现代控制理论基础
Fundamentals of Modern Control Theory

Chapter 11. The Design of State Variable Feedback Systems

LI Yin-Ya

School of Automation
Nanjing University of Science & Technology

October 30, 2019

Contents

1. Introduction
2. Controllability
3. Observability
4. Full-state feedback control design
5. Ackermann's formula
6. Observer design
7. Compensator design: Integrated full-state feedback and observer
8. Reference inputs
9. Optimal control systems
10. Internal model design
11. Design example: Automatic test system
12. State variable design using MATLAB
13. Sequential design example: Disk drive read system
14. Summary

Introduction

Preview

- present a system test for controllability and observability;
- introduce the pole placement design technique with state variable feedback;
- use Ackermann's formula to determine the feedback gain and place the poles at the desired locations;
- utilize an observer to estimate the state variable when the full state is not available;
- apply Ackermann's formula to design the observer;
- obtain state variable compensator by connecting the full-state feedback law to the observer;
- consider the design of the optimal control systems, and tracking problems with internal model design;
- revisit the disk drive read system.

Introduction

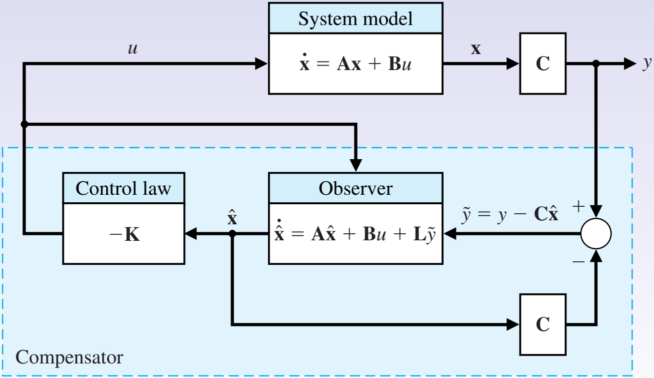
For a control system, the time-domain method can also be utilized to design a suitable compensation scheme with the control signal $u(t)$, which is a function of the measurable state variables.

The three steps for control design:

1. Assume all the state variable are measurable, and use them in a **full-state feedback control law**;
2. Construct an **observer** to estimate the state variables that are not directly sensed and available as outputs;
3. Connect the observer properly to the full-state feedback control law.

Introduction

A **compensator** refers to the state-variable controller comprised of full-state feedback control law and the observer. One compensator form is depicted in the following figure.



Controllability

It is important to note that a system must be **completely controllable** and **completely observable** to allow the flexibility to place all the closed-loop system poles arbitrarily.

These two concepts were introduced by Rudolph Kalman in 1960s, who was a central figure in the development of mathematical system theory upon which much of the subject of state variable methods rest.

The Kalman Filter named after his name, was instrumental in the successful of Apollo moon landing.

The concept and theorem of controllability

For the n dimensional system

$$\dot{x} = Ax + Bu, \quad x \in R^n$$

Definition: Completely controllable

A system is **completely controllable** if there exists an unconstrained control $u(t)$ that can transfer any initial state $x(t_0)$ to any other desired location $x(T)$ in a finite time, $t_0 \leq t \leq T$.

Theorem

The system is completely controllable if and only if rank $P_c = n$ (P_c is a full rank matrix) holds for the controllability matrix

$$P_c = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

Corollary

The SI system is completely controllable if and only if the determinant $|P_c| \neq 0$ holds for the controllability matrix P_c.

Extension of the concept of controllability

If a system is not completely controllable, but where the state variables that cannot be controlled are inherently stable, the system is classified as **stabilizable** (可镇定的).

Kalman state-space decomposition provides a mechanism for partitioning the state-space so that it becomes clear which state variables are controllable and which are not.

If the system is stabilizable, the control system design can proceed as usual in the controllable subspace, for the other state variables in uncontrollable subspace are stable themselves.

Review of stability

Consider state differential equation of SSR

$$\dot{x} = Ax + Bu$$

The eigenvalue λs of A are the solutions of characteristic equation

$$\det(\lambda I - A) = 0$$

Definition: Stability of SSR

The system of SSR is **asymptotically stable**, if the unforced state responses $x(t) = \Phi(t)x(0) = e^{At}x(0) \to 0$ as $t \to \infty$.

Often the term of asymptotically stable is simplified as "stable".

Theorem

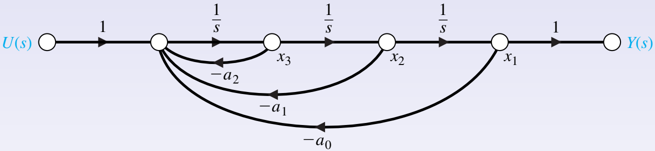
If all the eigenvalues or the poles of system matrix A locate in the left half-plane if and only if the system of SSR is asymptotically stable.

Ex 11.1 Controllability of a system

Consider the system

$$\begin{aligned}
\dot{x} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u \\
y &= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x + 0u
\end{aligned}$$

its signal-flow graph in phase variable canonical form



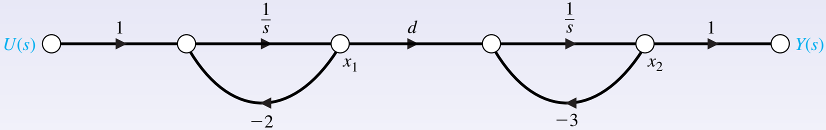
The controllability matrix is $P_c = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -a_2 \\ 1 & -a_2 & (a_2^2 - a_1) \end{pmatrix}$, the determinant $|P_c| = -1 \neq 0$, hence the system is controllable for any a_0,a_1,a_2.

Ex 11.2 Controllability of a state system

Consider the system

$$\begin{aligned}\dot{x}_1 &= -2x_1 + u \\ \dot{x}_2 &= dx_1 - 3x_2 \\ y &= x_2\end{aligned}$$

its signal-flow graph in phase variable canonical form



The controllability matrix P_c is $P_c = \begin{pmatrix} 1 & -2 \\ 0 & d \end{pmatrix}$, the determinant $|P_c| = d$, hence the system is controllable only when d is nozero.

Observability

Observability refers to the ability to estimate the state variable values with output signal $y(t)$.

The concept and theorem of observability

For the n dimensional system

$$\dot{x} = Ax + Bu,\ y = Cx,\quad x \in R^n$$

Definition: Completely observable

A system is **completely observable**, if for the given input signal $u(t)$, the initial state $x(0)$ can be determined from the observation history of $y(t)$, $0 \leq t \leq T$.

Theorem

The system is completely observable, if and only if rank $P_o{=}n$ (i.e P_o is a full rank matrix) holds for the observability matrix $P_o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$

Corollary

The SISO system is completely observable if and only if the determinant $|P_o| \neq 0$ holds for the observability matrix P_o.

Extension of the concept of observability

If a system is not completely observable, but the state variables that cannot be observable are inherently stable, the system is classified as **detectable** (可检测的).

Kalman state-space decomposition provides a mechanism for partitioning the state-space so that it becomes clear which state variables are observable and which are not.

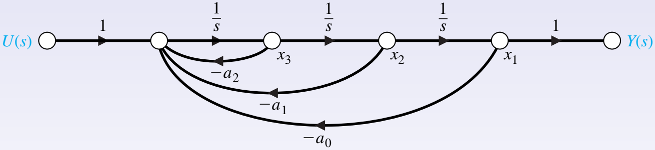
If the system is stabilizable and detectable, the control system design can proceed as usual in the controllable and observable subspace, for the other state variables in the supplementary space are stable themselves.

Ex 11.1 Observability of a system

Reconstider the system in Ex 11.1

$$\begin{aligned}\dot{x} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x + 0u\end{aligned}$$

its signal-flow graph in phase variable canonical form



The observability matrix is $P_o = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, the determinant $|P_o| = 1 \neq 0$, so $\text{rank}(P_o) = 3$ hence the system is observable.

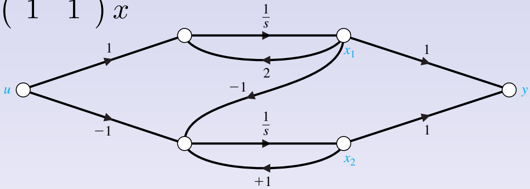
Ex 11.4 Observability of a two-state system

Consider the system

$$\dot{x} = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix} x + \begin{pmatrix} 1 \\ -1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 1 \end{pmatrix} x$$

its signal-flow graph is



The controllability matrix is $P_c = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}$, and the determinant $|P_c| = 0$, so the system is not controllable.

The observability matrix is $P_o = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, and the determinant $|P_o| = 0$, hence the system is not observable.

Note that $y = x_1 + x_2$, and $\dot{x}_1 + \dot{x}_2 = x_1 + x_2$, thus, the combination of state variables do not depend on $u(t)$; and x_1,x_2 can not be determined from $y(t)$ independently.

Full-state feedback control design

The full-state feedback design process is listed as below:

1. assume that the full-state feedback controller is in the form of

$$u(t) = -Kx$$

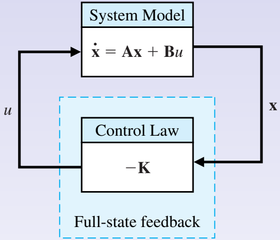
   where the complete state x is available by some means for all t.

2. determine the gain matrix K such than the eigenvalues of matrix $(A-BK)$ are placed at desired locations in the left half of s-plane.

Feedback controller design

Consider the system $\dot{x} = Ax + Bu$ and the full-state feedback controller $u = -Kx$.

The full-state feedback block diagram is illustrated in the figure



Then the closed-loop system is given by

$$\dot{x} = Ax + Bu = (A - BK)x$$

The corresponding characteristic equation is

$$\det\left[\lambda I - (A - BK)\right] = 0$$

Feedback controller design

If all poles of the characteristic equation

$$\det\left[\lambda I - (A - BK)\right] = 0$$

are placed at desired locations in the left-half of s-plane, the corresponding performance specifications will be satisfied for the closed loop system

$$\dot{x} = (A - BK)x$$

Theorem

For given SSR, the controller gain K can always be determined to place all the poles at any desired locations in the left-half of s-plane, if and only if the system is completely controllable.

对于系统SSR，确定控制器增益矩阵 K ，将闭环极点配置在 s 平面任意期望位置的充要条件是：系统完全能控。

Feedback controller design

The addition of a reference input can be considered as

$$u(t) = -Kx + Nr$$

where $r(t)$ is the **reference input**.

So the closed-loop system with a reference input is given by

$$\dot{x} = Ax + Bu = (A - BK)x + BNr$$

When $r(t) = 0$ for all $t > t_0$, the control design problem is known as **regulator design problem** (调节器设计问题).

Specification indices

The performance specifications are selected as

- The settling time is

$$T_s = \frac{4}{\zeta\omega_n}$$

- The percent overshoot is

$$P.O. = 100e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

or

$$\sigma\% = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100\%$$

Ex 11.5 Design of a third-order system

Consider the third-order system

$$y''' + 5y'' + 3y' + 2y = u(t)$$

and select the state with phase variables

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix}$$

then the phase state canonical form is given as

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -5 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x \end{aligned}$$

If state feedback controller given as

$$u(t) = -Kx(t), \quad K = \begin{pmatrix} k_1 & k_2 & k_3 \end{pmatrix}$$

Ex 11.5 Design of a third-order system

the closed-loop system is

$$\dot{x} = (A - BK)x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 - k_1 & -3 - k_2 & -5 - k_3 \end{pmatrix} x$$

and the characteristic polynomial is

$$\Delta(\lambda) = \det\left[\lambda I - (A - BK)\right] = \lambda^3 + (5 + k_3)\lambda^2 + (3 + k_2)\lambda + (2 + k_1)$$

For a rapid response with a low overshoot, choose a desired characteristic equation such as

$$\Delta^*(\lambda) = (\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2)(\lambda + \zeta\omega_n)$$

with damping ratio $\zeta = 0.8$ ($\sigma\% \approx 1.5\%$), and natural frequency $\omega_n = 6$ rad/s such that the settling time (with a 2% criterion)

$$T_s = \frac{4}{\zeta\omega_n} \approx 1$$

Ex 11.5 Design of a third-order system

and the desired characteristic polynomial is

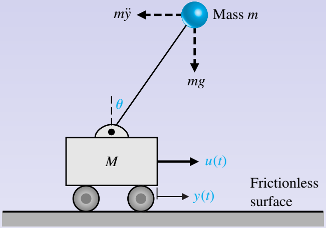
$$\Delta^*(\lambda) = (\lambda^2 + 9.6\lambda + 36)(\lambda + 4.8) = \lambda^3 + 14.4\lambda^2 + 82.08\lambda + 172.8$$

Comparing the characteristic polynomial of the state feedback control system

$$\Delta(\lambda) = \det[\lambda I - (A - BK)] = \lambda^3 + (5 + k_3)\lambda^2 + (3 + k_2)\lambda + (2 + k_1)$$

then it yields

$$\begin{array}{rcccl} 5 + k_3 & = & 14.4 & \Rightarrow & k_3 = 9.4 \\ 3 + k_2 & = & 82.08 & \Rightarrow & k_2 = 79.08 \\ 2 + k_1 & = & 172.8 & \Rightarrow & k_1 = 170.8 \end{array}$$



Choose the state variable as

$$\begin{aligned} x &= \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix}^T \\ &= \begin{pmatrix} y & y' & \theta & \theta' \end{pmatrix}^T \end{aligned}$$

the state equations are written as

$$\begin{aligned} M\dot{x}_2 + ml\dot{x}_4 - u(t) &= 0 \\ \dot{x}_2 + \ l\dot{x}_4 - \ gx_3 &= 0 \\ \dot{x}_1 &= x_2 \\ \dot{x}_3 &= x_4 \end{aligned}$$

$x_3 = \theta$ and $x_4 = \dot{\theta}$ can be measured with potentiometer and tachometer generator,respectively. $x_1 = y$ and $x_2 = \dot{y}$ can be measured by suitable sensor such as accelerator.

Thus the full-state feedback controller $u(t) = -Kx$ can be adopted to stabilize the inverted pendulum control system.

Ex11.6 Inverted pendulum control

In order to illustrate the utilization of state variable feedback, now consider the reduced system, where the action of the mass of the cart is simplified as force caused by its acceleration $u(t) = \dot{x}_2$.

From the second and 4th equations in original model, we see

$$l\dot{x}_4 - \ gx_3 = -\dot{x}_2 \Leftrightarrow -u(t), \text{ and } \ \dot{x}_3 = x_4$$

so the reduced system is

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 1 \\ g/l & 0 \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ -1/l \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x \end{aligned}$$

Ex11.6 Inverted pendulum control

The reduced system is

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 1 \\ g/l & 0 \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ -1/l \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x \end{aligned}$$

It is ease to see that the open loop system is unstable, since the characteristic equation

$$\Delta(\lambda) = \lambda^2 - \frac{g}{l} = 0$$

with one root in the right-hand of s-plane.

Let the feedback controller be

$$u(t) = -Kx = -\begin{pmatrix} k_1 & k_2 \end{pmatrix} x = -k_1 x_3 - k_2 x_4$$

so the closed-loop system is

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ \frac{g+k_1}{l} & \frac{k_2}{l} \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$$

Ex11.6 Inverted pendulum control

The closed-loop characteristic polynomial is given by

$$\Delta(\lambda) = \det\left[\lambda I - (A - BK)\right] = \lambda^2 - \frac{k_2}{l}\lambda - \frac{1}{l}(g + k_1)$$

For the desired characteristic polynomial of closed-loop system

$$\Delta^*(\lambda) = \lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = \lambda^2 + 16\lambda + 100$$

with damping ratio $\zeta = 0.8$, and natural frequency $\omega_n = 10$ rad/s such that the settling time $T_s = 4/\zeta\omega_n = 0.5$s.

Comparing the two polynomials $\Delta(\lambda)$ and $\Delta^*(\lambda)$, it yields

$$-\frac{k_2}{l} = 2\zeta\omega_n, \ \ -\frac{1}{l}(g + k_1) = \omega_n^2$$

hence $k_2 = -16l,\ k_1 = -100l - g.$

Ackermann's formula

For the completely controllable SISO system and the given desired characteristic polynomial, the gain matrix K of full-state feedback controller $u = -Kx$ can be determined directly with Ackermann formula.

Ackermann's formula

Consider the SISO system

$$\dot{x} = Ax + Bu,\ y = Cx, u \in R, y \in R$$

for the desired characteristic polynomial

$$q(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$$

then the full-state feedback gain matrix can be given as

$$K = \begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix} P_c^{-1} q(A)$$

where

$$q(A) = A^n + a_1 A^{n-1} + \cdots + a_{n-1}A + a_n I$$

and

$$P_c = \begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix}$$

Ex 11.7 Second-order system

Consider the system $G(s) = Y(s)/U(s) = 1/s^2$ and to determine the feedback gain with the poles being placed at $s = -1 \pm j$, therefore

$$q(\lambda) = \lambda^2 + 2\lambda + 2$$

Select the phase state variables as $x_1 = y$ and $x_2 = y'$, the state differential equation is

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

and controllability matrix $P_c = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and hence $q(A) = A^2 + 2A + 2I = \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}$. Therefore the gain matrix of full-state feedback controller $u = -Kx$ is

$$K = \begin{pmatrix} 0 & 1 \end{pmatrix} P_c^{-1} q(A) = \begin{pmatrix} 2 & 2 \end{pmatrix}$$

Observer design

If the system is complete observable, then all the state variables can be estimated by a dynamic process—observer with the output.

Full-state observer will provide estimations of all the state variables.

Since some state variables can be directly measured from output, so it is possible to design an observer just provide the estimations of the other part of state variables, the resulting observer known as **reduced-order observer**.

Design of full-state observer

Consider the system

$$\dot{x} = Ax + Bu, y = Cx \in R^{r\times 1}$$

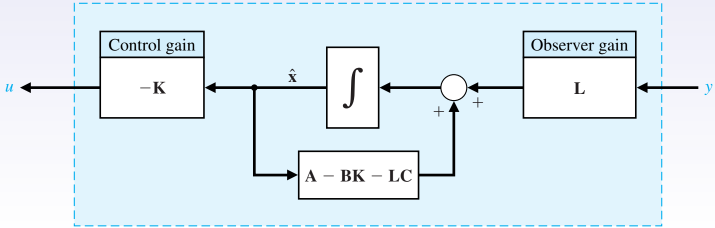
According to Luenberger, the full-state observer is given by

$$\begin{aligned}\dot{\hat{x}} = &\quad A\hat{x} + Bu + L(y - C\hat{x}) \\ = &\quad (A - LC)\hat{x} + Bu + Ly\end{aligned}$$

where $\hat{x}$ is the **estimation** of the state x; $L \in R^{n\times r}$ is the **observer gain matrix** which is to be determined.

Note that the full-state observer starts from an known $\hat{x}(0)$.

The block diagram model of full-state observer is



Design of full-state observer

The goal of the observer is to provide an estimation $\hat{x}$ so that the **estimation error** $e(t) = x(t) - \hat{x}(t) \to 0$, as $t \to \infty$.

Taking the time-derivative of $e(t)$, substituting into the state and observer equalities, to see

$$\begin{aligned} \dot{e} &= \dot{x} - \dot{\hat{x}} \\ &= Ax + Bu - [A\hat{x} + Bu + L(y - C\hat{x})] \\ &= (A - LC)e \end{aligned}$$

Hence for any initial error $e(0) = x(0) - \hat{x}(0)$, the estimation error is given as

$$e(t) = e^{(A-LC)t}e(0)$$

Theorem

The **estimation error** $e(t) = x(t) - \hat{x}(t) \to 0$, as $t \to \infty$, if and only if the pair $\begin{pmatrix} A & C \end{pmatrix}$ is complete observable.

Design of full-state observer

Therefore the observer design process reduces to that finding observer gain L such that all poles of the observer characteristic equation

$$\det\left[I\lambda - (A - LC)\right] = 0$$

lie at any pointed locations in the left-half s-plane to guarantee desired stability of **estimation error** $e(t)$.

Ex 11.8 Second-order system observer design

Consider
$$\begin{aligned}\dot{x} &= \begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x\end{aligned}$$

determine the observer gain matrix $L = \begin{pmatrix} l_1 & l_2 \end{pmatrix}^T$ such that the desired observer characteristic polynomial

$$\Delta_d(\lambda) = \lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = \lambda^2 + 16\lambda + 100$$

with damping ratio $\zeta = 0.8$, and natural frequency $\omega_n = 10$rad/s resulting the settling time $T_s = 4/\zeta\omega_n = 0.5$s.

It is ease to verify the system is completely observable.

Computing the actual characteristic polynomial, yields

$$\begin{aligned}\det\left[I\lambda - (A - LC)\right] &= \det\left[I\lambda - \begin{pmatrix} -l_1 + 2 & 3 \\ -l_2 - 1 & 4 \end{pmatrix}\right] \\ &= \lambda^2 + (l_1 - 6)\,\lambda + (-4l_1 + 3l_2 + 11)\end{aligned}$$

Equating the corresponding coefficient in both characteristic polynomials, obtains

Ex 11.8 Second-order system observer design

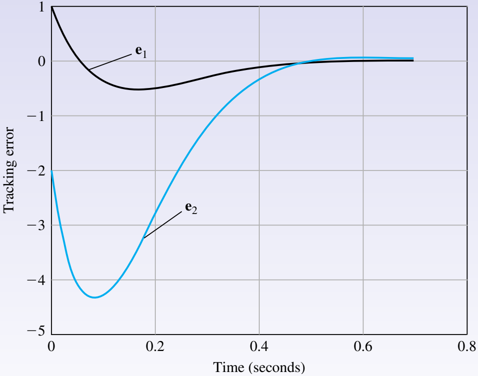
$$\begin{aligned} l_1 - 6 &= 16 \\ (-4l_1 + 3l_2 + 11) &= 100 \end{aligned}$$

therefore the observer gain matrix is $L = \begin{pmatrix} 22 \\ 59 \end{pmatrix}$. So the observer is given by

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(y - C\hat{x}) \\ &= \begin{pmatrix} -20 & 3 \\ -60 & 4 \end{pmatrix} \hat{x} + Bu + Ly \end{aligned}$$

Ex 11.8 Second-order system observer design

The response of the estimation error with an initial error of $e(0) = \begin{pmatrix} 1 & -2 \end{pmatrix}^T$ is shown in the following figure.



Ackermann's formula for observer design

Consider the completely observable SISO system

$$\dot{x} = Ax + Bu,\ y = Cx, u \in R, y \in R$$

with desired observer characteristic ploynomial

$$p(\lambda) = \lambda^n + \beta_1 \lambda^{n-1} + \cdots + \beta_{n-1}\lambda + \beta_n$$

Then the observer gain matrix is given as

$$L = p(A) P_o^{-1} \begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix}^T$$

where

$$p(A) = A^n + \beta_1 A^{n-1} + \cdots + \beta_{n-1} A + \beta_n I$$

and

$$P_o = \begin{pmatrix} C^T & A^T C^T & \cdots & (A^T)^{n-1} C^T \end{pmatrix}^T$$

Ex 11.9 Second-order system observer design using Ackermann's formula

Consider
$$\begin{aligned} \dot{x} &= \begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x \end{aligned}$$

determine the observer gain matrix L such that the desired characteristic equation $p(\lambda) = \lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2$ with damping ratio $\zeta = 0.8$, and natural frequency $\omega_n = 10$ rad/s and resulting that the settling time $T_s = 4/\zeta\omega_n = 0.5$s.

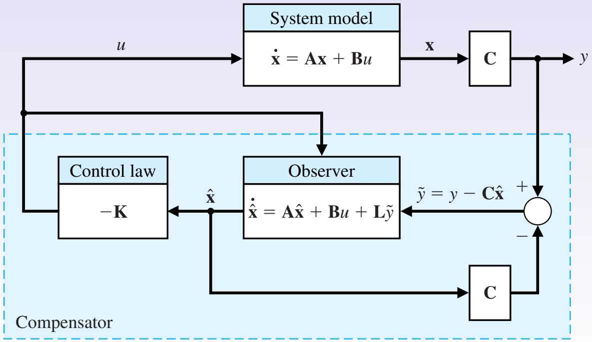
So $p(A) = A^2 + 16A + 100I = \begin{pmatrix} 133 & 66 \\ -22 & 177 \end{pmatrix}$, the observability matrix is $P_o = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}, P_o^{-1} = \begin{pmatrix} 1 & 0 \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix}$.

Therefore the observer gain matrix is given as
$$L = p(A)P_o^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 22 \\ 59 \end{pmatrix}$$

The result is identical to that of Ex 11.8.

Compensator design: Integrated full-state feedback and observer

The state variable **compensator** is constructed by appropriately connecting the full-state feedback controller to the observer.



Introduction

The strategy to design the compensator:

1. Design the state feedback controller $u(t) = -Kx(t)$ such that the roots of $\det\left[I\lambda - (A - BK)\right] = 0$ lies at prescribed locations in left-half s-plane, as if the complete state $x(t)$ is accessible.
2. Design an observer to provide state estimation $\hat{x}$ with the observer gain matrix such that the roots of $\det\left[I\lambda - (A - LC)\right] = 0$ lies at prescribed locations in left-half s-plane.
3. Employ state estimation in the state feedback controller in place of $x(t)$, the actual state feedback controller is given as $u(t) = -K\hat{x}(t)$.

It will be verified that a suitable compensator guarantee the stability of the overall closed-loop system, i.e. the state $x(t) \to 0$ and estimation error $e(t) \to 0$ as $t \to \infty$ with desired transient performances.

Compensator: the observer with $u = -K\hat{x}$

The full-state feedback

The system under consideration

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

The full-state feedback closed-loop system with controller $u = -Kx$ is

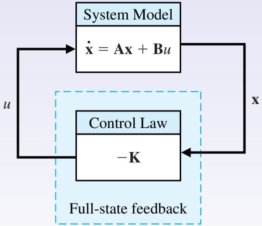


Fig: Full-state feedback block diagram

The full-state observer

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + Bu + L(y - C\hat{x}) \\ &= (A - LC)\hat{x} + Bu + Ly\end{aligned}$$

Taking $u(t) = -K\hat{x}(t)$ of the place of $u(t)$ in the observer, yields the **compensator system**

$$\begin{aligned}\dot{\hat{x}} &= (A - BK - LC)\hat{x} + Ly \\ u &= -K\hat{x}(t)\end{aligned}$$

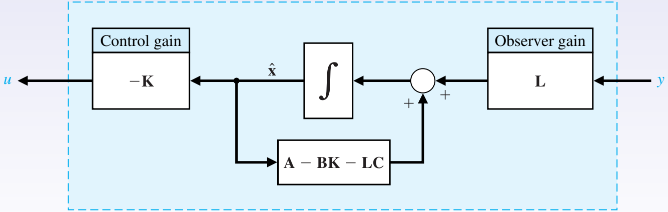


Fig: Full-state observer

Compensator: the observer with $u = -K\hat{x}$

The compensator system will run simultaneously with the full-state feedback system, so the overall system is given as

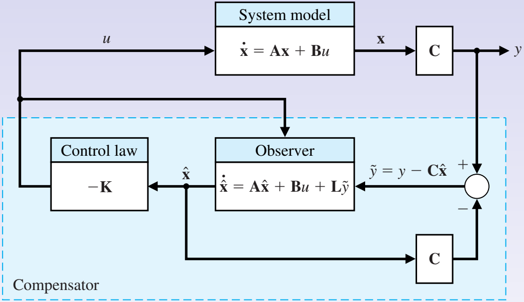


Fig: State feedback control with observer

The state of overall system could be

$$X = \begin{pmatrix} x \\ x - \hat{x} \end{pmatrix} = \begin{pmatrix} x \\ e \end{pmatrix}$$

Compensator: the observer with $u = -K\hat{x}$

Stability of overall system

Let $X = \begin{pmatrix} x \\ e \end{pmatrix}$, then the estimation error differential equation

$$\begin{aligned} \dot{e} &= \dot{x} - \dot{\hat{x}} = Ax + Bu - [(A - LC)\hat{x} + Bu + Ly] \\ &= (A - LC)e \end{aligned}$$

the state differential equation

$$\begin{aligned} \dot{x} &= Ax + Bu = Ax - BK\hat{x} \\ &= (A - BK)x + BKe \end{aligned}$$

The overall state variable feedback close-loop system

$$\begin{pmatrix} \dot{x} \\ \dot{e} \end{pmatrix} = \begin{pmatrix} A - BK & BK \\ 0 & A - LC \end{pmatrix} \begin{pmatrix} x \\ e \end{pmatrix}$$

then the characteristic polynomial is equal to

$$\Delta(\lambda) = \det\left[I\lambda - (A - BK)\right] \det\left[I\lambda - (A - LC)\right]$$

The overall closed-loop characteristic polynomial is

$$\Delta(\lambda) = \det\left[I\lambda - (A - BK)\right]\det\left[I\lambda - (A - LC)\right]$$

So the stability of the overall system is totally determined by the design of full-state feedback control and observer.

The fact that the controller gain matrix K and observer gain matrix L can be designed independently is an illustration of the **separation principle** (分离原理).

The design procedure is summarized as follows:

1. Determining K such that all roots of $\det[I\lambda - (A - BK)] = 0$ placed at prescribed locations in the left half-plane to meet the control system design specifications.
   If the system is completely controllable, one can place the poles arbitrarily in the s-plane.
2. Determining L such that $\det[I\lambda - (A - LC)] = 0$ has roots in the left half-plane to achieve acceptable observer performance.
   The ability to place the observer poles arbitrarily in the s-plane is guaranteed if the system is completely observable.
3. Connect the observer to the full-state feedback controller with

$$u = -K\hat{x}(t)$$

Ex 11.10 Compensator design for the inverted pendulum

Let $x = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix}^T = \begin{pmatrix} y & y' & \theta & \theta' \end{pmatrix}^T$, the system is written as

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{g}{l} & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{Ml} \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} x \end{aligned}$$

The parameters as specified as $l = 0.098$ m, $g = 9.8$ m/s^2, $m = 0.825$ kg, $M = 8.085$ kg, then

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 100 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0.123\,69 \\ 0 \\ -1.262\,1 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} x \end{aligned}$$

Ex 11.10 Compensator design for the inverted pendulum

The controllability matrix

$$\begin{aligned} P_c &= \begin{pmatrix} B & AB & A^2B & A^3B \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0.123\,69 & 0 & 1.262\,1 \\ 0.123\,69 & 0 & 1.262\,1 & 0 \\ 0 & -1.262\,1 & 0 & -126.21 \\ -1.262\,1 & 0 & -126.21 & 0 \end{pmatrix} \end{aligned}$$

Then $\det(P_c) = 196.5$, so the system is completely controllable.

The observability matrix

$$P_o = \begin{pmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Then $\det(P_o) = 1$, so the system is completely observable.

Ex 11.10 Compensator design for the inverted pendulum

Step 1: Design the full-state feedback controller
Suppose the desired closed-loop system characteristic equation is

$$q(\lambda) = (\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2)(\lambda^2 + 20 \cdot 2\zeta\omega_n\lambda + 20^2 \cdot \omega_n^2)$$

where $\zeta = 0.8$ and $\omega_n = 0.5$ resulting the response settling time $T_s = 10$ s with **separation value** 20 between the dominant poles $\lambda = -0.4 \pm 0.3i$ and nondominant poles $\lambda = -8 \pm 6i$.

Ex 11.10 Compensator design for the inverted pendulum

Using the Akermann's formula, the feedback gain matrix

$$
\begin{aligned}
K &= \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} P_c^{-1} q(A) = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \times \\
& \quad \begin{pmatrix} 0 & 0.12369 & 0 & 1.2621 \\ 0.12369 & 0 & 1.2621 & 0 \\ 0 & -1.2621 & 0 & -126.21 \\ -1.2621 & 0 & -126.21 & 0 \end{pmatrix}^{-1} \times \\
& \quad \begin{pmatrix} 25.0 & 84.0 & -213.05 & -16.8 \\ 0.0 & 25.0 & -1764.0 & -213.05 \\ 0.0 & 0.0 & 21330.0 & 1764.0 \\ 0.0 & 0.0 & 1.764 \times 10^5 & 21330.0 \end{pmatrix} \\
&= \begin{pmatrix} -2.2509 & -7.5629 & -169.03 & -14.052 \end{pmatrix}
\end{aligned}
$$

Ex 11.10 Compensator design for the inverted pendulum

Step2, Observer design

The goal is to achieve an accurate estimation as fast as possible without resulting in too large a gain matrix L.

If there are significant levels of measurement noise, then the magnitude of L should be kept correspondingly low to avoid amplify the measure noise.

For design purpose, the **separation** between the desired closed-loop system poles and the observer poles is set from 2 to 10.

Now select the desired observer characteristic equation in the form

$$p(\lambda) = (\lambda^2 + c_1\lambda + c_2)^2$$

where $c_1 = 32$, $c_2 = 711.11$, with $-16.0 \pm 21.33i$.

Ex 11.10 Compensator design for the inverted pendulum

Using the Akermann's formula, the observer gain matrix

$$
\begin{aligned}
L &= \begin{pmatrix} 5056\,80 & 45511. & -2546.\,2 & -64 \\ 0 & 5056\,80 & -51911. & -2546.\,2 \\ 0 & 0 & 7603\,00 & 51911. \\ 0 & 0 & 5191\,100 & 7603\,00 \end{pmatrix} \\
&\quad \times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\
&= p(A)P_0^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 64.0 \\ 2546.\,2 \\ -5.1911 \times 10^4 \\ -7.\,6030 \times 10^5 \end{pmatrix}
\end{aligned}
$$

Ex 11.10 Compensator design for the inverted pendulum

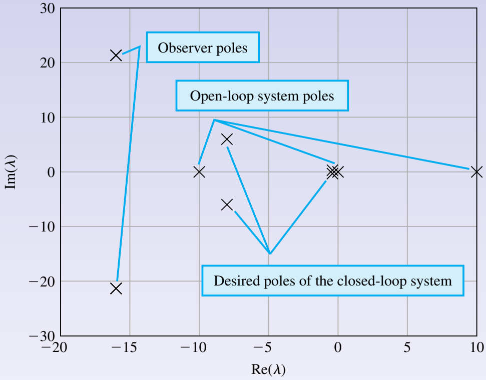


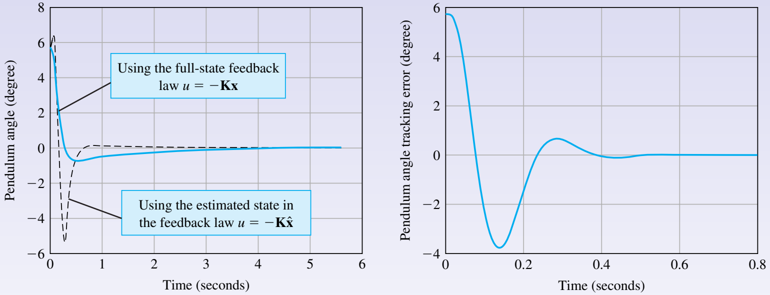
Fig: System pole map

Step 3: Compensator design

Connect the observer to the full-state feedback controller with

$$u(t) = -K\hat{x}(t)$$

The simulation result is shown in the figure with initial condition $x(0) = \begin{pmatrix} y & y' & \theta & \theta' \end{pmatrix}^T = \begin{pmatrix} 0 & 0 & 5.72\pi/180 & 0 \end{pmatrix}^T$ and $\hat{x}(0) = 0$



The figure shows that the closed-loop performance is deteriorated when utilizing the state estimation of the observer, since it takes time for the observer to provide accurate state estimation.

Reference inputs

In the preceding section, the state feedback control design is discussed without consideration of reference inputs(i.e $r(t) = 0$), the corresponding compensators are called **regulators** (调节器).

Since reference input tracking or command following is also an important aspect of feedback design, now consider how to design the state variable feedback compensator with a reference signal $r(t)$, such that the overall output $y(t)$ tracks $r(t)$ with zero steady-state tracking error, i.e $[y(t) - r(t)] \to 0$ as $t \to \infty$.

There are many different techniques that can be employed to permit the tracking of a reference input, the **internal model design** (内模设计) technique will be discussed in the latter section.

The compensator with reference inputs

Consider the system with a reference input $r(t)$

$$\dot{x} = Ax + Bu, \ y = Cx$$

Now the full-state feedback controller is given as

$$u = -Kx + Nr$$

where N is an adjustable scalar for MIMO system.

The full-state observer

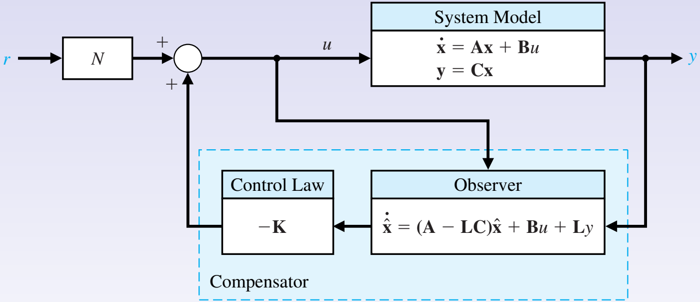
$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(y - C\hat{x}) \\ &= (A - LC)\hat{x} + Bu + Ly \end{aligned}$$

Taking $u(t) = -K\hat{x}(t) + Nr(t)$ of the place of $u(t)$ in the observer, yields the compensator system

$$\begin{aligned} \dot{\hat{x}} &= (A - BK - LC)\hat{x} + Ly + BNr \\ u &= -K\hat{x}(t) + Nr(t) \end{aligned}$$

The compensator with reference inputs

The overall system with the reference input is depicted in the figure.



The compensator with reference inputs

The estimation error differential equation is the same as that in regulator

$$\begin{aligned} \dot{e} &= \dot{x} - \dot{\hat{x}} = Ax + Bu - [(A - LC)\hat{x} + Bu + Ly] \\ &= (A - LC)e \end{aligned}$$

The state feedback closed-loop system

$$\begin{aligned} \dot{x} &= (A - BK)x + BKe + BNr \\ y &= Cx \end{aligned}$$

So design procedures for a compensator with reference inputs are almost the same as that of regulator, and now there is a parameter N to be adjusted so that $[y(t) - r(t)] \to 0$ as $t \to \infty$.

Optimal Control Systems

Systems adjusted to provide the minimum **performance index** are called **optimal control systems**.

The desired performance can be readily stated in terms of time-domain performance indices, such as the minimum overshoot, rising time and settling time for a unit step input.

For SSR, the performance index in integral form can be expressed as

$$J = \int_0^{t_f} g(x(t), u(t), t)dt$$

where t_f is the final time, x is the state vector, u is the control vector.

Thus the goal of the optimal control system is to design the controller $u(t) = u[x(t)]$ such that the performance index J is minimized.

Performance index: integral of quadratic form

Let the performance index of integral quadratic form in term of $x(t)$ and $u(t)$

$$J = \int_0^{\infty} \left(x^T I x + \lambda u^T u \right) dt$$

account for the system **stored energy** and the **expenditure of the energy** of control signal, where I is the identity matrix, and λ is a scalar weighting factor presenting the relative importance.

Performance index: integral of quadratic form

For control system

$$\dot{x} = Ax + Bu, \; y = Cx$$

design optimal controller

$$u = -Kx$$

such that the performance index

$$J = \int_0^\infty \left(x^T I x + \lambda u^T u\right) dt$$

is minimized, where $x(t)$ is the state response starting from initial value $x(0)$ of the closed-loop system

$$\dot{x} = (A - BK)x, \; y = Cx$$

If the optimization is realized, then the integral are convergent with $x(t) \to 0$ as $t \to \infty$, therefore the closed-loop system with optimal controller is stable.

The optimal controller design

For the closed-loop system

$$\dot{x} = (A - BK)x =: Hx, y = Cx$$

Substituting $u = -Kx$ into the performance index J, to have

$$\begin{aligned} J &= \int_0^\infty \left(x^T I x + \lambda x^T K^T K x\right) dt \\ &= \int_0^\infty x^T \left(I + \lambda K^T K\right) x dt =: \int_0^\infty x^T Q x dt \end{aligned}$$

where Q denotes $Q = I + \lambda K^T K$, a symmetrical matrix.

Suppose that there is a **symmetrical matrix** P such that

$$\frac{d}{dt}\left(x^T P x\right) = -x^T Q x$$

or

$$\begin{aligned} -x^T Q x &= \dot{x}^T P x + x^T P \dot{x} \\ &= x^T H^T P x + x^T P H x \\ &= x^T \left(H^T P + P H\right) x \end{aligned}$$

The optimal controller design

or

$$H^T P + PH = -Q$$

Then, it follows

$$\begin{aligned} J &= \int_0^\infty x^T Q x dt = -\int_0^\infty \frac{d}{dt}\left(x^T P x\right) dt \\ &= x^T(0) P x(0) - x^T(\infty) P x(\infty) \\ &= x^T(0) P x(0) \end{aligned}$$

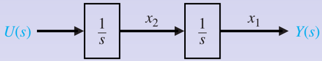
hence $x(t) \to 0$ as $t \to \infty$.

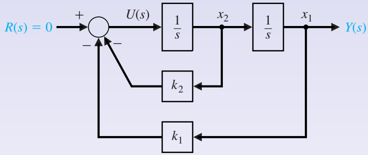
Therefore the design of optimal controller $u = -Kx$ is transformed to the optimization of $J = x^T(0)Px(0)$, where P is a function of under determined K.

The optimal control design steps are listed as follows:

1. Determine the matrix P from equation $H^T P + PH = -Q$;
2. Minimize $J = x^T(0)Px(0)$ by adjusting K.

Ex 11.11 State variable feedback





Consider the open loop system, then SSR is given as

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

For the performance index

$$J = \int_0^\infty x^T I x \; dt$$

with $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, determine the optimal controller

$$u = -Kx = -\begin{pmatrix} k_1 & k_2 \end{pmatrix} x$$

under the initial condition $x(0) = \begin{pmatrix} 1 & 1 \end{pmatrix}^T$.

Ex 11.11 State variable feedback

The closed-loop system matrix is

$$H = A - BK = \begin{pmatrix} 0 & 1 \\ -k_1 & -k_2 \end{pmatrix}$$

Set symmetrical matrix

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}$$

then solve it from the algebraic equation with $Q = I$

$$H^T P + PH = -Q$$

or the expanded form

$$\begin{pmatrix} 0 & -k_1 \\ 1 & -k_2 \end{pmatrix} \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} + \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -k_1 & -k_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Ex 11.11 State variable feedback

Completing the matrix multiplication and addition, we have

$$\begin{pmatrix} -2k_1p_{12} & p_{11} - k_1p_{22} - k_2p_{12} \\ p_{11} - k_1p_{22} - k_2p_{12} & 2p_{12} - 2k_2p_{22} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Comparing the corresponding matrix elements, we obtain

$$\begin{aligned} -2k_1p_{12} &= -1 \\ p_{11} - k_1p_{22} - k_2p_{12} &= 0 \\ 2p_{12} - 2k_2p_{22} &= -1 \end{aligned}$$

Solving these equations for p_{11},p_{12} and p_{22}, we obtain

$$p_{11} = \frac{k_1 + k_1^2 + k_2^2}{2k_1k_2}, \quad p_{12} = \frac{1}{2k_1}, \quad p_{22} = \frac{1 + k_1}{2k_1k_2}$$

Ex 11.11 State variable feedback

Then for initial condition $x(0) = \begin{pmatrix} 1 & 1 \end{pmatrix}^T$, the performance index is

$$
\begin{aligned}
J &= \int_0^\infty x^T I x dt = x(0)^T P x(0) \\
&= \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= p_{11} + 2p_{12} + p_{22} = \frac{1 + 2k_1 + k_2 + k_1^2 + k_2^2}{2k_1 k_2}
\end{aligned}
$$

Let $k_1 = 1$, then

$$J = \frac{4 + k_2 + k_2^2}{2k_2}$$

Ex 11.11 State variable feedback

To minimize J as the function of k_2, take the derivative with respect to k_2 and set it equal to zero

$$\begin{aligned}\frac{\partial J}{\partial k_2} &= \frac{1}{2}\frac{(1+2k_2)\,k_2 - \left(4+k_2+k_2^2\right)}{k_2^2} \\ &= \frac{1}{2}\frac{k_2^2-4}{k_2^2} = 0\end{aligned}$$

Therefore when $k_2 = 2$, $J_{\min} = 2.5$, thus an optimal controller can be given as

$$u^*(t) = -\begin{pmatrix} 1 & 2 \end{pmatrix} x = -x_1 - 2x_2$$

The characteristic equation of the closed-loop system is

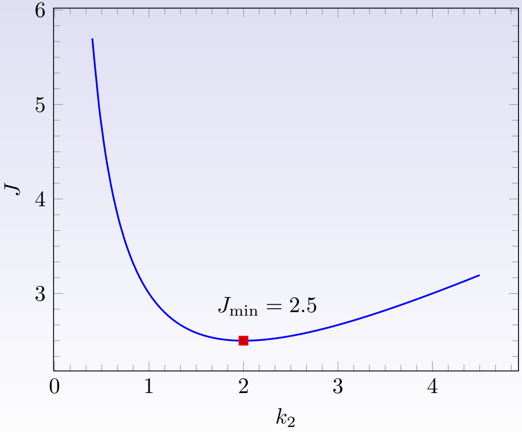
$$\det(\lambda I - H) = \lambda^2 + 2\lambda + 1$$

with repeated roots -1, hence the closed-loop system is stable.

Ex 11.11 State variable feedback

We recognize that the system is optimal only for the specific set of initial conditions.

The curve of the performance index as the function of k_2 shows that the system is not very sensitive to changes in k_2.



Ex 11.12 Determination of an optimal control system

Consider the system again in Ex 11.11

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x \end{aligned}$$

For the performance index

$$J = \int_0^\infty x^T I x \; dt$$

with

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

determine the optimal control $u = -Kx = -\begin{pmatrix} k_1 & k_2 \end{pmatrix} x$ for $k_1 = k_2 = k$ under the initial condition $x(0) = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$.

The closed-loop system matrix is

$$H = A - BK = \begin{pmatrix} 0 & 1 \\ -k & -k \end{pmatrix}$$

The symmetrical matrix solution $P = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}$ for the algebraic equation $H^T P + PH = -Q$ with $Q = I$ is given by

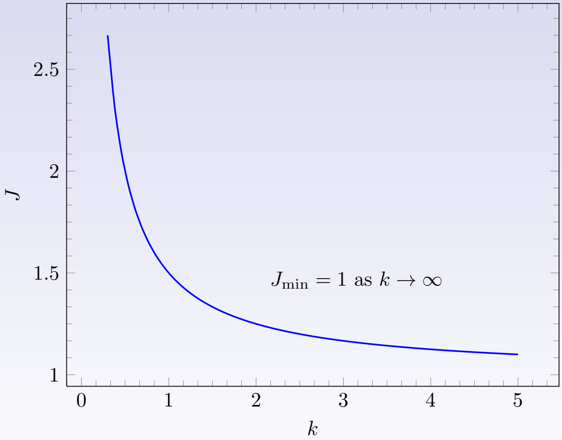
$$p_{11} = \frac{k_1 + k_1^2 + k_2^2}{2k_1 k_2} = \frac{1 + 2k}{2k}, p_{12} = \frac{1}{2k}, p_{22} = \frac{1 + k}{2k^2}$$

Then for the initial condition $x(0) = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$, the performance index

$$\begin{aligned} J &= \int_0^\infty x^T I x dt = x(0)^T P x(0) \\ &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = p_{11} = 1 + \frac{1}{2k} \end{aligned}$$

Ex 11.12 Determination of an optimal system

The minimum value of $J_{\min} = 1$ is obtained when $k \to \infty$. A plot of J versus k is shown in the following figure.



Ex 11.12 Determination of an optimal system

So the optimization will cause the feedback signal

$$u = -Kx = -kx_1 - kx_2$$

to be very large.

In actual control system, the magnitude of $u(t)$ will be restricted, such as $|u(t)| \leq 50$. So the maximum acceptable value of k is $k_{\max} = 50$, for the initial condition $x(0) = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$

$$J_{\min} = 1 + \frac{1}{2k_{\max}} = 1.01$$

which is very close to the absolute minimum of J.

This example shows that if the expenditure of the control signal energy not presented in the performance index, the system optimization may result in very large magnitudes of the controller gain matrix.

Ex 11.13 Optimization with control energy constraints

Consider the system once again in Ex 11.11

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x \end{aligned}$$

for the performance index

$$J = \int_0^\infty \left(x^T I x + \lambda u^T u\right) dt$$

with

$$Q = \left(I + \lambda K^T K\right) = \begin{pmatrix} 1 + \lambda k^2 & \lambda k^2 \\ \lambda k^2 & 1 + \lambda k^2 \end{pmatrix}$$

determine the optimal control $u = -Kx = -\begin{pmatrix} k & k \end{pmatrix} x$ under the initial condition $x(0) = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$.

Ex 11.13 Optimization with control energy constraints

The closed-loop system matrix

$$H = A - BK = \begin{pmatrix} 0 & 1 \\ -k & -k \end{pmatrix}$$

Set symmetrical matrix solution $P = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}$, solve it from algebraic equation

$$H^T P + PH = -Q$$

or the expanded form

$$\begin{pmatrix} 0 & -k \\ 1 & -k \end{pmatrix} \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} + \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -k & -k \end{pmatrix} = - \begin{pmatrix} 1+\lambda k^2 & \lambda k^2 \\ \lambda k^2 & 1+\lambda k^2 \end{pmatrix}$$

Completing the matrix multiplication and addition, to have

$$\begin{pmatrix} -2kp_{12} & p_{11} - kp_{12} - kp_{22} \\ p_{11} - kp_{12} - kp_{22} & 2p_{12} - 2kp_{22} \end{pmatrix} = - \begin{pmatrix} 1+\lambda k^2 & \lambda k^2 \\ \lambda k^2 & 1+\lambda k^2 \end{pmatrix}$$

Ex 11.13 Optimization with control energy constraints

Comparing the corresponding matrix elements, to obtain simultaneous equations

$$\begin{aligned} 2kp_{12} &= 1 + \lambda k^2 \\ -p_{11} + kp_{22} + kp_{12} &= \lambda k^2 \\ -2p_{12} + 2kp_{22} &= 1 + \lambda k^2 \end{aligned}$$

and solving these equation for p_{11},p_{12} and p_{22}, we obtain

$$p_{11} = \frac{1 + 2k + \lambda k^2}{2k},\ p_{12} = \frac{1 + \lambda k^2}{2k},\ p_{22} = \frac{(1+k)\left(1 + \lambda k^2\right)}{2k^2}$$

Then for initial condition $\begin{pmatrix} 1 & 0 \end{pmatrix}^T$, the performance index is

$$J = x(0)^T P x(0) = p_{11} = \frac{1 + 2k + \lambda k^2}{2k}$$

Let $\lambda = 1$, then

$$J = \frac{1 + 2k + k^2}{2k}$$

Ex 11.13 Optimization with control energy constraints

Take the derivative with respect to k and set it equal to zero

$$\frac{\partial J}{\partial k} = \frac{1}{2}\frac{(2+2k)\,k - \left(1+2k+k^2\right)}{k^2} = \frac{k^2-1}{2k^2} = 0$$

Therefore when $k=1$ and J is minimized with $J_{\min}=2$, thus an optimal controller can be given as

$$u^*(t) = -\begin{pmatrix} 1 & 1 \end{pmatrix} x = -x_1 - x_2$$

The characteristic equation of the closed-loop system is

$$\det(\lambda I - H) = \lambda^2 + \lambda + 1$$

with roots $-\frac{1}{2} \pm \frac{1}{2}i\sqrt{3}$, hence the closed-loop system is stable.

In this case the optimization value of J obtained here is considerably greater than that of the previous case, since the expenditure of control energy is included.

Internal model design

Internal model design provides asymptotic tracking of a reference input with zero steady-state error.

The reference inputs include steps, ramps and sinusoids.

For a step input, zero steady-state error can be achieved with a type-one system, this idea is formalized here by introducing an **internal model** of the reference input in the compensator.

Internal models of reference inputs

Consider system

$$\dot{x} = Ax + Bu, \; y = Cx$$

and the reference input generated by the linear system

$$\dot{x}_r = A_r x_r, \quad r(t) = d_r x_r$$

For steps

$$\dot{x}_r = 0, \quad r(t) = x_r$$

For ramps

$$\dot{x}_r = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x_r, \quad r(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} x_r$$

For sinusoids with frequency ω

$$\dot{x}_r = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} x_r, \quad r(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} x_r$$

Internal model design to step reference

Design controller to enable the tracking of a step reference with zero steady-state error.

The step reference is generated by

$$\dot{x}_r = 0, r(t) = x_r, \text{ or } \dot{r}(t) = 0$$

Tracking error is defined as

$$e(t) = y(t) - r(t)$$

its derivative

$$\begin{aligned} \dot{e}(t) &= \dot{y}(t) - \dot{r}(t) \\ &= \dot{y}(t) = C\dot{x} \end{aligned}$$

Define two intermediate variables

$$z =: \dot{x};\ w =: \dot{u}$$

then

$$\begin{aligned} \dot{e} &= Cz \\ \dot{z} &= Az + Bw \end{aligned}$$

Internal model design to step reference

or

$$\begin{pmatrix} \dot{e} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 & C \\ 0 & A \end{pmatrix} \begin{pmatrix} e \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ B \end{pmatrix} w$$

If the pair $\left(\begin{pmatrix} 0 & C \\ 0 & A \end{pmatrix}, \begin{pmatrix} 0 \\ B \end{pmatrix} \right)$ is completely controllable, there is a feedback of the form

$$w(t) = -K_1 e(t) - K_2 z(t)$$

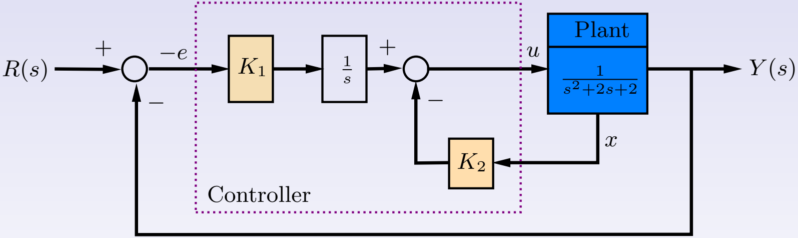
such that the closed-loop system is stable, i.e. $e(t) \to 0$ as $t \to \infty$.

The corresponding controller input, founded by integrating the intermediate control signal $w(t)$

$$u(t) = -K_1 \int_0^t e(\tau) d\tau - K_2 x(t)$$

Internal model design to step references

The system with the compensator including reference input internal model in loop is shown as below:



Ex 11.14 Internal model design for a unit step input

Let's consider a plant given by

$$\begin{aligned}\dot{x} &= \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x\end{aligned}$$

the intermediate state variables are $e = y - r, z = \dot{x}, \omega = \dot{u}$, then

$$\begin{aligned}\begin{pmatrix} \dot{e} \\ \dot{z} \end{pmatrix} &= \begin{pmatrix} 0 & C \\ 0 & A \end{pmatrix} \begin{pmatrix} e \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ B \end{pmatrix} \omega \\ &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -2 \end{pmatrix} \begin{pmatrix} e \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \omega\end{aligned}$$

its controllability matrix

$$P_c = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -2 & 2 \end{pmatrix}, \det(P_c) = -1$$

so the system is completely controllable.

Ex 11.14 Internal model design for a unit step input

Suppose the characteristic equation is

$$q(\lambda) = (\lambda + 10)(\lambda + 1 + j)(\lambda + 1 - j) = (\lambda + 10)(\lambda^2 + 2\lambda + 2)$$

then

$$q(\mathcal{A}) = q\left[\begin{pmatrix} 0 & C \\ 0 & A \end{pmatrix}\right] = \begin{pmatrix} 20 & 20 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

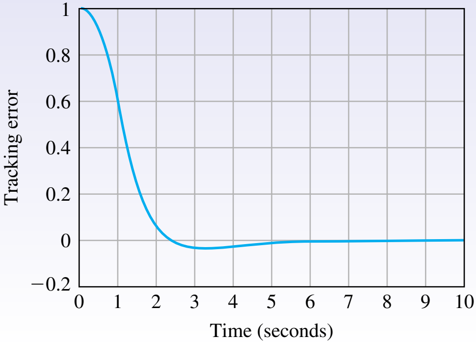
From the Ackermann's formula, the controller gain is

$$\begin{aligned} \begin{pmatrix} K_1 & K_2 \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} P_c^{-1} q(\mathcal{A}) \\ &= \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 20 & 20 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 20 & 20 & 10 \end{pmatrix} \end{aligned}$$

Therefore the control input given by the compensator with step reference internal model is

$$u(t) = -20 \int_0^t e(\tau) d\tau - \begin{pmatrix} 20 & 10 \end{pmatrix} x(t)$$

So for any initial tracking error $e(0)$, it is guaranteed that $e(t) \to 0$ as $t \to \infty$. The asymptotic stability of the tracking error $e(t)$ is illustrated in the following figure.



Internal model design to ramp references

Design a controller to enable the tracking of a ramp reference with zero steady-state error. The ramp reference is generated by

$$\dot{x}_r = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x_r, r(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} x_r, \text{ or } \ddot{r}(t) = 0$$

Tracking error is defined as $e(t) = y(t) - r(t)$, and its second order derivative is

$$\begin{aligned} \ddot{e}(t) &= \ddot{y}(t) - \ddot{r}(t) \\ &= \ddot{y}(t) = C\ddot{x} \end{aligned}$$

Define two intermediate variables $z =: \ddot{x}; \ w =: \ddot{u}$, then

$$\begin{aligned} \dot{e} &= \dot{e} \\ \ddot{e} &= Cz \\ \dot{z} &= Az + Bw \end{aligned}$$

Internal model design to ramp references

or

$$\begin{pmatrix} \dot{e} \\ \ddot{e} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & C \\ 0 & 0 & A \end{pmatrix} \begin{pmatrix} e \\ \dot{e} \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix} w$$

If the pair $\left(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & C \\ 0 & 0 & A \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix} \right)$ is completely controllable, there exists the feedback of the form

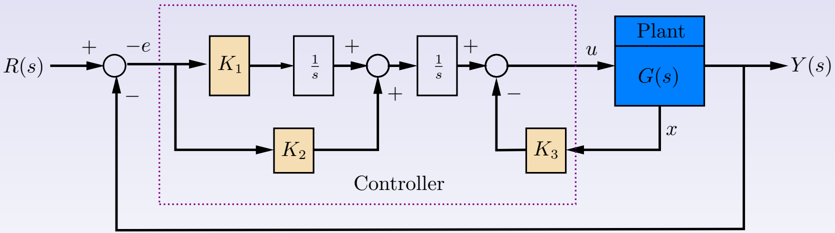
$$\omega = -K_1 e(t) - K_2 \dot{e}(t) - K_3 z(t)$$

such that the closed-loop system is stable, i.e. $e(t) \to 0$ as $t \to \infty$.

The corresponding controller input, founded by integrating the intermediate control signal $w(t)$ twice

$$u(t) = -K_1 \int_0^t \int_0^\tau e(\upsilon) d\upsilon d\tau - K_2 \int_0^t e(\tau) d\tau - K_3 x(t)$$

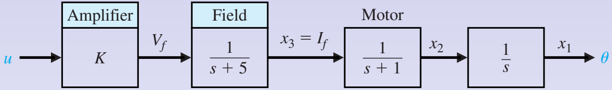
The system with the compensator including reference input internal model in loop showing in the following figure.



Design example: Automatic test system

Design example:Automatic test system

The open loop of the DC motor with mounted encoder wheel



The associated SSR

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -5 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ K \end{pmatrix} u$$

it is completely controllable for $K \neq 0$, since

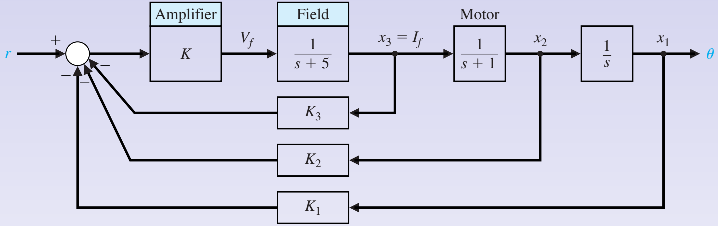
$$P_c = \begin{pmatrix} 0 & 0 & K \\ 0 & K & -6K \\ K & -5K & 25K \end{pmatrix}, \det(P_c) = -K^3$$

Let the full-state controller in the form

$$u = -\begin{pmatrix} K_1 & K_2 & K_3 \end{pmatrix} x + r = -K_1x_1 - K_2x_2 - K_3x_3 + r$$

Design example:Automatic test system

Then the close-loop block diagram is



Select the gain so that the step response with settling time $T_s \leq 2$ s, and an overshoot $P.O. \leq 4.0$.

Using the performance specification formulae

$$T_s \approx \frac{4}{\zeta\omega_n} \leq 2, \;\; 100\exp\left[\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right] \leq 4.0$$

to find that $\zeta \geq \frac{\ln 25}{\sqrt{\pi^2+(\ln 25)^2}} = 0.715\,65$, $\omega_n \geq 2.794\,7$.

Design example:Automatic test system

Suppose the characteristic equation is

$$\begin{aligned} q(\lambda) &= (\lambda + 10.62)(\lambda + 3.69 + 3.0j)(\lambda + 3.69 - 3.0j) \\ &= (\lambda + 10.62)\left(\lambda^2 + 7.38\lambda + 22.616\right) \end{aligned}$$

then

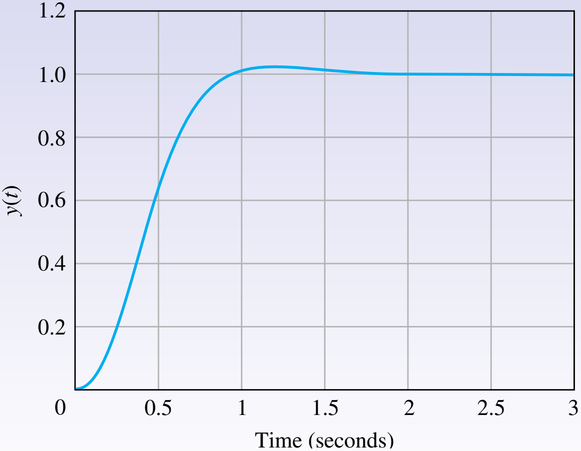
$$q(A) = \begin{pmatrix} 240.18 & 83.992 & 12.0 \\ 0.0 & 156.19 & 23.992 \\ 0.0 & 0.0 & 60.224 \end{pmatrix}$$

From the Ackermann's formula, the controller gain is given as

$$\begin{aligned} \mathcal{K} &= \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} P_c^{-1} q(A) \\ &= \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{5}{K} & \frac{5}{K} & \frac{1}{K} \\ \frac{6}{K} & \frac{1}{K} & 0 \\ \frac{1}{K} & 0 & 0 \end{pmatrix} q(A) \\ &= \begin{pmatrix} \frac{240.18}{K} & \frac{83.992}{K} & \frac{12.0}{K} \end{pmatrix} \end{aligned}$$

Design example:Automatic test system

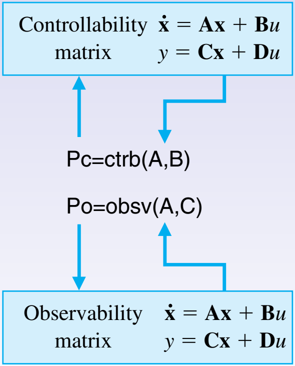
The step response of the closed-loop system with $K = 240$ is shown in the following figure.



State variable design using MATLAB

Matlab functions for controllability and observability

Controllability and observability of a SSR can be checked with Matlab functions `ctrb` and `obsv`, their usage is shown in the figure



Ex 11.15 Satellite trajectory control

The perturbation system of the circular equatorial orbit

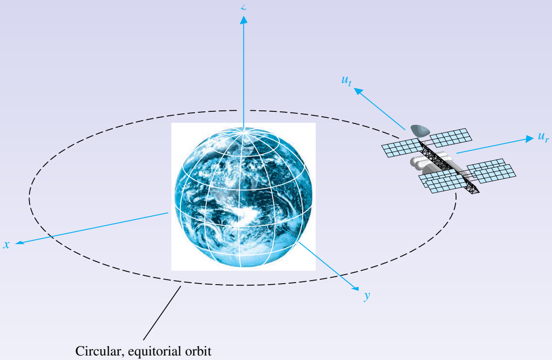
$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} u_r + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u_t$$

where u_r input from the radial thruster; u_t input from the tangential thruster; $\omega = 0.0011$ rad/s.

The goal of satellite trajectory control is to design full-state controllers u_r and u_t such that the closed-loop system is stable.

Ex 11.15 Satellite trajectory control

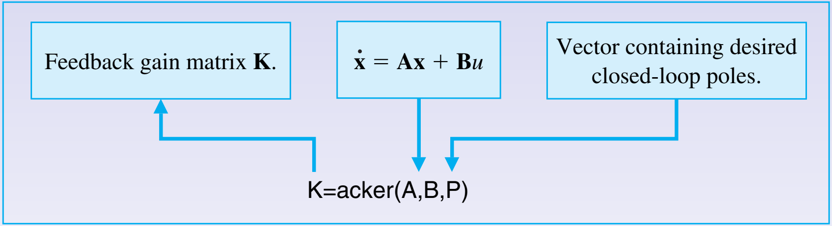
The satellite in an circular equatorial orbit is shown in figure



It can be verified with Matlab function `ctrb` that the satellite is not completely controllable when the tangential thruster fails ($u_t = 0$); but the satellite is completely controllable when only the tangential thruster is operational properly ($u_r = 0$).

Ex11.17 Second-order system design using the `acker` function

The usage of Matlab function `acker` is shown in the figure



For the system

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

the desired close-loop pole locations are $s_{1,2} = -1 \pm j$.

Ex11.17 Second-order system design using the `acker` function

Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and $P = \begin{pmatrix} -1+j \\ -1-j \end{pmatrix}$, then by using acker function, obtain the controller gain matrix $K = \begin{pmatrix} 2 & 2 \end{pmatrix}$ of controller $u = -Kx$.

The Matlab scripts are shown as follows:

```
A=[0 1;0 0];
B=[0;1];
P=[-1+j; -1-j];
K=acker(A,B,P)
```

```
K =

    2     2
```

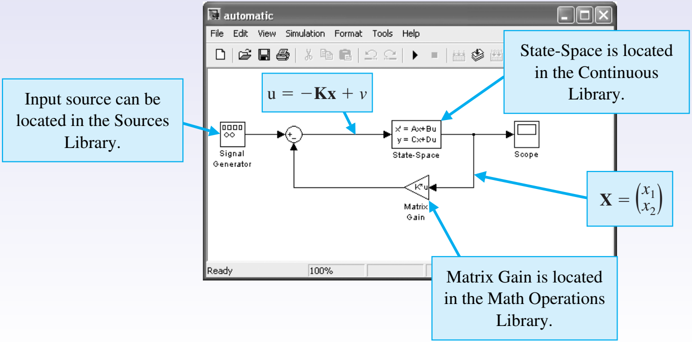
The feedback gain matrix.

State-space system simulation using Simulink

Take following system as an example to illustrate the usage of Simulink in system simulation

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \end{pmatrix} u$$



State-space system simulation using Simulink

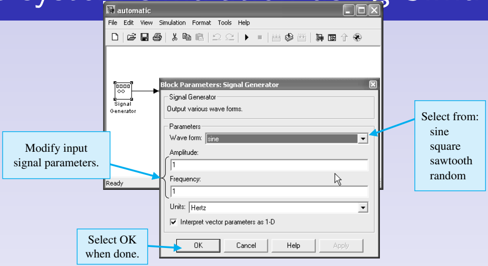


Fig: Select the reference input on the Signal Generator block

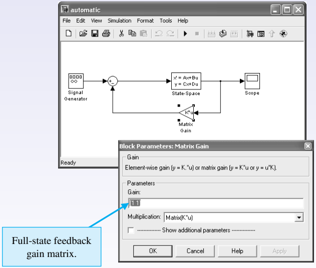


Fig: Incorporate the feedback into simulation block diagram

State-space system simulation using Simulink

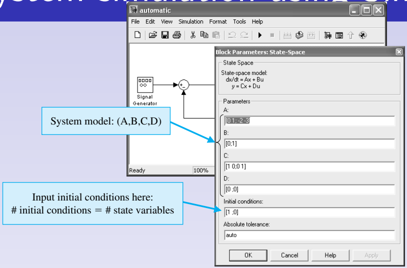


Fig: Define the state variable model's coefficients

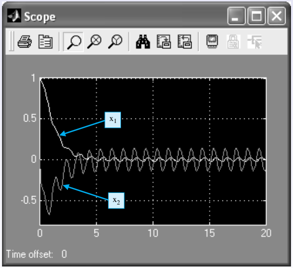


Fig: The result of simulation showing the state variable history

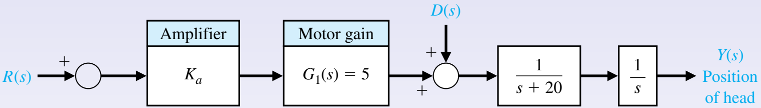
Sequential design example: Disk drive read system

Disk drive read system

The open-loop model of head control system

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 1 \\ 0 & -20 \end{pmatrix} x + \begin{pmatrix} 0 \\ 5K_a \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x \end{aligned}$$

is shown in



Select state variables $x_1 = y(t)$ and $x_2 = \dot{y}(t)$, the full-state feedback controller is given in the form with $K_1 = 1$

$$u(t) = -Kx = -\begin{pmatrix} K_1 & K_2 \end{pmatrix} x = -\begin{pmatrix} 1 & K_2 \end{pmatrix} x$$

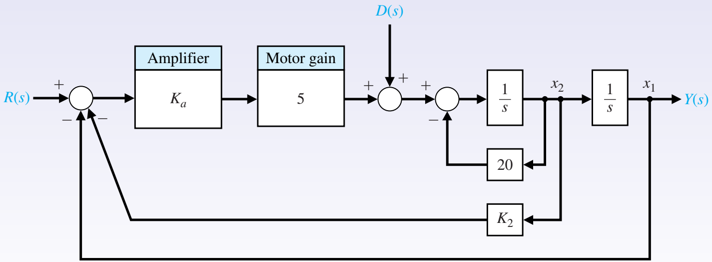
such that $y(t)$ accurately track the command $r(t)$.

Disk drive read system

The closed-loop system is given as

$$
\begin{aligned}
\dot{x} &= \begin{pmatrix} 0 & 1 \\ -5K_1K_a & -(20+5K_2K_a) \end{pmatrix} x + \begin{pmatrix} 0 \\ 5K_a \end{pmatrix} r \\
y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x
\end{aligned}
$$

and the corresponding block diagram is shown in the figure



Disk drive read system

then the actual closed-loop characteristic equation is

$$q(\lambda) = \lambda^2 + (20 + 5K_2K_a)\lambda + 5Ka = 0$$

Suppose the desired closed-loop characteristic equation is

$$q^*(\lambda) = \lambda^2 + 250\lambda + 192090 = 0$$

Therefore, we require that

$$\begin{aligned} 5K_a &= 19290 \qquad \Rightarrow \qquad K_a = 3858 \\ 20 + 5K_2K_a &= 250 \qquad \Rightarrow \qquad K_2 = 0.012 \end{aligned}$$

Summary

1. In this chapter, the design of control system in the time domain is examined.
2. The three-step design procedure for constructing state variable compensators is presented.
3. The optimal design of a system using state variable feedback is considered with an integral performance index.
4. Finally, internal model design is discussed.