Peter Lorenz

1.1) $f(x) = \frac{1}{2}(\log(x))^2$, domain $f = \{x \mid x > 0\}$

$$f'(x) = 2 \cdot \frac{1}{2}(\log(x))^1 \cdot \frac{1}{x}$$

$$= \underbrace{\log(x)}_{u} \cdot \underbrace{\frac{1}{x}}_{v}$$

$u' = \frac{1}{x}$

$v' = -x^{-2}$

Produktregel: $u \cdot v' + u' \cdot v$:

$$= \log(x) \cdot (-x^{-2}) + \frac{1}{x} \cdot \frac{1}{x} = -\frac{\log(x)}{x^2} + \frac{1}{x^2} =$$

$$= \frac{1}{x^2}(1 - \log(x))$$

$$f'(x) \stackrel{!}{=} 0$$

$$\frac{1}{x^2}(1-\log(x)) = 0 \quad | \cdot x^2$$

$$1 - \log(x) = 0 \quad | -1 \; | \cdot(-1)$$

$$\log(x) = 1$$

$$e^{\log(x)} = e^1$$

$$\underline{\underline{x = e^1}}$$

A: Diese Funktion ist nicht konvex, da sie ab ~~einem~~ der Nullstelle in den Minusbereich geht.

1.2) $f(x) = x_1 \cdot x_2 \,, \qquad x \in \mathbb{R}^2$

$$H_f(x) := \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x) \right) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_1 x_2}(x) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2 \partial x_2} \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Wenn die Hessematrix semi positiv definit ist, dann ist die Funktion konvex.

$\det(H_f(x)) < 0$ ... dann ist die Funktion f indefinit

$\det(H_f(x)) = (0 \cdot 0) - (1 \cdot 1) = \underline{-1}$ indefinit und daher nicht konvex.

1.3) $f(x) = \frac{x_1^2}{x_2} \,, \quad x \in \mathbb{R}^2 \,, \quad x_2 > 0$

$$\nabla f(x) = \begin{pmatrix} 2 \frac{x_1}{x_2} \\ -1 \cdot \frac{x_1^2}{x_2^2} \end{pmatrix}$$

$$\nabla^2 f(x) = \begin{pmatrix} \frac{2}{x_2} & -2 \cdot \frac{x_1}{x_2^2} \\ -2 \cdot \frac{x_1}{x_2^2} & 2 \cdot x_1^2 \cdot x_2^{-3} \end{pmatrix} = \begin{pmatrix} \frac{1}{x_2} & -\frac{x_1}{x_2^2} \\ -\frac{x_1}{x_2^2} & x_1^2 \cdot x_2^{-3} \end{pmatrix}$$

$$\det(\nabla^2 f(x)) = \frac{1}{x_2} \cdot (x_1^2 \cdot x_2^{-3}) - \left(\left(-\frac{x_1}{x_2^2}\right) \cdot \left(-\frac{x_1}{x_2^2}\right) \right) =$$

$$= \frac{x_1^2}{x_2 \cdot x_2^{+3}} - \frac{x_1^2}{x_2^4} = \frac{x_1^2}{x^4} - \frac{x_1^2}{x_4} = \underline{\underline{0}} \checkmark$$

~~Def funkt~~ Die Funktion f ist konvex.

Peter LORENZ

2.1) $f(x) = \frac{1}{2}\|A\cdot x - b\|^2 \quad , x\in\mathbb{R}^n, b\in\mathbb{R}^m, A\in\mathbb{R}^{m\times n}$

·) $m = n = 2$

$$f(x) = \frac{1}{2}\left\|\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}\cdot\begin{pmatrix} x_1 \\ x_2\end{pmatrix} - \begin{pmatrix} b_1 \\ b_2\end{pmatrix}\right\|^2 =$$

$$= \frac{1}{2}\left\|\begin{pmatrix} a_{11}\cdot x_1 + a_{12}\cdot x_2 \\ a_{21}\cdot x_1 + a_{22}\cdot x_2\end{pmatrix} - \begin{pmatrix} b_1 \\ b_2\end{pmatrix}\right\|^2 =$$

$$= \frac{1}{2}\left(\cancel{\sqrt{(a_{11}x_1 + a_{12}\cdot x_2 - b_1)^2 + (a_{21}x_1 + a_{22}\cdot x_2 - b_2)^2}}\right)^{\cancel{2}} =$$

$$\frac{\partial}{\partial x_1} f(x) = \frac{1}{\cancel{2}}\left(\cancel{2}(a_{11}\cdot x_1 + a_{12}\cdot x_2 - b_1)\cdot a_{11} + \cancel{2}(a_{21}x_1 + a_{22}x_2 - b_2)a_{21}\right)$$

$$= (a_{11}x_1 + a_{12}x_2 - b_1)a_{11} + (a_{21}x_1 + a_{22}x_2 - b_2)a_{21}$$

$$\frac{\partial}{\partial x_2} f(x) = (a_{11}x_1 + a_{12}x_2 - b_1)a_{12} + (a_{21}x_1 + a_{22}x_2 - b_2)a_{22}$$

$$\nabla f(x) = \begin{pmatrix} \frac{\partial}{\partial x_1} f(x) \\ \frac{\partial}{\partial x_2} f(x)\end{pmatrix} = \begin{pmatrix} (a_{11} \;\; a_{21})\begin{pmatrix} a_{11}x_1 + a_{12}x_2 - b_1 \\ a_{21}x_2 + a_{22}x_2 - b_2\end{pmatrix} \\ (a_{12} \;\; a_{22})\left(\text{——} \;\; " \;\; \text{——} \right)\end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22}\end{pmatrix}\begin{pmatrix} a_{11}x_1 + a_{12}x_2 - b_1 \\ a_{21}x_2 + a_{22}x_2 - b_2\end{pmatrix}$$

$$= A^T \cdot (A\cdot x - b)$$

$$\nabla^2 f(x) = A^T(A \cdot x - b)$$
$$= A^T \cdot A \cdot x - A^T \cdot b$$

$$\frac{\partial^2}{\partial x_1^2} f(x) = \begin{pmatrix} a_{11} & a_{21} \end{pmatrix} \cdot \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$$

$$\frac{\partial^2}{\partial x_2^2} f(x) = \begin{pmatrix} a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$$

$$\frac{\partial^2}{\partial x_1 x_2} f(x) = \begin{pmatrix} a_{11} & a_{21} \end{pmatrix} \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$$

$$\frac{\partial^2}{\partial x_2 x_1} f(x) = \begin{pmatrix} a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$$

$$\nabla^2 f(x) = \begin{pmatrix} \frac{\partial^2}{\partial x_1^2} & \frac{\partial^2}{\partial x_1 \partial x_2} \\ \frac{\partial^2}{\partial x_2 x_1} & \frac{\partial^2}{\partial x_2^2} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} a_{11} & a_{21} \end{pmatrix}\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} & \begin{pmatrix} a_{11} & a_{21} \end{pmatrix}\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} \\ \begin{pmatrix} a_{12} & a_{22} \end{pmatrix}\begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix} & \begin{pmatrix} a_{12} & a_{22} \end{pmatrix}\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} \end{pmatrix} =$$

$$= \begin{pmatrix} \begin{pmatrix} a_{11} & a_{21} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \\ \begin{pmatrix} a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} a_{21} & a_{11} \\ a_{22} & a_{21} \end{pmatrix} \end{pmatrix} =$$

$$= \underline{\underline{A^T A}}$$

Generalized: $f(x) = \frac{1}{2} \| A \cdot x - b \|^2$

$$\nabla f(x) = \frac{1}{2} \cdot 2 \, A^T (A \cdot x - b)$$

$$\nabla^2 f(x) = A^T (A \cdot x - b)$$
$$= A^T \cdot A \cdot x - A^T b$$
$$= ((A^T \cdot A)^T)^T \cdot x - A^T b$$
$$= (A^T \cdot A)^T = \underline{\underline{A^T \cdot A}}$$

2.2) $f(x) = \frac{1}{2} \| A(x + \alpha y) - b \|^2$

$$f(\alpha) = \frac{1}{2} \left\| \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \alpha \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right) - \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \right\|^2 =$$

$$= \frac{1}{2} \left\| \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \begin{pmatrix} x_1 + \alpha y_1 \\ x_2 + \alpha y_2 \end{pmatrix} - \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \right\|^2 =$$

$$= \frac{1}{2} \left\| \begin{matrix} (a_{11} \cdot (x_1 + \alpha y_1) + a_{12} \cdot (x_2 + \alpha y_2) - b_1) \\ (a_{21} (x_1 + \alpha y_1) + a_{22} (x_2 + \alpha y_2) - b_2) \end{matrix} \right\|^2 =$$

$$\frac{\partial}{\partial \alpha} f(\alpha) = \frac{1}{2} \left(\sqrt{(a_{11}(x_1 + \alpha y_1) + a_{12}(x_2 + \alpha y_2) - b_1)^2 + (a_{21}(x_1 + \alpha y_1) + a_{12}(x_2 + \alpha y_2) - b_2)^2} \right)^2 =$$

$$= \frac{1}{2}\left((a_{11}(x_1 + \alpha y_1) + a_{12}(x_2 + \alpha y_2) - b_1)^2 + (a_{21}(x_1 + \alpha y_1) + a_{12}(x_2 + \alpha y_2) - b_2)^2\right)$$

$$= \frac{1}{2}\left(2(\quad \text{—} " \text{—} \quad)(a_{11} y_1 + a_{12} y_2) + 2(\text{—} " \text{—})(a_{21} y_1 + a_{22} y_2)\right) =$$

$$= \left(\begin{matrix} \boxed{(a_{11} y_1 + a_{12} y_2)} (a_{11}(x_1 + \alpha y_1) + a_{12}(x_2 + \alpha y_2) - b_1) + \\ \boxed{(a_{21} y_1 + a_{22} y_2)} (a_{12}(x_1 + \alpha y_1) + a_{22}(x_2 + \alpha y_2) - b_2) \end{matrix} \right) =$$

$$= \left(\left((a_{11} \;\; a_{12}) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right) \left((a_{21} \;\; a_{22}) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right) \right) =$$

$$= \left(\left((y_1 \;\; y_2) \begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} \right) \left((y_1 \;\; y_2) \begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix} \right) \right) =$$

$$= (y_1 \;\; y_2) \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = y^T A^T \text{ — } ①$$

$$A(x + \alpha y) - b \text{ — } ②$$

$$\Rightarrow \underline{\underline{y^T A^T (A(x + \alpha y) - b)}}$$

$$\frac{\partial}{\partial \alpha} f(x) = y^T A^T \left(A(x + \alpha y) - b \right) =$$

$$= y^T A^T A \cdot (x + \alpha y - y^T A^T b) =$$

$$= y^T A^T A (x + \alpha y) - y^T A^T b =$$

$$= y^T A^T A x + y^T A^T A \alpha y - y^T A^T \cdot b =$$

$$= y^T A^T A x + y^T A^T A \alpha y - y^T A^T \cdot b =$$

$$= \quad + y^T A^T A y \alpha - \ldots$$

$$\frac{\partial^2}{\partial \alpha^2} f(x) = \underline{\underline{y^T A^T A \cdot y}}$$

c) Generalize for Arbitrary n

$$f(x) = \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}$$

$$\frac{\partial}{\partial x_j} f(x) = \frac{1}{n} f(x) \cdot \frac{1}{x_j}$$

$$\frac{\partial^2}{\partial x_j^2} f(x) = \frac{1}{n}\left(\underbrace{\frac{1}{n} \cdot \left(\prod_{i=1}^{n} x_j\right)^{\frac{1}{n}} \cdot \frac{\prod_{i=1}^{n} x_i}{x_j}}_{u'} \cdot \underbrace{\frac{1}{x_j}}_{v} + \underbrace{\left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}}_{u} \cdot (-1) \underbrace{\frac{1}{x_j^2}}_{v'} \right) =$$

$$\frac{\partial^2}{\partial x_j^2} f(x) = \frac{1}{n}\left(\frac{1}{n} f(x) \cdot \frac{1}{x_j x_j} + (-1) f(x) \cdot \frac{1}{x_j x_j}\right)$$

$$\frac{\partial^2}{\partial x_j^2} f(x) = \frac{1}{n} f(x) \cdot \left(\left(\frac{1}{n} - 1\right) \cdot \frac{1}{x_j x_j}\right)$$

$$= \frac{1}{n} \left(\left(\frac{1}{n} - 1\right)\right) \cdot f(x) \cdot \frac{1}{x_j x_j}$$

$j \neq k$

$$\frac{\partial^2}{\partial x_j x_k} f(x) = \frac{1}{n} \cdot \frac{1}{n} \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n} - 1} \frac{\prod_{i=1}^{n} x_i}{x_k} \cdot \frac{1}{x_j}$$

$$\cancel{= \frac{1}{n^2}}$$

$$\frac{\partial^2}{\partial^2 x_j^2} f(x) = \frac{1}{n} \left(\frac{1}{n} - 1\right) f(x) \frac{1}{x_j x_j} = \frac{1}{n} \left(\frac{1}{n} - 1\right) f(x) \frac{1}{x_j x_j}$$

$$\frac{\partial^2}{\partial x_j \partial x_k} f(x) = \frac{1}{n^2} f(x) \frac{1}{x_j x_k} = \frac{1}{n} \left(\frac{1}{n}\right) f(x) \frac{1}{x_j x_k}$$

$$\nabla^2 f(x) = \frac{1}{n} f(x) \cdot \begin{pmatrix} \left(\frac{1}{n} - 1\right) \frac{1}{x_1^2} & \frac{1}{n} \cdot \frac{1}{x_1 x_2} & \cdots & \frac{1}{n} \cdot \frac{1}{x_j x_k} \\ \vdots & & \ddots & \vdots \\ \left(\frac{1}{n}\right) \frac{1}{x_n x_1} & \cdots & \cdots & \left(\frac{1}{n} - 1\right) \cdot \frac{1}{x_n^2} \end{pmatrix}$$