Math 444 - Homework 1

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Claim 1.1.14 If $f : A \to B$ and E, F are subsets of A then $f(E \cup F) = f(E) \cup f(F)$ and $f(E \cap F) \subseteq f(E) \cap f(F)$

Proof. In order to prove $f(E \cup F) = f(E) \cup f(F)$ it suffices to show $f(E \cup F) \subseteq f(E) \cup f(F)$ and $f(E) \cup f(F) \subseteq f(E \cup F)$. First, suppose $y \in f(E \cup F)$ then there exists a $x \in (E \cup F)$ so that $y = f(x)$. If $y \in f(E)$, then $x \in E$. Likewise for F, if $y \in f(F)$ and $x \in F$. Hence we have shown, $f(E \cup F) \subseteq f(E) \cup f(F)$. Now we must show $f(E) \cup f(F) \subseteq f(E \cup F)$. Next, suppose $y \in f(E \cup F)$ so either $y \in f(E)$ or $y \in f(F)$. Since, $f(E) \subseteq f(E \cup F)$ and $f(F) \subseteq f(E \cup F)$, it must be true that $y \in f(E \cup F)$. Thus, we have shown each set contains the other so $f(E \cup F) = f(E) \cup f(F)$.

Clearly, $(E \cap F) \subseteq E$ and $(E \cap F) \subseteq F$, it follows that $f(E \cap F) \subseteq f(E)$ and $f(E \cap F) \subseteq f(F)$. So $f(E \cap F) \subseteq f(E) \cap f(F)$ □

Claim 1.1.22 Let $f : A \to B$ and $g : B \to C$ be functions.
If $g \circ f$ is injective, then f is injective.
If $g \circ f$ is surjective, then g is surjective.

Proof. First, we will show if $g \circ f$ is injective, then f is injective. Let $x_1, x_2 \in A$.
So $g \circ f = g(f(x_1)) = g(f(x_2)) = g \circ f(x_2)$ which implies $x_1 = x_2$ by the injectivity of $g \circ f$.
Hence, f must be injective.
Next, assume that $g \circ f$ is surjective. Let $c \in C$. Since $g \circ f$ is surjective there exists an $a \in A$ so that $(g \circ f)(a) = g(f(a)) = c$. If we let $y = f(a)$ then $g(y) = c$. Therefore, g must be surjective. □

Claim 1.2.7 $5^{2n} - 1$ is divisible by 8 for all $n \in \mathbb{N}$

Proof. Let $P(n)$ be the statement that 8 divides $5^{2n} - 1$ for any natural number n.
$P(1)$ holds since $5^2 - 1 = 24$ and 8 divides 24.
Assume for all $n \leq k$ that $5^{2n} - 1$ is divisible by 8.
Consider $n = k + 1$.

$$\begin{aligned}
5^{2(k+1)} - 1 &= 5^{2k+2} - 1 \\
&= 5^{2k} * 5^2 - 1 \\
&= 5^{2k} * 24 \\
&= 8 * (3 * 5^{2k}) \\
&= 8 * m
\end{aligned}$$

Hence, $P(n)$ holds true for all $n \in \mathbb{N}$ so our claim must be true. $\square$

Claim 1.2.17 Find the largest natural number m so that $n^3 - n$ is divisible by m for all $n \in \mathbb{N}$

Proof. Suppose $f(n) = n^3 - n$. $f(1) = 0, f(2) = 6, f(3) = 24, f(4) = 62, f(5) = 120, ...$ I claim that $m = 6$ is the largest number that divides $n^3 - n$ for all n and will prove so using an inductive argument.
Consider $n = 1$. Since $f(1) = 0$ and $6|0$ our base case holds true.
Now assume that 6 divides $n^3 - n$ for all $n \leq p$ where $p \in \mathbb{N}$ and is greater than 1.
Consider $n = p + 1$.

$$\begin{aligned} n^3 - n &= (p+1)^3 - (p+1) \\ &= (p^3 + 3p^2 + 3p + 1) - (p+1) \\ &= (p^3 - p) + (3p^2 + 3p) \end{aligned}$$

By our inductive hypothesis $p^3 - p$ is divisible by 6 and now consider $3p^2 + 3p$

$$3p^2 + 3p = 3(k)(k+1).$$

Now either k or $k + 1$ is even so we can factor a 2 out of k or $k + 1$ and it clearly becomes divisible by 6. Thus, we have shown that 6 is the largest number that divides $n^3 - n$ for all n . $\square$

Claim 1.3.12 If a set S has n elements, then $\mathcal{P}(S)$ has 2^n elements

Proof. If S is a set with one element, then $P(S)$ is a set with two elements. Namely, the only element in S and $\emptyset$. Suppose for all sets S with $n = 1, 2, ..., k$ elements that $P(S)$ contains 2^n elements. Now consider any set S with cardinality $n = k + 1$. We split S into the union of a set of size k and a set of size 1. By our inductive hypothesis, our set of size k has a powerset of size 2^k and by our base case the set with 1 element has a powerset of size 2. Now, we make combinations of every element in both sets and we have 2^k and 2^1 elements. Hence there are $2^k * 2 = 2^{k+1}$ elements in our powerset of S which has $k + 1$ elements. $\square$

Claim 1.3.9a Suppose S, T are sets that $T \subseteq S$. If S is countable, then T is countable.

Proof. By assumption our S is countable so by Definition 1.3.6 it is either denumerable or finite. Hence, we must consider these two cases. If S is a finite, then T is a finite set by Theorem 1.3.5. and therefore countable.
Now consider the case where S is countably infinite. By lemma 2 there exists a surjection $f : N \to S$. Since $T \subseteq S$ all $t \in T$ is also in S. It follows that for all $t \in T$ that there exists an $n \in \mathbb{N}$ so that $f(n) = t$. Hence, there is a surjection from $\mathbb{N}$ onto S. $\square$