

# Math 444 - Homework 9

Cameron Dart

August 13, 2017

**Claim 5.3.2** let  $I := [a, b]$  and let  $f : I \rightarrow \mathbb{R}, g : I \rightarrow \mathbb{R}$  be continuous on  $I$ . Let  $E := \{x \in I : f(x) = g(x)\}$ . If  $(x_n) \subseteq E$  and  $x_n \rightarrow x_0$  then  $x_0 \in E$

*Proof.* Suppose  $(x_n) \subseteq E$ , we have  $f(x_n) = g(x_n)$  for all  $n$ . Since  $f, g$  are continuous on  $I$ , we can apply the SCC,

$$\lim_{n \rightarrow \infty} f((x_n)) = \lim_{n \rightarrow \infty} g((x_n)) \implies f(\lim_{n \rightarrow \infty} x_n) = g(\lim_{n \rightarrow \infty} x_n) \implies f(x_0) = g(x_0) \in E$$

□

**Claim 5.3.5**  $p(x) := x^4 + 7x^3 - 9$  has at least two real roots

*Proof.* First, we show that  $p(x)$  is continuous on  $\mathbb{R}$ . If  $c \in \mathbb{R}$ , we have

$$\lim_{x \rightarrow c} p(x) = \lim_{x \rightarrow c} x^4 + 7x^3 - 9 = c^4 + 7c^3 - 9 = p(c)$$

Thus,  $p$  is continuous on  $\mathbb{R}$ .

Now suppose  $x_0 = -8$ ,  $x_1 = 0$  and  $x_2 = 2$ . Calculate  $p(x_0) = 503$ ,  $p(x_1) = -9$  and  $p(x_2) = 63$ . Clearly,  $p(x_0) > 0 > p(x_1)$  and  $p(x_1) < 0 < p(x_2)$ . So by the Location of Roots Theorem there exist two real numbers  $c_1, c_2$  such that  $c_1 = c_2 = 0$ . Hence,  $p(x)$  has at least two real roots. □

**Claim 5.3.17** Suppose  $f : [0, 1] \rightarrow \mathbb{R}$  is continuous and has only rational values, then  $f$  is constant. Suppose  $f : [0, 1] \rightarrow \mathbb{R}$  is continuous and has only irrational values, then  $f$  is constant.

*Rational Proof.* Let  $x, y \in [0, 1]$  and without the loss of generality suppose  $f(x) \neq f(y)$  and seek a contradiction. If  $f(x) \neq f(y)$ , then the density theorem states there exists an irrational number  $k$  so that  $f(x) < k < f(y)$ . But  $f$  is continuous and by the Bolzano Intermediate Value Theorem there must exist some  $m \in [0, 1]$  so that  $f(m) = k$ . However, this contradicts our assumption that  $f$  only takes on rational numbers. So it must be true that  $f$  is a constant function. □

*Irrational Proof.* A similar proof to the rational follows for an irrational  $f$ . Let  $x, y \in [0, 1]$  and suppose  $f(x) \neq f(y)$ . Since  $f(x) \neq f(y)$ , then there exists a rational number  $k$  in  $f(x) < k < f(y)$  by the density theorem. Hence, by the continuity of  $f$  and the Bolzano Intermediate Value Theorem,  $f$  takes a rational value at some point in  $[0, 1]$ . Thus, we have arrived at our contradiction and  $f$  must be constant. □

**Claim 5.4.9** If  $f$  is uniformly continuous on  $A \subseteq \mathbb{R}$ , and  $|f(x)| \geq k > 0$  for all  $x \in A$ , then  $1/f$  is continuous.

*Proof.* Given  $\epsilon > 0$  and  $u \in A$  choose  $\delta(\epsilon, u)$  so that if  $x \in A$  and  $|x - u| < \delta(\epsilon, u)$ , then  $|f(x) - f(u)| < k^2\epsilon$

$$\left| \frac{1}{f(x)} - \frac{1}{f(u)} \right| = \left| \frac{f(x) - f(u)}{f(x)f(u)} \right| \leq \left| \frac{f(x) - f(u)}{k^2} \right| < \frac{k^2\epsilon}{k^2} = \epsilon$$

Hence,  $1/f$  is uniformly continuous on  $A$ . □

**Claim 5.4.10** If  $f$  is uniformly continuous on a bounded subset  $A$  of  $\mathbb{R}$ , then  $f$  is bounded on  $A$ .

*Proof.* Suppose  $f$  is not bounded on  $A$  and seek contradiction. Since  $f$  is not bounded there exists a sequence  $x_n \in A$  so that  $\lim_{n \rightarrow \infty} f(x_n) = \infty$ . But  $A$  is bounded so by Bolzano-Weierstrass it has a convergence subsequence  $x_{i_n}$ . Since it is convergent, it is Cauchy but the Cauchy Criterion. It follows by **Theorem 5.4.7** that  $f(x_{i_n})$  is also Cauchy. Hence,  $f(x_{i_n})$  is bounded. Thus, we have arrived at a contradiction since  $\lim_{n \rightarrow \infty} f(x_n) = \infty$  but must be bounded since  $f(x_{i_n})$  is bounded. □

**Claim 5.4.12** If  $f$  is continuous on  $[0, \infty)$  and uniformly continuous on  $[a, \infty)$  for some positive constant  $a$ , then  $f$  is uniformly continuous.

*Proof.* Consider  $[0, a]$  a closed bounded interval on  $\mathbb{R}$ . Since  $f$  is continuous on  $[0, \infty)$ , it is continuous on  $[0, a]$  because  $a < \infty$ . It follows by the Uniform Continuity Theorem,  $f$  must be uniformly continuous on the closed bounded interval  $[0, a]$ . If  $[0, a]$  and  $[a, \infty)$ , it implies that  $f$  is uniformly continuous on  $[0, \infty)$  □