Measure

Definition A ***sigma algebra*** $\mathscr{F}$ of subsets of Ω:

- $\Omega \in \mathscr{F}$
- close under union and intersection
- close under complement

Definition A set function μ on $\mathscr{F}$ is a ***measure*** if (Probability measure if $\mu(\Omega) = 1$):

- $\mu(A) \in [0, \infty]$ for all $A \subset \Omega$
- $\mu(\emptyset) = 0$
- close under complement
- $\mu(\bigcup_k^\infty A_k) = \sum_k^\infty \mu(A_k)$ where $\{A_k\}$ is disjoint sequence in $\mathscr{F}$ and $\bigcup_k^\infty A_k \in \mathscr{F}$

Definition ***Outer measure*** μ^* is function defined on all subsets of Ω:

- $\mu^*(A) \in [0, \infty]$ for all $A \subset \Omega$
- $\mu^*(\emptyset) = 0$
- μ^* is monotone: $\mu^*(A) \leq \mu^*(B)$ if $A \subset B$
- μ^* is countably subadditive: $\mu^*(\bigcup_n A_n) \leq \sum_n \mu^*(A_n)$

Definition ***Lebesgue measure*** λ_k on $\mathscr{R}^k$ *(Note: R denote euclidean space. $\mathscr{R}$ denotes sigma field on euclidean space.)*

- $\lambda_k(\{(x_1, ..., x_k) | a_i < x_i < b_i\}) = \prod_{i=1}^k (b_i - a_i)$
- $\lambda_k(A) = \lambda_k(A + x)$ (translation invariance)
- $\lambda_k(TA) = |\det T| \lambda_k(A)$, T is linear and nonsingular.
- λ_k is regular (finite measure to bounded set)

Definition Function T between two measure spaces $(\Omega, \mathscr{F})$ and $(\Omega', \mathscr{F}')$ is ***measurable*** $\mathscr{F}/\mathscr{F}'$ if $\forall A \in \mathscr{F}', T^{-1}A \in \mathscr{F}$. We say T is ***measurable*** $\mathscr{F}$ if it is measurable $\mathscr{F}/\mathscr{R}^1$.

Definition ***Probability Space*** is denoted as $(\Omega, \mathscr{F}, P)$

Definition A ***random variable*** on $(\Omega, \mathscr{F}, P)$is a real-valued function $X = X(\omega)$ measurable $\mathscr{F}$. A ***random vector*** is mapping from Ω to R^k that is measurable $\mathscr{F}$. e.g. $X(\omega) = (X_1(\omega), ..., X_k(\omega))$.(pdf)

Definition The ***distribution of random variable*** X is the probability measure μ on $(R^1, \mathscr{R}^1)$ defined by

$$\mu(A) = P[X \in A], \quad A \in \mathscr{R}^1$$

($P[X \in A]$ means $P[\omega : X(\omega) \in A]$. (cdf)

Definition The ***distribution function of random variable*** X is

$$F(x) = \mu(-\infty, x] = P[X \leq x]$$

. If F is right-continuous, and non-decreasing, there is a random variable X on some $(\Omega, \mathscr{F}, P)$corresponding to F.

Integration

Let f, g be real measurable function on $(\Omega, \mathscr{F}, P)$

Definition The ***definite integral*** is denoted:

$$\int f d\mu = \int_\Omega f(\omega) d\mu(\omega) = \int_\Omega f(\omega) \mu(d\omega)$$

is defined by

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu$$

and

$$\int f^\pm d\mu = \sup \sum_i \left[\inf_{\omega \in A_i} f^\pm(\omega) \right] \mu(A_i)$$

Where $\{A_i\}$ is a finite decomposition of Ω into $\mathscr{F}$-sets.

Properties General integral:

- Monotonicity
- Linearity
- (Monotone Convergence)
  if $0 \leq f_n \uparrow f$ almost everywhere then $\int f_n d\mu \uparrow \int f d\mu$
- (Fatou's lemma)
  $\int \liminf_n f_n d\mu \leq \liminf_n \int f_n d\mu, f_n \geq 0$.
- (Dominated Convergence THM)
  $|f_n| \leq g$ almost everywhere (g integrable), $f_n \to f$ almost everywhere, then f, f_n integrable and $\int f_n d\mu \to \int f d\mu$
- f, g integrable and $\int_A f d\mu = \int_A g d\mu$ for all $A in \mathscr{F}$, then $f = g$ almost everywhere.

Definition If δ is nonnegative measurable function, define measure v

$$v(A) = \int_A \delta d\mu, A \in \mathscr{F}$$

. Then v is said to have ***density*** δ wrt to μ.

Properties $\int f dv = \int f \delta d\mu$

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Definition ***Transformatin of Measure***: Given measurable mapping T between $(\Omega, \mathscr{F})$ and $(\Omega', \mathscr{F}')$. For a measure μ on $\mathscr{F}$, define set function μT^{-1} on $\mathscr{F}'$ by

$$\mu T^{-1}(A') = \mu(T^{-1}A'), \forall A' \in \mathscr{F}'$$

Definition ***Change of variable***: Suppose f is real function on Ω' measurable $\mathscr{F}'$, so fT is real function on Ω measurable $\mathscr{F}$.
f is nonnegative or (integratable wrt $\mu T^{-1} \Leftrightarrow fT$ integrable wrt μ). The following hold:

$$\int_\Omega f(T\omega) \mu(d\omega) = \int_{\Omega'} f(\omega') \mu T^{-1}(d\omega')$$

$$\int_{T^{-1}A'} f(T\omega) \mu(d\omega) = \int_{A'} f(\omega') \mu T^{-1}(d\omega')$$

Probability Basics

Properties (Probability measure)

- $p(A \vee B) = p(A) + p(B) - p(A \wedge B)$
- $p(A, B) = p(A \wedge B) = p(A|B)p(B)$
- $p(A) = \sum\limits_b p(A, B) = \sum\limits_b p(A|B = b)p(B = b)$
- $p(X_{1:D}) = p(X_1)p(X_2|X_1)p(X_3|X_2, X_1)...p(X_D|X_{1:D-1})$
- $p(A|B) = \frac{p(A,B)}{p(B)}$ if $p(B) > 0$
- $p(X = x|Y = y) = \frac{p(X=x,Y=y)}{p(Y=y)} = \frac{p(X=x)p(Y=y|X=x)}{\sum_{x'} p(X=x')p(Y=y|X=x')}$
- $X \perp Y \iff p(X, Y) = p(X)p(Y)$
- $X \perp Y|Z \iff p(X, Y|Z) = p(X|Z)p(Y|Z)$
- $X \perp Y|Z \iff p(x, y|z) = g(x, z)h(y, z)\ \forall x, y, z\ s.t\ p(z) > 0$
- $F(q) = p(X \leq q)$
- $f(x) = \frac{d}{dx}F(x)$
- $P(a < X \leq b) = F(b) - F(a) = \int_a^b f(x)dx$
- $Unif(x|a, b) = \frac{1}{b-a}I(a \leq x \leq b)$
- $cov[X, Y] = E[(X - EX)(Y - EY)] = E[XY] - E[X][Y]$