Math 444 - Homework 3

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Claim 2.4.2 If $S := \left\{\frac{1}{n} - \frac{1}{m} : m, n \in \mathbb{N}\right\}$, then $\sup S = 1$ and $\inf S =$ -1

Proof. First, note $0 < \frac{1}{n} \leq 1$ and $0 < \frac{1}{m} \leq 1$. It follows that,

$$\frac{1}{n} - \frac{1}{m} \geq \frac{1}{n} - 1 > -1$$
$$\frac{1}{n} - \frac{1}{m} \leq 1 - \frac{1}{m} < 1$$

Thus, $-1 < \frac{1}{n} - \frac{1}{m} < 1$. $\square$

Claim 2.4.2a If $a > 0$, $aS = \{as : s \in S\}$, then $\inf(aS) = a \inf S$

Proof. Let $u = \sup S$ which means $s \leq u$ for all $s \in S$ and it is an upper bound of S. Since $a > 0$ it follows that $as \leq au$ for all s. Which shows that au is an upper bound for aS, so $\sup(aS) \leq a \sup S$.
Now let $v = \sup(aS)$ and $as \leq v$ for all $s \in S$. Since $a > 0$, we can divide both sides by a and we have $s \leq \frac{v}{a}$ for all s. This implies that $\frac{v}{a}$ is an upper bound of S and that $a \sup S \leq v = \sup(aS)$
The result of the two inequalities is our desired result $a \sup S = \sup aS$ $\square$

Claim 2.4.4b Let $b < 0$ and $bS = \{bs : s \in S\}$

$$\sup(bS) = b \inf S. \text{ Let } u = \sup S$$

Proof. Let $x \in bS$ so $\frac{x}{b} \in S$. $\sup S \geq \frac{x}{b}$ since $\sup S$ is an upper bound for S. It follows, $b \sup S \leq x$ and $b \sup S$ is a lower bound for bS. Now let u be a lower bound for bS. If $s \in S$, then $u \leq bs \implies \frac{u}{b} \geq s$. Thus, $\frac{u}{b}$ is an upper bound for S, and $\frac{u}{b} \leq \sup S$. Hence, $u \leq b \sup S$ and $b \sup S = \inf bS$ $\square$

Claim 2.4.8 Let X be a nonempty set, and f, g be defined on X and have bounded ranges in $\mathbb{R}$.

$$\sup f(x) + g(x) : x \in X \leq \sup f(x) : x \in X + \sup g(x) : x \in X$$

Proof. Let $u = \sup f$ and $v = \sup g$. $f(x) \leq u$ and $g(x)$ for all $x \in X$ by definition of supremum. $(f+g)(x) = f(x) + g(x) \leq u + v = \sup f + \sup g$. So $u + v$ is an upper bound for $f(x) + g(x)$. $u + v$ is also a supremum so $\sup\{f(x) + g(x)\} \leq \sup f + \sup g$ $\square$

Claim 2.5.3 Suppose S is a nonempty bounded subset of $\mathbb{R}$ and $I_S = [\inf S, \sup S]$.

(i) $S \subseteq I_S$

(ii) If J is any bounded interval containing S, $I_S \subseteq J$

Proof. Let $x \in S$. Since S is nonempty and bounded there exist $u = \inf S, v = \sup S$ so that $u \leq s$ and $s \leq v$ for all $s \in S$. $\inf S \leq x \leq \sup S$ by definition of infimum and supremum. It follows that $x \in [\inf S, \sup S] = I_S$ so $S \subseteq I_S$. $\square$

Proof. Let $s \in I_S$. Suppose $J \subseteq [a, b]$ where $a \leq \inf S$ and $b \geq \sup S$. Since $s \in I_S$ we know $\inf S \leq s \leq \sup S$. Combining our two inequalities we get,

$$a \leq \inf S \leq s \leq \sup S \leq b$$

Therefore, $s \in J$ and it follows that $I_S \subseteq J$. $\square$