Homework 3

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Theorem 3.16. Let $n \in \mathbb{N}$. $\sum_{i=1}^{n} i^3 = (\frac{n(n+1)}{2})^2$

Proof. Let $P(n)$ be the function that for any $n \in \mathbb{N}, P(n)$ satisfies the conditions in the theorem above

Consider $P(1)$

$1^3 = (\frac{1(1+1)}{2})^2$
$1 = 1$

Assume $P(n)$ holds true for all $n \leq k$ where $k \in \mathbb{N}$
Now consider the case for $P(k+1)$

$$\begin{aligned}
\sum_{i=1}^{k+1} i^3 &= \left(\sum_{i=1}^{k} i^3\right) + (k+1)^3 \\
&= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 && \text{(by inductive hypothesis)} \\
&= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\
&= \frac{k^4 + 2k^3 + k^2}{4} + k^3 + 3k^2 + 3k + 1 \\
&= \frac{k^4 + 2k^3 + k^2}{4} + \frac{4k^3 + 12k^2 + 12k + 4}{4} \\
&= \frac{k^4 + 6k^3 + 13k^2 + 12k + 4}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4} \\
&= \left(\frac{(k+1)(k+2)}{2}\right)^2
\end{aligned}$$

Thus it holds for $k+1$ and the inductive hypothesis holds true.
By induction it is true that for all $n \geq 1$. QED $\square$

Theorem 3.17. Let $n \in \mathbb{N}$. $\sum_{i=1}^{n} i(i+1) = (\frac{n(n+1)(n+2)}{3})$

Proof. Let $n = 1$
$1(1{+}1) = \left(\frac{(1(1+1)(1+2))}{3}\right)$
$2 = 2$
Assume $\forall n \leq k$ the theorem holds true
Now let $n = k + 1$

$$\begin{aligned}
\sum_{i=1}^{k+1} i(i+1) &= \left(\sum_{i=1}^{k} i(i+1)\right) + (k+1)(k+2) \\
&= \left(\frac{(k)(k+1)(k+2)}{3}\right) + (k+1)(k+2) && \text{(by our IH)} \\
&= \left(\frac{k^3 + 6k^2 + 11k + 6}{3}\right) + \left(\frac{3k^2 + 9k + 6}{3}\right) && \text{(expand)} \\
&= \left(\frac{k^3 + 9k^2 + 21k + 12}{3}\right) && \text{(combine like terms)} \\
&= \left(\frac{(k+1)(k+2)(k+3)}{3}\right) && \text{(factor)}
\end{aligned}$$

Thus it holds for $k + 1$ and the inductive hypothesis holds true.
By induction it is true that for all $n \geq 1$. QED ☐

Theorem 3.19. $\forall k \in \mathbb{N}$, $x < y \implies x^{2k-1} < y^{2k-1}$

Proof. $x < y \implies x^{2-1} < y^{2-1}$
$x < y \implies x < y$

Assume $\forall k \leq n + 1$ the inequality holds true
Let $k = n + 1$
$x < y \implies x^{2k+1-1} < y^{2k+2-1}$
$x < y \implies x^{2k+1} < y^{2k+1}$
$x < y \implies x * x^{2k} < y * y^{2k}$
By our inductive hypothesis $x^{2k} < y^{2k}$ holds true. Since this is true, and $x < y$ is also true, in our conditional it must be true that $x^{2k-1} < y^{2k-1}$
Thus it holds for $k + 1$ and the inductive hypothesis is true.
By induction it is true that for all $n \geq 1$. QED ☐

Theorem 3.28. For $n \in \mathbb{N}$ find a prove a formula for $\sum_{i=1}^{n} \frac{1}{i(i+1)}$

Proof. $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$
let $i = 1$
$\sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{i}{i+1}$

$\frac{1}{2} = \frac{1}{2}$

Now assume it holds true for all $i \leq k$
Let $i = k + 1$

$$\begin{aligned}
\sum_{i=1}^{k+1} \frac{1}{i(i+1)} &= \left(\sum_{i=1}^{k} \frac{1}{i(i+1)}\right) + \frac{k+1}{(k+1)(k+2)} \\
&= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} && \text{(by IH)} \\
&= \frac{(k)(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k^2 + 2k + 1}{(k+2)(k+1)} \\
&= \frac{(k+1)^2}{(k+2)(k+1)} \\
&= \frac{(k+1)}{(k+2)}
\end{aligned}$$

let $p = k + 1$

$$= \frac{p}{p+1}$$

Thus it holds for $k + 1$ and the inductive hypothesis holds true.
By induction it is true that for all $n \geq 1$. QED $\square$

Theorem 3.29. For $n \in \mathbb{N}$ find a prove a formula for $\sum_{i=1}^{n}(2i - 1)$

Proof. $\sum_{i=1}^{n}(2i - 1) = n^2$
Let $n = 1$
$2n - 1 = n^2$
$2(1) - 1 = 1^2$
$1 = 1$

Assume this holds true $\forall n \geq k$ Consider the case where $n = k + 1$

$$\begin{aligned}
\sum_{i=1}^{k+1}(2i - 1) &= \left(\sum_{i=1}^{k}(2i - 1)\right) + (2(k+1) - 1) \\
&= (k^2) + (2k + 1) && \text{(by our inductive hypothesis)}
\end{aligned}$$

By our inductive hypothesis, the $k + 1^{th}$ term is equivalent to the sum of the first k terms in the series, plus the $k + 1$ term
Thus it holds for $k + 1$ and the inductive hypothesis holds true.
By induction it is true that for all $n \geq 1$. QED $\square$