Math 444 - Homework 9

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Claim 5.3.2 let $I := [a, b]$ and let $f : I \to \mathbb{R}$, $g : I \to \mathbb{R}$ be continuous on I. Let $E := \{x \in I : f(x) = g(x)\}$. If$(x_n) \subseteq E$ and $x_n \to x_0$ then $x_0 \in E$

Proof. Suppose $(x_n) \subseteq E$, we have $f(x_n) = g(x_n)$ for all n. Since f, g are continuous on I, we can apply the SCC,

$$\lim_{n\to\infty} f((x_n)) = \lim_{n\to\infty} g((x_n)) \implies f(\lim_{n\to\infty} x_n) = g(\lim_{n\to\infty} x_n) \implies f(x_0) = g(x_0) \in E$$

□

Claim 5.3.5 $p(x) := x^4 + 7x^3 - 9$ has at least two real roots

Proof. First, we show that $p(x)$ is continuous on R. If $c \in \mathbb{R}$, we have

$$\lim_{x\to c} p(x) = \lim_{x\to c} x^4 + 7x^3 - 9 = c^4 + 7c^3 - 9 = p(c)$$

Thus, p is continuous on $\mathbb{R}$.
Now suppose $x_0 = -8$, $x_1 = 0$ and $x_2 = 2$. Calculate $p(x_0) = 503$, $p(x_1) = -9$ and $p(x_2) = 63$. Clearly, $p(x_0) > 0 > p(x_1)$ and $p(x_1) < 0 < p(x_2)$. So by the Location of Roots Theorem there exist two real numbers c_1, c_2 such that $c_1 = c_2 = 0$. Hence, $p(x)$ has at least two real roots. □

Claim 5.3.17 Suppose $f : [0, 1] \to \mathbb{R}$ is continuous and has only rational values, then f is constant. Suppose $f : [0, 1] \to \mathbb{R}$ is continuous and has only irrational values, then f is constant.

Rational Proof. Let $x, y \in [0, 1]$ and without the loss of generality suppose $f(x) \neq f(y)$ and seek a contradiction. If $f(x) \neq f(y)$, then the density theorem states there exists an irrational number k so that $f(x) < k < f(y)$. But f is continuous and by the Bolzano Intermediate Value Theorem there must exist some $m \in [0, 1]$ so that $f(m) = k$. However, this contradicts our assumption that f only takes on rational numbers. So it must be true that f is a constant function. □

Irrational Proof. A similar proof to the rational follows for an irrational f. Let $x, y \in [0, 1]$ and suppose $f(x) \neq f(y)$. Since $f(x) \neq f(y)$, then there exists a rational number k in $f(x) < k < f(y)$ by the density theorem. Hence, by the continuity of f and the Bolzano Intermediate Value Theorem, f takes a rational value at some point in $[0, 1]$. Thus, we have arrived at our contradiction and f must be constant. □

Claim 5.4.9 If f is uniformly continuous on $A \subseteq \mathbb{R}$, and $|f(x)| \geq k > 0$ for all $x \in A$, then $1/f$ is continuous.

Proof. Given $\epsilon > 0$ and $u \in A$ choose $\delta(\epsilon, u)$ so that if $x \in A$ and $|x - u| < \delta(\epsilon, u)$, then $|f(x) - f(u)| < k^2\epsilon$

$$\left|\frac{1}{f(x)} - \frac{1}{f(u)}\right| = \left|\frac{f(x) - f(u)}{f(x)f(u)}\right| \leq \left|\frac{f(x) - f(u)}{k^2}\right| < \frac{k^2\epsilon}{k^2} = \epsilon$$

Hence, $1/f$ is uniformly continuous on A. $\square$

Claim 5.4.10 If f is uniformly continuous on a bounded subset A of $\mathbb{R}$, then f is bounded on A.

Proof. Suppose f is not bounded on A and seek contradiction. Since f is not bounded there exists a sequence $x_n \in A$ so that $\lim_{n\to\infty} f(x_n) = \infty$. But A is bounded so by Bolzano-Weirstrass it has a convergence subsequence x_{i_n}. Since it is convergent, it is Cauchy but the Cauchy Criterion. It follows by **Theorem 5.4.7** that $f(x_{i_n})$ is also Cauchy. Hence, $f(x_{i_n})$ is bounded. Thus, we have arrived at a contradiction since $\lim_{n\to\infty} f(x_n) = \infty$ but must be bounded since $f(x_{i_n})$ is bounded. $\square$

Claim 5.4.12 If f is continuous on $[0, \infty)$ and uniformly continuous on $[a, \infty)$ for some positive constant a, then f is uniformly continuous.

Proof. Consider $[0, a]$ a closed bounded interval on $\mathbb{R}$. Since f is continuous on $[0, \infty)$, it is continuous on $[0, a]$ because $a < \infty$. It follows by the Uniform Continuity Theorem, f must be uniformly continuous on the closed bounded interval $[0, a]$. If $[0, a]$ and $[a, \infty)$, it implies that f is uniformly continuous on $[0, \infty)$ $\square$