EP1108 - Special Relativity Part 2

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Velocity addition in Special Relativity

Assume that S' is moving with velocity v with respect to S. V_x, V_y, V_z are the velocity components in S and the primed quantities the velocities in S'.

$$V'_x = \frac{dx'}{dt'}$$
$$V'_y = \frac{dy'}{dt'}$$
$$V'_z = \frac{dz'}{dt'}$$

By differentiating the Lorentz transformations, one obtains

$$dx' = \frac{dx - vdt}{\sqrt{1 - v^2/c^2}}$$
$$dy' = dy; dz' = dz$$
$$dt' = \frac{dt - vdx/c^2}{\sqrt{1 - v^2/c^2}}$$

Velocity addition

$$
\begin{aligned}
V'_x &= \frac{dx - vdt}{dt - vdx/c^2} \\
&= \frac{dx/dt - v}{1 - (v/c^2)dx/dt} \\
V'_x &= \frac{V_x - v}{1 - vVx/c^2}
\end{aligned}
$$

Similarly

$$
\begin{aligned}
V'_y &= \frac{V_y\sqrt{1 - v^2/c^2}}{1 - vV_x/c^2} \\
V'_z &= \frac{V_z\sqrt{1 - v^2/c^2}}{1 - vV_x/c^2}
\end{aligned}
$$

At low velocities $v << c$, these reduce to Galilean transformations.

The inverse transformation is given by

$$V_x = \frac{V_x' + v}{1 + vVx'/c^2}$$

$$V_y = \frac{V_y'\sqrt{1 - v^2/c^2}}{1 + vV_x'/c^2}$$

$$V_z = \frac{V_z'\sqrt{1 - v^2/c^2}}{1 + vV_x'/c^2}$$

Example:
If $v_x' = c$, i.e. a flash of light emitted with velocity c, an observer in frame S will measure the velocity

$$V_x = \frac{c + v}{1 + cv/c^2} = c$$

Both observers determine the same value for the speed of light in accord with the second postulate.

Relativistic mass, momentum and Energy

Define $\gamma \equiv \frac{1}{\sqrt{1-v^2/c^2}}$ $\beta \equiv v/c$

- Relativistic Mass : $m = \gamma m_0$ where m_0 is the rest mass.
- Relativistic Momentum : $p = mv = \gamma m_0 v = \gamma \beta m_o c$
- Relativistic total energy $E = mc^2$ or $E = \gamma m_0 c^2$ (Mass and Energy are equivalent). Relation between energy and momentum can also be written as follows : $E^2 = p^2c^2 + m_o^2 c^4$
- For a massless particle (eg. photon) $E = pc$
- Relativistic Kinetic energy $= (m - m_0)c^2$
- Invariant mass or proper mass (used in relativistic collisions) $=(\sum E/c)^2 - (\sum p)^2$

In nuclear/Particle physics, mass of elementary particle is denoted in terms of its equivalent energy (mc^2) For rest mass of an electron $= 9.1 \times 10^{-31}$ kg. Its equivalent energy $=$ 0.5 MeV, where $1\text{eV} = 1.6 \times 10^{-19}$J

Derivation of Relativistic Kinetic Energy

K.E. is the work done in moving an object from rest to its state of motion.

$$K.E. = \int F dx$$

$$\implies K.E. = \int \frac{dp}{dt} dx$$

$$= \int \frac{d(mv)}{dt} v dt = \int \frac{d}{dt}(m_0 \gamma v) v dt$$

Integrating by parts

$$K.E. = m_0 \gamma v^2 - \int m_0 \gamma v \frac{dv}{dt} dt$$

$$= \gamma m_0 v^2 - m_0 \int \gamma v dv$$

$$= \gamma m_0 v^2 - m_0 \int_0^v \frac{v}{\sqrt{1 - v^2/c^2}} dv$$

Derivation of Relativistic Kinetic Energy (contd)

$$K.E. = \gamma m_0 v^2 + m_0 c^2 \left[\sqrt{1 - v^2/c^2}\right]_0^v$$

$$K.E. = m_0 \frac{v^2}{\sqrt{1 - v^2/c^2}} + m_0 c^2 \left[\sqrt{1 - v^2/c^2}\right]_0^v$$

$$= \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}} - m_0 c^2$$

$$= (m - m_0)c^2$$

where

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} = m_0 \gamma$$

Lorentz Four Vectors

One can extend the concept of a vector in 3-dimensional space to a relativistic four vector A_μ

A relativistic 4-vector (A_μ) is defined to be a set of 4-quantities (A_0,A_1,A_2,A_3), which have the same Lorentz Transformation properties as the space-time coordinates $x_\mu = (ct, x, y, z)$

- Scalar product of two Lorentz four-vectors (A_0,A) and (B_0,B) can be defined as

$$A.B = A_0 B_0 - (A_1 B_1 + A_2 B_2 + A_3 B_3)$$

  and is equal to $A'.B'$ and is said to be Lorentz-invariant.

- Length of a four-vector $\equiv (A.A)^{1/2}$ which has the same value in all inertial frames.

$$(A.A)^{1/2} = (A_0^2 - A_1^2 - A_2^2 - A_3^2)^{1/2}$$

Examples of Lorentz Four Vectors

Some examples of Lorentz four vectors are as follows

1. (ct, x, y, z)
2. $(E/c,\ p_x, p_y, p_z)$ where E and p are relativistic energy and momentum
3. $(\frac{1}{c}\frac{\partial}{\partial t}\ \text{-}\frac{\partial}{\partial x},\ \text{-}\frac{\partial}{\partial y},\ \text{-}\frac{\partial}{\partial z}\ )$ (They transform like a Lorentz four-vector)

Note that the Velocity (V_x, V_y, V_z) are not the components of a 4-vector.

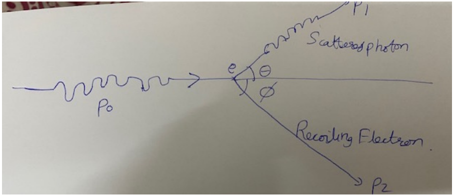
How to analyze relativistic collisions

Here are the rules for thumb for solving problems in relativistic dynamics:

1. One can apply principle of superposition to calculate the total momentum and total energy.
2. Total relativistic momentum (along each of the axis) must be the same before and after the collision (for same frame).
3. Total Energy before and after the collision must the same (for same frame).
4. Invariant mass is conserved.

Note (2) and (3) is strictly valid only if we analyze the dynamics in the same (lab) frame (before and after the collision). In some situations, one uses a different frame before and before collision (such as lab frame or Center of mass frame). In that case only (1) and (4) are valid.

Compton Scattering



Compton Scattering

A photon of momentum p_0 is incident upon a free electron with rest mass $= m$ at rest. After the collision, the photon has a momentum p_1, while the electron recoils with the momentum p_2. Calculate the scattering angle θ *(Note that rest mass here is defined as m and not m_0)*

Before the Collision:

Photon momentum (X-axis) $p_0 = E_0/c = h\nu_0/c = \frac{h}{\lambda_0}$

Electron Energy $= mc^2$; Electron momentum $= 0$ (X-axis)

Photon Energy $= E_0 = \frac{hc}{\lambda_0}$

Analysis of Compton Scattering

Before the Collision:

Total momentum (along X-axis)= p_0

Total momentum (along Y-axis) = 0

Total energy = $mc^2 + E_0$

After the Collision:

Total momentum (along X-axis)= $p_1 \cos(\theta) + p_2 \cos(\phi)$

Total momentum (along Y-axis) = $p_1 \sin(\theta) - p_2 \sin(\phi)$

Total Energy=$E_1 + \sqrt{m^2c^4 + p_2^2c^2}$ where $E_1 = p_1 c$ and p_2 is the electron momentum after the collision.

$$p_1 \sin(\theta) - p_2 \sin(\phi) = 0 \tag{1}$$

$$p_1 \cos(\theta) + p_2 \cos(\phi) = p_0 \tag{2}$$

$$E_0 + mc^2 = E_1 + (m^2c^4 + p_2^2c^2)^{1/2} \tag{3}$$

Eqns. (1) and (2) give us:

$$p_2^2 = p_0^2 + p_1^2 - 2p_0p_1 \cos(\theta) \tag{4}$$

Compton Scattering (contd)

Kinetic Energy of the electron after collision =
$(E - mc^2) = (m^2c^4 + p_2^2c^2)^{1/2} - mc^2$
From Eq (3), RHS = $(E_0 - E_1) = c(p_0 - p_1)$
Therefore,

$$(m^2c^4 + p_2^2c^2)^{1/2} - mc^2 = c(p_0 - p_1)$$

$$p_2^2 = (p_0 - p_1)^2 + 2mc(p_0 - p_1) \tag{5}$$

Combining (4) and (5)

$$\begin{aligned} mc(p_0 - p_1) &= p_0 p_1 (1 - \cos\theta) \\ &= 2p_0 p_1 \sin^2(\theta/2) \end{aligned}$$

Multiplying both sides by $h/(mcp_0p_1)$ and using $\lambda_0 = h/p_0$ and $\lambda_1 = h/p_1$

Compton Equation

$$\boxed{\Delta\lambda = \lambda_1 - \lambda_0 = 2\lambda_c \sin^2(\theta/2)}$$

where λ_c is the Compton wavelength is equal to h/mc This equation is called Compton equation or Equation of Compton scattering.

- The recoil electrons predicted by Compton's theory were observed in experiments in 1920s by Bethe, Wilson, Geiger etc.
- The energy of the recoil electron was measured by Bless in accord with Compton's theory.
- In astrophysics, one has situations where energy of electron is greater than the photon and in that case energy is transferred from electron to photon. This phenomena is known as "Inverse Compton" Scattering. This causes a distortion of CMB black body spectrum for a galaxy cluster along the line of sight. This is called "Sunyaev-Zeldovich" effect.

Example Problem (1)

Calculate the total distance travelled by a muon with total energy 100 GeV (assuming rest mass of 100 MeV) if its rest lifetime is 2 μ sec

Example Problem (2)

In a collision between a proton at rest (rest mass equal to m_p) and a moving proton, another particle of rest mass M is produced, in addition to the two protons. Find the minimum total energy the moving proton must have in order for this process to take place. (Hint: The minimum total energy corresponds to the case where all the particles after the collision are at rest in the Center of Mass frame where total momentum is equal to 0)