Stationarity

STAT 1321/2320

Kiran Nihlani

Department of Statistics
University of Pittsburgh

Outline

Stationarity

$\{X_t\}$.

k - variables.

$t_1, \ldots, t_k$.

Strict Stationarity

$\{X_t\}$ is **strictly stationary** if

lag h.

$$P(X_{t_1} \leq x_1, \ldots, X_{t_k} \leq x_k) = P(X_{t_1+h} \leq x_1, \ldots, X_{t_k+h} \leq x_k)$$

for all k, $t_1, \ldots, t_k$, $x_1, \ldots, x_k$, and h.

- This implies that shifting the time axis does not affect the joint distribution.
- This is too strong a condition for almost applications.
- We will use a milder version that only focuses on second order properties.

Recap - Mean and Covariance of Random Variables

second-order characteristics.

- For a random variable X, the mean is given by

$$\mu_1' = \mu_X = E(X) = \begin{cases} \sum_x x \cdot f(x) & \text{if } X \text{ is discrete} \\ \int_x x \cdot f(x)dx & \text{if } X \text{ is continuous} \end{cases}$$

- For two random variables X and Y, the covariance is given by

$$\mu_{1,1} = Cov(X, Y) = E[(X - \mu_X)(Y - \mu_Y)]$$

  - Note: $Cov(X, X) = Var(X) = E\left[(X - \mu_X)^2\right] = \mu_2 = \sigma^2$

Mean and Autocovariance

for all values of "t."

Let $\{X_t\}$ be a time series with $E(X_t^2) < \infty$.

- The **mean function** of $\{X_t\}$ is

$$\mu_t = \mu_{xt} = E[X_t]$$

- The **autocovariance function** of $\{X_t\}$ is

$$\gamma_x(s, t) = Cov(X_s, X_t) = E[(X_s - \mu_s)(X_t - \mu_t)]$$

for all integers s and t.

When no possible confusion exists about which time series we are referring to, we will drop the 'x' subscript.

Autocovariance

$\gamma_x(s,t)$: linear dependence
X_s , X_t

- Measures the linear dependence between two points on the same series observed at different times.
  - A smooth series may have a larger autocovariance (even for distant points, s and t) as compared to a choppy one.
- $\gamma_x(s,t) = \gamma_x(t,s)$

  $cov(X,Y) = cov(Y,X)$.
- $\gamma_x(s,t) = 0$ does not imply independence (just no linear dependence).
- For $s = t$,

$$\gamma_x(t,t) = Cov(X_t, X_t) = E[(X_t - \mu_{xt})^2] = Var(X_t)$$

$Var(X) = cov(X,X)$.

Stationarity

$\{x_t\}$, μ_{xt} or $\mu_t = \mu$

Weak Stationarity

$\{X_t\}$ is **weakly stationary** if

1. the mean function, μ_t, is constant and independent of t, and
2. For each h, $\gamma_x(t+h, t)$ is independent of t.

$cov(x_s, x_t)$, $|t-s|$

$cov(x_{t+h}, x_t) = \gamma(h)$.

Note:

- Whenever we use the term stationary we shall mean weakly stationary unless indicated otherwise.
- Strict stationarity along with $E(X_t^2) < \infty$ (for all t) implies weak stationarity.

Weak Stationarity - Condition 2

Condition 2: For each h, $\gamma_x(t+h,t)$ is independent of t.

- Autocovariance $\gamma_x(t+h,t)$ as formulated is a function of two variables, t and h.
- For a stationary series, the dependence reduces to just one variable, h, which is the **lag** between the time points.
- Therefore, for a stationary series, the autocovariance can be rewritten as:

$$\gamma_x(h) = \gamma(h,0) = \gamma(t+h,t)$$

general notation

$t = 0$.

$\gamma_x(h)$ is the value of the autocovariance function at lag h.

Autocorrelation Function (ACF)

Let $\{X_t\}$ be a time series with $E(X_t^2) < \infty$. The **autocorrelation function** is defined as:

$$\rho_X(s,t) = \frac{\gamma_X(s,t)}{\sqrt{\gamma_X(s,s)\gamma_X(t,t)}}$$ ← autocovariance.

for all integers s and t.

Note: If $\{X_t\}$ is a stationary time series, then the autocorrelation function can be written by substituting $s = t + h$ in the above formula

$$\rho_X(t+h,t) = \rho_X(h) = \boxed{\frac{\gamma_X(h)}{\gamma_X(0)}}$$

$$\rho_x(s,t) = \frac{\gamma_x(s,t)}{\sqrt{\gamma_x(s,s)\ \gamma_x(t,t)}}$$

$s = t+h.$

N^r: $\gamma_x(t+h, t) = \gamma_x(h).$

D^r: $[\gamma_x(t+h, t+h)][\gamma_x(t,t)]$

$\gamma_x(0) \cdot \gamma_x(0)$

$$\rho_x(t+h, t) = \frac{\gamma_x(h)}{\sqrt{\gamma_x(0)\ \gamma_x(0)}}$$

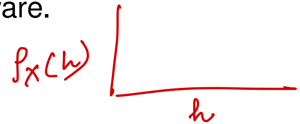
$$\rho_x(h) = \frac{\gamma_x(h)}{\gamma_x(0)}$$

More about Autocorrelation

$\rho_x(t+h, t)$ $h > 0$

x_t informs x_{t+h}

- The ACF measures the linear predictability of the series at time t, say x_t, using only the value x_s.
- $-1 \leq \rho(s, t) \leq 1$.
- Autocorrelation often results in a pattern/structure which is useful for model fitting.
  - We can visualize these patterns by plotting the ACF against the lag.
  - These plots can be drawn by hand or using a software.



Example 1 - iid noise

Let $\{X_t\}$ be an iid noise time series with mean 0 and finite variance σ^2. Find the mean, autocovariance, and autocorrelation function for this series. Also, check if this series is stationary.

- Mean Function: $\mu_{x_t} = \mu_t = E[X_t] = 0$ for all t. (constant).
- Autocovariance Function (ACvF):

$$\gamma_x(t+h, t) = E[(X_{t+h} - \cancel{\mu_{t+h}}^{\,0})(X_t - \cancel{\mu_t}^{\,0})] = E[X_{t+h} X_t]$$

$$= \begin{cases} E(X_t^2) = \sigma^2 & h = 0 \\ 0 & h \neq 0 \end{cases}$$

$$\operatorname{cov}(x, y) = E(xy) - E(x)E(y)$$

if $\operatorname{cov}(x, y) = 0$

$$E(xy) = E(x)E(y).$$

$$E(X_{t+h} X_t) = E(X_{t+h}) E(X_t).$$

$$= 0 \cdot 0 = 0.$$

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Autocovariance only depends on lag $h \implies X_t$ is stationary.

Example 1 - iid noise

- Autocorrelation Function (ACF):

$$\rho_x(t+h,t) = \frac{\gamma_x(t+h,t)}{\sqrt{\gamma_x(t+h,t+h)\gamma_x(t,t)}} = \begin{cases} \dfrac{\gamma_x(t,t)}{\sqrt{\gamma_x(t,t)\,\gamma_x(t,t)}} = \dfrac{\gamma_x(t,t)}{\gamma_x(t,t)} = 1 & h=0 \end{cases}$$

$$h \neq 0, \quad \frac{\gamma_x(t+h,t)}{\sqrt{(\quad)}} = 0$$

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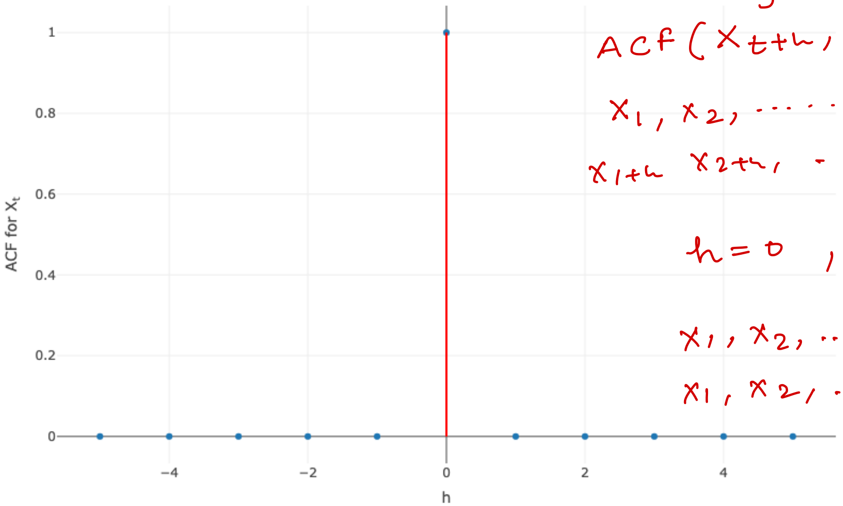
Example 1 - iid noise

$1 \quad h = 0$

$0 \quad h \neq 0$



$y \qquad x$

$ACF(X_{t+h}, X_t)$.

$x_1, x_2, \ldots\ldots$

$x_{1+h}, x_{2+h}, \ldots\ldots$

$h = 0, \quad ACF = 1$

$x_1, x_2, \ldots\ldots$

$x_1, x_2, \ldots\ldots$

(1) List of h values.

seq(-5, 5, by = 1)

(2) ACF values.

(a) If else statement

(b) Define a function.

function(h) { I(h==0) }

(3) Plot.

- scatter plot.
- segments.

segment(x_0, y_0, x_1, y_1)

add_segment.

geom_line.

Example 2 - Moving Average

Let $\{V_t\}$ be a 3 point moving average process of a white noise series (with finite variance σ^2). Find the mean, autocovariance, and autocorrelation function for this series. Also, check if this series is stationary.

$W_t \sim WN(0, \sigma^2)$.

$$V_t = \frac{1}{3}\left(W_{t-1} + W_t + W_{t+1}\right)$$

linearity property for expectations.

- Mean Function:

$$E(V_t) = E\left[\frac{1}{3}\left(W_{t-1} + W_t + W_{t+1}\right)\right] = \frac{1}{3}\left[E(W_{t-1}) + E(W_t) + E(W_{t+1})\right]$$

$$= \frac{1}{3}[0] = 0.$$

$= \mu_{V_t}$

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- Mean Function:

$$E(V_t) = E\left[\frac{1}{3}(W_{t-1} + W_t + W_{t+1})\right] = \frac{1}{3}[E(W_{t-1}) + E(W_t) + E(W_{t+1})]$$

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$\Longrightarrow$ Mean of the series is constant/independent of t

Example 2 - Moving Average

- Autocovariance Function (ACvF):

$$\gamma_v(t+h, t) = Cov(V_{t+h}, V_t)$$

$$= Cov\left[\frac{1}{3}(W_{t+h-1} + W_{t+h} + W_{t+h-1}), \frac{1}{3}(W_{t-1} + W_t + W_{t+1})\right]$$

Example 2 - Moving Average

$V_t = \sum a_i w_{t-i}$

- Autocovariance Function (ACvF):

$$\gamma_v(t+h,t) = Cov(V_{t+h}, V_t)$$

$$= Cov\left[\frac{1}{3}(W_{t+h-1} + W_{t+h} + W_{t+h+1}), \frac{1}{3}(W_{t-1} + W_t + W_{t+1})\right]$$

For $h = 0$

$$\gamma_v(t,t) = Cov\left[\underbrace{\frac{1}{3}(W_{t-1} + W_t + W_{t+1})}_{V_t}, \underbrace{\frac{1}{3}(W_{t-1} + W_t + W_{t+1})}_{V_t}\right]$$

$= \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) cov(w_{t-1}, w_{t-1}) + (1/3)(1/3)\, cov(w_t, w_t) + (1/3)(1/3)\, cov(w_{t+1}, w_{t+1})$

$= (3/9)\, \sigma^2$

$$V_t = \sum a_i w_i$$

$$V_{t+h} = \sum b_j w_i$$

$$\mathrm{cov}(V_t, V_{t+h})$$

$$= \sum_i a_i b_i \,\mathrm{var}(w_i) + \left[\sum_{i \neq j}\sum (a_i b_j + a_j b_i)\, \mathrm{cov}(w_i, w_j) \right]$$

$$= \sum_i a_i b_i \,\mathrm{var}(w_i)$$

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For $h = 0$

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$$= \frac{1}{9}\left[Cov(W_{t-1}, W_{t-1}) + Cov(W_t, W_t) + Cov(W_{t+1}, W_{t+1})\right]$$

Example 2 - Moving Average

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$$= Cov\left[\frac{1}{3}(W_{t+h-1} + W_{t+h} + W_{t+h-1}), \frac{1}{3}(W_{t-1} + W_t + W_{t+1})\right]$$

For $h = 0$

$$\gamma_v(t,t) = Cov\left[\frac{1}{3}(W_{t-1} + W_t + W_{t-1}), \frac{1}{3}(W_{t-1} + W_t + W_{t+1})\right]$$

$$= \frac{1}{9}\left[Cov(W_{t-1}, W_{t-1}) + Cov(W_t, W_t) + Cov(W_{t+1}, W_{t+1})\right] = \frac{3}{9}\sigma^2$$

Example 2 - Moving Average

For $h = 1$:

$$\gamma_v(t+1,t) = Cov(V_{t+1}, V_t) = Cov\left[\overbrace{\frac{1}{3}(W_t + W_{t+1} + W_{t+2})}^{V_{t+1}}, \overbrace{\frac{1}{3}(W_{t-1} + W_t + W_{t+1})}^{V_t}\right]$$

$$\tfrac{2}{9}\sigma^2 = \left(\tfrac{1}{3}\right)\left(\tfrac{1}{3}\right) cov(w_t, w_t) + \left(\tfrac{1}{3}\right)\left(\tfrac{1}{3}\right) cov(w_{t+1}, w_{t+1}) + \left[\left(\tfrac{1}{3}\right)\left(\tfrac{1}{3}\right) + \left(\tfrac{1}{3}\right)\left(\tfrac{1}{3}\right) cov(w_t, w_{t-1})\right] + \ldots$$

Similarly,

- For $h = -1, \gamma_v(t-1,t) =$ $\tfrac{2}{9}\sigma^2$.
- For $h = \pm 2, \gamma_v(t+h,t) =$
- For $|h| > 2, \gamma_v(t+h,t) =$

$h = 2$

$\operatorname{cov}(v_{t+2}, v_t)$

$= \operatorname{cov}\left[\frac{1}{3}\left[w_{t+1} + w_{t+2} + w_{t+3}\right), \frac{1}{3}(w_{t-1} + w_t + w_{t+1})\right]$

$= \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) \operatorname{cov}(w_{t+1}, w_{t+1}) = \frac{\sigma^2}{9}$

$h = 3,$

$\operatorname{cov}\left[\frac{1}{3}(w_{t+2} + w_{t+3} + w_{t+4}), \frac{1}{3}(w_{t-1} + w_t + w_{t+1})\right]$

$= 0.$

$$\gamma_x(t+h, t) = \begin{cases} \frac{3\sigma^2}{9} & h = 0 \\ \frac{2\sigma^2}{9} & h = \pm 1 \\ \frac{\sigma^2}{9} & h = \pm 2 \\ 0 & \text{o.w.} \end{cases}$$

Example 2 - Moving Average

For $h = 1$:

$$\gamma_v(t+1, t) = Cov(V_{t+1}, V_t) = Cov\left[\frac{1}{3}(W_t + W_{t+1} + W_{t+2}), \frac{1}{3}(W_{t-1} + W_t + W_{t+1})\right]$$

$$= \frac{1}{9}\left[Cov(W_{t+1}, W_{t+1}) + Cov(W_t, W_t)\right] = \frac{2}{9}\sigma^2$$

Similarly,

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Similarly,

- For $h = -1, \gamma_v(t-1, t) = \frac{2}{9}\sigma^2$
- For $h = \pm 2, \gamma_v(t+h, t) = \frac{1}{9}\sigma^2$
- For $|h| > 2, \gamma_v(t+h, t) = 0$

Autocovariance function is independent of t and only depends on h.
The mean is constant $\Longrightarrow$ stationarity.

Example 2 - Moving Average

- Autocorrelation Function (ACF):

$$\rho_x(t+h,t) = \frac{\gamma_x(t+h,t)}{\sqrt{\gamma_x(t+h,t+h)\gamma_x(t,t)}} = \begin{cases} \frac{\gamma_x(t,t)}{\sqrt{\gamma_x(t,t)\,\gamma_x(t,t)}} = 1 & h=0 \\ 2/3 & |h|=1 \\ 1/3 & |h|=2 \\ 0 & |h|>2 \end{cases}$$

$h=1$, $\dfrac{\gamma_x(t+1,t)}{\sqrt{\gamma_x(t+1,t+1)\;\gamma_x(t,t)}} = \dfrac{2/9\,\sigma^2}{\sqrt{(3/9\,\sigma^2)(3/9)\sigma^2}}$

$h = 2$

$$\rho(t+2, t) = \frac{\gamma_x(t+2, t)}{\sqrt{\gamma_x(t+2, t+2)\,\gamma_x(t, t)}}$$

$$= \frac{1/9\,\sigma^2}{\sqrt{\left(3/9\,\sigma^2\right)\left(3/9\,\sigma^2\right)}}$$

$$= 1/3$$

Example 2 - Moving Average

- Autocorrelation Function (ACF):

$$\rho_x(t+h,t) = \frac{\gamma_x(t+h,t)}{\sqrt{\gamma_x(t+h,t+h)\gamma_x(t,t)}} = \begin{cases} \frac{\gamma_x(t,t)}{\gamma_x(t,t)} = 1 & \text{for } h = 0 \\ \frac{\gamma_x(t+1,t)}{\sqrt{\gamma_x(t+1,t+1)\gamma_x(t,t)}} = \frac{2}{3} & \text{for } |h| = 1 \\ \frac{\gamma_x(t+2,t)}{\sqrt{\gamma_x(t+2,t+2)\gamma_x(t,t)}} = \frac{1}{3} & \text{for } |h| = 2 \\ 0 & \text{for } |h| > 2 \end{cases}$$

Example 2 - Moving Average

- ACF Plot

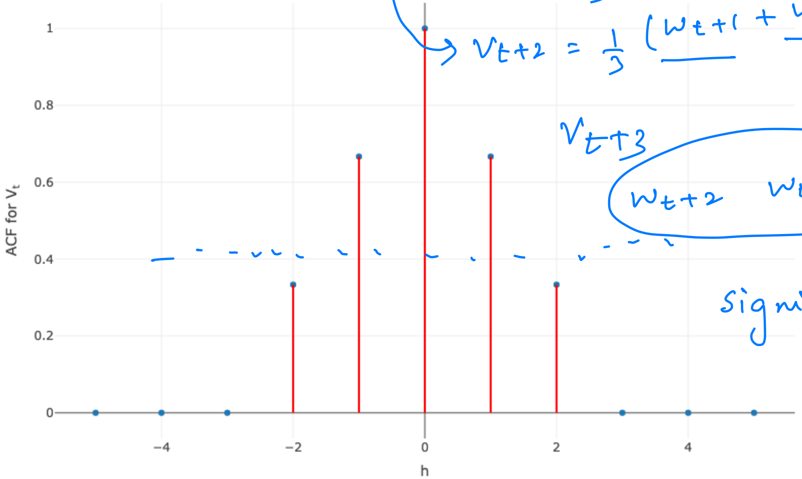
$V_t = \frac{1}{3}[w_{t-1} + w_t + w_{t+1}]$

$V_{t+1} = \frac{1}{3}(w_t + w_{t+1} + w_{t+2})$

$V_{t+2} = \frac{1}{3}(w_{t+1} + w_{t+2} + w_{t+3})$

V_{t+3}

w_{t+2} w_{t+3} w_{t+4}

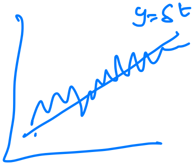


significant?

Example 3 - Random Walk with Drift

$X_t = \delta + X_{t-1} + W_t$.

Consider a random walk model given as



$$X_t = \delta t + \sum_{j=1}^{t} W_j, \qquad t = 1, 2, \ldots$$

where $W_t \sim WN(0, \sigma^2)$. Check stationarity of this series by calculating the mean and autocovariance function.

- Mean Function:

$$\mu_{xt} = E[X_t] = E\left[\delta t + \sum_{j=1}^{t} W_j\right] = E(\delta t) + E\left[\sum_{j=1}^{t} W_j\right]$$

$$= \delta t + \sum_{j=1}^{t} E(W_j)$$

$$= \delta t.$$

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where $W_t \sim WN(0, \sigma^2)$. Check stationarity of this series by calculating the mean and autocovariance function.

- Mean Function:

$$\mu_{xt} = E[X_t] = E\left[\delta t + \sum_{j=1}^{t} W_j\right] = \delta t + \sum_{j=1}^{t} E[W_j] = \delta t$$

For $\delta \neq 0$, mean is not constant and depends on $t \implies$ non-stationarity.

Example 3 - Random Walk with Drift

- Autocovariance Function, $\gamma_x(t+h,t) =$:

$$Cov(X_{t+h}, X_t) = Cov\left(\delta(t+h) + \sum_{j=1}^{t+h} W_j, \delta t + \sum_{j=1}^{t} W_j\right) = Cov\left(\sum_{j=1}^{t+h} W_j, \sum_{j=1}^{t} W_j\right)$$

$$Cov\left(\sum_{j=1}^{t+h} W_j, \sum_{j=1}^{t} W_j\right) = \begin{cases} cov\left(\sum_{j=1}^{t} w_j + \sum_{j=t+1}^{t+h} w_j, \sum_{j=1}^{t} w_j\right) & t+h > t \\ = t\, cov(w_j, w_j) \\ = t\, var(w_j) = \sigma^2 t. & t+h < t \end{cases}$$

$$y_1 = \sum_i a_i x_i \qquad y_2 = \sum_j b_j x_j$$

$$\mathrm{cov}(y_1, y_2)$$

$$= \sum_{\substack{i \\ i=j}} a_i b_i \,\underset{\mathrm{cov}(x_i, x_i)}{\mathrm{var}(x_i)} + \underbrace{\sum_{i \neq j}\sum (a_i b_j + a_j b_i)\, \mathrm{cov}(x_i, x_j)}_{0}$$

x_i's $\equiv w_i$

$a_i = 1$ $\qquad \sum_i \mathrm{cov}(w_i, w_i)$

$b = 1$

Example 3 - Random Walk with Drift

- Autocovariance Function, $\gamma_x(t+h,t) =:$

$$Cov(X_{t+h}, X_t) = Cov\left(\delta(t+h) + \sum_{j=1}^{t+h} W_j, \delta t + \sum_{j=1}^{t} W_j\right) = Cov\left(\sum_{j=1}^{t+h} W_j, \sum_{j=1}^{t} W_j\right)$$

$$Cov\left(\sum_{j=1}^{t+h} W_j, \sum_{j=1}^{t} W_j\right) = \begin{cases} Cov\left(\sum_{j=1}^{t} W_j + \sum_{j=t+1}^{t+h} W_j, \sum_{j=1}^{t} W_j\right) = t\sigma^2, & \text{if } t+h > t, \\ \end{cases}$$

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$$= \min\{t+h, t\}\, \sigma^2 \quad \leftarrow \text{function of } h \text{ and } t.$$

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$$= min\{t+h, t\}\sigma^2$$

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$$= min\{t+h, t\}\sigma^2 \text{ (dependent on } t \text{ for all } \delta \implies \text{ non-stationarity)}$$

Example 4 - MA(1) Process

An MA(1) Process is a first order moving average process given by

$$X_t = W_t + \theta W_{t-1}, \quad t = 0, \pm 1, \pm 2, \ldots$$

where $\{W_t\} \sim WN(0, \sigma^2)$ and θ is a constant.

Example 4 - MA(1) Process

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where $\{W_t\} \sim WN(0, \sigma^2)$ and θ is a constant.

- Mean Function:

$$\mu_{xt} = E[X_t] = E[w_t + \theta w_{t-1}] = E(w_t) + \theta E(w_{t-1})$$

$$= 0$$

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where $\{W_t\} \sim WN(0, \sigma^2)$ and θ is a constant.

- Mean Function:

$$\mu_{xt} = E[X_t] = E[W_t] + \theta W_{t-1}] = E[W_t \ + \theta E[W_{t-1}] = 0$$

$\Longrightarrow$ Mean of the series is constant/independent of t

Example 4 - MA(1) Process

$\gamma_x(t+h, t)$

$= E[(X_{t+h} - \mu_{t+h})(X_t - \mu_t)]$

$= E[X_{t+h} X_t]$

- Autocovariance Function:

$$\gamma_x(t+h,t) = E[X_{t+h}X_t] = E[(W_{t+h} + \theta W_{t+h-1})(W_t + \theta W_{t-1})]$$

$\mathrm{var}(W_t) = \sigma^2$

$= E(W_t^2) - \mu_t^2$

$$= \begin{cases} E(W_t + \theta W_{t-1})^2 & h = 0 \\ E(W_t^2) + \theta^2 E(W_{t-1}^2) + 2\theta E(W_t W_{t-1}) & h = 1 \\ = \sigma^2 + \theta^2\sigma^2 + 2\theta E(W_t)E(W_{t-1}) & \\ = \sigma^2(1+\theta^2) & \end{cases}$$

$h = 1$

$$\gamma_x(t+h, t) = E(X_{t+h} X_t)$$

$$= E\left[(w_{t+1} + \theta w_t)(w_t + \theta w_{t-1})\right]$$

$$= E(w_{t+1} w_t) + \theta E(w_{t+1} w_{t-1})$$

$$+ \theta E(w_t^2) + \theta^2 E(w_t w_{t-1})$$

$$= \theta E(w_t^2) = \theta \sigma^2$$

$h = 2$

$$E(X_{t+2} X_t) = E\left[(w_{t+2} + \theta w_{t+1})(w_t + \theta w_{t-1})\right]$$

Example 4 - MA(1) Process

$w_t + \theta w_{t-1}$

- Autocovariance Function:

$$\gamma_x(t+h,t) = E[X_{t+h}X_t] = E[(W_{t+h} + \theta W_{t+h-1})(W_t + \theta W_{t-1})]$$

$$= \begin{cases} E[W_t W_t] + \theta^2 E[W_{t-1} W_{t-1}] = (1+\theta^2)\sigma^2 & \text{for } h = 0 \\ \theta\sigma^2 & \text{for } |h| = 1 \\ 0 & \text{for } |h| > 1 \end{cases}$$

MA(2).

Example 4 - MA(1) Process

- Autocovariance Function:

$$\gamma_x(t+h,t) = E[X_{t+h}X_t] = E[(W_{t+h} + \theta W_{t+h-1})(W_t + \theta W_{t-1})]$$

$$= \begin{cases} E[W_t W_t] + \theta^2 E[W_{t-1} W_{t-1}] = (1+\theta^2)\sigma^2 & \text{for } h = 0 \\ \theta\sigma^2 & \text{for } |h| = 1 \\ 0 & \text{for } |h| > 1 \end{cases}$$

Autocovariance function is independent of t and only depends on h.
The mean is constant $\implies$ stationarity.

Example 4 - MA(1) Process

- Autocorrelation Function (ACF): $\rho_x(t+h,t) = \begin{cases} 1 & \text{for } h=0 \\ \frac{\theta}{1+\theta^2} & \text{for } |h|=1 \\ 0 & \text{for } |h|>1 \end{cases}$

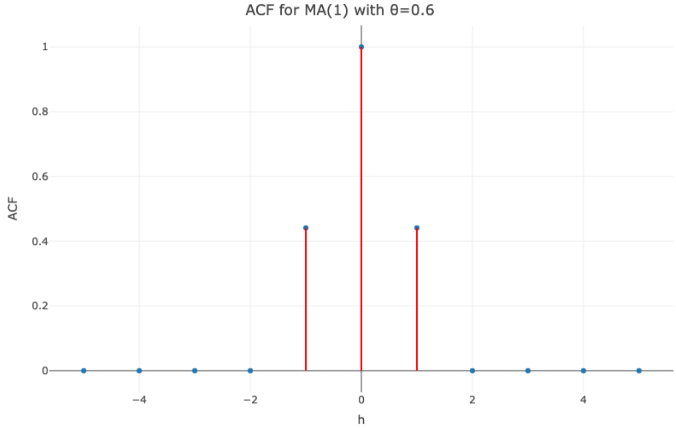
$|h| = 1,$

$$\rho_x(t+1,t) = \frac{\gamma_x(t+1,t)}{\sqrt{\gamma_x(t+1,t+1)\,\gamma_x(t,t)}} = \frac{\theta\sigma^2}{\sqrt{[(1+\theta^2)\sigma^2][(1+\theta^2)\sigma^2]}}$$

$$= \frac{\theta\sigma^2}{(1+\theta^2)\sigma^2} = \frac{\theta}{1+\theta^2}$$

Example 4 - MA(1) Process

- Autocorrelation Function (ACF): $\rho_x(t+h,t) = \begin{cases} 1 & \text{for } h = 0 \\ \frac{\theta}{1+\theta^2} & \text{for } |h| = 1 \\ 0 & \text{for } |h| > 1 \end{cases}$



$\theta = -0.6.$

Example 5 - AR(1) Process

$x_t = \phi x_{t-1} + w_t$

An AR(1) model is an autoregressive setup of order 1

$\mathrm{cov}(x_s, w_t)$ $s < t$

$= 0$

$$X_t = \phi X_{t-1} + W_t, \quad t = 0, \pm 1, \pm 2, \ldots$$

where $\{W_t\} \sim WN(0, \sigma^2)$ and W_t is uncorrelated with X_s for $s < t$. Assume that $\{X_t\}$ is stationary and $0 < |\phi| < 1$. Find the mean, autocovariance and autocorrelation functions for $\{X_t\}$.

Example 5 - AR(1) Process

$\mu = \phi \mu$

$\Rightarrow \mu(\phi - 1) = 0$

$\mu = 0$

An AR(1) model is an autoregressive setup of order 1

$$X_t = \phi X_{t-1} + W_t, \quad t = 0, \pm 1, \pm 2, \ldots$$

where $\{W_t\} \sim WN(0, \sigma^2)$ and W_t is uncorrelated with X_s for $s < t$. Assume that $\{X_t\}$ is stationary and $0 < |\phi| < 1$. Find the mean, autocovariance and autocorrelation functions for $\{X_t\}$.

- Mean function:

$$\mu_{xt} = E[X_t] = E[\phi X_{t-1} + W_t] = \phi E(X_{t-1}) + E(W_t)$$

$$\mu_{xt} = \phi \mu_{x(t-1)}$$

stationarity:
If mean is constant.

$$\mu_{xt} = \mu_{x(t-1)} = \mu \text{ (say)}$$

Example 5 - AR(1) Process

An AR(1) model is an autoregressive setup of order 1

$$X_t = \phi X_{t-1} + W_t, \quad t = 0, \pm 1, \pm 2, \ldots$$

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- Mean function:

$$\mu_{xt} = E[X_t] = E[\phi X_{t-1} + W_t] = \phi E[X_{t-1}] + E[W_t] = \phi \mu_{x(t-1)}$$

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Since X_t is stationary, the mean should be constant: $\mu_{xt} = \mu_{x(t-1)} = \mu_x$

Example 5 - AR(1) Process

An AR(1) model is an autoregressive setup of order 1

$$X_t = \phi X_{t-1} + W_t, \quad t = 0, \pm 1, \pm 2, \ldots$$

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Since X_t is stationary, the mean should be constant: $\mu_{xt} = \mu_{x(t-1)} = \mu_x$

$$\mu_x = \phi \mu_x \implies \mu_x = 0$$

Example 5 - AR(1) Process

$\gamma_x(t+h, t) = \gamma_x(h)$. $h > 0$

$t+h > t$.

- Autocovariance Function (ACvF):

For $h > 0$, $\gamma_x(h) = \gamma_x(-h) = Cov(X_{t+h}, X_t) = Cov(\phi X_{t+h-1} + W_{t+h}, X_t)$

$= \phi \, cov(X_{t+h-1}, X_t)$

$+ \underline{cov(W_{t+h}, X_t)}$

$= \phi \, cov(\underbrace{X_{t+h-1}, X_t}_{h-1})$

$= \phi \, \gamma_x(h-1)$

$= \phi \left[\phi \, \gamma_x(h-2) \right] = \ldots\ldots = \phi^h \gamma_x(0)$

Example 5 - AR(1) Process

- Autocovariance Function (ACvF):

For $h > 0$,
$$\gamma_x(h) = \gamma_x(-h) = Cov(X_{t+h}, X_t) = Cov(\phi X_{t+h-1} + W_{t+h}, X_t)$$
$$= \phi Cov(X_{t+h-1}, X_t) + Cov(W_{t+h}, X_t)$$

Example 5 - AR(1) Process

- Autocovariance Function (ACvF):

For $h > 0$,
$$\begin{aligned}\gamma_x(h) = \gamma_x(-h) = Cov(X_{t+h}, X_t) &= Cov(\phi X_{t+h-1} + W_{t+h}, X_t) \\ &= \phi Cov(X_{t+h-1}, X_t) + Cov(W_{t+h}, X_t) \\ &= \phi Cov(X_{t+h-1}, X_t) + 0\end{aligned}$$

Example 5 - AR(1) Process

- Autocovariance Function (ACvF):

For $h > 0$,

$$\begin{aligned}
\gamma_x(h) = \gamma_x(-h) = Cov(X_{t+h}, X_t) &= Cov(\phi X_{t+h-1} + W_{t+h}, X_t) \\
&= \phi Cov(X_{t+h-1}, X_t) + Cov(W_{t+h}, X_t) \\
&= \phi Cov(X_{t+h-1}, X_t) + 0 \\
&= \phi \gamma_x(h-1)
\end{aligned}$$

Example 5 - AR(1) Process

$= \phi^2 \gamma_x(0) + \sigma^2$

- Autocovariance Function (ACvF):

For $h > 0$,
$$\gamma_x(h) = \gamma_x(-h) = Cov(X_{t+h}, X_t) = Cov(\phi X_{t+h-1} + W_{t+h}, X_t)$$
$$= \phi Cov(X_{t+h-1}, X_t) + Cov(W_{t+h}, X_t)$$
$$= \phi Cov(X_{t+h-1}, X_t) + 0$$
$$= \phi \gamma_x(h-1) = \cdots = \phi^h \gamma_x(0)$$

$$\gamma_x(0) = Cov(X_t, X_t) = Cov(\phi X_{t-1} + W_t, \phi X_{t-1} + W_t)$$
$$= \phi^2 cov(X_{t-1}, X_{t-1}) + \phi\, cov(X_{t-1}, W_t)$$
$$+ \phi\, cov(W_t, X_{t-1}) + cov(W_t, W_t)$$

$$\gamma_x(0) = \phi^2 \operatorname{cov}(\underbrace{x_{t-1}, x_{t-1}}_{\text{lag } 0}) + \operatorname{cov}(\omega_t, \omega_t)$$

$$\gamma_x(0) = \phi^2 \gamma_x(0) + \sigma^2$$

$$[1 - \phi^2]\, \gamma_x(0) = \sigma^2$$

$$\Rightarrow \gamma_x(0) = \frac{\sigma^2}{1 - \phi^2}$$

Example 5 - AR(1) Process

- Autocovariance Function (ACvF):

For $h > 0$,

$$\begin{aligned}\gamma_x(h) = \gamma_x(-h) = Cov(X_{t+h}, X_t) &= Cov(\phi X_{t+h-1} + W_{t+h}, X_t) \\ &= \phi Cov(X_{t+h-1}, X_t) + Cov(W_{t+h}, X_t) \\ &= \phi Cov(X_{t+h-1}, X_t) + 0 \\ &= \phi \gamma_x(h-1) = \cdots = \phi^h \gamma_x(0)\end{aligned}$$

$$\gamma_x(0) = Cov(X_t, X_t) = Cov(\phi X_{t-1} + W_t, \phi X_{t-1} + W_t) = \phi^2 \gamma_x(0) + \sigma^2$$

$\gamma_x(0) = var(x_t) > 0$

$\frac{\sigma^2}{1-\phi^2} > 0$

$|\phi| < 1$

Example 5 - AR(1) Process

- Autocovariance Function (ACvF):

For $h > 0$,

$$\gamma_x(h) = \gamma_x(-h) = Cov(X_{t+h}, X_t) = Cov(\phi X_{t+h-1} + W_{t+h}, X_t)$$
$$= \phi Cov(X_{t+h-1}, X_t) + Cov(W_{t+h}, X_t)$$
$$= \phi Cov(X_{t+h-1}, X_t) + 0$$
$$= \phi \gamma_x(h-1) = \cdots = \phi^h \gamma_x(0)$$

$$\gamma_x(0) = Cov(X_t, X_t) = Cov(\phi X_{t-1} + W_t, \phi X_{t-1} + W_t) = \phi^2 \gamma_x(0) + \sigma^2$$

$$\implies \gamma_x(0) = \frac{\sigma^2}{1-\phi^2}$$

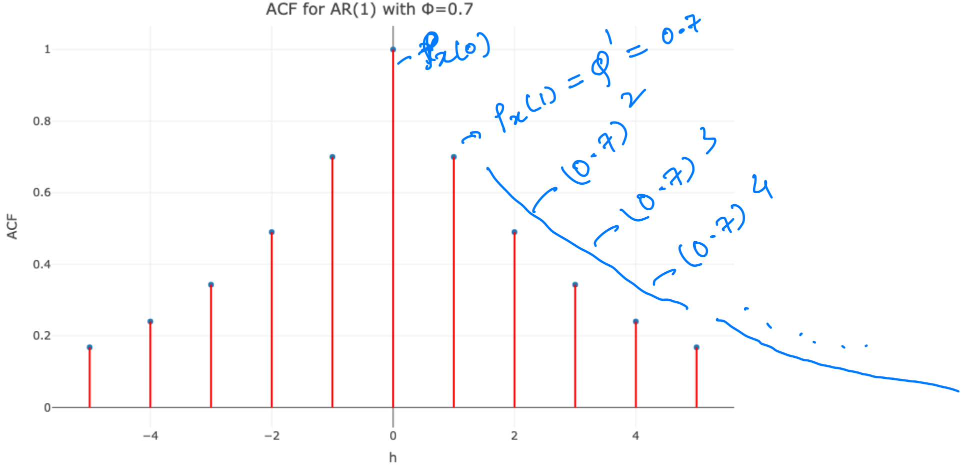
Example 5 - AR(1) Process

$\gamma_x(h) = \phi^{|h|} \gamma_x(0)$

- Autocorrelation Function (ACF): $\rho_x(h) = \frac{\gamma_x(h)}{\gamma_x(0)} = \phi^{|h|}$ for all h.

Example 5 - AR(1) Process

- Autocorrelation Function (ACF): $\rho_x(h) = \frac{\gamma_x(h)}{\gamma_x(0)} = \phi^{|h|}$ for all h.



Graphical Inspection

A stationary time series is a time series where there are no changes in the underlying system.

periodic trends.

$\gamma_x(t+h, t) = \gamma_x(h)$

variance: $h = 0$, $\gamma_x(0)$

- Constant mean (no trend)
- Constant variance (no heteroskedasticity)
- Constant autocorrelation structure
- No periodic component (no seasonality)

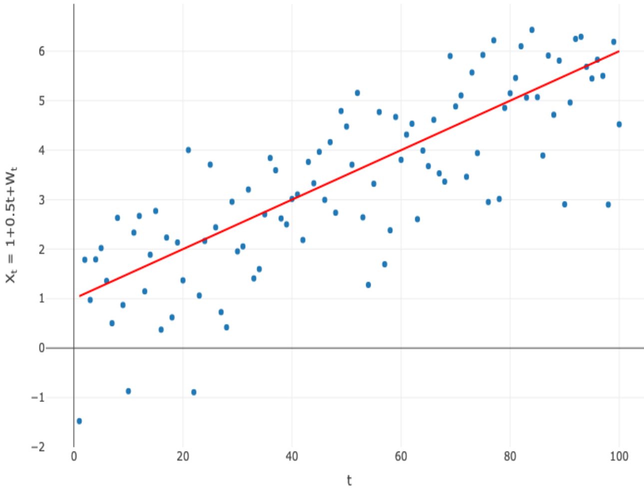
Can these be inferred from the mathematical definition we wrote earlier?

Models with Trend

- Consider a model with linear trend

$$X_t = 1 + 0.5t + W_t$$

where $\{W_t\}$ is iid $N(0, 1)$



Models with Trend

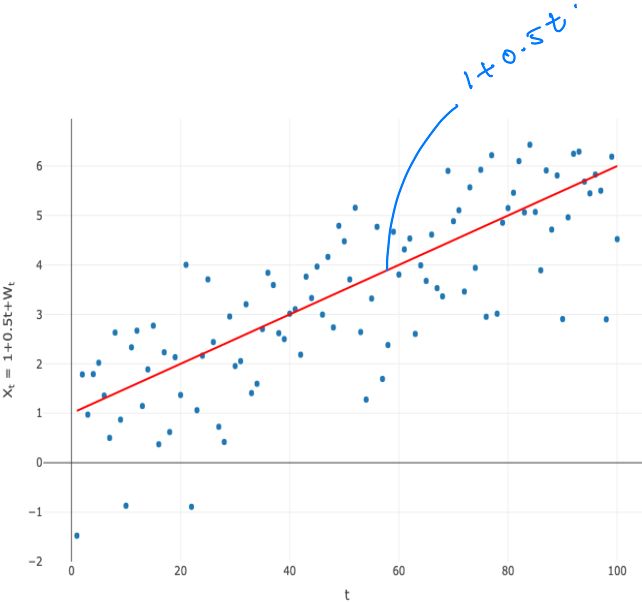
- Consider a model with linear trend

$$X_t = 1 + 0.5t + W_t$$

where $\{W_t\}$ is iid $N(0,1)$

- The mean function of the series is given by

$$\mu_{xt} = E[X_t] = 1 + 0.5t$$



Models with Trend

- Consider a model with linear trend

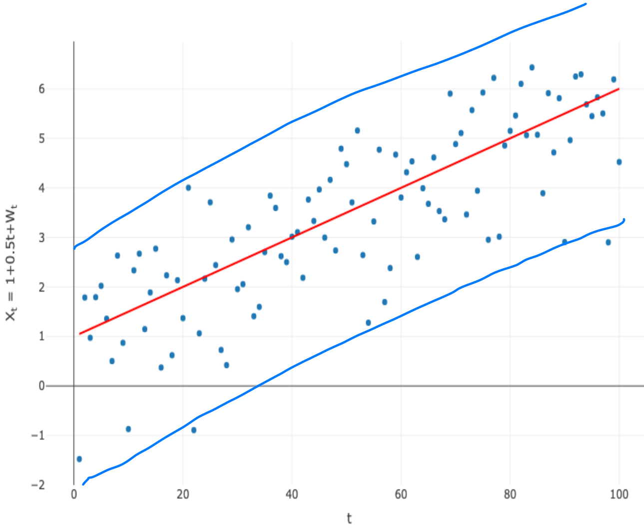
$$X_t = 1 + 0.5t + W_t$$

where $\{W_t\}$ is iid $N(0, 1)$

- The mean function of the series is given by

$$\mu_{xt} = E[X_t] = 1 + 0.5t$$

$\Longrightarrow$ non-stationarity since mean function is not constant.

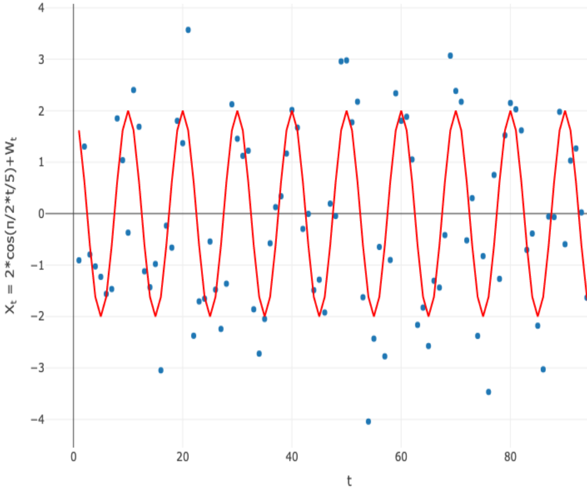


Model with Seasonality

- Consider a model with seasonality

$$X_t = 2\cos(\pi t/5) + W_t$$

where $\{W_t\}$ is iid $N(0, 1)$



Model with Seasonality

- Consider a model with seasonality

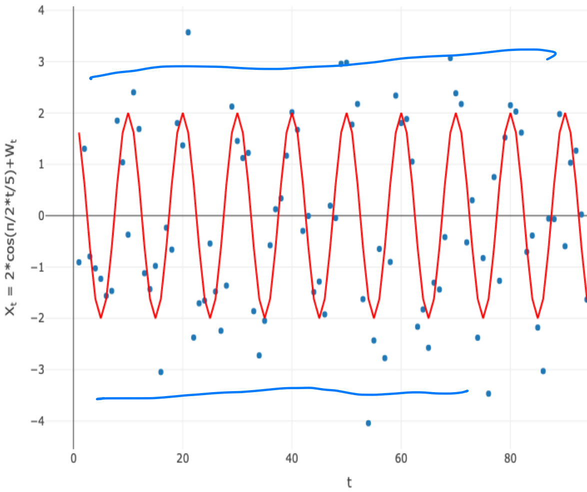
$$X_t = 2\cos(\pi t/5) + W_t$$

where $\{W_t\}$ is iid $N(0, 1)$

- The mean function of tho series is given by

$$\mu_{xt} = E[X_t] = 2\cos(\pi t/5)$$

seasonal trend.



Model with Seasonality

- Consider a model with seasonality

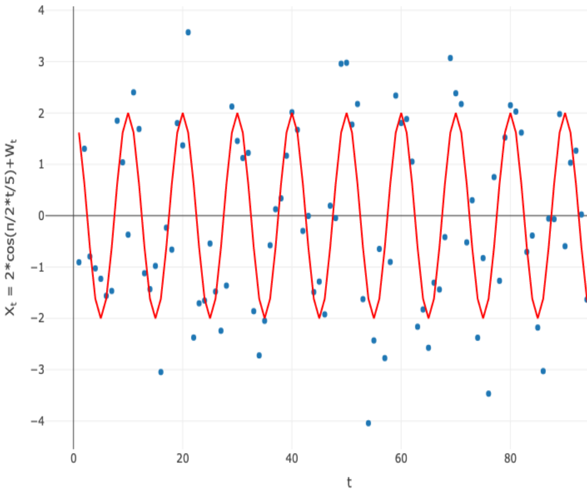
$$X_t = 2\cos(\pi t/5) + W_t$$

where $\{W_t\}$ is iid $N(0,1)$

- The mean function of tho series is given by

$$\mu_{xt} = E[X_t] = 2\cos(\pi t/5)$$

$\implies$ non-stationarity since mean function is not constant.



Models with Cycles

A model with some cyclical behavior and no trend or seasonality is stationary.

Choose 1 below and explain.

- True ✓
- False

Graphical Inspection - Stationarity

- In general, a stationary time series will have no predictable patterns in the long-term.
- Time plots will show the series to be roughly horizontal (although some cyclic behaviour is possible), with constant variance.
  - This can be verified by calculating the mean and variance over time.
- The time invariant autocorrelation structure can be verified using the ACF plots.

Review

a, b : constants.

- If $X_1, X_2, \ldots, X_n$ are random variables and

$$Y_1 = \sum_{i=1}^{n} a_i X_i \quad \text{and} \quad Y_2 = \sum_{i=1}^{n} b_i X_i$$

where $a_1, a_2, \ldots\ldots, a_n$ and $b_1, b_2, \ldots\ldots, b_n$ are constants, then

$$cov(Y_1, Y_2) = \sum_{i=1}^{n} a_i b_i \cdot var(X_i) + \sum_{i<j}\sum (a_i b_j + a_j b_i) \cdot cov(X_i, X_j)$$

$i \neq j$

indep.

$i = j$

- If $X_1, X_2, \ldots, X_n$ are random variables and

$$Y_1 = \sum_{i=1}^{n} a_i X_i \quad \text{and} \quad Y_2 = \sum_{i=1}^{n} b_i X_i$$

then

$$cov(Y_1, Y_2) = \sum_{i=1}^{n} a_i b_i \cdot var(X_i)$$