Homework 4

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Graph Theory

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Problem 1

Claim In the graph provided, we can get minimum spanning trees from Kruskal's and Prim's Algorithm.

Proof. Let xy denote the edge connecting the two vertices x and y. In Prim's, we always keep a connected component, starting with a single vertex and adding the next adjacent vertex with the smallest edge weight. Consider Prim's algorithm startin from vertex a.

Prim's Algorithm: ae, ef, fb, bc, cg, gh, hd, dp, pl, lk, ko, on, nj, ji, im

In Kruskal's, we do not keep a connected componented. Instead, at each stage, we look globally at the smallest edge and take it if it does not create a cycle in the current forest.

Kruskal's Algorithm: ae, bf, cg, dh, ad, am, mp, ef, pl, bc, ij, kl, ko, jn

□

Problem 2

Claim $\tau(K_{2,4}) =$ and $K_{2,4}$ has 2 nonisomorphic spanning trees

Proof. Let Q be the Laplacian Matrix for $K_{2,4}$

$$\begin{bmatrix} 4 & 0 & -1 & -1 & -1 & -1 \\ 0 & 4 & -1 & -1 & -1 & -1 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 & 0 \\ -1 & -1 & 0 & 0 & 2 & 0 \\ -1 & -1 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Suppose Q^* is the matrix resulting from deleting the $1st$ row and $1st$ column of Q

$$\begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 \\ -1 & 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & 0 & 2 \end{bmatrix}$$

□

By the **Matrix Tree Theorem** $\tau(K_{2,4}) = (-1)^2 \det(Q^*) = 32$. Hence, $K_{2,4}$ has 32 spanning trees.
Up to isomorphism there are two spanning trees of $K_{2,4}$. Specifically, we consider the partite set of size 2 in $K_{2,4}$. Let u, v be the vertices in the set previously described. Since our spanning tree must connect all vertices, the degrees of u, v cannot be 0. Suppose $d(u) = 2$. Then $d(v)$ must be 3 in order for our graph to be connected. Now suppose that $d(u) = 1$ this implies that $d(v) = 4$. If the vertex degrees for either case were switched, we would have isomorphic graphs. As a result, these are the only two isomorphic spanning trees of G.

Problem 3

Claim Let $G = (X, Y; E)$ be a bipartite simple graph with partite sets X and Y, where $|X| = |Y| = n$. Then $\alpha'(G) \geq \min, 2\delta(G)$

Proof. Assume $\alpha'(G) < \min(n, 2\delta(G))$ and seek contradiction. Immediately we can see that if G has any isolated vertices that $\delta(G) = 0$ and we reach a contradiction. Now consider a G with no isolated vertices. Hence, $\delta(G) \geq 1 \implies 2\delta(G) \geq 2$. If we have no isolated verices, then $\alpha'(G) = \beta(G)$, $\beta'(G) = \alpha(G)$, and $\alpha'(G) + \beta'(G) = 2n$.

$$\begin{aligned}
\alpha'(G) < \min(n, \delta(G)) &\implies \alpha'(G) + \beta'(G) < \min(n, 2\delta(G)) + \beta'(G) \\
&\implies 2n < \min(n, 2\delta(G)) + \alpha(G) \\
\text{Note, here we have an upperbound for } \alpha(G) \text{ of } n & \\
&\implies 2n < \min(n, 2\delta(G)) + n \\
&\implies n < \min(n, 2\delta(G))
\end{aligned}$$

For any smaller $\alpha(G)$ our left side of the inequality is greater than n and our final statement is vacuously false. Hence, we arrive at a contradiction and $\alpha'(G) \geq \min, 2\delta(G)$ □

Problem 4

Claim If A is a $0, 1$ matrix so that a *line* in A is a row or column and two 1's in A are independent if no line contains both of them. Then the max number of pairwise independent $1's$ is equal to the minimum number of lines covering all $1's$ in A.

Proof. Consider a matrix A that is a m by n where the rows of A correspond to the vertices of one partite set and the columns of A correspond to the other partite set. Label the vertices in our rows v_1 to v_m and the other partite set v_{m+1} to v_n. Two vertices, v_i, vj are adjacent if $a_{ij} = 1$. Otherwise, if $a_{ij} = 0$, then v_i and v_j are not connected.
The minimum number of lines needed to cover the 1s then is the number of rows and columns so that all the 1s are at some point present. By construction, selecting a row is the same as picking a vertex. So we are looking for the minimum vertex cover. 2 pairwise independent $1'$s are two edges that are completely vertex disjoint. The maximum number of these must then be the maximum matching. Thus, we are comparing the min vertex cover with max matching of a bigraph, which must be equal due to the Konig's Theorem. □

Problem 5

Claim The graph in 3.1.28 does not have a perfect matching.

Proof. By Tutte's theorem if there exists a $S \subseteq V(G)$ so that $o(G - S) > |S|$ then G does not contain a perfect matching. Consider the attached drawing of the graph that fixes a S such that the conditions above hold. As a result, G does not contain a perfect matching. □