Homework 4

Cameron Dart
Graph Theory

October 10, 2017

Problem 1

Claim For all odd $n \geq 3$ there exists an $n-$vertex tournament T such that every vertex in T is a king. There does not exist a tournament with 4 vertices that has this property.

Proof. Suppose our graph T has n vertices. Put these n vertices on a circle. For every vertex $u \in V(T)$ draw an edge from u to the next $\frac{n-1}{2}$ vertices on the circle. Repeat this for all n vertices. As a result, T has $\frac{n(n-1)}{2}$ edges and the underlying graph of T is K_n. So T must be a tournament. Suppose $u, v \in V(T)$. It must be the case that either $u \to v$ or there exists a $w \in N(u)$ so that $w \to v$. So by definition every vertex in T is a king.

Now assume there exists 4 vertex tournament T' where every vertex is a king. Construct T' in the same method as above. It follows that T' has 8 edges and we reach a contradiction. There does not exist a 4 vertex tournament such that every vertex is a king. □

Problem 2

Claim 1 Suppose G is a simple, connected graph with at least 3 vertices. Then the following claims are equivalent.

a) G is a tree

b) Every induced subgraph of G has a vertex of degree at most 1

c) For every vertex v in G, the number of components of $G - v$ equals the degree of v

Proof. $a \implies b$) Suppose G is a tree. G does not contain any cycles by definition so any induced subgraph of G must contain a leaf. Therefore, it has a vertex of degree 1. Now consider an induced subgraph of G that is not connected. Clearly, any isolated vertex will have degree zero. Hence, it is true that any induced subgraph of G contains a vertex of degree at most 1 and $a \implies b$.

$b \implies a$) Consider the contrapositive of the statement. If G is not a tree, then there exists an induced subgraph of G that has a all vertices degree of greater than 1. Since G is not a tree, there exists a cycle. Consider the induced subgraph that is a cycle in G. Clearly, every vertex in the induced cycle has degree greater than one. So it is true that $b \implies a$

$c \implies a$) Consider the contrapositive of $c \implies a$. If G is not a tree, then there exists a vertex $v \in V(G)$ such that the number of components of $G-v$ is not the degree of v. Since G is not a tree it has a cycle. Pick any vertex v in the cycle contained. $d_G(v)$ is at least two. If you delete v, then there still exists 1 component since the cycle is now a path. □

Problem 3

Claim Suppose G has vertex set $\{v_1, v_2, ..., v_n\}$ in which $n-1$ vertices $v_1, ..., v_{n-1}$ have the property such that $G - v_i$ for $i \in 1, 2, ..., n-1$ is a tree. G must either be a P_n or S_n

Proof. First, let G be a cycle of length n. Let $G' = G - v$ be the graph that deletes an arbitrary $v \in V(G)$. It is easy to see that G' is a path of length $n-1$ which is connected and acyclic. Therefore G' is a tree.

Now suppose that G is S_n and s is the vertex of degree $n-1$ in S_n. Choose our vertex set to be the independent set of size $n-1$ contained in S_n. Let $G' = G - v$ where v is in the independent set of G. G' must be connected since every vertex in our independent set is still adjacent s. Assume there is a cycle in G'. G' is bipartite so there cannot exist an odd cycle. If there was a cycle, it would be even and we would need to traverse two elements from our two independent sets, however, one of our independent sets contains just s. Hence, we reach a contradiction and there cannot exist a cycle in G'. It follows that G' is a tree because it is connected and acyclic.

Assume there exists another graph G with the properties above and seek contradiction. If a graph G has n vertices and n edges, then the degree sum is equal to $2n$. So if there exist multiple vertices with degree greater than 2, then there must be corresponding vertices with degree 0. Which contradicts o because we can choose a vertex set that contains isolated vertices. And it is not connected therefore not a tree. □

Problem 4

Claim For every $n \geq 2$ the minimum Wiener index of a $n-$vertex simple graph G with n edges is $2(n-1)^2 - 2$. The max value of $D(G)$ for a $5-$vertex tree is 40

Proof. The minimal Wiener index for $n-$vertex $n-1$ edge graph G is a star. The wiener index of a star is $(n-1)^2$. Adding a single edge to this star graph creates exactly one cycle and has the minimal Wiener index for a graph with n edges and n vertices. Namely, there are $(n-1)^2 - 1$ edges. Since we shortened the distance between the two vertices in our cycle that aren't the center of the star.
We have a maximal wiener index for a $n-$vertex graph when we consider P_n. Hence, P_5 has a maximal wiener index of $2(4) + 4(3) + 6(2) + 8(1) = 40$ by inspection. □

Problem 5

Claim There are $\frac{(n-2)!}{5!}(n-7)!$ prufer codes with one vertex of degree 6 and 6 leaves

Proof. The center appears 5 times in our prufer code so there are $\binom{n-2}{5}$ ways to place it in our prufer and $(n-7)!$ ways to order the rest of our vertices. It follows there are $\binom{n-2}{5}(n-7)!$ prufer trees with the given criteria. □