Stationary Time Series Models - Moving Average Models

STAT 1321/2320

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Outline

MA(1) Process - Review

$X_t = w_t + \theta w_{t-1}$

↳ zero mean. MA model

- An general MA(1) process can be written as

$$X_t = \underline{\mu} + W_t + \theta W_{t-1}, \qquad W_t \sim wn(0, \sigma^2)$$

- The mean function of the process, $\mu_t = \mu$.
- The autocovariance and autocorrelation functions are given by

$$\gamma(h) = \begin{cases} (1+\theta^2)\sigma^2 & h = 0 \\ \theta\sigma^2 & h = 1 \\ 0 & h > 1 \end{cases} \qquad \rho(h) = \begin{cases} 1 & h = 0 \\ \boxed{\frac{\theta}{1+\theta^2}} & h = 1 \\ 0 & h > 1 \end{cases}$$

- The process is stationary.

$$f(\theta) = \frac{\theta}{1+\theta^2}.$$

find "θ" where $f(\theta)$ is max./min.

$$f'(\theta) \overset{\text{set}}{=} 0.$$

$$\frac{(1+\theta^2) - \theta(2\theta)}{1+\theta^2} \overset{\text{set}}{=} 0.$$

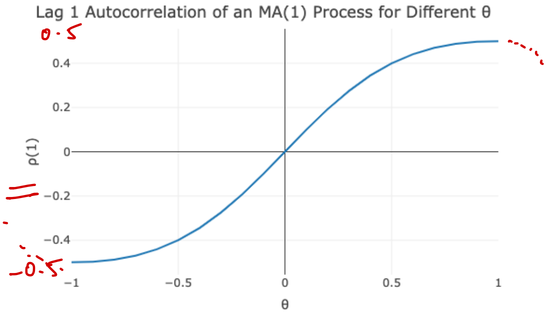
$$\boxed{\theta = \pm 1}$$

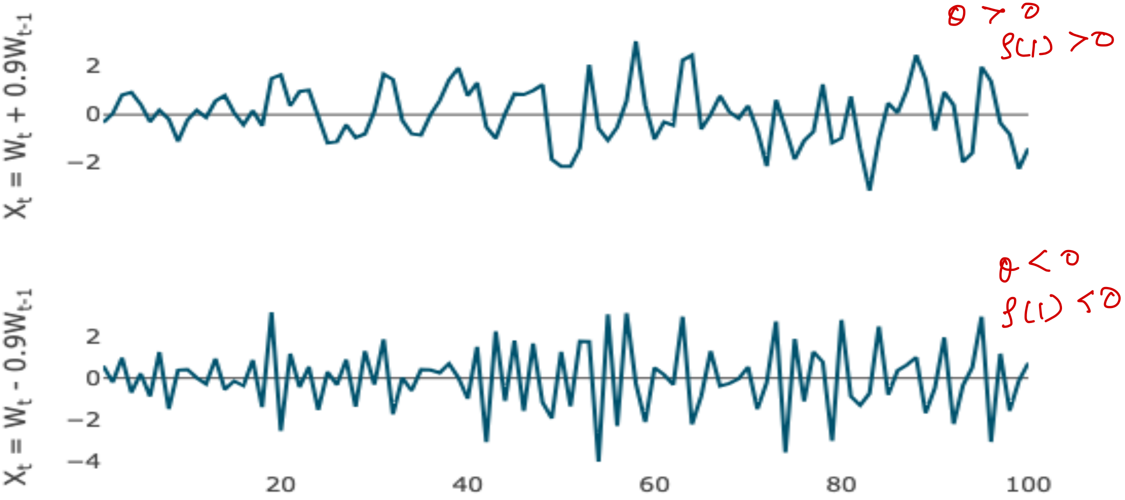
MA(1) Process - Autocorrelation

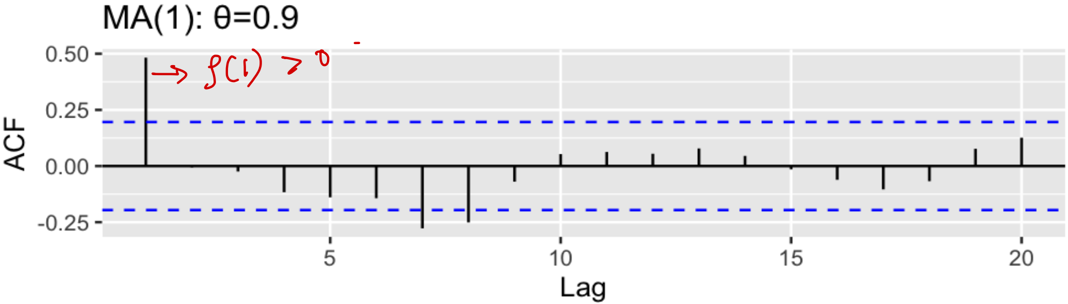
ϕ^{h}

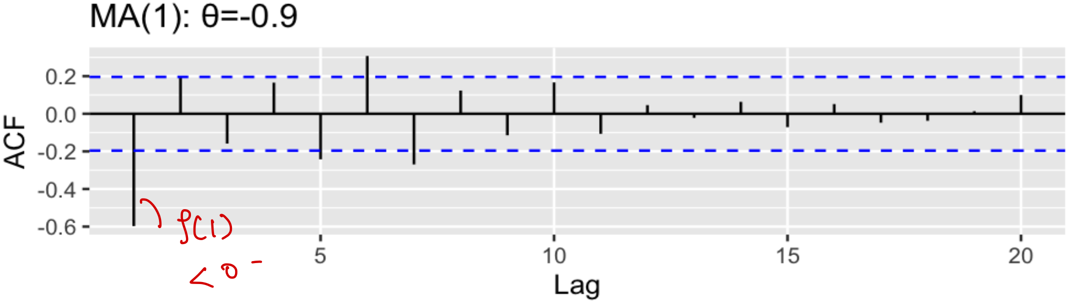
- X_t is only correlated with X_{t-1} but not $X_{t-2}, X_{t-3}, \ldots$
  - Contrast this with the case of the AR(1) model in which the correlation between X_t and X_{t-k} is never zero.

- It can also be shown that $|\rho(1)| \leq 1/2$.
  - The strongest positive correlation of 1/2 occurs when $\theta = 1$.
  - The strongest ~~positive~~ neg correlation of -1/2 occurs when $\theta = -1$.



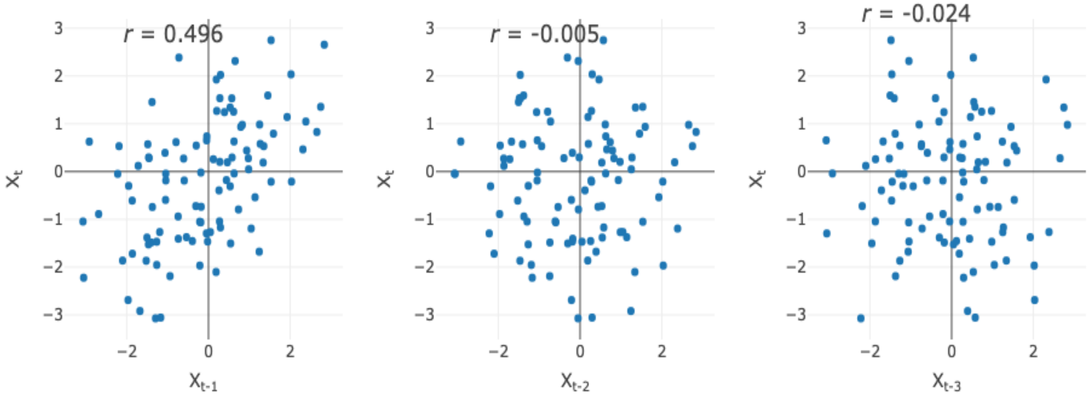

$X_t = W_t + 0.9W_{t-1}$
2
0
−2
$X_t = W_t - 0.9W_{t-1}$
2
0
−2
−4
20
40
60
80
100
$\theta > 0$
$\rho(1) > 0$
$\theta < 0$
$\rho(1) < 0$


MA(1): θ=0.9
0.50
0.25
0.00
-0.25
ACF
5
10
15
20
Lag
→ ρ(1) > 0


MA(1): θ=-0.9
0.2
0.0
-0.2
-0.4
-0.6
ACF
5
10
15
20
Lag
ρ(1) < 0

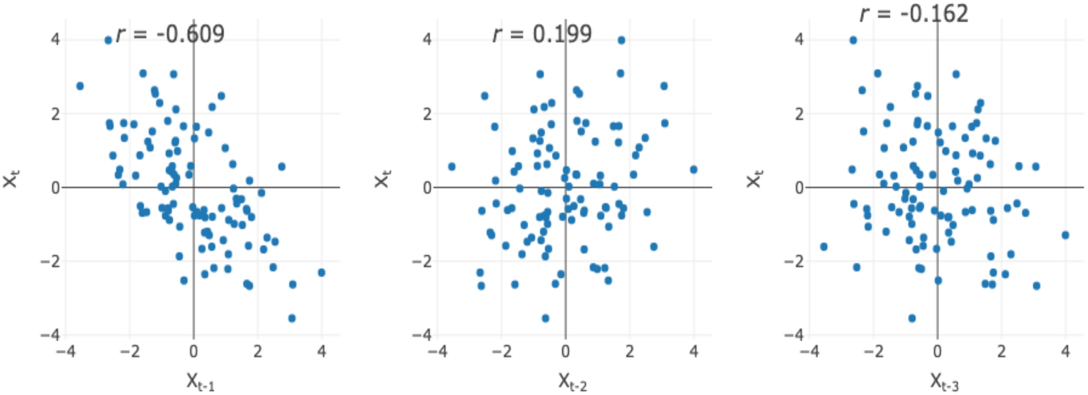
Scatter Plots with Lagged Series: MA(1) with $\theta = 0.9$

- Theoretically, the ACF should be $\theta/(1+\theta^2) = 0.497$ for $h = 1$ and 0 for $h > 1$.



Scatter Plots with Lagged Series: MA(1) with $\theta = -0.9$

- Theoretically, the ACF should be $\theta/(1+\theta^2) = -0.497$ for $h = 1$ and 0 for $h > 1$.



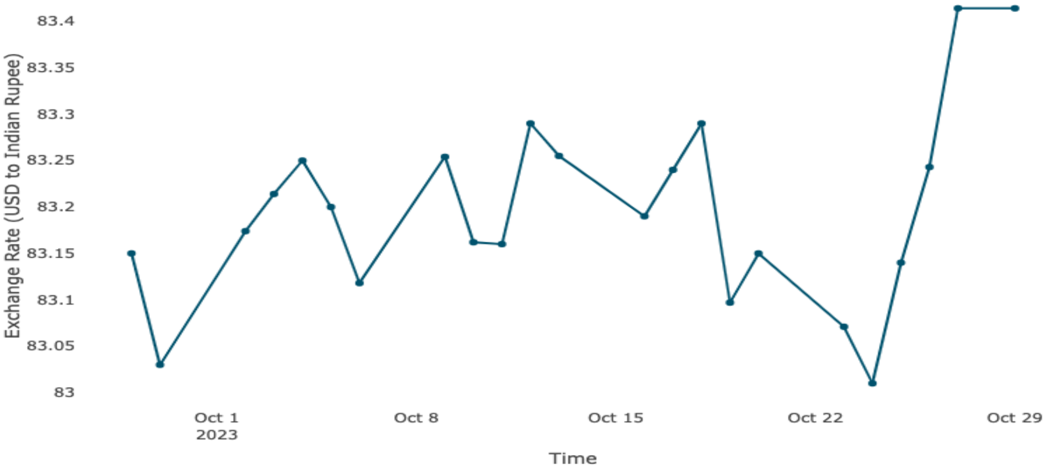
MA Models in Real Life

- An AR model finds more inuitive applications than the MA model because of its setup.
- An MA model is entirely made up of white noise terms that fluctuate around the mean μ.
  - These white noise terms may be thought of as innovations or shocks in the system.
  - These shocks may be auto-correlated (only) in the short run.
  - Examples?

lag 1.

Example - Exchange Rate

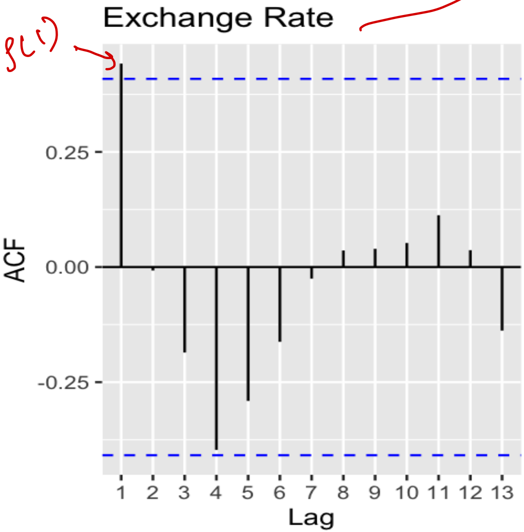
MA(1) model.

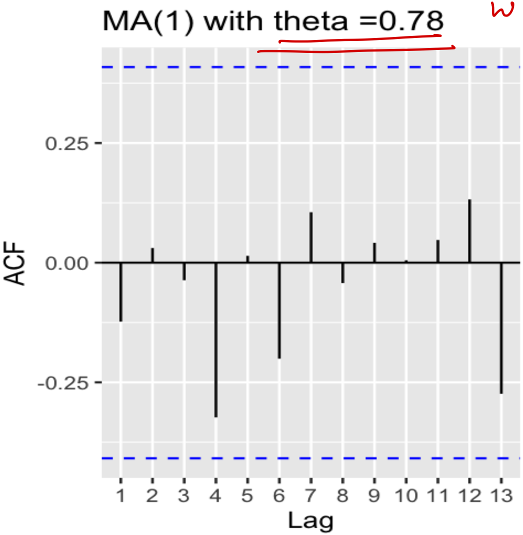


Example - Exchange Rate

MA(1) model -

W N .





Non-uniqueness of MA Models

θ, invertibility.

- Consider two models A and B:

$$A : X_t = W_t + \theta W_{t-1} \qquad B : X_t = W_t + \frac{1}{\theta} W_{t-1}$$

strike that.

- They have the same ~~auto-covariance and~~ auto-correlation structures for a given value of σ_w^2 and form of white noise.
- For $\theta = 5$ and $\sigma_w^2 = 1$

Model A.

$$\gamma(h) = \begin{cases} 26, & h = 0 \\ 5, & h = 1 \\ 0, & h > 1 \end{cases} \qquad \rho(h) = \begin{cases} 1, & h = 0 \\ 5/26, & h = 1 \\ 0, & h > 1 \end{cases}$$

Non-uniqueness of MA Models

- Consider two models A and B:

$$A : X_t = W_t + 5 W_{t-1} \qquad W_t \sim \text{ iid } N(0,1)$$

$$B : X_t^* = W_t^* + \frac{1}{5} W_{t-1}^* \qquad W_t^* \sim \text{ iid } N(0,25)$$

- These models are theoretically the same because of normality for white noise.
- In real life, we observe X_t and X_t^*, and not W_t and W_t^* so we cannot distinguish between these two models.
  - We will have to choose only one of them.

Duality between AR and MA Models

AR(1) model.

$x_t = \phi x_{t-1} + w_t$

- Recall that a causal AR model can be written as

$$X_t = \sum_{j=0}^{\infty} \psi_j W_{t-j} = \sum_{j=0}^{\infty} \phi^j W_{t-j} = \boxed{W_t + \phi W_{t-1} + \phi^2 W_{t-2} + \ldots}$$

- The last representation is basically an MA model of infinite order.
- Can an MA process be represented as an AR process?

$$x_t = \underbrace{w_t + \phi w_{t-1} + \phi^2 w_{t-2} + \phi^3 w_{t-3} + \ldots}_{\text{MA}(\infty)}$$

MA(1) Process as an AR Process

w_{t-1}
$= x_{t-1} - \theta w_{t-2}.$

- Consider an MA(1) process

$$X_t = W_t + \theta W_{t-1} \implies W_t = X_t - \theta W_{t-1}$$

- Use the recursion and continue the substitution infinitely-often

w_{t-1}

$w_{t-2} = x_{t-2} - \theta w_{t-3}$

$$
\begin{aligned}
W_t = X_t - \theta W_{t-1} &= X_t - \theta[X_{t-1} - \theta W_{t-2}] \\
&= X_t - \theta X_{t-1} + \theta^2 W_{t-2} \\
&\vdots \\
w_t \;&= X_t - \theta X_{t-1} + \theta^2 X_{t-2} - \theta^3 X_{t-3} + \cdots
\end{aligned}
$$

$x_t = (\theta x_{t-1} + \theta^2 x_{t-2} + \theta^3 x_{t-3} + \cdots) + w_t$

Invertibility

X_t ~~t~~ MA(1).

AR(∞)

- Rewriting we get,

$$X_t = \theta X_{t-1} - \theta^2 X_{t-2} + \theta^3 X_{t-3} + \cdots + W_t$$

causal.

- If $|\theta| < 1$, the MA(1) model has been **inverted** into an infinite-order AR model.
- So the MA(1) model is invertible if and only if $|\theta| < 1$.
- In general, an MA model would be invertible if the roots of the MA characteristic equation all exceed 1 in absolute value.
  - In other words, invertibility of MA models is similar to stationarity of AR models.

$$X_t = \theta X_{t-1} - \theta^2 X_{t-2} + \theta^3 X_{t-3} + \cdots$$

$|\theta| < 1$:

$|\theta| > 1$

Invertibility and the Nonuniqueness Problem

- We can solve the non-uniqueness problem of MA processes by restricting attention only to invertible MA models.
- There is only one set of coefficient parameters that yield an invertible MA process with a particular autocorrelation function.

$\theta = 2$.

- Example: Consider two models A and B:

$W_t + \theta W_{t-1}$ $\qquad\qquad$ $W_t + 1/\theta \, W_{t-1}$

$$A : X_t = W_t + \underline{\underline{2}} W_{t-1} \qquad\qquad B : X_t = W_t + 0.5 W_{t-1}$$

$|0.5| < 1$

Which model would you prefer?

$|2| > 1$ $\qquad$ $\rightarrow$ invertible.

$\Rightarrow$ not invertible.

Invertibility

$X_t \to w_t$.

A linear process $\{X_t\}$ is invertible (strictly, a invertible function of $\{W_t\}$), if there is a

$$\pi(B) = \pi_0 + \pi_1 B + \pi_2 B^2 + \cdots$$

with $\sum_{j=0}^{\infty} |\pi_j| < \infty$ such that

$$W_t = \pi(B) X_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}$$

Characteristic Equation for an MA(1) Model

- An MA(1) model can be written as:

charac polynomial.

$$X_t = W_t + \theta W_{t-1} = \overbrace{(1 + \theta B)}W_t = \theta(B) W_t$$

- $\theta(B) = 1 + \theta B$ is the characteristic polynomial. The characteristic equation can then be written as

$$\theta(z) = 1 + \theta z = 0 \quad \Rightarrow \quad z_1 = -1/\theta.$$

- The model would be invertible if absolute value of the root of the above equation $z_1 = -1/\theta$ is greater than 1.

$$|z_1| = |-1/\theta| > 1 \implies |\theta| < 1$$

MA(2) Process

Stationary. $\boxed{\dfrac{\gamma(h)}{\gamma(0)}} = \rho(h)$

- An general MA(2) process can be written as

$$X_t = \mu + W_t + \theta_1 W_{t-1} + \theta_2 W_{t-2}, \qquad W_t \sim wn(0, \sigma^2)$$

- The mean function of the process, $\mu_t = \mu$
- The autocovariance and autocorrelation functions are given by

$$\gamma(h) = \begin{cases} (1 + \theta_1^2 + \theta_2^2)\sigma^2 & h = 0 \checkmark \\ (\theta_1 + \theta_1\theta_2)\sigma^2 & h = 1 \checkmark \\ \theta_2\sigma^2 & h = 2 \\ 0 & h > \cancel{1}\, 2 \end{cases} \qquad \rho(h) = \begin{cases} 1 & h = 0 \\ \frac{\theta_1 + \theta_1\theta_2}{1 + \theta_1^2 + \theta_2^2} & h = 1 \\ \frac{\theta_2}{1 + \theta_1^2 + \theta_2^2} & h = \cancel{1}\, 2 \\ 0 & h > \cancel{1}\, 2 \end{cases}$$

MA(2) Process

Prove the autocovariance and autocorrelation results on the previous slide.

$$\gamma(0) = \text{cov}(x_t, x_t) = \text{Var}(x_t)$$

$$x_t = \mu + \theta_1 w_{t-1} + \theta_2 w_{t-2} + w_t$$

$$\text{Var}(x_t) = \text{Var}\left[\theta_1 w_{t-1} + \theta_2 w_{t-2} + w_t\right]$$

$$= \theta_1^2 \text{var}(w_{t-1}) + \theta_2^2 \text{var}(w_{t-2}) + \text{var}(w_t)$$

$$= \left[1 + \theta_1^2 + \theta_2^2\right] \sigma^2$$

$$\gamma(1) = \operatorname{cov}(X_t, X_{t-1})$$

$$= \operatorname{cov}\left[\mu + w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2},\ \mu + w_{t-1} + \theta_1 w_{t-2} + \theta_2 w_{t-3}\right]$$

$$= \theta_1 \operatorname{cov}(w_{t-1}, w_{t-1}) + \theta_1\theta_2 \operatorname{cov}(w_{t-2}, w_{t-2})$$

$$= [\theta_1 + \theta_1\theta_2]\,\sigma^2$$

$$\gamma(2) = \operatorname{cov}(X_t, X_{t-2})$$

$$= \operatorname{cov}\left[\mu + w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2},\ \mu + w_{t-2} + \theta_1 w_{t-3} + \theta_2 w_{t-4}\right]$$

$$= \theta \operatorname{cov}(w_{t-2}, w_{t-2}) = \theta\sigma^2$$

$$\gamma(h) = 0, \quad h > 2$$

MA(2) Process - Invertibility

- An MA(2) model can be written as:

$$X_t = W_t + \theta_1 W_{t-1} + \theta_2 W_{t-2} = (1 + \theta_1 B + \theta_2 B^2) W_t = \theta(B) W_t$$

- $\theta(z) = 1 + \theta_1 z + \theta_2 z^2$ is the characteristic polynomial. The characteristic equation can then be written as

$$\theta(z) = 1 + \theta_1 z + \theta_2 z^2 = 0$$

- The model would be invertible if absolute value of the roots of the above equation is greater than 1. This is analogous to checking

$$\theta_1 + \theta_2 > -1, \theta_2 - \theta_1 > -1, |\theta_2| < 1$$

Example

$\theta(z) \overset{\text{set}}{=} 0 \qquad |z| > 1 \qquad$ polyroot.

Check the invertibility and stationarity of the following models:

1. $X_t = W_t - 2.1W_{t-1} = (1 - 2.1B)W_t \Rightarrow \theta(z) = 1 - 2.1z \overset{\text{set}}{=} 0.$

   $|z_1| = \left|\frac{1}{2.1}\right| < 1$

   not-invertible

2. $X_t = W_t + 0.3W_{t-1} = (1 + 0.3B)W_t$

   $\to \theta(z) = 1 + 0.3z \overset{\text{set}}{=} 0$

   $|z_1| = \left|\frac{-1}{0.3}\right| > 1$

3. $X_t = W_t - W_{t-1} + 0.21W_{t-2}$
4. $X_t = W_t - 0.1W_{t-1} - 0.56W_{t-2}$
5. $X_t = W_t - 0.8W_{t-1} + 0.5W_{t-2}$

$\frac{0.8 \pm \sqrt{0.64 - 2}}{(2 * 0.5)} = 0.8 \pm \sqrt{-1.36}$

$= 0.8 \pm i\sqrt{1.36}$

inv.

$|0.8 + i\sqrt{1.36}| = (0.8)^2 + 1.36$

$$X_t = w_t - w_{t-1} + 0.21\, w_{t-2}$$

$$= (1 - B + 0.21 B^2)\, w_t$$

$$\theta(z) = 1 - z + 0.21 z^2$$

$$= 1 - 0.3z - 0.7z + 0.21 z^2$$

$$= (1 - 0.3z) - 0.7z(1 - 0.3z)$$

$$= (1 - 0.7z)(1 - 0.3z) \overset{\text{set}}{=} 0.$$

$$z_1 = \frac{1}{0.7}, \quad z_2 = \frac{1}{0.3}.$$

$|z_1| > 1$ $\quad$ $|z_2| > 1$ $\quad$ invertibl

$$X_t = w_t - 0.1 w_{t-1} - 0.56 w_{t-2}$$

$$= (1 - 0.1B - 0.56B^2) w_t$$

$$\phi(z) = 1 - 0.1z - 0.56z^2$$

$$= 1 - 0.8z + 0.7z - 0.56z^2$$

$$= (1 - 0.8z) + 0.7z(1 - 0.56z)$$

$$= (1 + 0.7z)(1 - 0.8z) \underset{\text{set}}{=} 0.$$

$$z_1 = \frac{-1}{0.7}, \quad z_2 = \frac{1}{0.8}$$

$|z_1| > 1$, $|z_2| > 1$ invertible.

MA(q) Process

- An MA(q) process is given as:

$$X_t = W_t + \theta_1 W_{t-1} + \theta_2 W_{t-2} + \cdots + \theta_q W_{t-q}$$

with MA characteristic polynomial

$$\theta(z) = 1 + \theta_1 z + \theta_2 z^2 + \cdots + \theta_q z^q$$

and corresponding MA characteristic equation

$$1 + \theta_1 z + \theta_2 z^2 + \cdots + \theta_q z^q = 0$$

↳ q roots.

MA(q) Process - Invertibility and Stationarity

- An MA(q) process would be invertible if and only if the roots of the characteristic equation exceed 1 in absolute value.
- An MA(q) process is stationary for all values of θ_j.

Example

Consider the following two models

$$X_t = W_t - 2.2W_{t-1} + 0.4W_{t-2};\ W_t \sim N(0, \sigma_w^2 = 2)$$

$$X_t^* = W_t^* - 0.7W_{t-1}^* + 0.1W_{t-2}^*;\ W_t^* \sim N(0, \sigma_{w^*}^2 = 8)$$

These models have the same autocovariance function. Which model would you pick and why?

$\phi(z) = 1 - 2.2z + 0.4z^2 = 0$

$= 1 - 2z - 0.2z + 0.4z^2$ set 0.

$z_1 = \frac{1}{2}, \frac{1}{0.2}$

$\rightarrow$ check for invertibicity.

$|\frac{1}{2}| < 1$

$$\phi(z) = 1$$