Peter Lorenz

1.1) $f(x) = \frac{1}{2}(\log(x))^2$, domain $f = \{x \mid x > 0\}$

$$f'(x) = 2 \cdot \frac{1}{2}(\log(x))^1 \cdot \frac{1}{x}$$

$$= \underbrace{\log(x)}_{u} \cdot \underbrace{\frac{1}{x}}_{v}$$

$u' = \frac{1}{x}$

$v' = -x^{-2}$

Produktregel: $u \cdot v' + u' \cdot v$:

$$= \log(x) \cdot (-x^{-2}) + \frac{1}{x} \cdot \frac{1}{x} = -\frac{\log(x)}{x^2} + \frac{1}{x^2} =$$

$$= \frac{1}{x^2}(1 - \log(x))$$

$$f'(x) \overset{!}{=} 0$$

$$\frac{1}{x^2}(1 - \log(x)) = 0 \quad | \cdot x^2$$

$$1 - \log(x) = 0 \quad | -1 \; | \cdot(-1)$$

$$\log(x) = 1$$

$$e^{\log(x)} = e^1$$

$$\underline{\underline{x = e^1}}$$

A: Diese Funktion ist nicht konvex, da sie ab ~~einem~~ der Nullstelle in den Minusbereich geht.

1.2) $f(x) = x_1 \cdot x_2$, $x \in \mathbb{R}^2$

$$H_f(x) := \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_1 x_2}(x) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2 \partial x_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Wenn die Hessematrix semi positiv definit ist, dann ist die Funktion konvex.

$\det(H_f(x)) < 0$ ... dann ist die Funktion f indefinit

$\det(H_f(x)) = (0 \cdot 0) - (1 \cdot 1) = \underline{-1}$ indefinit und daher nicht konvex.

1.3) $f(x) = \frac{x_1^2}{x_2}$, $x \in \mathbb{R}^2$, $x_2 > 0$

$$\nabla f(x) = \begin{pmatrix} 2\frac{x_1}{x_2} \\ -1 \cdot \frac{x_1^2}{x_2^2} \end{pmatrix}$$

$$\nabla^2 f(x) = \begin{pmatrix} \frac{2}{x_2} & -2 \cdot \frac{x_1}{x_2^2} \\ -2 \cdot \frac{x_1}{x_2^2} & 2 \cdot x_1^2 \cdot x_2^{-3} \end{pmatrix} = \begin{pmatrix} \frac{1}{x_2} & -\frac{x_1}{x_2^2} \\ -\frac{x_1}{x_2^2} & x_1^2 \cdot x_2^{-3} \end{pmatrix}$$

$$\det(\nabla^2 f(x)) = \frac{1}{x_2} \cdot (x_1^2 \cdot x_2^{-3}) - \left(\left(-\frac{x_1}{x_2^2}\right) \cdot \left(-\frac{x_1}{x_2^2}\right)\right) =$$

$$= \frac{x_1^2}{x_2 \cdot x_2^{+3}} - \frac{x_1^2}{x_2^4} = \frac{x_1^2}{x^4} - \frac{x_1^2}{x_4} = \underline{\underline{0}} \checkmark$$

~~Def fkt~~ Die Funktion f ist konvex.

Peter LORENZ

2.1) $f(x) = \frac{1}{2} \|A \cdot x - b\|^2 \quad , x \in \mathbb{R}^n, b \in \mathbb{R}^m, A \in \mathbb{R}^{m \times n}$

•) $m = n = 2$

$$f(x) = \frac{1}{2} \left\| \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \right\|^2 =$$

$$= \frac{1}{2} \left\| \begin{pmatrix} a_{11} \cdot x_1 + a_{12} \cdot x_2 \\ a_{21} \cdot x_1 + a_{22} \cdot x_2 \end{pmatrix} - \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \right\|^2 =$$

$$= \frac{1}{2} \left(\sqrt{(a_{11} x_1 + a_{12} \cdot x_2 - b_1)^2 + (a_{21} x_1 + a_{22} \cdot x_2 - b_2)^2} \right)^2 =$$

$$\frac{\partial}{\partial x_1} f(x) = \frac{1}{2} \left(2 (a_{11} \cdot x_1 + a_{12} \cdot x_2 - b_1) \cdot a_{11} + 2 (a_{21} x_1 + a_{22} x_2 - b_2) a_{21} \right)$$

$$= (a_{11} x_1 + a_{12} x_2 - b_1) a_{11} + (a_{21} x_1 + a_{22} x_2 - b_2) a_{21}$$

$$\frac{\partial}{\partial x_2} f(x) = (a_{11} x_1 + a_{12} x_2 - b_1) a_{12} + (a_{21} x_1 + a_{22} x_2 - b_2) a_{22}$$

$$\nabla f(x) = \begin{pmatrix} \frac{\partial}{\partial x_1} f(x) \\ \frac{\partial}{\partial x_2} f(x) \end{pmatrix} = \begin{pmatrix} (a_{11} \;\; a_{21}) \begin{pmatrix} a_{11} x_1 + a_{12} x_2 - b_1 \\ a_{21} x_2 + a_{22} x_2 - b_2 \end{pmatrix} \\ (a_{12} \;\; a_{22}) \left(\text{---} \;\; '' \;\; \text{---} \right) \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} a_{11} x_1 + a_{12} x_2 - b_1 \\ a_{21} x_2 + a_{22} x_2 - b_2 \end{pmatrix}$$

$$= A^T \cdot (A \cdot x - b)$$

$$\nabla^2 f(x) = A^T(A \cdot x - b)$$
$$= A^T \cdot A \cdot x - A^T \cdot b$$

$$\frac{\partial^2}{\partial x_1^2} f(x) = \begin{pmatrix} a_{11} & a_{21} \end{pmatrix} \cdot \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$$

$$\frac{\partial^2}{\partial x_2^2} f(x) = \begin{pmatrix} a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$$

$$\frac{\partial^2}{\partial x_1 x_2} f(x) = \begin{pmatrix} a_{11} & a_{21} \end{pmatrix} \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$$

$$\frac{\partial^2}{\partial x_2 x_1} f(x) = \begin{pmatrix} a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$$

$$\nabla^2 f(x) = \begin{pmatrix} \frac{\partial^2}{\partial x_1^2} & \frac{\partial^2}{\partial x_1 \partial x_2} \\ \frac{\partial^2}{\partial x_2 x_1} & \frac{\partial^2}{\partial x_2^2} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} a_{11} & a_{21} \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} & \begin{pmatrix} a_{11} & a_{21} \end{pmatrix} \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} \\ \begin{pmatrix} a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix} & \begin{pmatrix} a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} \end{pmatrix} =$$

$$= \begin{pmatrix} \begin{pmatrix} a_{11} & a_{21} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \\ \begin{pmatrix} a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{11} \\ a_{22} & a_{21} \end{pmatrix} \end{pmatrix} =$$

$$= \underline{\underline{A^T A}}$$

Generalized: $f(x) = \frac{1}{2} \| A \cdot x - b \|^2$

$$\nabla f(x) = \frac{1}{2} \cdot 2 A^T (A \cdot x - b)$$

$$\nabla^2 f(x) = A^T (A \cdot x - b)$$
$$= A^T \cdot A \cdot x - A^T b$$
$$= ((A^T \cdot A)^T)^T \cdot x - A^T b$$
$$= (A^T \cdot A)^T = \underline{\underline{A^T \cdot A}}$$

2.2) $f(x) = \frac{1}{2} \| A(x + \alpha y) - b \|^2$

$$f(\alpha) = \frac{1}{2} \left\| \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \alpha \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right) - \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \right\|^2 =$$

$$= \frac{1}{2} \left\| \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \begin{pmatrix} x_1 + \alpha y_1 \\ x_2 + \alpha y_2 \end{pmatrix} - \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \right\|^2 =$$

$$= \frac{1}{2} \left\| \begin{matrix} (a_{11} \cdot (x_1 + \alpha y_1) + a_{12} \cdot (x_2 + \alpha y_2) - b_1) \\ (a_{21} (x_1 + \alpha y_1) + a_{22} (x_2 + \alpha y_2) - b_2) \end{matrix} \right\|^2 =$$

$$\frac{\partial}{\partial \alpha} f(\alpha) = \frac{1}{2} \left(\sqrt{(a_{11} (x_1 + \alpha y_1) + a_{12} (x_2 + \alpha y_2) - b_1)^2 + (a_{21} (x_1 + \alpha y_1) + a_{12} (x_2 + \alpha y_2) - b_2)^2} \right)^2 =$$

$$= \frac{1}{2} \left((a_{11} (x_1 + \alpha y_1) + a_{12} (x_2 + \alpha y_2) - b_1)^2 + (a_{21} (x_1 + \alpha y_1) + a_{12} (x_2 + \alpha y_2) - b_2)^2 \right)$$

$$= \frac{1}{2} \left(2 (\text{---}\, " \,\text{---}) (a_{11} y_1 + a_{12} y_2) + 2 (\text{---}\, " \,\text{---}) (a_{21} y_1 + a_{22} y_2) \right) =$$

$$= \left(\begin{matrix} \boxed{(a_{11} y_1 + a_{12} y_2)} (a_{11} (x_1 + \alpha y_1) + a_{12} (x_2 + \alpha y_2) - b_1) + \\ \boxed{(a_{21} y_1 + a_{22} y_2)} (a_{12} (x_1 + \alpha y_1) + a_{22} (x_2 + \alpha y_2) - b_2) \end{matrix} \right) =$$

$$= \left(\left((a_{11} \;\; a_{12}) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right) \left((a_{21} \;\; a_{22}) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right) \right) =$$

$$= \left(\left((y_1 \;\; y_2) \begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} \right) \left((y_1 \;\; y_2) \begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix} \right) \right) =$$

$$= (y_1 \;\; y_2) \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = y^T A^T \text{ — } ①$$

$$A(x + \alpha y) - b \text{ — } ②$$

$$\Rightarrow \underline{\underline{y^T A^T (A (x + \alpha y) - b)}}$$

$$\frac{\partial}{\partial \alpha} f(x) = y^T A^T (A(x + \alpha y) - b) =$$

$$= y^T A^T A \cdot (x + \alpha y - y^T A^T b) =$$

$$= y^T A^T A (x + \alpha y) - y^T A^T b \cancel{x} =$$

$$= y^T A^T A x + y^T A^T A \alpha y - y^T A^T \cdot b =$$

$$= y^T A^T A x + y^T A^T A \alpha y - y^T A^T \cdot b =$$

$$= \qquad + y^T A^T A y \alpha - \ldots$$

$$\frac{\partial^2}{\partial \alpha^2} f(x) = \underline{\underline{y^T A^T A \cdot y}}$$

2.3) $f(x) = \left(\prod_{i=1}^{n} x_i \right)^{\frac{1}{n}}$

$\cancel{\frac{\partial}{\partial x_1} f(x) = \frac{1}{n}}$

•) $n=2$

$f(x) = (x_1 x_2)^{\frac{1}{2}}$

$\nabla f(x_1) = \frac{1}{2}(x_1 x_2)^{-\frac{1}{2}} \cdot x_2$

$\nabla f(x_2) = \frac{1}{2}(x_1 x_2)^{-\frac{1}{2}} \cdot x_1$

$\frac{\nabla^2 f(x)}{\partial x_1} = u \cdot v' + u' \cdot v = \frac{1}{2}(x_1 x_2)^{-\frac{1}{2}} \cdot 1 + \left(-\frac{1}{4}(x_1 x_2)^{-\frac{3}{2}}\right) x_2$

$\frac{\nabla^2 f(x)}{\partial x_2} =$ —— " —— $= \frac{1}{2}(x_1 x_2)^{-\frac{1}{2}} \cdot 1 + \left(-\frac{1}{4}(x_1 x_2)^{-\frac{3}{2}}\right) \cdot x_1$

•) $n=3$

$f(x) = (x_1 x_2 x_3)^{\frac{1}{3}}$

$\nabla f(x_1) = \frac{1}{3}(x_1 x_2 x_3)^{-\frac{2}{3}} \cdot x_2 x_3$

$\nabla f(x_2) = \frac{1}{3}(x_1 x_2 x_3)^{-\frac{2}{3}} \cdot x_1 x_3$

$\nabla f(x_3) = \frac{1}{3}(x_1 x_2 x_3)^{-\frac{2}{3}} \cdot x_1 x_2$

$\frac{\nabla^2 f(x)}{\partial x_1} = u \cdot v' + u' \cdot v = \frac{1}{3}(x_1 x_2 x_3)^{-\frac{2}{3}} \cdot 1 + \left(-\frac{2}{9}\right)(x_1 x_2 x_3)^{-\frac{5}{3}} \cdot x_2 x_3$

$\frac{\nabla^2 f(x)}{\partial x_2} =$ —— " —— $= \frac{1}{3}(x_1 x_2 x_3)^{-\frac{2}{3}} \cdot 1 - \frac{2}{9}(x_1 x_2 x_3)^{-\frac{5}{3}} \cdot x_1 x_3$

$\frac{\nabla^2 f(x)}{\partial x_3} =$ —————— " —————— $\cdot x_1 x_2$

c) Generalize for Arbitrary n

$$f(x) = \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}$$

$$\frac{\partial}{\partial x_j} f(x) = \frac{1}{n} f(x) \cdot \frac{1}{x_j}$$

$$\frac{\partial^2}{\partial x_j^2} f(x) = \frac{1}{n}\Big(\underbrace{\frac{1}{n} \cdot \left(\prod_{i=1}^{n} x_j\right)^{\frac{1}{n}} \cdot \frac{\prod_{i=1}^{n} x_i}{x_j}}_{u'} \cdot \underbrace{\frac{1}{x_j}}_{v} + \underbrace{\left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}}_{u} \cdot \underbrace{(-1)\frac{1}{x_j^2}}_{v'}\Big) =$$

$$\frac{\partial^2}{\partial x_j^2} f(x) = \frac{1}{n}\left(\frac{1}{n} f(x) \cdot \frac{1}{x_j x_j} + (-1) f(x) \cdot \frac{1}{x_j^2 x_j}\right)$$

$$\frac{\partial^2}{\partial x_j^2} f(x) = \frac{1}{n} f(x) \cdot \left(\left(\frac{1}{n} - 1\right) \cdot \frac{1}{x_j x_j}\right)$$

$$= \frac{1}{n} \left(\left(\frac{1}{n} - 1\right)\right) \cdot f(x) \cdot \frac{1}{x_j x_j}$$

$j \neq k$

$$\frac{\partial^2}{\partial x_j x_k} f(x) = \frac{1}{n} \cdot \frac{1}{n} \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}-1} \frac{\prod_{i=1}^{n} x_i}{x_k} \cdot \frac{1}{x_j}$$

$$= \cancel{\frac{1}{n^2}}$$

$$\frac{\partial^2}{\partial^2 x_j^2} f(x) = \frac{1}{n} \left(\frac{1}{n} - 1\right) f(x) \frac{1}{x_j x_j} = \frac{1}{n} \left(\frac{1}{n} - 1\right) f(x) \frac{1}{x_j x_j}$$

$$\frac{\partial^2}{\partial x_j \partial x_k} f(x) = \frac{1}{n^2} f(x) \frac{1}{x_j x_k} = \frac{1}{n} \left(\frac{1}{n}\right) f(x) \frac{1}{x_j x_k}$$

$$\nabla^2 f(x) = \frac{1}{n} f(x) \cdot \begin{pmatrix} \left(\frac{1}{n} - 1\right)\frac{1}{x_1^2} & \frac{1}{n} \cdot \frac{1}{x_1 x_2} & \cdots & \frac{1}{n} \cdot \frac{1}{x_j x_k} \\ \vdots & \ddots & & \vdots \\ \left(\frac{1}{n}\right) \frac{1}{x_n x_1} & \cdots & \cdots & \left(\frac{1}{n} - 1\right) \cdot \frac{1}{x_n^2} \end{pmatrix}$$