Math 444 - Homework 2

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Claim 2.1.4 If $a \in \mathbb{R}$ satisfies $a * a = a$ then $a = 0$ or $a = 1$.

Proof. Suppose $a \in \mathbb{R}$. By **Theorem 2.1.8a** either $a^2 > 0$ or $a = 0$ and $a^2 = 0$. Hence, $a * a = a$ is satisfied if $a = 0$.
Additionally, using our hypothesis, **M3**, and **M4**, we get

$$aa = a \implies aa\left(\frac{1}{a}\right) = a\left(\frac{1}{a}\right) \implies a = 1$$

Thus we have shown if $aa = a$, then $a = 0$ or $a = 1$. $\square$

Claim 2.1.8a If x, y are rational numbers, then $x + y$ and xy are rational numbers.
If x is a rational number and y is irrational, then $x + y$ is irrational.
If, in addition, $x \neq 0$, then xy is an irrational number.

Proof. Suppose $m, n \in \mathbb{Q}$. By definition of rational number $m = \frac{a}{b}$ and $n = \frac{k}{l}$ where $a, b, k, l \in \mathbb{Z}$ and $b, l \neq 0$. We know $\frac{1}{b}$ and $\frac{1}{l}$ exist by **M4**

$$mn = \frac{a}{b}\frac{k}{l} = ab^{-1}kl^{-1} = (ak)(b^{-1}l^{-1})$$

Since b, l are nonzero integers and integers are closed under multiplication we can show,

$$(ak)(b^{-1}l^{-1}) = \frac{ak}{bl} = \frac{s}{t} \in \mathbb{Q}$$

Hence, rational numbers are closed under addition.
Now consider $m + n$

$$n + m = \frac{a}{b} + \frac{k}{l} = \frac{al}{lb} + \frac{kb}{lb} = \frac{al + kb}{lb} \in \mathbb{Q}$$

$\square$

Claim 2.1.8b Suppose $x \in \mathbb{Q}$ and $y \in \mathbb{R} \backslash \mathbb{Q}$. Then $x + y$ and xy are in $\mathbb{R} \backslash \mathbb{Q}$.

Proof. Suppose to contradiction that $x + y$ is rational. Since Q is a field $-x$ exists and the sum of two rational numbers is a rational. However, we arrive at a contradiction because $x + y - x = y$ which is an irrational number. Hence, the addition of an irrational and a rational number is irrational.
Now suppose x is nonzero and to contradiction that xy is rational. First, let $x = \frac{c}{d}$ for $c, d \neq 0$ and $r = \frac{a}{b}$

$$\begin{aligned} xy &= r \in \mathbb{Q} \\ \frac{c}{d}y &= \frac{a}{b} \\ y &= \frac{ac}{bd} \end{aligned}$$

Thus, we have arrived at a contradiction. Hence the product of a rational and irrational is irrational. □

Claim 2.2.5 If $a < x < b$ and $a < y < b$, then $|x - y| < b - a$.

Proof. By definition $|x - y| < b - a \implies -(b - a) < x - y < b - a$. Consider the first expression

$$a < x < b \tag{1}$$

If we multiply $a < y < b$ by -1 we get

$$-b < -y < -a \tag{2}$$

Now we add $(1) + (2)$.

$$a - b < x - y < b - a \implies -(b - a) < x - y < b - a = |x - y| < (b - a)$$

So our claim holds true. □

Claim 2.2.17 If $a, b \in \mathbb{R}$ and $a \neq b$, then there exists ϵ-neighborhoods U of a and V of b such that $U \cap V = \emptyset$.

Proof. Without the loss of generality, assume $a < b$. Suppose $x = a + \epsilon$ the largest element in U and $y = b - \epsilon$ the largest element in V. We can say that $U \cap V = \emptyset$ if $x < y$.

$$\begin{aligned} x &< y \\ a + \epsilon &< b - \epsilon \\ a + 2\epsilon &< b \\ 2\epsilon &< b - a \\ \epsilon &< \frac{b - a}{2} \end{aligned}$$

Hence, $U \cap V = \emptyset$ for any $\epsilon \geq \frac{b-a}{2}$. □

Claim 2.3.9 Let $S \subseteq \mathbb{R}$ be nonempty. If $u = \sup S$, then for every $n \in N$ the number $u - 1/n$ is not an upper bound of S but $u + 1/n$ is an upper bound of S.

Proof. For any $n \in \mathbb{N}$. $\frac{1}{n} > 0$.

$$\begin{aligned} \frac{1}{n} &> 0 \\ u + \frac{1}{n} &> u \\ u &> u - \frac{1}{n} \end{aligned}$$

Additionally,

$$\begin{aligned} \frac{1}{n} &> 0 \\ u + \frac{1}{n} &> n \end{aligned}$$

So by definition $u - 1/n$ is not an upper-bound and $u + 1/n$ is an upper-bound for S. $\square$

Claim 2.3.11 Suppose S be a bounded set in $\mathbb{R}$ and let S_0 be a nonempty subset of S. Show $\inf S \leq \inf S_0 \leq \sup S_0 \leq \sup S$

Proof. First, we will show $\inf S \leq \inf S_0$. Since S, S_0 are bounded below, there exists w, w_0 that are infimums of S, S_0 respectively. If S_0 is contained in S, then w has to be a lower bound for S_0 by definition of infimum. Hence, $\inf S \leq \inf S_0$. Let $s_0 \in S_0$. It follows that $\inf S_0 \leq s_0 \leq \sup S_0$. Hence, $\inf S_0 \leq \sup S_0$. Finally, let that $u = \sup S$ and $u_0 = \sup S_0$. Notice that every element in S_0 is also an element of S. Thus, u_0 must be less than or equal to the least upper bound for S, or u. So $\sup S_0 \leq \sup S$. Hence, $\inf S \leq \inf S_0 \leq \sup S_0 \leq \sup S$ $\square$