n variables $x_1, x_2, ..., x_n$:

$$C = f(x_1, x_2, ..., x_n) = c_1x_1 + c_2x_2 + \cdot + c_nx_n.$$

The function f is a real-valued function whose domain is a subset of $\mathbb{R}^n$. Sometimes we will use vector notation to write such functions more compactly: If $x = (x_1, x_2, ..., x_n)$, we often write $f(x)$in place of $f(x_1, x_2, ..., x_n)$. With this notation we can rewrite the function defined in Equation 3 as

$$f(x) = c \cdot x,$$

where $c = (c_1, c_2, ..., c_n)$ and $c \cdot x$ denotes the dot product of the vectors c and x in V_n.

In view of the one-to-one correspondence between points $(x_1, x_2, ..., x_n)$ in $\mathbb{R}^n$ and their position vectors $x = (x_1, x_2, ..., x_n)$ in V_n, we have three ways of looking at a function f defined on a subset of $\mathbb{R}^n$:

1. As a function of n real variables $x_1, x_2, ..., x_n$.
2. As a function of a single point variable $(x_1, x_2, ..., x_n)$
3. As a function of a single vector variable $x = (x_1, x_2, ..., x_n)$

We will see that all three points of view are useful.

3 Limits and continuity

Limits

Let $u = (x, y) \in \mathbb{R}^2$. Then we write

$$\|u\| = \sqrt{x^2 + y^2}.$$

As you know, this is the Euclidean norm of u.

Let $D \subseteq \mathbb{R}^2$ and $p = (a,b) \in \mathbb{R}^2$. The point p is called a limitpoint or accumulation point of D if D includes points $u = (x,y)$ arbitrarily close to p, i.e.,

$$\forall\, r > 0\ \exists\, u \in D:\ 0 < \|p - u\| < r.$$

Limit at a point

Let f be a real valued function defined on $D \subseteq \mathbb{R}^2$ and $p = (a,b) \in \mathbb{R}^2$ be a limitpoint of D. Then we say that L is the limit of $f(u)$ as $u = (x,y) \in D$ tends to p if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\forall\, u = (x,y) \in D,\ 0 < \|u - p\| < \delta \Rightarrow |f(x,y) - L| < \varepsilon.$$

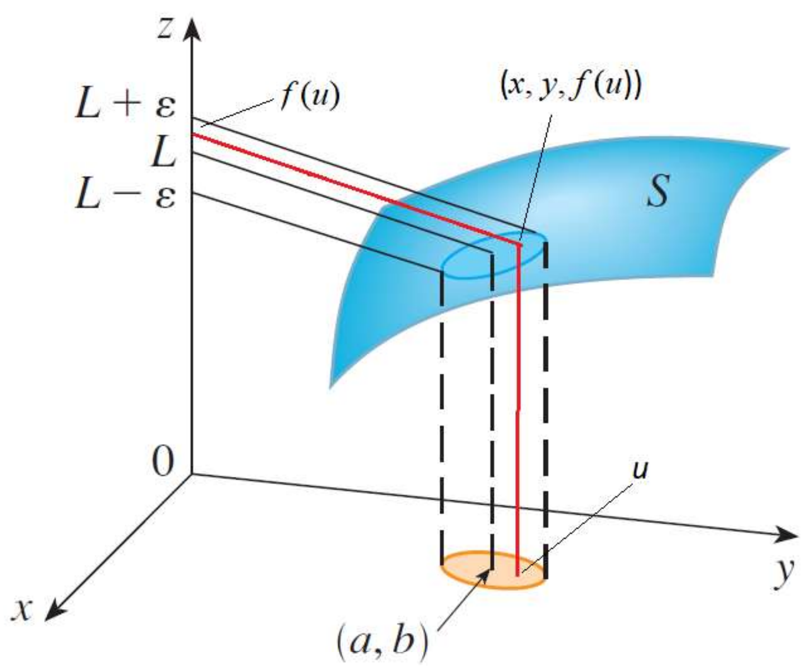

Figure 11: Limit at a point u.

In this case, we write

$$\lim_{u \to p} f(u) = L.$$

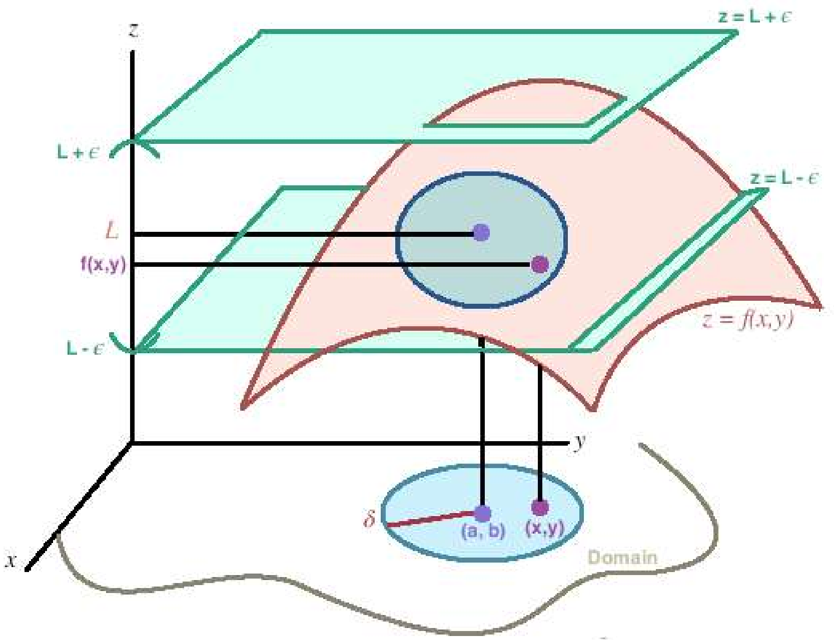

Figure 12: Limit at a point u.

Other notations for the limit in the above definition are

$$\lim_{(x,y)\to(a,b)} f(x,y) = L \quad \text{and} \quad f(x,y) \to L \text{ as } (x,y) \to (a,b).$$

The above definition says that

Intuitively

The distance between $f(u)$ and L can be made arbitrarily small by making the distance from u to p sufficiently small (but not 0).

Verifying a Limit by the Definition

Example 9. Show that

$$\lim_{(x,y)\to(a,b)} f(x,y) = a.$$

Solution. Let $f(x,y) = x$ and $L = a$. We need to show that for each $\varepsilon > 0$ if $(x,y) \neq (a,b)$ and

$$\|(x,y) - (a,b)\| < \delta, i.e., \sqrt{(x-a)^2 + (y-b)^2} < \delta,$$

then

$$|f(x,y) - L| < \varepsilon.$$

Now, if $(x,y) \neq (a,b)$ and

$$\|(x,y) - (a,b)\| < \delta,$$

then we have

$$\begin{aligned} |f(x,y) - L| &= |x - a| = \sqrt{(x-a)^2} \\ &\leq \sqrt{(x-a)^2 + (y-b)^2} \\ &= \|(x,y) - (a,b)\| \\ &< \delta. \end{aligned}$$

So, choosing $\delta = \varepsilon$, we obtain

$$|f(x,y) - L| < \varepsilon. \qquad \blacksquare$$

Limit laws:

Limits of functions of several variables have the same properties regarding sums, differences, products, and quotients as do limits of functions of single variables.

Let $u = (x,y)$ and $p = (a,b)$. If $\lim_{u \to p} f(u) = L$, $\lim_{u \to p} g(u) = M$, then the following sum, product, and quotient rules, and squeeze theorem hold.

Limit laws

(a) $\lim_{u \to p} (f(u) + g(u)) = L + M$

(b) $\lim_{u \to p} (f(u)g(u)) = LM$

(c) $\lim_{u \to p} \frac{f(u)}{g(u)} = \frac{L}{M} \; (M \neq 0).$

Some of these properties are used in the next example.

Example 10. Evaluate

$$\lim_{(x,y) \to (1,2)} \frac{3x^2 y}{x^2 + y^2}.$$

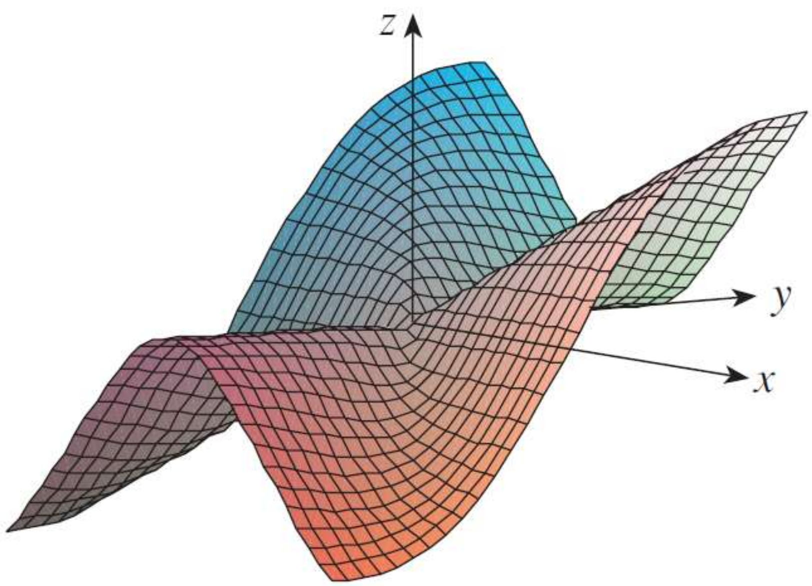

Figure 13: $\frac{3x^2y}{x^2+y^2}$

.

Solution.

We observe that

$$\lim_{(x,y)\to(1,2)} 3x^2y = 5(1)^2(2) = 6,$$
$$\lim_{(x,y)\to(1,2)} (x^2+y^2) = 1^2 + 2^2 = 5.$$

Because the limit of a quotient is equal to the quotient of the limits (and the denominator is not 0), we have

$$\lim_{(x,y)\to(1,2)} \frac{3x^2y}{x^2+y^2} = \frac{6}{5}. \qquad \blacksquare$$

Example 11. Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$, if exists.

Solution. In this case, the limits of the numerator and of the denominator are both 0, and so we cannot evaluate the limit by using operations on limits as in the previous example. However, it seems reasonable that the limit might be 0, because if $y = ax$, then

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{3ax^3}{x^2(1+a^2)} = 0.$$

So, we try applying the definition to $L = 0$.

For it we observe that

$$\frac{x^2}{x^2+y^2} \le 1, \qquad y \le |y| \le \sqrt{x^2+y^2}.$$

Now, suppose that $\|(x,y)-(0,0)\| < \delta$. Then

$$\sqrt{x^2+y^2} = \|(x,y)\| < \delta.$$

Now, put $f(x,y) = \frac{3x^2y}{x^2+y^2}$. For $(x,y) \neq (0,0)$ we have

$$\begin{aligned}
|f(x,y) - 0| = \left|\frac{3x^2y}{x^2+y^2}\right| &= 3|y|\frac{x^2}{x^2+y^2} \\
&\leq 3|y| = 3\sqrt{y^2} \\
&\leq 3\sqrt{x^2+y^2} \\
&< 3\delta \\
&= \varepsilon. \qquad (\text{choosing } \delta = \varepsilon/3)
\end{aligned}$$

Therefore,

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0. \qquad \blacksquare$$

Squeeze theorem

It makes easier sometimes to find the limit applying the following theorem, in the case when it is not possible to find the limit by using the above operations.

Squeeze theorem

Let

$$\lim_{u\to p} f(u) = L, \quad \lim_{u\to p} g(u) = M.$$

If

$$\lim_{u\to p} f(u) = \lim_{u\to p} g(u) \text{ and } f(u) \leq h(u) \leq g(u),$$

then $\lim\limits_{u\to p} h(u)$ exists and equals L which equals M.

Example 12. Find $\lim\limits_{(x,y)\to(0,0)} \dfrac{3x^2y}{x^2+y^2}$, if exists.

Solution. Let $y = ax$. Then

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{3ax^3}{x^2(1+a^2)} = 0.$$

This shows that the limit along any line through the origin is 0. Although this doesn't prove that the given limit is 0, we begin to suspect that the limit does exist and is equal to 0. we prove it.

Since $y^2 \geq 0$, we have

$$x^2 \leq x^2 + y^2 \Rightarrow \frac{x^2}{x^2+y^2} \leq 1$$

$$\Rightarrow \quad 0 \leq \frac{3x^2|y|}{x^2+y^2} \leq 3\sqrt{y^2}$$

$$\Rightarrow \quad \lim_{(x,y)\to(0,0)} 0 \leq \lim_{(x,y)\to(0,0)} \frac{3x^2|y|}{x^2+y^2} \leq \lim_{(x,y)\to(0,0)} 3\sqrt{y^2}$$

However, we have

$$\lim_{(x,y)\to(0,0)} 0 = 0, \qquad \lim_{(x,y)\to(0,0)} 3\sqrt{y^2} = 0.$$

Therefore, by the Squeeze Theorem,

$$\lim_{(x,y)\to(0,0)} \frac{3x^2|y|}{x^2+y^2} = 0.$$

Using the property that $-|c| \leq c \leq |c|$ for any real number c, we obtain

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0. \qquad \blacksquare$$

The definition refers only to the distance between u and p. It does not refer to the direction of approach. That means, if the limit exists, then $f(u)$ must approach the same limit no matter how u approaches p.

For some functions, it is easy to recognize that a limit does not exist. For instance, it is clear that the limit

$$\lim_{(x,y)\to(0,0)} \frac{1}{x^2+y^2}$$

does not exist because the values of $f(x,y)$ increase without bound as (x,y) approaches along any path (see the figure given below.

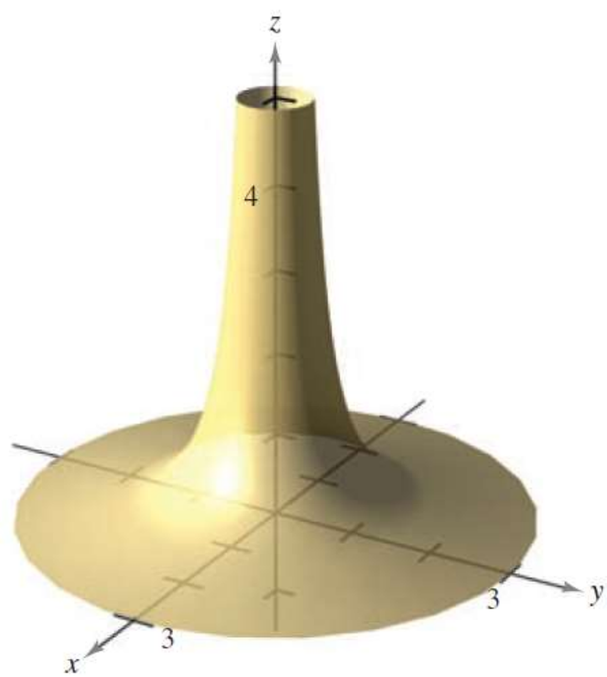

Figure 14: $\lim_{(x,y)\to(0,0)} \frac{1}{x^2+y^2}$ does not exist.

For other functions, it is not so easy to recognize that a limit does not exist. However, in many cases, the following criterion is very helpful.

Nonexistence criterion for limits

Let $f(u) \to L_1$ as $u \to p$ along a path C_1 and $f(u) \to L_2$ as $u \to p$ along a path C_2. If $L_1 \neq L_2$, then the limit $\lim_{u \to p} f(u)$ does not exist.

Thus, by this criterion, if we can find two different paths of approach along which the function $f(u)$ has different limits, then it follows that $\lim_{u \to p} f(u)$ does not exist.

Example 13. Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ does not exist.

Solution. Let's approach $(0,0)$ first along the x-axis and then along the y-axis.

Example 14. If $f(x,y) = \dfrac{xy}{x^2 + y^2}$, does the limit $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ exist?

Solution. Let's approach $(0,0)$ first along the x-axis and along the y-axis. Then approach $(0,0)$ along another line, say, $x = y$ for all $x \neq 0$. No!

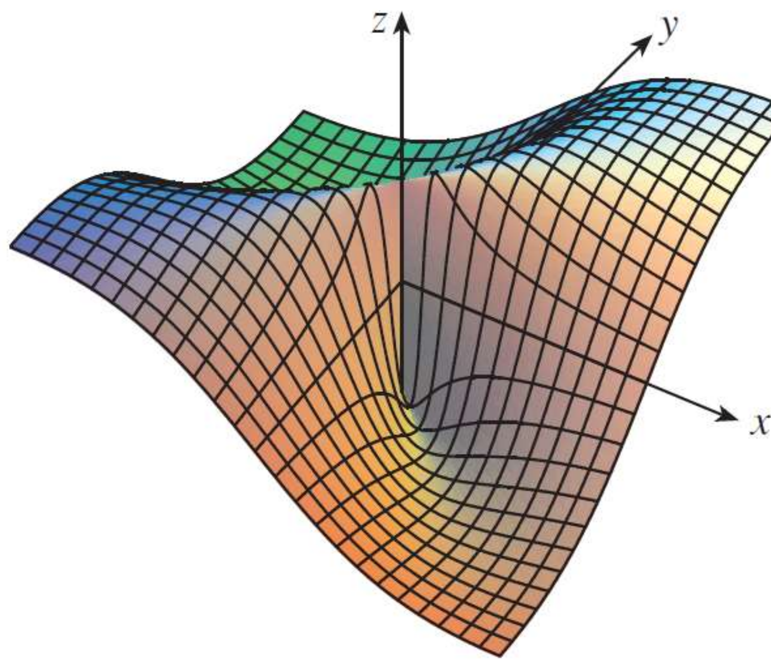

Figure 15: $f(x, y) = \dfrac{xy}{x^2 + y^2}$

Example 15. If $f(x, y) = \dfrac{xy^2}{x^2 + y^4}$, does the limit $\lim_{(x,y)\to(0,0)} f(x, y)$ exist?

Solution. Let's approach $(0, 0)$ along any nonvertical line $y = mx$ through the origin. But if we approach $(0, 0)$ Then approach $(0, 0)$ along along the parabola $x = y^2$, then $f(x, y) = 1/2$. No!

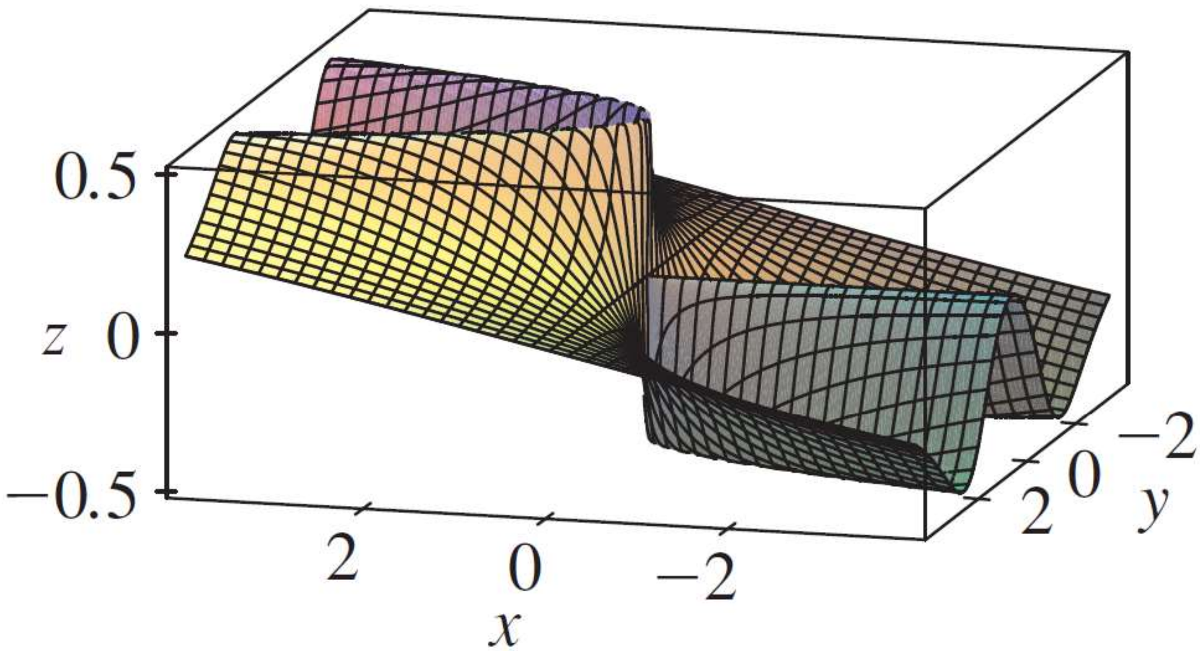

Figure 16: $f(x, y) = \dfrac{xy^2}{x^2 + y^4}$

Continuity

Continuity at a point

A function f of two variables is called **continuous at a point** (a, b) in a set $D \subseteq \mathbb{R}^2$ if the following conditions are satisfied:

1. $f(a, b)$ exists.
2. $\displaystyle\lim_{(x,y)\to(a,b)} f(x, y)$ exists.
3. $\displaystyle\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.

We say f is **continuous on a set** $D \subseteq \mathbb{R}^2$ if f is continuous at every point (a, b) in D.

The intuitive meaning of continuity is that if the point (x, y) changes by a small amount, then the value of $f(x, y)$ changes by a small amount. This means that a surface that is the graph of a continuous function has no hole or break.

Using the properties of limits, you can see that sums, differences, products, and quotients of continuous functions are continuous on their domains. Moreover, the following property related to a composition of two functions also holds:

Interchange of the limit and function

Let $\lim_{(x,y)\to(a,b)} f(x, y) = L$. If $F(t)$ is a continuous function at $t = L$, then

$$\lim_{(x,y)\to(a,b)} F(f(x, y)) = F(L) = F(\lim_{(x,y)\to(a,b)} f(x, y)).$$

That is, for continuous functions, we may interchange the limit and function composition operations.

Let's use these facts to give examples of continuous functions.

It is easy to show that

$$\lim_{(x,y)\to(a,b)} x = a, \qquad \lim_{(x,y)\to(a,b)} y = b, \qquad \lim_{(x,y)\to(a,b)} c = c.$$

These limits show that the functions $f(x, y) = x$, $g(x, y) = y$, and $h(x, y) = c$ are continuous everywhere on $\mathbb{R}^2$.

A **polynomial function of two variables** is a sum of terms of the form $cx^m y^n$, where c is a constant and m and n are nonnegative integers. It follows that all polynomials are continuous on $\mathbb{R}^2$.

Likewise, any rational function $f(x,y) = \dfrac{P(x,y)}{Q(x,y)}$, where $P(x,y), Q(x,y)$ are polynomials and $Q(x,y) \neq 0$ is continuous on its domain because it is a quotient of continuous functions $P(x,y)$ and $Q(x,y)$.

Example 16. Evaluate $\lim\limits_{(x,y)\to(a,b)} (x^2y^3 - x^3y^2 + 3x + 2y)$.

Solution. ...

Example 17. Discuss the continuity of the function

$$f(x,y) = \frac{3x^2y}{x^2+y^2}.$$

Solution. ...

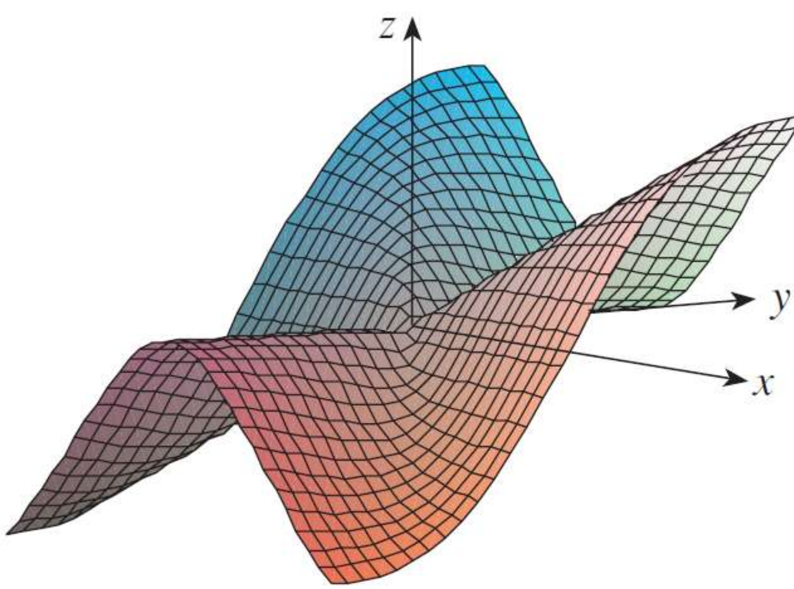

Figure 17: $f(x,y) = \dfrac{3x^2y}{x^2+y^2}$.

Example 18. Discuss the continuity of the function

$$f(x, y) = \begin{cases} \dfrac{3x^2y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Solution. ...

Example 19. If $f(x, y) = \dfrac{xy}{x^2 + y^2}$, does the limit $\displaystyle\lim_{(x,y)\to(0,0)} f(x, y)$ exist?

Solution. ...

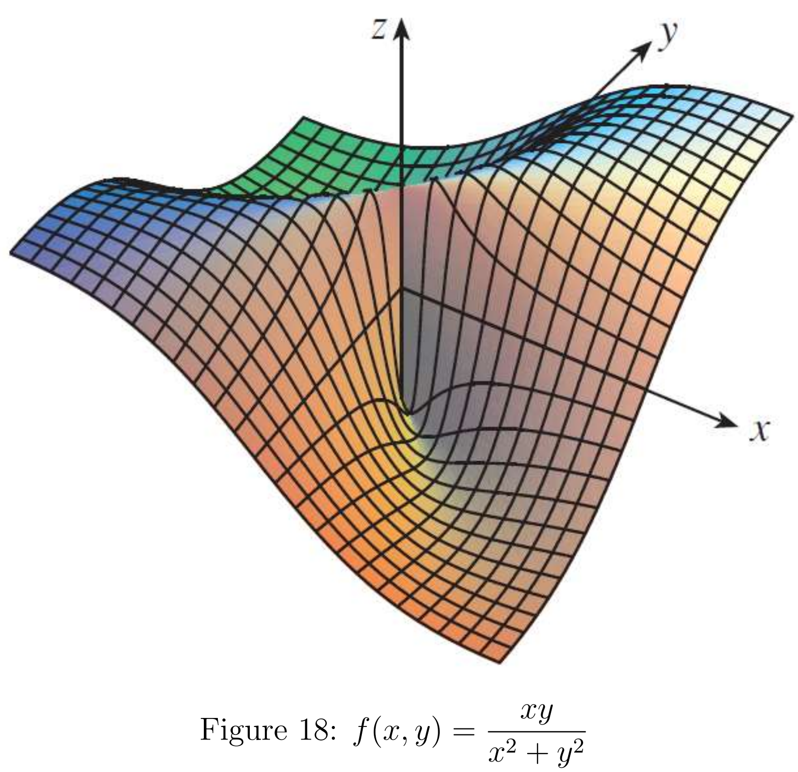

Figure 18: $f(x, y) = \dfrac{xy}{x^2 + y^2}$

Example 20. Where is the function $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ continuous?

Solution. ...

Example 21. Show that the function

$$f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

is discontinuous at the origin.

Solution.

Example 22. Where is the function $h(x,y) = \arctan(y/x)$ continuous?

Solution. The function $f(x,y) = y/x$ is a rational function and therefore continuous except on the line $x = 0$. The function $g(t) = \arctan t$ is continuous everywhere. So the composite function

$$g(f(x,y)) = \arctan(y/x) = h(x,y)$$

is continuous except where $x = 0$. The graph in the following figure shows the break in the graph of h above the x-axis.

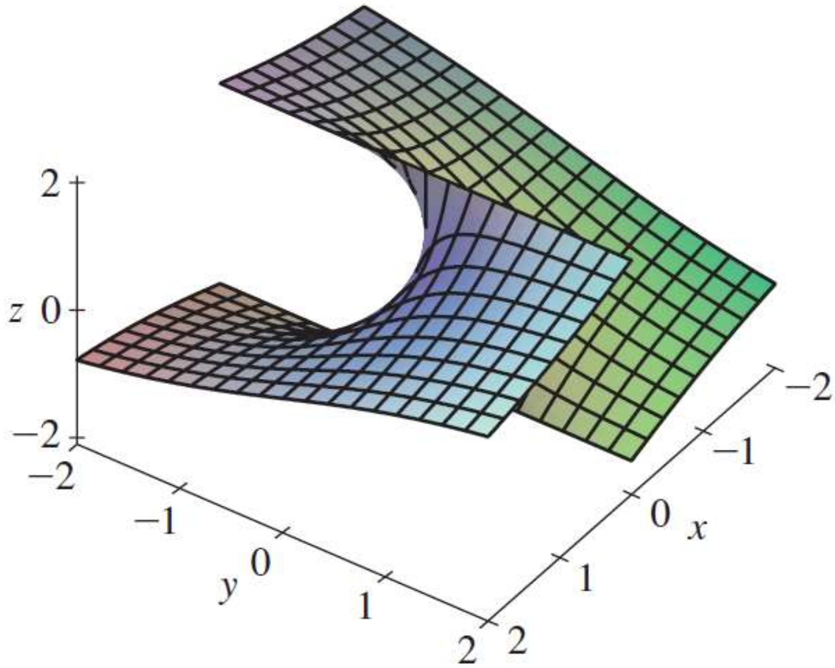

Figure 19: The function $h(x, y) = arctan(y/x)$
is discontinuous where $x = 0$.

Everything that we have done in this section can be extended to functions of three or more variables. The notation

$$\lim_{(x,y,z)\to(a,b,c)} f(x, y, z) = L$$

means that the values of $f(x, y, z)$ approach the number as the point (x, y, z) approaches the point (a, b, c) along any path in the domain of f. The function f is continuous at (a, b, c) if

$$\lim_{(x,y,z)\to(a,b,c)} f(x, y, z) = f(a, b, c)$$

For instance, the function

$$f(x, y, z) = \frac{1}{x^2 + y^2 + z^2 - 1}$$

is a rational function of three variables and so is continuous at every point in $\mathbb{R}^2$ except where $x^2 + y^2 + z^2 = 1$. In other words, it is discontinuous on the sphere with center at the origin and of radius 1.