CS 289A – Spring 2023 – Homework 4

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1 Honor Code

I did not collaborate with any students. I did refer to ChatGPT frequently.

"I certify that all solutions are entirely in my own words and that I have not looked at another student's solutions. I have given credit to all external sources I consulted."

Signed Colin Skinner

Signature ____________________ Date 3/10/2023

2 Logistic Regression with Newton's Method

2.1

$$\nabla_{\mathbf{w}} J(\mathbf{w}) = \nabla_{\mathbf{w}} \left(-\mathbf{y} \cdot \ln \mathbf{s} - (\mathbf{1} - \mathbf{y}) \cdot \ln(\mathbf{1} - \mathbf{s})\right)$$

$$= -(\nabla_{\mathbf{w}} \mathbf{y}) \ln \mathbf{s} - (\nabla_{\mathbf{w}} \ln \mathbf{s})\mathbf{y} - (\nabla_{\mathbf{w}}(\mathbf{1} - \mathbf{y})) \ln(\mathbf{1} - \mathbf{s}) - (\nabla_{\mathbf{w}} \ln(\mathbf{1} - \mathbf{s}))(\mathbf{1} - \mathbf{y})$$

$$= -(\nabla_{\mathbf{w}} \ln \mathbf{s})\mathbf{y} - (\nabla_{\mathbf{w}} \ln(\mathbf{1} - \mathbf{s}))(\mathbf{1} - \mathbf{y})$$

$$= -(\nabla_{\mathbf{w}} \mathbf{s} \cdot \nabla_{\mathbf{s}} \ln \mathbf{s})\mathbf{y} - (\nabla_{\mathbf{w}}(\mathbf{1} - \mathbf{s}) \cdot \nabla_{\mathbf{s}} \ln(\mathbf{1} - \mathbf{s}))\,(\mathbf{1} - \mathbf{y})$$

$$= -(\nabla_{\mathbf{w}} \mathbf{s} \cdot \mathrm{diag}\,(1/s_i)\,\mathbf{1})\mathbf{y} + (\nabla_{\mathbf{w}} \mathbf{s} \cdot \mathrm{diag}(1/(1 - s_i))\mathbf{1})\,(\mathbf{1} - \mathbf{y})$$

Note that

$$\nabla_{\mathbf{w}} s_i = \nabla_{\mathbf{w}} s(\mathbf{x}_i \cdot \mathbf{w})$$

$$= \nabla_{\mathbf{w}}(\mathbf{x}_i \cdot \mathbf{w}) \nabla_{\gamma} s(\gamma)$$

where $\gamma = \mathbf{x}_i \cdot \mathbf{w}$

$$= \nabla_{\gamma} s(\gamma) \mathbf{x}_i$$

$$= \frac{\partial}{\partial \gamma} \left(1 + e^{-\gamma}\right)^{-1} \mathbf{x}_i$$

$$= -\left(1 + e^{-\gamma}\right)^{-2} (-e^{-\gamma})\mathbf{x}_i$$

$$= \frac{e^{-\gamma}}{(1+e^{-\gamma})^2}\mathbf{x}_i$$

$$= \frac{1}{1+e^{-\gamma}}\left(1-\frac{1}{1+e^{-\gamma}}\right)\mathbf{x}_i$$

$$= s_i\left(1-s_i\right)\mathbf{x}_i$$

Therefore

$$\nabla_{\mathbf{W}}\mathbf{s} = \begin{bmatrix}\nabla_{\mathbf{W}}s_1 & \nabla_{\mathbf{W}}s_2 & ... & \nabla_{\mathbf{W}}s_n\end{bmatrix}^T$$

$$= \begin{bmatrix}s_1\left(1-s_1\right)\mathbf{x}_1 & s_2\left(1-s_2\right)\mathbf{x}_2 & ... & s_n\left(1-s_n\right)\mathbf{x}_n\end{bmatrix}^T$$

$$= \begin{bmatrix} s_1\left(1-s_1\right)\mathbf{x}_1^T \\ s_2\left(1-s_2\right)\mathbf{x}_2^T \\ . \\ . \\ . \\ s_n\left(1-s_n\right)\mathbf{x}_n^T \end{bmatrix}$$

$$= \begin{bmatrix} | & | & & | \\ \mathbf{x}_1 & \mathbf{x}_2 & ... & \mathbf{x}_n \\ | & | & & | \end{bmatrix}\begin{bmatrix} s_1\left(1-s_1\right) & & & & \\ & . & & & \\ & & . & & \\ & & & . & \\ & & & & s_n\left(1-s_n\right) \end{bmatrix}$$

$$= X^T\mathrm{diag}(s_i\left(1-s_i\right))$$

and therefore

$$-(\nabla_{\mathbf{w}}\mathbf{s}\cdot\mathrm{diag}\left(1/s_i\right)\mathbf{1})\mathbf{y} + (\nabla_{\mathbf{w}}\mathbf{s}\cdot\mathrm{diag}(1/(1-s_i))\mathbf{1})\left(\mathbf{1}-\mathbf{y}\right)$$

$$= -(X^T\mathrm{diag}(s_i\left(1-s_i\right))\mathrm{diag}\left(1/s_i\right)\mathbf{1})\mathbf{y} + \left(X^T\mathrm{diag}(s_i\left(1-s_i\right))\mathrm{diag}(1/(1-s_i))\mathbf{1}\right)\left(\mathbf{1}-\mathbf{y}\right)$$

$$= -X^T \mathrm{diag}(1 - s_i)\mathbf{y} + X^T \mathrm{diag}(s_i)(\mathbf{1} - \mathbf{y})$$

$$= -X^T \left(\mathrm{diag}(1 - s_i)\mathbf{y} - \mathrm{diag}(s_i)(\mathbf{1} - \mathbf{y})\right)$$

$$= -X^T \left(\mathrm{diag}(y_i)\mathbf{1} - \mathrm{diag}(y_i s_i)\mathbf{1} - (\mathrm{diag}(s_i)\mathbf{1} - \mathrm{diag}(y_i s_i)\mathbf{1})\right)$$

$$= -X^T \left(\mathrm{diag}(y_i)\mathbf{1} - \mathrm{diag}(s_i)\mathbf{1}\right)$$

$$\boxed{= X^T \left(\mathbf{s} - \mathbf{y}\right)}$$

2.2

$$\nabla^2_{\mathbf{w}} J(\mathbf{w}) = \nabla_{\mathbf{w}} \left(\nabla_{\mathbf{w}} J(\mathbf{w})\right)$$

$$= \nabla_{\mathbf{w}} X^T \left(\mathbf{s} - \mathbf{y}\right)$$

$$= \left(\nabla_{\mathbf{w}} \left(\mathbf{s} - \mathbf{y}\right)\right) X$$

$$= \left(\nabla_{\mathbf{w}} \mathbf{s} - \nabla_{\mathbf{w}} \mathbf{y}\right) X$$

From 2.1 we have $\nabla_{\mathbf{w}} \mathbf{s} = X^T \text{diag}(s_i\,(1 - s_i))$ so

$$= \boxed{X^T \text{diag}(s_i\,(1 - s_i)) X}$$

2.3

When iterating through Newton's method We want the $(t+1)th$ update to be given by

$$\mathbf{w}^{(t+1)} = \mathbf{w}^{(t)} + \mathbf{e}^{(t)}$$

where $\mathbf{e}^{(t)}$ is an error term such that

$$(\nabla^2_{\mathbf{w}} J(\mathbf{w}))\mathbf{e}^{(t)} = -\nabla_{\mathbf{w}} J(\mathbf{w})$$

Let

$$\mathrm{diag}(s_i\,(1-s_i)) = \Gamma$$

Then we have

$$\nabla^2_{\mathbf{w}} J(\mathbf{w}) = X^T \Gamma X$$

Which is a real, symmetric matrix. Also, note that

$$s_i(1-s_i) = \frac{e^{-\gamma}}{(1+e^{-\gamma})^2}$$

$$> 0, \quad \forall \gamma \in \mathbb{R}$$

Since the Hessian of $J(\mathbf{w})$ is a real, symmetric matrix that has an eigendecomposition with eigenvalue matrix Γ, and since each eigenvalue $s_i(1-s_i) > 0$, then $\det(\nabla^2_{\mathbf{w}} J(\mathbf{w})) > 0$ and $\nabla^2_{\mathbf{w}} J(\mathbf{w})$ is invertible. Therefore,

$$(\nabla^2_{\mathbf{w}} J(\mathbf{w}))\mathbf{e}^{(t)} = X^T \Gamma X \mathbf{e}^{(t)}$$

$$= -X^T(\mathbf{s}^{(t)} - \mathbf{y})$$

and

$$\mathbf{e}^{(t)} = (X^T \Gamma X)^{-1} X^T(\mathbf{y} - \mathbf{s}^{(t)})$$

For logistic regression this gives

$$\boxed{\mathbf{w}^{(t+1)} = \mathbf{w}^{(t)} - (X^T \Gamma X)^{-1} X^T(\mathbf{y} - \mathbf{s}^{(t)})}$$

2.4

(a)

$$\mathbf{s}^{(0)} = \begin{bmatrix} 0.9478 & 0.8808 & 0.8022 & 0.5250 \end{bmatrix}^T$$

(b)

$$\mathbf{w}^{(1)} = \begin{bmatrix} -1.0953 & 0.4377 & 1.6783 \end{bmatrix}^T$$

(c)

$$\mathbf{s}^{(1)} = \begin{bmatrix} 0.9435 & 0.8695 & 0.9185 & 0.7435 \end{bmatrix}^T$$

(d)

$$\mathbf{w}^{(2)} = \begin{bmatrix} -1.1878 & -0.1315 & 3.3667 \end{bmatrix}^T$$

3 Wine Classification with Logistic Regression

3.1

For some weight parameter $\lambda > 0$ of the l_2 penalty, and for some $\mathbf{w}' = \mathbf{w} - \begin{bmatrix} 0 & 0 & \dots & \alpha \end{bmatrix}^T$, where $\alpha \in \mathbb{R}$ is the bias, and where $\epsilon > 0$ is the learning rate, the update rule for gradient descent using a logistic regression loss function with l_2 penalty is (source: lecture 3 notes)

$$\boxed{\mathbf{w}^{(t+1)} = \mathbf{w}^{(t)} - \epsilon(X^T(\mathbf{s} - \mathbf{y}) + \lambda \mathbf{w}')}$$

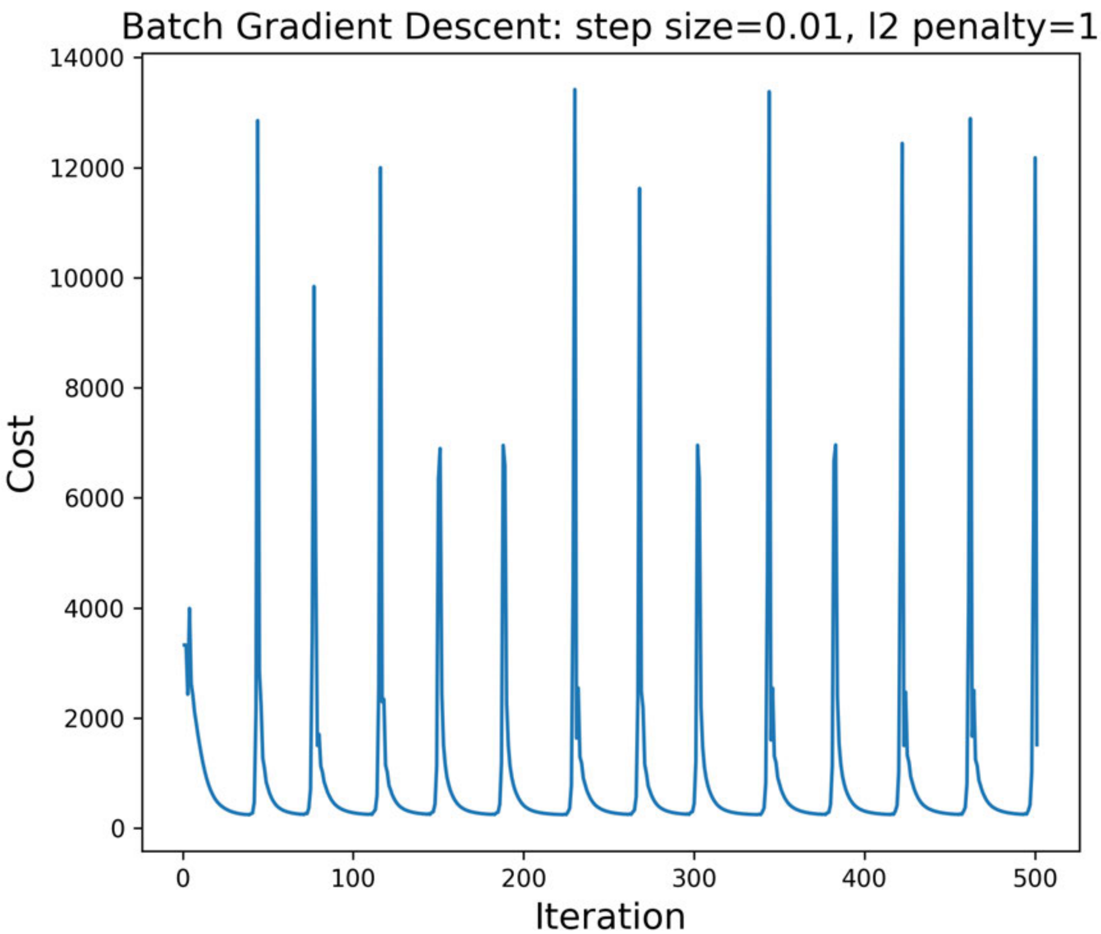
3.2

Ultimately, for the hyperparameters I chose the step size $\epsilon = 0.001$ and the l_2 penalty $\lambda = 0.01$. I tuned the hyperparameters first by doing ten-fold cross validation with the following values for my step size and l_2 penalty

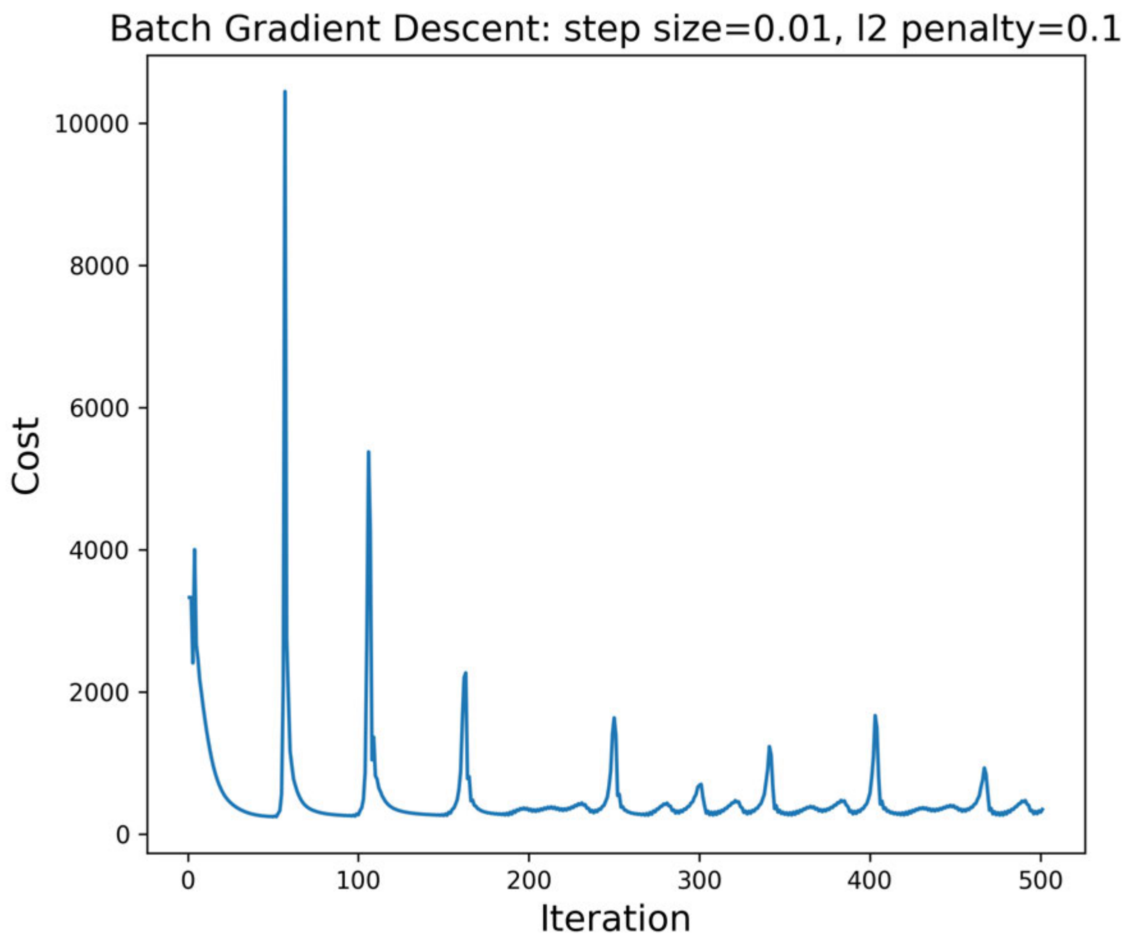
$$\epsilon = [0.1, 0.01, 0.001, 0.0001]$$

$$\lambda = [10, 1, 0.1, 0.01]$$

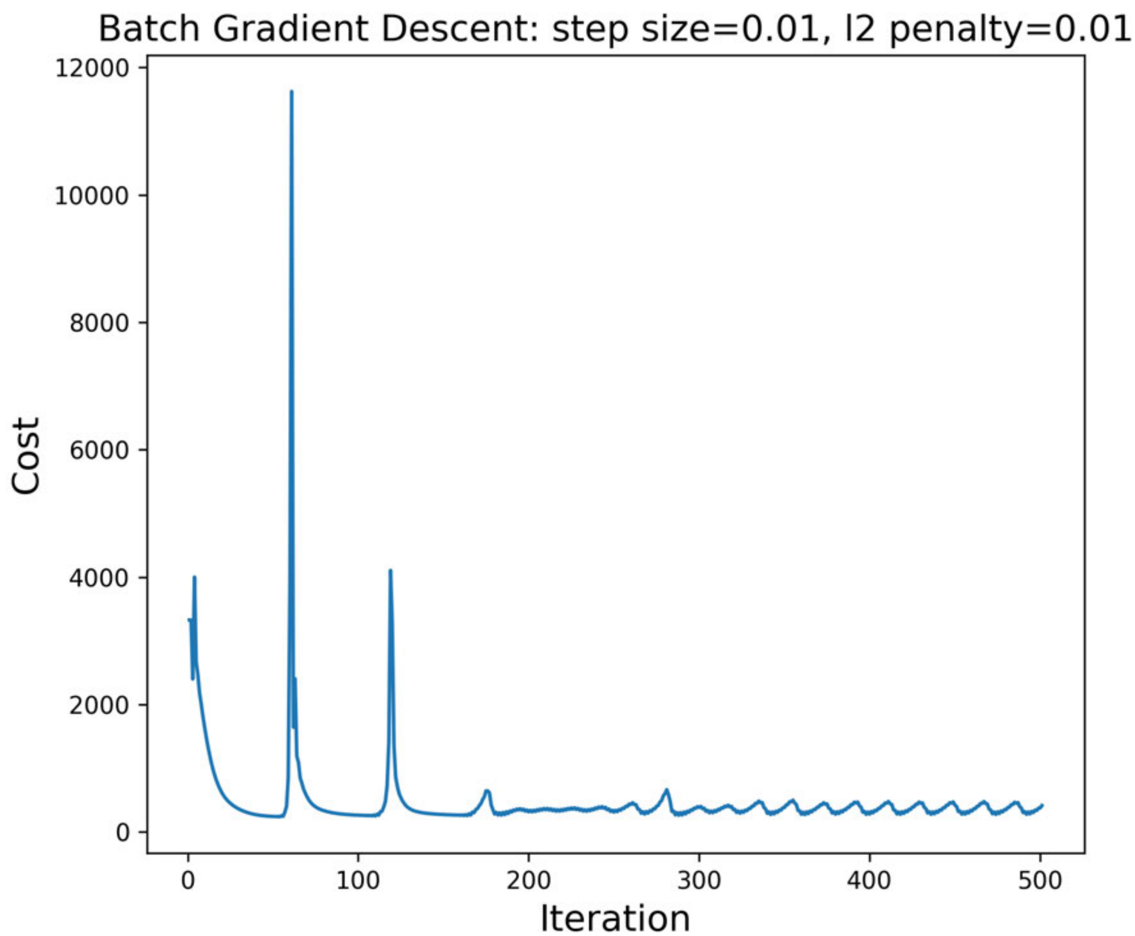
After testing all combinations, I looked at the two which had the lowest cost and AUC. I generated the following plots by doing an 80/20 split and tracking the the cost of the training data per iteration.



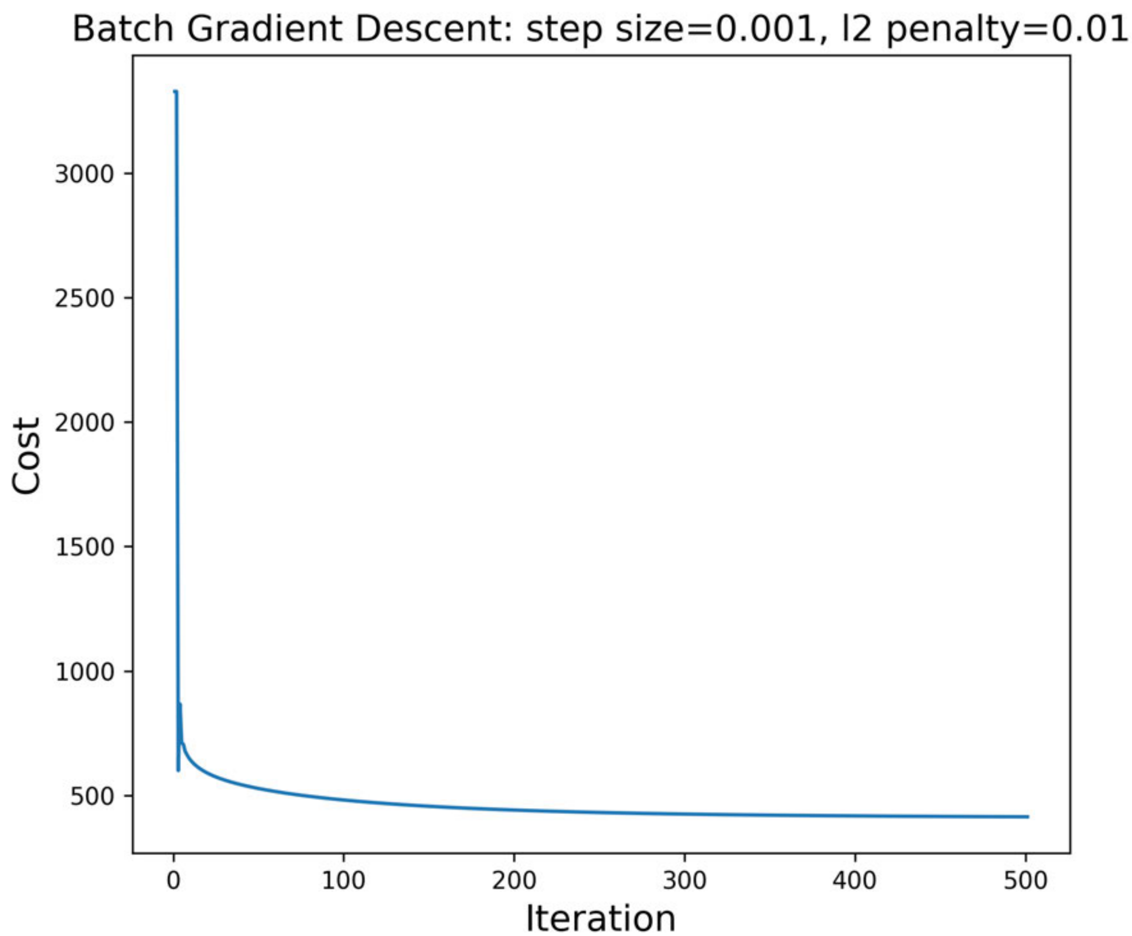
This has some strange oscillations so I decided not go with that.



The relationship still has oscillations with a higher penalty, but not as bad. Let's test some other lambdas



Maybe we can get rid of the oscillations by decreasing the step size:



This looks the best as the cost function monotonically decreases after the first few iterations until it appears to reach a minimum. Here is the code I used for this section (also in code appendix)(NOTE: copying this code and trying to run it will likely not work as I had to add some carriage returns to some lines so that they do not go beyond the margin; any attempt to run this code should use the .ipynb file submitted with this assignment):

```
import pandas as pd
import numpy as np
import matplotlib.pyplot as plt
from math import e
from math import log
import scipy.io as sc
from scipy.special import expit, logit
from sklearn.model_selection import KFold
from sklearn.metrics import roc_auc_score
from sklearn.model_selection import train_test_split
```

```
%matplotlib inline

def logistic_fn(x,w):

    return expit(np.dot(x,w))

#returns the mean and stdev of the
#features of the input data as a list of tuples

def get_fit(data):

    stats = []

    for i in range(data.shape[1]):

        stats+=[(data[:,i].mean(), data[:,i].std())]

    return stats

#normalizes the features of a dataset by subtracting the training mean
#and dividing by the training stdev; the stats variable
#should be a list of tuples

def transform(data,stats):

    for i in range(data.shape[1]):

        data[:,i] = (data[:,i]-stats[i][0])/stats[i][1]

    return data

#update function for gradient descent with l2 penalty
def logr_update(data, labels, s_vals, weights, step, penalty):

    return weights+step*(np.matmul(data.T,(labels - s_vals)) - penalty*weights)

#append a column of ones for the fictitious dimension
def add_fic(data):

    if data.ndim ==1:

        data=data.reshape(1,data.shape[0])

    return np.append(data,np.ones((data.shape[0],1)),axis=1)
```

```
#calculates the cost of the logistic regression cost function with l2 penalty
def logr_cost(data, labels, s_vals, weights, penalty):

    log_s = [log(x+1e-10) for x in s_vals]

    log_s_comp = [log((1-x)+1e-10) for x in s_vals]

    return -np.dot(labels,log_s)-np.dot((1-labels),log_s_comp)
    +penalty*np.linalg.norm(weights)


# Define the decision function for
def predict(x, weights):

    #calculate the logistic function with the
    score = logistic_fn(x,weights)

    return  (score >= 0.5).astype(int)

#Performs batch gradient descent with a fixed number of iterations
def bgd(data, labels, step, penalty,iterations):

    #initialize the weight with the zero vector
    weight = np.zeros(data.shape[1])

    s_vals = logistic_fn(data,weight)

    #initialize the cost with the first weight vector of all zeros
    cost=[logr_cost(data, labels, s_vals, weight, penalty)]



    for i in range(iterations):

        #creat the w' vector for the update and calculating the cost
        w_p = np.append(weight[:-1],0)

        weight = logr_update(data,labels,s_vals,w_p, step, penalty)

        cost+=[logr_cost(data, labels,s_vals, w_p, penalty)]

        s_vals = logistic_fn(data,weight)

    return weight,cost
```

```
data=sc.loadmat('data.mat')
labels=data['y'].reshape(len(data['y']))
test = data['X_test']
x = data['X']
k_fold = KFold(n_splits=10, random_state = 42, shuffle=True)

steps = [0.1, 0.01, 0.001, 0.0001]  # step size values to try
lambdas = [10, 1,0.1, 0.01]  # l2 penlaty values to try

# Iterate through each combination of hyperparameters to optimize the model
for step in steps:
    for penalty in lambdas:

        # Initialize list for cross-validation scores
        cv_costs = []
        cv_auc_roc = []

        # Iterate through each fold of the data
        for train_indices, val_indices in k_fold.split(x_train):

            # Split the data into training and validation sets
            X_train, y_train = x[train_indices], labels[train_indices]
            X_val, y_val = x[val_indices], labels[val_indices]

            # normalize the training and validation data by
            #the training data values
            fit = get_fit(X_train)
            X_train = transform(X_train,fit)
            X_train = add_fic(X_train) #add column of 1s for the bias

            X_val = transform(X_val,fit)
            X_val = add_fic(X_val) #add column of 1s for bias

            weights, cost = bgd(X_train, y_train, step=step,
            penalty=penalty, iterations=500)

            # Calculate the cost function on the validation set

            s_vals = logistic_fn(X_val,weights)
            cost_val = logr_cost(X_val, y_val, s_vals, weights, penalty=penalty)

            # Make predictions on the validation set
            y_pred = predict(X_val, weights)

            # Calculate the AUC-ROC score on the validation set
```

```
            auc_roc = roc_auc_score(y_val, y_pred)

            # Append the cross-validation scores to the lists
            cv_costs.append(cost_val)
            cv_auc_roc.append(auc_roc)

        # Compute the mean and standard deviation of the cross-validation scores
        cv_mean_cost = np.mean(cv_costs)
        cv_std_cost = np.std(cv_costs)
        cv_mean_auc_roc = np.mean(cv_auc_roc)
        cv_std_auc_roc = np.std(cv_auc_roc)


        print(f'step size={step}, lambda={penalty}, mean CV cost={cv_mean_cost:.3f},
        std={cv_std_cost:.3f},
        mean CV AUC-ROC={cv_mean_auc_roc:.3f}, std={cv_std_auc_roc:.3f}')

x_train, x_val, y_train, y_val = train_test_split(x,labels,
test_size = 0.2, random_state=42)

#normalize the data
fit = get_fit(x_train)
x_train = transform(x_train,fit)
x_train = add_fic(x_train) #add column of 1s for the bias

x_val = transform(x_val,fit)
x_val = add_fic(x_val) #add column of 1s for bias

w_final, cost = bgd(x_train, y_train, 0.001, 0.01, 500)

plt.figure(figsize=(7,6))
plt.plot(np.linspace(1,501,501),cost)
plt.xlabel('Iteration', fontsize=15)
plt.ylabel('Cost', fontsize=15)
plt.title('Batch Gradient Descent: step size=0.001, l2 penalty=0.01', fontsize=15)
plt.savefig('Q3.2 0001001.png',dpi=300)
```

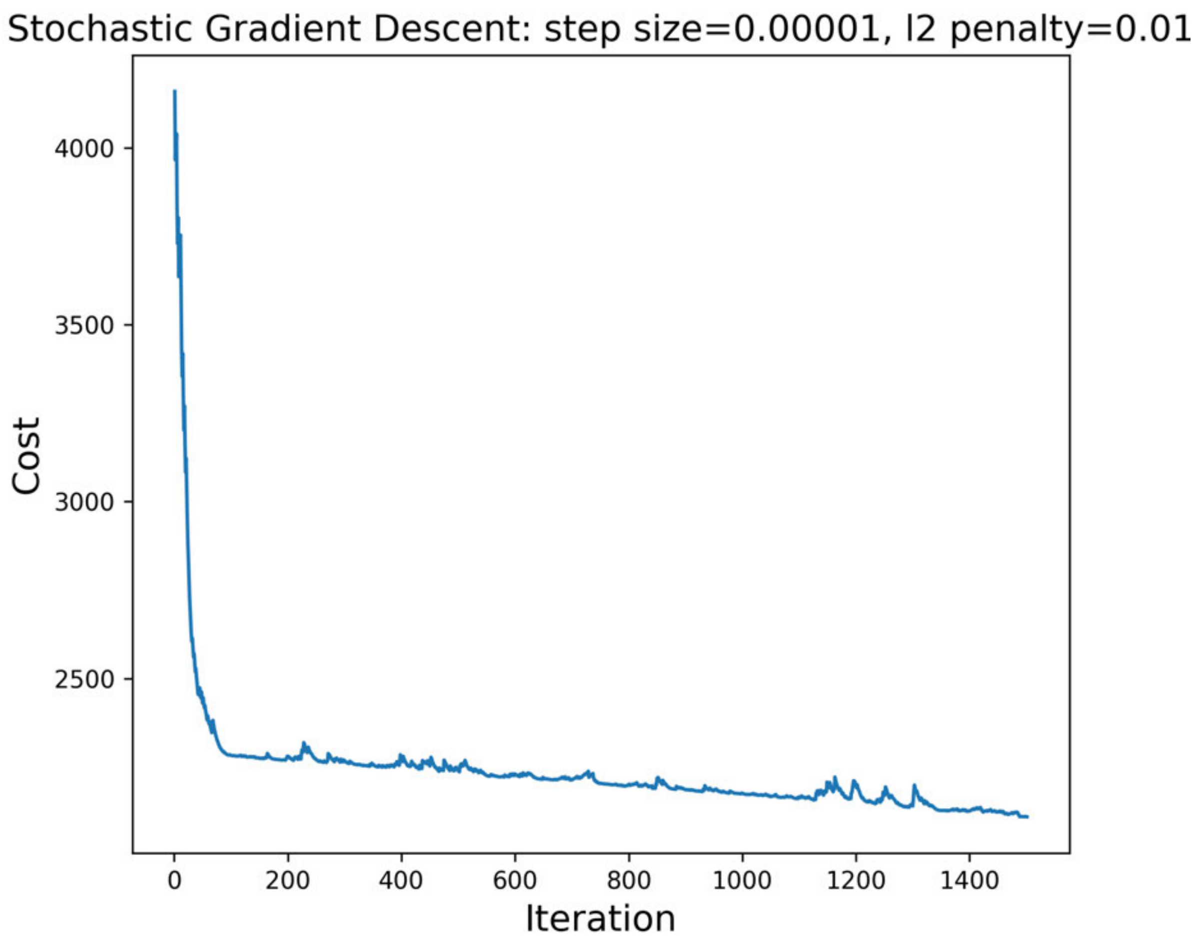
3.3

The update rule for SGD is (source: lecture 3 notes)

$$\boxed{\mathbf{w}^{(t+1)} = \mathbf{w}^{(t)} - \epsilon((y_i - s(\mathbf{x}_i^T\mathbf{w}))\mathbf{x}_i - \lambda\mathbf{w}')}$$

3.4

For SGD, the step size had to be very small to get a cost function that was mostly decreasing with iteration. I used a step size of 0.00001 and l_2 penalty of 0.001. I trained the SGD model by doing a shuffle and split of the data (here the split was only used to control the number of iterations as the validation data were not used for anything). I used a random seed of 42 to generate random integers which were then fed to the 'train_test_split' function from the sklearn library.
My batch gradient descent model seemed to be heading towards convergence faster (though I did not do enough iterations with either model to observe convergence). This could be due to the SGD model using single training examples at a time. I tried to give a bit more functionality to the model to have it do mini-batches and over several epochs, but unfortunately ran out of time before I could get it to be functional.



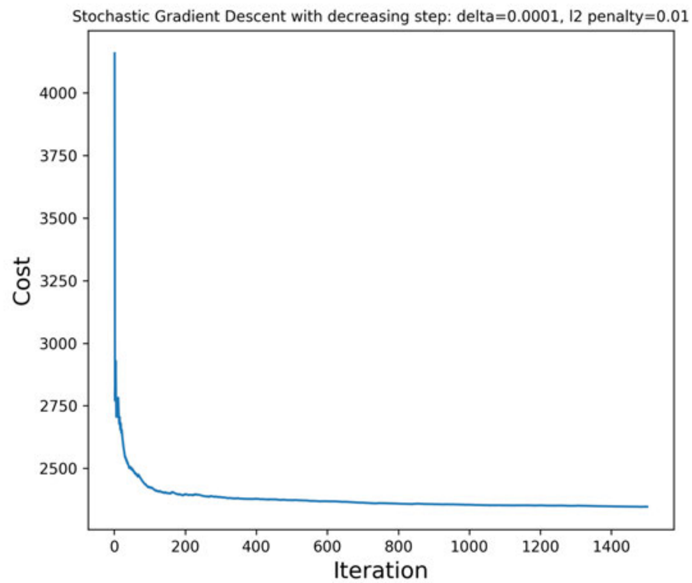

```
import random
```

```
#update function for stochastic gradient descent with l2 penalty
def sgd_update(data, labels, weight, step, penalty):

    #creat the w' vector for the update and calculating the cost
    w_p = np.append(weight[:-1],0)

    return weight+step*((labels -logistic_fn(data,weight))*data - penalty*w_p)

#calculates the cost of the logistic regression cost function with l2 penalty
def sgd_cost(data, labels, weights, penalty):

    s_vals = logistic_fn(data,weights)

    log_s = [log(x+1e-10) for x in s_vals]

    log_s_comp = [log((1-x)+1e-10) for x in s_vals]

    return -np.dot(labels,log_s)-np.dot((1-labels),
    log_s_comp)+penalty*np.linalg.norm(weights)

# Performs stochastic gradient descent
def sgd(data, labels, step, penalty, epochs, batch_size):

    data = add_fic(data)

    # Initialize the weight with the zero vector
    weight = np.zeros(data.shape[1])

    # Initialize the cost with the first weight vector of all zeros
    cost = [sgd_cost(data, labels, weight, penalty)]

    random.seed(42)

    for epoch in range(epochs):

        rstate = random.randint(0,data.shape[0])

        # Shuffle the data and labels for the epoch
        x_train, x_val, y_train, y_val = train_test_split(data,
        labels, test_size = 0.75, random_state=rstate)

        for i in range(0, x_train.shape[0], batch_size):

            # Select the mini-batch of data and labels
            x = x_train[i:i+batch_size]
```

```
            y = y_train[i:i+batch_size]

            # Compute the logistic function and update the weight
            weight = sgd_update(x[0], y, weight, step, penalty)


            # Compute the cost for the mini-batch and store it
            cost += [sgd_cost(data, labels, weight, penalty)]


    return weight, cost

weight, cost = sgd(x, labels, step=0.00001, penalty=0.01, epochs=1, batch_size=1)
```

3.5

Adding the feature of decreasing step size appears to stabilize the oscillations and the model converges slightly faster.



3.6

My Kaggle score is 0.93951. My Kaggle username is Colin Skinner.

4 A Bayesian Interpretation of Lasso

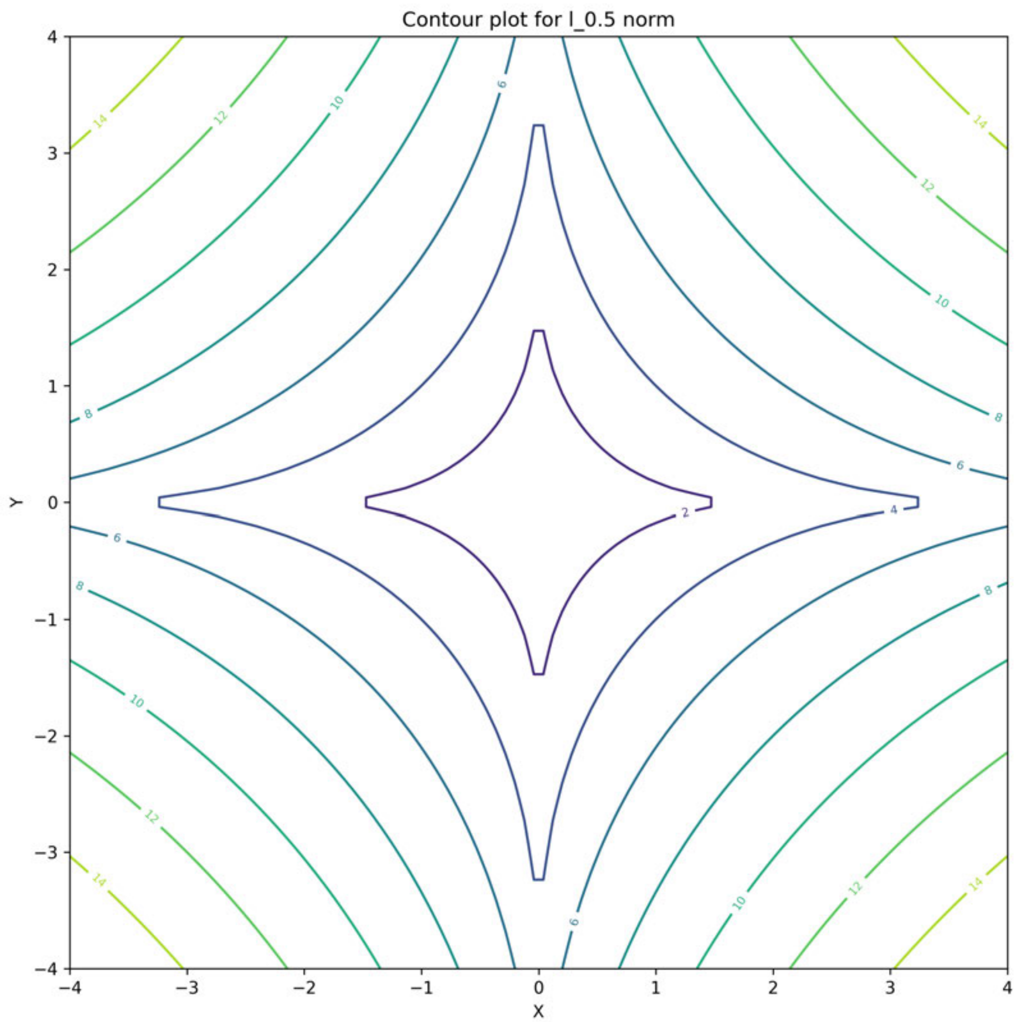
4.1

4.2

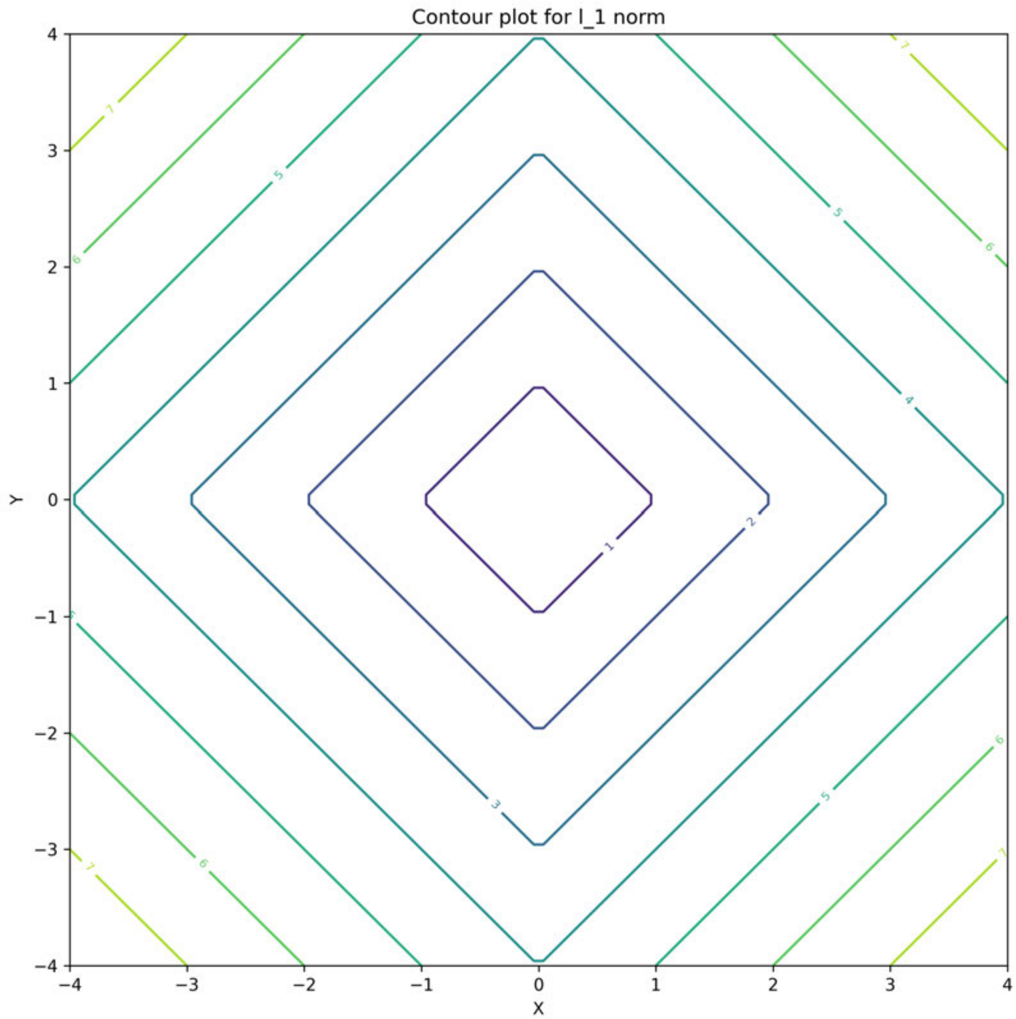
5 1-regularization, 2-regularization, and Sparsity

5.1

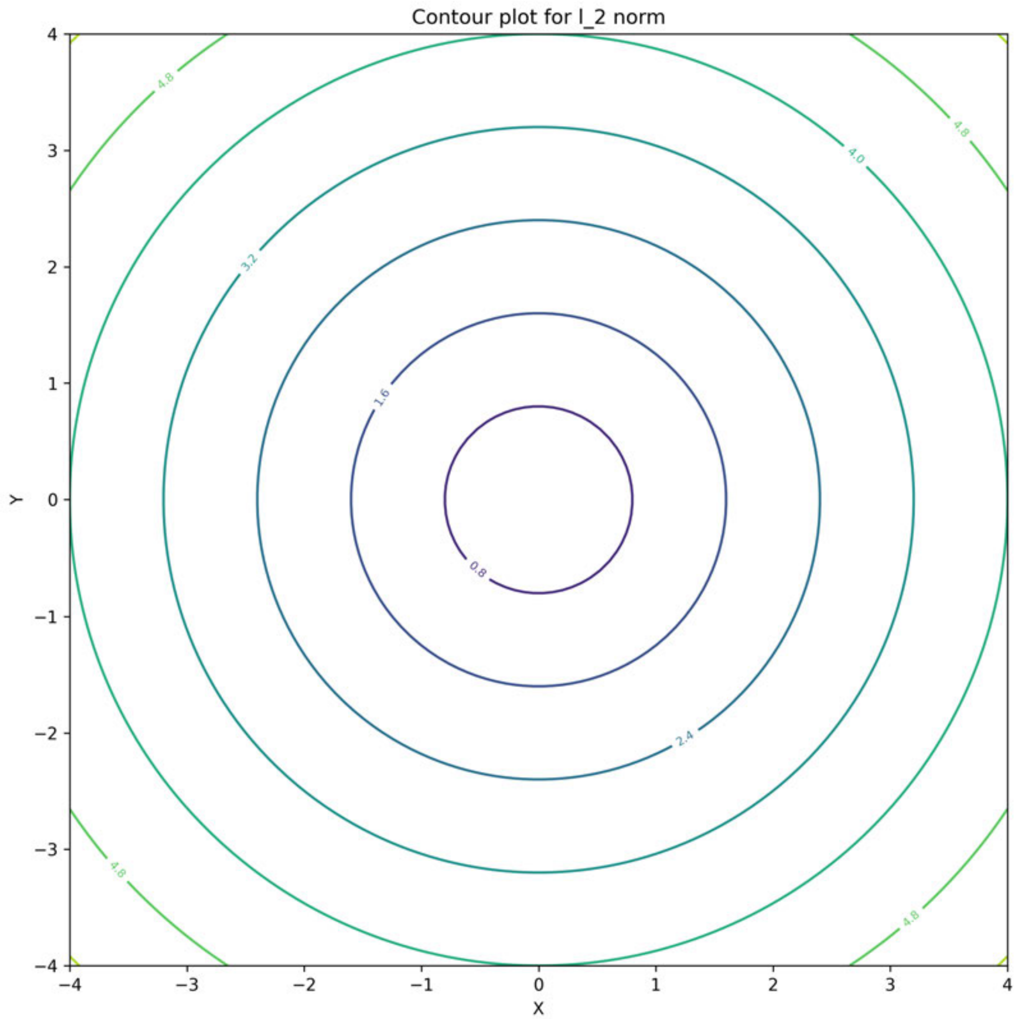
5.1.1 $l_{0.5}$-norm



5.1.2 l_1-norm



5.1.3 l_2-norm


Contour plot for l_2 norm
4.8
4.8
4.0
3.2
1.6
0.8
2.4
4.8
4.8
Y
X
4
3
2
1
0
−1
−2
−3
−4
−4
−3
−2
−1
0
1
2
3
4

5.2

5.3

5.4

5.5

In [1]:
```python
import pandas as pd
import numpy as np
import matplotlib.pyplot as plt
from math import e
from math import log
import scipy.io as sc
from scipy.special import expit, logit
%matplotlib inline
```

In [2]:
```python
def logistic_fn(x,w):

    return expit(np.dot(x,w))
```

In [3]:
```python
def update(x,w,y,s_vals):
    hess_inv = np.linalg.inv(np.matmul(np.matmul(x.T,np.diag(s)),x))
    diff = np.matmul(x.T,(y - s))
    return w - np.matmul(hess_inv,diff)
```

Problem 2.4 (do not submit)

In [100...
```python
x = np.array([[0.2, 3.1, 1],
              [1.0, 3.0, 1],
              [-0.2, 1.2, 1],
              [1.0, 1.1, 1]])
```

In [101...
```python
y = np.array([1,1,0,0])
```

In [108...
```python
w0 = [-1, 1, 0]
```

In [109...
```python
s0=[]
for i in range(x.shape[0]):
    s0+=[s(x[i],w0)]
```

In [119...
```python
s0 = logistic_fn(x,w0)
```

In [120...
```python
s0
```

Out[120]:
```
array([0.94784644, 0.88079708, 0.80218389, 0.52497919])
```

In [111...
```python
w1 = update(x,w0,y,s0)
```

In [112...
```python
w1
```

Out[112]:
```
array([-1.09534796,  0.43773381,  1.67830773])
```

In [113...
```python
s1 = logistic_fn(x,w1)
```

In [118...
```python
s1
```

Out[118]:
```
array([0.94354568, 0.86945643, 0.91853884, 0.74354326])
```

Q3.2

In [6]:
```
from sklearn.model_selection import KFold
from sklearn.metrics import roc_auc_score
from sklearn.model_selection import train_test_split
```

In [338…
```
#returns the mean and stdev of the features of the input data as a list of tuples

def get_fit(data):

    stats = []

    for i in range(data.shape[1]):

        stats+=[(data[:,i].mean(), data[:,i].std())]

    return stats
```

In [339…
```
#normalizes the features of a dataset by subtracting the training mean and dividing by
#should be a list of tuples

def transform(data,stats):

    for i in range(data.shape[1]):

        data[:,i] = (data[:,i]-stats[i][0])/stats[i][1]

    return data
```

In [340…
```
#update function for gradient descent with l2 penalty
def logr_update(data, labels, s_vals, weights, step, penalty):

    return weights+step*(np.matmul(data.T,(labels - s_vals)) - penalty*weights)
```

In [341…
```
#append a column of ones for the fictitious dimension
def add_fic(data):

    if data.ndim ==1:

        data=data.reshape(1,data.shape[0])

    return np.append(data,np.ones((data.shape[0],1)),axis=1)
```

In [342…
```
#calculates the cost of the logistic regression cost function with l2 penalty
def logr_cost(data, labels, s_vals, weights, penalty):

    log_s = [log(x+1e-10) for x in s_vals]

    log_s_comp = [log((1-x)+1e-10) for x in s_vals]

    return -np.dot(labels,log_s)-np.dot((1-labels),log_s_comp)+penalty*np.linalg.norm(
```

In [348…
```
# Define the decision function for
def predict(x, weights):
```

```python
    #calculate the logistic function with the
    score = logistic_fn(x,weights)

    return  (score >= 0.5).astype(int)
```

In [349...

```python
#Performs batch gradient descent with a fixed number of iterations
def bgd(data, labels, step, penalty,iterations):

    #initialize the weight with the zero vector
    weight = np.zeros(data.shape[1])

    s_vals = logistic_fn(data,weight)

    #initialize the cost with the first weight vector of all zeros
    cost=[logr_cost(data, labels, s_vals, weight, penalty)]



    for i in range(iterations):

        #creat the w' vector for the update and calculating the cost
        w_p = np.append(weight[:-1],0)

        weight = logr_update(data,labels,s_vals,w_p, step, penalty)

        cost+=[logr_cost(data, labels,s_vals, w_p, penalty)]

        s_vals = logistic_fn(data,weight)

    return weight,cost
```

In [350...

```python
data=sc.loadmat('data.mat')
```

In [351...

```python
labels=data['y'].reshape(len(data['y']))
```

In [352...

```python
test = data['X_test']
```

In [353...

```python
x = data['X']
```

In [354...

```python
k_fold = KFold(n_splits=10, random_state = 42, shuffle=True)
```

In [355...

```python
steps = [0.1, 0.01, 0.001, 0.0001]  # step size values to try
lambdas = [10, 1,0.1, 0.01]  # L2 penlaty values to try
```

In [356...

```python
# Iterate through each combination of hyperparameters to optimize the model
for step in steps:
    for penalty in lambdas:

        # Initialize list for cross-validation scores
        cv_costs = []
        cv_auc_roc = []

        # Iterate through each fold of the data
        for train_indices, val_indices in k_fold.split(x_train):

            # Split the data into training and validation sets
            X_train, y_train = x[train_indices], labels[train_indices]
```

```
        X_val, y_val = x[val_indices], labels[val_indices]

        # normalize the training and validation data by the training data values
        fit = get_fit(X_train)
        X_train = transform(X_train,fit)
        X_train = add_fic(X_train) #add column of 1s for the bias

        X_val = transform(X_val,fit)
        X_val = add_fic(X_val) #add column of 1s for bias

        weights, cost = bgd(X_train, y_train, step=step, penalty=penalty, iteratic

        # Calculate the cost function on the validation set

        s_vals = logistic_fn(X_val,weights)
        cost_val = logr_cost(X_val, y_val, s_vals, weights, penalty=penalty)

        # Make predictions on the validation set
        y_pred = predict(X_val, weights)

        # Calculate the AUC-ROC score on the validation set
        auc_roc = roc_auc_score(y_val, y_pred)

        # Append the cross-validation scores to the lists
        cv_costs.append(cost_val)
        cv_auc_roc.append(auc_roc)

    # Compute the mean and standard deviation of the cross-validation scores
    cv_mean_cost = np.mean(cv_costs)
    cv_std_cost = np.std(cv_costs)
    cv_mean_auc_roc = np.mean(cv_auc_roc)
    cv_std_auc_roc = np.std(cv_auc_roc)


    print(f'step size={step}, lambda={penalty}, mean CV cost={cv_mean_cost:.3f}, s
```

```
step size=0.1, lambda=10, mean CV cost=6231.396, std=211.044, mean CV AUC-ROC=0.702,
std=0.045
step size=0.1, lambda=1, mean CV cost=535.677, std=305.692, mean CV AUC-ROC=0.963, st
d=0.024
step size=0.1, lambda=0.1, mean CV cost=301.604, std=178.119, mean CV AUC-ROC=0.974,
std=0.009
step size=0.1, lambda=0.01, mean CV cost=154.027, std=91.185, mean CV AUC-ROC=0.983,
std=0.007
step size=0.01, lambda=10, mean CV cost=239.888, std=105.991, mean CV AUC-ROC=0.958,
std=0.036
step size=0.01, lambda=1, mean CV cost=38.726, std=12.382, mean CV AUC-ROC=0.988, std
=0.004
step size=0.01, lambda=0.1, mean CV cost=33.457, std=13.728, mean CV AUC-ROC=0.985, s
td=0.007
step size=0.01, lambda=0.01, mean CV cost=34.857, std=9.045, mean CV AUC-ROC=0.983, s
td=0.004
step size=0.001, lambda=10, mean CV cost=100.352, std=11.567, mean CV AUC-ROC=0.978,
std=0.006
step size=0.001, lambda=1, mean CV cost=53.272, std=11.855, mean CV AUC-ROC=0.981, st
d=0.004
step size=0.001, lambda=0.1, mean CV cost=44.340, std=12.029, mean CV AUC-ROC=0.980,
std=0.004
step size=0.001, lambda=0.01, mean CV cost=43.333, std=12.052, mean CV AUC-ROC=0.980,
std=0.004
step size=0.0001, lambda=10, mean CV cost=111.886, std=12.561, mean CV AUC-ROC=0.966,
std=0.004
step size=0.0001, lambda=1, mean CV cost=71.700, std=12.772, mean CV AUC-ROC=0.967, s
td=0.004
step size=0.0001, lambda=0.1, mean CV cost=67.046, std=12.796, mean CV AUC-ROC=0.967,
std=0.004
step size=0.0001, lambda=0.01, mean CV cost=66.573, std=12.799, mean CV AUC-ROC=0.96
7, std=0.004
```

Based on the above metrics there are a couple of competing combinations to check based on a low cost and high AUC:

step size=0.01, lambda=1, and step size=0.01, lambda=0.1

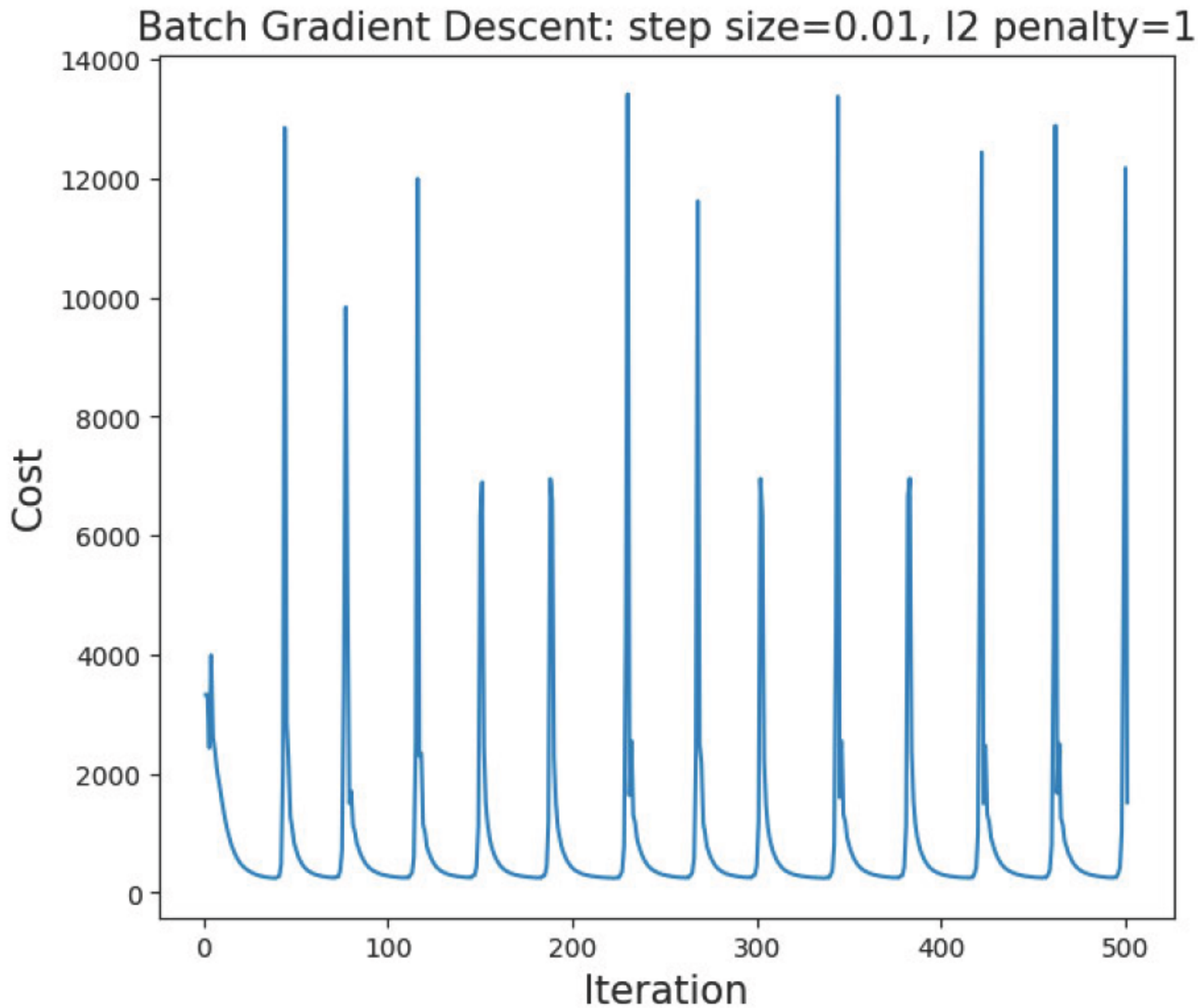
In [357…
```
x_train, x_val, y_train, y_val = train_test_split(x,labels, test_size = 0.2, random_st
```

In [358…
```
#normalize the data
fit = get_fit(x_train)
x_train = transform(x_train,fit)
x_train = add_fic(x_train) #add column of 1s for the bias

x_val = transform(x_val,fit)
x_val = add_fic(x_val) #add column of 1s for bias
```

In [359…
```
w_final, cost = bgd(x_train, y_train, 0.01, 1, 500)
```

In [360…
```
plt.figure(figsize=(7,6))
plt.plot(np.linspace(1,501,501),cost)
plt.xlabel('Iteration', fontsize=15)
plt.ylabel('Cost', fontsize=15)
plt.title('Batch Gradient Descent: step size=0.01, l2 penalty=1', fontsize=15)
# plt.savefig('Q3.2 0011.png',dpi=300)
```

Out[360]: Text(0.5, 1.0, 'Batch Gradient Descent: step size=0.01, l2 penalty=1')



This has some strage oscillations. Let's not go with that.

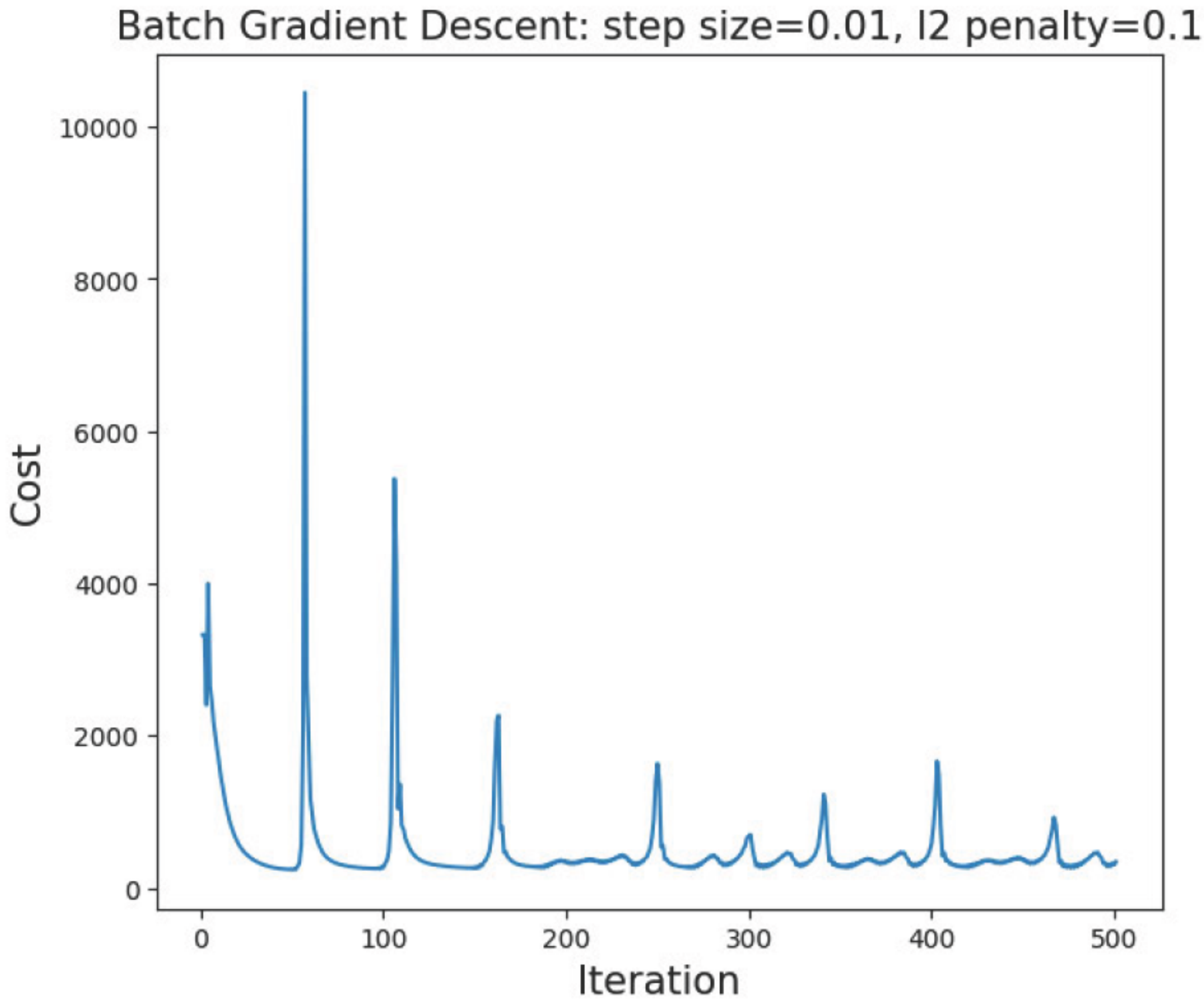
In [361…

```
w_final, cost = bgd(x_train, y_train, 0.01, 0.1, 500)
```

In [362…

```
plt.figure(figsize=(7,6))
plt.plot(np.linspace(1,501,501),cost)
plt.xlabel('Iteration', fontsize=15)
plt.ylabel('Cost', fontsize=15)
plt.title('Batch Gradient Descent: step size=0.01, l2 penalty=0.1', fontsize=15)
# plt.savefig('Q3.2 00101.png',dpi=300)
```

Out[362]: Text(0.5, 1.0, 'Batch Gradient Descent: step size=0.01, l2 penalty=0.1')



Still has oscillations, but not as bad. Let's test some other lambdas

In [363…

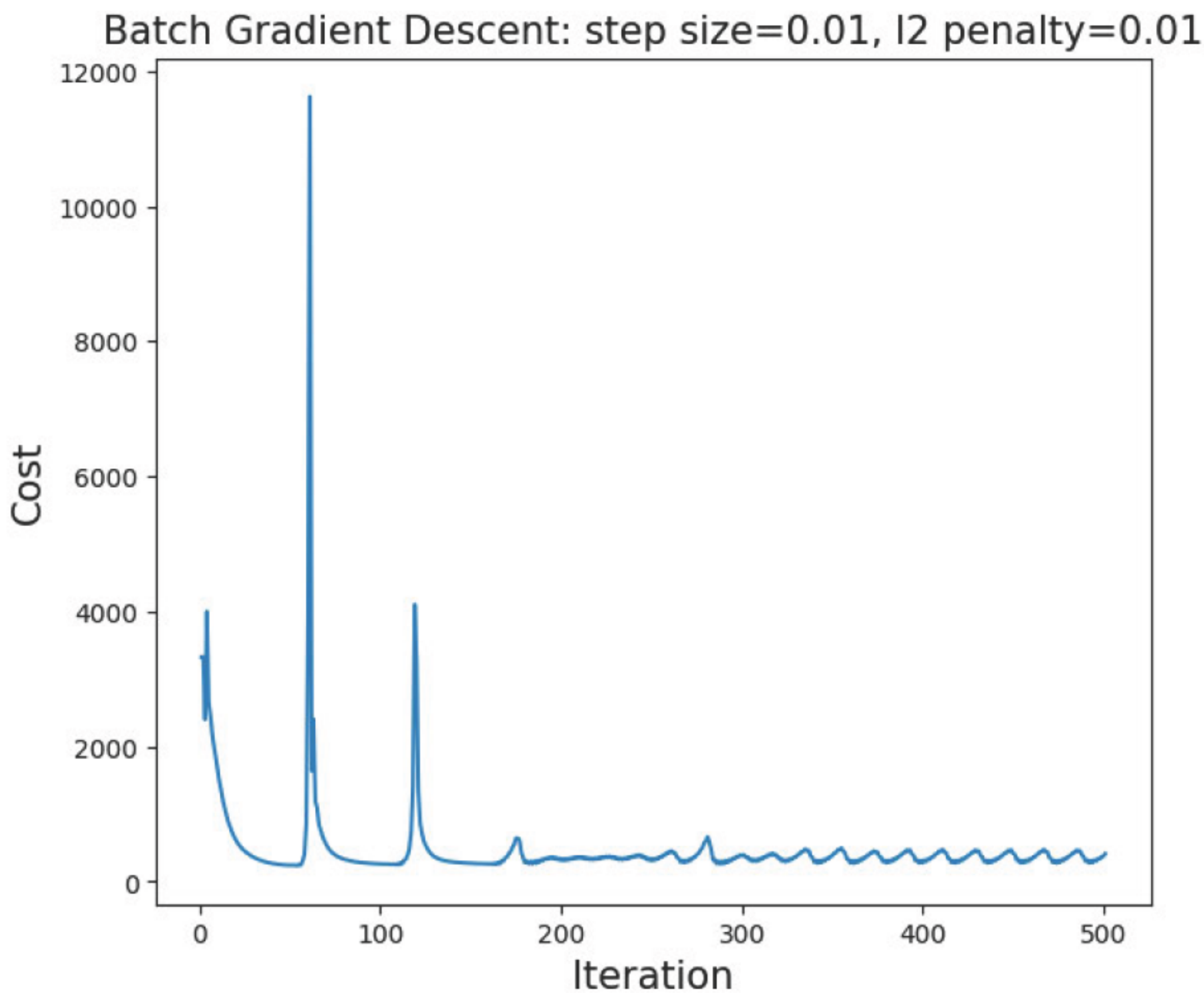
```
w_final, cost = bgd(x_train, y_train, 0.01, 0.01, 500)
```

In [364…

```
plt.figure(figsize=(7,6))
plt.plot(np.linspace(1,501,501),cost)
plt.xlabel('Iteration', fontsize=15)
plt.ylabel('Cost', fontsize=15)
plt.title('Batch Gradient Descent: step size=0.01, l2 penalty=0.01', fontsize=15)
# plt.savefig('Q3.2 001001.png',dpi=300)
```

Out[364]:

```
Text(0.5, 1.0, 'Batch Gradient Descent: step size=0.01, l2 penalty=0.01')
```

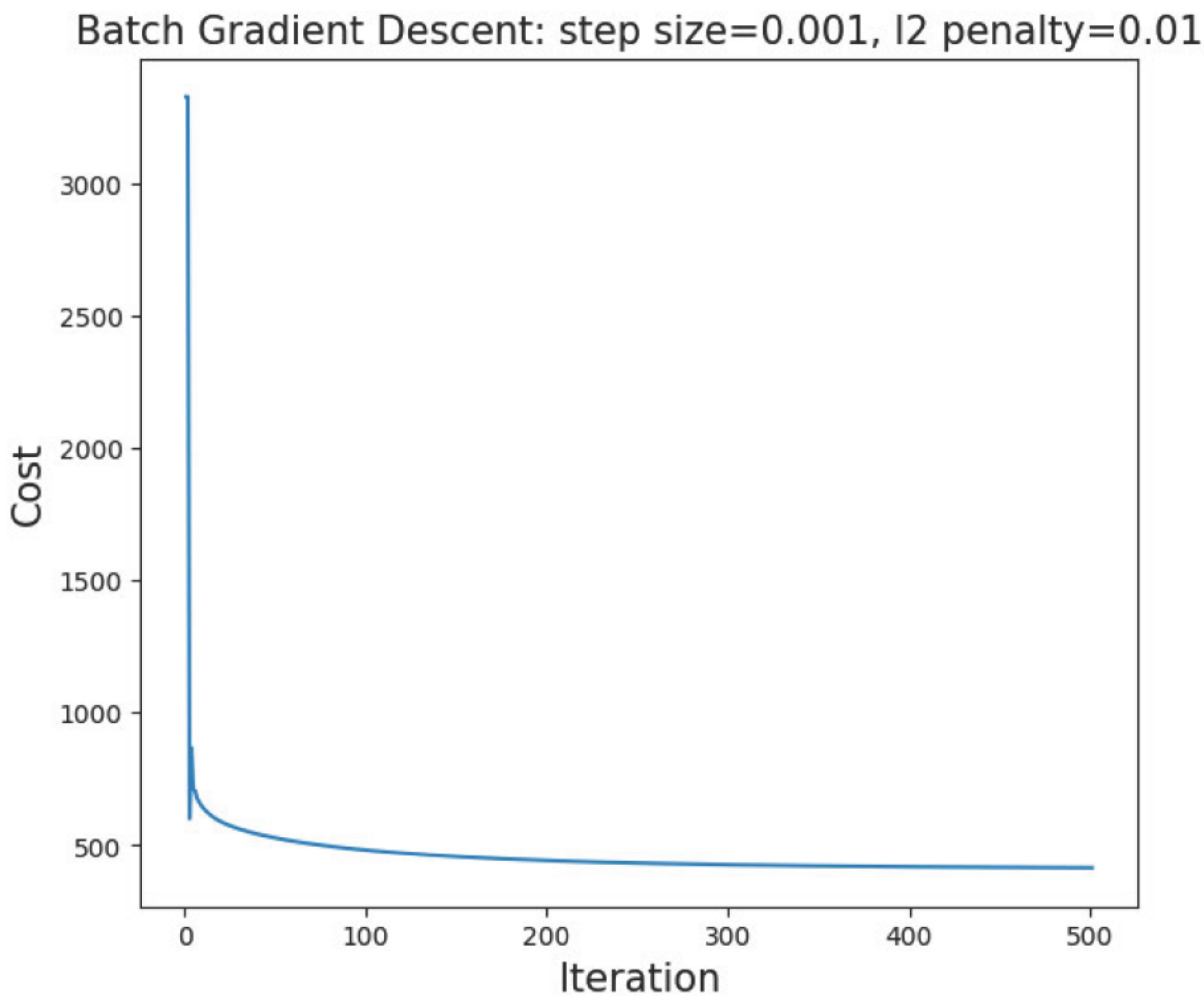


Maybe we can get rid of the oscillations by decreasing the step size

In [365…
```
w_final, cost = bgd(x_train, y_train, 0.001, 0.01, 500)
```

In [366…
```
plt.figure(figsize=(7,6))
plt.plot(np.linspace(1,501,501),cost)
plt.xlabel('Iteration', fontsize=15)
plt.ylabel('Cost', fontsize=15)
plt.title('Batch Gradient Descent: step size=0.001, l2 penalty=0.01', fontsize=15)
# plt.savefig('Q3.2 0001001.png',dpi=300)
```

Out[366]:
```
Text(0.5, 1.0, 'Batch Gradient Descent: step size=0.001, l2 penalty=0.01')
```



Q3.3

In [22]:
```
import random
```

In [215…
```
#update function for stochastic gradient descent with l2 penalty
def sgd_update(data, labels, weight, step, penalty):

    #creat the w' vector for the update and calculating the cost
    w_p = np.append(weight[:-1],0)

    return weight+step*((labels -logistic_fn(data,weight))*data - penalty*w_p)
```

In [242…
```
#calculates the cost of the logistic regression cost function with l2 penalty
def sgd_cost(data, labels, weights, penalty):

    s_vals = logistic_fn(data,weights)

    log_s = [log(x+1e-10) for x in s_vals]

    log_s_comp = [log((1-x)+1e-10) for x in s_vals]

    return -np.dot(labels,log_s)-np.dot((1-labels),log_s_comp)+penalty*np.linalg.norm(
```

In [300…

```python
# Performs stochastic gradient descent
def sgd(data, labels, step, penalty, epochs, batch_size):

    data = add_fic(data)

    # Initialize the weight with the zero vector
    weight = np.zeros(data.shape[1])

    # Initialize the cost with the first weight vector of all zeros
    cost = [sgd_cost(data, labels, weight, penalty)]

    random.seed(42)

    for epoch in range(epochs):

        rstate = random.randint(0,data.shape[0])

        # Shuffle the data and labels for the epoch
        x_train, x_val, y_train, y_val = train_test_split(data,labels, test_size = 0.

        for i in range(0, x_train.shape[0], batch_size):

            # Select the mini-batch of data and labels
            x = x_train[i:i+batch_size]

            y = y_train[i:i+batch_size]

            # Compute the logistic function and update the weight
            weight = sgd_update(x[0], y, weight, step, penalty)


            # Compute the cost for the mini-batch and store it
            cost += [sgd_cost(data, labels, weight, penalty)]


    return weight, cost
```

In [303…

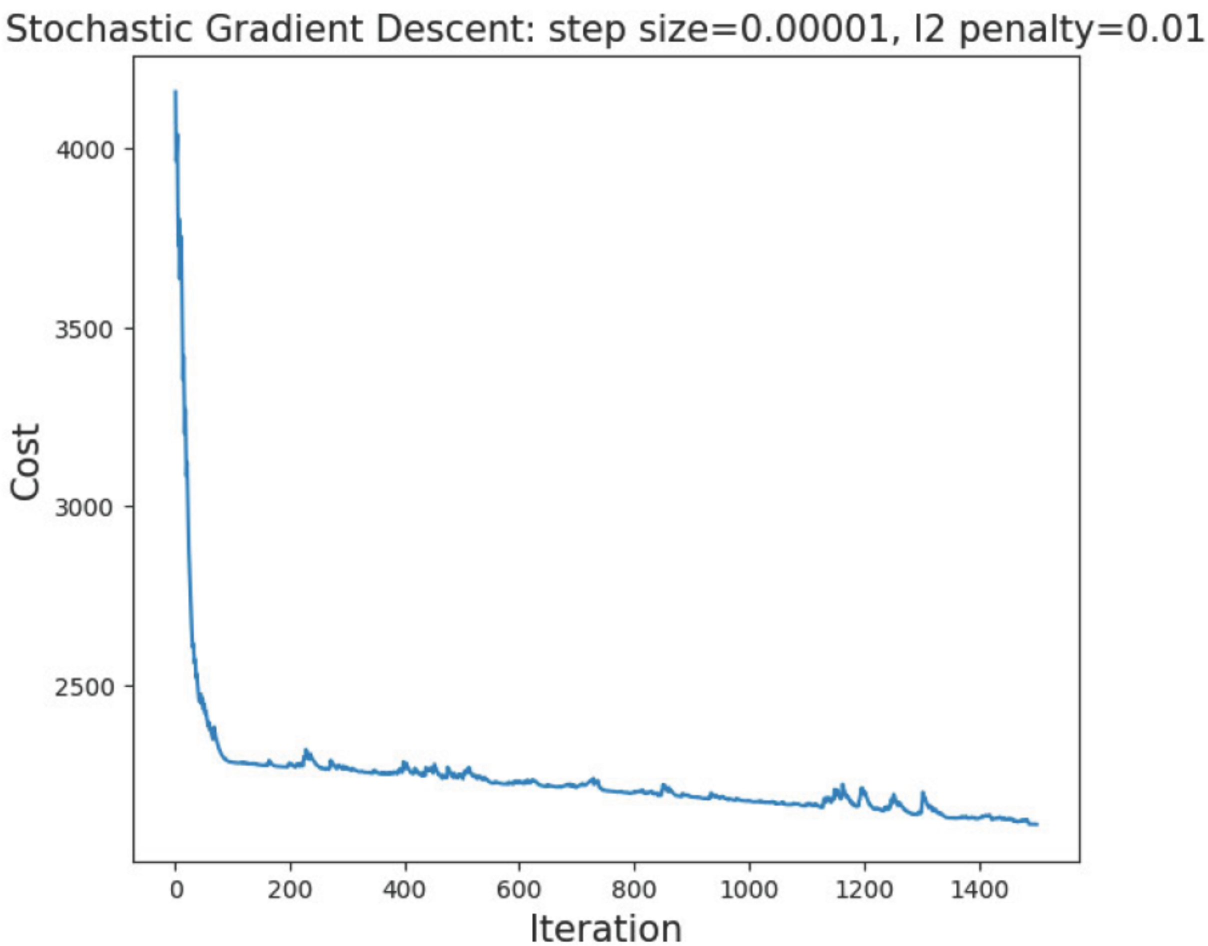
```python
weight, cost = sgd(x, labels, step=0.00001, penalty=0.01, epochs=1, batch_size=1)
```

In [304…

```python
plt.figure(figsize=(7,6))
plt.plot(np.linspace(1,len(cost),len(cost)),cost)
plt.xlabel('Iteration', fontsize=15)
plt.ylabel('Cost', fontsize=15)
plt.title('Stochastic Gradient Descent: step size=0.00001, l2 penalty=0.01', fontsize=
# plt.savefig('Q3.4 SGD.png',dpi=300)
```

Out[304]:

```
Text(0.5, 1.0, 'Stochastic Gradient Descent: step size=0.00001, l2 penalty=0.01')
```



Q3.5

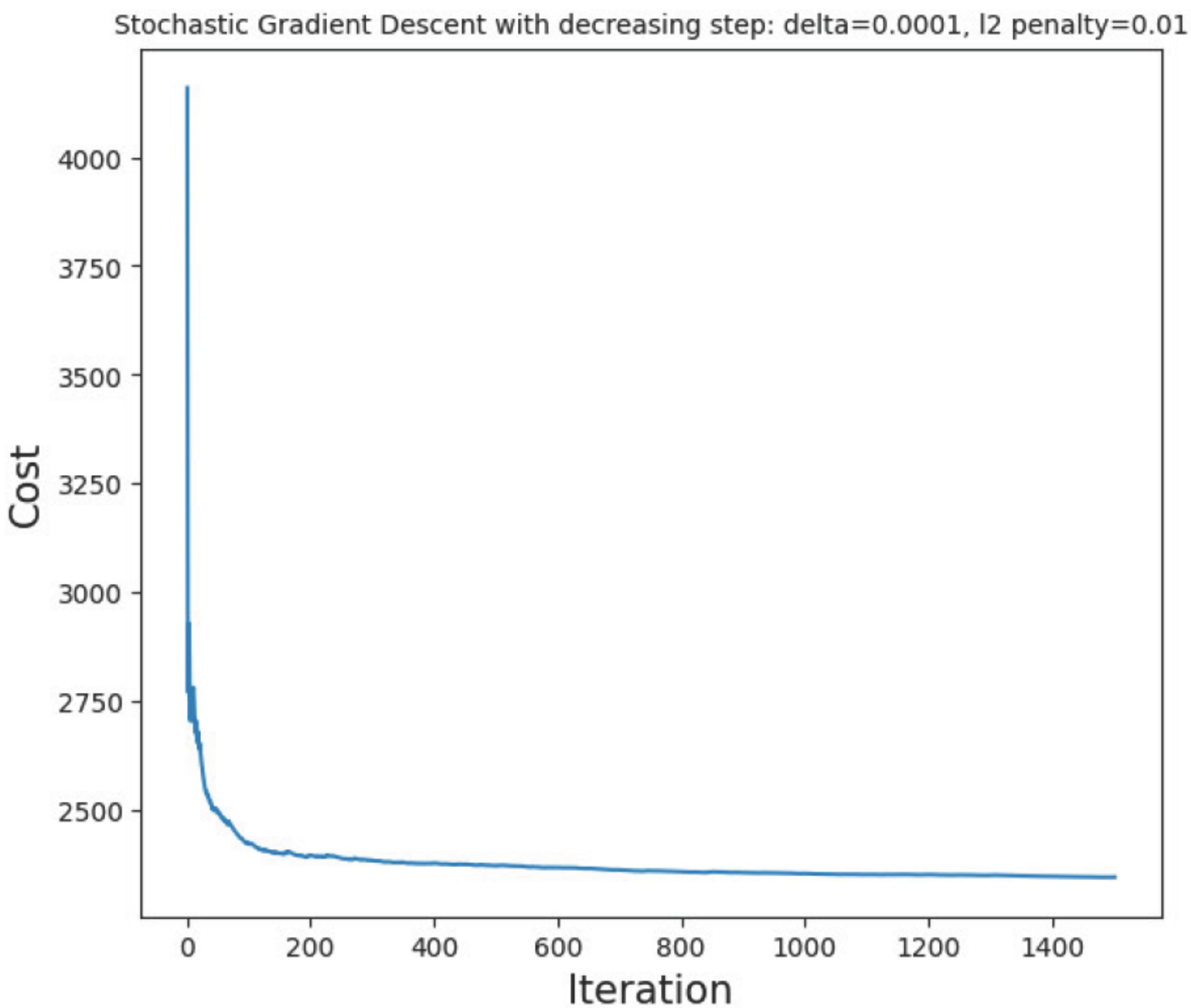
In [322…

```
# Performs stochastic gradient descent
def sgd_step(data, labels, delta, penalty, epochs, batch_size):

    data = add_fic(data)

    # Initialize the weight with the zero vector
    weight = np.zeros(data.shape[1])

    # Initialize the cost with the first weight vector of all zeros
    cost = [sgd_cost(data, labels, weight, penalty)]

    random.seed(42)

    for epoch in range(epochs):

        rstate = random.randint(0,data.shape[0])

        # Shuffle the data and labels for the epoch
        x_train, x_val, y_train, y_val = train_test_split(data,labels, test_size = 0.

        for i in range(0, x_train.shape[0], batch_size):

            # Select the mini-batch of data and labels
            x = x_train[i:i+batch_size]
```

```python
            y = y_train[i:i+batch_size]

            # Compute the logistic function and update the weight
            weight = sgd_update(x[0], y, weight, delta/(i+1), penalty)


            # Compute the cost for the mini-batch and store it
            cost += [sgd_cost(data, labels, weight, penalty)]


    return weight, cost
```

In [331…
```python
weight, cost = sgd_step(x, labels, delta=0.0001, penalty=0.01, epochs=1, batch_size=1)
```

In [336…
```python
plt.figure(figsize=(7,6))
plt.plot(np.linspace(1,len(cost),len(cost)),cost)
plt.xlabel('Iteration', fontsize=15)
plt.ylabel('Cost', fontsize=15)
plt.title('Stochastic Gradient Descent with decreasing step: delta=0.0001, l2 penalty=
plt.savefig('Q3.5 SGD.png',dpi=300)
```



Q3.6

In [369…

```
#normalize the data
fit = get_fit(x)
x_train = transform(x,fit)
x_train = add_fic(x_train) #add column of 1s for the bias
```

In [370…

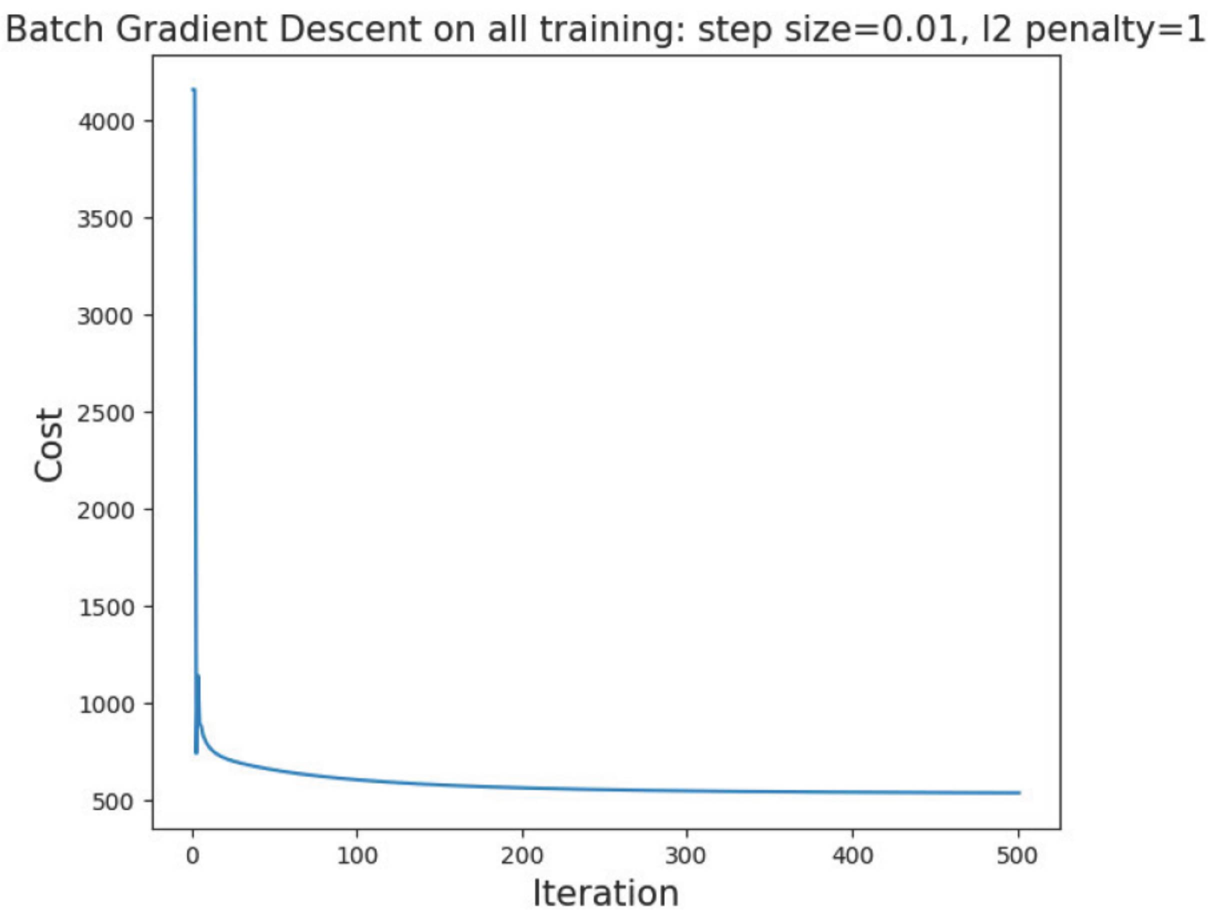
```
w_final, cost = bgd(x_train, labels, 0.001, 0.01, 500)
```

In [371…

```
plt.figure(figsize=(7,6))
plt.plot(np.linspace(1,len(cost),len(cost)),cost)
plt.xlabel('Iteration', fontsize=15)
plt.ylabel('Cost', fontsize=15)
plt.title('Batch Gradient Descent on all training: step size=0.01, l2 penalty=1', font
# plt.savefig('Q3.2 0011.png',dpi=300)
```

Out[371]:

```
Text(0.5, 1.0, 'Batch Gradient Descent on all training: step size=0.01, l2 penalty=
1')
```



In [375…

```
#normalize the test data
x_test = transform(test,fit)
x_test = add_fic(x_test) #add column of 1s for the bias
```

In [376…

```
y_pred = predict(x_test, w_final)
```

In [377…

```
y_pred
```

Out[377]:
```
array([0, 1, 1, 0, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0,
       0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 0, 0, 0, 0,
       0, 0, 1, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0,
       0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0,
       1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
       1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 1,
       0, 1, 1, 1, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0,
       0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 1, 1, 0, 0, 0, 0, 1, 1, 0,
       1, 0, 1, 0, 1, 1, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 1,
       0, 1, 0, 0, 1, 0, 1, 1, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0,
       0, 1, 0, 0, 1, 1, 0, 1, 1, 0, 0, 0, 0, 0, 1, 0, 1, 1, 1, 1, 1, 0,
       0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0,
       1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0,
       0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0,
       0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 1, 0, 0, 1, 0,
       0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0,
       1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 1, 1, 0,
       1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 0, 1, 0,
       1, 0, 0, 1, 0, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
       0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 0, 0, 0, 1, 1, 0, 0, 0, 1, 0, 0, 1,
       0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0,
       0, 1, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0,
       0, 0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 1, 0])
```

In [378…
```python
df = pd.DataFrame({'Id': np.linspace(1,len(y_pred),len(y_pred),dtype=int), 'Category':
df.to_csv('wine_preds.csv',index=False)
```

Q5.1

In [395…
```python
def grid_gen(xlim,ylim, num):

    # Define the range of x and y values
    x_min, x_max = -xlim, xlim
    y_min, y_max = -ylim, ylim
    num_points = num  # number of points in each direction

    # Create a 1D array of x values and y values
    x = np.linspace(x_min, x_max, num_points)
    y = np.linspace(y_min, y_max, num_points)

    # Create a 2D grid of points using the meshgrid function
    X, Y = np.meshgrid(x, y)

    return X,Y
```

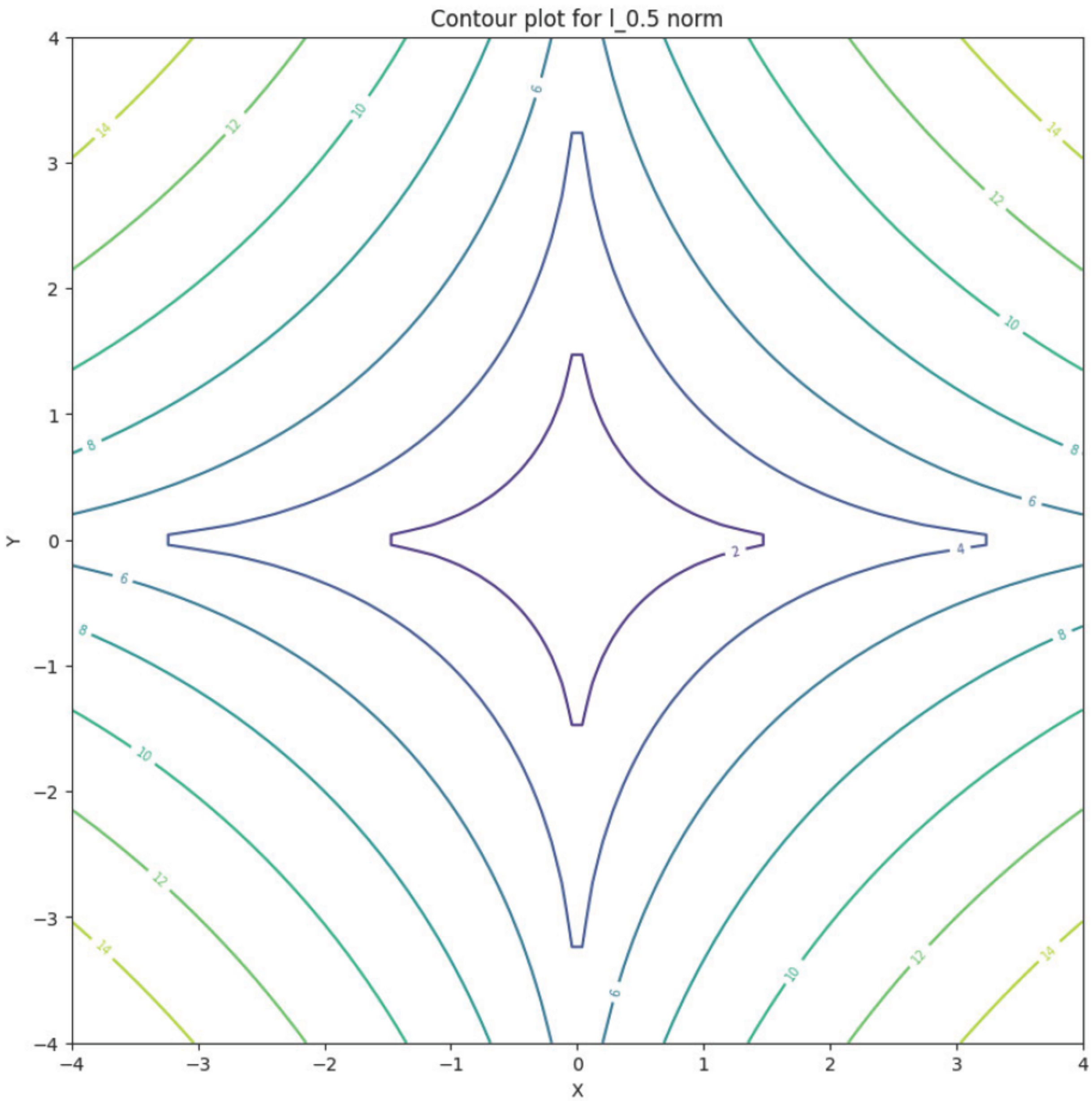
In [398…
```python
def f_eval(X,Y,p):

    return (np.abs(X)**p + np.abs(Y)**p)**(1/p)
```

In [408…
```python
def contour(p,xlim=None,ylim=None,num=None,f=None,grid=None,flag=True):

    if flag==True:
        X,Y = grid_gen(xlim,ylim,num)
        norm = f_eval(X,Y,p)

    else:
```

```
        norm=f

    fig = plt.figure(figsize=(10,10))
    cs = plt.contour(X,Y,norm)
    plt.clabel(cs,inline_spacing=5, fontsize=7)
    plt.xlabel('X')
    plt.ylabel('Y')
    plt.title('Contour plot for l_{} norm'.format(p))
    plt.savefig('l_{}_norm_plot.png'.format(p), dpi=300)
```

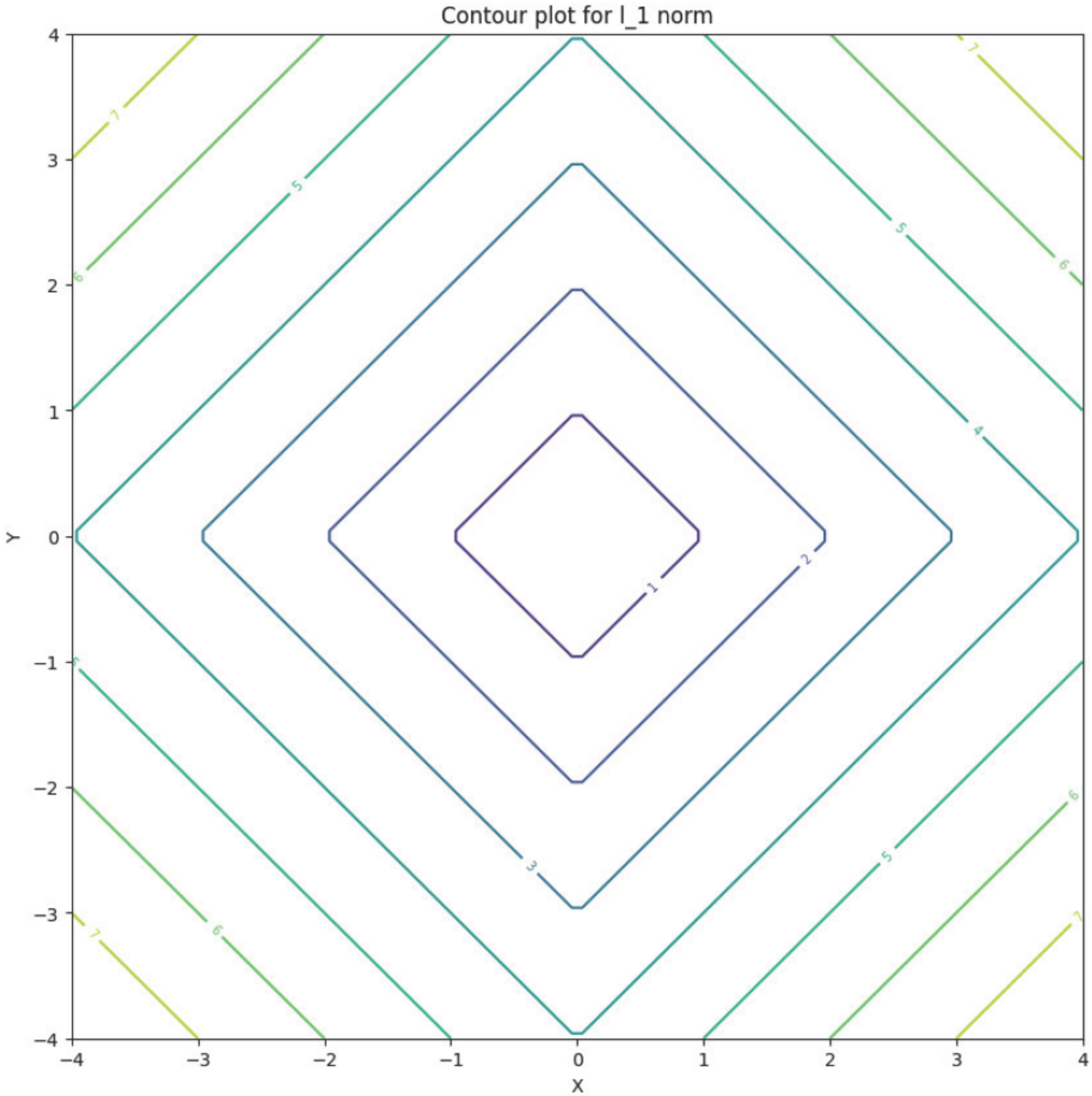
In [410…

```
contour(0.5,4,4,100)
```



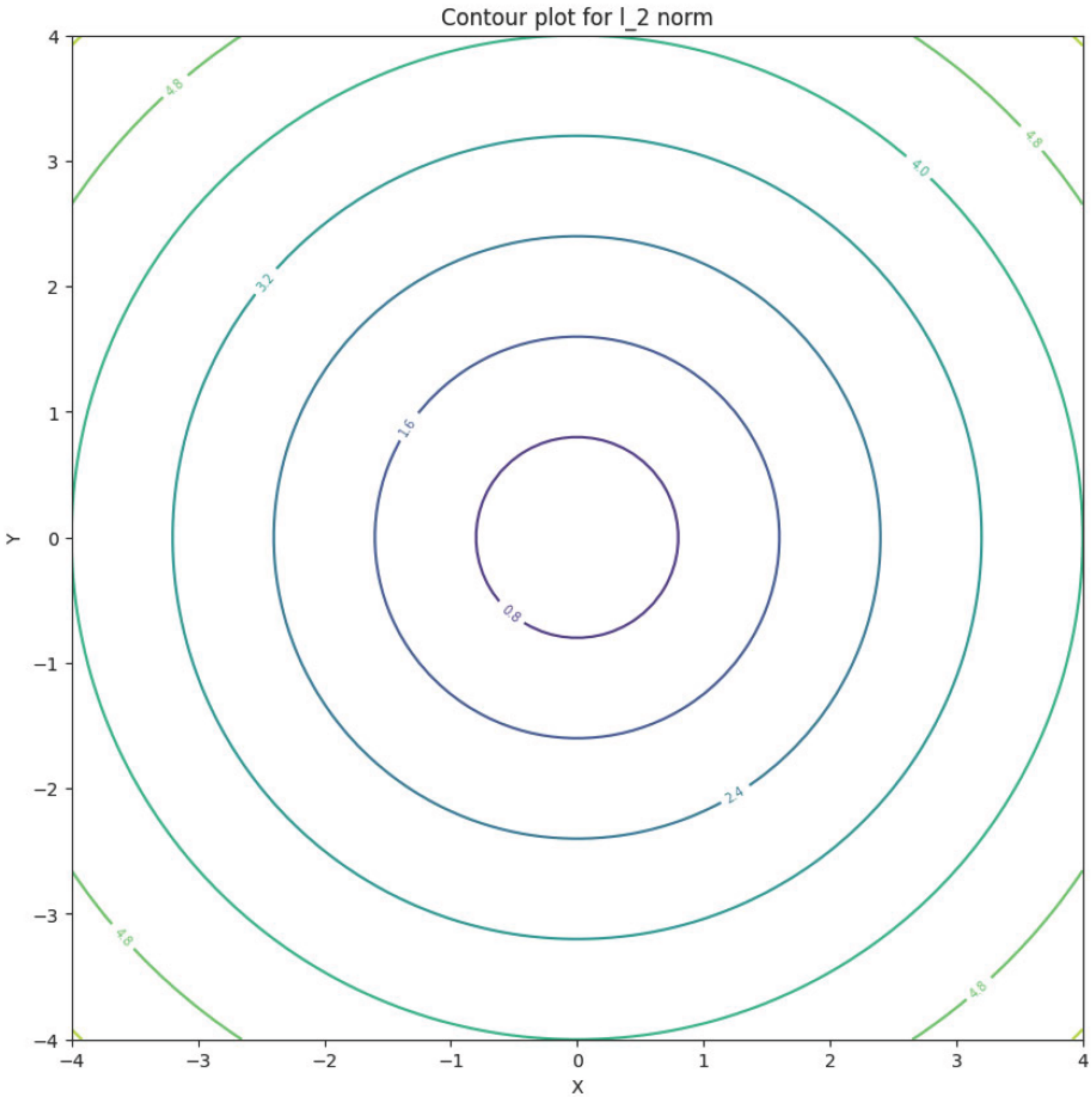
In [411…

```
contour(1,4,4,100)
```



In [412…

```
contour(2,4,4,100)
```



In [413…

```
import sys

print("Python version:", sys.version)
```

```
Python version: 3.7.15 (default, Nov 24 2022, 18:44:54) [MSC v.1916 64 bit (AMD64)]
```

In [414…

```
print("NumPy version:", np.__version__)
```

```
NumPy version: 1.21.5
```

In [416…

```
import sklearn
print("scikit-learn version:", sklearn.__version__)
```

```
scikit-learn version: 1.0.2
```

In [ ]:

```
```