Vector Calculus

We know that $\int_a^b f(x)\, dx$ deals the integration along x-axis. i.e. $y=0$

Similarly, $\int_c^d f(y)\, dy$ deals with integration along y-axis i.e. $x=0$.

In this chapter, ~~we~~ we extend this notion (concept) along curve.

Smooth curve

A curve or function is said to be smooth if it is continuous and differentiable at each point.

In particular, $y = \sin x$, $y = \cos x$ are smooth in the interval $[-\pi, \pi]$

Eg let $F = \frac{1}{t}\vec{i} + 2t\vec{j} + \cos t\,\vec{k}$

Then the value of F exists for all t except $t=0 \in \mathbb{R}$

$\therefore$ F is smooth in $\mathbb{R} - \{0\}$

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Fact:
The curve $\vec{F} = F_1(x,y,z)\vec{i} + F_2(x,y,z)\vec{j} + F_3(x,y,z)\vec{k}$ always represent space vector.
In particular, the function
$$F = 3xy\,\vec{i} + 3xyz\,\vec{j} + 3x^2yz\,\vec{k}$$
represent space curve

Smooth Surface

A surface is said to be smooth if it has a unique tangent at each point.

In other words, a surface $z = f(x,y)$ is called smooth if $z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ are exist and continuous at each point.

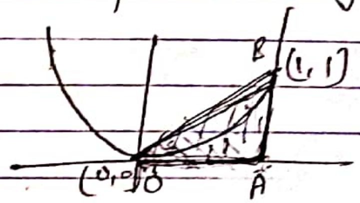
Line integral

Let us consider a vector point function
$$\vec{F} = F_1(x,y,z)\vec{i} + F_2(x,y,z)\vec{j} + F_3(x,y,z)\vec{k}$$
defined on a smooth curve C, which is represented by $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

then the line integral of F along the curve C is denoted by $\int_C \vec{F}\cdot d\vec{r}$

Note
$$\int_C \vec{F}\cdot d\vec{r} = \int_C (F_1\vec{i} + F_2\vec{j} + F_3\vec{k})\cdot(dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k})$$
$$= \int_C (F_1dx + F_2dy + F_3dz)$$

Q) Evaluate the integral
$$\int_C xy\,dx + (x+y)\,dy$$
i) along the line through (0,0) to (1,1)
ii) along the parabola $y = x^2$ from (0,0) to (



iii) along the line OA & then AB (OAB).

Soln
1) Eqn of OB is
$$y - 0 = \frac{1-0}{1-0}(x-0)$$
or, $y = x$
$\therefore dy = dx$

$$\therefore \int_{OB} (xy)\,dx + (x+y)\,dy$$

$$= \int_{OB} x^2\,dx + 2x\,dx$$

$$= \int_0^1 x^2\,dx + 2x\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^1 + 2\left[\frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{3} + 1$$

$$= \frac{4}{3}$$

ii) $y = x^2$

$$\frac{dy}{dx} = 2x$$

$$dy = 2x\,dx$$

So

$$\int (xy)\,dx + (x+y)\,dy$$

$$= \int x^3\,dx + (x+x^2)\,2x\,dx$$

$$= \int_0^1 x^3\,dx + 2x^2\,dx + 2x^3\,dx$$

$$= \left[\frac{x^4}{4}\right]_0^1 + 2\left[\frac{x^3}{3}\right]_0^1 + 2\left[\frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{4} + \frac{2}{3} + \frac{1}{2}$$

$$= \frac{3 + 8 + 6}{12}$$

$$= \frac{17}{12}$$

iii) Along OA then AB

~~Eqn of OA is $y = 0$.~~

Eqn. of AB is $x = 1$.

$$\therefore \int_{OAB} xy\,dx + (x+y)\,dx$$

$$= \int_{OA} \underbrace{xy\,dx + (x+y)\,dy}_{y=0\ \therefore\ dy=0} + \int_{AB} \underbrace{xy\,dx + (x+y)\,dy}_{x=1,\ dx=0}$$

$$= \int_0^1 x\cdot 0\,dx + (x+0)0 + \int 1\cdot y\cdot 0 + (1+y)\,dy$$

$$= \int_0^1 0\,dx + \int_0^1 dy + y\,dy$$

$$= 0 + \left[y\right]_0^1 + \left[\frac{y^2}{2}\right]_0^1$$

$$= 0 + 1 + 1/2$$

$$= 3/2$$

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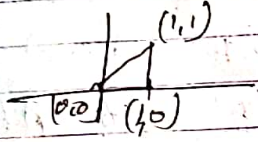
Thus from (i), (ii) & (iii), we see that the integration of above given function are different. However, initial point & terminal point are same.

Q) Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = y^2\vec{i} + x^2\vec{j}$

and C is the straight line joining (0,0) to (1,0) and from (1,0) to (1,1)



Soln

$\oint_C \vec{F}\cdot d\vec{r}$

$= \oint_C (y^2\vec{i} + x^2\vec{j})(dx\vec{i} + dy\vec{j})$

$= y^2 dx + x^2 dy$

Now

For st. ~~line~~ line joining (0,0) to (1,0).

Equation of st. line,

$y - 0 = \frac{1-0}{1-0}(x-0)$

$y = x$

$dy = dx$

So,

$\int_C \vec{F}\cdot d\vec{r}$

$= \int_C y^2 dx + x^2 dy$

$= \int_0^1 x^2 dx + x^2 dx$

$= 2\left[\frac{x^3}{3}\right]_0^1$

$= \frac{2}{3}$

again

for st. line joining (1,0) to (1,1)

Equn of st. line,

~~y=0, y=1~~

$y - 0 = \frac{1-0}{1-1}(x-1)$

$y = 0$

$x = 1$

=

$\phi = f(x,y,z)$

Theorem (Line Integration independent of path)

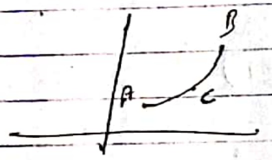
The line integral $\int_C \vec{F}\cdot d\vec{r}$ of continuous function $\vec{F}$ is independent of path C if $\vec{F} = \nabla\phi$ for some scalar value function ϕ.

Proof

Suppose, $\vec{F} = \nabla\phi$ for some scalar value function ϕ.

Our claim; $\int_C \vec{F}\cdot d\vec{r}$ is independent of path C.

Let C be the path joining A & B.



Now $\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla\phi \cdot d(\vec{i}x+\vec{j}y+\vec{k}z)$

$$= \int_C \left(i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}\right)\cdot(\vec{i}dx+\vec{j}dy+\vec{k}dz) \quad \left[\nabla = i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right]$$

$$= \int_C \left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right)$$

$$= \int_C d\phi$$

$$= \int_A^B d\phi$$

$$= [\phi]_A^B$$

$$= \phi(B) - \phi(A)$$

This shows that $\int_C \vec{F}\cdot d\vec{r}$ depends only on initial point A & final point B, no the path.

Irrotational Vector

A vector function $\vec{F}$ is called irrotational if $\nabla\times\vec{F} = \vec{0}$

Q. Is the vector $\vec{F} = (x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}$ irrotational?

Sol:

$$\nabla\times\vec{F} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\times\left((x^2-y^2)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}\right)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2-y^2) & (y^2-zx) & (z^2-xy) \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(z^2-xy) - \frac{\partial}{\partial z}(y^2-2x)\right]$$
$$-\vec{j}\left[\frac{\partial}{\partial x}(z^2-xy) - \frac{\partial}{\partial z}(x^2-yz)\right]$$
$$+\vec{k}\left[\frac{\partial}{\partial x}(y^2-2x) - \frac{\partial}{\partial y}(x^2-yz)\right]$$
$$= \vec{i}[(0-x)-(0-x)] - \vec{j}[(0-y)-(0-y)]$$
$$+\vec{k}[(0-z)-(0-z)]$$
$$= \vec{0}$$

Q. In the above example $\vec{F} = (x^2-yz)\vec{i} + (y^2-2x)\vec{j} + (z^2-xy)\vec{k}$, show that $\int_c \vec{F}\cdot d\vec{r}$ is independent of path.

~~Le~~ sol^n

We show that $\vec{F} = \nabla\phi$ for some scalar valued function ϕ.

now, $\vec{F} = \nabla\phi$ ~~$\vec{a}$~~

or, $(x^2-yz)\vec{i} + (y^2-2x)\vec{j} + (z^2-xy)\vec{k} = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$

on Equating the corresponding component

$$\frac{\partial\phi}{\partial x} = x^2-yz, \quad \frac{\partial\phi}{\partial y} = y^2-2x \quad \& \quad \frac{\partial\phi}{\partial z} = z^2-xy$$

now

$$\phi_1 = \frac{x^3}{3} - xyz + c_1, \quad \phi_2 = \frac{y^3}{3} - xyz + c_2, \quad \phi_3 = \frac{z^3}{3} - xyz + c_3$$

$\therefore$ required function

$$\phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{z^3}{3} - 3xyz + c$$

Formula

If $x = x(t)$, $y = y(t)$, $z = z(t)$ then,

$$\int_c \vec{F}\cdot d\vec{r} = \int_c (F_1\vec{i} + F_2\vec{j} + F_3\vec{k})\cdot(\vec{i}\,dx + \vec{j}\,dy + \vec{k}\,dz)$$

$$= \int_c (F_1\,dx + F_2\,dy + F_3\,dz)$$

$$= \int_a^b \left(F_1\frac{dx}{dt} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt}\right)dt$$

Q) Evaluate $\int_C \vec{F}\cdot d\vec{r}$

where, $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$
& $\vec{r} = a\cos t\,\vec{i} + b\sin t\,\vec{j} + e^t\vec{k}$
from $t=0$ to $t=\pi/2$

Soln
Here, $F_1 = yz$, $F_2 = zx$, $F_3 = xy$
$x = a\cos t$, $y = b\sin t$, $z = e^t$

now $\frac{dx}{dt} = -a\sin t$, $\frac{dy}{dt} = b\cos t$, $\frac{dz}{dt} = e^t$

Now
$$\int_C \vec{F}\cdot d\vec{r}$$
$$= \int_0^{\pi/2} \Big[(b\sin t\, e^t(-a\sin t)) + (e^t\cdot a\cos t)\, b\cos t + (a\cos t\, b\sin t)e^t\Big] dt$$
$$= \int_0^{\pi/2} -abe^t\sin^2 t + abe^t\cos^2 t + 2abe^t \frac{\sin t\cos t}{2}\, dt$$
$$= ab\int_0^{\pi/2} \left[e^t\cos 2t + \frac{1}{2}e^t\sin 2t\right] dt$$

$= 0$

Q) Evaluate: $\int_C \vec{F}\times d\vec{r}$

when, $\vec{F} = xy\vec{i} - 2\vec{j} + x^2\vec{k}$
& C is the curve
$x = t^2$, $y = 2t$, $z = t^3$
from $t=0$ to $t=1$

Soln
$\vec{r} = t^2\vec{i} + 2t\vec{j} + t^3\vec{k}$
$d\vec{r} = (2t\vec{i} + 2\vec{j} + 3t^2\vec{k})\,dt$

Now
$$\vec{F}\times d\vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ xy & -2 & x^2 \\ 2t & 2 & 3t^2 \end{vmatrix}$$
$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t^3 & -t^3 & t^4 \\ 2t\cdot dt & 2dt & 3t^2\cdot dt \end{vmatrix}$$
$$= \vec{i}\,(-3t^5\cdot dt - 2t^4\cdot dt) - \vec{j}\,(6t^5\cdot dt - 2t^5\cdot dt) + \vec{k}\,(4t^3\cdot dt + 2t^4\cdot dt)$$
$$= \vec{i}\,(-3t^5 - 2t^4)\,dt - \vec{j}\,(4t^5\cdot dt) + \vec{k}\,(4t^3 + 2t^4)\,dt$$

$$= \int_{C_1} \vec{F} \times d\vec{r}$$

$$= \int_0^1 \left(\qquad \right] dt$$

$$= \left[\frac{-3t^6}{6} - \frac{2t^5}{5}\right]_0^1 \vec{i} + -\left[\frac{4t^6}{6}\right]_0^1 \vec{j}$$

$$+ \left[\frac{4t^4}{4} - \frac{2t^5}{5}\right]_0^1 \vec{k}$$

$$= \frac{-9}{10}\vec{i} - \frac{2}{3}\vec{j} + \frac{7}{5}\vec{k}$$

Q. Evaluate, $\int_C \phi \, d\vec{r}$ where
$\phi = 2xyz^2$ & C is the curve
$x = t^2, \; y = 2t, \; z = t^3$
from $t=0$ to $t=1$

ans: $\frac{8}{11}\vec{i} + \frac{4}{5}\vec{j} + \vec{k}$

Divergence & Curve

let $\vec{V} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$.
where,
F_1, F_2, F_3 are the function of x, y & z.
Here, $\vec{F}$ is called vector function of scalar variable x, y & z.

Then, the divergence of $\vec{F}$ is denoted by $\nabla \cdot \vec{F}$ and is defined by

$$\nabla \cdot \vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot (F_1\vec{i} + F_2\vec{j} + F_3\vec{k})$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= \sum \frac{\partial F_i}{\partial X}, \quad i = 1,2,3, \; X \text{ varies } x, y, z$$

which is scalar point function.
Here, ∇ (nebla) is called vector different operator.

Curl of vector function

same

Then the curl of $\vec{F}$ is denoted by $\nabla \times \vec{F}$ and is defined by