Math 444 - Homework 4

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Claim 3.1.5b

$$\lim_{n\to\infty} \frac{2n}{n+1} = 2$$

Proof. Suppose $\epsilon > 0$. Then,

$$\left|\frac{2n}{n+1} - 2\right| = \left|\frac{2n - 2(n+1)}{n+1}\right| = \left|\frac{-2}{n+1}\right| = \frac{2}{n+1} < \frac{2}{n} \tag{1}$$

Let $K \in \mathbb{N}$ with $K > \frac{2}{\epsilon}$. If $n \geq K$, then

$$n > \frac{2}{\epsilon} \implies \frac{2}{n} < \epsilon \implies \left|\frac{2n}{n+1} - 2\right| < \epsilon \tag{2}$$

Thus, $\frac{2n}{n+1}$ converges to 2 by the definition of limit. □

Claim 3.1.18 If $\lim x_n = x > 0$, then there exists $k \in \mathbb{N}$ so that if $n \geq k$, then $\frac{x}{2} < x_n < 2x$.

Proof. Suppose $\epsilon = \frac{x}{2}$. Since $x_n \to x$, it must be true that there exists a $K \in \mathbb{N}$ so that if $n \geq K$, then $|x_n - x| < \epsilon$.

$$\begin{aligned} |x_n - x| < \epsilon &\implies -\epsilon < x_n - x < \epsilon \\ &\implies x - \epsilon < x_n < x + \epsilon \\ &\implies \frac{x}{2} < x_n < \frac{3}{2}x < 2x \end{aligned}$$

So for all $n \geq K$ it holds that $\frac{x}{2} < x_n < 2x$. □

Claim 3.2.7 If b_n is bounded and $\lim a_n = 0$, then $\lim(a_n b_n) = 0$

Proof. We cannot use **Theorem 3.2.3** since it requires a function to be convergent, not bounded. A convergent sequence is always bounded but the converse is not necessarily true. As a counter example consider the sequence $(-1)^n$ for all $n \in \mathbb{N}$ is bounded by $[-1, 1]$ but does not converge to either.
Since b_n is bounded, we know that there exists some $M \in \mathbb{R}$ so that $|b_n| \leq M$ for all n. It follows that $a_n b_n \leq M|a_n|$ Let K be a natural number so that $\frac{1}{K} < \frac{\epsilon}{M}$ and consider the following,

$$|a_n b_n - 0| = |a_n b_n = |a_n||b_n| \leq M|a_n| < \frac{M\epsilon}{M} = \epsilon \tag{3}$$

So by definition $\lim a_n b_n = 0$ for all $n \geq K$. □

Claim 3.2.12 If $0 < a < b, \lim\left(\frac{a^{n+1}+b^{n+1}}{a^n+b^n}\right) = b$

Proof.

$$\begin{aligned}
\lim\left(\frac{a^{n+1}+b^n}{a^n+b^n}\right) &= \frac{\lim b + a\frac{a^n}{b^n}}{\lim 1 + \frac{a^n}{b^n}} \\
&= \frac{b + a\lim\frac{a^n}{b^n}}{1 + \lim\frac{a^n}{b^n}}
\end{aligned}$$

Now consider $\lim\frac{a^n}{b^n}$. If $0 < a < b$, then $0 < \frac{a}{b} < 1$. So $\lim\left(\frac{a}{b}\right)^n = 0$.

$$\frac{b + \lim\frac{a^n}{b^n}}{1 + \lim\frac{a^n}{b^n}} = \frac{b+0}{1+0} = b$$

Hence, $\lim\left(\frac{a^{n+1}+b^{n+1}}{a^n+b^n}\right) = b$ □