EP1108 - Special Relativity Part 1

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References for Special Relativity

There are tons of references for special relativity (from basic to intermediate to advanced). In this class we shall focus on special relativity covered in Modern Physics books. However, for students fascinated with the subject, some advanced references shall also be provided. References for relativity this course are as follows:

- Concepts of Modern Physics by Arthur Beiser
- Elements of Modern Physics by S.H. Patil
- Introduction to Modern Physics by H.S. Mani and G.K. Mehta
- http://galileo.phys.virginia.edu/classes/252/home.html
- https://www.youtube.com/watch?v=toGH5BdgRZ4 (Lenny Susskind youtube lectures on Special Relativity)
- https://arxiv.org/abs/1511.02121 (Advanced)
- Relativity and Common Sense by Hermann Bondi (Advanced and also includes General Relativity)
- Introduction to Special Relativity by Robert Resnick (Advanced)

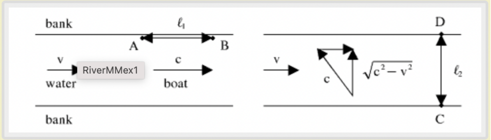
Prelude- Warmup to Michelson- Morley experiment

The picture of how light behaved circa 1880s

- Light behaves like a wave.
- It propagates in a medium, which permeates all space called "ether", and has a fixed velocity with respect to ether
- If we can determine velocity of Earth with respect to the ether, we can calculate the velocity of light in any direction as seen by an earth bound observer.

Michelson and Morley designed an ingenuous experiment to determine the speed of this "Ether wind".
Note however that there were some hints that this picture of "light propagating in a medium called ether" is not correct, as it was in conflict with electromagnetism (i.e. Maxwell's equations)

Analogy with boat flowing along river



Boat across river

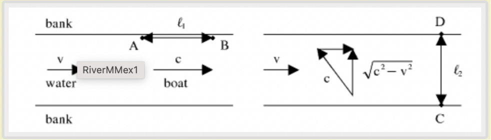
Consider a boat sailing across a rover flowing with a speed v to the right and boat moves with a velocity c

Velocity of boat sailing downstream w.r.t bank = c+v

Velocity of boat sailing upstream w.r.t bank = c-v

Total time for boat to go from A to B and back = $\frac{l_1}{c-v} + \frac{l_1}{c+v}$

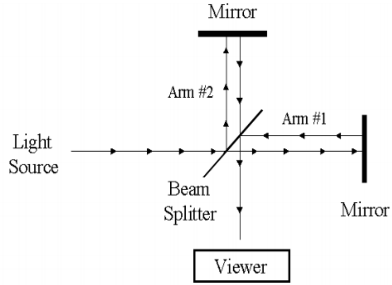
Boat flowing perpendicular to the river



Boat across river

If boat travels at right angles across the river, the velocity V wrt the bank is given by $\sqrt{c^2 - v^2}$
Total time required by boat travel the distance l_2 from C to D and back is given by $\frac{2l_2}{\sqrt{c^2 - v^2}}$
Credit for above picture
https://thespectrumofriemannium.wordpress.com/2012/06/07/log012-michelson-morley-experiment/

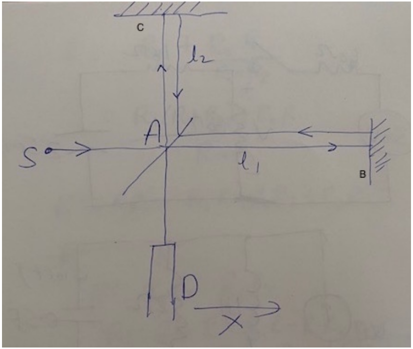
Layout of Michelson-Morley experiment



Michelson-Morley experiment

A pulse of light is directed at an angle of 45 degrees at a beam splitter (half transparent, half silvered), so that half of the pulse goes through glass and the other half is reflected. Both these beams are reflected from the distant mirrors and then again to beamsplitter. They are again half transmitted and half reflected and reflected part goes to the viewer.

Michelson Morley experiment analysis



Michelson-Morley experiment

If there is an ether pervading throughout space, we move through it with the same speed (v) as Earth's motion around the Sun (30 km/sec). Assume speed of light is denoted by c

Difference in path lengths

The time taken for light to travel from A to B and back is given by

$$t_1 = \frac{l_1}{c - v} + \frac{l_1}{c + v}$$

The time for light to travel from A to C and back is equal to

$$t_2 = \frac{2l_2}{\sqrt{c^2 - v^2}}$$

Difference in time (Δ) is given by:

$$\Delta = t_1 - t_2 = \frac{2l_1 c}{c^2 - v^2} - \frac{2l_2}{\sqrt{c^2 - v^2}}$$

Path difference = $c\Delta$

If the apparatus is turned by 90°, the roles of l_1 and l_2 are interchanged and

$$\Delta' = t_1' - t_2' = \frac{2l_1}{\sqrt{c^2 - v^2}} - \frac{2l_2 c}{c^2 - v^2}$$

The shift in interference fringe at D is given by

$$\begin{aligned}
\delta &= c\frac{(\Delta' - \Delta)}{\lambda} \\
&= \frac{2(l_1 + l_2)}{\lambda}\left[\frac{1}{\sqrt{1 - v^2/c^2}} - \frac{1}{1 - v^2/c^2}\right] \\
&\approx -\frac{(l_1 + l_2)}{\lambda}\left(\frac{v^2}{c^2}\right)
\end{aligned}$$

In M-M experiment, $l_1 + l_2 \approx 20m$ and $\lambda \approx 5.9 \times 10^{-7}$ m.
$\implies \delta \approx 0.37$. However, no such shift was observed. Therefore, the null result of the M-M experiment rendered untenable the ether hypothesis.

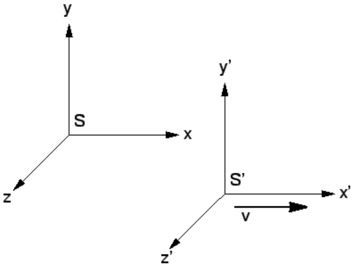
Postulates of Special Relativity

The special theory of relativity was formulated by Einstein in 1905. It is based upon the following postulates:

1. The law of physics may be expressed in the same form in all frames of references moving with a constant velocity with respect to one another. These frames are also called *inertial frames*.
2. The speed of light in free space has the same value for all observers regardless of their state of motion.

The first postulate also expresses the absence of any universal frame of reference.

Galilean Transformations



Consider an inertial frame S' moving with respect to inertial frame S at constant velocity v. Event in S $\equiv (x,y,z)$. What will be the co-ordinates of this event in the frame S' (x',y',z',t')?

$$
\begin{aligned}
x' &= x - vt \\
y' &= y \\
z' &= z \\
t' &= t
\end{aligned}
$$

Problems with Galilean Transformations

To convert the velocity components in S frame to S'

$$v'_x = \frac{dx'}{dt'} = v_x - v$$
$$v'_y = \frac{dy'}{dt'} = v_y$$
$$v'_z = \frac{dz'}{dt'} = v_z$$

This violates the second postulate of special relativity, which assumes the same speed of light c in all frames of reference. If the speed of light in the x-direction is frame S to be c, in the S' system it will $c' = c - v$.
They are also not consistent with theory of electromagnetism postulated by Maxwell in 1865.

Galilean to Lorentz Transformations

We explain how the Galilean transformations can be extended to incorporate laws of special relativity. Assume

$$x' = k(x - vt) \tag{1}$$

. Since laws of Physics have the same form in both S and S', one can write x in terms of x' and t' in the same way by replacing v with $-v$

$$x = k(x' + vt') \tag{2}$$

Since the motion of S' wrt S is only along X-direction, $y' = y$ and $z' = z$ Plugging Eq. 1 into Eq. 2, we get:

$$x = k^2(x - vt) + kvt'$$

$$t' = kt + \left(\frac{1 - k^2}{kv}\right) x \tag{3}$$

Derivation of Lorentz Transformations

At instant $t = 0$ the origins of the two frames coincide and $t' = 0$
Suppose a flare is set off at the common origin of S and S' at $t = 0$ and the observers in each system proceed to measure the speed at which light spreads out. From second postulate of relativity both observers must find the same speed equal to c. Therefore

$$x = ct \tag{4}$$
$$x' = ct' \tag{5}$$

Substituting (1) and (3) in (5)

$$k(x - vt) = c\left[kt + \left(\frac{1-k^2}{kv}\right)x\right] \tag{6}$$
$$\implies x = ct\left[\frac{1 + v/c}{1 - (1/k^2 - 1)\frac{c}{v}}\right] \tag{7}$$

Lorentz Transformations

Comparing (4) and (7)

$$\left[\frac{1+v/c}{1-(1/k^2-1)\frac{c}{v}}\right] = 1$$

$\implies k = \frac{1}{\sqrt{1-v^2/c^2}}$. Therefore, Lorentz transformation equations are as follows

$$x' = \frac{1}{\sqrt{1-v^2/c^2}}(x - vt) \tag{8}$$

$$t' = \frac{1}{\sqrt{1-v^2/c^2}}(t - vx/c^2) \tag{9}$$

$$y' = y \tag{10}$$

$$z' = z \tag{11}$$

Note that if the frame S' is moving in Y and Z directions, then y' and z' will not be the same as y and z

Inverse Lorentz Transformations

One can go from S' to S using

$$x = \frac{1}{\sqrt{1 - v^2/c^2}}(x' + vt') \tag{12}$$

$$t = \frac{1}{\sqrt{1 - v^2/c^2}}(t' + vx'/c^2) \tag{13}$$

$$y = y' \tag{14}$$

$$z = z' \tag{15}$$

Two observations of Lorentz transformations

- Measurements of *time* as well as position depend upon the frame of reference of the observer so that two events which occur simultaneously in one frame at different places need not be simultaneous in another.
- Lorentz transformations reduce to Galilean transformations when $v << c$

Lorentz-Fitzgerald Length Contraction

Consider a rod lying along the X-axis of a frame of reference S. If x_1 and x_2 are coordinates of the end of the rod, length of the rod is $L_0 = x_2 - x_1$

Qt: What will be the length of rod (L) measured from the frame S'?

$$x_1 = \frac{x_1' + vt'}{\sqrt{1 - v^2/c^2}}$$
$$x_2 = \frac{x_2' + vt'}{\sqrt{1 - v^2/c^2}}$$

So

$$\begin{aligned} L_0 &= x_2 - x_1 \\ &= \frac{x_2' - x_1'}{\sqrt{1 - v^2/c^2}} \end{aligned}$$

By definition: $L = x_2' - x_1'$

Therefore,

$$L_0 = \frac{L}{\sqrt{1 - v^2/c^2}}$$

or $\boxed{L = L_0\sqrt{1 - v^2/c^2}}$

Since the relative velocity only appears as v^2, length contraction is symmetric between S and S'.

Time dilation

Clock moving with respect to an observer appears to tick less rapidly than they do when they are at rest. This is called *time dilation*.

Imagine a clock at the point x' in the moving frame S'. When an observer in S' finds the time is equal to t_1' , the observer in S will measure it at t_1. Then we get

$$t_1 = \frac{t_1' + \frac{vx'}{c^2}}{\sqrt{1 - v^2/c^2}}$$

For the observer in S', after a time interval t_0, observer in moving system finds that time is now t_2' according to his clock $t_0 = t_2' - t_1'$. The observer in S measures the same interval to be

$$t_2 = \frac{t_2' + \frac{vx'}{c^2}}{\sqrt{1 - v^2/c^2}}$$

.

The total elapsed duration in S is equal to

$$t = t_2 - t_1 = \frac{t_2' - t_1'}{\sqrt{1 - v^2/c^2}}$$

Therefore

$$\boxed{t = \frac{t_0}{\sqrt{1 - v^2/c^2}}}$$

A stationary clock measures a longer time interval between events occurring in a moving frame of reference than does a clock in the moving frame.

Time dilation example-Decay of muons

Lifetime of a muon (known as μ-mesons in older books) in its rest frame is equal to 2μs. Its speed relative to the Earth is equal to 2.994×10^8 seconds. Muons are produced in Earth's atmosphere, about 6 km from the surface of the Earth. Non-relativistically, a muon would cover a distance of $l = v\tau$ of 600 m and should never reach the Earth. However, many such muons are detected on surface of the Earth. We can analyze this from two points of view

- *From Earth*: Muon lifetime undergoes time dilation and is given by $t = \tau/\sqrt{1 - v^2/c^2} \approx 32\mu s$ The distance travelled by muon $= vt \approx 9.6$ km.
- *From muon*: Earth moves towards the muon. Distance from muon to the Earth undergoes length contraction as seen from the muon. The distance as seen from the muon $=$ $z' = z\sqrt{1 - v^2/c^2} \approx 375m$. Therefore, the Earth reaches the muon after a time $t = z'/v \approx 1.25\mu s < \tau$, i.e. shorter than muon lifetime.

This experiment has actually been done Ref: *Phys. Lett 55B, 420 (1975)*