Math 444 - Homework 7

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Claim 4.1.7

$$\lim_{x\to c} x^3 = c^3$$

Proof. Given $\epsilon > 0$ we aim to find a $\delta > 0$ so that if $|x-c| < 1$ then

$$-1 < x - c < 1 \implies c - 1 < x < c + 1$$

It follows that, $x^2 < (c+1)^2 = (c^2+2c+1)$ and $xc < (c+1)c = c^2 + c$

$|x^3 - c^3| = |x-c||x^2+xc+c^2| < |x-c|((c^2+2c+1)+(c^2+c)+c^2) = |x-c|(3c^2+3c^2+1)$

Let $\delta = \inf(1, \frac{\epsilon}{(3c^2+3c^2+1)})$

$$|x^3-c^3| = |x-c||x^2+xc+c^2| < \frac{\epsilon}{(3c^2+3c^2+1)}(3c^2+3c^2+1) = \epsilon$$

Thus, we have shown that there exists a way of choosing a $\delta(\epsilon) = \inf(1, \frac{\epsilon}{(3c^2+3c^2+1)}) > 0$ for any $\epsilon > 0$, we can infer

$$\lim_{x\to c} x^3 = c^3$$

□

Proof. Let $f : A \to \mathbb{R}$ so that $f(x) := x^3$. Let x_n be any sequence in A that converges to c. So by the sequence theorems we know that x_n^3 converges to c^3. Hence, $\lim_{x\to c} x^3 = c^3$. □

Claim 4.1.11b

$$\lim_{x\to 6} \frac{x^2-3x}{x+3} = 2$$

Proof. Let $f(x) := \frac{x^2-3x}{x+3}$

$$|f(x) - 2| = \left|\frac{x^2-3x}{x+3} - 2\right| = \left|\frac{(x-6)(x+1)}{(x+3)}\right|$$

We bound the coefficient of $|x-6|$ by the condition $5 < x < 7$. So we have,

$$\left|\frac{x^2-3x}{x+3} - 2\right| < \frac{4}{5}|x-6|$$

Given $\epsilon > 0$, we choose $\delta(\epsilon) = \inf(1, \frac{5\epsilon}{4})$. Then if $0 < |x-2| < \delta(\epsilon)$, we have $|f(x) - 2| \leq \frac{4}{5}|x-6| < \epsilon$. Since $\epsilon > 0$ is arbitrary, the assertion is proved. □

Claim 4.1.15 If $f = \mathbb{R} \to \mathbb{R}$ be defined by setting $f(x) := x$ if x is rational and $f(x) = 0$ if x is irrational. f has a limit at $x = 0$ and if $c \neq 0$ then f does not have a limit at c.

Proof. Given $\epsilon > 0$ choose $\delta(\epsilon) = \epsilon$ so that if $x \in A$ and $0 < |x| < \delta$ then

$$|f(x) - 0| = |f(x)| = |x| < \epsilon \text{ if x is rational, otherwise } 0 < \epsilon$$

Hence, $|f(x)| < \epsilon$ and the limit of $\lim_{x \to 0} f(x) = 0$
Suppose $c \in \mathbb{R}$ so that $c \neq 0$. Let x_n, y_n be two subsequences converging to c so that $x_n \in \mathbb{R} - \mathbb{Q}$ and $y_n \in \mathbb{Q}$ for all $n \in \mathbb{N}$. So $f(x_n)$ converges to 0 and $f(y_n)$ converges to $f(y_n)$. Hence, $\lim_{x \to c} x_n \neq \lim_{x \to c} y_n$. Thus, f does not have a limit at c. □

Claim 4.2.5 Let f, g be defined on $A \subseteq \mathbb{R}$ to $\mathbb{R}$ and let c be a cluster point of A. If f is bounded on a neighborhood of c and that $\lim_{x \to c} g = 0$, then $\lim_{x \to c} fg = 0$

Proof. Let x_n be any sequence in A that converges to c so that if $x_n \neq c$ for all $n \in \mathbb{N}$, $f(x_n)$ converges to some $L \in \mathbb{R}$ by the Sequential Criterion. Since f is bounded on a neighborhood of c, there exists $M \in \mathbb{R}$ so that $|f(x)| \leq M$ for all $x \in A \cap V_\delta(c)$. Hence, f is bounded by M. So using that the limit of products is the product of limits we have,

$$\lim_{x \to c} fg = \lim_{x \to c} f \lim_{x \to c} g = L * 0 = 0$$

□

Claim 4.2.10 There exists f, g such that f and g do not have limits at a point c but so that fg and $f + g$ has a limit at c.

Proof. Let $f(x) = 0$ for all $x \in \mathbb{Q}$ and $f(x) = 1$ for all $x \in \mathbb{R} - \mathbb{Q}$. Let $g(x) = 1$ for all $x \in \mathbb{Q}$ and $g(x) = 0$ for all $x \in \mathbb{R} - \mathbb{Q}$. Neither $\lim_{x \to 0} f(x)$ nor $\lim_{x \to 0} g(x)$ exists. However, fg and $f + g$ are both constant functions. Thus, their limits as they approach 0 exists. □

Claim 4.2.14 Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, and $c \in \mathbb{R}$ be a cluster point of A. If $\lim_{x \to c} f$ exists, then $\lim_{x \to c} |f| = |\lim_{x \to c} f|$

Proof. Since $\lim_{x \to c} f(x)$ exists, it must be true that $\lim_{x \to c} f(x) = L$ for some $L \in \mathbb{R}$. Given $\epsilon > 0$ let $\delta > 0$ so that $|f(x) - L| < \epsilon$ whenever $|x - c| < \delta$. Note, for all $a, b \in \mathbb{R}$ it holds that $|a| \leq |a - b| + |b| \implies |a| - |b| \leq |a - b|$. Hence, $||f(x)| - |L|| \leq |f(x) - L| < \epsilon \implies \lim_{x \to c} |f(x)| = |L| \implies \lim_{x \to c} |f| = |\lim_{x \to c} f|$ □