German University in Cairo

الجامعة الألمانية بالقاهرة

Discrete Mathematics

MATH-305/501

Dr. Ahmed M. H. Abdelfattah

ahmed.abdelfattah@guc.edu.eg

Computer Science and Engineering, GUC

http://shams.academia.edu/AhmedAbdelFattah

#2 (Arguments in PL)

Logical Equivalences

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Answer:

1. Construct the truth table of $\neg (p \wedge q) \leftrightarrow (\neg p \vee \neg q)$
2. Check whether it is a tautology

   if "Yes", then $\neg (p \wedge q)$ and $(\neg p \vee \neg q)$ are logically equivalent.

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p	q	$p \wedge q$	$\neg(p \wedge q)$ ①	$\neg p$	$\neg q$	$\neg p \vee \neg q$ ②	① $\leftrightarrow$ ②
T	T	T	**F**	F	F	**F**	T
T	F	F	**T**	F	T	**T**	T
F	T	F	**T**	T	F	**T**	T
F	F	F	**T**	T	T	**T**	T

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1. Any material implication has the form $p \to q$, where p and q are propositions.
   E.g. $A \to B$ and $(R \vee S) \to (T \wedge \neg S)$ are both forms of implication.

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   E.g. $A \to B$ and $(R \vee S) \to (T \wedge \neg S)$ are both forms of implication.

2. Recall that the contrapositive of $p \to q$ has the form $\neg q \to \neg p$.

Example: **Show that both a material implication and its contrapositive are logically equivalent.**

Answer:

1. Any material implication has the form $p \to q$, where p and q are propositions.

   E.g. $A \to B$ and $(R \lor S) \to (T \land \neg S)$ are both forms of implication.

2. Recall that the contrapositive of $p \to q$ has the form $\neg q \to \neg p$.

   E.g. The contrapositive of $A \to B$ is $\neg B \to \neg A$

   E.g. The contrapositive of $(R \lor S) \to (T \land \neg S)$ is $\neg(T \land \neg S) \to \neg(R \lor S)$

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   E.g. $A \to B$ and $(R \vee S) \to (T \wedge \neg S)$ are both forms of implication.

2. Recall that the contrapositive of $p \to q$ has the form $\neg q \to \neg p$.

   E.g. The contrapositive of $A \to B$ is $\neg B \to \neg A$

   E.g. The contrapositive of $(R \vee S) \to (T \wedge \neg S)$ is $\neg(T \wedge \neg S) \to \neg(R \vee S)$

3. We need therefore to check whether or not $(p \to q) \leftrightarrow (\neg q \to \neg p)$ is a tautology

   if "Yes", then a material implication $p \to q$ and its contrapositive statement $\neg q \to \neg p$ are logically equivalent.

p	q	$p \to q$	$\neg q \to \neg p$	$(p \to q) \leftrightarrow (\neg q \to \neg p)$
T	T	T		
T	F	F		
F	T	T		
F	F	T		

Important Equivalence Rules

- Commutativity:
  1. $p \vee q \equiv q \vee p$.
  2. $p \wedge q \equiv q \wedge p$.
- Associativity:
  1. $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  2. $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- De Morgan's laws:
  1. $\neg(p \vee q) \equiv \neg p \wedge \neg q$
  2. $\neg(p \wedge q) \equiv \neg p \vee \neg q$
- Distributive Properties:
  1. $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
  2. $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
- Implication: $p \rightarrow q \equiv \neg p \vee q$
- Double Negation: $\neg\neg p \equiv p$
- Equivalence: $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

Tautologies, Contradictions, and Contingencies

A **tautology** is a statement that is true under any truth assignment to its variables

Example: $p \vee \neg p$ $(p \wedge p) \leftrightarrow p$ $p \to (q \to p)$

A **contradiction** is a statement that is false under any truth assignment to its variables

Example: $p \wedge \neg p$ $p \leftrightarrow \neg(p \vee p)$

A statement is **satisfiable** if it is not a contradiction.

Example: p

A statement is **falsifiable** if it is not a tautology.

Example: $p \to q$

A statement is a **contingency** if it is both falsifiable and satisfiable.

Example: $p \vee (q \wedge \neg r)$

Tautologies, Contradictions, and Contingencies

Exercise: Complete the following table

	Tautology	Satisfiable	Contingency
$p \to \neg p$	x	✓	
$p \vee \neg p$	✓	✓	
$p \wedge \neg p$			x
$p \vee q$			✓
$p \to q$		✓	
$p \wedge q \to q$	✓	✓	

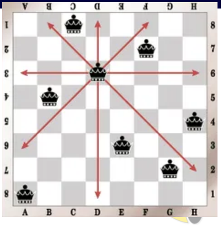
Tautologies, Contradictions, and Contingencies

Exercise: Complete the following table

	Tautology	Satisfiable	Contingency
$p \to \neg p$	x	✓	
$p \lor \neg p$	✓	✓	
$p \land \neg p$			x
$p \lor q$			✓
$p \to q$		✓	
$p \land q \to q$	✓	✓	

Logic Applications:

- Web searching.
- System configurations.
- Logic Puzzles, the n-queens puzzles, Sudoku, etc.
- Automatic theorem-proving: see "Resolution".
- Logic Gate Design.



Logical Vs. Human: NL conditional $\neq$ Material conditional

The 4-Card Puzzle: Deductive Reasoning and the Selection Task

Rules: ① Each card has a **letter** on one side and a **number** on the other.

② A card that has D on one side must have 3 on the other.

③ A response that identifies a card that need not be inverted, or that fails to identify a card that needs to be inverted, is incorrect.

Given: Four cards showing D, K, 3, and 7

Task: *To verify the rule's truth:* **which cards should only be checked?**

Result: Any ideas?

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Given: Four cards showing D, K, 3, and 7

Task: *To verify the rule's truth:* **which cards should only be checked?**

Result: Not even 10% of participants found the correct solution (D and 7).

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The 4-Card Puzzle: Deductive Reasoning and the Selection Task

Rules: ① Each card shows a **person** on one side and a **beverage** on the other.

② If someone is drinking **alcohol**, someone must be ≥ 21.

③ A response that identifies a card that need not be inverted, or that fails to identify a card that needs to be inverted, is incorrect.

Given: Four cards showing **beer**, **cola**, 25, and 16

Task: *To verify the rule's truth:* **which cards should only be checked?**

Result: Participants behave significantly better (**beer** and **16**) in social contexts.

Other variants: Turn as few cards as possible to prove the rule

1. If there is a vowel at one side, there will be an even number at the other side.
2. If a card shows an even number on one face, then its opposite face is red.

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Propositions: A statement is a sentence that can be assigned exactly one of the two truth values (*tertium non datur*)

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A statement is falsifiable if it is not a tautology.
A statement is a contingency if it is both falsifiable and satisfiable.

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Entscheidungsproblem: Is there always an algorithm (i.e., a definite procedure) that can decide whether or not a statement is true?

Smullyan's[1] Puzzle of the Politicians:

A certain convention numbered 100 politicians.

- Each politician was either crooked or honest.
- We are given the following two facts.
  1. At least one of the politicians was honest.
  2. Given any two of the politicians, at least one of the two was crooked.
- Can it be determined from these two facts how many of the politicians were honest and how many were crooked?

src: https://raymondsmullyan.com/

Solution:

- Only one honest politician.
- Why?

  Write your argument here:

[1] Visit https://raymondsmullyan.com/ or read (Rosen, 2018, pp. 21) for more about Raymond Smullyan, the modern-day Lewis Carroll, the master of logic puzzles.

Arguments

- An argument is a *pair* $(\mathcal{P}, q)$ that consists of:
  1. $\mathcal{P} = \{p_1, p_2, \ldots, p_n\}$ is a finite set of statements called hypotheses (or premises), $\{p_1, p_2, \ldots, p_n\}$, and
  2. q is a statement called the conclusion, q.

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- The hypotheses and conclusion are commonly displayed as

$$\begin{array}{l} p_1 \\ p_2 \\ \vdots \\ p_n \\ \hline \therefore\ q \end{array}$$

or represented in PL as implications:

$$p_1 \wedge p_2 \wedge \cdots \wedge p_n \rightarrow q$$

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or represented in PL as implications:

$$p_1 \wedge p_2 \wedge \cdots \wedge p_n \rightarrow q$$

Example:

$$\begin{array}{l} U \wedge V \\ U \rightarrow \neg R \\ \neg V \vee S \\ \hline \therefore\ S \wedge \neg R \end{array}$$

Here, $U \wedge V$, $U \rightarrow \neg R$, and $\neg V \vee S$ are the premises, whereas $S \wedge \neg R$ is the conclusion.

The argument as an implication: $(U \wedge V) \wedge (U \rightarrow \neg R) \wedge (\neg V \vee S) \rightarrow (S \wedge \neg R)$

Valid Arguments

Definition

An argument is valid if the conclusion is true whenever all hypotheses are true.

Argument: $(D \to T) \land D \;\to\; T$

If I drive to work, then I will arrive tired	$D \to T$
I drive to work	D
$\therefore$ I will arrive tired	$\therefore\ T$

Argument: $W \land (W \to R) \;\to\; R$

The British PM is a woman	W
If the British PM is a woman, then the British PM is a mother	$W \to R$
$\therefore$ The British PM is a mother	$\therefore\ R$

Valid Arguments

An argument is valid if the conclusion is true whenever all hypotheses are true $\equiv$
An argument is valid if it is a tautology

Are the following arguments valid?

1.

Example (Argument1)

The British PM is a woman
If the British PM is a woman, then the British PM is a mother

---

$\therefore$ The British PM is a mother

2.

Example (Argument2)

The British PM is either a man or a woman
If the British PM is a mother, then the British PM is a woman

---

$\therefore$ The British PM is a man

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Are the following arguments valid?

1. Valid

Example (Argument1)

The British PM is a woman	W
If the British PM is a woman, then the British PM is a mother	$W \to R$
$\therefore$ The British PM is a mother	$\therefore\ R$

2. Invalid

Example (Argument2)

The British PM is either a man or a woman	$M \vee W$
If the British PM is a mother, then the British PM is a woman	$R \to W$
$\therefore$ The British PM is a man	$\therefore\ M$

More on Valid Arguments

- The truth of the conclusion has to *follow from* the truth of the hypotheses.
- There should be no world in which the hypotheses are all true and the conclusion is false.
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Example

Prove that $\begin{array}{l} P \to Q \\ P \\ \hline \therefore\ Q \end{array}$ is a valid argument.

Notes:

- This argument is known as modus ponens (usually abbreviated MP).
- To show its validity, simply prove that $((P \to Q) \land P) \to Q$ is a tautology.

Validity Vs. Truth

- Statements can be true or false
- Arguments can be valid or invalid
- **Tautologies:** A tautology is a statement that's true under any truth assignment.
- **Logical fallacies:** invalid arguments that are based on contingencies.

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Validity is all about the form:

If you Invest in the stock market, then you get Rich	$I \to R$
If you get Rich, then you become Happy	$R \to H$
$\therefore$ If you Invest in the stock market, then you become Happy	$\therefore I \to H$

So, no matter what is being said, $(I \to R) \wedge (R \to H) \to (I \to H)$ is a valid argument.

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Take Care: An argument can be VALID, despite its conclusion being FALSE!

Smoking is healthy	H
If smoking is healthy, then cigarettes are prescribed by physicians	$H \rightarrow P$
$\therefore$ Cigarettes are prescribed by physicians	$\therefore P$

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If smoking is healthy, then cigarettes are prescribed by physicians	$H \rightarrow P$
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Take Care: An INVALID argument is a fallacy

If you solve every problem, then you will learn discrete mathematics	$S \rightarrow M$
You learned discrete mathematics.	M
$\therefore$ You solve every problem.	$\therefore S$

This Slide is for Rent:

Formal Proofs

Instead of using truth tables, we can produce a formal proof that a given argument is valid (i.e., that the conclusion follows from the hypotheses).

A formal proof is a *sequence* of statements in which each statement is either a hypothesis or the result of applying a predefined set of derivation rules on earlier statements in the proof.

- For the proof to be correct, derivation rules must be truth-preserving—they must represent valid argument.

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Derivation Rules

- Derivation rules come in two flavors.
  1. Equivalence rules allow statements and sub-statements to be substituted by logically-equivalent statements.
  2. Inference rules allow new statements to be derived from previous statements in the proof.

Inference:

The Form of a Formal Proof

p_1	(hypothesis)
p_2	(hypothesis)
$\vdots$	$\vdots$
p_n	(hypothesis)
s_1	(obtained by applying a derivation rule to earlier statements)
s_2	(obtained by applying a derivation rule to earlier statements)
$\vdots$	$\vdots$
$\therefore\ q$	(obtained by applying a derivation rule to earlier statements)

- **Inference** is the name given to the reasoning process, by which we assert or deny the truth of a conclusion on the basis of other beliefs (premises) assumed to be true.
- Corresponding to every inference, one[2] can formulate a group of statements as an argument leading to the conclusion in question.

[2] Inference is a subjective process.

Important Equivalence Rules (again)

- Commutativity:
  1. $p \vee q \equiv q \vee p$.
  2. $p \wedge q \equiv q \wedge p$.
- Associativity:
  1. $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  2. $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- De Morgan's laws:
  1. $\neg(p \vee q) \equiv \neg p \wedge \neg q$
  2. $\neg(p \wedge q) \equiv \neg p \vee \neg q$
- Distributive Properties:
  1. $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
  2. $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
- Implication: $p \to q \equiv \neg p \vee q$
- Double Negation: $\neg\neg p \equiv p$
- Equivalence: $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

Important Inference Rules

Modus Ponens (**MP**)	$\begin{array}{l} p \\ p \to q \\ \hline \therefore\ q \end{array}$	Modus Tollens (**MT**)	$\begin{array}{l} \neg q \\ p \to q \\ \hline \therefore\ \neg p \end{array}$
Hypothetical Syllogism (**HS**)	$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore\ p \to r \end{array}$	Disjunctive Syllogism (**DS**)	$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore\ q \end{array}$
Addition	$\begin{array}{l} p \\ \hline \therefore\ p \vee q \end{array}$	Simplification	$\begin{array}{l} p \wedge q \\ \hline \therefore\ p \end{array}$
Conjunction	$\begin{array}{l} p \\ q \\ \hline \therefore\ p \wedge q \end{array}$	Resolution	$\begin{array}{l} p \vee q \\ \neg p \vee r \\ \hline \therefore\ q \vee r \end{array}$

See Table 1 in (Rosen, 2018, pp. 76)

Example

Example (**N.B.** You can show that $A \wedge (A \rightarrow B) \wedge (B \rightarrow \neg C) \rightarrow \neg C$ is a tautology using truth tables; whence the validity of the argument.)

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Prove that $A \wedge (A \to B) \wedge (B \to \neg C) \to \neg C$

① A (hypothesis)

② $A \to B$ (hypothesis)

③ $B \to \neg C$ (hypothesis)

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④ B (from 1 and 2 using MP)

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Prove that $A \wedge (A \to B) \wedge (B \to \neg C) \to \neg C$

① A (hypothesis)

② $A \to B$ (hypothesis)

③ $B \to \neg C$ (hypothesis)

④ B (from 1 and 2 using MP)

⑤ $\neg C$ (from 3 and 4 using MP)

Example

Example (Without using TT)

Prove that $A \wedge (B \to C) \wedge ((A \wedge B) \to (D \vee \neg C)) \wedge B \to D$

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Example (Without using TT)

Prove that $A \wedge (B \to C) \wedge ((A \wedge B) \to (D \vee \neg C)) \wedge B \to D$

① A (hypothesis)

② $B \to C$ (hypothesis)

③ $(A \wedge B) \to (D \vee \neg C)$ (hypothesis)

④ B (hypothesis)

Example

Example (Without using TT)

Prove that $A \wedge (B \to C) \wedge ((A \wedge B) \to (D \vee \neg C)) \wedge B \to D$

1. A (hypothesis)
2. $B \to C$ (hypothesis)
3. $(A \wedge B) \to (D \vee \neg C)$ (hypothesis)
4. B (hypothesis)
5. C (2, 4, MP)

Example

Example (Without using TT)

Prove that $A \wedge (B \to C) \wedge ((A \wedge B) \to (D \vee \neg C)) \wedge B \to D$

① A (hypothesis)

② $B \to C$ (hypothesis)

③ $(A \wedge B) \to (D \vee \neg C)$ (hypothesis)

④ B (hypothesis)

⑤ C (2, 4, MP)

⑥ $A \wedge B$ (1, 4, conjunction)

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③ $(A \wedge B) \to (D \vee \neg C)$ (hypothesis)

④ B (hypothesis)

⑤ C (2, 4, MP)

⑥ $A \wedge B$ (1, 4, conjunction)

⑦ $D \vee \neg C$ (3, 6, MP)

Example

Example (Without using TT)

Prove that $A \wedge (B \to C) \wedge ((A \wedge B) \to (D \vee \neg C)) \wedge B \to D$

①	A	(hypothesis)
②	$B \to C$	(hypothesis)
③	$(A \wedge B) \to (D \vee \neg C)$	(hypothesis)
④	B	(hypothesis)
⑤	C	(2, 4, MP)
⑥	$A \wedge B$	(1, 4, conjunction)
⑦	$D \vee \neg C$	(3, 6, MP)
⑧	$\neg C \vee D$	(7, commutativity)

Example

Example (Without using TT)

Prove that $A \wedge (B \to C) \wedge ((A \wedge B) \to (D \vee \neg C)) \wedge B \to D$

① A (hypothesis)

② $B \to C$ (hypothesis)

③ $(A \wedge B) \to (D \vee \neg C)$ (hypothesis)

④ B (hypothesis)

⑤ C (2, 4, MP)

⑥ $A \wedge B$ (1, 4, conjunction)

⑦ $D \vee \neg C$ (3, 6, MP)

⑧ $\neg C \vee D$ (7, commutativity)

⑨ $C \to D$ (8, implication)

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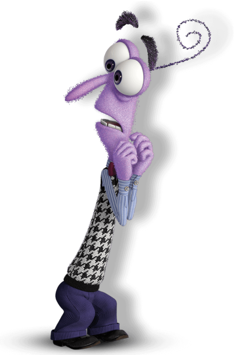
⑩	D	(5, 9, MP)



Example

Is the argument

$$\begin{array}{l} U \land V \\ U \to \neg R \\ \neg V \lor S \\ \hline \therefore\ S \land \neg R \end{array}$$

valid?

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Answer: YES; the argument is valid!

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- Show that: $p \to q \to r \leftrightarrow p \wedge q \to r$.
- Show that: $(p \vee q) \wedge (\neg p \vee r) \to (q \vee r)$.

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  - Non-clauses can be replaced by ≥ 1 equivalent statements that are clauses.

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THANK YOU

References: I

Rosen, K. (2018). *Discrete Mathematics and Its Applications*. McGraw-Hill Education, 8th edition.