, x_2-z_2, y_2)$

we get,

$x_1 + z_1 = x_2 + z_2$

$x_1 - z_1 = x_2 - z_2$

$y_1 = y_2$

Solving we get,

$x_1 = x_2$, $y_1 = y_2$, $z_1 = z_2$

Solving we get,

$u = v$ [which is unique]

Hence, T is one-one.

(ii) T is onto :-

let $v = (x, y, z) \in \mathbb{R}^3$ $\exists$ an element $u = (r, s, t) \in \mathbb{R}^3$ such that $T(u) = v$

or, $T(r, s, t) = (x, y, z)$

or, $(r+t, r-t, s) = (x, y, z)$

Here we get,

$r+t = x$

$r-t = y$

$s = z$

Date
Page

Solving we get,

$r = \frac{x+y}{2}$, $s = z$, $t = \frac{x-y}{2}$

So,

$T(r, s, t) = (x, y, z)$

$T\left(\frac{x+y}{2}, z, \frac{x-y}{2}\right) = (x, y, z)$

$(r, s, t) = \left(\frac{x+y}{2}, z, \frac{x-y}{2}\right) \in \mathbb{R}^3$

$\therefore$ T is one-one and onto.

Hence, linear transformation is invertible.

Now,

$T^{-1}(x, y, z) = (r, s, t)$

$T^{-1}(x, y, z) = \left(\frac{x+y}{2}, z, \frac{x-y}{2}\right)$ #

Linear map Associated with a matrix

Date
Page

(1) Find the matrix representation of the following linear transformation relative to the usual basis (standard basis for $\mathbb{R}^n$, $n = 1, 2, 3$.

(i) $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x, 2y)$

Solution:-

Basis $= \{(0,1), (1,0)\}$

let $v_1 = (1, 0)$

$v_2 = (0, 1)$

Now,

$T(v_1) = T(1,0) = (1,0)$

$T(v_2) = T(0,1) = (0,2)$

We can write, $T(v_1) = (1,0) = 1.(1,0) + 0(0,1)$

$T(v_2) = (0,2) \Rightarrow 0(1,0) + 2(0,1)$

The matrix associated with linear map T

= Transpose of co-efficient of matrix.

$$= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}_{2\times 2} \quad \#$$

Date
Page

(ii) $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x,y,z) = (x,0,z)$

Solution:-

Basis $= \{(1,0,0), (0,1,0), (0,0,1)\}$

let $v_1 = (1,0,0)$

$v_2 = (0,1,0)$

$v_3 = (0,0,1)$

Now,

$T(v_1) = (1,0,0) = 1(1,0,0) + 0(0,1,0) + 0(0,0,1)$

$T(v_2) = (0,0,0) = 0(1,0,0) + 0(0,1,0) + 0(0,0,1)$

$T(v_3) = (0,0,1) = 0(1,0,0) + 0(0,1,0) + 1(0,0,1)$

The matrix associated with linear map T.

= Transpose of co-efficient of matrix.

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$$

Date
Page

(iii) $T: \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x,y,z) = (x,z)$

Solution:-

Basis$(\mathbb{R}^3) = \{(1,0,0), (0,1,0), (0,0,1)\}$

let $v = (x,y,z) \in \mathbb{R}^3$, So,

$$T(v) = \begin{bmatrix} x \\ z \end{bmatrix}$$

$$T(x,y,z) = \begin{bmatrix} 1x + 0.y + 0z \\ 0x + 0.y + 1z \end{bmatrix}$$

$$T(x,y,z) = x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T(x,y,z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$T(v) = AV$$

$$\therefore A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \#$$

(iii) $T: \mathbb{R}^1 \to \mathbb{R}^2$ defined by $T(x) = (x, 0)$

Solution:-

Basis of $\mathbb{R}^1 = \{(1)\}$

let $v = x \in \mathbb{R}^1$ so

$$T(v) = \begin{pmatrix} x \\ 0 \end{pmatrix}$$

$$T(x) = \begin{bmatrix} 1 \cdot x \\ 1 \cdot 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} [x]$$

$$T(v) = AV$$

$$\therefore A = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \#$$

(iv) $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x, y) = (x, y, x+y)$

Solution:-

Basis of $\mathbb{R}^2 = \{(1,0), (0,1)\}$

let $v = (x, y) \in \mathbb{R}^2$

Then,

$$T(v) = \begin{bmatrix} x \\ y \\ x+y \end{bmatrix}$$

$$T(x,y) = \begin{bmatrix} 1x + 0y \\ 0x + 1y \\ 1x + 1y \end{bmatrix}$$

$$T(x,y) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$\cancel{T(x,y) =}$

$T(v) = Av$

$\therefore A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ #

Assignment - 2

Eigen values and Eigen Vectors

Q1 Write down the (i) characteristics polynomials, (ii) the characteristic equation, (iii) the characteristic value (iv) multiplicity of the characteristic value and (v) the characteristic vector of the matrix.

(a) $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Solution:-

matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Now, we have,

characteristic polynomials $= |A - \lambda I|$

$$= \left| \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right|$$

$$= \begin{vmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix}$$

$$= \lambda^2 - 1$$

Then,

characteristic equation is

$\lambda^2 - 1 = 0$

$(\lambda + 1)(\lambda - 1) = 0$

Here,

$\lambda + 1 = 0$ $\qquad$ $\lambda - 1 = 0$

$\lambda = -1$ $\qquad$ $\lambda = 1$

We get, $\lambda = -1, 1$

$\therefore$ characteristic value is $-1, 1$.

Clearly, there is no repeation in the value of λ.
Hence, multiplicity of characteristic value is 1.

For eigen vector,
let $X = \begin{pmatrix} x \\ y \end{pmatrix}$ be the eigen vector of the matrix A. Then,

$$(A - \lambda I) X = 0$$

$$\begin{bmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \quad \text{——— (1)}$$

When $\lambda = -1$

$$\begin{bmatrix} 1-(-1) & 0 \\ 0 & -1-(-1) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\begin{bmatrix} 1+1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore x = 0$

$y = k$ (say) $\quad \therefore X = \begin{pmatrix} 0 \\ k \end{pmatrix} = k \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

where, $k \neq 0$

when $\lambda = 1$

$$\begin{bmatrix} 1-1 & 0 \\ 0 & -1-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ -y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore y = 0$, $x = k$ (say)

$\therefore X = \begin{pmatrix} k \\ 0 \end{pmatrix} = k \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

where, $k \neq 0$.

$\therefore$ When $\lambda = -1$, the eigen vector is $X = k \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ where $k \neq 0$.

$\therefore$ When $\lambda = 1$, the eigen vector is $X = k \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ where $k \neq 0$.

(6) $A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$.

Solution :-

Matrix $A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$

Characteristic matrix $= A - \lambda I$

$$= \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix}$$

Characteristic polynomials $= |A - \lambda I|$

$$= \begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix}$$

$$= (5-\lambda)(2-\lambda) - 4$$

$$= 10 - 7\lambda + \lambda^2 - 4$$

$$= \lambda^2 - 7\lambda + 6$$

$$= \lambda^2 - 6\lambda - \lambda + 6$$

$$= \lambda(\lambda - 6) - 1(\lambda - 6)$$

$$= (\lambda - 6)(\lambda - 1)$$

Characteristic equation is

$(\lambda - 6)(\lambda - 1) = 0$

Hence $\lambda = 6, 1$

Clearly, there is no repetition in the value of λ.
Hence, multiplicity of characteristic value is 1.

For eigen vector,
let $X = \begin{pmatrix} x \\ y \end{pmatrix}$ be the eigen vector of matrix A.

Then, we have,

$$(A - \lambda I) X = 0$$

$$\begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \quad \text{----} \; (1)$$

When $\lambda = 6$, we get,

$$\begin{bmatrix} 5-6 & 4 \\ 1 & 2-6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\begin{bmatrix} -x+4y \\ x-4y \end{bmatrix} = 0$$

Here

$$x - 4y = 0$$
$$x = 4y$$

&

let $y = k$ then,

$$x = 4k$$

So, $X = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4k \\ k \end{pmatrix} = k \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

where k is non-zero scalar. i.e $k \neq 0$.

when $\lambda = 1$, we get,

$$\begin{bmatrix} 5-1 & 4 \\ 1 & 2-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4x + 4y \\ x + y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$4x + 4y = 0$

$x + y = 0$

$x = k$

$x = -y$ (say) [Here $k \neq 0$]

$y = -k$

Then, $X = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} k \\ -k \end{pmatrix} = k \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

where $k \neq 0$.

$\therefore$ When $\lambda = 6$, the eigen vector is $X = k \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, where, $k \neq 0$.

$\therefore$ When $\lambda = 1$, the eigen vector is $X = k \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, where, $k \neq 0$

Exercise :-

(1) Write down the characteristic values, and the characteristic vectors of the matrix

Ans

(a) $A = \begin{pmatrix} 0 & 3 \\ 2 & -1 \end{pmatrix}$

Solution :-

matrix $A = \begin{pmatrix} 0 & 3 \\ 2 & -1 \end{pmatrix}$

We have, let characteristic value be λ.

characteristic matrix $= A - \lambda I$

$$= \begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\lambda & 3 \\ 2 & -1-\lambda \end{bmatrix}$$

Characteristic polynomials $= |A - \lambda I|$

$$= \begin{vmatrix} -\lambda & 3 \\ 2 & -1-\lambda \end{vmatrix}$$

$$= +\lambda + \lambda^2 - 6$$

$$= \lambda^2 + \lambda - 6$$

$$= \lambda^2 + 3\lambda - 2\lambda - 6$$

$$= \lambda(\lambda+3) - 2(\lambda+3)$$

$$= (\lambda+3)(\lambda-2)$$

characteristic equation is

$$(\lambda+3)(\lambda-2) = 0$$

$\therefore \lambda = -3, +2$

So the characteristic value is $-3, +2$.

For eigen vector,
let $X = \begin{pmatrix} x \\ y \end{pmatrix}$ be the eigen vector of matrix A.
So we get,

$$(A - \lambda I)X = 0$$

$$\begin{bmatrix} -\lambda & 3 \\ 2 & -1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\begin{bmatrix} -\lambda x + 3y \\ 2x - (1+\lambda)y \end{bmatrix} = 0 \quad \text{------} \quad (1)$$

When $\lambda = -3$, (1) become,

$$\begin{bmatrix} +3x + 3y \\ 2x + 2y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Here, $3x + 3y = 0$
$2x + 2y = 0$

$\Rightarrow x + y = 0$
$x = -y$
let $y = k$ Then $x = -k$

So, $X = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -k \\ k \end{pmatrix} = k\begin{pmatrix} -1 \\ 1 \end{pmatrix}, k \neq 0$

$\therefore X = k\begin{pmatrix} -1 \\ 1 \end{pmatrix}, k \neq 0$

Again, when $\lambda = 2$, (1) become,

$$\begin{bmatrix} -2x + 3y \\ 2x - 3y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Here,

$$2x - 3y = 0$$
$$2x = 3y$$

let $y = k$ then $x = \frac{3k}{2}$

So, $X = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{3k}{2} \\ k \end{pmatrix} = k \begin{pmatrix} 3/2 \\ 1 \end{pmatrix}$

$$X = \frac{k}{2} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = k' \begin{pmatrix} 3 \\ 2 \end{pmatrix} \text{ where } k' \neq 0$$

$\therefore$ When $\lambda = -3$, the eigen vector is
$X = k \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ where $k \neq 0$.

$\therefore$ When $\lambda = 2$, the eigen vector is
$X = k' \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ where $k' \neq 0$.

#

(b) $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

Solution :-

Matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

Let characteristic value be λ.

So,

characteristic matrix $= A - \lambda I$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}$$

characteristic polynomials $= |A - \lambda I|$

$$= \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)^2 - 1$$

$$= \lambda^2 - 4\lambda + 4 - 1$$

$$= \lambda^2 - 4\lambda + 3$$

$$= \lambda^2 - 3\lambda - \lambda + 3$$

$$= \lambda(\lambda - 3) - 1(\lambda - 3)$$

$$= (\lambda - 3)(\lambda - 1)$$

characteristic equation is

$(\lambda - 3)(\lambda - 1) = 0$

$\therefore \lambda = 3, 1$

Hence, characteristic value $= 3, 1$.

For eigen vector,

let $X = \begin{pmatrix} x \\ y \end{pmatrix}$ be eigen vector of matrix A.

Then,

$(A - \lambda I) X = 0$

$$\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \quad \text{----} (1)$$

When $\lambda = 1$, (1) become,

$$\begin{bmatrix} 2-1 & 1 \\ 1 & 2-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\begin{bmatrix} x+y \\ x+y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Here,

$x + y = 0$

$x = -y$

let $y = k$ then $x = -k$

So, $X = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -k \\ k \end{pmatrix} = k \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $k \neq 0$.

$\therefore X = k \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $k \neq 0$.

Again, when $\lambda = 3$, ① becomes,

$$\begin{bmatrix} 2-3 & 1 \\ 1 & 2-3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\begin{bmatrix} -x+y \\ x-y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Here,

$$x - y = 0$$

$$x = y$$

let $y = k$, then $x = k$.

So,

$$X = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} k \\ k \end{pmatrix} = k\begin{pmatrix} 1 \\ 1 \end{pmatrix}, k \neq 0.$$

$$\therefore X = k\begin{pmatrix} 1 \\ 1 \end{pmatrix}, k \neq 0.$$

$\therefore$ When $\lambda = 1$, the eigen vector is

$$X = k\begin{pmatrix} -1 \\ 1 \end{pmatrix}, k \neq 0.$$

$\therefore$ When $\lambda = 3$, the eigen vector is

$$X = k\begin{pmatrix} 1 \\ 1 \end{pmatrix}, k \neq 0.$$

#

Eigen values and eigen vectors of the 3×3 matrix

(1) Find the eigen values and eigenvectors of the following matrices:-

(a) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Solution:-

matrix $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

let λ be characteristic value.

so,

characteristic matrix $= (A - \lambda I)$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix}$$

characteristic polynomials $= |A - \lambda I|$

$$= \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)(\lambda^2 - 1)$$

characteristic equation is.

$$(1-\lambda)(\lambda^2-1) = 0$$

$$(\lambda-1)(\lambda+1)(\lambda-1) = 0$$

Hence,

$$\lambda = 1, 1, -1$$

So characteristic value $= 1, 1, -1$

For eigen vector,

Let $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be eigen vector of matrix A.

Then,

$$(A - \lambda I)(X) = 0$$

$$\begin{pmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{---} (1)$$

When, $\lambda = 1$, (1) become,

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -x+y \\ x-y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Here,

$$x - y = 0 \qquad x = y$$

Let $y = k$ then $x = k$ let $z = k$

So, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} k \\ k \\ k \end{pmatrix} = k \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, k \neq 1$

Again, $\lambda = -1$, (1) become,

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x + y \\ x + y \\ 2z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Here, $x + y = 0$
$x + y = 0$
$2z = 0$

So, $z = 0$, $x = -y$ let $y = k$, then $x = -k$

$\therefore X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -k \\ k \\ 0 \end{pmatrix} = k \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, k \neq 0.$

$\therefore$ When $\lambda = 1$, the eigen vector is $X = k \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, k \neq 1.$

When $\lambda = -1$, the eigen vector is $X = k \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, k \neq 0.$

CamScanner

(6) $\begin{pmatrix} 2 & 1 & 2 \\ 0 & -1 & 3 \\ 0 & 1 & 1 \end{pmatrix}$

Solution:-

matrix $A = \begin{pmatrix} 2 & 1 & 2 \\ 0 & -1 & 3 \\ 0 & 1 & 1 \end{pmatrix}$

let λ be characteristic vector.

So, characteristic matrix $= (A - \lambda I)$

$$= \begin{pmatrix} 2 & 1 & 2 \\ 0 & -1 & 3 \\ 0 & 1 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2-\lambda & 1 & 2 \\ 0 & -1-\lambda & 3 \\ 0 & 1 & 1-\lambda \end{pmatrix}$$

characteristic polynomial $= |A - \lambda I|$

$$= \begin{vmatrix} 2-\lambda & 1 & 2 \\ 0 & -1-\lambda & 3 \\ 0 & 1 & 1-\lambda \end{vmatrix}$$

$$= (2-\lambda)\{\lambda^2 - 1 - 3\}$$

$$= (2-\lambda)(\lambda^2 - 4)$$

characteristic equation is

$(2-\lambda)(\lambda^2-4) = 0$

$(\lambda-2)(\lambda+2)(\lambda-2) = 0$

$\therefore \lambda = 2, -2, 2.$

For eigen vector,

let $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be eigen vector of matrix A.

So,

$$(A - \lambda I) X = 0$$

$$\begin{bmatrix} 2-\lambda & 1 & 2 \\ 0 & -1-\lambda & 3 \\ 0 & 1 & 1-\lambda \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad ----(1)$$

When $\lambda = 2$, (1) become,

$$\begin{bmatrix} 2-2 & 1 & 2 \\ 0 & -1-2 & 3 \\ 0 & 1 & 1-2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 0 & -3 & 3 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here,

$$y + 2z = 0$$
$$-3y + 3z = 0$$
$$y - z = 0$$

By solving we get, $y = 0$, $z = 0$ let $x = k$

Then,

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} k \\ 0 \\ 0 \end{pmatrix} = k \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad k \neq 0$$

Again, when $\lambda = -2$, (1) become,

$$\begin{bmatrix} 2+2 & 1 & 2 \\ 0 & -1+2 & 3 \\ 0 & 1 & 1+2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

~~Here~~

$$\begin{bmatrix} 4x + y + 2z \\ y + 3z \\ y + 3z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here,

$4x + y + 2z = 0$

$y + 3z = 0$

$y + 3z = 0$

Solving we get, let, $x = k$

$x = k$, $y = -12k$, $z = 4k$.

So, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} k \\ -12k \\ 4k \end{pmatrix} = k \begin{pmatrix} 1 \\ -12 \\ 4 \end{pmatrix}$, $k \neq 0$.

$\therefore$ When $\lambda = 2$, eigen vector $X = k(1, 0, 0)$ where $k \neq 0$.

$\therefore$ When $\lambda = -2$, eigen vector $X = k(1, -12, 4)$ where $k \neq 0$.

(c) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

Solution :-

Matrix $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

let, characteristic value be λ.

So,

characteristic matrix $= A - \lambda I$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2-\lambda & 0 & 0 \\ 0 & 4-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{pmatrix}$$

characteristic polynomials $= |A - \lambda I|$

$$= \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 4-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix}$$

$$= (2-\lambda)(4-\lambda)(3-\lambda)$$

characteristic equation is

$(2-\lambda)(4-\lambda)(3-\lambda) = 0$

solving we get,

$\lambda = 2, 4, 3$

For eigen vector,

let $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be eigen vector of matrix A.

So, $(A - \lambda I) \cdot X = 0$

$$\begin{bmatrix} 2-\lambda & 0 & 0 \\ 0 & 4-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{--- (1)}$$

When $\lambda = 2$, (1) becomes,

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 2y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Here, $y = 0$, $z = 0$

let $x = k$.

Then,

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} k \\ 0 \\ 0 \end{pmatrix} = k \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \; k \neq 0.$$

when $\lambda = 4$, (1) becomes,

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2x \\ 0 \\ -z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here, $-2x = 0$ i.e $x = 0$

$-z = 0$ i.e $z = 0$

Let $y = k$.

Then, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ k \\ 0 \end{pmatrix} = k \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $k \neq 0$.

when $\lambda = 3$, (1) become,

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here, $-x = 0$

$y = 0$

Let $z = k$.

Then, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ k \end{pmatrix} = k \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $k \neq 0$.

when, $\lambda = 2$, eigen vector $X = k\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

when, $\lambda = 4$, eigen vector $X = k\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

when, $\lambda = 3$, eigen vector $X = k\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ #

Date
Page

6) $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

Solution :-

matrix $A = \begin{pmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{pmatrix}$

let λ be characteristic value. So,

characteristic matrix $= A - \lambda I$

$$= \begin{pmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{pmatrix}$$

characteristic polynomial $= |A - \lambda I|$

$$= \begin{vmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{vmatrix}$$

$$= (3-\lambda)(2-\lambda)(5-\lambda)$$

characteristic equation is

$(3-\lambda)(2-\lambda)(5-\lambda) = 0$

Solving, we get,

$\lambda = 3, 2, 5$

For eigen ~~matrix~~ vectors

let $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be eigen vector of matrix A.

So, $(A - \lambda I)X = 0$

$$\begin{bmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{---- (1)}$$

When, $\lambda = 3$, (1) becomes,

$$\begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} y + 4z \\ -y + 6z \\ 2z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$y + 4z = 0$

$-y + 6z = 0$

$2z = 0$

$\therefore z = 0, \quad y = 0, \quad x = k$ (let)

So, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix} = k \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; k \neq 0.$

When $\lambda = 2$, ① becomes,

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x+y+4z \\ 6z \\ 3z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here,

$x+y+4z=0$

$6z=0$, $3z=0$ So, $z=0$

$x+y=0$.

$x=-y$ Let $y=k$, then $x=-k$

So,

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -k \\ k \\ 0 \end{pmatrix} = k \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, k \neq 0$$

When $\lambda = 5$, ① becomes,

$$\begin{bmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2x+y+z \\ 0-3y+6z \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here, $-2x+y+4z=0$

$-3y+6z=0$

CamScanner

(c) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Solution,

Matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Let λ be characteristic value. So,

characteristic matrix $= A - \lambda I$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{pmatrix}$$

characteristic polynomial $= |A - \lambda I|$

$$= \begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)(1-\lambda)(1-\lambda)$$

characteristic equation is

$$(1-\lambda)(1-\lambda)(1-\lambda) = 0$$

So, $\lambda = 1, 1, 1$.

For eigen vector,

Let $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be eigen vector of matrix A.

So, $(A - \lambda I) X = 0$

$$\begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{---- (1)}$$

When, $\lambda = 1$, (1) becomes,

$$\begin{bmatrix} 1-1 & 1 & 1 \\ 0 & 1-1 & 1 \\ 0 & 0 & 1-1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} y+z \\ z \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here, $y + z = 0$

$z = 0$

$\therefore y = 0$

Let $x = k$.

Hence,

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} k \\ 0 \\ 0 \end{pmatrix} = k \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ where } k \neq 0.$$

Cayley - Hamilton Theorem for

Date
Page

Verify the Cayley - Hamilton Theorem for the following matrices. Also find the inverse of each:-

(a) $A = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$

Solution:-

matrix $A = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$

let λ be characteristic value. So,

characteristic matrix = $(A - \lambda I)$

$$= \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2-\lambda & -1 \\ 1 & 3-\lambda \end{pmatrix}$$

characteristic equation is

$|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$(2-\lambda)(3-\lambda) + 1 = 0$

$\lambda^2 - 5\lambda + 6 + 1 = 0$

$\lambda^2 - 5\lambda + 7 = 0$

CamScanner

Date
Page

To verify cayley Hamilton theorem, we show the square matrix C satisfies its characteristic equation $|C - \lambda I| = 0$ i.e.

C satisfies $\lambda^2 - 5\lambda + 7 = 0$

we show,

$C^2 - 4C + 7I = 0$

$$\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} - 5\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} + 7\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 0$$

$$\begin{pmatrix} 4-1 & -2-3 \\ 2+3 & -1+9 \end{pmatrix} - \begin{pmatrix} 10 & -5 \\ 5 & 15 \end{pmatrix} + \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 3-10+7 & -5+5+0 \\ 5-5+0 & 8-15+7 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{(True)}$$

Hence, we can say that matrix A satisfies its characteristic equation

To find C^{-1}, Here $|C| = 7 \neq 0$

we have,

$$C^2 - 5C + 7I = 0$$
$$C^2 - 5C + 7CC^{-1} = 0$$
$$C(C - 5I + 7C^{-1}) = 0 \quad , C \neq 0$$
$$C - 5I + 7C^{-1} = 0$$
$$7C^{-1} = 5I - C$$
$$C^{-1} = \frac{1}{7}\left\{\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} - \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}\right\}$$

$$C^{-1} = \frac{1}{7}\begin{bmatrix} 5-2 & 1 \\ -1 & 5-3 \end{bmatrix}$$

$$C^{-1} = \begin{bmatrix} 3/7 & 1/7 \\ -1/7 & 2/7 \end{bmatrix} \quad \#$$

(b) $A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$

Solution:-

$A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$

let λ be characteristic value:

So, characteristic matrices $= A - \lambda I$

characteristic equation is $|A - \lambda I| = 0$

or, $\left| \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = 0$

or, $\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$

~~or, $(2-\lambda)(4-4\lambda+\lambda^2-1)+1(\lambda-2+1)+(1-2+\lambda)=0$~~

~~or, $(2-\lambda)(\lambda^2-4\lambda+3)+\lambda-2-\lambda=0$~~

$(2-\lambda)((2-\lambda)^2-1)+1(\lambda-2+1)+(1-2+\lambda)=0$

$(2-\lambda)(4-4\lambda+\lambda^2-1)+\lambda-2+1+1-2+\lambda=0$

$(2-\lambda)(\lambda^2-4\lambda+3)+2(\lambda-1)=0$

$(2-\lambda)(\lambda^2-3\lambda-\lambda+3)+2(\lambda-1)=0$

$(2-\lambda)(\lambda-3)(\lambda-1)+2(\lambda-1)=0$

$(\lambda-1)\{(2-\lambda)(\lambda-3)+2\}=0$

$$(\lambda-1)\{2\lambda-6-\lambda^2+3\lambda+2\}=0$$

$$-(\lambda-1)\{\lambda^2-5\lambda+4\}=0$$

$$-(\lambda-1)\{\lambda^2-4\lambda-\lambda+4\}=0$$

$$(\lambda-1)(\lambda-4)(\lambda-1)=0$$

To verify cayley Hamilton theorem A must satisfies characteristic equation.

$$(A-I)(A-4I)(A-I)=0$$

$$\left[\begin{pmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{pmatrix}-\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right]\left[\begin{pmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{pmatrix}-\begin{bmatrix}4 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 4\end{bmatrix}\right]$$

$$\left[\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}-\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & & 1\end{bmatrix}\right]=0$$

$$\begin{bmatrix}1 & -1 & 1\\ -1 & 1 & -1\\ 1 & -1 & 1\end{bmatrix}\begin{bmatrix}-2 & -1 & 1\\ -1 & -2 & -1\\ 1 & -1 & -2\end{bmatrix}\begin{bmatrix}1 & -1 & 1\\ -1 & 1 & -1\\ 1 & -1 & 1\end{bmatrix}=0$$

$$\begin{bmatrix}-2+1+1 & -1+2-1 & 1+1-1\\ 2-1-1 & 1-2+1 & -1-1+2\\ -2+1+1 & -1+2-1 & 1+1-2\end{bmatrix}\begin{bmatrix}1 & -1 & 1\\ -1 & 1 & -1\\ 1 & -1 & 1\end{bmatrix}=0$$

$$\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & -1 & 1\\ -1 & 1 & -1\\ 1 & -1 & 1\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ (True)}$$

Hence, we can say that matrix A satisfies its characteristic equation.

To solve A^{-1},

At first $|A| = \begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix}$

$$= 2(4-1) + (-2+1) + (1-2)$$
$$= 2 \times 3 - 1 - 1$$
$$= 4$$

we have,

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$
$$A^3 - 6A^2 + 9A - 4I = 0$$
$$A(A^2 - 6A + 9I - 4A^{-1}) = 0 \qquad A \neq 0$$

$$A^2 - 6A + 9I - 4A^{-1} = 0$$

$$-4A^{-1} = 6A - A^2 + 10I$$

$$A^{-1} = \frac{1}{-4}\left[\begin{pmatrix} 12 & -6 & 6 \\ -6 & 12 & -6 \\ 6 & -6 & 12 \end{pmatrix} - \begin{pmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{pmatrix} - \begin{pmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix}\right]$$

$$= \frac{1}{-4}\begin{bmatrix} -3 & -1 & 1 \\ -1 & -3 & -1 \\ 1 & -1 & -3 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

#

Q $B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Solution:-

Matrix $B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

let λ be characteristic value.

so characteristic matrices $= B - \lambda I$

characteristic equation is $|B - \lambda I| = 0$

$$\left| \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(1-\lambda)(1-\lambda) = 0$$

To verify cayley Hamilton theorem B must satisfies characteristic equation.

$$(I-B)(I-B)(I-B) = 0$$

$$\Rightarrow \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right] \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right] \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right]$$

$= 0$

$$\Rightarrow \begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{(True)}$$

Hence, we can say that matrix B satisfies it characteristic equation.

To solve B^{-1},

At first,

$$|B| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1(1-0) = 1 \#$$

we have,

$$1 - 3B + 3B^2 - B^3 = 0$$

$$B(B^{-1} - 3I + 3B - B^2) = 0 \qquad A \neq 0$$

$$B^{-1} - 3I + 3B - B^2 = 0$$

$$B^{-1} = B^2 + 3I - 3B$$

Date
Page

$$B^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} + 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 3 \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 3 & 3 \\ 0 & 3 & 3 \\ 0 & 0 & 3 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore B^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

CamScanner

Assignment-3

Date 2080/01/03
Page

Eigen value / Eigen vector / Eigen decomposition

Q) Find the eigen values and eigen vectors of the following matrices. Are the eigen vectors orthogonal?

(a) $A = \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}$

Solution :-

$A = \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}$

Let λ be characteristic value then characteristic equation is

$|A - \lambda I| = 0$

$\left| \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = 0$

$\begin{vmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{vmatrix} = 0$

$(\lambda - 2)(\lambda + 6) - 9 = 0$

$\lambda^2 + 4\lambda - 12 - 9 = 0$

$\lambda^2 + 4\lambda - 21 = 0$

$\lambda^2 + 7\lambda - 3\lambda - 21 = 0$

$\lambda(\lambda + 7) - 3(\lambda + 7) = 0$

$(\lambda + 7)(\lambda - 3) = 0$

$\therefore \lambda = -7, 3$

Now, let v be eigen vector such that $v = \begin{pmatrix} x \\ y \end{pmatrix}$

So, $(A - \lambda I) v = 0$

$\Rightarrow \begin{pmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$ ——— (1)

when $\lambda_1 = -7$, we get, V_1 be eigen vector,

$$\Rightarrow \begin{pmatrix} +9 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} 9x_1 + 3y_1 \\ 3x_1 + y_1 \end{pmatrix} = 0$$

Here, $9x_1 + 3y_1 = 0$

$3x_1 + y_1 = 0$

$y_1 = -3x_1$

let $x = k_1$ then $y = -3k_1$

So, $V_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = k_1 \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ where $k \neq 0$.

Again, when $\lambda_2 = 3$, V_2 be eigen vector, we g

$$\Rightarrow \begin{pmatrix} -1 & 3 \\ 3 & -9 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} -x_2 + 3y_2 \\ 3x_2 - 9y_2 \end{pmatrix} = 0$$

Here, $-x_2 + 3y_2 = 0$

$3x_2 - 9y_2 = 0$

$x_2 = 3y_2$

let $y_2 = k_2$ then $x_2 = 3k_2$

So,

$V_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = k_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, where $k \neq 0$.

So, $V_1 = k_1' \begin{pmatrix} 1 \\ -3 \end{pmatrix}$, $V_2 = k_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

To show orthogonal, we have to show $v_1 \cdot v_2 = 0$.

So,

$$v_1 \cdot v_2 = k_1 \begin{pmatrix} 1 \\ -3 \end{pmatrix} \cdot k_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$= k_1 \cdot k_2 \, (3-3)$$

$$= k_1 \cdot k_2 \, (0)$$

$$= 0$$

$\therefore v_1 \cdot v_2 = 0$. Hence v_1 and v_2 are orthogonal eigen vector. //

(b) $B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

Solution :-

$B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

let λ be characteristic value then, characteristic equation is

$\Rightarrow |B - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = 0$

$\Rightarrow 4 - 5\lambda + \lambda^2 - 4 = 0$

$\Rightarrow \lambda^2 - 5\lambda = 0$

$\Rightarrow \lambda(\lambda - 5) = 0$

$\Rightarrow \lambda = 0, 5$

Now, let v be eigen vector where $v = \begin{pmatrix} x \\ y \end{pmatrix}$

So, $(B - \lambda I)v = 0$

$\Rightarrow \begin{pmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$ ---- (1)

When, $\lambda_1 = 0$, we get v_1 be eigen vector, so,

$\Rightarrow \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$

$\Rightarrow \begin{pmatrix} x_1 + 2y_1 \\ 2x_1 + 4y_1 \end{pmatrix} = 0$

Here, $x_1 + 2y_1 = 0$ $\Rightarrow x = -2y$

$2x_1 + 4y_1 = 0$

Let, $y_1 = k_1$, $x_1 = -2k_1$

So,

$v_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = k_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ where $k_1 \neq 0$

Again, when $\lambda_2 = 5$, v_2 be eigen vector, we get

So, $\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$

$\Rightarrow \begin{pmatrix} -4x_2 + 2y_2 \\ 2x - y_2 \end{pmatrix} = 0$

Here, $-4x_2 + 2y_2 = 0$

$2x_2 - y_2 = 0$ $\Rightarrow y_2 = 2x$.

Let $x = k_2$ $y = 2k$

So,

$v_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = k_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ where $k_2 \neq 0$.

So, $v_1 = k_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix}$, $v_2 = k_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Here, to ~~show~~ prove v_1 and v_2 are orthogonal, we have to show $v_1 \cdot v_2 = 0$.

So,

$$v_1 \cdot v_2 = k_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} \cdot k_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= k_1 \cdot k_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= k_1 \cdot k_2 (-2+2)$$

$$= k_1 \cdot k_2 (0)$$

$$= 0$$

$\therefore v_1 \cdot v_2 = 0$

Hence, v_1 and v_2 are orthogonal eigenvector.

Date
Page

(a) $A = \begin{pmatrix} 9 & -2 \\ -2 & 6 \end{pmatrix}$

Solution:

$A = \begin{pmatrix} 9 & -2 \\ -2 & 6 \end{pmatrix}$

let λ be characteristic value. Then, characteristic equation is

$|A - \lambda I| = 0$

$\begin{vmatrix} 9-\lambda & -2 \\ -2 & 6-\lambda \end{vmatrix} = 0$

$54 - 15\lambda + \lambda^2 - 4 = 0$

$\lambda^2 - 15\lambda + 50 = 0$

$\lambda^2 - 10\lambda - 5\lambda + 50 = 0$

$\lambda(\lambda - 10) - 5(\lambda - 10) = 0$

$(\lambda - 10)(\lambda - 5) = 0$

$\lambda = 10, 5.$

let v be eigen vector where $v = \begin{pmatrix} x \\ y \end{pmatrix}$

~~when~~ we have,

$(A - \lambda I) v = 0$

$\begin{pmatrix} 9-\lambda & -2 \\ -2 & 6-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$ --- (1)

when, $\lambda_1 = 10$, we have, v_1 is eigen vector,

$\Rightarrow \begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$

$\Rightarrow \begin{pmatrix} -1x - 2y \\ -2x - 2y \end{pmatrix} = 0$

Date
Page

Here, $-x - 2y = 0$

$-2x - 4y = 0$

$x = -2y$

Let $y = k_1$, $x = -2k_1$

So,

$$v_1 = \begin{pmatrix} x \\ y \end{pmatrix} = k_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} \text{ where } k_1 \neq 0.$$

Again, $\lambda_2 = 5$, we have, v_2 is eigen vector

$$\Rightarrow \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} 4x - 2y \\ -2x + y \end{pmatrix} = 0$$

Here, $4x - 2y = 0$

$-2x + y = 0 \quad \Rightarrow y = 2x.$

Let $x = k_2$ then, $y = 2k_2$

So,

$$v_2 = \begin{pmatrix} x \\ y \end{pmatrix} = k_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ where } k_2 \neq 0.$$

So, $v_1 = k_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ & $v_2 = k_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Now to prove v_1 & v_2 are orthogonal, we have to show $v_1 \cdot v_2 = 0$. So,

$$v_1 \cdot v_2 = k_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} \cdot k_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= k_1 k_2 (-2 + 2)$$

$$= k_1 \cdot k_2 (0)$$

$$= 0$$

$v_1 \cdot v_2 = 0$

Hence, v_1 and v_2 are orthogonal eigen vector. #

CamScanner

Date
Page

(2) Find the eigen vectors of the matrix $A = \begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}$. Are the eigen vectors orthogonal?

Solution:-

$$A = \begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}$$

let λ be characteristic value. Then, characteristic equation is,

$\Rightarrow |A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 3-\lambda & 2 \\ 4 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow 3 - 4\lambda + \lambda^2 - 8 = 0$

$\Rightarrow \lambda^2 - 4\lambda - 5 = 0$

$\Rightarrow \lambda^2 - 5\lambda + \lambda - 5 = 0$

$\Rightarrow \lambda(\lambda - 5) + 1(\lambda - 5) = 0$

$\Rightarrow (\lambda - 5)(\lambda + 1) = 0$

$\therefore \lambda = 5, -1,$

let $v = \begin{pmatrix} x \\ y \end{pmatrix}$ be eigen vector, So,

$(A - \lambda I) v = 0$

$\begin{pmatrix} 3-\lambda & 2 \\ 4 & 1-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$ ------ (1)

when $\lambda_1 = 5$, we get,

$\Rightarrow \begin{pmatrix} -2 & 2 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$

$\Rightarrow \begin{pmatrix} -2x + 2y \\ 4x - 4y \end{pmatrix} = 0$

Here, $-2x + 2y = 0$

$4x - 4y = 0$

$x = y$ let, $y = k_1$, $x = k_1$.

So, $v_1 = \begin{pmatrix} x \\ y \end{pmatrix} = k_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, where $k_1 \neq 0$

Again, when $\lambda_2 = -1$, we get,

$\Rightarrow \begin{pmatrix} 4 & 2 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$

$\Rightarrow \begin{pmatrix} 4x+2y \\ 4x+2y \end{pmatrix} = 0$

Here $4x+2y=0 \Rightarrow 2x+y=0 \Rightarrow y=-2x$

let $x = k_2$ then, $y = -2k_2$

So,

$v_2 = \begin{pmatrix} x \\ y \end{pmatrix} = k_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, where $k_2 \neq 0$

Here, $v_1 = k_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ & $v_2 = k_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Again,

$$v_1 \cdot v_2 = k_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot k_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$
$$= k_1 \cdot k_2 (1-2)$$
$$= -k_1 \cdot k_2$$
$$= k_3 \neq 0$$

ie $v_1 \cdot v_2 \neq 0$.

Hence, v_1 and v_2 are not orthogonal eigen vectors. //

Date
Page

(3) Show that the matrix $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ has complex eigen values. Find the corresponding eigen vectors.

Solution:-

$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Let λ be characteristic value. Then, characteristic equation is,

$\Rightarrow |B - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^2 + 1 = 0$

$\Rightarrow \lambda = \pm\sqrt{-1}$

$\Rightarrow \lambda = \pm i$

Now, let $v = \begin{pmatrix} x \\ y \end{pmatrix}$ be eigen vector. So,

$\Rightarrow (A - \lambda I) v = 0$

$\Rightarrow \begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$ ------ (1)

When, $\lambda_1 = i$, we get,

$\Rightarrow \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$

$\Rightarrow \begin{pmatrix} -ix + y \\ -x - iy \end{pmatrix} = 0$

Here, $-ix + y = 0$

$-x - iy = 0$

~~solving we get,~~

$y = +ix$

let $x = k_1$, then $y = ik$

so,

$$v_1 = \begin{pmatrix} x \\ y \end{pmatrix} = k_1 \begin{pmatrix} 1 \\ i \end{pmatrix} \text{ where } k_1 \neq 0.$$

Again, when $\lambda_2 = -i$, we get,

$$\Rightarrow \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} ix + y \\ -x + iy \end{pmatrix} = 0$$

Here, $ix + y = 0$

$-x + iy = 0$

$x = iy$

let $y = k_2$ then $x = ik$

so,

$$v_2 = \begin{pmatrix} x \\ y \end{pmatrix} = k_2 \begin{pmatrix} i \\ 1 \end{pmatrix} \text{ where } k_2 \neq 0.$$

$\therefore$ $v_1 = k_1 \begin{pmatrix} 1 \\ i \end{pmatrix}$ & $v_2 = k_2 \begin{pmatrix} i \\ 1 \end{pmatrix}$ are eigen vector of $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. #

(4) Consider the following matrices

(a) $A = \begin{pmatrix} 9 & -2 \\ -2 & 6 \end{pmatrix}$ (b) $C = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$ (c) $L = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$

(d) $E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Find the ~~matrices~~ eigen values and eigen vectors in each case.

Express the matrices ~~into~~ as the ~~forms~~

(i) product of three matrices in the form $A = PDP^{-1}$, where the symbol as usual.

(ii) Decompose the matrices A as the eigen value decomposition of the form $A = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T$ where the symbol as usual.

(a) Solution:-

$$A = \begin{pmatrix} 9 & -2 \\ -2 & 6 \end{pmatrix}$$

Let λ be characteristic value. Then, characteristic equation is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 9-\lambda & -2 \\ -2 & 6-\lambda \end{vmatrix} = 0$$

$$\Rightarrow 54 - 15\lambda + \lambda^2 - 4 = 0$$

$$\Rightarrow (\lambda - 10)(\lambda - 5) = 0$$

$$\Rightarrow \lambda = 10, 5$$

Let v be eigen vector, so,

$$(A - \lambda I) v = 0.$$

when, $\lambda_1 = 10$

$v_1 = k_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ where $k_1 \neq 0$

~~so~~

when, $\lambda_2 = 5$

$v_2 = k_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ where, $k_2 \neq 0$

Let u_1 and u_2 be orthonormal vector of v_1 & v_2 respectively.

so, $u_1 = \dfrac{v_1}{\|v_1\|} = \dfrac{k_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix}}{k_1\sqrt{4+1}} = \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$

$u_2 = \dfrac{v_2}{\|v_2\|} = \dfrac{k_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}}{k_2\sqrt{1+4}} = \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$

Here, $P = \begin{array}{c} \begin{array}{cc} u_1 & u_2 \end{array} \\ \begin{pmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{2} & 2/\sqrt{5} \end{pmatrix} \end{array}$; $P^T = \begin{pmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}$

Now, we have,

$D = P^{-1}AP$ also $A = PDP^{-1}$

so,

$D = P^{-1}AP$

$= P^T AP$ $[\because P^{-1} = P^T]$

$= \begin{pmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix} \begin{pmatrix} 9 & -2 \\ -2 & 6 \end{pmatrix} \begin{pmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}$

$= \dfrac{1}{5} \begin{pmatrix} -18-2 & 4+6 \\ 9-4 & -2+12 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}$

$= \dfrac{1}{5} \begin{pmatrix} 40+10 & -20+20 \\ -10+10 & 5+20 \end{pmatrix}$

$$= \frac{1}{5}\begin{pmatrix} 50 & 0 \\ 0 & 25 \end{pmatrix}$$

$$= \begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix}$$

$$\therefore D = \begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix}$$

$$\therefore A = PDP^{-1}$$

We know, $\lambda_1 = 10$, $\lambda_2 = 5$

$$A = \begin{pmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix} \begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}$$

$$= \begin{pmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix} \begin{pmatrix} 10 \times (-2/\sqrt{5}) & 10 \times 1/\sqrt{5} \\ 5 \times 1/\sqrt{5} & 5 \times 2/\sqrt{5} \end{pmatrix}$$

$$= \begin{pmatrix} (-2/\sqrt{5}) \times 10 \times (-2/\sqrt{5}) + 1/\sqrt{5} \times 5 \times 1/\sqrt{5} & (-2/\sqrt{5}) \times 10 \cdot 1/\sqrt{5} + (+1/\sqrt{5}) \times 5 \times 2/\sqrt{5} \\ 1/\sqrt{5} \times 10 \times (-2/\sqrt{5}) + 2/\sqrt{5} \times 5 \times 1/\sqrt{5} & 1/\sqrt{5} \times 10 \times 1/\sqrt{5} + 2/\sqrt{5} \times 5 \times 2/\sqrt{5} \end{pmatrix}$$

$$A = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T$$

$$= 10 \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} + 5 \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}$$

$$=$$

Now, we have to prove $A = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T$

LHS,

$$\lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T = 10\begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}\begin{pmatrix} -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} + 5\begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}\begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}$$

$$= 10\begin{pmatrix} +\frac{4}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{pmatrix} + 5\begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix}$$

$$= \begin{pmatrix} 8 & -4 \\ -4 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & -2 \\ -2 & 6 \end{pmatrix} = A$$

$\therefore A = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T$ ##

(b) $C = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$

Solution:-

Let λ be the characteristic value. Then,

Characteristic equation is,

$\Rightarrow |C - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^2 + \lambda - 2 - 4 = 0$

$\Rightarrow \lambda^2 + \lambda - 6 = 0$

$\Rightarrow (\lambda+3)(\lambda-2) = 0$

$\lambda = -3, 2$

Let v be eigen vector, $v = \begin{pmatrix} x \\ y \end{pmatrix}$

So, $(C - \lambda I) v = 0$

$\Rightarrow \begin{pmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{pmatrix} = 0$ ------ (1)

When, $\lambda_1 = -3$, equation (1) becomes,

$\Rightarrow \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$

$\Rightarrow \begin{pmatrix} 4x + 2y \\ 2x + y \end{pmatrix} = 0$

Here, $4x + 2y = 0$

$2x + y = 0$

$y = -2x$

Let $x = k_1$ $\therefore y = -2k_1$

$\therefore v_1 = k_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ where, $k_1 \neq 0$

When, $\lambda_2 = 2$, (1) become,

$$\Rightarrow \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} -x+2y \\ 2x-4y \end{pmatrix} = 0$$

Here, $-x+2y=0$

$2x-4y=0$

$\Rightarrow x = 2y$

Let $y = k_2$ Then $x = 2k_2$

So, $v_2 = k_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ ∵ where $k_2 \neq 0$

Let u_1 and u_2 be two orthonormal vectors of v_1 and v_2 - so,

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{k_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix}}{k_1\sqrt{5}} = \begin{pmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{pmatrix}$$

$$u_2 = \frac{v_2}{\|v_2\|} = \frac{k_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}}{k_2\sqrt{5}} = \begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$$

Here, $P = \begin{array}{c} \begin{array}{cc} u_1 & u_2 \end{array} \\ \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} \end{array}$, $P^T = \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}$

Now, we have,

$D = P^{-1}AP$ Also $A = PDP^{-1}$

So,

$$D = P^{-1}AP$$

$$= P^{T}AP \quad [\because P^{-1} = P^{T}]$$

$$= \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 1-4 & 2+4 \\ 2+2 & 4-2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} -3-12 & -6+6 \\ 4-4 & 8+2 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} -15 & 0 \\ 0 & 10 \end{bmatrix}$$

$$\therefore D = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\therefore A = PDP^{-1}$$

We know, $\lambda_1 = -3$, $\lambda_2 = 2$

Now, we have to prove $A = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T$

So,

$$\lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T = -3 \begin{pmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{pmatrix} + 2 \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$$

$$= -3 \begin{pmatrix} \frac{1}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{4}{5} \end{pmatrix} + 2 \begin{pmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{3}{5} & +\frac{6}{5} \\ \frac{6}{5} & -\frac{12}{5} \end{pmatrix} + \begin{pmatrix} \frac{8}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{2}{5} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-3+8}{5} & \frac{6+4}{5} \\ \frac{6+4}{5} & \frac{-12+2}{5} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{5}{5} & \frac{10}{5} \\ \frac{10}{5} & \frac{-10}{5} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} = A$$

$\therefore A = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T$ #

(c) $L = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$

Solution:-

Let λ be characteristic value. Then characteristic equation is

$\Rightarrow |L - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 3-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = 0$

$(3-\lambda)(1-\lambda) = 0$

$\therefore \lambda = 3, 1$

Let v be eigen vector. $v = \begin{pmatrix} x \\ y \end{pmatrix}$

So,

$\Rightarrow (L - \lambda I) v = 0$

$\Rightarrow \begin{pmatrix} 3-\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$ --- (1)

When $\lambda_1 = 3$, (1) becomes,

$\Rightarrow \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$

$\Rightarrow \begin{pmatrix} 0 \\ -2y \end{pmatrix} = 0$

Here, $y = 0$ let $x = k_1$

Then, $v_1 = \begin{pmatrix} x \\ y \end{pmatrix} = k_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ where $k_1 \neq 0$.

Date
Page

When, $\lambda_2 = 1$, (1) becomes,

$$\Rightarrow \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} 2x \\ 0 \end{pmatrix} = 0$$

$\therefore x = 0$ let $y = k_2$

Then,

$$v_2 = \begin{pmatrix} x \\ y \end{pmatrix} = k_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ where } k_2 \neq 0.$$

Let u_1 and u_2 be orthonormal vector of v_1 & v_2 respectively. So,

$$u_1 = \frac{v_1}{\|v_1\|} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$u_2 = \frac{v_2}{\|v_2\|} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

~~Above~~ Here, $P = \begin{array}{c} \begin{array}{cc} u_1 & u_2 \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$, $P^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

So,

$$D = P^{-1} A P$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

$\therefore D = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ & $A = PDP^{-1}$

We have, $\lambda_1 = 3$, $\lambda_2 = 1$

Now, we have to prove $C = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T$

So,

$$\lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T = 3\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix}$$

$$= 3\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

$\therefore C = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T$ #

Date
Page

(2) $E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

solution :-

let λ be characteristic value, then,
characteristic equation is

$\Rightarrow |E - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^2 - 1 = 0$

$\lambda = 1, -1$

let v be eigen vector, so,

$(E - \lambda I) v = 0$

$\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$ ——— (1)

when, $\lambda_1 = 1$, (1) becomes,

$\Rightarrow \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$

$\Rightarrow \begin{pmatrix} -x + y \\ x - y \end{pmatrix} = 0$

Here, $-x + y = 0$

$x - y = 0$

$\Rightarrow y = x$

let $y = k$, then $x = k$,

$\therefore v_1 = k_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ where, $k_1 \neq 0$

CamScanner

When, $\lambda_2 = -1$, (1) becomes

$$\Rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} x+y \\ x+y \end{pmatrix} = 0$$

Here, $x+y=0$

$x = -y$

let $y = k_2$ then $x = -k_2$

So,

$$v_2 = \begin{pmatrix} x \\ y \end{pmatrix} = k_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ where } k_2 \neq 0.$$

let u_1 and u_2 be orthogonal vector of v_1 & v_2 respectively. So,

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{k_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}}{k_1 \sqrt{2}} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$u_2 = \frac{v_2}{\|v_2\|} = \frac{k_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}}{k_2 \sqrt{2}} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$$

Here,

$$P = \begin{array}{cc} u_1 & u_2 \\ \end{array} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \text{ also } P^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Now, we have,

$$D = P^{-1} E P \quad \text{also } E = P D P^{-1}$$

$$= P^T E P \quad [\because P^{-1} = P^T]$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}+\frac{1}{2} & \frac{1}{2}-\frac{1}{2} \\ -\frac{1}{2}+\frac{1}{2} & -\frac{1}{2}-\frac{1}{2} \end{pmatrix}$$

$$\therefore D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\therefore E = PDP^{-1}$$

We know, $\lambda_1 = 1$, $\lambda_2 = -1$,

Now, we have to prove $A = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T$

$$\lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T = 1 \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} + (-1) \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}-\frac{1}{2} & \frac{1}{2}+\frac{1}{2} \\ \frac{1}{2}+\frac{1}{2} & \frac{1}{2}-\frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\therefore E = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T \quad \#$$

(5) Consider the following matrices:-

(a) $A = \begin{pmatrix} 1 & 4 & 3 \\ 4 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$ (b) $B = \begin{pmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{pmatrix}$

Find the eigen value and eigen vectors in each case.

Express the matrices as the

(i) Product of the three matrices of the form $A = PDP^{-1}$ (Diagonalize)

(ii) Decompose the matrices as the spectral value decomposition of the form $A = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \lambda_3 u_3 u_3^T$

(a) Solution:-

$$A = \begin{pmatrix} 1 & 4 & 3 \\ 4 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

Let λ be characteristic value, Then characteristic equation is,

$\Rightarrow |A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 4 & 3 \\ 4 & 1-\lambda & 0 \\ 3 & 0 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)^3 - 4(4-4\lambda) + 3(-3+\lambda) = 0$

$\Rightarrow (1-\lambda)^3 - 16 + 16\lambda - 9 + 9\lambda = 0$

$\Rightarrow (1-\lambda)^3 - 25(1-\lambda) = 0$

$\Rightarrow (1-\lambda)\{\lambda^2 - 2\lambda + 1 - 25\} = 0$

$\Rightarrow (1-\lambda)(\lambda^2 - 2\lambda - 24) = 0$

$\Rightarrow (1-\lambda)(\lambda^2 - 6\lambda + 4\lambda - 24) = 0$

$\Rightarrow (1-\lambda)(\lambda-6)(\lambda+4) = 0$

$\therefore \lambda = 1, 6, -4$

let v be eigen vector, so

$(A - \lambda I) v = 0$

$$\Rightarrow \begin{pmatrix} 1-\lambda & 4 & 3 \\ 4 & 1-\lambda & 0 \\ 3 & 0 & 1-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \quad \text{---} (1)$$

when, $\lambda_1 = 1$, (1) becomes,

$$\Rightarrow \begin{pmatrix} 0 & 4 & 3 \\ 4 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} 4y + 3z \\ 4x \\ 3x \end{pmatrix} = 0$$

Here, $4y + 3z = 0$

$4x = 0$

$\therefore x = 0$

$4y + 3z = 0$

$y = \frac{-3z}{4}$

when $z = k_1$, $y = -\frac{3}{4} k_1$

Then, $v_1 = k_1 \begin{pmatrix} 0 \\ -3/4 \\ 1 \end{pmatrix} = k_1' \begin{pmatrix} 0 \\ -3 \\ 4 \end{pmatrix}$ where $k_1 \neq 0$

when, $\lambda_2 = 6$, (1) becomes,

$$\Rightarrow \begin{pmatrix} -5 & 4 & 3 \\ 4 & -5 & 0 \\ 3 & 0 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} -5x + 4y + 3z \\ 4x - 5y \\ 3x - 5z \end{pmatrix} = 0$$

Here,

$$-5x + 4y + 3z = 0$$
$$4x - 5y = 0$$
$$3x - 5z = 0$$

Solving we get,

$$x = 5k_2, \quad y = 4k_2, \quad z = 3k_2$$

$$v_2 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = k_2 \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}, \text{ where } k_2 \neq 0.$$

when $\lambda_3 = -4$, (1) becomes,

$$\Rightarrow \begin{pmatrix} 5 & 4 & 3 \\ 4 & 5 & 0 \\ 3 & 0 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} 5x + 4y + 3z \\ 4x + 5y \\ 3x + 5z \end{pmatrix} = 0$$

Here,
$$5x+4y+3z=0$$
$$4x+5y=0$$
$$3x+5z=0$$

Solving we get,
$$x=-5k_3, \quad y=4k_3, \quad z=3k_3$$

So, $v_3 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = k_3 \begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix}$, where $k_3 \neq 0$.

Let u_1, u_2 and u_3 be orthonormal vector to v_1; v_2 and v_3 respectively. So,

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{25}} \begin{pmatrix} 0 \\ -3 \\ 4 \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ -3/5 \\ 4/5 \end{pmatrix}$$

$$u_2 = \frac{v_2}{\|v_2\|} = \frac{1}{5\sqrt{2}} \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 4/5\sqrt{2} \\ 3/5\sqrt{2} \end{pmatrix}$$

$$u_3 = \frac{v_3}{\|v_3\|} = \frac{1}{5\sqrt{2}} \begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} \\ 4/5\sqrt{2} \\ 3/5\sqrt{2} \end{pmatrix}$$

So,

$$P = \begin{array}{c} \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \\ \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{3}{5} & \frac{4}{5\sqrt{2}} & \frac{4}{5\sqrt{2}} \\ \frac{4}{5} & \frac{3}{5\sqrt{2}} & \frac{3}{5\sqrt{2}} \end{pmatrix} \end{array} \quad \text{and} \quad P^T = \begin{pmatrix} 0 & -\frac{3}{5} & \frac{4}{5} \\ \frac{1}{\sqrt{2}} & \frac{4}{5\sqrt{2}} & \frac{3}{5\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{4}{5\sqrt{2}} & \frac{3}{5\sqrt{2}} \end{pmatrix}$$

Now, we have,

$D = P^{-1}AP$ also, $A = PDP^{-1}$

$= P^TAP \quad [\because P^{-1} = P^T]$

$$= \begin{pmatrix} 0 & -\frac{3}{5} & \frac{4}{5} \\ \frac{1}{\sqrt{2}} & \frac{4}{5\sqrt{2}} & \frac{3}{5\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{4}{5\sqrt{2}} & \frac{3}{5\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 4 & 3 \\ 4 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{3}{5} & \frac{4}{5\sqrt{2}} & \frac{4}{5\sqrt{2}} \\ \frac{4}{5} & \frac{3}{5\sqrt{2}} & \frac{3}{5\sqrt{2}} \end{pmatrix}$$

$$\therefore D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -4 \end{pmatrix}$$

$\therefore A = PDP^{-1}$

We know, $\lambda_1 = 1$, $\lambda_2 = 6$, $\lambda_3 = -4$

Now we have to prove $A = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \lambda_3 u_3 u_3^T$.

So,

$$\lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \lambda_3 u_3 u_3^T = 1 \begin{pmatrix} 0 \\ -\frac{3}{5} \\ \frac{4}{5} \end{pmatrix} \begin{pmatrix} 0 & -\frac{3}{5} & \frac{4}{5} \end{pmatrix} + 6 \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{4}{5\sqrt{2}} \\ \frac{3}{5\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{4}{5\sqrt{2}} & \frac{3}{5\sqrt{2}} \end{pmatrix} + (-4) \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{4}{5\sqrt{2}} \\ \frac{3}{5\sqrt{2}} \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{4}{5\sqrt{2}} & \frac{3}{5\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 4 & 3 \\ 4 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} = A.$$

$\therefore A = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \lambda_3 u_3 u_3^T$

Date
Page

(b) $B = \begin{pmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{pmatrix}$

Solution:-

Let λ be characteristic value. Then,
characteristic equation is,

$\Rightarrow |B - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 6-\lambda & -2 & -1 \\ -2 & 6-\lambda & -1 \\ -1 & -1 & 5-\lambda \end{vmatrix} = 0$

$\Rightarrow (6-\lambda)(\lambda-3)(\lambda-8) = 0$

$\therefore \lambda = 3, 6, 8$

Let v be eigen vector, Then,

$\Rightarrow (B - \lambda I) v = 0$

$\Rightarrow \begin{pmatrix} 6-\lambda & -2 & -1 \\ -2 & 6-\lambda & -1 \\ -1 & -1 & 5-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$ --- (1)

when $\lambda_1 = 3$

$\begin{pmatrix} 3 & -2 & 1 \\ -2 & 3 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$

$\begin{pmatrix} 3x - 2y - z \\ -2x + 3y - z \\ -x - y + 2z \end{pmatrix} = 0$

Date
Page

Here, $3x - 2y - z = 0$
$-2x + 3y - z = 0$
$-x - y + 3z = 0$

Solving we get,
$x = k_1, \; y = k_1, \; z = k_1$

So,
$$v_1 = k_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{where } k_1 \neq 0.$$

When $\lambda_2 = 6$, (1) becomes,
$$\Rightarrow \begin{pmatrix} 0 & -2 & -1 \\ -2 & 0 & -1 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} -2y - z \\ -2x - z \\ -x - y - z \end{pmatrix} = 0$$

Here, $-2y - z = 0$
$-2x - z = 0$
$-x - y - z = 0$

Solving we get,
$x = k_2, \; y = k_2 \quad z = -2k_2$

$$v_2 = k_2 \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \quad \text{where, } k_2 \neq 0.$$

CS CamScanner

when $\lambda_3 = 8$, ① becomes,

$$\Rightarrow \begin{pmatrix} -2 & -2 & -1 \\ -2 & -2 & -1 \\ -1 & -1 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} -2x & -2y & -z \\ -2x & -2y & -z \\ -x & -y & -3z \end{pmatrix} = 0$$

Here, $-2x - 2y - z = 0$
$-2x - 2y - z = 0$
$-x - y - 3z = 0$

Solving we get,
$x = k_3 \quad y = -k_3 \quad , z = 0$

So,
$$v_3 = k_3 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad \text{where } k_3 \neq 0$$

Let u_1, u_2, u_3 be orthonormal vector to v_1, v_2 and v_3 respectively. So,

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$u_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \end{pmatrix}$$

$$u_3 = \frac{v_3}{\|v_3\|} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\\0\end{pmatrix}$$

So,

$$P = \begin{array}{c} \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \\ \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \end{pmatrix} \end{array} \quad \text{and} \quad P^T = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

Now we have,

$D = P^{-1}BP$ also $B = PDP^{-1}$

$= P^T B P \quad [\because P^{-1} = P^T]$

$$= \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \end{pmatrix}$$

$$\therefore D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

$\therefore B = PDP^{-1}$

We know $\lambda_1 = 3$, $\lambda_2 = 6$, $\lambda_3 = 8$

Now we have to prove: $B = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \lambda_3 u_3 u_3^T$

For that,

$$\lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \lambda_3 u_3 u_3^T = 3\begin{pmatrix}\frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\end{pmatrix} + 6\begin{pmatrix}\frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{6}}\\ -\frac{2}{\sqrt{6}}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}}\end{pmatrix} + 8\begin{pmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\\ 0\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\end{pmatrix}$$

$$= \begin{pmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{pmatrix} = B$$

$$\therefore B = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \lambda_3 u_3 u_3^T \quad \#$$

Assignment-4

Date
Page

1) Decide the definiteness of the following quadratic form using eigen values:-

(a) $Q(x) = x_1^2 + 10x_1x_2 + x_2^2$

comparing with $ax_1^2 + 2b.x_1x_2 + cx_2^2$

$a = 1$, $b = 5$, $c = 1$

So,

$D_1 = 1$, $D_2 = \frac{ac - b^2}{a}$

$= \frac{1 - 5^2}{1} = -24$

Here $D_1 > 0$, $D_2 < 0$, Hence $Q(x)$ is indefinite.

Date
Page

(b) $Q(x) = 4x_1^2 - 4x_1x_2 + 4x_2^2$

comparing with $ax_1^2 + 2bx_1x_2 + cx_2^2$, we get,

$a = 4$, $b = -2$, $c = 4$

So,

$D_1 = a = 4$

$$D_2 = ac - b^2$$
$$= 4 \times 4 - (-2)^2$$
$$= 16 - 4$$
$$= 12$$

Here $D_1 > 0$, $D_2 > 0$, Hence Q is positive

(c) $Q(x) = -x_1^2 - 2x_1x_2 - x_2^2$

comparing with $ax_1^2 + 2bx_1x_2 + cx_2^2$, we get,

$a = -1$, $b = -1$, $c = -1$

So,

$D_1 = a = -1$

$$D_2 = ac - b^2$$
$$= 1 - 1$$
$$= 0$$

Here, $D_1 < 0$ and $D_2 = 0$, hence, $Q(x)$ is negative semi-definite.

CamScanner

Date
Page

(6) Find a change of variable $X = PY$ that removes the cross product term $x_1 x_2$ in the quadratic form. Also find formula and new quadratic form. Find the value of Y if $X = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ in each case,

(a) $5x_1^2 - 4x_1x_2 + 5x_2^2$

Solution

$Q(X) = 5x^2 - 4x_1x_2 + 5x_2^2$

Above can be written as $X^T A X$ i.e.

$$(x_1 \quad x_2)\begin{pmatrix} 5 & -2 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Here, $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $\quad A = \begin{pmatrix} 5 & -2 \\ -2 & 5 \end{pmatrix}$

Let λ be characteristic value. Then,

characteristic equation is,

$\Rightarrow |A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 5-\lambda & -2 \\ -2 & 5-\lambda \end{vmatrix} = 0$

$\Rightarrow (5-\lambda)^2 - 4 = 0$

$\Rightarrow \lambda^2 - 10\lambda + 25 - 4 = 0$

$\Rightarrow \lambda^2 - 7\lambda - 3\lambda + 21 = 0$

$\Rightarrow (\lambda - 7)(\lambda - 3) = 0$

$\therefore \lambda_1 = 7$ & $\lambda_2 = 3$

Let v be eigen vector. so :-

$(A-\lambda I)v = 0$

$$\begin{pmatrix} 5-\lambda & -2 \\ -2 & 5-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad \text{----} (1)$$

when $\lambda_1 = 7$, (1) becomes,

$$\Rightarrow \begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$\Rightarrow -2x_1 - 2x_2 = 0$

$\Rightarrow x_1 + x_2 = 0$

$\Rightarrow x_1 = -x_2$

let $x_2 = k$, then $x_1 = -k$

$$v_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} k \\ -k \end{pmatrix} = k\begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ where, } k \neq 0$$

when $\lambda_2 = 3$, (1) becomes,

$$\Rightarrow \begin{pmatrix} 5-3 & -2 \\ -2 & 5-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$\Rightarrow 2x_1 - 2x_2 = 0$

$\Rightarrow x_1 = x_2$

let $x_2 = k$, $x_1 = k$

so, $$v_2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} k \\ k \end{pmatrix} = k\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ where } k \neq 0$$

Let u_1 & u_2 be orthonormal vector.

$$u_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \qquad u_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad P^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

In order to remove the cross term $x_1 x_2$ in $Q(X)$ let us chang the variable

$X = PY$

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 7 & 0 \\ 0 & 3 \end{pmatrix}$$

we have, let $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$Q(X) = X^T P X = Y^T D Y$

$$x^2 - 8x_1x_2 - 5x^2 = (y_1 \; y_2) \begin{pmatrix} 7 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= 7y_1^2 + 3y_2^2 \quad \#$$

We have,

$X = PY$

$Y = P^{-1} X$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{---} \; (*)$$

Here, $(x_1 \; x_2) = (2 \; 1)$

replacing x_1 & x_2 in $(*)$, we get,

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} \end{pmatrix} \quad \#$$

(b) $x_1^2 + 10x_1x_2 + x_2^2$

Solution,

$Q(X) = x_1^2 + 10x_1x_2 + x_2^2$

Above can be written as X^TAX i.e.

$$(x_1 \quad x_2)\begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Here, $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix}$

Let λ be eigen value. Then,

characteristic equation is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 5 \\ 5 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - 2\lambda + 1 - 25 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda - 24 = 0$$

$$\Rightarrow (\lambda - 6)(\lambda + 4) = 0$$

Here, $\lambda = 6$, $\lambda = -4$

Let v be eigen vector. So,

$$(A - \lambda I)v = 0$$

$$\begin{pmatrix} 1-\lambda & 5 \\ 5 & 1-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad \text{---} \quad (1)$$

When $\lambda_1 = 6$, (1) becomes

$$\Rightarrow \begin{pmatrix} -5 & 5 \\ 5 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\Rightarrow -5x_1 + 5x_2 = 0$$

$$\Rightarrow x_1 = +x_2$$

Let $x_2 = k$ $\quad x_1 = +k$

Then, $v_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = k\begin{pmatrix} +1 \\ 1 \end{pmatrix}$ where $k \neq 0$.

When $\lambda_2 = -4$, (1) becomes,

$$\Rightarrow \begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\Rightarrow 5x_1 + 5x_2 = 0$$

$$\Rightarrow x_1 = -x_2$$

let $x_1 = k$, then $x_2 = -k$.

Then,

$$v_2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = k \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{where } k \neq 0$$

Let u_1, u_2 be orthonormal vectors,

$$u_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$u_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Now,

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \qquad P^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Diagonal matrix $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & -4 \end{pmatrix}$

In order to remove the cross term in $Q(x)$ let us change the variable $X = PY$

we have, let $Y = (y_1 \; y_2)$

$$Q(X) = X^T P X = Y^T D Y$$

$$x_1^2 + 10x_1x_2 + x_2^2 = (y_1 \; y_2) \begin{pmatrix} 6 & 0 \\ 0 & -4 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= 6y_1^2 - 4y_2^2$$

#

We have,

$X = PY$

$Y = P^{-1}X$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{----} \quad (*)$$

Here, $(x_1 \quad x_2) = (2 \quad 1)$

Then, $(*)$ becomes,

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad \#$$

(c) $4x_1^2 - 4x_1x_2 + 4x_2^2$

Solution,

$Q(X) = 4x_1^2 - 4x_1x_2 + 4x_2^2$

Above can be written as X^TAX, i.e.

$$(x_1 \quad x_2) \begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Here, $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $A = \begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix}$

Let λ be eigen value. Then,

characteristic equation is

$|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 4-\lambda & -2 \\ -2 & 4-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda^2 - 8\lambda + 16 - 4 = 0$

$\Rightarrow \lambda^2 - 8\lambda + 12 = 0$

$\Rightarrow (\lambda - 6)(\lambda - 2) = 0$

$\therefore \lambda = 6 \quad , \quad \lambda = 2$

Let v be eigen vector. So,

$(A - \lambda I) v = 0$

$$\begin{pmatrix} 4-\lambda & -2 \\ -2 & 4-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad ----(1)$$

When $\lambda_1 = 6$, (1) becomes,

$$\Rightarrow \begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} -2x_1 - 2x_2 \\ -2x_1 - 2x_2 \end{pmatrix} = 0$$

$\Rightarrow x_1 + x_2 = 0$

$\Rightarrow x_1 = -x_2$

When, $x_2 = k \quad x_1 = -k$

$$v = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = k \begin{pmatrix} -1 \\ 1 \end{pmatrix} \text{ where } k \neq 0$$

When, $\lambda_2 = 4$, (1) becomes,

$$\Rightarrow \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} 2x_1 - 2x_2 \\ -2x_1 + 2x_2 \end{pmatrix} = 0$$

$\Rightarrow x_1 - x_2 = 0$

$\Rightarrow x_1 = x_2$

Let $x_1 = k \quad x_2 = k$

$$v = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = k \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ where } k \neq 0.$$

Let u_1 & u_2 be two orthonormal vectors. Then,

$$u_1 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad \& \quad u_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Now,

$$P = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \quad \& \quad P^{-1} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Diagonal matrix $(D) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}$

In order to remove the cross term $x_1 x_2$ in $Q(x)$ let us change the variable $X = PY$.

Let $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$$Q(X) = X^T P X = Y^T D Y$$

$$4x_1^2 - 4x_1x_2 + 4x_2^2 = (y_1 \;\; y_2) \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= 6y_1^2 + 2y_2^2 \quad \#$$

We have,

$X = PY$

$Y = P^{-1}X$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{——— } (*)$$

Here, $(x_1 \;\; x_2) = (2 \;\; 1)$

Then, $(*)$ becomes,

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} \end{pmatrix} \quad \#$$

Date
Page

(b) $x_1^2 - 6x_1x_2 + 9x_2^2$

Solution:-

$Q(X) = x_1^2 - 6x_1x_2 + 9x_2^2$

Above can be written as X^TAX i.e.

$$(x_1 \ x_2)\begin{pmatrix} 1 & -3 \\ -3 & 9 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Here, $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $A = \begin{pmatrix} 1 & -3 \\ -3 & 9 \end{pmatrix}$

Let λ be eigen value. Then characteristic equation is

$\Rightarrow |A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & -3 \\ -3 & 9-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^2 - 10\lambda + 9 - 9 = 0$

$\Rightarrow \lambda^2 - 10\lambda = 0$

$\Rightarrow \lambda(\lambda - 10) = 0$

$\therefore \lambda = 10, \lambda = 0$.

Let v be eigen vector. So,

$(A - \lambda I)v = 0$

$$\begin{pmatrix} 1-\lambda & -3 \\ -3 & 9-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad \text{----} (1)$$

When $\lambda_1 = 10$, (1) becomes

$\Rightarrow \begin{pmatrix} -9 & -3 \\ -3 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$\Rightarrow \begin{pmatrix} -9x_1 - 3x_2 \\ -3x_1 - 1x_2 \end{pmatrix} = 0$

$\Rightarrow 3x_1 + x_2 = 0 \qquad \Rightarrow -3x_1 = +x_2$

Let $x_1 = k$ then $x_2 = -3k$

Then, $v_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = k\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ where $k \neq 0$.

When $\lambda = 0$, ① becomes,

$\Rightarrow \begin{pmatrix} 1 & -3 \\ -3 & 9 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

$\Rightarrow \begin{pmatrix} x_1 - 3x_2 \\ -3x_1 + 9x_2 \end{pmatrix} = 0$

$\Rightarrow x_1 - 3x_2 = 0 \quad \Rightarrow x_1 = 3x_2$

Then, let $x_2 = k$, $x_1 = 3k$

$v_2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = k\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ where, $k \neq 0$.

Let u_1 & u_2 be an orthonormal vectors,

$u_1 = \begin{pmatrix} \frac{1}{\sqrt{10}} \\ -\frac{3}{\sqrt{10}} \end{pmatrix}$ & $u_2 = \begin{pmatrix} \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{pmatrix}$

Now,

$P = \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ -\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix}$ & $P^{-1} = \begin{pmatrix} \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix}$

Diagonal matrix $(D) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 10 & 0 \\ 0 & 0 \end{pmatrix}$

In order to remove the cross term x_1x_2 in $Q(x)$, let us change the variable $X = PY$

we have, let $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$Q(x) = X^TAX \Rightarrow Y^TDY$

$x_1^2 - 6x_1x_2 + 9x_2^2 = (y_1\ y_2)\begin{pmatrix} 10 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

Date
Page

$= 10y_1^2$

CamScanner