现代控制理论基础
Fundamentals of Modern Control Theory

Chapter 3. State Variable Models

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Introduction

Preview

In CCS, a linear time-invariant physical system is described with a transfer function model, which is equivalent to an nth-order ODE with zero initial value.

Transfer function model provides a practical approach to design and analysis, and allows to use block diagrams to interconnect subsystem.

In MCS, above system is described with an alternative model—state variable model.

The state variable model is a set of first-order ODEs, derived from above nth-order ODE by choosing a set variables—state variable, which is readily for computer solution and analysis.

In this chapter, the relationship between these two models are investigated, and several physical systems are presented and analyzed.

Introduction

Definition: time-varying control system

One or more of the parameters of the system may vary as a function of time t.
时变控制系统是指系统有一个或多个参数随时间变化的系统。

Definition: multivariable system

A system with several input and output signals(multi-input and multi-output system).
多变量系统是指有多个输入多个输出信号的系统。

Definition: time-domain

The mathematical domain that incorporates the response and model of a system in terms of time, t.
时域是指一种数学域，与频域相区别；采用时间 t 和时间响应来描述系统。

Introduction

- Availability of digital computers make it is convenient to study the physical system in time-domain formulation.
- The time-domain approaches can be utilized also for nonlinear, time-varying, and multivariable systems.
- Time-domain model is the essential basis for modern control theory.

The state variables of a dynamic system

Introduction to the concepts of state and state variable.

Definition: The state of a system

A set of variables such that the knowledge of the variables and the input functions will provide the future state and output of the system with the equation describing its dynamics.

系统状态是指系统在时间域中的行为或运动性息的集合，可以用一组变量来刻画，只要知道了这组变量的当前取值、输入信号和系统的动态方程，就能够完全确定系统的未来状态和输出响应。

Definition: State variables

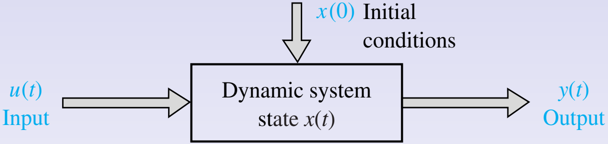
The state variables describe the future response of a system, given the present state, the excitation inputs, and the equations describing the dynamics.

状态变量是指描述系统状态的变量的集合，通常是指能完全表征系统未来运动状态的数目最小的一组变量。

如给定了 $t = t_0$ 时刻这组变量的取值和 $t \geq t_0$ 时刻的输入函数，以及系统的动态方程，则系统在 $t \geq t_0$ 时刻的行为就能完全确定。这样一组变量就称为状态变量。

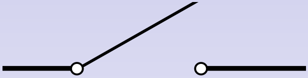
The state of a system

The state variables of a dynamic system



Generally, The **state** denoted with a n-dimensional vector $x = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T$, and $x_1,\ x_2, \cdots, x_n$ are the **state variables**, maybe not unique.

The state of a switch



The state x of the switch has only one variable, simply denoted with x.

And state variable x can assume two possible values, that is

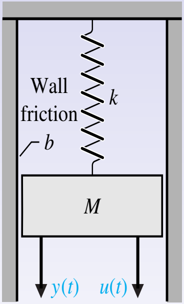
$$x = \begin{cases} \text{on} \\ \text{off} \end{cases}$$

Input u also has two possible values, that is

$$u = \begin{cases} \text{push} \\ \text{pull} \end{cases}$$

If state variable value at the present t_0 is known, when the specified input $u(t)$ is applied, the future state value $x(t)$ will be determined.

A spring-mass-damper system



The second-order ODE is given by

$$M\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = u(t)$$

The state $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ with two variables is sufficient to describe this system.
The variables x_1 and x_2 denote the position $y(t)$ and velocity $\dot{y}(t)$ of the mass at time t,respectively, thus

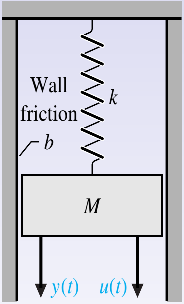
$$x_1(t) = y(t),\ x_2(t) = \frac{dy(t)}{dt}$$

Therefore,

$$\frac{dx_1}{dt} = x_2\ ,\frac{dx_2}{dt} = \frac{d^2y}{dt^2}$$

A spring-mass-damper system

Writing above second-order ODE in terms of the state variables, a set of two first-order ODEs, namely **state differential equation** follows



$$x_2 = \frac{dx_1}{dt}$$

$$M\frac{dx_2}{dt} + bx_2 + kx_1 = u(t)$$

or

$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = -\frac{k}{M}x_1 - \frac{b}{M}x_2 + \frac{1}{M}u$$

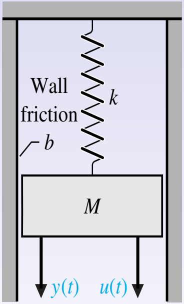
The **output equation** is given as

$$y = x_1$$

A spring-mass-damper system

Since $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, it follows $\dot{x} = \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix}$

The compact form of **state differential equation**



$$\frac{dx_1}{dt} = x_2$$
$$\frac{dx_2}{dt} = -\frac{k}{M}x_1 - \frac{b}{M}x_2 + \frac{1}{M}u$$

is given as

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{b}{M} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} u$$

The compact form of **output equation**

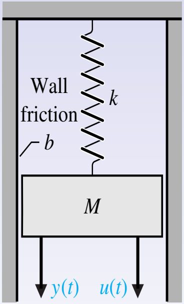
$$y = x_1$$

is given as

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

A spring-mass-damper system

Put **state differential equation** and **output equation** together to obtain state variable model or state-space representation



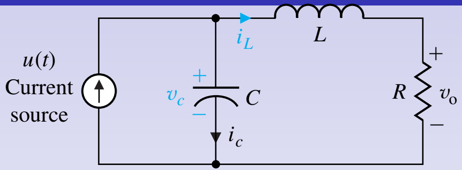
$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{b}{M} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

Furthermore,

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

where $A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{b}{M} \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $D = 0$.

An RLC circuit

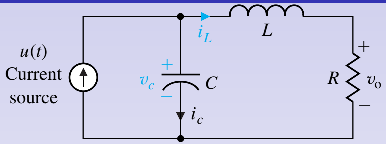


Let $x_1 = v_c(t)$, the capacitor voltage; $x_2 = i_L(t)$, the inductor current. The state $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ is sufficient to characterize the RLC circuit, since the stored energy of the network can be given in terms of x_1, x_2

$$\mathcal{E} = \frac{1}{2}L \cdot x_2^2 + \frac{1}{2}C \cdot x_1^2 = \frac{1}{2}L \cdot i_L^2 + \frac{1}{2}C \cdot v_c^2$$

For a passive RLC network, the number of required state variables is equal to the number of independent energy-storage elements; for general system equal to the number of integrators contained.

An RLC circuit



Let $x_1 = v_c(t)$, the capacitor voltage; $x_2 = i_L(t)$, the inductor current; $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$, the state of the RLC network.

From Kirchhoff's current law, the rate of change of capacitor voltage is given by

$$i_c = C\frac{dv_c}{dt} = u(t) - i_L, \text{or} \quad i_c = C\frac{dx_1}{dt} = u(t) - x_2$$

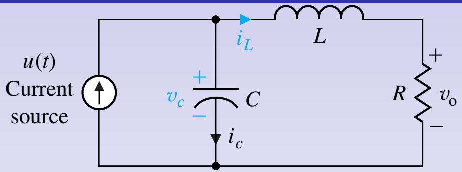
From Kirchhoff's voltage law, the rate of change of inductor current is given by

$$L\frac{di_L}{dt} = -Ri_L + v_c, \text{or} \;\; L\frac{dx_2}{dt} = -Rx_2 + x_1$$

The output of the network is represented by

$$v_0 = Ri_L, \text{or} \;\; y = v_0 = Rx_2$$

An RLC circuit



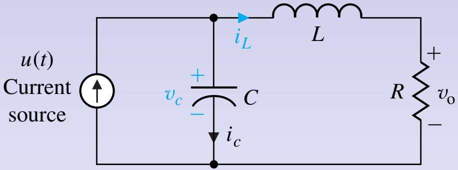
Put above equations in term of x_1, x_2 together, the state variable model of the RCL network is obtained

$$\frac{dx_1}{dt} = -\frac{1}{C}x_2 + \frac{1}{C}u(t)$$
$$\frac{dx_2}{dt} = \frac{1}{L}x_1 - \frac{R}{L}x_2$$
$$y = Rx_2$$

It's compact form is

$$\dot{x} = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R}{L} \end{bmatrix} x + \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & R \end{bmatrix} x + 0 \cdot u$$

The alternative state variable model for the RLC circuit



Let $x_1 = v_c(t)$, the capacitor voltage; $x_2 = v_L(t)$, the inductor voltage instead of the inductor current $i_L(t)$.

So

$$x_2 = v_L(t) = v_c - Ri_L, \text{or } x_2 = x_1 - Ri_L$$

Substitute x_1, x_2 into following 2 equations from Kirchhoff's law

$$i_c = C\frac{dv_c}{dt} = u(t) - i_L, \quad L\frac{di_L}{dt} = -Ri_L + v_c$$

an alternative state variable model is given by

$$\dot{x} = \begin{bmatrix} -\frac{1}{RC} & \frac{1}{RC} \\ -\frac{1}{RC} & \frac{1}{RC} - \frac{R}{L} \end{bmatrix} x + \begin{bmatrix} \frac{1}{C} \\ \frac{1}{C} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & -1 \end{bmatrix} x + 0 \cdot u$$

Summary of state variable model

- State is a set of variables, which number is just enough to describe the system.
- In terms of the state variables, the nth-order ODE for system is rewritten as a set of first-order ODEs, that is **state differential equation**.
- The system output also can be represented in terms of the state variables, that is **output equation**.
- A State variable model or State-space representation is composed of SDE(state differential equation) and OE(output equation), the matrix form is

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

- The state variables could be chosen in different ways, but the variable number is same, so a system may have several state variable models for it.
- Usually choose easily measured physical variables as state variables.

The state differential equation

Regarding the solution formula of scalar state differential equation, the solution of state differential equation in matrix form is obtained.

In the preceding section, the state variable models are proposed to describe the actual systems.

State variable model now called the **state-space representation** or **dynamic equation** is comprised of the state differential equation

$$\dot{x} = Ax + Bu$$

and the output equation

$$y = Cx + Du$$

where $A = (a_{ij})_{n\times n}$, the state matrix; $B = (b_{ij})_{n\times m}$, the input matrix; $C = (c_{ij})_{p\times n}$, the output matrix; $D = (d_{ij})_{p\times m}$, the direct transmission matrix; the input $u \in R^m$, the output $y \in R^p$,

the state vector $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in R^n$, the state space.

Solution formula of the first-order ODE

For the the first-order differential equation

$$\dot{x} = ax + bu$$

where $x(t)$ and the given $u(t)$ are scalar functions of t.

Taking Laplace transformation of above equation, to have

$$sX(s) - x(0) = aX(s) + bU(s)$$

therefore

$$X(s) = \frac{x(0)}{s-a} + \frac{b}{s-a}U(s)$$

The inverse Laplace transformation of above equation results in the solution formula

$$\begin{aligned} x(t) &= e^{at}x(0) + \int_0^t e^{a(t-\tau)} bu(\tau)d\tau \\ &= \exp(at)x(0) + \int_0^t \exp\left[a(t-\tau)\right] bu(\tau)d\tau \end{aligned}$$

Solution formula for state equation

For general state different equation

$$\dot{x} = Ax + Bu$$

its solution should be in the similar form, say

$$x(t) = \exp(At)x(0) + \int_0^t \exp\left[A(t-\tau)\right] Bu(\tau)d\tau$$

where I is the $n \times n$ identity matrix, and for e^{At} is defined as

$$e^{At} = \exp(At) = I + At + \frac{A^2t^2}{2!} + \cdots + \frac{A^kt^k}{k!} + \cdots$$

converges for all finite t and any A.

Solution formula for state equation

To see the formula
$$x(t) = \exp(At)x(0) + \int_0^t \exp\left[A(t-\tau)\right] Bu(\tau)d\tau$$
is the solution to the state differential equation
$$\dot{x} = Ax + Bu$$

Taking Laplace transformation on both side of the equation, and rearranging to have
$$X(s) = (sI - A)^{-1} x(0) + (sI - A)^{-1} BU(s)$$
where denote $\Phi(s) = (sI - A)^{-1}$, so $\Phi(t) = \mathcal{L}^{-1}\left[(sI - A)^{-1}\right]$.

Taking the inverse Laplace transformation and using the property of convolution, to obtain
$$\begin{aligned} x(t) &= \Phi(t)x(0) + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau \\ &= \Phi(t)x(0) + \int_0^t \Phi(\tau)Bu(t-\tau)d\tau \end{aligned}$$

Thus we have $\Phi(t) = \mathcal{L}^{-1}\left[(sI - A)^{-1}\right] = e^{At}$

The solution formula

$$x(t) = \Phi(t)x(0) + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau$$

$$\Phi(t) = \mathcal{L}^{-1}\left[(sI-A)^{-1}\right] = e^{At}$$

shows that the `matrix exponential function` $\Phi(t) = e^{At}$ determines the `unforced response` (零输入响应) $x(t) = \Phi(t)x(0)$ of SDE with $u(t) = 0$.

So

$$\Phi(t) = e^{At}$$

called the `state transition matrix` （状态转移矩阵）of the system.

state transition matrix

Writing the unforced response

$$x(t) = \Phi(t)x(0)$$

in state variable form

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} = \begin{pmatrix} \phi_{11}(t) & \phi_{12}(t) & \cdots & \phi_{1n}(t) \\ \phi_{21}(t) & \phi_{22}(t) & \cdots & \phi_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n1}(t) & \phi_{n2}(t) & \cdots & \phi_{nn}(t) \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_n(0) \end{pmatrix}$$

where the term $\phi_{ij}(t)$ is the response of state variable $x_i(t)$ to $x_j(0)$ while $x_k(0) = 0, k \neq j$.

$\phi_{ij}(t)$恰好对应于除第 j 个状态变量外，其它状态变量初值为零时的状态响应.

$$x_i(t) = \phi_{ij}(t)x_j(0)$$

This relationship will be used to evaluate the coefficients $\phi_{ij}(t)$ of $\Phi(t)$ with the signal-flow graph and Mason's signal-flow gain formula.

Signal-flow graph and block diagram models

Utilizing Mason's signal-flow gain formula to establish the state-space representation of the system from its transfer function model.

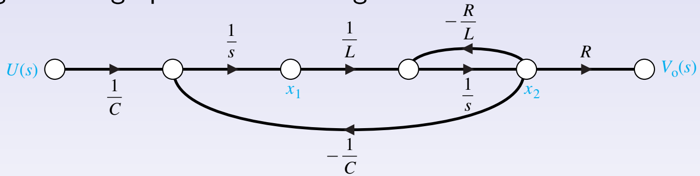
In the same way, to obtain the transfer function model from the state-space representation of the system.

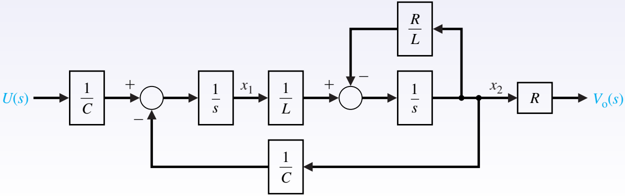
The relationship between SSR and TF

For the RLC circuit

$$\dot{x}_1 = -\frac{1}{C}x_2 + \frac{1}{C}u(t)$$
$$\dot{x}_2 = \frac{1}{L}x_1 - \frac{R}{L}x_2$$
$$v_0 = Rx_2$$

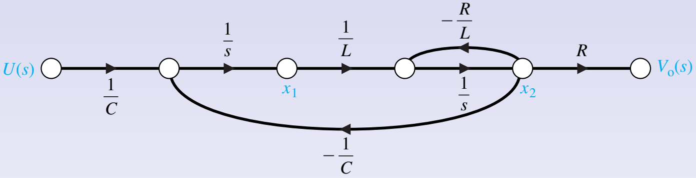
its signal-flow graph and block diagram are





The relationship between SSR and TF

For the RLC circuit, its signal-flow graph is



From the mason's formula, the TF is found to be

$$\frac{V_0(s)}{U(s)} = \frac{\frac{R}{LCs^2}}{1-(-\frac{R}{Ls})-(-\frac{1}{LCs^2})} = \frac{R}{LCs^2+RCs+1}$$

Procedures from TF to SSR with Manson's formula

For the RLC network is ease to obtain its SSR, but it is often difficult for many "large" actual systems.

We already know how to derive the TF of system, so the procedures to derive the state-space representation can be given as below:

1. derive the transfer function model;
2. derive signal-flow graph(not unique) with Manson's formula;
3. choose state variable(not unique);
4. then derive several key canonical forms (标准型) of the state-space representation.

Review of Mansion's signal-flow gain formula

For the signal-flow graph model, the gain relates the output $Y(s)$ to input $U(s)$ is given in the form

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\Sigma_k P_k \Delta_k}{\Delta}$$

The k`th path gain` P_k, continuous succession of branches that traversed in arrow direction and with no node encountered more than twice.

A `loop`, a closed path in which no node encountered more than twice.

Review of Mansion's signal-flow gain formula

The determinant of the graph Δ, where L_n all different loop gains, $L_i, i = m, q, r, t, \cdots$, is the gains of nontouching loops

$$\Delta = 1 - \sum L_n + \sum L_m L_q - \sum L_r L_s L_t + \cdots$$

The `cofactor` Δ_k, is the determinant with the loops touching P_k removed.

When all the loops are touching and all the forward paths touch the loops, the gain formula reduces to

$$G(s) = \frac{\Sigma_k P_k}{1 - \sum L_n}$$

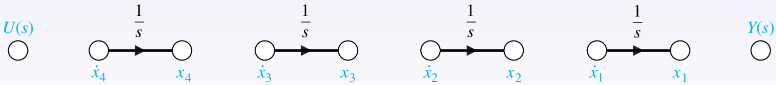
Taking the 4th-order TF as an illustration example

$$\text{Consider} \quad G(s) = \frac{Y(s)}{U(s)} = \frac{b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$
$$= \frac{b_0 s^{-4}}{1 + a_3 s^{-1} + a_2 s^{-2} + a_1 s^{-3} + a_0 s^{-4}}$$

Noting the system is 4th-order, hence there are 4 integrators for the system, and identify 4 state variables x_1,x_2,x_3,x_4.

Recalling formular $G(s) = \frac{\Sigma_k P_k}{1-\sum L_n}$, to see the graph should have 4 touching loops with gain $-a_3 s^{-1}$, $-a_2 s^{-2}$, $-a_1 s^{-3}$, $-a_0 s^{-4}$, respectively.

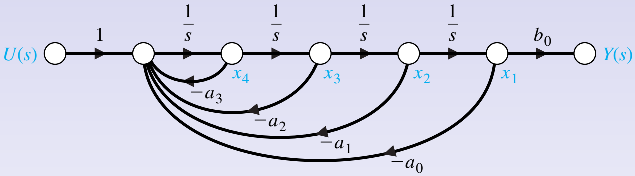
In the graph, flow graph nodes and integrators are in series interconnection



Taking the 4th-order TF as an illustration example

Consider $$G(s) = \frac{b_0 s^{-4}}{1 + a_3 s^{-1} + a_2 s^{-2} + a_1 s^{-3} + a_0 s^{-4}}$$

The signal-flow graph



Choosing the outputs of the 4 integrators as the state variables x_1,x_2,x_3,x_4 marked in the graph, hence satisfy

$$\begin{aligned}
y &= b_0 x_1 \\
\dot{x}_1 &= x_2 \leftrightarrow y'/b_0 \\
\dot{x}_2 &= x_3 \leftrightarrow y''/b_0 \\
\dot{x}_3 &= x_4 \leftrightarrow y'''/b_0 \\
\dot{x}_4 &= -a_0 x_1 - a_1 x_2 - a_2 x_3 - a_3 x_4 + u
\end{aligned}$$

Taking the 4th-order TF as an illustration example

Consider $G(s) = \dfrac{b_0 s^{-4}}{1 + a_3 s^{-1} + a_2 s^{-2} + a_1 s^{-3} + a_0 s^{-4}}$

$$\begin{aligned} y &= b_0 x_1 \\ \dot{x}_1 &= x_2 \leftrightarrow y\prime / b_0 \\ \dot{x}_2 &= x_3 \leftrightarrow y'' / b_0 \\ \dot{x}_3 &= x_4 \leftrightarrow y''' / b_0 \\ \dot{x}_4 &= -a_0 x_1 - a_1 x_2 - a_2 x_3 - a_3 x_4 + u \end{aligned}$$

therefore the state-space representation is given by

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} b_0 & 0 & 0 & 0 \end{bmatrix} x \end{aligned}$$

Taking the 4th-order TF as an illustration example II

Consider
$$
\begin{aligned}
G(s) &= \frac{Y(s)}{U(s)} = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0} \\
&= \frac{b_3 s^{-1} + b_2 s^{-2} + b_1 s^{-3} + b_0 s^{-4}}{1 + a_3 s^{-1} + a_2 s^{-2} + a_1 s^{-3} + a_0 s^{-4}}
\end{aligned}
$$

The numerator terms represent other 3 forward-path factors added to the new graph, so the signal-flow graph (**Phase variable canonical form**) is

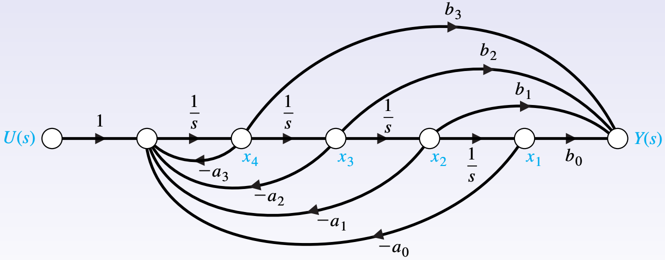


Fig: Phase variable canonical form

Taking the 4th-order TF as an illustration example II

Consider $G(s) = \dfrac{Y(s)}{U(s)} = \dfrac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$

Choosing state variables called phase variables

$$\begin{aligned} y &= y_1 + b_1 x_2 + b_2 x_3 + b_3 x_4, \quad y_1 = b_0 x_1 \\ \dot{x}_1 &= x_2 \leftrightarrow y_1'/b_0 \\ \dot{x}_2 &= x_3 \leftrightarrow y_1''/b_0 \\ \dot{x}_3 &= x_4 \leftrightarrow y_1'''/b_0 \\ \dot{x}_4 &= -a_0 x_1 - a_1 x_2 - a_2 x_3 - a_3 x_4 + u \end{aligned}$$

therefore the SSR of **Phase variable canonical form**(相变量标准型) is given by

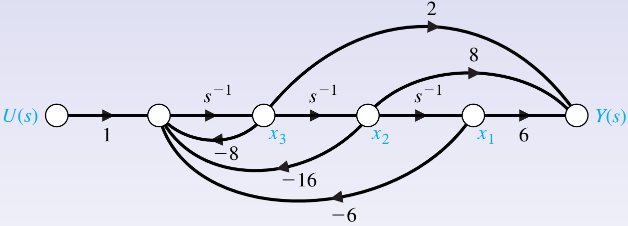
$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u \\ y &= \begin{pmatrix} b_0 & b_1 & b_2 & b_3 \end{pmatrix} x \end{aligned}$$

Ex3.1 Phase variable canonical form

Consider the TF of a control system (Fig 3.11)

$$\frac{Y(s)}{U(s)} = \frac{2s^2+8s+6}{s^3+8s^2+16s+6} = \frac{2s^{-1}+8s^{-2}+6s^{-3}}{1+8s^{-1}+16s^{-2}+6s^{-3}}$$

its graph in **phase variable canonical form** is



then the SSR is founded to be

$$\begin{aligned}\dot{x} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -16 & -8 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u \\ y &= \begin{pmatrix} 6 & 8 & 2 \end{pmatrix} x\end{aligned}$$

The signal-flow graph with another structure for TF

Consider again

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

$$= \frac{b_3 s^{-1} + b_2 s^{-2} + b_1 s^{-3} + b_0 s^{-4}}{1 + a_3 s^{-1} + a_2 s^{-2} + a_1 s^{-3} + a_0 s^{-4}}$$

to derive the signal-flow graph with another structure, the integrators not in series interconnection

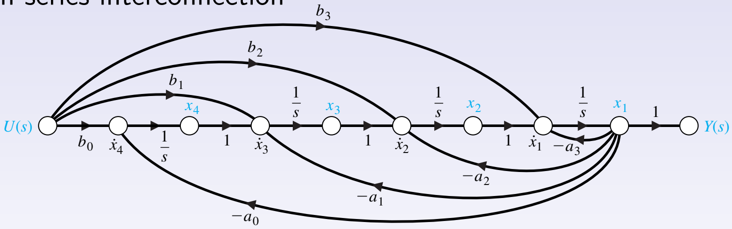
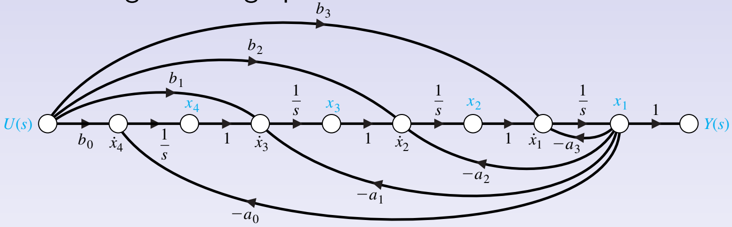


Fig: Input feedforward canonical form

Since the forward-path gains are obtained by feeding forward $U(s)$, so the graph is called the **input feedforward canonical form** (输入前馈标准型).

The signal-flow graph with another structure for TF

For
$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$
to derive the signal-flow graph with another structure



The state variables are chosen as

$$\begin{aligned}
x_4 &= \int(b_0 u - a_0 y)dt & \dot{x}_4 &= b_0 u - a_0 x_1 \\
x_3 &= \int(b_1 u - a_1 y + x_4)dt & \dot{x}_3 &= b_1 u - a_1 x_1 + x_4 \\
x_2 &= \int(b_2 u - a_2 y + x_3)dt & \dot{x}_2 &= b_2 u - a_2 x_1 + x_3 \\
x_1 &= \int(b_3 u - a_3 y + x_2)dt & \dot{x}_1 &= b_3 u - a_3 x_1 + x_2 \\
y &= x_1 & y &= x_1
\end{aligned}$$

The signal-flow graph with another structure for TF

For $$G(s) = \frac{Y(s)}{U(s)} = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

the state variables are chosen as

$$\begin{array}{ll} x_4 = \int (b_0 u - a_0 y) dt & \dot{x}_4 = b_0 u - a_0 x_1 \\ x_3 = \int (b_1 u - a_1 y + x_4) dt & \dot{x}_3 = \ b_1 u - a_1 x_1 + x_4 \\ x_2 = \int (b_2 u - a_2 y + x_3) dt & \dot{x}_2 = \ b_2 u - a_2 x_1 + x_3 \\ x_1 = \int (b_3 u - a_3 y + x_2) dt & \dot{x}_1 = \ b_3 u - a_3 x_1 + x_2 \\ y = x_1 & y = x_1 \end{array}$$

rearranging to obtain SSR of **input feedforward canonical form**(输入前馈标准型)

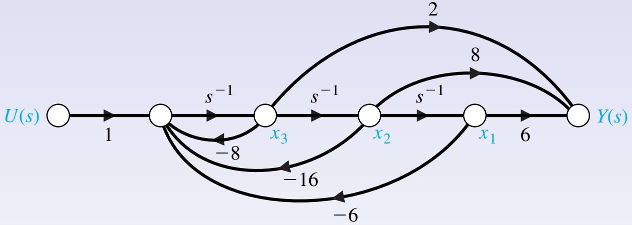
$$\begin{aligned} \dot{x} &= \begin{pmatrix} -a_3 & 1 & 0 & 0 \\ -a_2 & 0 & 1 & 0 \\ -a_1 & 0 & 0 & 1 \\ -a_0 & 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} b_3 \\ b_2 \\ b_1 \\ b_0 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} x \end{aligned}$$

Ex3.1 Phase variable canonical form

Recall the TF of the control system (in Fig.11)

$$\frac{Y(s)}{U(s)} = \frac{2s^2+8s+6}{s^3+8s^2+16s+6} = \frac{2s^{-1}+8s^{-2}+6s^{-3}}{1+8s^{-1}+16s^{-2}+6s^{-3}}$$

its graph in phase variable canonical form is



then the SSR is founded to be

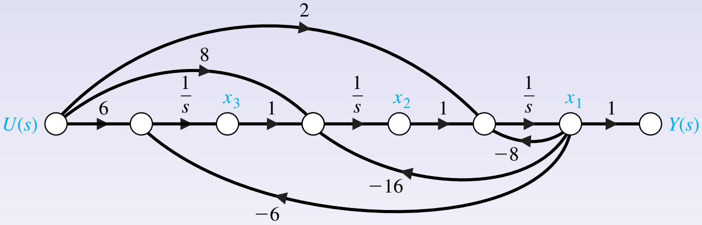
$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -16 & -8 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u \\ y &= \begin{pmatrix} 6 & 8 & 2 \end{pmatrix} x \end{aligned}$$

Ex3.1 Input feedforward canonical form

The TF of system shown in Fig 3.11 is

$$\frac{Y(s)}{U(s)} = \frac{2s^2+8s+6}{s^3+8s^2+16s+6} = \frac{2s^{-1}+8s^{-2}+6s^{-3}}{1+8s^{-1}+16s^{-2}+6s^{-3}}$$

its graph in `input feedforward canonical form` is



then the SSR is founded to be

$$\begin{aligned} \dot{x} &= \begin{pmatrix} -8 & 1 & 0 \\ -16 & 0 & 1 \\ -6 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 2 \\ 8 \\ 6 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x \end{aligned}$$

Summary

- It is simple to establish the state-space representation from signal-flow graphs in **phase variable canonical form** or **input feedforward canonical form**.
- The state variables are chosen as the outputs of the integrators, but the state variables of the two canonical forms are not identical, although the state variable numbers are equal to the order of the TF.
- One set of state variables may be related to other set of state variables by linear transformation, such as $z = Mx$, M is an invertible matrix.
- From both two canonical forms, no need to factor the numerator or denominator polynomial to obtain the SSR, thus we can establish the SSR from very complex TF and avoid the tedious effort to the factoring of polynomials.

Alternative signal-flow graph and block diagram models

Basing on one of two canonical forms of signal-flow graph in preceding section, two alternative forms of signal-flow graph are proposed.

For the block diagram models, where each block represents a physical device or variable, if the phase variable canonical form of each block is separately determined, then the overall signal-flow graph is obtained, called the `physical state variable signal-flow graph` (物理状态变量流图).

Alternative signal-flow graph and block diagram models

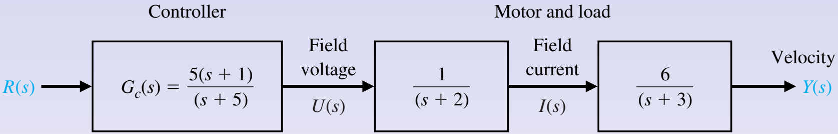
Using the partial fraction expansion of TF, thus large systems is separated into the parallel of many small systems, then decoupled s-tate differential matrix equation is obtained, the corresponding signal-flow graph is called the `decoupled state variable signal-flow graph` (解耦状态变量流图).

The model called **diagonal canonical form** (对角标准型),if its state matrix is a diagonal matrix. A model can always be transformed in to diagonal canonical form,if it possesses distinct poles.

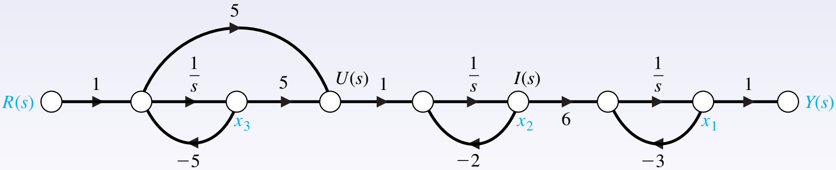
The model called **Jordan canonical form** (约当标准型),if its state matrix is a Jordan matrix. A model can always be transformed in to Jordan canonical form.

Alternative SFG of block diagram model

For the open loop DC Motor

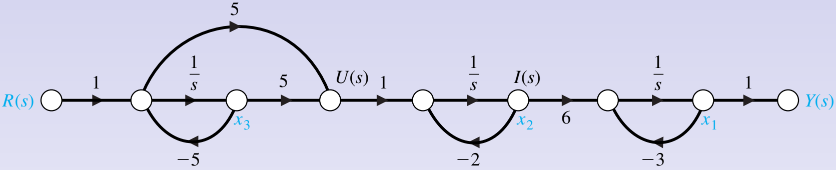


combining the phase state models of the blocks, to obtain the physical state variable flow graph



Alternative SFG of block diagram model

Consulting the graph



Choose state variables as

$x_1 = y;\ \dot{x}_1 = -3x_1 + 6x_2$
$x_2 = i(t);\ \dot{x}_2 = -2x_2 + u = -2x_2 - 20x_3 + 5r$
$u = 5x_3 + 5(r - 5x_3) \Rightarrow x_3 = r(t)/4 + u(t)/20;\ \dot{x}_3 = -5x_3 + r$

So the SSR is

$$\begin{aligned} \dot{x} &= \begin{pmatrix} -3 & 6 & 0 \\ 0 & -2 & -20 \\ 0 & 0 & -5 \end{pmatrix} x + \begin{pmatrix} 0 \\ 5 \\ 1 \end{pmatrix} r \\ y &= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x \end{aligned}$$

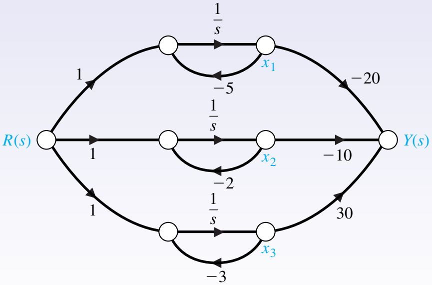
Alternative SFG of the partial fraction expansion of TF

For the partial fraction expansion of $\frac{Y(s)}{U(s)}$

$$\frac{Y(s)}{U(s)} = \frac{30(s+1)}{(s+5)(s+2)(s+3)} \Rightarrow \frac{k_1}{(s+5)} + \frac{k_2}{(s+2)} + \frac{k_3}{(s+3)}$$

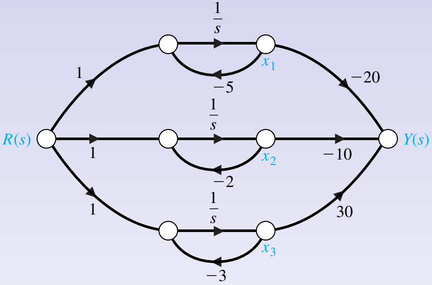
$k_1 = \frac{30(s+1)(s+5)}{[(s+5)(s+2)(s+3)]}|_{s=-5} = -20,\ k_2 = -10,\ k_3 = 30$

Then the decoupled state variable flow graph is



Alternative SFG of the partial fraction expansion of TF

The decoupled state variable flow graph is



The SSR in `diagonal canonical form` is obtained

$$
\begin{aligned}
\dot{x} &= \begin{pmatrix} -5 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} r \\
y &= \begin{pmatrix} -20 & -10 & 30 \end{pmatrix} x
\end{aligned}
$$

Ex 3.2 Spread of an epidemic disease

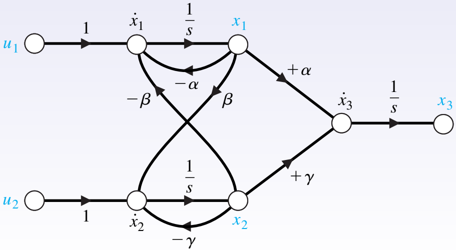
The spread of an epidemic disease can be described by

$\dot{x}_1 = -\alpha x_1 - \beta x_2 + u_1$
$\dot{x}_2 = \beta x_1 - \gamma x_2 + u_2$
$\dot{x}_3 = \alpha x_1 + \gamma x_2$

where x_1 susceptible group; x_2 infected group;x_3 healthy group.

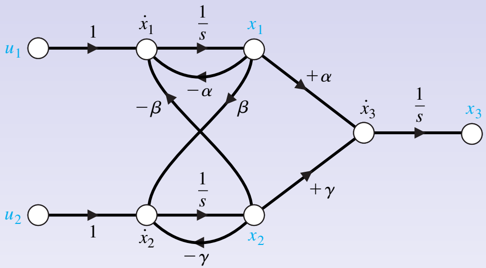
For a closed population $u_1 = u_2 = 0$, determine the condition on α, β, γ so that the disease is eliminated from the population, i.e. state x tends to equilibrium $\begin{pmatrix} 0 & 0 & x_{30} \end{pmatrix}^T$.

From the signal-flow graph, with 3 loops, 2 of them are nontouching



Ex 3.2 Spread of an epidemic disease

From the signal-flow graph, with 3 loops, 2 of them are nontouching



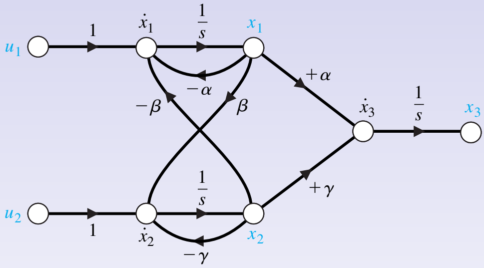
obtain the graph determinant

$$\Delta(s) = 1 - (-\alpha s^{-1} - \gamma s^{-1} - \beta^2 s^{-2}) + (\alpha\gamma s^{-2})$$

then the characteristic equation is

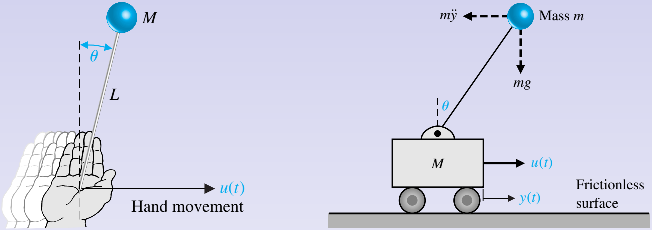
$$q(s) = s^2\Delta(s) = s^2 + (\alpha + \gamma)s + (\beta^2 + \alpha\gamma) = 0$$

Ex 3.2 Spread of an epidemic disease



Therefore when the roots are in the left-hand plane, i.e $(\alpha + \gamma) > 0$, and $(\beta^2 + \alpha\gamma) > 0$, then system is stable and the unforced state responses tend to the equilibrium $\begin{pmatrix} 0 & 0 & x_{30} \end{pmatrix}^T$.

Ex3.3 Inverted pendulum control



The state variables are expressed in terms of the angular rotation $\theta(t)$ and the car position $y(t)$. The goal of system control is to keep the mass m in an upright position by applying horizontal force $u(t)$ on the car. Assuming $m \ll M$ and small $\theta(t)$, so that the state equations are linear.

Recalling the assumptions, the sum of the forces in horizontal direction is

$$My'' + ml\theta'' - u(t) = 0$$

the sum of the torques about the pivot point is

$$mly'' + ml^2\theta'' - mgl\theta = 0$$

Ex3.3 Inverted pendulum control

Choose the state variable as

$$x = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix}^T = \begin{pmatrix} y & y' & \theta & \theta' \end{pmatrix}^T,$$

the state equations are written as

$$\begin{aligned} M\dot{x}_2 + ml\dot{x}_4 - u(t) &= 0 \\ \dot{x}_2 + \ l\dot{x}_4 - \ gx_3 &= 0 \\ \dot{x}_1 &= x_2 \\ \dot{x}_3 &= x_4 \end{aligned}$$

Solving for $l\dot{x}_4$ in the first equation and substituting into the second one, obtain

$$(M - m)\dot{x}_2 + mgx_3 = u(t)$$

or

$$\begin{aligned} \dot{x}_2 &= -\frac{mgx_3}{(M-m)} + \frac{u(t)}{(M-m)} \\ &\approx -\frac{mg}{M}x_3 + \frac{1}{M}u(t) \end{aligned}$$

Ex3.3 Inverted pendulum control

Then substituting $\dot{x}_2$ into the second equation, obtain

$$\begin{aligned} \dot{x}_4 &= \frac{Mg}{(M-m)l}x_3 - \frac{u(t)}{(M-m)l} \\ &\approx \frac{g}{l}x_3 - \frac{1}{Ml}u(t) \end{aligned}$$

Finally the SSR in matrix form is given by

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{g}{l} & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{Ml} \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} x \end{aligned}$$

The transfer function from the state equation

With signal-flow graph model, the state-space representation is obtained from a TF.

Now the problem is, for a given system of SSR, to determine its transfer function.

The transfer functions of state-space representations

For a system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

the Laplace transfromation of state and output equation is

$$\begin{aligned} sX(s) &= AX(s) + BU(s) \\ Y(s) &= CX(s) + DU(s) \end{aligned}$$

Reordering the first equation, obtain

$$(sI - A)X(s) = BU(s)$$

then

$$X(s) = (sI - A)^{-1}BU(s) = \Phi(s)BU(s)$$

where $\Phi(s) = (sI - A)^{-1}$.

The transfer functions of state-space representations

Substituting $X(s)$ into the output equation, obtain

$$Y(s) = C(sI - A)^{-1}BU(s) + DU(s) = C\Phi(s)BU(s) + DU(s)$$

Therefore the transfer function $G(s) = Y(s)/U(s)$ of the system is

$$G(s) = C(sI - A)^{-1}B + D = C\Phi(s)B + D$$

For SISO system, $G(s)$ is a scalar function, otherwise a matrix function.

Ex3.4 Transfer function of an RLC circuit

The SSR of the RLC circuit

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R}{L} \end{pmatrix} x + \begin{pmatrix} \frac{1}{C} \\ 0 \end{pmatrix} u \\ y &= \begin{pmatrix} 0 & R \end{pmatrix} x \end{aligned}$$

then

$$(sI - A) = \begin{pmatrix} s & \frac{1}{C} \\ -\frac{1}{L} & s + \frac{R}{L} \end{pmatrix}$$

Therefore

$$\Phi(s) = (sI - A)^{-1} = \frac{1}{\Delta(s)} \begin{pmatrix} s + \frac{R}{L} & -\frac{1}{C} \\ \frac{1}{L} & s \end{pmatrix}$$

where the `determinant` $\Delta(s) = |sI - A| = s^2 + \frac{R}{L}s + \frac{1}{LC}$,

Then the TF is

$$\begin{aligned} G(s) &= C\Phi(s)B = \frac{1}{\Delta(s)} \begin{pmatrix} 0 & R \end{pmatrix} \begin{pmatrix} s + \frac{R}{L} & -\frac{1}{C} \\ \frac{1}{L} & s \end{pmatrix} \begin{pmatrix} \frac{1}{C} \\ 0 \end{pmatrix} \\ &= \frac{1}{\Delta(s)} \frac{R}{LC} = \frac{\frac{R}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \end{aligned}$$

The time response and the state transition matrix

The state transition matrix or system unforced time response still can be derived with its signal-flow graph, before completing state-space representation.

Viewpoints of state transition matrix

It is desirable to obtain the time response of the state variables, to examine the performance of the system. The time responses are clearly represented by solution formula

$$x(t) = \Phi(t)x(0) + \int_0^t \Phi(t-\tau)Bu(\tau)dt$$

where $\Phi(t) = \mathcal{L}^{-1}\left[\Phi(s)\right] = \mathcal{L}^{-1}\left[(sI-A)^{-1}\right]$, so the problem focuses on the evaluation of $\Phi(t)$.

There are several algebraic methods to evaluate $\Phi(t)$, and here we mention two of them.

- direct evaluation of the sum of the exponential series

$$\Phi(t) = \exp(At) = \sum_{k=0}^{\infty} \frac{A^k}{k!} t^k$$

- evaluation matrix inversion of $\Phi(s) = (sI-A)^{-1}$, and taking inverse Laplace transformation of $\Phi(s)$. But matrix inversion process was thought to be tedious for higher-order system.

Viewpoints of state transition matrix

Now consider the method by utilizing Mason's signal-flow gain formula. In order to see clearly, take a second-order system as an illustration

$$X(s) = \Phi(s)x(0) = \begin{pmatrix} \phi_{11}(s) & \phi_{12}(s) \\ \phi_{21}(s) & \phi_{22}(s) \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$$

or

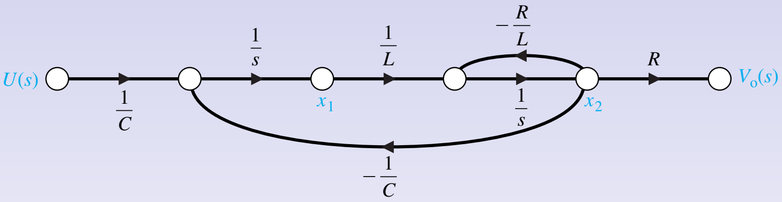
$$\begin{aligned} X_1(s) &= \phi_{11}(s)x_1(0) + \phi_{12}(s)x_2(0) \\ X_2(s) &= \phi_{21}(s)x_1(0) + \phi_{22}(s)x_2(0) \end{aligned}$$

When $x_2(0) = 0$, $\phi_{11}(s)$ and $\phi_{21}(s)$ act as the transfer function relating $X_1(s)$, $X_2(s)$ to $x_1(0)$, respectively.

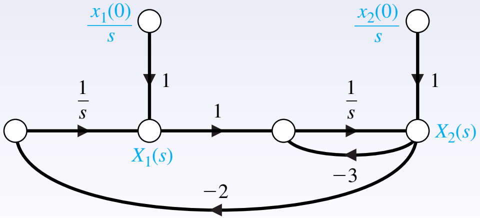
So $\phi_{11}(s)$, $\phi_{21}(s)$, and similarly $\phi_{12}(s)$, $\phi_{22}(s)$, can be evaluated with Manson's formula on the signal-flow graph limited to the part of that relating $x(0)$ and $x(t)$.

Ex3.5 Evaluation of the state transition matrix

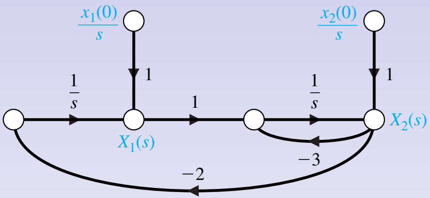
Consider the RLC network of Fig3.4, the original SFG is



When $R = 3, L = 1, C = 1/2$, the SFG related $X(s)$ and $x(0)$ is



Ex3.5 Evaluation of the state transition matrix



The path gain between $x_1(t)$ and $x_1(0)$

$$\phi_{11}(s) = \frac{1 \cdot (1 + 3s^{-1})}{1 + 3s^{-1} + 2s^{-2}} \frac{1}{s} = \frac{s+3}{s^2 + 3s + 2}$$

Similarly,

$$\phi_{12}(s) = \frac{-2s^{-1} \cdot 1}{1 + 3s^{-1} + 2s^{-2}} \frac{1}{s} = \frac{-2}{s^2 + 3s + 2}$$

$$\phi_{21}(s) = \frac{s^{-1} \cdot 1}{1 + 3s^{-1} + 2s^{-2}} \frac{1}{s} = \frac{1}{s^2 + 3s + 2}$$

$$\phi_{22}(s) = \frac{1 \cdot 1}{1 + 3s^{-1} + 2s^{-2}} \frac{1}{s} = \frac{s}{s^2 + 3s + 2}$$

Ex3.5 Evaluation of the state transition matrix

So the overall state transition matrix in Laplace transformation form is

$$\begin{aligned} \Phi(s) &= \frac{1}{s^2+3s+2}\begin{pmatrix} s+3 & -2 \\ 1 & s \end{pmatrix} \\ &= \begin{pmatrix} \frac{2}{s+1}+\frac{-1}{s+2} & \frac{-2}{s+1}+\frac{2}{s+2} \\ \frac{1}{s+1}+\frac{-1}{s+2} & \frac{-1}{s+1}+\frac{2}{s+2} \end{pmatrix} \end{aligned}$$

then state transition matrix is

$$\Phi(t) = \mathcal{L}^{-1}\left[\Phi(s)\right] = \begin{pmatrix} 2e^{-t}-e^{-2t} & -2e^{-t}+2e^{-2t} \\ e^{-t}-e^{-2t} & -e^{-t}+2e^{-2t} \end{pmatrix}$$

The state response with initial values $x_1(0) = x_2(0) = 1$, and $u(t) = 0$,

$$x(t) = \Phi(t)x(0) = \begin{pmatrix} e^{-2t} \\ e^{-2t} \end{pmatrix}$$

Note that the state response is linear combination of exponential functions like $e^{\lambda_i t}$, where λ_is are the eigenvalues of $\Phi(s)$.

A discrete-time evaluation of the time response

The state response of SSR can be obtained by utilizing a discrete-time approximation(离散时间近似).

The time axis is divided with sufficient small time increment T which much less than the time constant of the system, then state variables are evaluated at the successive time intervals $t = 0, T, 2T, 3T, \cdots$, with reasonable accuracy.

Discrete-time approximation

Consider the state differential equation

$$\dot{x} = Ax + Bu$$

The definition of time derivative of state $x(t)$ is

$$\dot{x}(t) = \lim_{\Delta t \longrightarrow 0} \frac{x(t + \Delta t) - x(t)}{\Delta t}$$

For a given sufficiently small constant T, there is a reasonable approximation to the derivative

$$\dot{x}(t) \approxeq \frac{x(t+T) - x(t)}{T}$$

Hence

$$\frac{x(t+T) - x(t)}{T} \approxeq Ax(t) + Bu(t)$$

furthermore

$$x(t+T) \approxeq (I + TA)x(t) + TBu(t)$$

holds at the successive discret time intervals $t = 0, T, 2T, \cdots$.

or

$$
\begin{aligned}
x\left[(k+1)T\right] &\approx (I+TA)x(kT)+TBu(kT) \\
&\approx \Psi(T)x(kT)+TBu(kT)
\end{aligned}
$$

where $\Psi(T) =: (I+TA)$, $t = kT, k = 0, 1, 2, 3, \cdots$.

For the brevity, $x(kT)$, $u(kT)$ are denoted with $x(k)$ and $u(k)$, respectively, so the above approximation equation is rewritten as the **discrete-time approximation**（离散时间近似）:

$$
x(k+1) \approx \Psi(T)x(k) + TBu(k), k = 0, 1, 2, 3, \cdots
$$

Starting from initial value $x(0)$, the state variable $x(t)$ can be evaluated, one by one, at the successive time intervals. This recurrence operation is called as `Euler's method`.

Other integration approaches, such as the popular `Runge-Kutta method` (龙格-库塔法), are much more effective to evaluate the time response of the systems.

Ex3.6 Response of the RLC network

Consider the specified RLC network with $R = 3, L = 1, C = 1/2$, then *time constant* is 0.5 $(-1, -2$, the poles)

$$\dot{x} = \begin{pmatrix} 0 & -2 \\ 1 & -3 \end{pmatrix} x + \begin{pmatrix} 2 \\ 0 \end{pmatrix} u, \Psi(T) = (I + TA) = \begin{pmatrix} 1 & -2T \\ T & 1-3T \end{pmatrix}$$

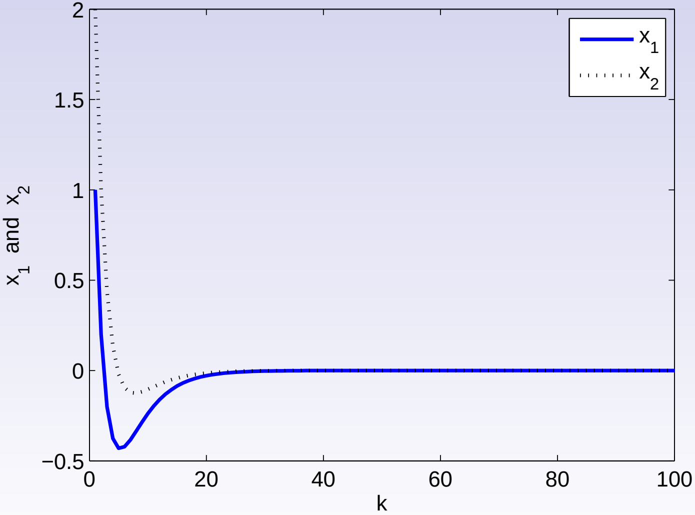
The discrete-time approximation

$$x(k+1) \approxeq \begin{pmatrix} 1 & -2T \\ T & 1-3T \end{pmatrix} x(k) + \begin{pmatrix} 2T \\ 0 \end{pmatrix} u(k),\ T < 0.5/2$$

The Matlab scrip for evaluation of the example is

```
% p159 ex3.6%%m159ex3d6
clear A=[0 -2;1 -3]; B=[2;0]; u=0; T=0.2; x=[1;2]; kk=1:1:100; xk=x;
for k=1:1:99
    x=(eye(2)+T*A)*x+T*B*u;
    xk=[xk x];
end plot(kk,xk(1,:),kk,xk(2,:)),legend('x_1','x_2')
```

Ex3.6 Response of the RLC network



Ex 3.7 Time response of an epidemic

For the specified epidemic system when $\alpha = \beta = \gamma = 1$:

$$\dot{x} = \begin{pmatrix} -1 & -1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} u$$

$$\Psi(T) = (I + TA) = \begin{pmatrix} 1-T & -T & 0 \\ T & 1-T & 0 \\ T & T & 1 \end{pmatrix}$$

Then the discrete-time approximation is

$$x(k+1) \approxeq \begin{pmatrix} 1-T & -T & 0 \\ T & 1-T & 0 \\ T & T & 1 \end{pmatrix} x(k) + \begin{pmatrix} T & 0 \\ 0 & T \\ 0 & 0 \end{pmatrix} u(k), T < 1/2$$

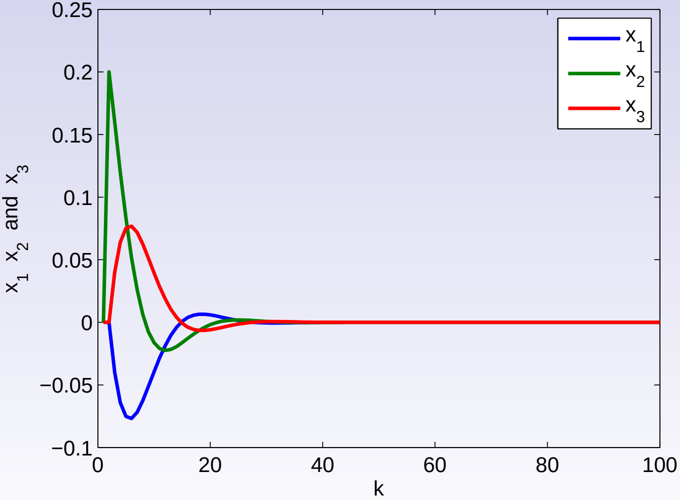
Ex 3.7 Time response of an epidemic

The Matlab scrip for $u_1 \equiv 0$; $u_2(0) = 1, u_2(k) = 0$, for $k \geq 1$, is listed as below:

```
% ex3.6 mex3d7.m
clear A=[-1 -1 0;1 -1 0;1 1 0]; B=[1 0;0 1;0 0]; T=0.2;
x=[0;0;0];kk=1:1:100; xk=x; for k=1:1:99
    if k==1
        u=[0;1];x=(eye(3)+T*A)*x+T*B*u;
    else
        u=[0;0]; x=(eye(3)+T*A)*x+T*B*u;
    end
     xk=[xk x];
end plot(kk,xk(1,:),kk,xk(2,:),kk,xk(3,:)),legend('x_1','x_2','x_3')
```

Negative values of x_1 are obtained as a result of an inadequate model, the actual physical value cannot become negative.

Ex 3.7 Time response of an epidemic



The discrete-time approximation of nonlinear system

Similar to linear system, consider the nonlinear system

$$\dot{x} = f(x, u, t)$$

Since

$$\dot{x} \approxeq \frac{x(t+T) - x(t)}{T} =: \frac{x(k+1) - x(k)}{T}, \quad t = kT,\ k = 0, 1, 2, \cdots$$

then the discrete-time approximation is give as

$$x(k+1) = x(k) + Tf(x(k), u(k), k),\ k = 0, 1, 2, \cdots$$

Ex3.8 Improved model of an epidemic

The system is modified as

$$\begin{aligned}\dot{x}_1 &= -\alpha x_1 - \beta x_1 x_2 + u_1(t)\\ \dot{x}_2 &= \beta x_1 x_2 - \gamma x_2 + u_2(t)\\ \dot{x}_3 &= \alpha x_1 + \gamma x_2\end{aligned}$$

The Euler's discrete time approximation

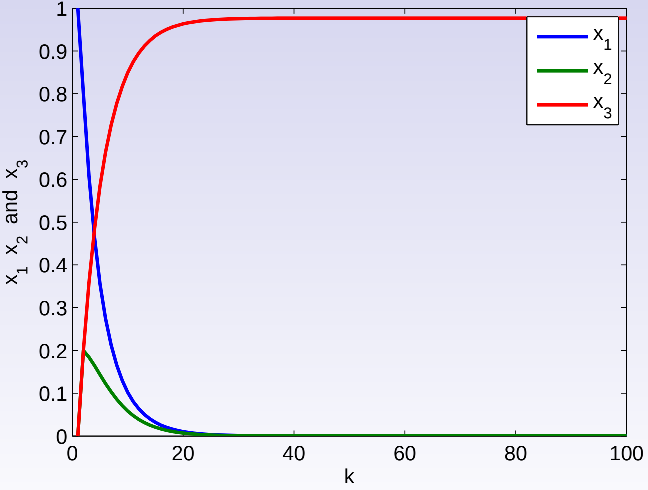
$$\begin{aligned}x_1(k+1) &= x_1(k) + T\left[-\alpha x_1(k) - \beta x_1(k)x_2(k) + u_1(k)\right]\\ x_2(k+1) &= x_2(k) + T\left[\beta x_1(k)x_2(k) - \gamma x_2(k) + u_2(k)\right]\\ x_3(k+1) &= x_3(k) + T\left[\alpha x_1(k) + \gamma x_2(k)\right]\end{aligned}$$

Ex3.8 Improved model of an epidemic

The Matlab scrip for $u_1 \equiv 0$; $u_2(0) = 1, u_2(k) = 0$, for $k \geq 1$, is listed as below:

```
 % ex3.8 mex3d8.m
clear alpha=1;beta=1;gamma=1;T=0.2;kk=1:1:100;
 x1=1;x2=0;x3=0 ; xk=[x1;x2;x3];u1=0;
for k=1:1:99
    if k==1
        u2=1;
    else
        u2=0;
    end
    x1=x1+T*(-alpha*x1-beta*x1*x2+u1);
    x2=x2+T*(beta*x1*x2-gamma*x2+u2);
    x3=x3+T*(alpha*x1+gamma*x2);
    xk=[xk [x1;x2;x3]];
end plot(kk,xk(1,:),kk,xk(2,:),kk,xk(3,:)),legend('x_1','x_2','x_3')
```

Ex3.8 Improved model of an epidemic



Summary

The evaluation of the time response of the state variables of linear systems is readily accomplished by using either (1) the transition matrix approach or (2) the discrete-time approximation.

The transition matrix of linear system is readily obtained from the signal-flow graph state model.

For a nonlinear system, the discrete-time approximation provides a suitable approach, and the discrete-time approximation method is particularly useful if a digital computer is used for numerical calculation.

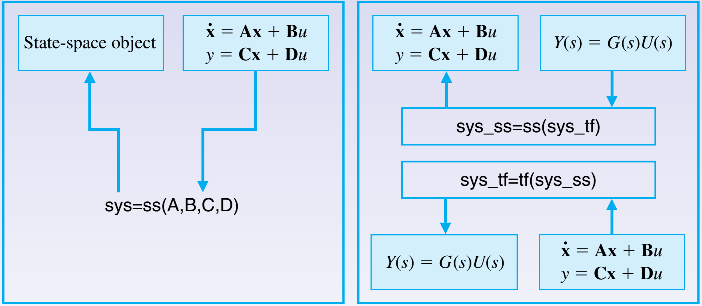
Analysis of state variable models using Matlab

The time-domain method utilizes a state-space representation which determined with 4 suitable matrices A, B, C, D.

Since the main computational unit in MATLAB is the matrix, the SSR lends itself well to the MATLAB environment.

Important Matlab functions useful to the analysis SSR

They are **tf,ss,expm,lsim**. The usage of function tf and ss are shown in following figure

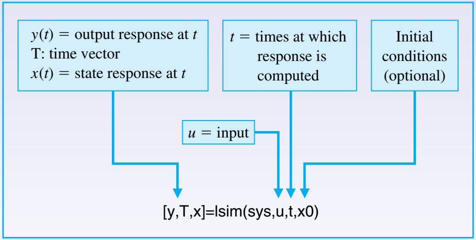


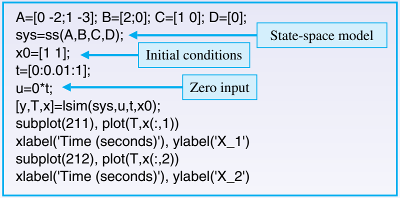
The function `expm` evaluates the state transition matrix at time t, such as `expm(A*t)`.

Important Matlab functions useful to the analysis SSR

The function `lsim` evaluates the time response of SSR at a given successive time intervals, with input $u(t)$ starting from initial value $x(0)$, t is the time vector representing the successive time intervals.

The usages are shown in the following figures.





Sequential design example: disk drive read system

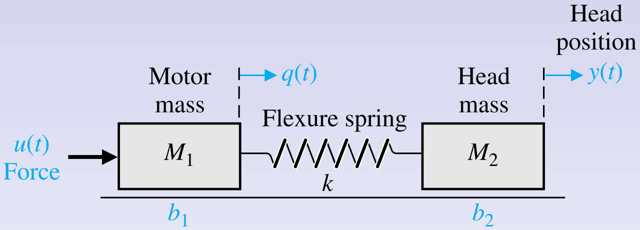
Advanced disks have stringent requirements on the accuracy of the reader head position and of the movement from one track to another.

A state variable model of the disk drive system will be developed and influence of the flexure mount to the system will be considered.

Under the proper control, the position of the head $y(t)$ can be accurately controlled with the force $u(t)$ generated by the DC motor.

SSR of disk drive read system

The disk drive read system is a two-mass system with spring flexure



The following equations characterize Mass M_1, M_2

$$\begin{aligned} M_1 q'' + b_1 q' + k(q - y) &= u \\ M_2 y'' + b_2 y' + k(y - q) &= 0 \end{aligned}$$

Choose the state variables as

$$x_1 = q, x_2 = y; \quad x_3 = q', x_4 = y'$$

The two second-order equations written in term of the state variables

$$\begin{aligned} M_1 \dot{x}_3 + b_1 x_3 + k(x_1 - x_2) &= u \\ M_2 \dot{x}_4 + b_2 x_4 - k(x_1 - x_2) &= 0 \end{aligned}$$

SSR of disk drive read system

or

$$\begin{aligned} \dot{x}_3 &= \frac{-k}{M_1}x_1 + \frac{-k}{M_1}x_2 - \frac{b_1}{M_1}x_3 + \frac{1}{M_1}u \\ \dot{x}_4 &= \frac{k}{M_2}x_1 + \frac{k}{M_2}x_2 - \frac{b_2}{M_2}x_4 \end{aligned}$$

The system output is chosen as the position rate of of Mass 2, namely $v = y' = x_4$.

For $k = 10$, M_1, M_2,b_1, b_2 specified in Table 3.3, the SSR is developed as

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -500 & 500 & -20.5 & 0 \\ 20000 & -20000 & 0 & -8.2 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 50 \end{pmatrix} u \\ v &= \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} x \end{aligned}$$

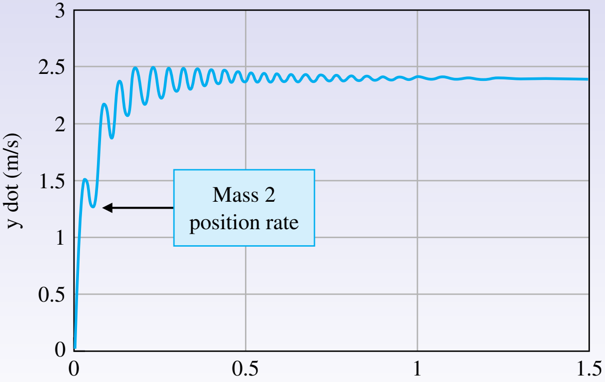
SSR of disk drive read system

The Matlab script to evaluate the unit (1kg) step response is

```
% mex3d37.m
% Model Parameters
k=10; M1=0.02; M2=0.0005; b1=410e-03; b2=4.1e-03; t=[0:0.001:1.5];
% State Space Model
A=[0 0 1 0;0 0 0 1;-k/M1 k/M1 -b1/M1 0; k/M2 -k/M2 0 -b2/M2];
B=[0;0;1/M1;0]; C=[0 0 0 1]; D=[0]; sys=ss(A,B,C,D);
% Simulated step Response
step(sys,t); xlabel('Time (seconds)'), ylabel('y dot (m/s)')
```

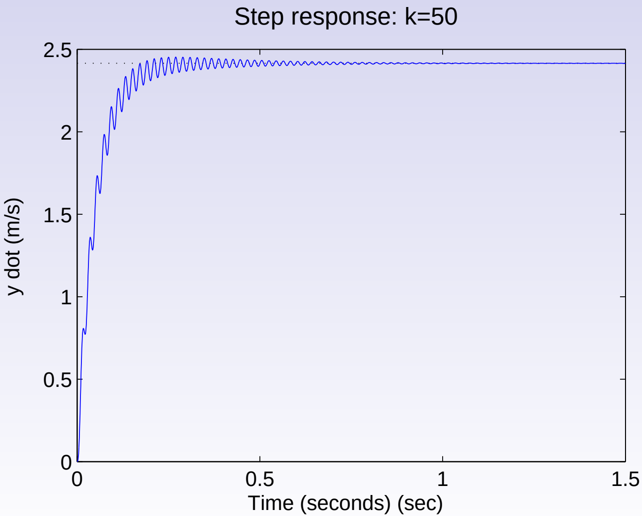
SSR of disk drive read system

The step response with $k = 10$ is shown as below and it shows that a more rigid flexure is required.



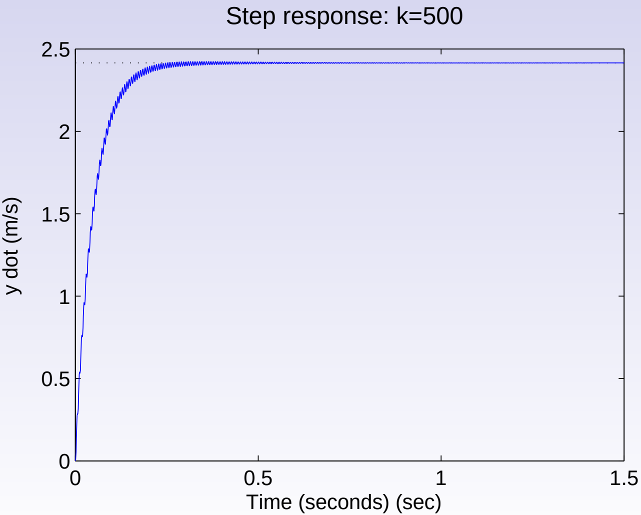
SSR of disk drive read system

The step response with $k = 50$ is shown as below :



SSR of disk drive read system

The step response with $k = 500$ is shown as below:



Summary of the chapter

- Consider the description and analysis of systems in time domain.
- Discuss the concept of state and state variable.
- Select the variables describing the stored energy, and note the nonuniqueness of a set of state variable.
- Discuss state differential equation and its solution.
- Consider two alternative signal-flow graph models to represent the TF.
- Note that Mason's formula makes it ease to obtain SFG, then SSR.
- Discuss time response, the state transition matrix and how to obtain it by using Manson's formula.
- Consider discrete-time approximation of the response of system including that for nonlinear and time-varying system.
- Use Matlab in analysis of SSR.
- Develop SSR for the sequential design example, disk drive read system.