4 Partial derivatives

It is worthwhile to note that a function f of two or more variables does not have a unique rate of change because each variable may affect f in different ways.

For example, the current I in a circuit is a function of both voltage V and resistance R given by Ohm's Law:

$$I(V, R) = \frac{V}{R}.$$

The current I is increasing as a function of V but decreasing as a function of R.

The partial derivatives are the rates of change with respect to each variable separately. A function $f(x, y)$ of two variables has two partial derivatives, denoted f_x and f_y , defined by the following limits (if they exist):

Partial derivatives

$$f_x(a, b) = \lim_{h\to 0} \frac{f(a + h, b) - f(a, b)}{h}$$
$$f_y(a, b) = \lim_{k\to 0} \frac{f(a, b + k) - f(a, b)}{k}$$

Thus, f_x is the derivative of $f(x, b)$ as a function of x alone, and f_y is the derivative of $f(a, y)$ as a function of y alone. The Leibniz notation for partial derivatives is

$$\frac{\partial f}{\partial x} = D_x f = f_x, \qquad \frac{\partial f}{\partial y} = D_y f = f_y,$$

$$\left.\frac{\partial f}{\partial x}\right|_{(a,b)} = f_x(a,b), \qquad \left.\frac{\partial f}{\partial y}\right|_{(a,b)} = f_y(a,b).$$

To compute partial derivatives, all we have to do is remember that the partial derivative with respect to x is just the ordinary derivative of the function f of a single variable that we get by keeping y fixed. Thus we have the following rule.

Rules for Finding Partial Derivatives

Let $z = f(x,y)$.

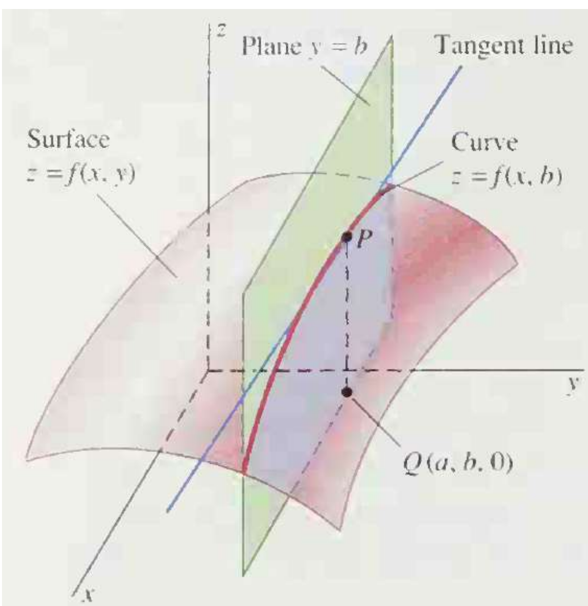
1. To find f_x, regard y as a constant and differentiate $f(x,y)$ with respect to x.
2. To find f_y, regard x as a constant and differentiate $f(x,y)$ with respect to y.

4.1 Interpretations of Partial Derivatives

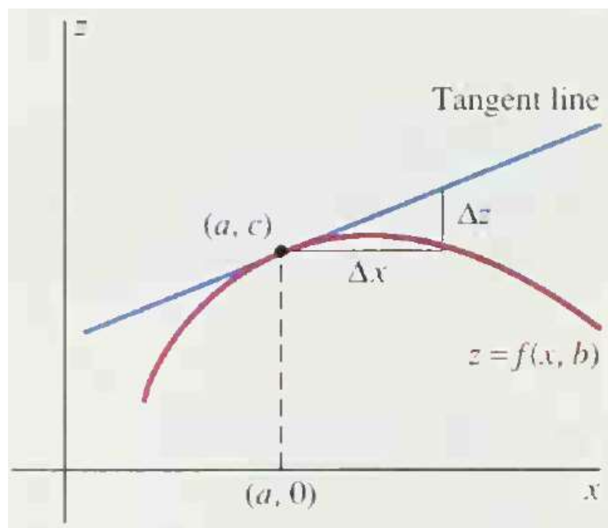
The intersection of a surface $z = f(x, y)$ with a vertical plane $y = b$ that is parallel to the xz-coordinate plane. Along the intersection curve, the x-coordinate varies but the y-coordinate is constant: $y = b$ at each point, because the curve lies in the vertical plane $y = b$.

x-curve on a surface

A curve of intersection of $z = f(x, y)$ with a vertical plane parallel to the xz-plane is called an x-curve on the surface.



(a) An x-curve and its tangent line.



(b) Projection into the xz-plane of the x-curve through $P(a, b, c)$ and its tangent line.

Figure (a) shows a point $P(a.b, c)$ in the surface $z = f(x, y)$, the x-curve through P and the line tangent to this x-curve at P. Figure (b) shows the parallel projection of the vertical plane $y = b$ onto the xz-plane itself. We can now "ignore" the presence of $y = b$ and regard $z = f(x, b)$ as a function of the single variable x. The slope of the line tangent to the original x-curve through P (see Fig. (a)) is equal to the slope of the tangent line in Fig. (b). But by familiar single-variable calculus, this latter slope is given by

$$\frac{\partial z}{\partial x} = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h} = f_x(a,b).$$

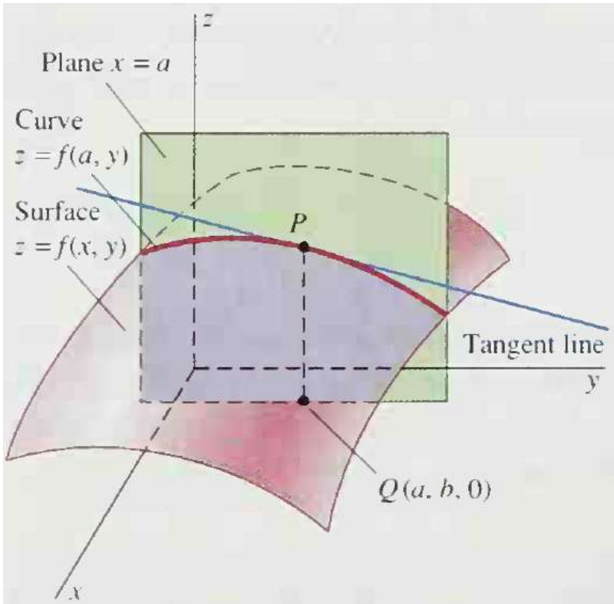
Thus, we see that the geometric meaning of f_x is this:

The value $\partial z/\partial y = f_x(a,b)$ is the slope of the line tangent at $P(a,b,c)$ to the x-curve through P on the surface $z = f(x,y)$ as shown in Figure (a).

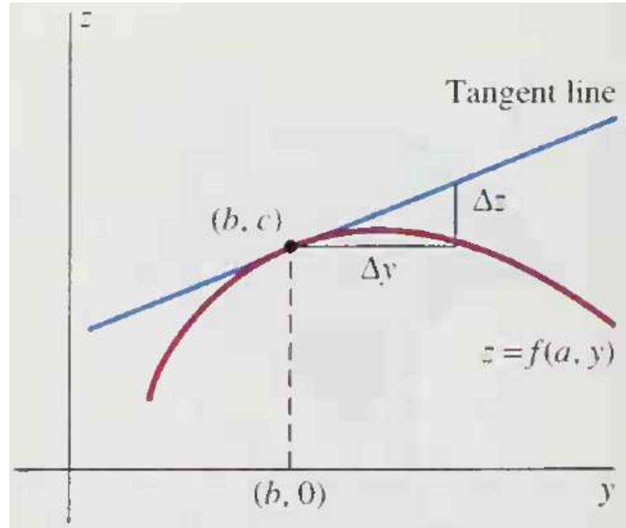
We proceed in much the same way to investigate the geometric meaning of partial derivative f_x.

y-curve on a surface

A curve of intersection of a surface $z = f(x,y)$ with a vertical plane parallel to the yz-plane is called a y-curve on the surface.



(c) A y-curve and its tangent line.



(d) Projection into the yz-plane of the y-curve through $P(a, b, c)$ and its tangent line.

Figure (c) shows a point $P(a.b, c)$ in the surface $z = f(x, y)$, the y-curve through P and the line tangent to this y-curve at P. Figure (d) shows the parallel projection of the vertical plane $x = a$ onto the yz-plane itself. We can now "ignore" the presence of $x = a$ and regard $z = f(a, y)$ as a function of the single variable y. The slope of the line tangent to the original y-curve through P (see Fig. (c)) is equal to the slope of the tangent line in Fig. (d). But by familiar single-variable calculus, this latter slope is given by

$$\frac{\partial z}{\partial y} = \lim_{k \to 0} \frac{f(a, b + k) - f(a, b)}{k} = f_y(a, b).$$

Thus, we see that the geometric meaning of f_x is this:

If $x = x_0$ then $z = f(x_0, y)$ represents the curve formed by intersecting the surface $z = f(x, y)$ with the plane $x = x_0$ as shown in Figure (c).

We put the two geometric interpretations together for the comparison purpose.

Geometric interpretation

- If $y = y_0$ then $z = f(x, y_0)$ represents the curve formed by intersecting the surface $z = f(x, y)$ with the plane $y = y_0$.
- If $x = x_0$ then $z = f(x_0, y)$ represents the curve formed by intersecting the surface $z = f(x, y)$ with the plane $x = x_0$.

Partials evaluated at a point

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} \quad - \quad \text{the slope of the tangent line } L_1 \text{ to the curve } f(x, y_0) \text{ at } (x_0, y_0).$$

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} \quad - \quad \text{the slope of the tangent line } L_2 \text{ to the curve } f(x_0, y) \text{ at } (x_0, y_0).$$

Informally, the values of f_x and f_y at a point denote the slopes of the surface in the x- and y-directions at the point, respectively.

Problem 7.

(a) For $f(x, y) = 9x^2y - 3x^5y$, find f_x and f_y.

(b) For $f(x, y) = xe^{x^2y}$, find f_x and f_y, and evaluate each at the point $(1, \ln 2)$.

Problem 8. Let $f(x, y) = \sqrt{3x + 2y}$.

(a) Find the slope of the surface $z = f(x, y)$ in the x-direction at the point (4, 2).

(b) Find the slope of the surface $z = f(x, y)$ in the y-direction at the point (4, 2).

Problem 9. Let $z = \sin(y^2 - 4x)$.

(a) Find the rate of change of z with respect to x at the point (2, 1) with y held fixed.

(b) Find the rate of change of z with respect to y at the point (2, 1) with x held fixed.

Example 23 (Implicit partial differentiation:)**.** Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if z is defined implicitly as a function of x and y by the equation

$$x^3 + y^3 + z^3 + 6xyz = 1.$$

Solution. ...

4.2 Functions of More Than Two Variables

Problem 10. Find f_x, f_y, and f_z, if $f(x, y, z) = z\ln(x^2 y \cos z)$.

4.3 Higher Derivatives

Suppose that f is a function of two variables x and y. Since the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are also functions of x and y, these functions may themselves have partial derivatives. This gives rise to four possible second-order partial derivatives of f, which are defined by

Differentiate twice with respect to x.

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}$$

Differentiate twice with respect to y.

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}$$

Differentiate first with respect to y and then with respect to x.

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{xy}$$

Differentiate first with respect to x and then with respect to y.

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{yx}$$

The last two cases are called the mixed second-order partial derivatives or the mixed second partials. Also, the derivatives $\partial f/\partial x$ and $\partial f/\partial y$ are often called the first-order partial derivatives when it is necessary to distinguish them from higher-order partial derivatives.

Similar conventions apply to the second-order partial derivatives of a function of three variables.

Warning

Observe that the two notations for the mixed second partials have opposite conventions for the order of differentiation. In the "∂" notation the derivatives are taken right to left, and in the "subscript" notation they are taken left to right.

Example 24. Find the second partial derivatives of

$$f(x, y) = x^3 + x^2y^3 + 2y^2.$$

Solution. ...

The following theorem, which was discovered by the French mathematician Alexis Clairaut (1713 - 1765), gives conditions under which we can assert that $f_{xy} = f_{yx}$.

Theorem 4.1. *Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then*

$$f_{xy}(a, b) = f_{yx}(a, b).$$

Proof.

For small values of $h \neq 0$, consider the difference

$$\Delta(h) = [f(a+h, b+h) - f(a+h, b)] - [f(a, b+h) - f(a, b)].$$

Notice that if we let $g(x) = f(x, b + h) - f(x, b)$, then

$$\Delta(h) = g(a + h) - g(a).$$

By the Mean Value Theorem, there is a number c between a and $a+h$ such that

$$g(a+h)-g(a) = g'(c)h = h[f_x(c, b+h) - f_x(c, b)].$$

Applying the Mean Value Theorem again, this time to f_x, we get a number d between b and $b+h$ such that

$$f_x(c, b+h) - f_x(c, b) = f_{xy}(c, d)h$$

Combining these equations, we obtain

$$\Delta(h) = h^2 f_{xy}(c, d)$$

If $h \to 0$, then $(c, d) \to (a, b)$, so the continuity of f_{xy} at (a, b) gives

$$\lim_{h\to 0} \frac{\Delta(h)}{h^2} = \lim_{(c,d)\to(a,b)} f_{xy}(c, d) = f_{xy}(a, b)$$

Similarly, by writing

$$\Delta(h) = [f(a+h, b+h) - f(a, b+h)] - [f(a+h, b) - f(a, b)].$$

and using the Mean Value Theorem twice and the continuity of f_{yx} at (a, b) , we obtain

$$\lim_{h\to 0} \frac{\Delta(h)}{h^2} = f_{yx}(a, b)$$

It follows that

$$f_{xy}(a, b) = f_{yx}(a, b). \quad \blacktriangleleft$$

Problem 11. Let $f(x, y) = e^x \cos y$. Confirm that the mixed second-order partial derivatives of f are the same

Partial derivatives of order 3 or higher

Example 25. Calculate f_{xxyz} if $f(x, y, z) = \sin(3x + yz)$.

5 Partial Differential Equations

A **partial differential equation** (PDE) is a differential equation involving functions of several variables and their partial derivatives.

5.1 Laplace's equation

Laplace's equation

The partial differential equation of the form:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is called **Laplace's equation**. Solutions of this equation are called **harmonic functions**.

Laplace's equations play a role in problems of heat conduction, fluid flow, and electric potential.

Example 26. Show that the function $u(x, y) = e^x \sin y$ is a solution of Laplace's equation.

Solution. ...