Discrete Mathematics

MATH-305

Dr. Ahmed M. H. Abdelfattah

ahmed.abdelfattah@guc.edu.eg

Computer Science and Engineering, GUC

http://shams.academia.edu/AhmedAbdelFattah

#3 (Predicate Calculus & FoL)

Objectives

By the end of this section you will be able to

1. Use the formal symbols of predicate logic.
2. Find the truth value in some interpretation of an expression in predicate logic.
3. Use predicate logic to represent English language sentences.

A Simpler Example Motivating FOL

- Consider the argument

  All men are mortal
  Adam is a man
  ___
  $\therefore$ Adam is mortal

- Is it valid?

  Yes! But why?

- Consider another argument

  All men are mortal
  Some men are wise
  ___
  $\therefore$ Some mortals are wise

- Is it valid?

  Yes! But why?

Even More . . .

- Consider trying to represent the knowledge that all GUC members are smart.
- In propositional logic:

$$A \wedge B \wedge C \wedge D \wedge \cdots$$

  where
  - A is "Alaa is smart"
  - B is "Bahaa is smart"
  - C is "Cherry is smart"
  - D is "Dahlia is smart"
  - etc.
- At least two problems:
  - Very inefficient representation.
  - Misses important generalizations.

What Do We Need?

- We need a more expressive language.
- One that allows us to go deeper than the statement level.
- One that allows us to explicitly represent
  - Expressions like "all" and "some", which are called quantifiers.
  - Properties like "GUC member" or "smart".
  - Variables standing for arbitrary individuals.
- The (first-order) predicate logic is a language that gives us all that (and more).

n-ary Predicates and Constants

- Properties are denoted by predicates.
- Particular individuals (entities) are denoted by constants.

Example

- "Ahmed is smart" could be represented as $Smart(A)$
  - $Smart$ is a predicate denoting the property of being smart. (Note that a meaningless string like $Dsfg$ is as good a predicate as $Smart$.)
  - A is a constant denoting the individual named "Ahmed".
- Using this notation, "Bahaa is smart" could be represented as $Smart(B)$, where B denotes Bahaa.

The predicate *Smart* is a unary predicate: it denotes a property of a single individual.

Example (A general n-ary predicate denotes a relation between n individuals.)

- "Cherry likes Computer Science" could be represented as $L(C, CS)$

  Where C denotes Cherry, CS denotes computer science, and L denotes the binary liking-relation:

  $L(x, y)$ is a predicate that denotes the sentence: "x likes y"

Interpretations

- The same logical statement may mean different things depending on how it is interpreted.
- In propositional logic, an interpretation is a simple assignment of truth values to the atomic statements (a row in the truth table).
- In **predicate logic**, an interpretation has to specify a set of entities, called the domain of interpretation (or the domain of discourse).
  - It should specify which property/relation is denoted by each predicate and which entity is denoted by each constant.

Ex. What would $L(C, CS)$ possibly mean under different interpretations?

The Universal Quantifier $\forall$

Quantification expresses the extent to which a predicate is true over a range of elements.

- Let the domain of interpretation be the set of all GUC students.
- The statement "all GUC students are smart" could be represented as $\forall x[Smart(x)]$.
- Here $\forall$ is a universal quantifier and x is a universally quantified variable.
- The expression $Smart(x)$ is the scope of the quantifier.

The Existential Quantifier $\exists$

Quantification expresses the extent to which a predicate is true over a range of elements.

- The statement “some GUC student is smart" could be represented as $\exists x[Smart(x)]$
- This could alternatively be read as "there exists a smart GUC student" or "there is at least one smart GUC student".
- Here, $\exists$ is an existential quantifier and x is an existentially quantified variable.

Examples with Quantifiers

- Let $R(x)$ denote "x is an odd integer < 8"
  - $R(1)$ denotes "1 is an odd integer < 8" (True)
  - $R(8)$ denotes "8 is an odd integer < 8" (False)
  - $R(-3)$ denotes "-3 is an odd integer < 8" (True)
- Let $P(x)$ denote "x is a prime number"
  - For integers, $\exists x \; P(x)$ is true and $\forall x \; P(x)$ is false

  Q: *What might* $\exists x \; (R(x) \to P(x))$ *and* $\forall x \; (P(x) \to R(x))$ *mean?*
- The sentence "$Q(x) : x + 1 > 4$" is a predicate, where x is a real number
  - $\forall x \; Q(x)$ is false (check $Q(\sqrt{2})$).
  - $\exists x \; Q(x)$ is true.
- The statement $\exists y \; (y + 2 = y)$ is false.
- Let $\forall n \; T(n)$ be "For all positive integers n, $n^2 + 41n + 41$ is a prime number", then $\neg(\forall n \; T(n)) \equiv \exists n \; \neg T(n)$.

Expanding the Domain

- What if the domain is the set of everything in the world?
- "All GUC students are smart" can no longer be represented as $\forall x[Smart(x)]$.
- Rather,

$$\forall x[GUC(x) \rightarrow Smart(x)]$$

  where GUC denotes the property of being a GUC student.
- What about "some GUC student is smart"?

$$(\exists x)[GUC(x) \wedge Smart(x)]$$

- Do you see why they are different?

Translation

When translating (e.g., from English) to predicate logic

- One needs to specify what the domain is.
- One needs to provide the meanings of all predicates and constants.
- In most interesting cases, the scope of a universal quantifier is an implication.

  . . . and the scope of an existential quantifier is a conjunction.

Translation Basics: A typical/usual translation scheme:

- Proper nouns $\leftrightarrow$ Constants.
- Verbs (verb to-be is special, though) $\leftrightarrow$ Predicate symbols (of arity 1, 2 or 3).
- Nouns $\leftrightarrow$ Unary predicate symbols.
- Adjectives $\leftrightarrow$ Unary predicate symbols.
- Prepositions $\leftrightarrow$ Predicate symbols of arity 2 or more.

Simple Cases for Translation

Example

- Fido is a dog.
  - $Dog(Fido)$
- Fido is a black dog.
  - $Dog(Fido) \wedge Black(Fido)$
- Fido likes Lacy.
  - $Likes(Fido, Lacy)$

Translations including Quantification

- - All lions are brave.
  - Every lion is brave.
  - Each lion is brave.
  - $\forall x[Lion(x) \rightarrow Brave(x)]$
- - Some lion is brave.
  - There is a brave lion.
  - A lion exists that is also brave.
  - $\exists x[Lion(x) \wedge Brave(x)]$
- What is wrong with
  - $\forall x[Lion(x) \wedge Brave(x)]$ (for "All lions are brave") and
  - $\exists x[Lion(x) \rightarrow Brave(x)]$ (for "Some lion is brave")

  respectively?

Free and Bound Variables

- A variable is BOUND if it is universally- or existentially- quantified.
- A variable is FREE if it is not bound.
- We (agree to) take a statement containing a free variable to be neither true nor false.
- A language of predicate logic has to specify which symbols are variables and which are constants (or predicates).

Example

- $L(x, CS)$ is neither true nor false (if x is a variable).
- $\forall x[L(x, y)]$ is neither true nor false (if y is a variable).

Scope and Ambiguity

- Every man loves a woman.

$$\forall x \left(Man(x) \Rightarrow \exists y \left(Woman(y) \ \wedge \ Loves(x, y) \right) \right)$$

- There is a woman loved by all men.

$$\exists y \left(Woman(y) \wedge \forall x \left(Man(x) \ \Rightarrow \ Loves(x, y) \right) \right)$$

Counting and Uniqueness

There is exactly one brave lion.

- $\exists x \left[Lion(x) \wedge Brave(x) \wedge \forall y \left[\left(Lion(y) \wedge Brave(y) \right) \Rightarrow Eq(x, y) \right] \right]$

Validity in Predicate Logic

- In *propositional logic*, truth tables are used to determine validity (whether a statement is a tautology.)
- For a statement with n atomic sub-statements, we need to consider 2^n possibilities.

i.e. A statement is a tautology if it is true in all such possibilities.

- In *predicate logic*, an interpretation corresponds to a row in the propositional logic truth table.

i.e. A statement is a tautology if it is true under all interpretations.

- The number of possible interpretations is ∞
  and there is no algorithm to determine validity in general.
- For individual statements, we can provide arguments for, or counter examples to, their validity.

Exercise 1

Example

$$\neg(\forall x[P(x)]) \to \exists x[\neg P(x)]$$

- This is a valid statement.
- The only way that $\forall x[P(x)]$ would be false is if there is some x such that $P(x)$ is false.
- That is, $\exists x[\neg P(x)]$ must be true.

Exercise 2

Example

$$(\exists x[P(x)] \wedge \exists x[Q(x)]) \rightarrow \exists x[P(x) \wedge Q(x)]$$

- This is *not* valid.
- Counter example: domain is $\mathbb{Z}$, $P(x)$ means that x is even, and $Q(x)$ means that x is odd.
- The antecedent is true, but the consequent is false.
- Note that the converse is valid.

Some Equivalences containing Quantifiers

Theorem 3, page 60, Kolman et al. (2009):

Theorem

1. $\neg(\forall x\ P(x)) \equiv \exists x\ \neg P(x)$
2. $\neg(\exists x\ P(x)) \equiv \forall x\ \neg P(x)$
3. $\exists x\ (P(x) \lor Q(x)) \equiv \exists x\ P(x) \lor \exists x\ Q(x)$
4. $\forall x\ (P(x) \land Q(x)) \equiv \forall x\ P(x) \land \forall x\ Q(x)$
5. $\exists x\ (P(x) \to Q(x)) \equiv \forall x\ P(x) \to \exists x\ Q(x)$

HOWEVER,

1. $\exists x\ (P(x) \land Q(x)) \not\equiv \exists x\ P(x) \land \exists x\ Q(x)$

yet: $\exists x\ (P(x) \land Q(x)) \to \exists x\ P(x) \land \exists x\ Q(x)$ is a tautology

2. $((\forall x\ P(x)) \lor (\forall x\ Q(x))) \not\equiv \forall x\ (P(x) \lor Q(x))$

yet: $((\forall x\ P(x)) \lor (\forall x\ Q(x))) \to \forall x\ (P(x) \lor Q(x))$ is a tautology

Prove that the following is INVALID:

1. $\exists x\ P(x) \wedge \exists x\ Q(x) \longrightarrow \exists x\ \big(P(x) \wedge Q(x)\big)$.[1]

[1] But remember that $\exists x\ \ (P(x) \wedge Q(x)) \rightarrow \exists x\ P(x) \wedge \exists x\ Q(x)$ is a tautology!

Prove that the following is INVALID:

1. $\forall x\ \big(P(x) \vee Q(x)\big) \longrightarrow \Big((\forall x\ P(x)) \vee (\forall x\ Q(x)) \Big)$.[2]

[2] But remember that $((\forall x\ P(x)) \vee (\forall x\ Q(x))) \rightarrow \forall x\ \ (P(x) \vee Q(x))$ is a tautology!

Sample Exercises:

Which of the following is/are the negation of "**All lions are brave**"?

1. **All lions are not brave**?
2. **Some lion is not brave**?
3. **Some lion is brave**?
4. **A lion exists that is also brave**?
5. **There is a lion that is not brave**?

Prove your answer.

Sample Exercises:

What is the negation of "$\forall x \Big(\mathsf{M}(x) \to \exists y \big(\mathsf{W}(y) \ \wedge \ \mathsf{L}(x,y) \big) \Big)$"?

Example

Translate the following statements using a suitable language of predicate logic.

1. Every magic pond has a golden fish in it.
   - What about: There is a golden fish in every magic pond?
2. In any magic pond, the only fish are golden fish.

First, the interpretation:

- Assume the domain to include everything that we can conceive of.
- Define the following predicates.
  - $MP(x)$: x is a magic pond.
  - $F(x)$: x is a fish.
  - $GF(x)$: x is a golden fish. (There is something tricky about this one)
  - $In(x, y)$: x is in y.

And then the translation . . .

1. *Every magic pond has a golden fish in it.*

   $\forall x[MP(x) \rightarrow \exists y[GF(y) \wedge In(y,x)]]$

   for all x in the domain **do**
     if x is a magic pond **then**
       there exists a y in the domain such that
         y is a golden fish and y is in x

2. *In any magic pond, the only fish are golden fish.*

   $\forall x[MP(x) \rightarrow \forall y[(F(y) \wedge In(y,x)) \rightarrow GF(y)]]$

   for all x in the domain **do**
     if x is a magic pond **then**
       for all y in the domain **do**
         if y is a fish and y is in x **then**
           y is a golden fish

Reasoning in Predicate Logic

- In predicate logic, we do not have the equivalent of truth tables to easily prove the validity of an argument.
- We must rely on derivations (formal proofs).
- Predicate logic inherits all of the derivation rules of propositional logic.
- It adds a couple of more rules to deal with quantifiers.

Remember: Free and Bound Variables

- A variable is BOUND if it is universally- or existentially- quantified.
- A variable is FREE if it is not bound.
- We (agree to) take a statement containing a free variable to be neither true nor false.
- A language of predicate logic has to specify which symbols are variables and which are constants (or predicates).

Example

- $L(x, CS)$ is neither true nor false (if x is a variable).
- $\forall x[L(x, y)]$ is neither true nor false (if y is a variable).

Using Only Derivation Rules of Propositional Logic

Example

Prove that the following argument is valid.

$$[P(c) \vee \neg[\exists y Q(y)]] \wedge \exists y Q(y) \wedge [P(c) \rightarrow \forall x P(x)] \rightarrow \forall x P(x)$$

① $P(c) \vee \neg[\exists y Q(y)]$ (hypothesis)

② $\exists y Q(y)$ (hypothesis)

③ $P(c) \rightarrow \forall x P(x)$ (hypothesis)

④ $\neg[\exists y Q(y)] \vee P(c)$ (1, commutativity)

⑤ $\exists y Q(y) \rightarrow P(c)$ (4, implication)

⑥ $P(c)$ (2,5, Modus Ponens)

⑦ $\forall x P(x)$ (3,6, Modus Ponens)

Remember: Predicate Logic Equivalences

- $\neg\forall x(P(x)) \equiv \exists x(\neg P(x))$
- $\neg\exists x(P(x)) \equiv \forall x(\neg P(x))$
- $\forall x(P(x) \land Q(x)) \equiv (\forall x(P(x)) \land \forall x(Q(x)))$
- $\exists x(P(x) \lor Q(x)) \equiv (\exists x(P(x)) \lor \exists x(Q(x)))$

Note!

- $\forall x(P(x) \lor Q(x)) \not\equiv (\forall x(P(x)) \lor \forall x(Q(x)))$
- $\exists x(P(x) \land Q(x)) \not\equiv (\exists x(P(x)) \land \exists x(Q(x)))$

Predicate Logic Rules of Inference

- There are four rules, two for each quantifier.
  1. Universal instantiation.
  2. Existential instantiation.
  3. Universal generalization.
  4. Existential generalization.
- We have to be careful when using these rules.
  - Well, "EXTREMELY careful" actually!

Universal Instantiation

$$\frac{\forall x \Phi[x]}{\Phi[c]}$$

Rationale

If it's true for all entities in the domain, then it's true for any particular entity in the domain.

- $\Phi[x]$ is a statement where x occurs.
- c is a constant.
- $\Phi[c]$ is the result of replacing the free occurrences of x in $\Phi[x]$ by c.
- The free occurrences of x in $\Phi[x]$ are those bound by $\forall$ in $\forall x \Phi[x]$.

Exercise

Example

Prove that the following argument is valid. (Where s is a constant.)

$$(\forall x[H(x) \to M(x)] \wedge H(s)) \to M(s)$$

① $\forall x[H(x) \to M(x)]$ (hypothesis)

② $H(s)$ (hypothesis)

③ $H(s) \to M(s)$ (1, universal instantiation)

④ $M(s)$ (2,3, modus ponens)

Existential Generalization

$$\frac{\Phi[c]}{\exists x \Phi[x]}$$

Rationale:

If it's true for a particular entity in the domain, then there is an entity in the domain for which it's true.

- c is a constant.
- x does not occur in $\Phi[c]$.
  - Otherwise, variable capture takes place.

Exercise

Prove that the following argument is valid.

Example

$$\forall x P(x) \rightarrow \exists x P(x)$$

① $\forall x P(x)$ (hypothesis)

② $P(c)$ (1, universal instantiation)

③ $\exists x P(x)$ (2, existential generalization)

Another Example

Example

Something's wrong with the following proof. Can you tell what it is?

① $P(a) \to \forall x Q(x, a)$ (hypothesis)

② $\exists x[P(x) \to \forall x Q(x, x)]$ (1, existential generalization)

Variable Capture!

Existential Instantiation

$$\frac{\exists x \Phi[x]}{\Phi[c]}$$

Rationale

We temporarily introduce a name for the individual of which $\exists x \Phi[x]$ holds.

- The name must not refer to anyone that we already know.
- c is a constant (traditionally called a Skolem constant).
- Restriction:
  - c has not been previously used in the proof.
  - c does not occur in the conclusion.

Exercise

Example

The following are legitimate steps within some proof.

① $\forall x[P(x) \rightarrow Q(x)]$ (hypothesis)

② $\exists y P(y)$ (hypothesis)

③ $P(a)$ (2, existential instantiation)

④ $P(a) \rightarrow Q(a)$ (1, universal instantiation)

⑤ $Q(a)$ (3, 4, modus ponens)

- Note! The above is NOT a proof of the validity of the argument $(\forall x[P(x) \rightarrow Q(x)] \wedge \exists y P(y)) \rightarrow Q(a)$.
- Recall that a Skolem constant cannot occur in the conclusion.
- Note that swapping steps 3 and 4 yields a wrong derivation.

Universal Generalization

$$\frac{\Phi[c]}{\forall x \Phi[x]}$$

- c is a constant.
- c does not occur in the hypotheses or the conclusion.
- Any Skolem constant in Φ was introduced into the derivation strictly before c.

Rationale

If it's true for an arbitrary entity, then it is true for all entities.

- We use this rule all the time, whenever we need to prove that some property is true of all elements in a set.
- The restrictions ensure the arbitrariness of x.

Exercise

Example

Prove that the following argument is valid.

$$[\ \forall x[P(x) \rightarrow Q(x)] \wedge \forall x P(x)\ ] \rightarrow \forall x Q(x)$$

① $\forall x[P(x) \rightarrow Q(x)]$ (hypothesis)

② $\forall x P(x)$ (hypothesis)

③ $P(a) \rightarrow Q(a)$ (1, universal instantiation)

④ $P(a)$ (2, universal instantiation)

⑤ $Q(a)$ (3, 4, modus ponens)

⑥ $\forall x Q(x)$ (5, universal generalization)

Another Exercise

Example

Something's wrong with the following proof. Can you tell what it is?

① $P(a)$ (hypothesis)

② $\forall x[P(x)]$ (1, universal generalization)

a **occurs in a hypothesis!**

Yet Another Exercise

Example

Something's wrong with the following proof. Can you tell what it is?

① $\forall x[\exists y[P(x,y)]]$ (hypothesis)

② $\exists y[P(a,y)]$ (1, universal instantiation)

③ $P(a,c)$ (2, existential instantiation)

④ $\forall x[P(x,c)]$ (3, universal generalization)

The Skolem constant c was not introduced before a!

GUC

German University in Cairo

الجامعة الألمانية بالقاهرة

THANK YOU

References: I

Kolman, B., Busby, R. C., and Ross, S. C. (2009). *Discrete Mathematical Structures*. Pearson/Prentice Hall.