Math 444 - Homework 4

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Claim 3.5.6 There exists a non cauchy sequence that satisfies $\lim_{n\to\infty} |x_{n+1} - x_n| < 0$.

Proof. Consider the sequence x_n defined as follows,

$$x_n = \sum_{n=0}^{\infty} \frac{1}{n}$$

Clearly,

$$\lim_{n\to\infty} \frac{1}{n} \to 0 \text{ and } \lim_{n\to\infty} \frac{1}{n+1} \to 0 \text{ so } \lim_{n\to\infty} |x_{n+1} - x_n| = 0$$

But x_n diverges so there must exist a non cauchy sequence that satisfies $\lim_{n\to\infty} |x_{n+1} - x_n| < 0$. □

Claim 3.5.7 If x_n is a cauchy sequence so that $x_n \in \mathbb{Z}$, then x_n is constant

Proof. Assume to contradiction that x_n is a non constant cauchy sequence contained in $\mathbb{Z}$. Since x_n is cauchy by definition for all $\epsilon > 0$ there exists $k = k(\epsilon)$ so that if $m, n \geq k$, then $|x_m - x_n| < \epsilon$. Since x_n is contained in $\mathbb{Z}$, $|x_m - x_n| >= 1$ for all x_m, x_n where $x_m \neq x_n$. This contradicts our previous statement that $|x_m - x_n| < \epsilon$ for all $\epsilon > 0$. Thus, it must be true that $x_m = x_n$ and x_n is constant. □

Claim 3.7.3b

$$\sum_{n=0}^{\infty} = \frac{1}{(\alpha+n)(\alpha+n+1)} = \frac{1}{\alpha} > 0, \text{if } \alpha > 0.$$

Proof.

$$\begin{aligned}
\sum_{n=0}^{\infty} = \frac{1}{(\alpha+n)(\alpha+n+1)} &= \frac{1}{(\alpha+1)} - \frac{1}{(\alpha+n+1)} \\
&= \frac{1}{\alpha} - \cancel{\frac{1}{\alpha+1}} + \cancel{\frac{1}{\alpha+1}} - \cancel{\frac{1}{\alpha+2}} + ... + \cancel{\frac{1}{\alpha+n}} - \frac{1}{\alpha+n+1} \\
&= \frac{1}{\alpha} - \frac{1}{\alpha+n+1} \\
\lim_{n\to\infty} \frac{1}{\alpha} - \frac{1}{\alpha+n+1} &= \frac{1}{\alpha} \quad \forall \alpha > 0
\end{aligned}$$

□

Claim 3.7.8

Proof. □

Claim 3.7.11 If Σa_n is convergent, then Σa_n^2 must converge.

Proof. Suppose Σa_n converges to some $L \in R$. Note,

$$\sum_{n=0}^{\infty} a_n^2 \leq \left(\sum_{n=0}^{\infty} a_n \right)^2 = L^2$$

Let $a_n' = (\sum_{n=0}^{\infty} a_n)^2$ and apply the ratio test to a_n and a_n'

$$\lim_{n \to \infty} \frac{a_n'}{a_n} = \frac{L^2}{L} = L$$

Thus, the ratio test says, a_n' must converge since a_n converges. Lastly, a_n^2 converges when we apply the comparison test to it with a_n'. □