Stationary Time Series Models - Autoregressive Models

STAT 1321/2320

Kiran Nihlani

Department of Statistics
University of Pittsburgh

Outline

AR(1) Process - Review

- A zero mean AR(1) process can be written as

$$X_t = \phi X_{t-1} + W_t, \qquad \boxed{|\phi| < 1}$$

- The mean function of the process, $\mu_t = 0$
- The autocovariance and autocorrelation functions are given by

$$\gamma(h) = \begin{cases} \frac{\sigma^2}{1-\phi^2} & h = 0 \\ \phi^{|h|}\left(\frac{\sigma^2}{1-\phi^2}\right) & h \neq 0 \end{cases} \qquad \rho(h) = \begin{cases} 1 & h = 0 \\ \phi^{|h|} & h \neq 0 \end{cases}$$

exponential decay.

- The process is stationary.

Autocorrelations for AR(1) Model

$ACF = \phi^{|h|}$

- Since $|\phi| < 1$, the autocorrelation gets closer to zero (weaker) as the number of lags increases.
  - If $0 < \phi < 1$, all the autocorrelations are positive.
  - Value of the process is associated with very recent values much more than with values far in the past.
- If $-1 < \phi < 0$, the lag-1 autocorrelation is negative, and the signs of the autocorrelations alternate from positive to negative over the further lags.
- For ϕ near 1, the overall graph of the process will appear smooth, while for ϕ near -1, the overall graph of the process will appear jagged.

Simulated AR(1) Models

Simulate two AR(1) models given by

$$X_t = 0.9X_{t-1} + W_t \qquad\qquad X_t = -0.9X_{t-1} + W_t$$

where $W_t \sim N(0, 1)$.

Explore the time series plots, ACF plots, and lagged scatter plots for these.

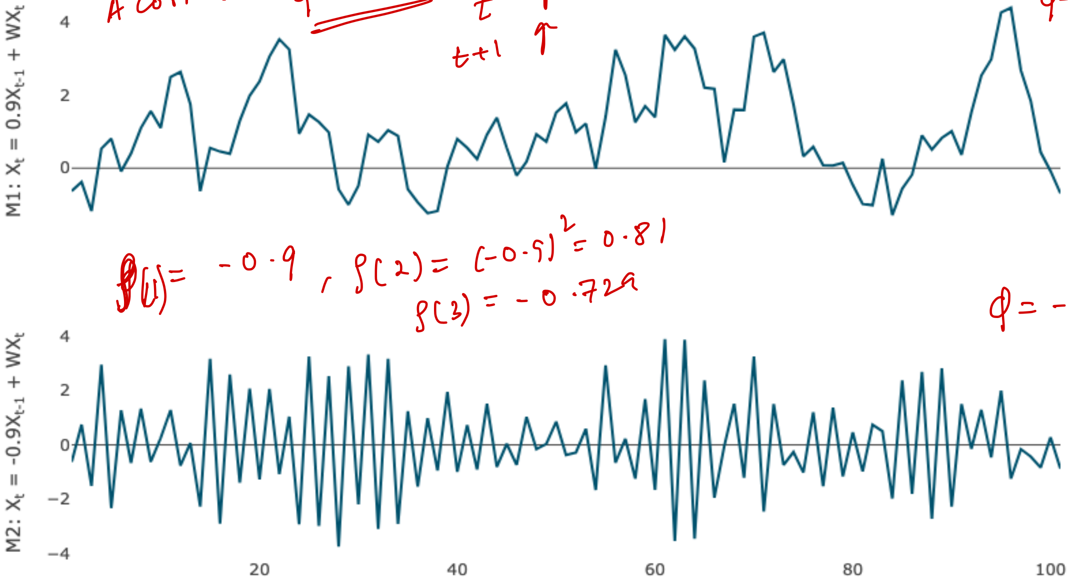
A corr : $\rho(h) \geq 0$; $t \uparrow$ $t+1 \uparrow$

$\phi = 0.9$

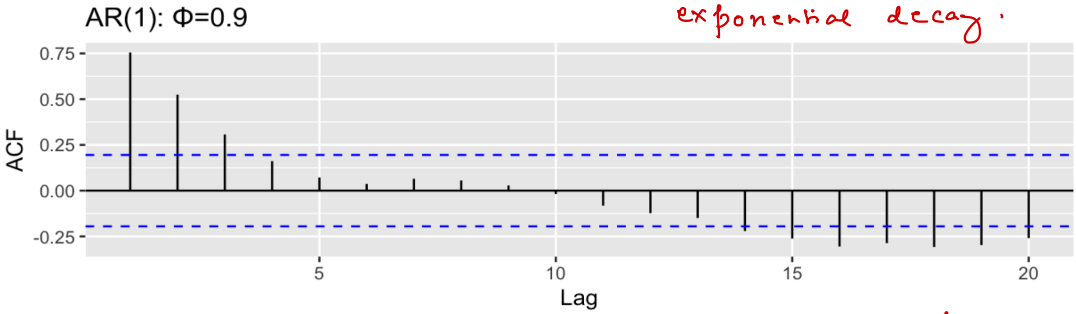
$\rho(1) = -0.9$, $\rho(2) = (-0.9)^2 = 0.81$

$\rho(3) = -0.729$

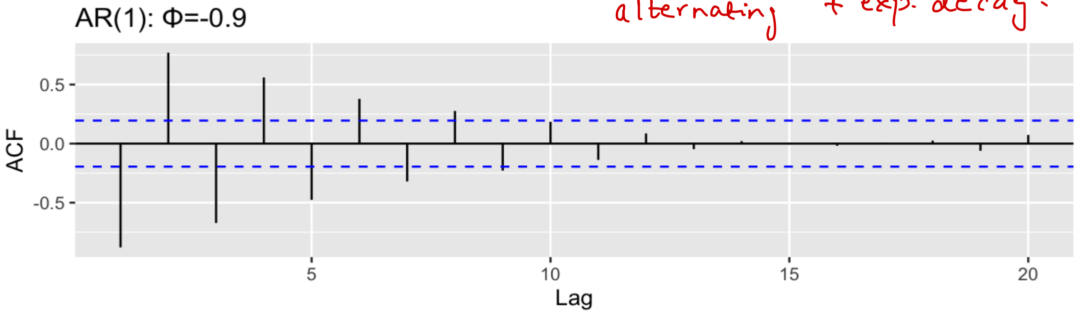
$\phi = -0.9$



exponential decay.



alternating + exp. decay.



Scatter Plots with Lagged Series: AR(1) with $\phi = 0.9$

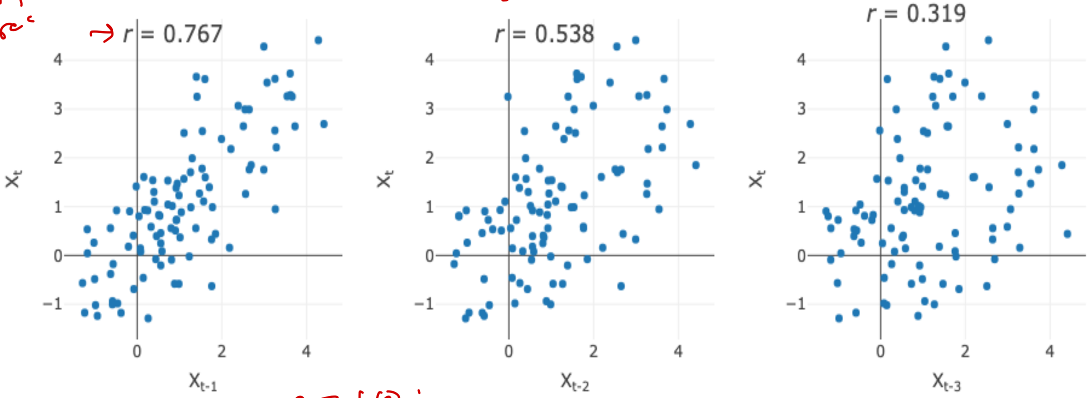
- Theoretically, the ACF should be $\phi^{|h|} = 0.9^{|h|}$ for $|h|$=1, 2, 3.

$\rho = 0.729.$

sample corre

$\rho = 0.9$

$\rho = 0.81$



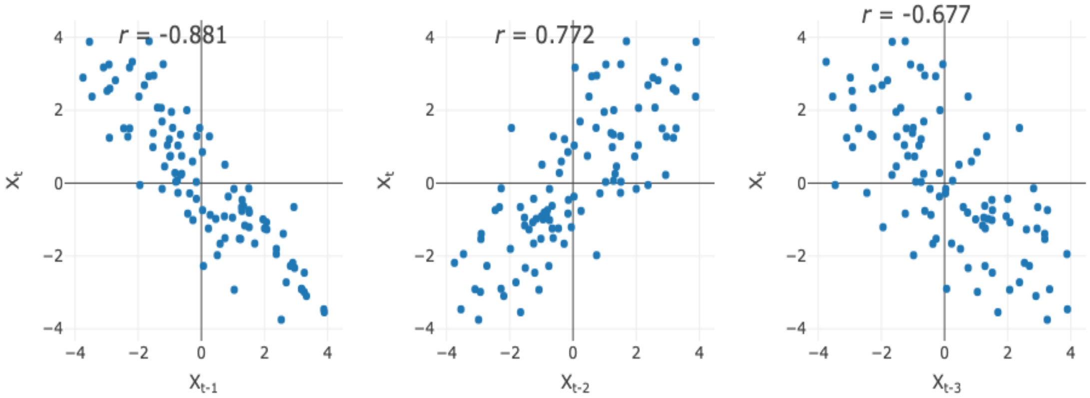
$n = 100.$

Scatter Plots with Lagged Series: AR(1) with $\phi = -0.9$

- Theoretically, the ACF should be $\phi^{|h|} = 0.9^{|h|}$ for $|h|$=1, 2, 3.

$\rho = -0.729$

$\rho = -0.9$ $\rho = 0.81$



X_t : AR(1) model.

$$X_t = \phi X_{t-1} + w_t.$$

$$y = \beta_1 X + \varepsilon_t.$$

y: X_t, $\text{lm}(X_t \sim X_{t-1})$

x: X_{t-1}

AR(1) Process with Non-Zero Mean

$X_t \;,\; E(X_t) = \mu$

- Consider a stationary AR(1) process that has mean μ, then the AR process can be written as

$x_t \to X_t - \mu$.

$$\begin{aligned} X_t - \mu &= \phi(X_{t-1} - \mu) + W_t \\ \implies X_t &= (1-\phi)\mu + \phi X_{t-1} + W_t \\ \implies X_t &= \phi_0 + \phi X_{t-1} + W_t \end{aligned}$$

- This is like fitting a regression model with intercept.

mean for your time series.

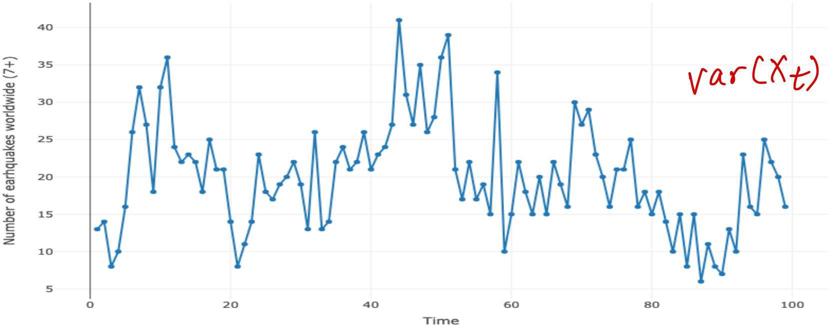
Example- Earthquake

AR(1) model
: stationary.

- Data is collected on number of earthquakes with magnitude greater than or equal to 7.
- Summarized data is available in quakes.dat file on Canvas.

mean : indep of "t" / constant

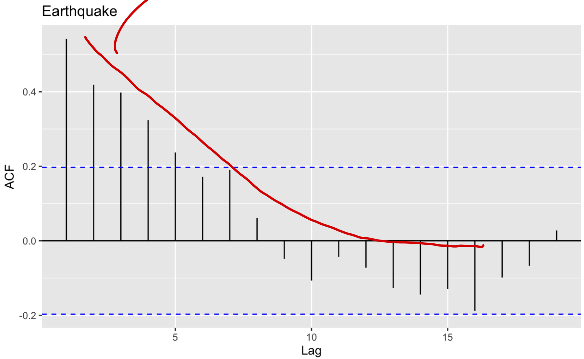
var(X_t) should be indep of "t" ??

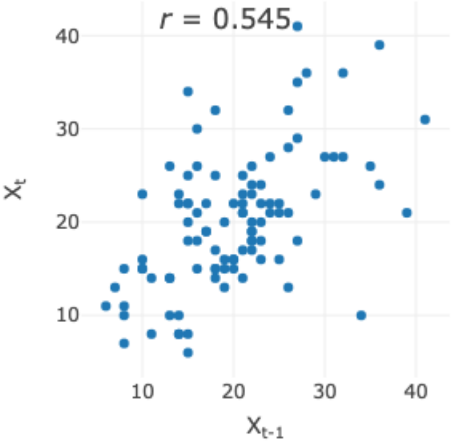


Earthquake - AR Model

x_t x_{t-2}

AR model is probable -





Earthquake - Linear Regression

$X_t = \phi_0 + \phi X_{t-1} + \varepsilon_t$.

- Fit a linear regression model with X_t as the response and X_{t-1} as the explanatory variable.

SLR : $r^2 = R\text{-sq}$.

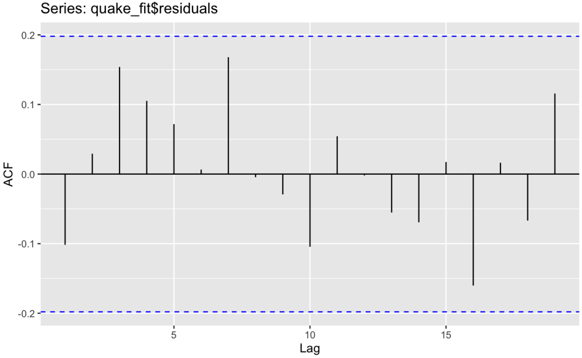
ϕ_0

	Estimate	Std. Error	t value	Pr(>|t|)
Intercept	9.19070	1.81924	5.052	2.08e-06 ***
X_{t-1}	0.54339	0.08528	6.372	6.47e-09 ***

ϕ

- AR(1) with non-zero mean is a probable model for the data.
- R-squared is pretty low at approximately 29%.

$\text{corr}(X_t, X_{t-1}) = \phi$



Causality

B - Stationarity.

A linear process $\{X_t\}$ is causal (strictly, a causal function of $\{W_t\}$), if there is a

$$\psi(B) = \psi_0 + \psi_1 B + \psi_2 B^2 + \cdots$$

polynomial with Back shift.

with $\sum_{j=0}^{\infty} |\psi_j| < \infty$ such that

$$(\psi_0 + \psi_1 B + \psi_2 B^2 + \cdots) W_t$$

$$= \psi_0 W_t + \psi_1 W_{t-1} + \psi_2 W_{t-2} + \cdots$$

$$X_t = \psi(B) W_t = \sum_{j=0}^{\infty} \psi_j W_{t-j}$$

This means the current status only relates to the past events, not the future.

X_t $\quad$ W_t

3 point:

MA smoother:

$$V_t = \frac{w_{t-1} + w_t + w_{t+1}}{3}$$

- Linear process
- not a causal process

$$V_t = \sum \psi_j w_{t-j} \qquad \sum |\psi_j| < \infty$$

AR(1) as a Linear Process

$x_t = \phi x_{t-1} + w_t$

$x_t - \phi x_{t-1} = w_t.$

- An AR(1) process is

$$X_t = \phi X_{t-1} + W_t \implies (1 - \phi B) X_t = W_t$$

$$\implies X_t = (1 - \phi B)^{-1} W_t$$

Correction

$$\implies X_t = (1 + \phi B + \phi^2 B^2 + \phi^3 B^3 + \cdots) W_t$$

$$\implies X_t = \sum_{j=0}^{\infty} \phi^j W_{t-j}$$

- This shows AR(1) is a linear process with $\mu = 0$ and $\psi_j = \phi^j$ for $j \geq 0$ and 0, otherwise.

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 + \cdots$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

$$(1-\phi B)^{-1} = 1 + \phi B + \phi^2 B^2 + \phi^3 B^3 + \phi^4 B^4 + \cdots$$

$$X_t = \phi X_{t-1} + w_t$$

$$= \phi(\phi X_{t-2} + w_{t-1}) + w_t$$

$$= \phi^2 X_{t-2} + \phi w_{t-1} + w_t$$

$$= \cdots\cdots$$

$$= w_t + \phi w_{t-1} + \phi^2 w_{t-2} + \cdots$$

$$\psi_j = \begin{cases} \phi^j & \text{for } j \geq 0 \\ 0 & \text{otherwise.} \end{cases} \qquad \sum |\psi_j| < \infty$$

$1 + \phi + \phi^2 + \phi^3 + \cdots$

converge. $\longrightarrow \dfrac{1}{1-\phi}$ if $|\phi| < 1$

Geo. series:

$a + ar + ar^2 + \cdots$

$\longrightarrow \dfrac{a}{1-r}$ if $|r| < 1$

Causality of AR(1)

$$\sum_j |\phi^j| < \infty.$$

converge.

- An AR(1) process is

$$X_t = \sum_{j=0}^{\infty} \phi^j W_{t-j}$$

- When $\boxed{|\phi| < 1,}$ AR(1) process $\{X_t\}$ is a causal function of $\{W_t\}$.
- When $|\phi| > 1$, AR(1) process is not causal.

$|\phi| < 1 \longrightarrow$ stationary.

$$X_t = 0.5\, X_{t-1} + w_t.$$

$$\rightarrow X_t = 2\, X_{t-1} + w_t. \leftarrow$$

"explosive"

$$X_0 = 1$$

$$X_1 = 2 + w_t$$

$$X_2 = 4 + 2w_t + w_t$$

$$\gamma(t) = t\sigma^2,$$

Characteristic Polynomial and Equation

unit-root.

- An AR(1) process is

$$(1 - \phi B)X_t = W_t$$

$\Rightarrow \underbrace{\Phi(B)}_{\text{charac. polynomial}} X_t = W_t.$

- The characteristic polynomial here is:

$B \rightarrow z$

charac. polynomial.: $$\phi(z) = 1 - \phi z$$

and the corresponding characteristic equation is

$$\phi(z) = 0 \implies 1 - \phi z = 0$$

z : charac. root/ solution.

- Let z_1 be the root/solution of the above characteristic equation. The process is stationary if $|z_1| > 1$.

roots of the characteristic eqn.

$\Rightarrow \quad |\text{roots}| > 1$

$$\phi(z) = 1 - \phi z = 0$$

root: $\quad z_1 = 1/\phi.$

$$|z_1| > 1 \Rightarrow \frac{1}{|\phi|} > 1 \Rightarrow |\phi| < 1$$

(stationary)

Linear Process

$$X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j w_{t-j}$$

$w_t \sim wn(0, \sigma_w^2)$

$\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$

3 point mov. avg. smoother

$$v_t = \frac{1}{3}(w_{t-1} + w_t + w_{t+1}).$$

Causality

$$X_t = \psi(B) w_t$$

$$= \sum_{j=0}^{\infty} \psi_j w_{t-j}$$

$w_t \sim wn(0, \sigma_w^2)$

$\sum_{j=0}^{\infty} |\psi_j| < \infty$

AR(2) Process

$B \longrightarrow Z$

- An AR(2) Process is given as

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + W_t$$

where W_t is uncorrelated with X_s for $s < t$.

$cov(X_s, W_t) = 0$. $s < t$

- The process can also be written using the backshift operator, B, as

$$\underbrace{(1 - \phi_1 B - \phi_2 B^2) X_t = W_t}$$

Charac. polynomial.

polynomial. in B.

AR(2) Process - Characteristic Equation

$1 - \phi_1 B - \phi_2 B^2$

$B \to z$

$1 - \phi_1 z - \phi_2 z^2 = 0$.

- The characteristic equation for an AR(2) process is

$$1 - \phi_1 z - \phi_2 z^2 = 0$$

- The roots for the equation are given as:

$ax^2 + bx + c = 0$

$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

$$\frac{-\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{2\phi_2}$$

z_1, z_2

$|z_1| > 1$

$|z_2| > 1$

- The process would be stationary if absolute value of roots is greater than 1.
- This would hold only if

$$\phi_1 + \phi_2 < 1, \phi_2 - \phi_1 < 1, \text{ and } |\phi_2| < 1$$

$ax^2 + bx + c$

Discriminant : $b^2 - 4ac$

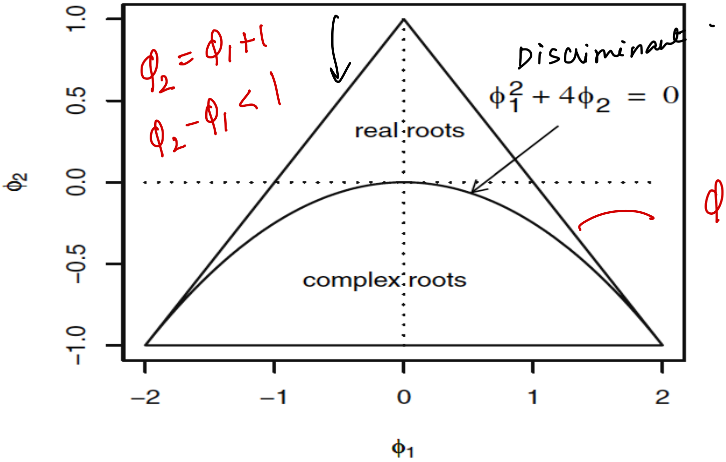
- $+ve$: real roots.
- $= 0$: equal roots.
- $-ve$: complex roots.

$1 - \phi_1 z - \phi_2 z^2$

$b^2 - 4ac = \boxed{\phi_1^2 + 4\phi_2}$

AR(2) Stationarity Parameter Region

$|\phi_2| < 1$



$\phi_2 + \phi_1 < 1$

Problems

Check if the following series are stationary

1. $X_t = \frac{1}{2}X_{t-1} + W_t$
2. $X_t = X_{t-1} - \frac{1}{4}X_{t-2} + W_t$
3. $X_t = -\frac{1}{4}X_{t-2} + W_t$

(a) $X_t = \frac{1}{2} X_{t-1} + W_t$.

AR(1) process, with $\phi = \frac{1}{2}$.

- $|\phi| < 1 \Rightarrow |1/2| = 1/2 < 1$
  $\Rightarrow$ stationarity.

- $X_t - \frac{1}{2} X_{t-1} = W_t$

  $$\underbrace{\left(1 - \frac{1}{2}B\right)}_{\text{charac. polynomial.}} X_t = W_t$$

$$\phi(z) = 1 - \frac{1}{2} z \overset{\text{set}}{=} 0$$

$$z_1 = 2$$

$|z_1| > 1 \Rightarrow$ stationarity.

② $X_t = X_{t-1} - \frac{1}{4} X_{t-2} + w_t$.

AR(2) process : $\phi_1 = 1$, $\phi_2 = -1/4$

- $\phi_1 + \phi_2 < 1$ $\quad 1 - 1/4 = \frac{3}{4} < 1$

$\phi_2 - \phi_1 < 1$ $\quad -1/4 - 1 = \frac{-5}{4} < 1$

$|\phi_2| < 1$ $\quad |-1/4| = \frac{1}{4} < 1$

$\Rightarrow$ stat.

- $(1 - B + \frac{1}{4} B^2) X_t = w_t$.

$\phi(z) = 1 - z + \frac{z^2}{4} \overset{\text{set}}{=} 0$

$\Rightarrow z^2 - 4z + 4 = 0$

$\Rightarrow (z-2)^2 = 0$.

$z_i = 2, 2$

$|z_i| > 1 \Rightarrow$ stat.

3 $X_t = -\frac{1}{4} X_{t-2} + w_t$.

AR(2) process with $\phi_1 = 0$, $\phi_2 = -1/4$

- $\phi_1 + \phi_2 < 1$ $\quad$ $0 - 1/4 = -1/4 < 1$

$\phi_2 - \phi_1 < 1$ $\quad$ $-1/4 - 0 = -1/4 < 1$

$|\phi_2| < 1$ $\quad$ $|-1/4| = 1/4 < 1$

$\Rightarrow$ stat.

- $\left(1 + \frac{B^2}{4}\right) X_t = w_t$.

$\Phi(z) = 1 + \frac{z^2}{4} \overset{\text{set}}{=} 0$.

$\Rightarrow z^2 = -4$

$z = \pm 2\sqrt{-1} = \pm 2i$

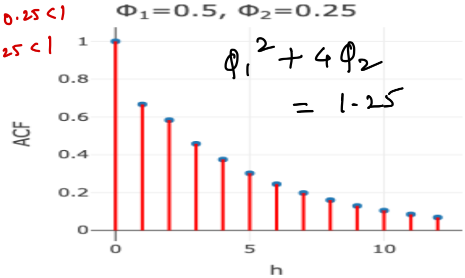
$|c + id| = \sqrt{c^2 + d^2}$

$\sqrt{0^2 + 2^2} = 2 \Rightarrow$ stat

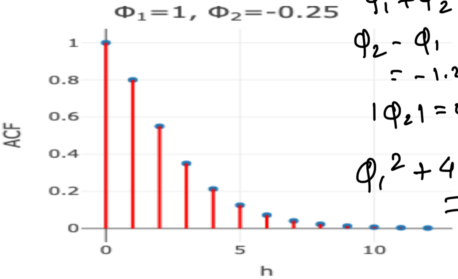
$\phi_1 + \phi_2 = 0.75 < 1$

$\phi_2 - \phi_1 = -0.25 < 1$

$|\phi_2| = 0.25 < 1$



$\phi_1^2 + 4\phi_2 = 1.25$



$\phi_1 + \phi_2 < 1$

$\phi_2 - \phi_1 = -1.25 < 1$

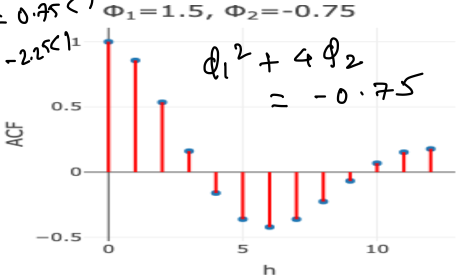
$|\phi_2| = 0.25 <$

$\phi_1^2 + 4\phi_2 = 0$

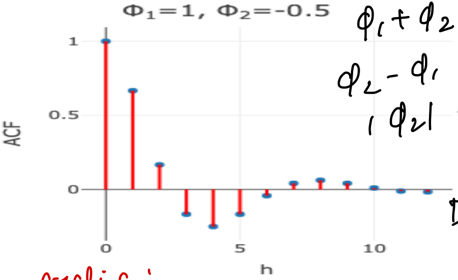
$\phi_1 + \phi_2 = 0.75 < 1$

$\phi_2 - \phi_1 = -2.25 < 1$

$|\phi_2| < 1$



$\phi_1^2 + 4\phi_2 = -0.75$



$\phi_1 + \phi_2 < 1$

$\phi_2 - \phi_1 < 1$

$|\phi_2| < 1$

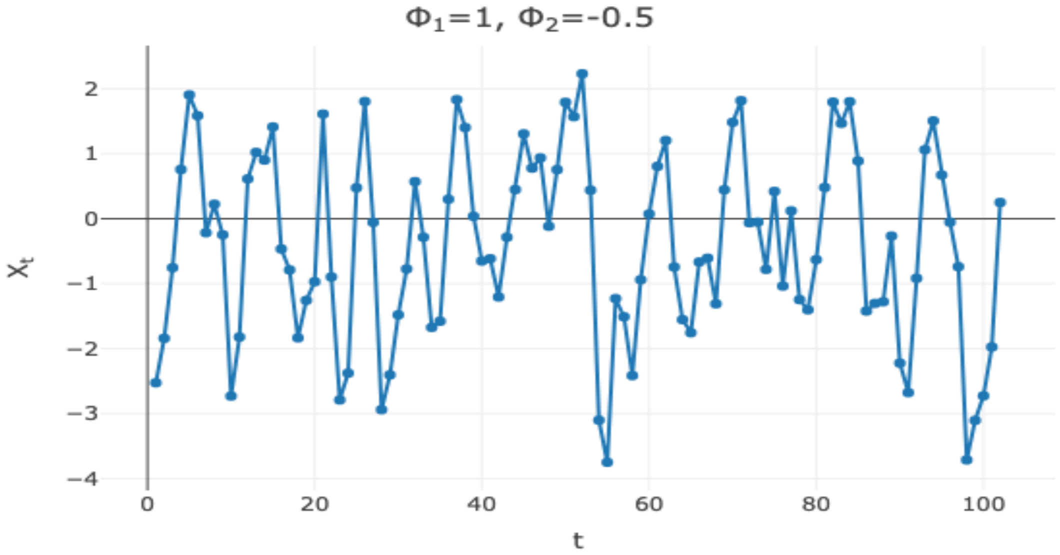
$D = -1$

quasi-cyclic

ACF of AR(2) - Observations

- The ACF pattern changes with values of ϕ_1 and ϕ_2.
- Even with the changes the ACF shows (sort of) an exponential decay.
- There is some (quasi) cyclical behavior in the ACF values that may need further discussion.
    - Positive Discriminant: The roots are real if $\phi_1^2 + 4\phi_2 > 0$ in which case the ACF decreases exponentially with h.
    - Negative Discriminant: The roots are complex if $\phi_1^2 + 4\phi_2 \cancel{>} < 0$ in which case the ACF turns out to be a damped sinusoidal wave.

AR(2) Process



AR(2) Process

For a stationary AR(2) process (with zero mean and W_t is uncorrelated with X_s for $s < t$), show that:

$$\gamma(0) = \left(\frac{1-\phi_2}{1+\phi_2}\right)\frac{\sigma_w^2}{(1-\phi_2)^2-\phi_1^2}$$

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + W_t .$$

$$\underset{\gamma(0)}{Var(X_t)} = \phi_1^2 \overset{\gamma(0)}{var(X_{t-1})} + \phi_2^2 \overset{\gamma(0)}{var(X_{t-2})} + 1^2 \overset{\sigma_w^2}{var(W_t)}$$

$$+ \underbrace{2\phi_1\phi_2 \, cov(X_{t-1}, X_{t-2})}_{2\phi_1\phi_2\gamma(1)} + 2\phi_1 \overset{0}{\cancel{cov(X_{t-1}, W_t)}} + 2\phi_2 \overset{0}{\cancel{cov(X_{t-2}, W_t)}}$$

$$\gamma(0) = \phi_1^2\gamma(0) + \phi_2^2\gamma(0) + 2\phi_1\phi_2\gamma(1) + \sigma_w^2$$

$$(1-\phi_1^2-\phi_2^2)\gamma(0) = 2\phi_1\phi_2\gamma(1) + \sigma_w^2 \quad \text{--- (1)}$$

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + W_t.$$

$h>0$, multiply both sides by X_{t-h}.

$$E(X_t X_{t-h}) = E(\phi_1 X_{t-1}X_{t-h} + \phi_2 X_{t-2}X_{t-h} + W_t X_{t-h})$$

$$\gamma(h) = \phi_1\gamma(h-1) + \phi_2\gamma(h-2) \quad \text{--- (2)}$$

Divide both sides by $\gamma(0)$

$$\rho(h) = \phi_1\rho(h-1) + \phi_2\rho(h-2) \quad \text{--- (3)}$$

Put $h=1$ in eqn (2)

$$\gamma(1) = \phi_1\gamma(0) + \phi_2\gamma(-1)$$

$$(1-\phi_2)\gamma(1) = \phi_1\gamma(0) \Rightarrow \gamma(1) = \frac{\phi_1\gamma(0)}{1-\phi_2}$$

$$(1-\phi_1^2-\phi_2^2)\,\gamma(0) = 2\phi_1\phi_2\gamma(1) + \sigma_w^2$$

Plug $\gamma(1) = \dfrac{\phi_1\gamma(0)}{1-\phi_2}$

$$(1-\phi_1^2-\phi_2^2)\,\gamma_0 - \frac{2\phi_1^2\phi_2\gamma(0)}{1-\phi_2} = \sigma_w^2$$

$$\Big[(\underbrace{1-\phi_1^2-\phi_2^2}_{(1-\phi_2^2)-\phi_1^2})(1-\phi_2) - 2\phi_1^2\phi_2\Big]\gamma_0 = (1-\phi_2)\sigma_w^2$$

$$\big((1-\phi_2)(1+\phi_2)-\phi_1^2\big)(1-\phi_2) - 2\phi_1^2\phi_2\Big]\gamma(0) = -''-$$

$$(1-\phi_2)^2(1+\phi_2) - \underbrace{\phi_1^2(1-\phi_2) - 2\phi_1^2\phi_2}\Big]\gamma(0) = -''-$$

$$(1-\phi_2)^2(1+\phi_2) - \underbrace{\phi_1^2 - \phi_1^2\phi_2}$$

$$\Big[(1-\phi_2)^2(1+\phi_2) - \phi_1^2(1+\phi_2)\Big]\gamma(0) = (1-\phi_2)\sigma_w^2$$

$$(1+\phi_2)\big[(1-\phi_2)^2-\phi_1^2\big]\gamma(0) = (1-\phi_2)\sigma_w^2$$

$X, Y.$

$$\operatorname{cov}(X,Y) = E(XY) - E(X)E(Y).$$

$$\operatorname{cov}(X,Y) = 0$$
$$\Rightarrow E(XY) = E(X)E(Y)$$

$X_1, X_2, \ldots, X_n \qquad Y = \sum_{i=1}^{n} a_i X_i$

$$\operatorname{Var}(Y) = \operatorname{Var}\left(\sum_{i=1}^{n} a_i X_i\right)$$
$$= \sum_{i=1}^{n} a_i^2 \operatorname{var}(X_i) + 2 \sum_{i<j}\sum a_i a_j \operatorname{cov}(X_i, X_j)$$

AR(2) Process

$h=0: \quad \rho(0) = 1$

For a stationary AR(2) process (with zero mean and W_t is uncorrelated with X_s for $s < t$), show that:

$$\rho(1) = \frac{\phi_1}{1-\phi_2} \qquad \rho(2) = \frac{\phi_2(1-\phi_2)+\phi_1^2}{1-\phi_2}$$

$\rightarrow \rho(h) = \phi_1 \rho(h-1) + \phi_2 \rho(h-2)$

$h=1: \quad \rho(1) = \phi_1 \rho(0) + \phi_2 \rho(-1)$

$(1-\phi_2)\rho(1) = \phi_1 \Rightarrow \rho(1) = \frac{\phi_1}{1-\phi_2}$

$h=2: \quad \rho(2) = \phi_1 \rho(1) + \phi_2 \rho(0)$

Example

$\rho(1) = \dfrac{\phi_1}{1-\phi_2}$, $\rho(2) = \dfrac{\phi_2(1-\phi_2) + \phi_1^2}{1-\phi_2}$

Consider the AR(2) process given by

$$X_t = X_{t-1} - \frac{1}{2}X_{t-2} + W_t \leftarrow$$

Is this process stationary? If so, what is its ACF?

$$\left(1 - B + \frac{B^2}{2}\right) X_t = W_t .$$

$$\phi(z) = 1 - z + \frac{z^2}{2} \overset{\text{set}}{=} 0$$

complex roots.

Discriminant: $\phi_1^2 + 4\phi_2$: $1^2 + 4(-1/2) = -1 < 0$

$$\phi(z) = 1 - z + \frac{z^2}{2} \overset{\text{set}}{=} 0.$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = 1/2$

$b = -1$

$c = 1$

$$\frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times \frac{1}{2}}}{2 \times \frac{1}{2}}$$

$$= 1 \pm \sqrt{-1} = 1 \pm i$$

$$|1 \pm i| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

AR(p) Process

- An AR(p) process is given as:

$$X_t = \overbrace{\phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p}}^{\text{`p' previous values'}} + W_t$$

with AR characteristic polynomial

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p$$

and corresponding AR characteristic equation

$$1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p = 0$$

AR(p) Process - Stationarity

AR(p)

$\phi_2 - \phi_1 < 1$

Assume W_t is uncorrelated with X_s for $s < t$

- A stationary solution exists for the AR(p) process if and only if the p roots of the AR characteristic equation each exceed 1 in absolute value (modulus).
- It can be shown that for the roots to be greater than 1 in modulus, it is necessary, but not sufficient, that both

$$\phi_1 + \phi_2 + \cdots + \phi_p < 1 \quad \text{and} \quad |\phi_p| < 1$$

lag p

$p = 2$

$\phi_1 + \phi_2 < 1$, $|\phi_2| < 1$