现代控制理论基础
Fundamentals of Modern Control Theory

Chapter 13. Digital Control systems

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Introduction

Preview

A digital computer may serve as a compensator or a controller in a feedback control system.

Since the computer receives and processes data only at specific intervals, then the time series of signal and the discrete system are derived.

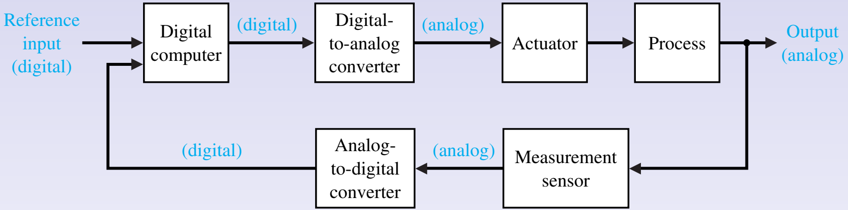
The time series, called **sampled data**, can be transformed to the s-domain, then to the z-domain by the relation $z = e^{sT}$.

The z-transform of transfer function is used to analyze the stability and transient response of digital control system.

The root locus method is utilized to determine the location of the roots of the characteristic equation of digital control system.

Introduction of the chapter

The block diagram of a single-loop digital control system



The digital computer receives the error in digital form and operates on the signal in digital form, and provides an output also in digital form.

A **digital control system** uses digital signals and digital computer to control a process.

离散系统的基本概念

基本概念

Definition：连续系统

控制系统中所有信号都是时间变量的连续函数的系统。

Definition：离散系统

控制系统中有一处或多处信号是一串脉冲序列或数字序列的系统。称系统中离散信号为脉冲序列形式的离散系统为采样控制系统；称系统中离散信号为数字序列形式的离散系统为数字控制系统（计算机控制系统）。

基本概念

Definition：采样与采样器

称把连续信号转变为脉冲序列的过程为采样。实现采样过程的装置称为采样器（采样开关）。

Definition：复现与保持器

称把脉冲序列转变为连续信号的过程为信号复现过程。实现复现过程的装置称为保持器。

Definition：采样周期

对连续信号在时间轴上按一定间隔进行采样，该采样间隔为采样周期。称采样间隔恒定不变的采样为周期采样；若采样间隔是时变的，称为非周期采样；若采样间隔是随机的，称为随机采样。

基本概念

Definition：A/D转换器

把连续的模拟信号转换为离散数字信号的装置。A/D转换包括采样过程和量化过程(编码过程)。

Definition：D/A转换器

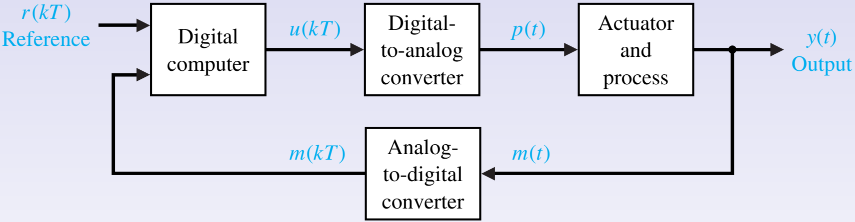
把离散数字信号转换为连续的模拟信号的装置。D/A转换包括解码过程和复现过程。

如果A/D转换器有足够的字长来表示数字信号，且量化单位足够小，故由量化引起的幅值误差可忽略。此外，若认为采样编码过程瞬时完成，可用理想脉冲序列代替数字信号。相应地A/D转换器可以用理想采样开关来表示。

Sampled-data systems

Sampling period and sampled data

Assume that all numbers that enter or leave the computer at the same fixed period, T, called the sampling period (采样周期).



In above system, the reference input is a sequence of sampled values $r(kT)$. Similarly, the variables $r(kT)$, $m(kT)$ and $u(kT)$ are discrete signals in contrast to continuous time function $r(t)$, $m(t)$ and $u(t)$.

Sampling period and sampled data

Sampled data

Sampled data (or a discrete signal) are datas obtained from the system variables only at discrete intervals, and denoted as $x(kT)$.

仅在时间点上获得的系统变量 $x(t)$ 的数据，被称为采样数据或离散信号，记作 $x(kT)$.

Sampler

Sampler

A sampler is basically a switch that closes every T seconds for one instant of time.

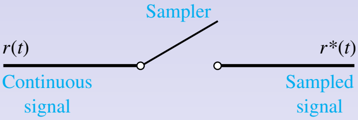
采样器就是一个每间隔 T 秒瞬时闭合一次的采样开关。其功能就是将连续信号转换为脉冲序列（离散信号）。

Ideal Sampler

若采样器在采样时，其采样瞬时的脉冲幅值，等于相应采样瞬时连续信号的幅值，且采样持续时间趋于零，称该采样器为理想采样器（理想采样开关）。

Sampler

Consider the following ideal sampler



the input is $r(t)$ and output $r^*(t)$ is called the **sampled data series (采样序列)**.

At current sample time $t = nT$, the present value of $r^*(t)$ is $r(kT)$, then

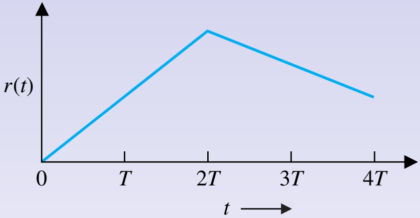
$$r^*(t) = r(kT)\delta(t - kT)$$

For all time of t, the sampled data series can be denoted as

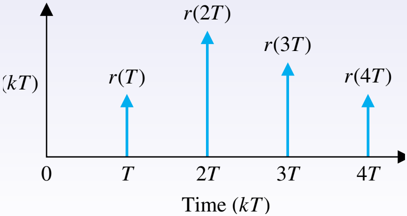
$$r^*(t) = \sum_{k=0}^{\infty} r(kT)\delta(t - kT)$$

Depiction of a sampled signal

For the input $r(t)$,

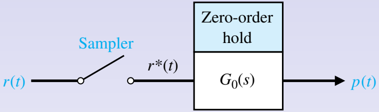


the sampled signal $r^*(t)$ is a series of impulses represented by vertical arrows with magnitude $r(kT)$.

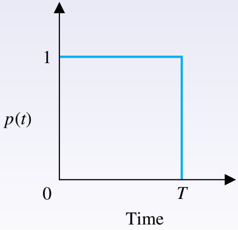


Zero-order hold

The **Zero-order hold** (零阶保持器) is the counterpart of the sampler, the **Zoh** converts the discrete signal into continuous time signal.

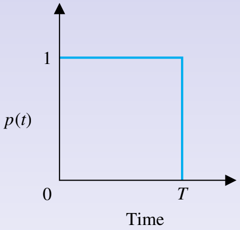


For $kT \leq t < (k+1)T$, Zoh takes the value $r(kT)$ and holds it constant.



Zero-order hold

The **Zero-order hold** (零阶保持器) 在理想单位脉冲信号 $\delta(t)$ 作用下其输出



可表示为两个单位阶跃函数之和:

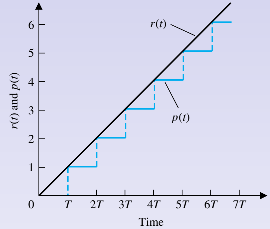
$$p(t) = 1(t) - 1(t - T)$$

Thus, the transfer function of Zoh is

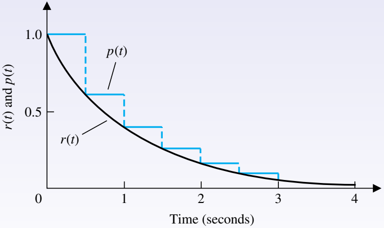
$$G_0(s) = \frac{1 - e^{-sT}}{s}$$

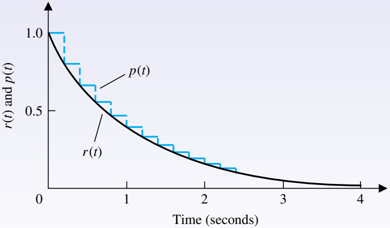
The responses of sampler and Zoh

The response of an sampler and a Zoh for a ramp input.



The responses for an exponential input are





The output of Zoh will approach to the input as T tends to zero.

采样信号的频谱与香农采样定理

令连续信号 $r(t)$ 的采样信号为 $r^*(t)$ ，由采样信号

$$r^*(t) = \sum_{k=0}^{\infty} r(kT)\delta(t-kT)$$

可得采样信号的傅氏变换

$$r^*(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} r[j(\omega + k\omega_s)]$$

其中 T 为采样周期， $\omega_s = 2\pi/T$ 为采样角频率。定义 ω_h 为连续信号频谱 $|r(j\omega)|$ 中的最大角频率。则

采样信号的频谱 $|r^*(j\omega)|$

采样信号的频谱 $|r^*(j\omega)|$ ，是以采样角频率 ω_s 为周期的无穷多个频谱之和。每一个频谱与连续信号的频谱 $|r(j\omega)|$ 形状一致，仅在幅值上变化了 $1/T$ 倍。

采样信号的频谱与香农采样定理

Theorem: 香农采样定理

如果采样器的输入信号 $r(t)$ 具有有限带宽，并且有直到 ω_h 的频率分量，则使信号 $r(t)$ 完满地从采样信号 $r^*(t)$ 中恢复过来的采样周期 T （采样频率 ω_s ）必须满足如下条件：

$$T \leq \frac{2\pi}{2\omega_h} \quad (\omega_s \geq 2\omega_h)$$

Precision of digital control system

The precision of the digital computer and the associated signal converters is limited.

Precision is the degree of exactness or discrimination with which a quantity is stated.

The precision of the computer is limited by **a finite word length**.

The precision of the analog-to digital converter is limited by the ability to store its output, then the converted signal includes an **amplitude quantization error** (幅值量化误差).

When the quantization error and the error due to a computer's finite word length are small relative to the amplitude of the signal, the precision limitation can be neglected.

The z-transform

Definition of the z-transform

The output $r^*(t)$ of ideal sampler, is a series of impulses with magnitudes $r(kT)$

$$r^*(t) = \sum_{k=0}^{\infty} r(kT)\delta(t - kT),\ t \geq 0$$

Using Laplace transform, to have

$$\mathcal{L}[r^*(t)] = \sum_{k=0}^{\infty} r(kT)\mathcal{L}[\delta(t - kT)] = \sum_{k=0}^{\infty} r(kT)e^{-kTs}$$

Now obtain an infinite series of the factors of e^{Ts} and its powers, define

$$z = e^{Ts}$$

then a new transform based on Laplace transform is derived , called the z-**transform**.

Definition of the z-transform

The z-transform of $r^*(t)$

$$R(z) = Z[r(t)] = Z[r^*(t)] = \sum_{k=0}^{\infty} r(kT) z^{-k}$$

The z-transform of the unit step function

$$u(t) = 1, u^*(t) = \sum_{k=0}^{\infty} 1 \cdot \delta(t - kT)$$

$$\begin{aligned} Z[u(t)] &= Z[u^*(t)] = \sum_{k=0}^{\infty} 1 \cdot z^{-k} \\ &= \frac{1}{1 - z^{-1}} = \frac{z}{z - 1} \end{aligned}$$

In general, the z-transform of function $f(t)$, is

$$F(z) = Z[f(t)] = Z[f^*(t)] = \sum_{k=0}^{\infty} f(kT) z^{-k}$$

Ex 13.1 Transform of an exponential

The z-transform of $f(t) = e^{-at}, t \geq 0,$

$$
\begin{aligned}
F(z) &= Z[e^{-at}] = \sum_{k=0}^{\infty} e^{-akT} \cdot z^{-k} \\
&= \sum_{k=0}^{\infty} \left[e^{-aT} z^{-1}\right]^k \\
&= \frac{1}{1 - e^{-aT} z^{-1}} = \frac{z}{z - e^{-aT}}
\end{aligned}
$$

Ex Transform of the unit ramp

The z-transform of $f(t) = t, t \geq 0$,

$$F(z) = Z[t] = \sum_{k=0}^{\infty} kT \cdot z^{-k}$$

From

$$\begin{aligned} F(z) &= T(1z^{-1} + 2z^{-2} + 3z^{-3} + \cdots) \\ z^{-1}F(z) &= T(1z^{-2} + 2z^{-3} + 3z^{-4} + \cdots) \end{aligned}$$

Minus two sides of both equalities, to see

$$\begin{aligned} (1 - z^{-1})F(z) &= T(1z^{-1} + 1z^{-2} + 1z^{-3} + \cdots) \\ &= T\frac{z^{-1}}{1 - z^{-1}} \end{aligned}$$

Therefore

$$\begin{aligned} F(z) &= Z[t] = T\frac{z^{-1}}{(1 - z^{-1})^2} \\ &= \frac{Tz}{(z - 1)^2} \end{aligned}$$

Ex Transform of a sinusoid $f(t) = \sin \omega t$

Utilize the known z-transform of $Z[e^{-at}] = \frac{z}{z-e^{-aT}}$ and the equality

$$\sin \omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

to see

$$\begin{aligned}
Z\left[\sin \omega t\right] &= \frac{1}{2j}\left(Z\left[e^{j\omega t}\right] - Z\left[e^{-j\omega t}\right]\right) \\
&= \frac{1}{2j}\left(\frac{z}{z-e^{j\omega T}} - \frac{z}{z-e^{-j\omega T}}\right) \\
&= \frac{1}{2j}\frac{z(e^{j\omega T} - e^{-j\omega T})}{z^2 - z(e^{j\omega T} + e^{-j\omega T}) + 1} \\
&= \frac{1}{2j}\frac{2jz\sin \omega T}{z^2 - 2z\cos \omega T + 1} \\
&= \frac{z \sin \omega T}{z^2 - 2z\cos \omega T + 1}
\end{aligned}$$

A table of z-transform and Laplace transform of some important time functions is given in Table 13.1.

Properties of the z-transform

- $Z\left[\alpha x_1(t) + \beta x_2(t)\right] = \alpha X_1(z) + \beta X_2(z)$
- $Z\left[x(t - nT)\right] = z^{-n} X(z)$
- $Z\left[x(t + nT)\right] = z^{n}\left[X(z) - \sum_{k=0}^{n-1} x(kT) z^{-k}\right]$
- $Z\left[t x(t)\right] = -Tz \frac{d}{dz} X(z)$
- $Z\left[e^{-at} x(t)\right] = X(ze^{aT})$
- $\lim_{z\to\infty} X(z) = x(0)$, if the limit exists
- $\lim_{z\to 1} (z - 1)X(z) = x(\infty)$, if $x(\infty)$ exists
- $$Z\left[G_0(s)G(s)\right] = (1 - z^{-1}) Z\left[G(s)/s\right]$$

  where $G_0(s) = (1 - e^{-sT})/s$ is the transfer function of the Zoh
- Denotes $R^*(s) = \mathcal{L}[r^*(t)]$, thus the z-transform of $r^*(t)$

  $$R(z) = Z\left[R(s)\right] = R^*(s)\big|_{e^{sT}=z}$$
- $Z\left[G(s)R^*(s)\right] = G(z)R(z)$

z 反变换

z 反变换，是已知 z 变换的表达式 $X(z)$ ，求取相应的离散序列 $x(kT)$ 的过程。记为

$$x(kT) = Z^{-1}[X(z)]$$

常用 z 反变换方法有：

1. 部分分式法（查表法）
2. 幂级数法
3. 反演积分法（留数法）

z 反变换: 部分分式法（查表法）

部分分式法（查表法）只能对部分 $X(z)$ 进行反变换。设 $X(z)$ 无重极点，故可将 $X(z)/z$ 通过留数定理展开为

$$\frac{X(z)}{z} = \sum_{i=1}^{m} \frac{A_i}{z - z_i} \quad \Rightarrow \quad X(z) = \sum_{i=1}^{m} \frac{A_i z}{z - z_i}$$

然后逐项查表得到

$$x_i(kT) = Z^{-1}\left[\frac{A_i z}{z - z_i}\right], i = 1, 2, \cdots, m$$

最后得到采样序列

$$x^*(t) = \sum_{k=0}^{\infty} \sum_{i=1}^{m} x_i(kT)\delta(t - kT)$$

z 反变换: 幂级数法

将 $X(z)$ 表示为两个多项式之比

$$X(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}}, \quad m \leq n$$

再对上式做综合除法可得

$$X(z) = c_0 + c_1 z^{-1} + \cdots + c_k z^{-k} + \cdots = \sum_{k=0}^{\infty} c_k z^{-k}$$

可得采样序列

$$x^*(t) = \sum_{k=0}^{\infty} c_k \delta(t - kT)$$

z 反变换: 留数法

$X(z)$ 对应的采样序列为

$$x(kT) = \sum_{i=1}^{m} \mathrm{Res}[X(z)z^{k-1}]_{z \to z_i}$$

其中 $\mathrm{Res}[X(z)z^{k-1}]_{z \to z_i}$ 表示函数 $X(z)z^{k-1}$ 在 z_i 处的留数。若 z_i 为单根，则

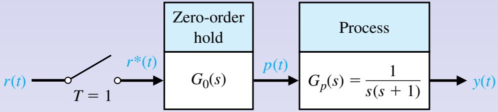
$$\mathrm{Res}[X(z)z^{k-1}]_{z \to z_i} = \lim_{z \to z_i} (z - z_i)X(z)z^{k-1}$$

若 z_i 为 m 重根，则

$$\mathrm{Res}[X(z)z^{k-1}]_{z \to z_i} = \frac{1}{(m-1)!} \lim_{z \to z_i} \frac{d^{m-1}(z - z_i)^m X(z)z^{k-1}}{dz^{m-1}}$$

Ex 13.3. Transfer function of an open-loop system

Consider the open-loop with a Zoh



The transfer function of Zoh is

$$G_0(s) = \frac{(1 - e^{-sT})}{s}$$

The s transfer function of the output $y(t)$ is

$$Y(s) = G_0(s)G_p(s)R^*(s)$$

then the z transfer function of the output $y(t)$ is

$$\begin{aligned} Y(z) &= Z\left[Y(s)\right] = Z\left[G_0(s)G_p(s)R^*(s)\right] \\ &= Z\left[G_0(s)G_p(s)\right]R(z) \end{aligned}$$

Ex 13.3. Transfer function of an open-loop system

Therefore the z transfer function of the open-loop is

$$
\begin{aligned}
G(z) &= \frac{Y(z)}{R(z)} = Z\left[G_0(s)G_p(s)\right] = (1-z^{-1})Z\left[G_p(s)/s\right] \\
&= (1-z^{-1})Z\left[\frac{1}{s^2(s+1)}\right] = (1-z^{-1})Z\left[\frac{1}{(s+1)} - \frac{1}{s} + \frac{1}{s^2}\right] \\
&= (1-z^{-1})\left[\frac{z}{z-e^{-T}} - \frac{z}{z-1} + \frac{Tz}{(z-1)^2}\right] \\
&= \frac{z-1}{z} z \frac{(z-1)^2 - (z-1)\left(z-e^{-T}\right) + T\left(z-e^{-T}\right)}{\left(z-e^{-T}\right)(z-1)^2} \\
&= \frac{(z-1)^2 - (z-1)\left(z-e^{-T}\right) + T\left(z-e^{-T}\right)}{\left(z-e^{-T}\right)(z-1)} \\
&= \frac{\left(T+e^{-T}-1\right)z + \left(1-e^{-T}-Te^{-T}\right)}{\left(z-e^{-T}\right)(z-1)}
\end{aligned}
$$

Ex 13.3. Transfer function of an open-loop system

When $T = 1$, $r(t) = \delta(t)$, then the unit impulse response is

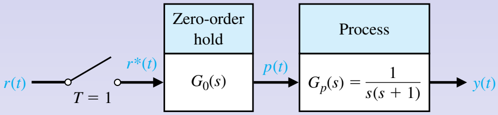
$$
\begin{aligned}
Y(z) &= \frac{e^{-1}z + \left(1 - 2e^{-1}\right)}{z^2 - (1 + e^{-1})z + e^{-1}} = \frac{0.367\,88z + 0.264\,24}{z^2 - 1.\,367\,9z + 0.367\,88} \\
&= 0.367\,88z^{-1} + 0.767\,46z^{-2} + 0.914\,48z^{-3} + 0.968\,57z^{-4} + \cdots
\end{aligned}
$$

which obtained by dividing the denominator into the numerator:

$$
\begin{array}{r|lllll}
 & 0.367\,88z^{-1} & +0.767\,46z^{-2} & +0.914\,48z^{-3} & +0.968\,57z^{-4} & +\cdots \\
\cline{2-6}
z^2 - 1.\,367\,9z + 0.367\,88 & 0.367\,88z & +0.264\,24 & & & \\
 & 0.367\,88z & -0.503\,22 & 0.13534z^{-1} & & \\
\cline{2-6}
 & & 0.767\,46 & -0.13534z^{-1} & & \\
 & & 0.767\,46 & -1.04980z^{-1} & 0.282\,33z^{-2} & \\
\cline{3-6}
 & & & 0.914\,48z^{-1} & -0.28233z^{-2} & \\
 & & & 0.914\,48z^{-1} & -1.2509z^{-2} & 0.336\,42z^{-3} \\
\cline{4-6}
 & & & & 0.968\,57z^{-2} & -0.336\,42z^{-3}
\end{array}
$$

The TF in the z domain of the open-loop system

The following system



and a fictitious output-sampler operates synchronously with the same sampling period.

The TF in the z domain of the open-loop system is

$$\frac{Y(z)}{R(z)} = G(z) = Z[G_0 G_p(s)]$$

represented by

The TF in the z domain of the open-loop system

Definition：脉冲传递函数的定义

对于线性定常系统，在零初始条件下，脉冲传递函数定义为：系统输出采样序列的 z 变换与输入采样序列的 z 变换之比。

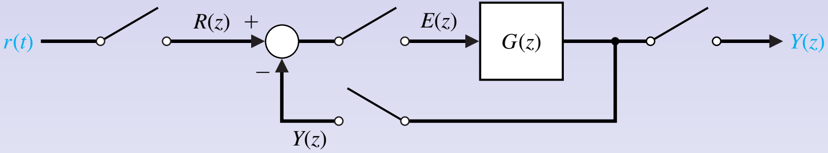
$$G(z) = \frac{Z(y^*(t))}{Z(r^*(t))} = \frac{Y(z)}{R(z)}$$

零初始条件：当 $t \leq 0$ 时，$r(0), r(-T), \cdots$ 与 $y(0), y(-T), \cdots$ 均等于零。

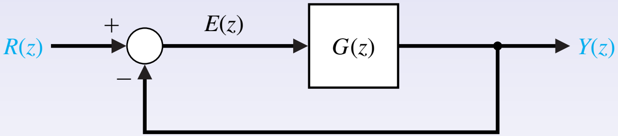
Closed-loop feedback sampled-Data systems

TF closed-loop sampled-data system

The closed-loop sampled-data system is



the corresponding system in z domain



The TF of the closed-loop sampled-data system

$$T(z) = \frac{Y(z)}{R(z)} = \frac{G(z)}{1 + G(z)}$$

Ex 13.4. TF closed-loop sampled-data system

The TF $G(z)$ of the open-loop system is given as that in Ex 13.3 for $T = 1$s, that is

$$G(z) = \frac{0.3678z + 0.2644}{z^2 - 1.3678z + 0.3678}$$

then

$$T(z) = \frac{Y(z)}{R(z)} = \frac{G(z)}{1 + G(z)} = \frac{0.3678z + 0.2644}{z^2 - z + 0.6322}$$

and

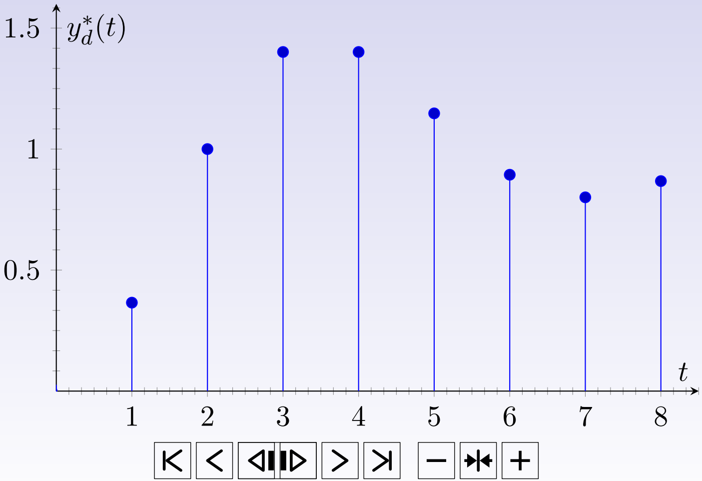
$$\begin{aligned} Y(z) &= T(z)R(z) = \frac{0.3678z + 0.2644}{z^2 - z + 0.6322}\frac{z}{z-1} \\ &= 0.3678z^{-1} + 1.0z^{-2} + 1.3997z^{-3} \\ &\quad +1.3997z^{-4} + 1.147z^{-5} + 0.8943z^{-6} + \cdots \end{aligned}$$

hence

$$\begin{aligned} y^*(t) &= 0.3678\delta(t-T) + 1.0\delta(t-2T) + 1.3997\delta(t-3T) \\ &\quad +1.3997\delta(t-4T) + 1.147\delta(t-5T) + 0.8943\delta(t-6T) + \cdots \end{aligned}$$

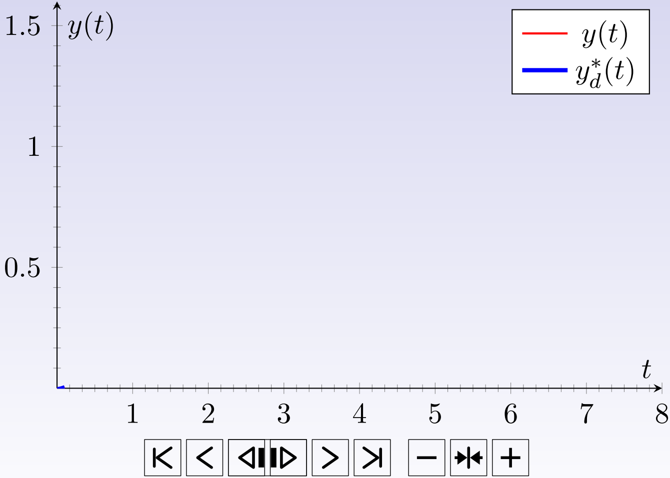
Ex 13.4. TF closed-loop sampled-data system

The response of the system with a output-sampler are shown as follows in animation:



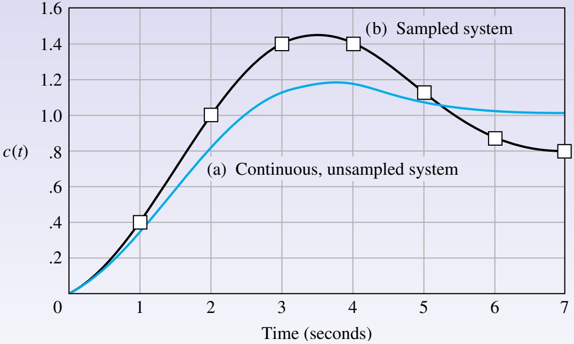
Ex 13.4. TF closed-loop sampled-data system

The responses are shown as follows in animation:



Ex 13.4. TF closed-loop sampled-data system

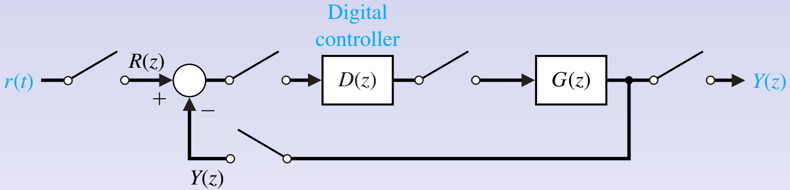
The responses of the closed-loop sampled-data system and the original continuous time system areas follows:



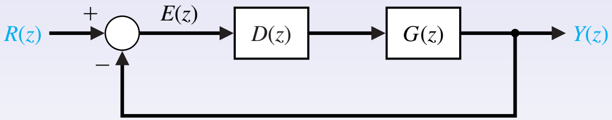
We see that the performance of sampled-data system is deteriorated.

TF of closed-loop system with a digital controller

The closed-loop sampled-data system with a digital controller



and the corresponding sampled-data z-transform model



The TF of the closed-loop sampled-data system

$$T(z) = \frac{Y(z)}{R(z)} = \frac{D(z)G(z)}{1 + D(z)G(z)}$$

Stability analysis in the z-plane

For the linear time invariant system, if all poles lie in the left-half s-plane, then it is stable.

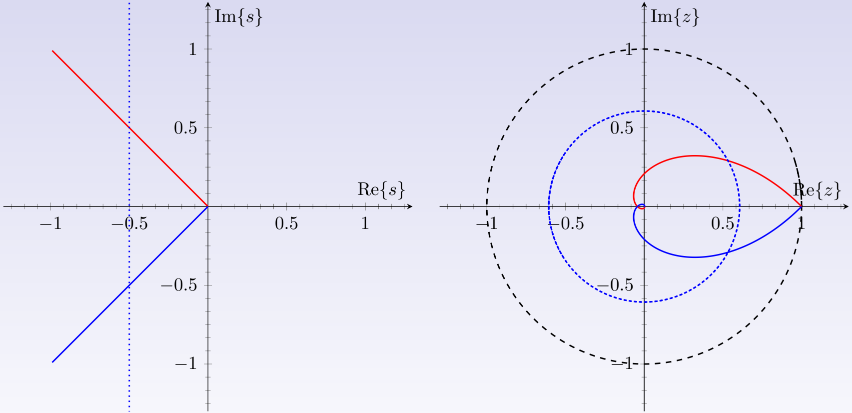
For the corresponding sampled data system, related with $z = e^{sT}$, is stable if $|z| = |e^{sT}| < 1$.

Stability of sampled data system

A sampled data system is stable if all the poles of the z-domain transfer function lie within the unit circle of the z-plane.

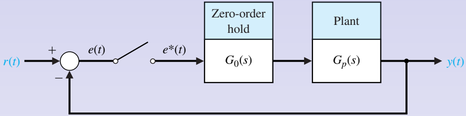
若采样系统的所有极点均位于 z 平面的单位圆内，则该系统是稳定的。

Stability analysis in the z-plane



Ex13.5 Stability of a sampled data system

Consider the system with $G_p = \frac{K}{s(s+1)}$ and $T = 1$s.



Recalling the results obtained in Ex 13.3, to have

$$G(z) = K\frac{0.3678z + 0.2644}{z^2 - 1.3678z + 0.3678} = K\frac{az + (1-2a)}{z^2 - (1+a)z + a}$$

where $a = 0.3678$, $K > 0$. Then the closed loop TF is

$$\begin{aligned} T(z) &= \frac{Y(z)}{R(z)} = \frac{G(z)}{1+G(z)} \\ &= \frac{K\left(az + (1-2a)\right)}{z^2 + \left[Ka - (1+a)\right]z + K(1-2a) + a} \end{aligned}$$

then the characteristic equation

$$q(z) = z^2 + \left[Ka - (1+a)\right]z + K(1-2a) + a$$

Ex13.5 Stability of a sampled data system

Statement

The 2nd-order polynomials equation $q(z) = 0$ has all its roots within the unit circle if and only if the inequalities hold

$$|q(0)| < 1,\ q(1) > 0,\ q(-1) > 0$$

With this statement, when

$$\begin{aligned}
|K(1-2a)+a| &< 1 \Rightarrow K < \frac{1-a}{(1-2a)} = 2.3911 \\
1 + [Ka-(1+a)] + K(1-2a) + a &> 0 \Rightarrow K > 0 \\
1 - [Ka-(1+a)] + K(1-2a) + a &> 0 \Rightarrow K < \frac{2+2a}{(3a-1)} = 26.3972
\end{aligned}$$

hold, i.e $0 < K < 2.3911$, the closed-loop sampled data system is stable.

Stability of a sampled data system：w 变换法

w 域中的劳斯稳定判据

若采样系统 z 域的特征多项式为 $q(z)$，对其作如下双线性变换

$$z = \frac{w+1}{w-1} \quad \Longleftrightarrow \quad w = \frac{z+1}{z-1}$$

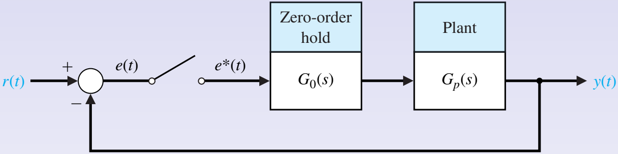
可得 w 域的特征多项式为 $p(w)$，根据 w 域中的特征方程系数，可以直接应用劳斯稳定判据判断采样系统的稳定性。

Performance of a sampled-data, second-order system

The relationships between performance and K, T are illustrated with a second-order system.

Stability performance of a 2nd-order system

Consider the system



with

$$G_p(s) = \frac{K}{s(\tau s + 1)}$$

Stability performance of a 2nd-order system

The open loop z domain TF is

$$
\begin{aligned}
G(z) &= Z[G_0(s)G_p(s)] = \left(1 - z^{-1}\right) Z[G_p(s)/s] \\
&= (1 - z^{-1}) Z\left[\frac{K}{s^2(\tau s + 1)}\right] \\
&= K(1 - z^{-1}) Z\left[\frac{\tau}{(s + 1/\tau)} - \frac{\tau}{s} + \frac{1}{s^2}\right] \\
&= K(1 - z^{-1}) \left[\frac{\tau z}{z - e^{-T/\tau}} - \frac{\tau z}{z - 1} + \frac{T z}{(z - 1)^2}\right] \\
&= K \frac{\tau (z - 1)^2 - \tau (z - 1)(z - E) + T (z - E)}{(z - 1)(z - E)} \\
&= K \frac{(T - \tau + \tau E) z + (-TE + \tau - E\tau)}{z^2 - (1 + E) z + E}
\end{aligned}
$$

where $E = e^{-T/\tau}$.

Stability performance of a 2nd-order system

Then the closed loop z domain TF is

$$T(z) = \frac{G(z)}{1+G(z)}$$

$$= \frac{K\left[\left(T-\tau\left(1-E\right)\right)z+\tau\left(1-E\right)-TE\right]}{z^2+z\{K\left[\left(T-\tau\left(1-E\right)\right)\right]-\left(1+E\right)\}+K\left[\tau\left(1-E\right)-TE\right]+E}$$

hence the characteristic equation

$$q(z) = z^2+z\{K\left[\left(T-\tau\left(1-E\right)\right)\right]-\left(1+E\right)\}+K\left[\tau\left(1-E\right)-TE\right]+E$$

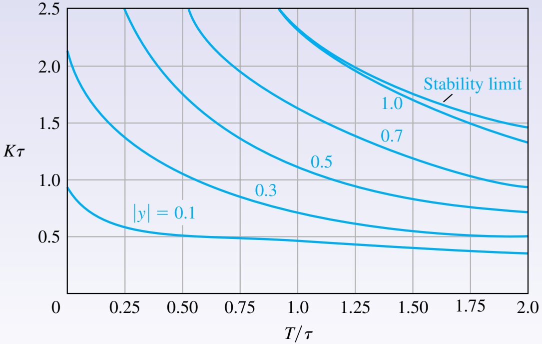
From the statement and assumption $T \ll \tau$, the closed loop sampled data system is stable $\Leftrightarrow$ the following inequalities hold

$$\begin{aligned}
|q(0)| &= |K\left[\tau\left(1-E\right)-TE\right]+E| < 1 \\
&\Rightarrow K\tau < \frac{1-E}{(1-E)-\frac{T}{\tau}E} \\
q(1) &= KT(1-E) > 0, \\
q(-1) &= 2+2E+K\left[2\tau(1-E)-T(1+E)\right] > 0,
\end{aligned}$$

Stability performance of a 2nd-order system

With the first inequality $K\tau < \frac{1-E}{(1-E)-\frac{T}{\tau}E}$, the maximum gain permissable for $\tau = 1$ is given in the table:

	T/τ	0	0.1	0.5	1	2
maximum	$K\tau$	∞	20.34	4.36	2.39	1.456

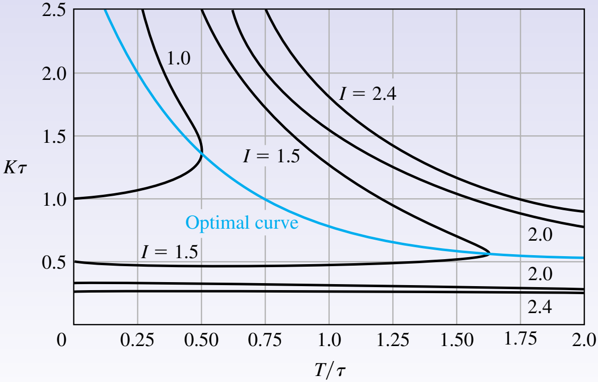


The maximum overshoot $|y|$ respects to $K\tau$ vs T/τ for the input $1(t)$ is shown in the above figure.

Stability performance of a 2nd-order system

For the given performance criterion, integral of squared error

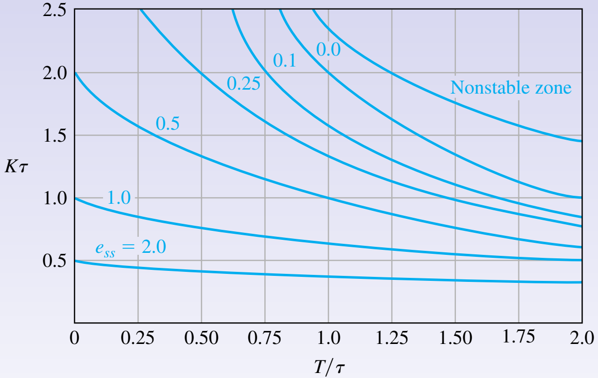
$$I = \frac{1}{\tau}\int_0^\infty e^2(t)dt$$



The minimization of I for $T/\tau = 0.75$, requires $K\tau = 1$.

Stability performance of a 2nd-order system

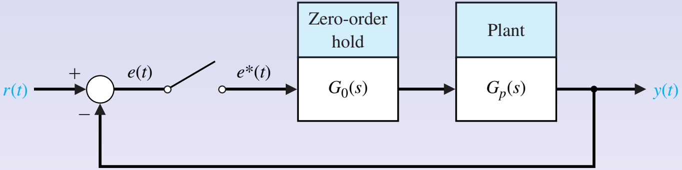
The steady-state error for a unit ramp $r(t) = t$



For a given T/τ, the steady-state error can be reduced by increasing $K\tau$, but doing so will cause greater overshoot and settling time for a step input.

Ex 13.6 Design of a sampled system

Consider the closed-loop sampled system



with

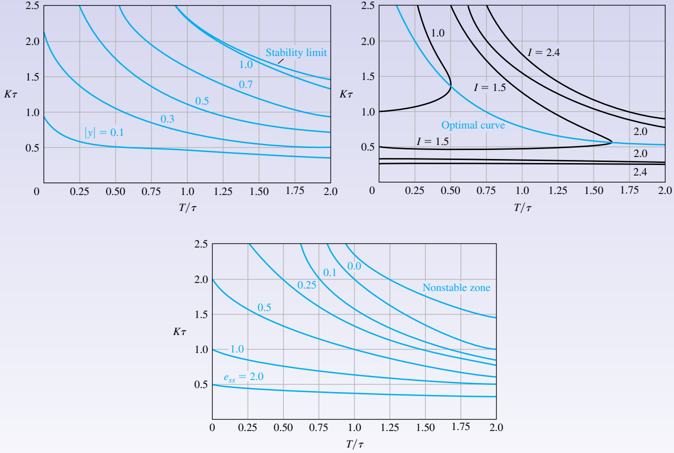
$$G_p(s) = \frac{K}{s\,(0.1s+1)\,(0.005s+1)}$$

to select T and K for suitable performance.

As an approximation, the effects of $\tau_2 = 0.005$s is neglected, because it is only 5% of the primary time constant $\tau_1 = 0.1$s.

Ex 13.6 Design of a sampled system

Then regarding the figure



for $|y| = 30\%$, select $T/\tau = 0.25, T = 0.025$s, yielding $K\tau \approx 1.4$; the stead-state error $e_{ss} \approx 0.6$; the performance criterion $I < 1.0$.

离散系统稳态误差的计算

对于闭环稳定的有误差采样的离散系统，其稳态误差可以通过 z 变换的终值定理来求解：

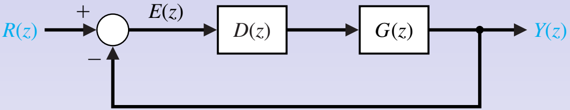
$$e(\infty) = \lim_{t\to\infty} e^*(t) = \lim_{z\to 1}(z-1)E(z) = \lim_{z\to 1}(1-z^{-1})E(z)$$

其中：$E(z) = Z[e^*(t)]$.

Closed-loop systems with digital computer compensation

Performance improvement with digital controller

A closed-loop sampled system with a digital controller



The closed-loop TF is

$$\frac{Y(z)}{R(z)} = T(z) = \frac{D(z)G(z)}{1 + D(z)G(z)}$$

and the TF relating error to input

$$\frac{E(z)}{R(z)} = \frac{1}{1 + D(z)G(z)}$$

Recalling the results obtained in Ex 13.5 where $T = 1$s, the open-loop TF is

$$G(z) = \frac{0.3768(z + 0.7189)}{(z - 1)(z - 0.3678)}$$

Performance improvement with digital controller

To cancel the pole of $G(z)$ at $z = 0.3678$, select

$$D(z) = \frac{K(z-0.3678)}{(z+r)}$$

furthermore

$$D(z) = \frac{1.359(z-0.3678)}{(z+0.240)}$$

then

$$D(z)G(z) = 0.50\frac{(z+0.7189)}{(z-1)(z+0.240)}$$

The the unit step response of uncompensated closed-loop system

$$\begin{aligned} Y(z) &= \frac{G(z)}{1+G(z)}R(z) \\ &= \frac{0.3768(z+0.7189)}{(z-1)(z-0.3678)+0.3768(z+0.7189)}\frac{z}{z-1} \\ &= 0.376\,8z^{-1} + 1.021\,1z^{-2} + 1.418\,9z^{-3} + 1.401\,7z^{-4} + \\ &\quad 1.130\,5z^{-5} + 0.872\,79z^{-6} + 0.790\,58z^{-7} + 0.873\,71z^{-8} + \cdots \end{aligned}$$

Performance improvement with digital controller

then

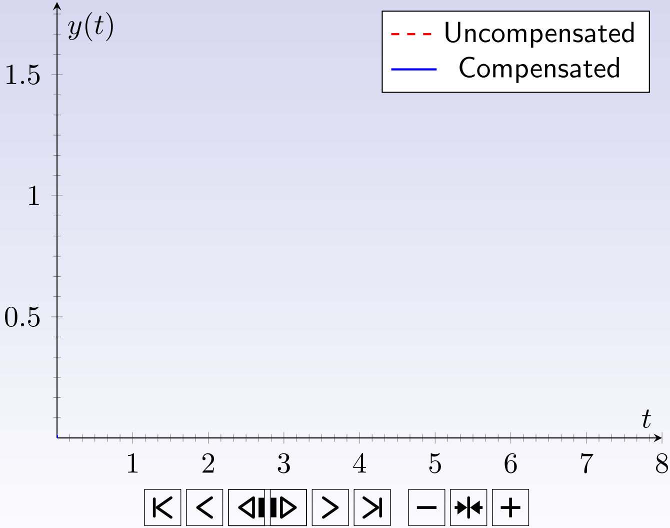
$$\begin{aligned} y^*(t) &= 0.376\,8\delta(t-T) + 1.021\,1(t-2T) + 1.418\,9(t-3T) + \\ &\quad 1.401\,7(t-4T) + 1.130\,5(t-5T) + 0.872\,79(t-6T) \\ &\quad +0.790\,58(t-7T) + 0.873\,71(t-8T) + \cdots \end{aligned}$$

The error response of the compensated closed-loop system

$$\begin{aligned} E(z) &= \frac{R(z)}{1+D(z)G(z)} \\ &= \frac{(z-1)(z+0.240)}{(z-1)(z+0.240)+0.5(z+0.7189)}\frac{z}{z-1} \\ &= 1.0 + 0.5z^{-1} + 0.010\,55z^{-2} - 5.698\,2\times10^{-2}z^{-3} \\ &\quad -1.607\,6\times10^{-2}z^{-4} + 2.626\,9\times10^{-3}z^{-5} + \\ &\quad 2.603\,2\times10^{-3}z^{-6} + 3.630\,5\times10^{-4}z^{-7} + \cdots \end{aligned}$$

Performance improvement with digital controller

The responses are shown as follows in animation:



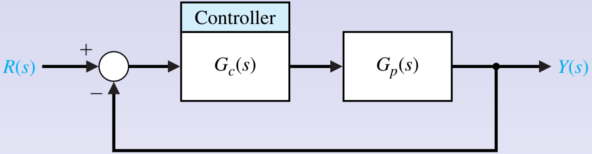
The methods for determining digital controller

It is beyond the objective of this book to discuss all the extensive design methods for the compensator $D(z)$, two methods are considered:

1. The $G_c(s)$ to $D(z)$ conversion method.
2. The root locus z-plane method.

The $G_c(s)$ to $D(z)$ conversion method

1. Determining a controller $G_c(s)$ for a given plant $G_p(s)$ for the system



2. Converting the controller in the form

$$G_c(s) = K\frac{s+a}{s+b}$$

to $D(z)$ for a given sampling period T with the conversion formula

$$D(z) = C\frac{z-A}{z-B}$$

where

$$A = e^{-aT}, \quad B = e^{-bT}, \quad C\frac{1-A}{1-B} = K\frac{a}{b}$$

Ex 13.7 Design to meet a phase margin specification

Consider a system with a plant

$$G_p(s) = \frac{1740}{s(0.25s+1)}$$

From the approach introduced in Section 10.4, select lead compensator as

$$G_c(s) = \frac{5.6(s+50)}{(s+312)}$$

to achieve a phase margin of 45° with a crossover frequency $\omega_c = 125$ rad/s.

Setting $T = 0.001$ s, obtain

$$A = e^{-0.05} = 0.951, B = e^{-0.312} = 0.732, C = 4.908\,4$$

then the digital controller is given as

$$D(z) = 4.908\,4 \frac{z-0.951}{z-0.732}$$

Principle in determining a sampling period T

Usually select T as

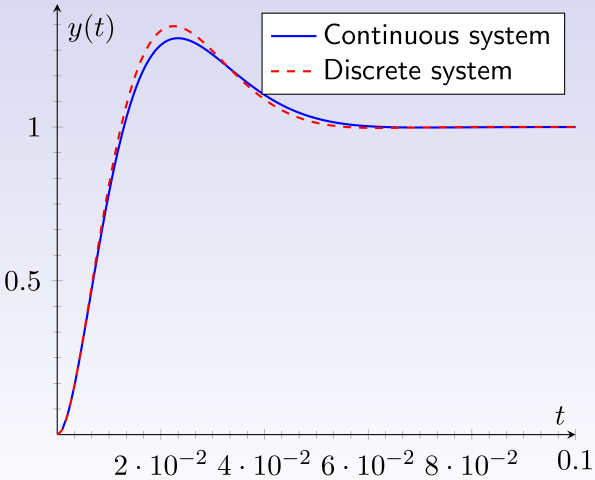
$$T \approx \frac{1}{10}\frac{2\pi}{\omega_B}$$

where ω_B is the bandwidth of the closed-loop continuous system.

In Ex 13.7 $\omega_B = 180$ rad/s$>$ $\omega_c = 125$ rad, T should be chosen as $T = 0.1\frac{2\pi}{180} \approx 0.003$ s, note that $T = 0.001$ s is adopted in the Ex 13.7.

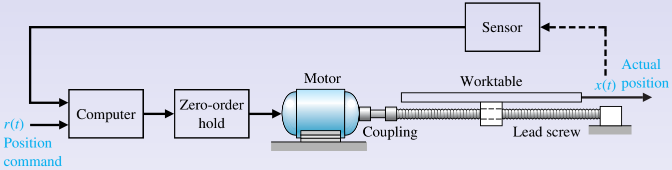
Ex 13.7. The response of the closed-loop system

The responses of continuous and discrete systems are shown as follows:

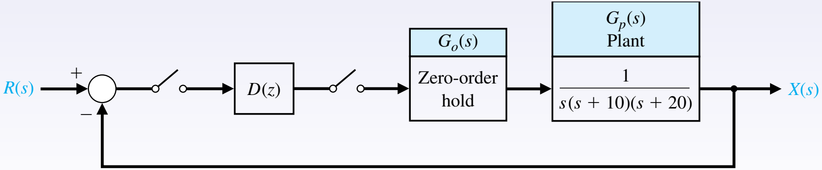


The design of a Worktable motion control system

The table motion control system

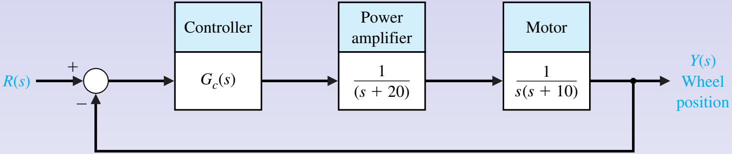


the system block diagram model



The design of the Worktable motion control system

The continuous time controller in the loop is shown in the figure



where the plant is

$$G_p(s) = \frac{1}{s(s+10)(s+20)}$$

The performance for two controllers

Compensator $G_c(s)$	K	Percent overshoot	Settling time	Rise time
K	700	5.0	1.12s	0.40s
$K\frac{s+11}{s+62}$	8000	5.0	0.60s	0.25s

The design of the Worktable motion control system

For continuous time controller

$$G_c(s) = 8000\frac{s+11}{s+62}$$

let sampling period $T = 0.01$s$\ll T_r = 0.25$s, then the digital controller is given as

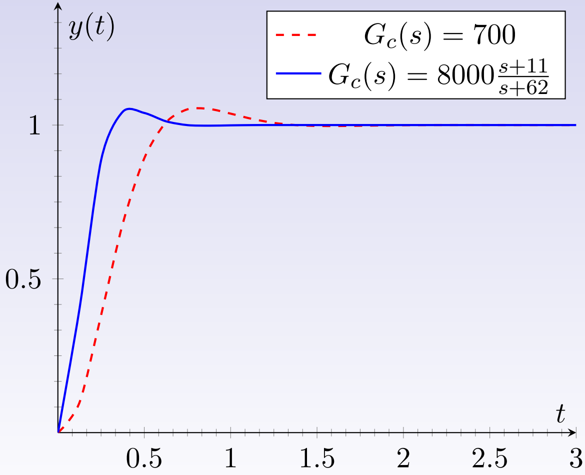
$$D(z) = C\frac{z-A}{z-B}$$

where $A = e^{-0.11} = 0.895\,8$, $B = e^{-0.62} = 0.537\,9$,$C = K\frac{a(1-B)}{b(1-A)} = 6294.\,5$.

The response is expected to be very similar to that obtained for the continuous system.

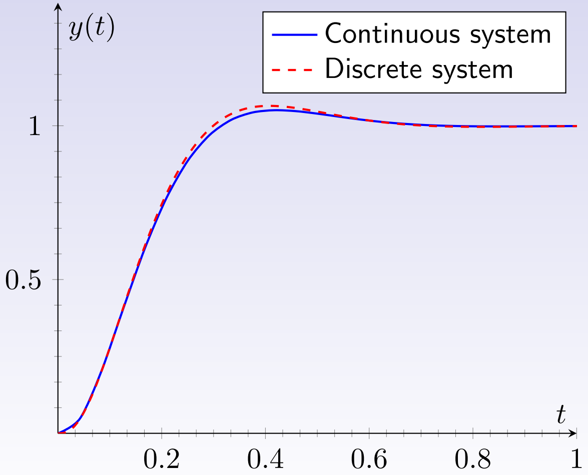
The design of the Worktable motion control system

The responses are shown as follows:



The design of the Worktable motion control system

The responses of continuous and discrete systems are shown as follows:



The root locus of digital control systems

Root locus in the z-plane

1. The root locus starts at the poles and progresses to the zeros.
2. The root locus lies on a section of the real axis to the left of an odd number of poles and zeros.
3. The root locus is symmetrical with respect to the real axis.
4. The root locus satisfies

$$1 + KG(z)D(z) = 0$$

5. The root locus may break away from the real, axis and may reenter the real axis. The points are determined from the equation

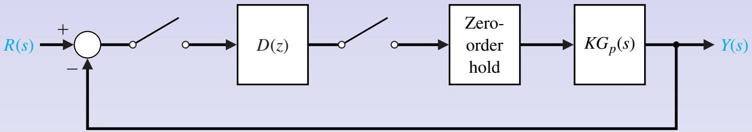
$$K = -\frac{1}{G(z)D(z)} =: F(z)$$

by solving the real solution $z = \sigma$ from equation

$$\frac{d}{d\sigma}F(\sigma) = 0$$

Ex 13.8 Root locus of a second-order system

Consider the system with $D(z) = 1$ and $G_p(s) = 1/s^2$



then

$$
\begin{aligned}
G(z) &= (1 - z^{-1})Z[G_p(s)/s] = (1 - z^{-1})Z[1/s^3] \\
&= (1 - z^{-1})\frac{T^2}{2}\frac{z(z+1)}{(z-1)^3} = \frac{T^2}{2}\frac{(z+1)}{(z-1)^2}
\end{aligned}
$$

Let $T = \sqrt{2}$s and plot the root locus, where

$$KG(z) = K\frac{(z+1)}{(z-1)^2}$$

Ex 13.8 Root locus of a second-order system

Since the characteristic equation is

$$1 + KG(z) = 1 + K\frac{(z+1)}{(z-1)^2} = 0$$

solve for K to obtain

$$K = -\frac{(z-1)^2}{(z+1)} =: F(z)$$

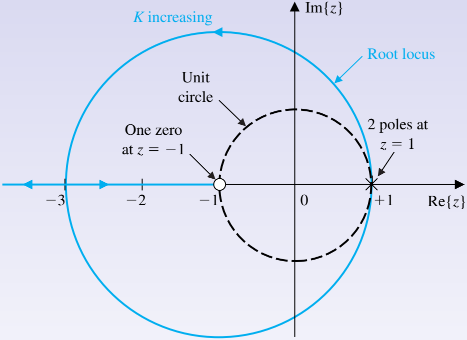
Then solve the equation

$$\begin{aligned} \frac{d}{dz}F(z) &= -\frac{2(z-1)(z+1) - (z-1)^2}{(z+1)^2} \\ &= -\frac{z^2 + 2z - 3}{(z+1)^2} = 0 \end{aligned}$$

to obtain real roots $z_1 = 1$ and $z_2 = -3$, therefore the locus leaves the repeated poles at $z_1 = 1$ (`breakaway point`) and reenters at $z_2 = -3$ (`entry point`).

Ex 13.8 Root locus of a second-order system

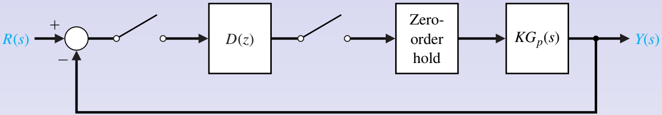
The rootlocus is shown in the following figure:



From the figure, to see that the system always has two roots outside the unit circle and is unstable for all $K > 0$.

Ex 13.9 Design of a digital compensator

Consider the system now with $D(z) = \frac{z-a}{z-b}$ and $G_p(s) = 1/s^2$



Let $T = \sqrt{2}$s, then

$$KD(z)G(z) = K\frac{(z-a)(z+1)}{(z-b)(z-1)^2}$$

Set $a = 1$ and $b = 0.2$, to have

$$KD(z)G(z) = K\frac{(z+1)}{(z-0.2)(z-1)}$$

Ex 13.9 Design of a digital compensator

Since characteristic equation is

$$1 + KD(z)G(z) = 1 + K\frac{(z+1)}{(z-0.2)(z-1)} = 0$$

solve for K to obtain

$$K = -\frac{(z-0.2)(z-1)}{(z+1)} =: F(z)$$

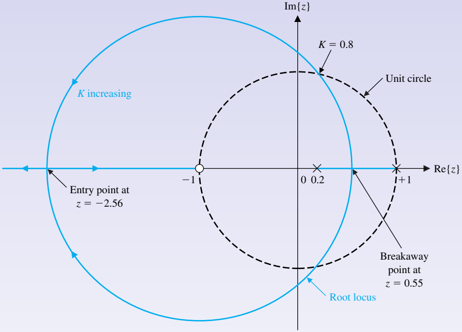
Then solve the equation

$$\begin{aligned}\frac{d}{dz}F(z) &= -\frac{(2z-1.2)(z+1)-(z-0.2)(z-1)}{(z+1)^2} \\ &= -\frac{z^2+2z-1.4}{(z+1)^2} = 0\end{aligned}$$

to obtain real roots $z_1 = 0.549\,19$ and $z_2 = -2.549\,2$, therefore the locus leaves the poles at $z_1 = 0.549\,19$ and reenters at $z_2 = -2.549\,2$.

Ex 13.9 Design of a digital compensator

From the root locus

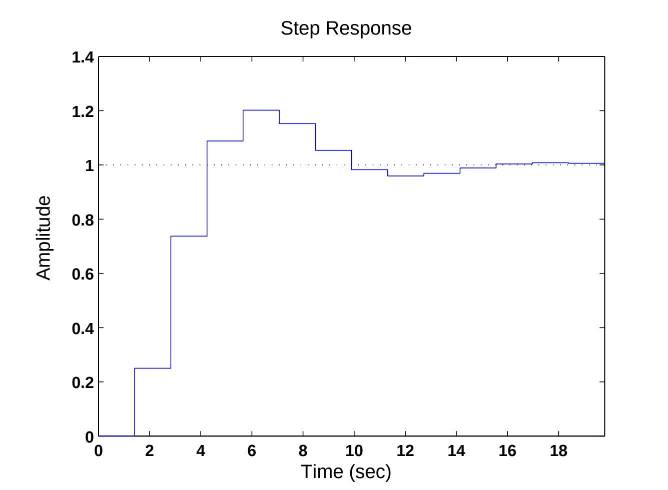


to see that the locus will transverse the unit circle from inside, so there exists a $K_{\max} > 0$, such that the systems is stable for $0 < K < K_{\max}$.

It can be verified that root locus is on the unit circle at $K \approx 0.8$, so $K_{\max} = 0.8$.

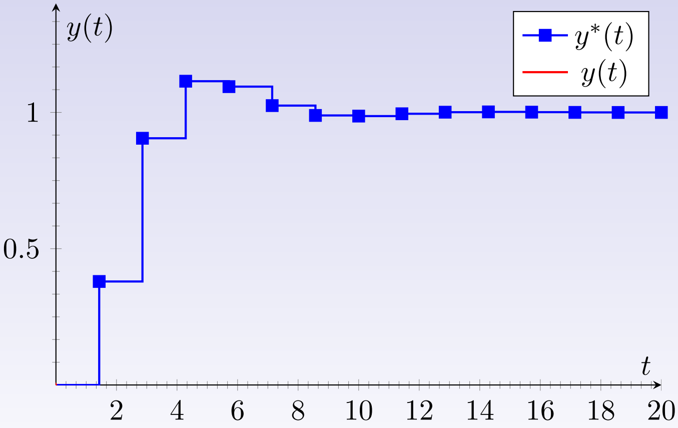
Ex 13.9 Design of a digital compensator

The unit step response is shown in the figure, where $K = 0.2059$.



Ex 13.9 Design of a digital compensator

The unit step response is shown in the figure, where $K = 0.2059$.



Ex 13.9 Design of a digital compensator

Improvement of design

Reconsider the system

$$KD(z)G(z) = K\frac{(z-a)(z+1)}{(z-b)(z-1)^2}$$

For more favorable system performance, the root locus can be improved by setting $a = 1$ and $b = -0.98$ such that

$$KD(z)G(z) = K\frac{(z+1)}{(z+0.98)(z-1)} \approx \frac{K}{(z-1)}$$

When $K = 1$,the transfer function of the closed loop system is

$$T(z) = \frac{Y(z)}{R(z)} = z^{-1}$$

Thus the response of the sampled system is the input step delayed by one sampling period.

The damping ratio versus pole distribution

The lines on the s-plane can be transformed into the z-plane, and the mapping between the s-plane and z-plane is obtained by the relation $z = e^{sT}$.

The lines of constant ζ on the s-plane are

$$s = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}, \quad 0 \leq \omega_n < \infty$$

Using the mapping of $z = e^{sT}$, we have

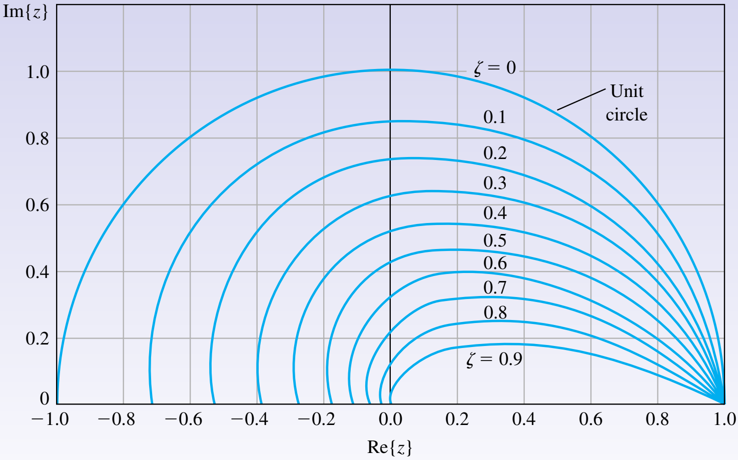
$$z = e^{-\zeta\omega_n T} e^{j\omega_n T\sqrt{1-\zeta^2}}$$

i.e.,

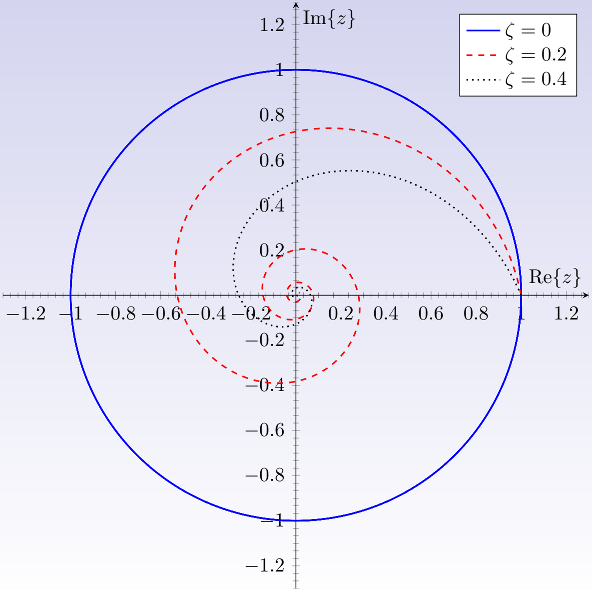
$$\begin{cases} \mathrm{Re}\{z\} = e^{-\zeta\omega_n T}\cos(\omega_n T\sqrt{1-\zeta^2}) \\ \mathrm{Im}\{z\} = e^{-\zeta\omega_n T}\sin(\omega_n T\sqrt{1-\zeta^2}) \end{cases}$$

where $0 \leq \omega_n T < \infty$.

The damping ratio versus pole distribution



The damping ratio versus pole distribution



Implementation of digital controllers

Implementation of digital controllers

Consider the PID controller with an s-domain TF

$$G_c(s) = \frac{U(s)}{X(s)} = K_1 + \frac{K_2}{s} + K_3 s$$

a digital implementation of $G_c(s)$ is determined by using a discrete approximation for the derivative and integration.

The discrete approximation for the derivative is given by

$$u(kT) = \dot{x}(kT) = \frac{1}{T}\left[x(kT) - x((k-1)T)\right]$$

so z-transform of the equation is

$$U(z) = \frac{1}{T}(1 - z^{-1})X(z) = \frac{z-1}{Tz}X(z)$$

Implementation of digital controllers

The discrete approximation for the integration is given by

$$u(kT) = u((k-1)T) + Tx(kT)$$

then z-transform of the equation is

$$U(z) = z^{-1}U(z) + TX(z)$$

that is

$$U(z) = \frac{1}{1-z^{-1}}TX(z) = \frac{Tz}{z-1}X(z)$$

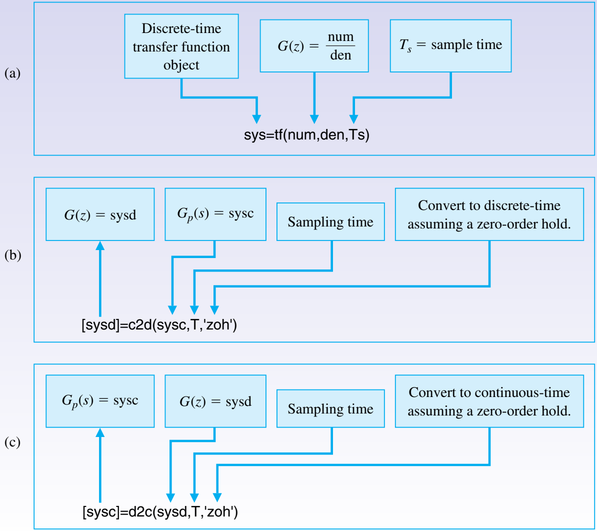
Hence, the z-domain TF of the PID controller is

$$G_c(z) = \frac{U(z)}{X(z)} = K_1 + K_2\frac{Tz}{z-1} + K_3\frac{z-1}{Tz}$$

Digital control systems using MATLAB

Function c2d and d2c

Function c2d converts continuous time systems to discrete time system, d2c is its counterpart.



an illustration of function c2d

```
% This script converts the transfer function
% Gp(s) = 1/s(s+1) to a discrete-time system
% with a sampling period of T=1 sec.
%
num=[1]; den=[1 1 0]; sysc=tf(num,den);
T=1;
[sysd]=c2d(sysc,T,'zoh');
```

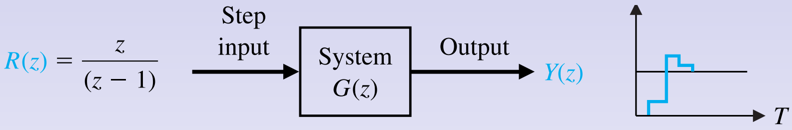
Sampling time

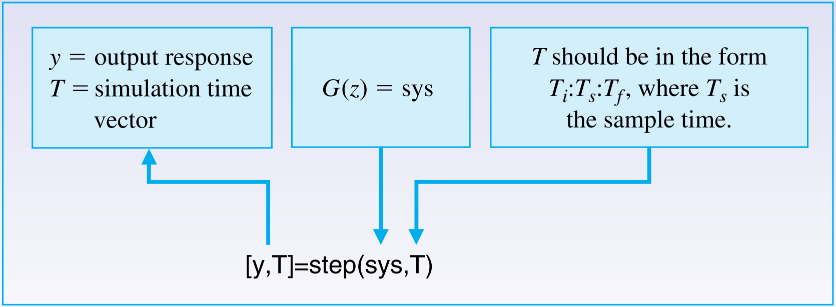
Zero-order hold

Output

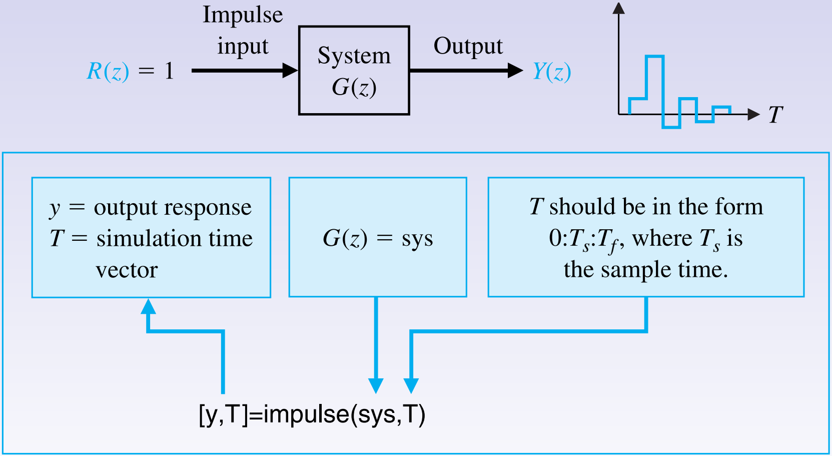
```
>>
Transfer function:
  0.3679 z + 0.2642
-------------------
z^2 - 1.368 z + 0.3679

Sampling time: 1
```

an illustration of function step

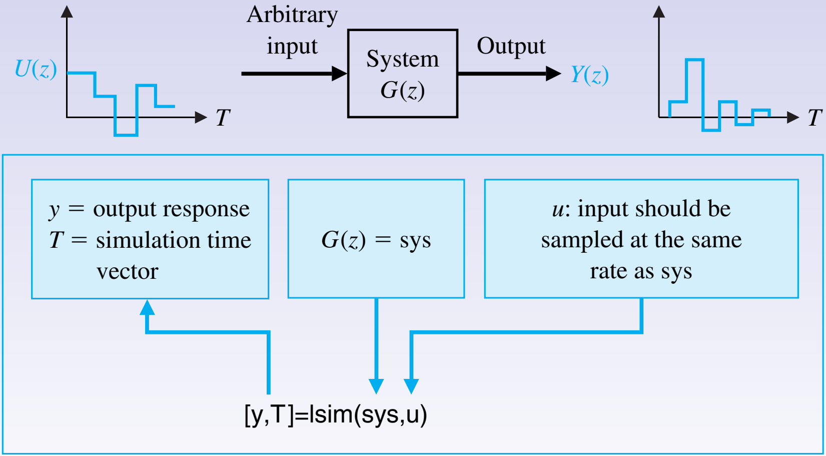




an illustration of function impulse



an illustration of function lsim



EX 13.10 Unit step response

For the plant

$$G(z) = \frac{0.3768(z + 0.7189)}{(z - 1)(z - 0.3678)}$$

let compensator be

$$D(z) = \frac{K(z - 0.3678)}{(z + 0.24)}$$

then the open-loop,

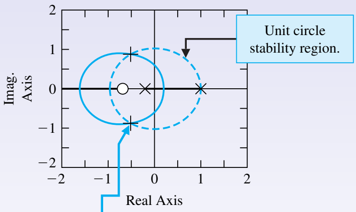
$$D(z)G(z) = K\frac{0.3768(z + 0.7189)}{(z - 1)(z + 0.24)}$$

determine K such that the closed-loop system is stable. The Matlab scripts is given as

EX 13.10 Unit step response

```
% This script generates the root locus for
% the sampled data system
%
%      K(0.3678)(z+0.7189)
%      ---------------------------
%        (z-1)(z+0.2400)
%
num=[0.3678 0.2644]; den=[1.0000 -0.7600 -0.2400]; sys=tf(num,den);
rlocus(sys);hold on
x=[-1:0.1:1];y=sqrt(1-x.^2);
plot(x,y,'--',x,-y,'--')
```

Plot unit circle.




```
>>rlocfind(sys)
Select a point in the graphics window

selected_point =
  -0.4787 + 0.8530i

ans =
  4.6390
```

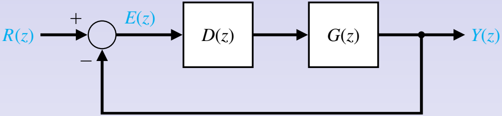
Determine K at the unit circle boundary.

$K = 4.639$

Sequential design example: Disk drive read system

The digital controller design of disk drive read system

The feedback disk drive read system with a digital controller is given in



where $T = 1\text{ms}$

$$
\begin{aligned}
G(z) &= Z\left[G_0(s)G_p(s)\right] = Z\left[\frac{1-e^{-sT}}{s}\frac{5}{s(s+20)}\right] \\
&= Z\left[\frac{1-e^{-sT}}{s}\frac{0.25}{s(0.05s+1)}\right] \\
&\approx Z\left[\frac{1-e^{-sT}}{s}\frac{0.25}{s}\right] = 0.25\left(1-z^{-1}\right)Z\left[\frac{1}{s^2}\right] \\
&= 0.25\left(1-z^{-1}\right)\frac{Tz}{(z-1)^2} = \frac{0.25T}{(z-1)}
\end{aligned}
$$

The digital controller design of disk drive read system

Let digital controller $D(z) = K$, the open-loop system is

$$D(z)G(z) = K\frac{0.25T}{(z-1)}$$

When $K = 4000$, and noting $T = 1$ms$= 0.001$s, obtain the TF of the closed-loop system

$$T(z) = \frac{D(z)G(z)}{1 + D(z)G(z)} = z^{-1}$$

therefore the step response is the input step delayed by one sampling period.

Summary

1. A digital computer may serve as a compensator or a controller in a feedback control system.
2. Since the computer receives and processes data only at specific intervals, then the time series of signal and the discrete system are derived.
3. The time series, called **sampled data**, can be transformed to the s-domain, then to the z-domain by the relation $z = e^{sT}$.
4. The z-transform of transfer function is used to analyze the stability and transient response of digital control system.
5. The root locus method is utilized to determine the location of the roots of the characteristic equation of digital control system.