Practice Exam II

EXAM A

1. Suppose $f: R \to R$ satisfies $|f(x) - f(y)| \leq K|x-y|$ for all x and y in R. Show that f is continuous at each c in R.

2. Evaluate $\lim\limits_{x \to 0} \dfrac{\sqrt{1+2x} - \sqrt{1+4x}}{x + 2x^2}$

3. Show that the following limits do not exist:

   (i) $\lim\limits_{x \to 0} (x + \operatorname{sgn} x)$ (ii) $\lim\limits_{x \to 3} f(x)$ where

$$f(x) = \begin{cases} x+4, & x > 3 \\ x^2 - 1, & x < 3 \end{cases}$$

4. Let $x_n = \sum\limits_{k=1}^{n} \dfrac{1}{\sqrt{k}}$. Is x_n a Cauchy sequence?

5. Evaluate $\lim\limits_{x \to 2} \dfrac{x+3}{x+5}$ using the definition of limit.

Answers, Hints, Etc.

1. Take $\delta = \dfrac{\varepsilon}{K}$

2. -1

3. (i) Consider $x_n = \dfrac{(-1)^n}{n}$ (ii) Consider $x_n = 3 + \dfrac{(-1)^n}{n}$

4. No ($p = \frac{1}{2}$)

5. $\frac{5}{7}$

EXAM B

1. (i) Define: c is a cluster point of A

   (ii) If c is not a cluster point of A and $c \in A$, show that any $f: A \to R$ is continuous at c.

2. Consider $f: R \to R$ given by
$$f(x) = \begin{cases} x & x \in Q \\ 0 & x \in R - Q. \end{cases}$$
   Where is f continuous? Justify.

3. If $c > 0$, show from the definition of limit that
$$\lim_{x \to c} \sqrt{x} = \sqrt{c}.$$

4. Evaluate $\displaystyle\lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1}$ using any results from class.

5. Suppose c is a cluster point of A, $f: A \to R$, $g: A \to R$, $\displaystyle\lim_{x \to c} f(x)$ exists and $\displaystyle\lim_{x \to c} g(x)$ does not exist. Show that $\displaystyle\lim_{x \to c} f(x) + g(x)$ does not exist.

Answers, Hints, Etc.

1. See class notes

2. f is continuous only at $x = 0$. Use SCC.

4. $\frac{1}{2}$

5. Write $g(x) = [f(x) + g(x)] - f(x)$ and argue by contradiction.