Arithmetic with Complex Numbers

Complex numbers use the special value i, which has the property that $i \times i = -1$. All of complex arithmetic follows from this property.

A complex number has two parts: real and imaginary. For example, in the number $2 + 3i$, the real part is 2 and the imaginary part is 3.

Ordinary real numbers are just complex numbers with a zero imaginary part:

$3 = 3 + 0i$

Addition and Subtraction

You can add or subtract two complex numbers by simply adding or subtracting their real parts, and adding or subtracting their imaginary parts:

$(2 + 3i) + (4 - 5i)$

$(2 + 3i) + (4 - 5i) = (2+4) + (3-5)i$

$(2 + 3i) + (4 - 5i) = 6 - 2i$

The general formulas for addition and subtraction are:

$(a + bi) + (c + di) = (a+c) + (b+d)i$

$(a + bi) - (c + di) = (a-c) + (b-d)i$

Multiplication

Multiplication is also straightforward, as long as you remember that $i \times i = -1$. For example,

$(2 + 3i) \times (4 - 2i)$

$(2 + 3i) \times (4 - 2i) = 2\times4 + 2\times(-2i) + (3i)\times4 + (3i)\times(-2i)$

$(2 + 3i) \times (4 - 2i) = 2\times4 - 2\times2i + 3\times4i - 3\times2(i\times i)$

$(2 + 3i) \times (4 - 2i) = 2\times4 + -2\times2i + 3\times4i + 3\times2$

$(2 + 3i) \times (4 - 2i) = (2\times4 + 3\times2) + (-2\times2 + 3\times4)i$

$(2 + 3i) \times (4 - 2i) = 14 + 8i$

The general formula for multiplication is:

$(a + bi) \times (c + di) = (ac - bd) + (ad + bc)i$

Conjugate

To do division, we need to define the conjugate of a complex number: it's just the number, with the imaginary part negated. For example:

conjugate($3 + 4i$) = $3 - 4i$

Division

How do you divide $(2 + 4i) / (1 + i)$?

First, multiply top and bottom by $(1 - i)$, which is conjugate of the denominator:

$(2 + 4i) / (1 + i) = [(2 + 4i)\times(1 - i)] / [(1 + i)\times(1 - i)]$

Then

$(2 + 4i) / (1 + i) = (6 + 2i) / 2$

$(2 + 4i) / (1 + i) = 3 + i$

Notice that multiplying the $(1 + i)$ by $(1 - i)$ yields an ordinary real number, with no imaginary part. Multiplying a number by its conjugate always yields a real number.

A bit of algebra will show that the general formula for division needs these steps:

numerator = ($a + bi$) × conjugate($c + di$)

denominator = ($c + di$) × ($c - di$)

$(a + bi) / (c + di)$ = numerator / denominator

In that last line, the numerator will be a complex number, but the denominator will be an ordinary real number, so the right-hand side simply involves dividing both parts of the numerator by the denominator.

Magnitude

We can also use conjugates to get the magnitude of a complex number:

magnitude($a + bi$)

magnitude($a + bi$) = $[(a + bi)\times(a - bi)]^{0.5}$

magnitude($a + bi$) = $\sqrt{(a^2 + b^2)}$