Homework 2

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Graph Theory

September 11, 2017

Claim 1 A graph is bipartite iff every subgraph H of G has an independent set of size at least $\frac{|V(H)|}{2}$

Proof. $\Longrightarrow$) Suppose G is a bipartite graph. So G must not contain any odd cycles. Since H is an arbitrary subgraph, we cannot introduce any cycles by removing any edges or vertices from a graph that does not initially contain any. Hence, any subgraph H must also be bipartite. Hence, H can be split into two independent sets. It follows that one of the independent sets must be at least $\frac{|V(H)|}{2}$.

$\Longleftarrow$) Suppose every subgraph H of G has an independent set of size at least $\frac{|V(H)|}{2}$. □

Let $n \geq 4$ and G be an $n-$vertex simple connected graph not containing $4-$vertex path P_4 as an induced subgraph

Claim 2.a If G is not complete bipartite, then G contains the cycle C_3

Proof. content... □

Claim 2.b If G has no vertex adjacent to all other vertices, then G has the cycle C_4 as an induced subgraph

Proof. content.. □

Claim 3 If G is a $5-$regular graph, then $E(G)$ cannot be partitioned in paths of length at least 7.

Proof. Suppose G is a 5-regular graph. Assume $E(G)$ can be partitioned into paths of length at least 7 and seek contradiction. Consider the degree sum formula

$$\sum_{v \in V(G)} d_G(v) = 2|E(G)|$$

$V(G)$ is even for G to be a simple $5-$regular graph. Otherwise, $|E(G)| = \frac{5(2k+1)}{2} \in \mathbb{N}$ for some integer k. Clearly, this is false. So $V(G)$ must be even.

Since $V(G) = 2k$ we can apply **Theorem 1.2.33** and the minimum trails that decompose G is the $\max\{1, k\}$.

Assume $V(G) = 2k$ for an integer k such that $k \geq 3$. If k was less than 2, G would not be simple. By **Theorem 1.2.26**, G cannot be Eulerian so there is not a single path that decomposes G. Hence, the minimum number of trails that decompose G cannot be 1. Suppose the minimum number of trails that decomposes G is k. If this is true, then our $5k$ edges must be decomposed into k trails. Which gives us trails of length $5 < 7$. So we reach a contradiction. Thus, $E(G)$ cannot be partitioned into paths of length at least 7. □