Optimization for Computer Science
Assignment 2

November 1, 2016

Submission: Upload your report as a pdf file (`MatrNr.pdf`) to the TU-Graz TeachCenter using the MyFiles extension (can also be your scanned-in hand written notes).
Deadline: November 8, 2016 at 23:58h.

1 Convexity

Prove or disprove the convexity/concavity of the following functions. Describe all your steps in your report. Hint: A twice differentiable function is convex iff the Hessian matrix is positive semidefinite.

1. $f(x) = \frac{1}{2}\left(\log(x)\right)^2, \quad \operatorname{dom} f = \{x \,|\, x > 0\}$
2. $f(x) = x_1 x_2, \quad x \in \mathbb{R}^2$
3. $f(x) = \frac{x_1^2}{x_2}, \quad x \in \mathbb{R}^2,\ x_2 > 0$

2 Derivatives

Calculate analytically the first and second derivatives of the following functions. Describe all your steps in your report.

1. $f(x) = \frac{1}{2}\|Ax - b\|^2, \quad x \in \mathbb{R}^m,\ b \in \mathbb{R}^n,\ A \in \mathbb{R}^{m \times n}$
   Calculate the gradient $\nabla f(x)$ and the Hessian $\nabla^2 f(x)$ in two steps: First, for $m = n = 2$ and then in general matrix form.
2. $f(\alpha) = \frac{1}{2}\|A(x + \alpha y) - b\|^2, \quad \alpha \in \mathbb{R},\ x, y \in \mathbb{R}^m,\ b \in \mathbb{R}^n,\ A \in \mathbb{R}^{m \times n}$
3. $f(x) = \left(\prod_{i=1}^n x_i\right)^{\frac{1}{n}}, \quad x \in \mathbb{R}^n_+$
   Hint: Start by explicitly writing down the cases for $n = 2$ and $n = 3$ and then generalize for arbitrary n. Consider diagonal and off-diagonal elements of the Hessian separately.

Bonus task: $f(x) = \|Dx\|^{\varepsilon}_{2,1,1}, \quad x \in \mathbb{R}^{M\times N},\ D \in \mathbb{R}^{M\times N\times 2}$
D is a linear operator that approximates the spatial gradient of an image $x \in \mathbb{R}^{M\times N}$ by finite differences. The action of the linear operator is defined as $(Dx)_{i,j,1} = x_{i,j+1} - x_{i,j}$ and $(Dx)_{i,j,2} = x_{i+1,j} - x_{i,j}$. Finally, we apply the ε-smoothed $2,1,1$-norm which, for a tensor $A \in \mathbb{R}^{M\times N\times K}$, is given by $\|A\|^{\varepsilon}_{2,1,1} = \sum_{n=1}^{N}\sum_{m=1}^{M}\sqrt{\sum_{k=1}^{K} A^2_{m,n,k} + \varepsilon^2}$