Math 444 - Homework 4

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Claim 3.3.2 Given $x_{n+1} := 2 - \frac{1}{n}$ where $x_1 > 1$, x_{n+1} is bounded, monotone and converges to 1

Proof. First, we prove that x_{n+1} is monotonically decreasing by showing $x_n > x_{n+1}$. Note, if $x_1 > 1$, then $\frac{1}{x_1} < 1$.

$$x_n - x_{n+1} = x_n - (2 - \frac{1}{x_n}) = x_n + \frac{1}{x_n} - 2 = \frac{x_n^2 + 1}{x_n} - 2 > 0 \tag{1}$$

Next, we prove by induction on n that x_{n+1} is bounded below by 1. Consider $n = 1$. So $x_2 = 2 - \frac{1}{x_1} > 1$. Assume it holds true for all $n \leq k$ that $x_{n+1} > 1$. Now let $n = k$, so $x_{k+1} = 2 - \frac{1}{x_n} > 1$. Lastly, we show that x_{n+1} is bounded above by 2. Again we use the fact that $\frac{1}{x_n} < 1$

$$x_{n+1} = 2 - \frac{1}{x_n} < 2 \tag{2}$$

Thus, we have shown $1 < x_{n+1} < 2$ and x_n is monotonically decreasing so by the Monotone Convergence Theorem, it converges to its infimum of 1 $\square$

Claim 3.3.10 The series $y_n := \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{2n}$ converges.

Proof. If, $y_n := \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{2n}$, then

$$y_{n+1} := \frac{1}{n+2} + \frac{1}{n+3} + ... + \frac{1}{2n+1} + \frac{1}{2n+2} \tag{3}$$

Now, we show that y_n is monotonically increasing by considering two arbitrary consecutive elements. $y_{n+1} - y_n$

$$y_{n+1} = \cancel{\frac{1}{n+2}} + \cancel{\frac{1}{n+3}} + ... + \cancel{\frac{1}{2n}} + \frac{1}{2n+1} + \frac{1}{2n+2} \tag{4}$$

$$-y_n = -\frac{1}{n+1} - \cancel{\frac{1}{n+2}} - ... - \cancel{\frac{1}{2n}} \tag{5}$$

So it follows that,

$$\begin{aligned} y_{n+1} - y_n &= \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} \\ &= \frac{1}{2n+1} - \frac{1}{2n+2} \\ &= \frac{1}{(2n+1)(2n+2)} > 0 \quad \forall n \in \mathbb{N} \end{aligned}$$

Thus, y_n is monotonically increasing. Next, we show that y_n is bounded above and below by comparing the series of the smallest and largest element in y_n with y_n itself.

$$y_n = \sum_{k=n+1}^{2n} \frac{1}{k} < \sum_{i=1}^{n} \frac{1}{n+1} = \frac{n}{n+1} = 1 - \frac{1}{n+1} < 1$$

$$y_n = \sum_{k=n+1}^{2n} \frac{1}{k} > \sum_{i=1}^{n} \frac{1}{2n} = \frac{n}{2n} = \frac{1}{2}$$

Finally, we apply the **MCT** and y_n converges □

Claim 3.4.4a The sequence $x_n = (1 - (1)^n + \frac{1}{n}))$ diverges

Proof. Assume to contradiction that x_n converges to L.

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} 1 - (-1)^n + \frac{1}{n} = \lim_{n\to\infty} 1 - \lim_{n\to\infty} (-1)^n + \lim_{n\to\infty} \frac{1}{n}$$

Since $(-1)^n$ diverges, it follows that x_n diverges as well. □

Claim 3.4.4b The sequence $x_n = \sin(\frac{\pi n}{4})$ diverges

Proof. Assume to contradiction that x_n converges. Suppose x_{n_j} is the subsequence of odd values in x_n. I claim there are two subsequences contained in x_{n_j} that both converge to different values. Which would mean x_{n_j} diverges. Namely, the odd residue classes of 8. Specifically, the residue classes of $1, 3 \mod 8$ converge to $\frac{\sqrt{2}}{2}$ then $5, 7 \mod 8$ converge to $-\frac{\sqrt{2}}{2}$. The four cases to consider are $\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}, \sin(\frac{3\pi}{4}) = \frac{\sqrt{2}}{2}, \sin(\frac{5\pi}{4}) = -\frac{\sqrt{2}}{2}$, and $\sin(\frac{7\pi}{4}) = -\frac{\sqrt{2}}{2}$. We only needed to consider these cases due to the cyclic nature of the sinusoidal function. Hence, x_{n_j} diverges. So it follows that our x_n diverges as well contradicting our earlier assumption that it converge. □

Claim 3.4.5 Let $X = (x_n), Y = (y_n), Z = (z_n)$ where $z_{2n-1} = x_n$ and $z_{2n} = y_N$. Z converges if and only if X, Y converge and $\lim X = \lim Y$.

Proof. If Z converges, then all of its subsequences converge with equal limits. Hence, z_{2n}, z_{2n-1} both converges. So it follows that x_n, y_n converge as well and $\lim x_n = \lim y_n = L$. Since $Y = (y_n) = (z_{2n})$ and $X = (x_n) = (z_{2n-1})$.
Next, we will show the converse is true as well.
Suppose $\lim x_n = \lim y_n = L$. Given $\epsilon > 0$ there exist $n_1, n_2 \in \mathbb{N}$ so that,

$$|x_n - L| < \epsilon \quad \forall n \geq n_1 \quad a$$

$$|y_n - L| < \epsilon \quad \forall n \geq n_2 \quad b$$

Let $n_0 = \max(n_1, n_2)$. So for all $n \geq (2n_0 + 1)$ if $n = 2k$, then $2k \geq 2n_0$ which implies $k \geq n_0 + \frac{1}{2} > n_0$. So if $n = 2k - 1$, then $2k - 1 \geq 2n_0 + 1$ and it follows that $k \geq n_0 + 1 \geq n_0$. In either case, from a, b we have $|x_n - L|$ and $|y_n - L|$. So for $n \geq 2n_0 + 1$ it must be true that $|z_n - L| < \epsilon$. Hence, $\lim z_n$ exits and equals L.
Thus, our claim holds true. $\square$

Claim 3.4.11 If $x_n \geq 0$ for all $n \in \mathbb{N}$ and that $\lim((-1)^n x_n)$ exists, then $((-1)^n x_n)$ converges.

Proof. Let $y_n = ((-1)^n x_n)$ and $\lim y_n = L$. So $y_{2n} = x_2 n$ and $y_{2n-1} = -x_{2n-1}$. Note, the limits of the subsequences must be equal since y_n converges. So $y_{2n} = y_{2n-1}$. But if $x_{2n} \implies y \geq 0$ and $-x_{2n-1} \leq 0 \implies y \leq 0$. Which implies that $\lim_{n\to\infty} y_n = 0$ and that for all $\epsilon > 0$, there exists a $k = k(\epsilon)$ so that if $n \neq k$, then $|y_n| = |(-1)^n x_n| = |x_n| < \epsilon$ for all $n \geq n_0$. Therefore, x_n converges to 0 and our claim holds true.. $\square$