# Unit 4: Double Integrals in General Regions

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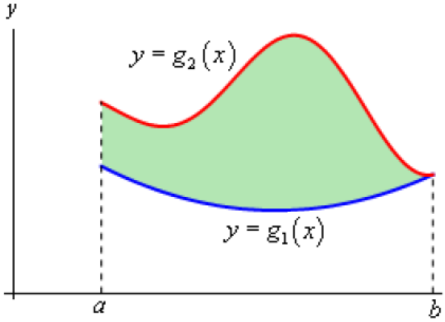
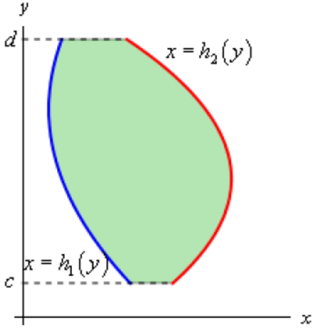
# Summary

To this point, we restricted our attention to rectangular domains (in some cases, triangular domains). Now we shall treat the more general case of domains.

When $D$ is a region between two curves in the $xy$-plane, we can evaluate double integrals over $D$ as iterated integrals.

There are two types of regions that we need to look at. Here is a sketch of both of them.



We will often use set builder notation to describe these regions.

**Types of regions**

**Type I:** $D = \{(x, y)|\ a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$

**Type II:** $D = \{(x, y)|\ h_1(x) \leq x \leq h_2(x), c \leq y \leq d\}$

This notation indicates that we are going to use all the points, $(x, y)$, in which both of the coordinates satisfy the two given inequalities.

**Area of a plane region**

Although we usually think of double integrals as representing volumes, it is worth noting that we can express the area of a domain $D$ in the plane as the double integral of the constant function $f(x, y) = 1$:

$$\text{Area}(D) = \iint_D 1\, dA. \tag{1}$$

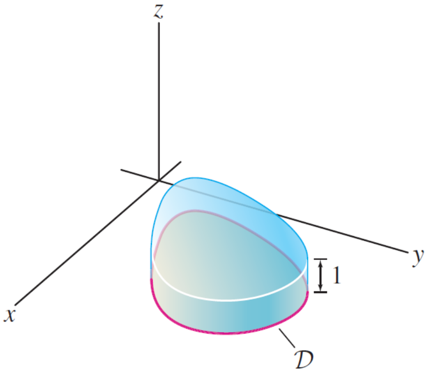



Figure 1:

Indeed, as we see in Figure 1, the the area of $D$ is equal to the volume of the "cylinder" of height 1 with $D$ as base. More generally, for any constant $C$,

$$\iint_D C \, dA = C \text{ Area}(D).$$

## Conceptual insight

Equation (1) tells us that we can approximate the area of a domain $D$ by a Riemann sum for

$$\iint_D 1\, dA.$$

In this case, $f(x, y) = 1$, and we obtain a Riemann sum by adding up the areas $\Delta x_i \Delta y_j$ of those rectangles in a grid that are contained in $D$ or that intersects the boundary of $D$ (See the figure given below).

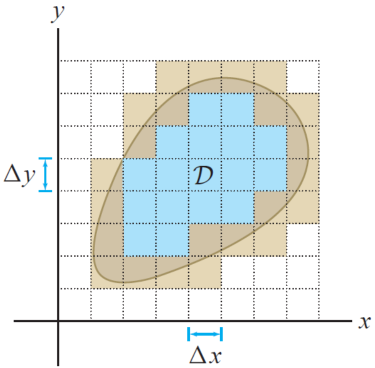



**Figure 2:** Approximation of $D$ by small rectangles.

The finer the grid, the better the approximation. The exact area is the limit as the sides of the rectangles tend to zero.

**Theorem (Area of a plane region of type I)**

*Let a plane region be given by*

$$D = \{(x, y) |\ a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\},$$

*where $g_1$ and $g_2$ are continuous functions on $[a, b]$. Then the area $A$ of $D$ is given by*

$$A = \int_a^b \int_{g_1(x)}^{g_2(x)} dydx.$$

*Proof.*
Consider the type I region
$D = \{(x, y)|\ a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$. We know that the area of $D$ is given by

$$\int_a^b (g_2(x) - g_1(x))\ dx.$$

We can view the expression $g_2(x) - g_1(x)$ as

$$g_2(x) - g_1(x) = \int_{g_1(x)}^{g_2(x)} 1\ dy,$$

*Proof...*
That means, we can express the area of $D$ as an iterated integral:

$$\begin{aligned} \text{Area of } D &= \int_a^b (g_2(x) - g_1(x))\, dx \\ &= \int_a^b \left( \int_{g_1(x)}^{g_2(x)} 1\, dy \right) dx \\ &= \int_a^b \int_{g_1(x)}^{g_2(x)} 1\, dydx. \end{aligned}$$

Using a process similar to that above, the area of a type II region $D$ could also be obtained. We have

$$\text{Area of } D = \int_c^d \int_{h_1(y)}^{h_2(y)} 1\, dxdy.$$

◀

**Theorem (Area of a plane region of type II)**

*Let a plane region be given by*

$$D = \{(x, y)|\ h_1(y) \leq x \leq h_2(y), c \leq y \leq d\},$$

*where $h_1$ and $h_2$ are continuous functions on $[c, d]$. Then the area $A$ of $D$ is given by*

$$A = \int_c^d \int_{h_1(y)}^{h_2(y)} dx\, dy.$$

**Example**

Find the area of the region enclosed by $y = 2x$ and $y = x^2$.

**Solution.**
We'll find the area of the region using both orders of integration.
*For the type I region*, we have

$$2x = x^2 \Rightarrow x = 0, 2.$$

Thus,

$$0 \leq x \leq 2, \quad x^2 \leq y \leq 2x.$$

Therefore, the required area is

$$\int_0^2 \int_{x^2}^{2x} 1 \, dy \, dx = \int_0^2 (2x - x^2) \, dx = \left(x^2 - \frac{1}{3}x^3\right)\Big|_0^2 = \frac{4}{3}.$$

**Solution...**

*For the type II region*, we have

$$x = \frac{1}{2}y, \quad x = \sqrt{y}.$$

We then have

$$\frac{1}{2}y = \sqrt{y}.$$

This implies that

$$y^2 = 4y \Rightarrow y = 0,\ 4.$$

Therefore, the required area is

$$\int_0^4 \int_{y/2}^{\sqrt{y}} 1\, dx\, dy = \int_0^4 (\sqrt{y} - y/2)\, dy = \left(\frac{2}{3}y^{3/2} - \frac{1}{4}y^2\right)\Big|_0^4 = \frac{4}{3}.$$

**theorem**

Let $f(x, y)$ be continuous.

1. If $D = \{(x, y) |\ a \le x \le b, g_1(x) \le y \le g_2(x)\}$, then
$$\iint_D f(x, y)\ dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\ dy\, dx.$$
2. If $D = \{(x, y) |\ h_1(x) \le x \le h_2(x), c \le y \le d\}$, then
$$\iint_D f(x, y)\ dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\ dx\, dy.$$

## Setting up Limits of Integration

To apply the above theorem, it is helpful to start with a two-dimensional sketch of the region $D$. It is not necessary to graph $f(x, y)$. For a type I region, the limits of integration can be obtained as follows:

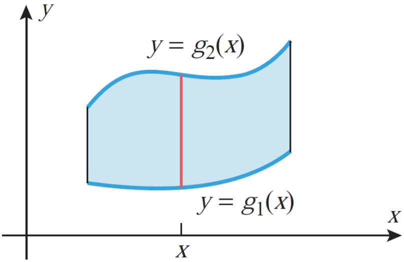



Fig. (a)

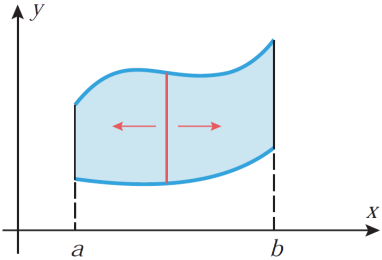



Fig. (b)

## Determining Limits of Integration: Type I Region

1. $x$ is held fixed for the first integration. We draw a vertical line through the region $D$ at an arbitrary fixed value $x$ (Figure (a)). This line crosses the boundary of $D$ twice. The lower point of intersection is on the curve $y = g_1(x)$ and the higher point is on the curve $y = g_2(x)$. These two intersections determine the lower and upper $y$-limits of integration over the type I region.
2. Imagine moving the line drawn in Step 1 first to the left and then to the right (Figure (b)). The leftmost position where the line intersects the region $D$ is $x = a$, and the rightmost position where the line intersects the region $D$ is $x = b$. This yields the limits for the $x$-integration over the type I region.

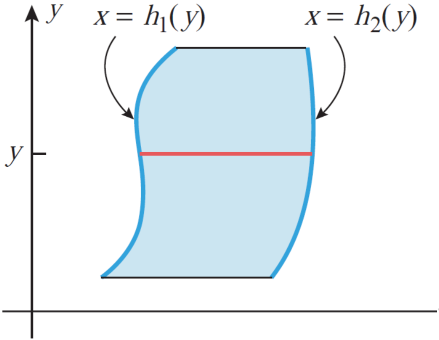



Fig. (c)

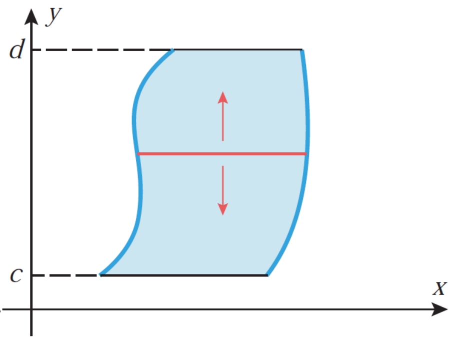



Fig. (d)

## Determining Limits of Integration: Type II Region

1. $y$ is held fixed for the first integration. We draw a horizontal line through the region $D$ at a fixed value $y$ (Figure (c)). This line crosses the boundary of $D$ twice. The leftmost point of intersection is on the curve $x = h_1(y)$ and the rightmost point is on the curve $x = h_2(y)$. These intersections determine the $x$-limits of integration over the type II region.
2. Imagine moving the line drawn in Step 1 first down and then up (Figure (d)). The lowest position where the line intersects the region $D$ is $y = c$, and the highest position where the line intersects the region $D$ is $y = d$. This yields the $y$-limits of integration over the type II region.

## Calculating a double integral over a type I region

**Example**

Example Evaluate $\iint_D (x + 2y)\, dA$, where $D$ is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

**Solution.**

We have

$$2x^2 = 1 + x^2 \Rightarrow x = \pm 1.$$

Then $y = 2$. Thus, the parabolas intersect at $(-1, 2)$ and $(1, 2)$. We see that the region $D$ is given by

$$D = \{(x, y) : 1 \leq x \leq 1, 2x^2 \leq y \leq 1 + x^2\}$$

## Solution...

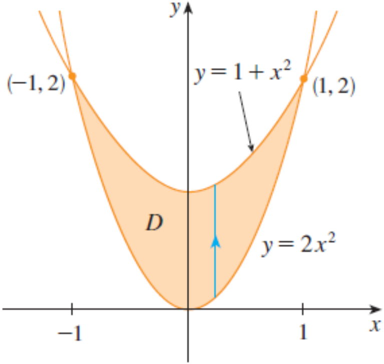



**Figure 3:** Type I region

We also see that the lower boundary is $y = 2x^2$ and the upper boundary is $y = 1 + x^2$. Therefore, we have

$$\begin{aligned}\iint_D (x+2y)\; dA &= \int_{-1}^{1}\int_{2x^2}^{1+x^2} (x+2y)\; dy\, dx\\ &= \frac{12}{15}.\end{aligned}$$

## Calculating a double integral over both type I and type II region

### Example

Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region $D$ in the $xy$-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

**Solution.**

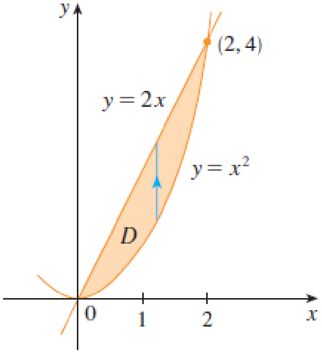



**Figure 4:** Type I region

From the figure we see that $D$ can be viewed as a type I region:

$$D = \{(x, y) : \ 0 \le x \le 2, x^2 \le y \le 2x\}$$

Therefore the volume under $z = x^2 + y^2$ and above $D$ is

$$\begin{aligned} V &= \iint_D (x^2 + y^2)\ dA \\ &= \int_0^2 \int_{x^2}^{2x} (x^2 + y^2)\ dy\, dx \\ &= \frac{216}{35}. \end{aligned}$$

**Solution.**

Alternatively,

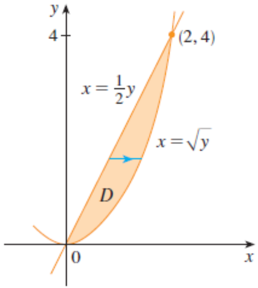



**Figure 5:** Type II region

From the figure we see that $D$ can also be written as a type II region:

$$D = \{(x, y) : \ 0 \leq y \leq 4, \frac{1}{2}y \leq x \leq \sqrt{y}\}$$

Therefore another expression for $V$ is

$$\begin{aligned} V &= \iint\limits_{D} (x^2 + y^2)\ dA \\ &= \int_0^4 \int_{\frac{1}{2}y}^{\sqrt{y}} (x^2 + y^2)\ dx\, dy \\ &= \frac{216}{35}. \end{aligned}$$

## Choosing the better description of a region

**Example**

Example Evaluate $\iint\limits_D xy\,dA$, where $D$ is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

**Solution...**

The region $D$ can be written as both type I and type II, but the description of $D$ as a type I region is more complicated because the lower boundary consists of two parts. Therefore we prefer to express $D$ as a type II region:

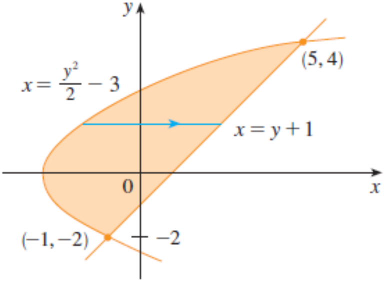



**Figure 6:** Type II region

**Solution...**

Then we have

$$D = \{(x, y) : \; -2 \leq y \leq 4, \tfrac{1}{2}y^2 - 3 \leq x \leq y + 1\}.$$

Thus,

$$\iint_D xy \; dA = \int_{-2}^{4} \int_{\frac{1}{2}y^2-3}^{y+1} xy \; dx \, dy$$
$$= 36.$$

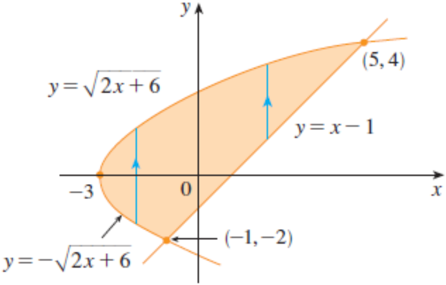



**Figure 7:** Type I region

If we had expressed $D$ as a type I region using Figure 12(a), then we would have obtained

$$\iint\limits_D xy\;dA = \int_{-3}^{-1}\int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} xy\;dy\,dx + \int_{-1}^{5}\int_{x-1}^{\sqrt{2x+6}} xy\;dy\,dx$$

# Reversing the order of integration

## Example

Example Find the iterated integral

$$\int_0^1 \int_1^x \sin(y^2)\; dy\, dx.$$

**Solution.**

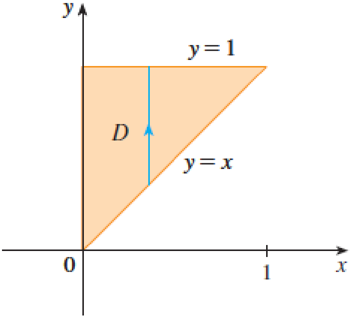
If we try to evaluate the integral as it stands, we are faced with the task of first evaluating $\int \sin(y^2)\ dy$. But it's impossible to do so in finite terms since $\int \sin(y^2)\ dy$ is not an elementary function. So we must change the order of integration. This is accomplished by first expressing the given iterated integral as a double integral.

**Solution...**

We have

$$\int_0^1 \int_x^1 \sin(y^2)\ dy\,dx = \iint\limits_D \sin(y^2)\ dA,$$

where $D = \{(x, y):\ 0 \le x \le 1, x \le y \le 1\}$. The sketch of this region $D$ is as follows:



**Solution...**

An alternative description of $D$ is as follows:

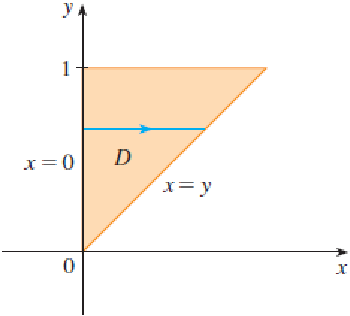



**Figure 9:** Type II region

**Solution...**

$$D = \{(x,y):\ 0 \le y \le 1, 0 \le x \le y\}$$

This enables us to express the double integral as an iterated integral in the reverse order:

$$\begin{aligned}\int_0^1 \int_x^1 \sin(y^2)\ dy\,dx &= \iint\limits_D \sin(y^2)\ dA \\ &= \int_0^1 \int_0^y \sin(y^2)\ dx\,dy \\ &= \tfrac{1}{2}(1-\cos 1).\end{aligned}$$

## Problem

Evaluate the integral:

$$\int_0^8 \int_{\sqrt[3]{y}}^2 \sqrt{x^4+1}\, dx\, dy.$$

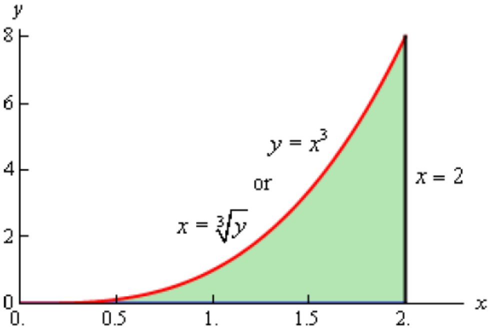



**Figure 10:** Domain of integration

## Properties of Double Integrals

Note that all first three of these properties are really just generalizations of properties of double integrals over rectangles.

1. $\iint\limits_D [f(x,y)+g(x,y)]\, dA = \iint\limits_D f(x,y)\, dA + \iint\limits_D g(x,y)\, dA$
2. If $c$ is a constant, then
$$\iint\limits_D cf(x,y)\, dA = c\iint\limits_D f(x,y)\, dA$$
3. If $f(x,y) \geq g(x,y)$ for all in $(x,y) \in D$, then
$$\iint\limits_D f(x,y)\, dA \geq \iint\limits_D g(x,y)\, dA$$

Assume that $D = D_1 \cup D_2$, where $D_1$ and $D_2$ don't overlap except perhaps on their boundaries. See the figure.

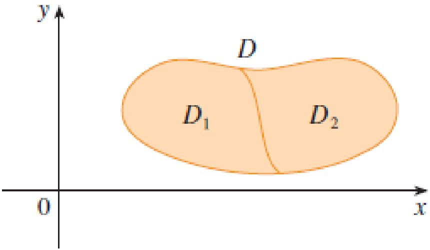



**Figure 11:**

Then

**4.** $\iint\limits_{D} [f(x,y)+g(x,y)]\ dA = \iint\limits_{D_1} f(x,y)\ dA + \iint\limits_{D_2} f(x,y)\ dA$

**5.**

$$\iint\limits_{D} 1\ dA = A(D),$$

where $A(D)$ is the area of $D$.

**6.** If $m \leq f(x,y) \leq M$ for all in $(x,y) \in D$, then

$$mA(D) \leq \iint\limits_{D} f(x,y)\ dA \leq MA(D)$$