Homework 2

Cameron Dart
Graph Theory

September 13, 2017

Claim 1 A graph is bipartite iff every subgraph H of G has an independent set of size at least $\frac{|V(H)|}{2}$

Proof. $\Longrightarrow$ Suppose G is a bipartite graph. Then G must not contain any odd cycles. We cannot introduce any cycles by removing any edges or vertices from a graph that does not initially contain any. Hence, any subgraph H must also be bipartite and therefore split into two independent sets. It follows that one of the independent sets must be at least $\frac{|V(H)|}{2}$.

$\Longleftarrow$ Consider the contrapositive, if G is not bipartite then there exists a subgraph H of G such that there is not an independent set of size at least $\frac{|V(H)|}{2}$. If G is not bipartite then there exists an odd cycle. Suppose H is the subgraph of G that is C_n for some odd integer n. H does not have an independent set that is at least $\frac{|V(H)|}{2}$. □

Claim 2.a Let $n \geq 4$ and G be an $n-$vertex simple connected graph not containing $4-$vertex path P_4 as an induced subgraph If G is not complete bipartite, then G contains the cycle C_3

Proof. Consider the contrapositive of our claim. If G does not contain the cycle, then G is complete bipartite. Since G is connected and there are no 3-cycles in G there exists $u, v, w \in V(G)$ such that the subgraph u, v, w is complete bipartite. Now consider $z \in V(G)$ that is connected to any of u, v, w. There must not be a 3-cycle between any three vertices of u, v, w, z. If z is connected to one of our independent sets in u, v, w then it must be connected to all elements in that independent set. Otherwise, we have an induced subgraph of P_4 which contradicts our original assumption. Hence, G is complete bipartite. □

Claim 2.b If G has no vertex adjacent to all other vertices, then G has the cycle C_4 as an induced subgraph

Proof. Suppose G has no vertex that is adjacent to all others vertices. Let $u = \Delta(G)$, v and w be any two vertices adjacent to u, and z is a vertex not adjacent to z. In order for our graph to be connected, z must connect to v or w. However, if it only connects to one, then we have an induced subgraph of P_4. So it must connect to both. Hence, u, v, w, z contains a 4-cycle. □

Claim 3 If G is a $5-$regular graph, then $E(G)$ cannot be partitioned in paths of length at least 7.

Proof. G Suppose G is a n vertex 5-regular graph. Assume that $E(G)$ can be partitioned into paths of length at least 7. G has $\frac{5n}{2}$ edges. Note, this means n must be even. Since paths must start and end at different vertices, we need $\frac{n}{2}$ paths to cover G. So if our paths need to be at least length 7 we need a minimum $\frac{7n}{2}$ edges. Since $\frac{7n}{2} > \frac{5n}{2}$ we reach out sought contradiction. □

Claim 4 Every cycle of length $2r$ in the hypercube Q_n is contained in a subcube of dimension at most r

Proof. Consider a hypercube Q_n and a cycle of length $2r$. In a hypercube, u, v are adjacent if their coordinates differ by 1. So if two vertices share a common neighbor, their coordinates differ by at most 2, and so on. So a path of length r differs in at most r coordinates. Hence, our path of length r spans at most r dimensions and our cycle is contained in some subcube of dimension at most r. □

Additionally, yes a cycle of length 8 can be contained in a subcube of dimension 2 and a cycle of length 6 cannot be contained in 2 dimensions.

Claim 5 G has 6 equivalence classes.

Proof. Initially there are 4! permutations of our 4 paths, however, we don't distinguish between the starting vertex x, y and the reverse cycles are equivalent so we are overcounting by a factor of 4. Hence, there are 6 equivalence classes. $\square$