Introduction to Bonus Certificates

Project structured products in practice

Iacob Jessica, Jordan François

Teacher : Prof. Aymeric Kalife

MIDO

Decembre 2024

Contents

1 An introduction to Bonus Certificates

Bonus certificates are participation products that feature full upside participation and a conditional capital protection as long as the underlying asset doesn't cross a predefined threshold. In most products, a fixed "bonus" will be paid to the holder of the certificate if the underlying asset didn't cross the preset barrier and if the performance of the underlying asset is lower than the predefined bonus. They are typically short term products with maturity ranging from two to four years.
Bonus certificates are constructed by means of a zero-strike call (i.e. a certificate) and a long down-and-out put option. The zero-strike call provides the participation (both to the upside and the downside), and the down-and-out put option generates the barrier and the bonus. More precisely, the strike of the down-and-out put option determines the level of the bonus and the out-strike determines the level of the barrier.

1.1 Some definitions

Let introduce some definitions.

Definition 1.1 : Participation Structured Products

Participation products provide the possibility to participate in the performance of an asset or basket of assets. These assets could be anything, from equities, funds, bonds, ETFs indices to a mix of those.

Definition 1.2 : Barrier Options

A barrier option is a type of derivative where the payoff depends on whether or not the underlying asset has reached or exceeded a predetermined price. They are considered exotic options. A barrier option's payoff is based on the underlying asset's price path. The option becomes worthless or may be activated upon the crossing of a price point barrier.

Definition 1.3 : Knock-in and Knock-out Options

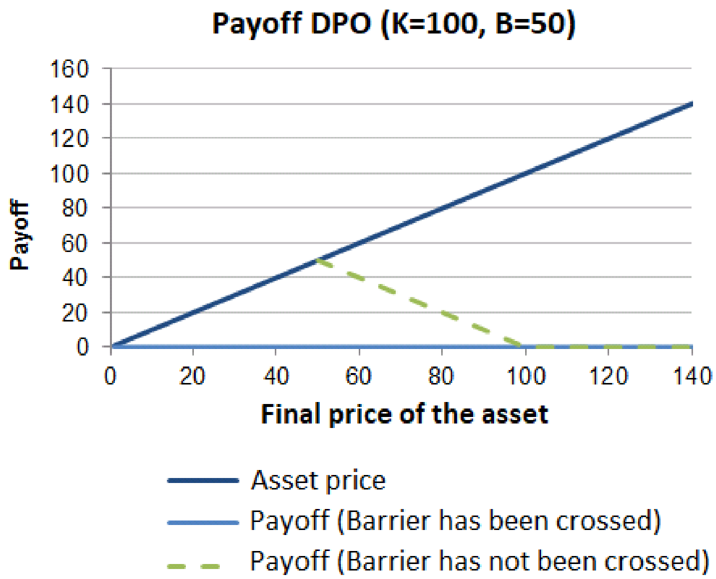
A barrier option can be a knock-out or a knock-in.
A knock-in option is a type of contract that is not an option until a certain price is met. So if the price is never reached, it is as if the contract never existed. However, if the underlying asset reaches a specified barrier, the knock-in option comes into existence.
A knock-out option ceases to exist if the underlying asset reaches a predetermined barrier during its life. It sets a cap on the level an option can reach in the holder's favor.
The difference between a knock-in and knock-out option is that a knock-in option comes into existence only when the underlying security reaches a barrier, while a knock-out option ceases to exist when the underlying security reaches a barrier.

Definition 1.4 : Down-and-Out Options

A down-and-out option is a barrier option, precisely a knock-out barrier option. It gives the holder the right, but not the obligation, to purchase or sell an underlying asset at a predetermined strike price, if the underlying asset's price does not go below a specified barrier during the option's life. Then, it expires worthless if the underlying reaches a certain price, limiting profits for the holder and limiting losses for the writer.

Example of Down-and-out Put:

Let a DOP (Downout Put) of maturity one year with strike $K = 100$ and a barrier at $B = 50$, then this option is sold at \$5 and a Put option with same strike and maturity is sold at \$15.
If we think the underlying price will fall, but no under \$50 until the maturity, we'll rather buy the Downout Put option. In the case that the underlying price cross the barrier (\$50), the contract is canceled and we say that the barrier has been crossed.
On the contrary, if the underlying price fall but no under \$50 until the maturity, the Downout Put option will bring benefice. For instance, if the underlying close at \$60 without having crossed the barrier, the gain is $100 - 60 - 5 = \$35$ which is (Strike - Final price - premium), the gain of a Put would be only $100 - 60 - 15 = \$25$.



Property 1.1 : Properties of a Down-and-Out Put

A DOP is influenced by the volatility: If the strike is far enough from the barrier, the implied volatility and the option's price are positively correlated. But if the strike is close to the barrier, a high volatility could push the underlying price across the barrier and then the contract will be canceled.
A high skew will increases the probability of the underlying to cross the barrier and the DOP will be cheaper.
A high maturity let a higher possibility to the underlying to cross the barrier then the DOP will be cheaper too.

Definition 1.5 : Zero-strike Call Option

A zero-strike call optiona is a financial contract that give the buyer the right—but not the obligation—to buy a stock, bond, commodity, or other asset or instrument at zero price within a specific period. A call seller must sell the asset if the buyer exercises the call.

1.2 How do Bonus Certificates work ?

Bonus certificates combine three advantages in one product. The investor benefits from rising prices of the underlying instrument, receives a sizeable bonus payment, and, in the case of falling prices, is protected up to the safety barrier. In case of an unexpected slump, the bonus payment is dropped, and the price of the underlying instrument is credited at the end of maturity.

The bonus level is set above the current price of the underlying at the issue of the certificate. The barrier is set below the initial value. If the specific certificate comes with a cap as well, it is set at or above the bonus level.

The redemption at the end of maturity hinges on the development of the underlying. The following two cases can occur:
1. If the underlying asset never falls to or below the barrier level during the term of the certificate, the investor receives at least a payment equaling the bonus level. If the price of the underlying is higher than the bonus level at the end of period date, he receives the higher payment out of the two.

2. If the underlying does fall to or below the barrier at least once during the term of the certificate, there will be no bonus payment. The investor gets the performance of the underlying paid out at the end of maturity. Depending on whether the price of the underlying is below or above the issue price, the investor suffers a loss or makes a profit.

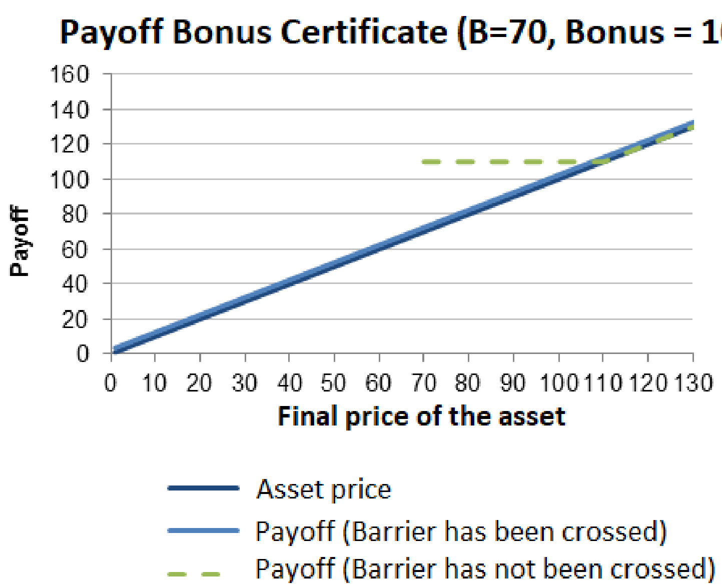
1.3 An exemple of Bonus Certificate

Let's introduce an example of bonus certificate in ordre to illustrate how this structured product works.

Consider a bonus certificate on an underlying S with maturity two years, a 70% barrier and a bonus of 10%. Assume that $S_0 = 100$ (the value of the underlying at time t=0). The investor will buy this structured product for 100.

If the underlying never fell under 30% of its initial value during the two years, at the maturity, the investor will gain the maximum between 110% of the initial value and the final value. For instance, if $S_T = 105$, the investor will gain 110 which is 110% of 100, if $S_T = 130$ the investor will gain 130 (the gain is $\max(S_T, 110\% \times S_0)$).

On the contrary, if during the two years, the underlying fell under 30% of its initial value, the down-and-out put in the bonus certificate is canceled. Only the 0 strike call is maintened, then if the underlying price is $S_T = 90$, the investor will have only 90. We say the investor has lost his bonus.



Property 1.2 : Parameters of a Bonus Certificate

More the barrier, volatility, maturity, dividends are higher and more the bonus will be higher as more risks are taken.

1.4 Pricing with Heston Model

In this section we will price a bonus certificate option numerically thanks to the Heston Model because in comparison to the Black-Scholes Model, the volatility is not longer constant, it's now stochastic and this will give us a more accurate simulation of the market.

1.4.1 Framework

Let $T > 0$, a time horizon and $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \leq t \leq T}, \mathbb{Q})$ a filtered space where $\mathbb{Q}$ is the risk neutral measure under which the discounted asset price process is a $\mathbb{Q}$-martingale. We consider the stochastic differential equation

$$dS_t = rS_t dt + \sqrt{V_t} S_t dW_t^1$$
$$dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t} dW_t^2,$$

where

$(W_t^1)_{t\in[0,T]}$ the $\mathbb{Q}$- Brownian motion of the asset price,

$(W_t^2)_{t\in[0,T]}$ the $\mathbb{Q}$- Brownian motion of the asset's price variance,

ρ the correlation coefficient for $(W_t^1)_{\in[0,T]}$ and $(W_t^2)_{\in[0,T]}$ $(d\langle W^1, W^2\rangle_t = \rho dt \ \forall t \in [0,T])$,

$(S_t)_{t\in[0,T]}$ asset price process,

$(\sqrt{V_t})_{t\in[0,T]}$ volatility process of the asset price,

σ volatility of the volatility,

r risk-free rate,

θ long-term price variance,

κ rate of reversion to the long-term price variance.

To simulate a Bonus Certificate with a Bonus (B) and a Barrier (H), we introduce the stopping time

$$\tau = \inf\{t \in [0,T],\ S_t \leq S_H\}$$

where S_H denote the price corresponding to the Barrier H, if the barrier is never crossed, we let $\tau = T$. Then, the payoff of a Bonus Certificate at maturity with the underlying S is

$$BC(T) = \max(S_T, S_0(1+B))\mathbb{1}_{\tau>T} + S_T\mathbb{1}_{\tau\leq T},$$

where $BC(t)$ denote the payoff of the Bonus Certificate at time t. Therefore, the price we are looking for is

$$BC(0) = \mathbb{E}^{\mathbb{Q}}[e^{-rT}BC(T)].$$

Because there is no explicit formula for this expectation, we will compute it numerically with Monte Carlo method, and the Greeks by finite differences method.

1.5 When to invest in a Bonus Certificate ?

Invest	**Don't invest**
the volatility is high, or expected to fall	the remaining maturity is more than three years
the skew is high or expected to fall	underlying falling
the dividend yield of the underlying stock or index is high or expected to fall	
underlying moving sideways or rising	

2 Sensitivity analysis

2.1 A little recap about the greeks

Greeks are measures of the sensitivity of a derivative's price (such as an option) to changes in an underlying parameter on which the value of the derivative is dependent (holding all other parameters fixed).
They are important tools in risk management and are necessary to assess risk and hedge properly. Indeed, since Greeks provide a set of measures of sensitivity of the value of a portfolio with respect to a small variation in a specific underlying parameter, component risks can be treated in isolation, and the portfolio can be rebalanced in order to achieve the desired exposure.

Delta: $\Delta = \frac{dV}{dS}$ **Gamma:** $\Gamma = \frac{d^2V}{dS^2}$ **Vega:** $\nu = \frac{dV}{d\sigma}$

Theta : $\theta = -\frac{dV}{d\tau}$ **Rho :** $\rho = \frac{dV}{dr}$ **Volga:** $= \frac{d^2V}{d\sigma^2}$ **Vanna :** $= \frac{dV}{dSdV}$

We will focus on the Delta, Gamma, Theta, Vega, and Volga.

2.2 Black-Scholes model vs Heston model

Black-Scholes model makes an assumption that stock returns are normally distributed with known mean and variance. It doesn't depend on the mean spot return and it can not be generalized by variation of the mean. One problem with the Black Scholes model is that it predicts a flat profile for the implied volatility surface, while empirical facts indicate that it is not constant as a function of exercise price, nor as a function of time to maturity. In fact, it is in practice often found to be curved in a way that resembles a smile.
The Heston model provides a model of stochastic volatility, which results in a graphical volatility smile. It assumes that the spot asset is correlated with volatility.

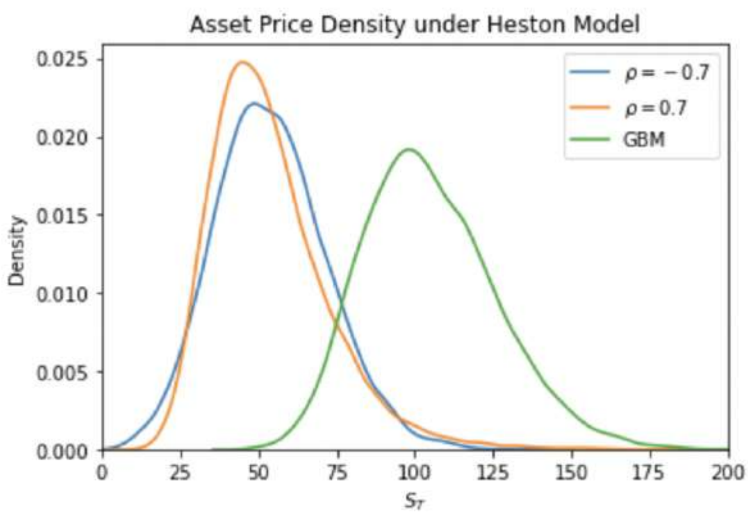


Figure 1: Heston and Black-Scholes model

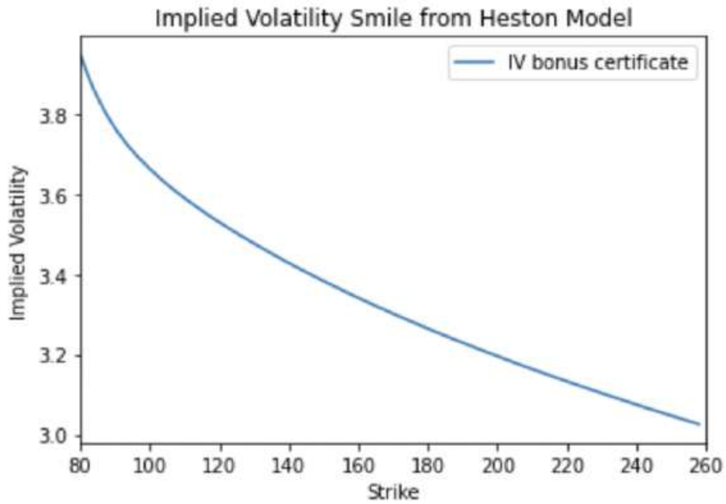


Figure 2: Smile with Heston model

The Heston model performs better than the Black Scholes model since the density function of the simulated path is much closer to the density function of the empirical log returns. Thus we will use Heston model to analyse the greeks of this option.

2.3 Price and sensitivities

In this part, we compute numerically with Monte Carlo methods the price of a Bonus Certificate with a barrier level fixed at \$70, and a bonus level fixed at \$110. The parameters used for the Heston Model are

$$V_0 = 0.25^2,\ \sigma = 0.2,\ \kappa = 3,\ \theta = 0.2^2,\ r = 0.05,\ \rho = 0.7,\ T = 1.$$

The asset price under the Heston Model will be simulated with the Euler Scheme method with a time discretization $t_0 = 0 < t_1 < ... < t_m = T$ with $m = 10^2$ and a constant mesh $\Delta t = \frac{T}{m}$. Every payoff will be approximated with Monte Carlo methods with $n = 10^4$ since

$$BC(0) = \lim_{n\to+\infty} \frac{e^{-rT}}{n} \sum_{i=1}^{n} \max(\bar{S}_T^{m,i}, S_0^i(1+B))\mathbb{1}_{\tau^i>T} + \bar{S}_T^{m,i}\mathbb{1}_{\tau^i\leq T}$$

where $(\bar{S}_T^{m,i})_{i\in\{1,\dots,n\}}$ are n independent simulation of S_T where for $i \in \{1,...,n\}$ and for all $k \in \{0,...,m-1\}$,

$$\begin{aligned}
\bar{S}_{t_{k+1}}^{m,i} &= \bar{S}_{t_k}^{m,i} + r\bar{S}_{t_k}^{m,i}\Delta t + \bar{S}_{t_k}^{m,i}\sqrt{\bar{V}_{t_k}^{m,i}}(W_{t_{k+1}}^{1,i} - W_{t_k}^{1,i}),\\
\bar{S}_{t_0}^{m,i} &= S_0
\end{aligned}$$

and

$$\begin{aligned}
\bar{V}_{t_{k+1}}^{m,i} &= \bar{V}_{t_k}^{m,i} + \kappa(\theta - \bar{V}_{t_k}^{m,i})\Delta t + \sigma\sqrt{\bar{V}_{t_k}^{m,i}}(W_{t_{k+1}}^{2,i} - W_{t_k}^{2,i}),\\
\bar{V}_{t_0}^{m,i} &= V_0.
\end{aligned}$$

We have also $\mathrm{Cov}(W_{t_k}^{i,1}, W_{t_k}^{i,2}) = \rho$ for all $i \in \{1,...,n\}$ and for all $k \in \{0,...,m-1\}$.

2.3.1 Price

Thanks to Monte Carlo methods, we can compute the price of the Bonus Certificate. If $S_T < \$70$, then the DOP do not longer exists and the Bonus Certificate worth only the asset, therefore we have a linear relationship between the asset value and the Bonus Certificate value if $S_T < \$70$. If $S_T > \$110$, the Bonus is not longer given, and we have also a linear relationship. The only interesting case is if $\$70 < S_T < \$110$, then if the barrier has not been crossed until maturity we earn the Bonus which is \$110, or we earn only the asset if the barrier has been crossed earlier, causing the price of the Bonus Certificate being slightly greater than S_T.

Figure 3: Premium of a bonus certificate

2.3.2 Delta

The Delta is the theoretical estimate of how much an option's value may change given a $1 move up or down in the underlying. The Delta value 0 represents an option where the premium barely moves relative to price changes in the underlying stock.

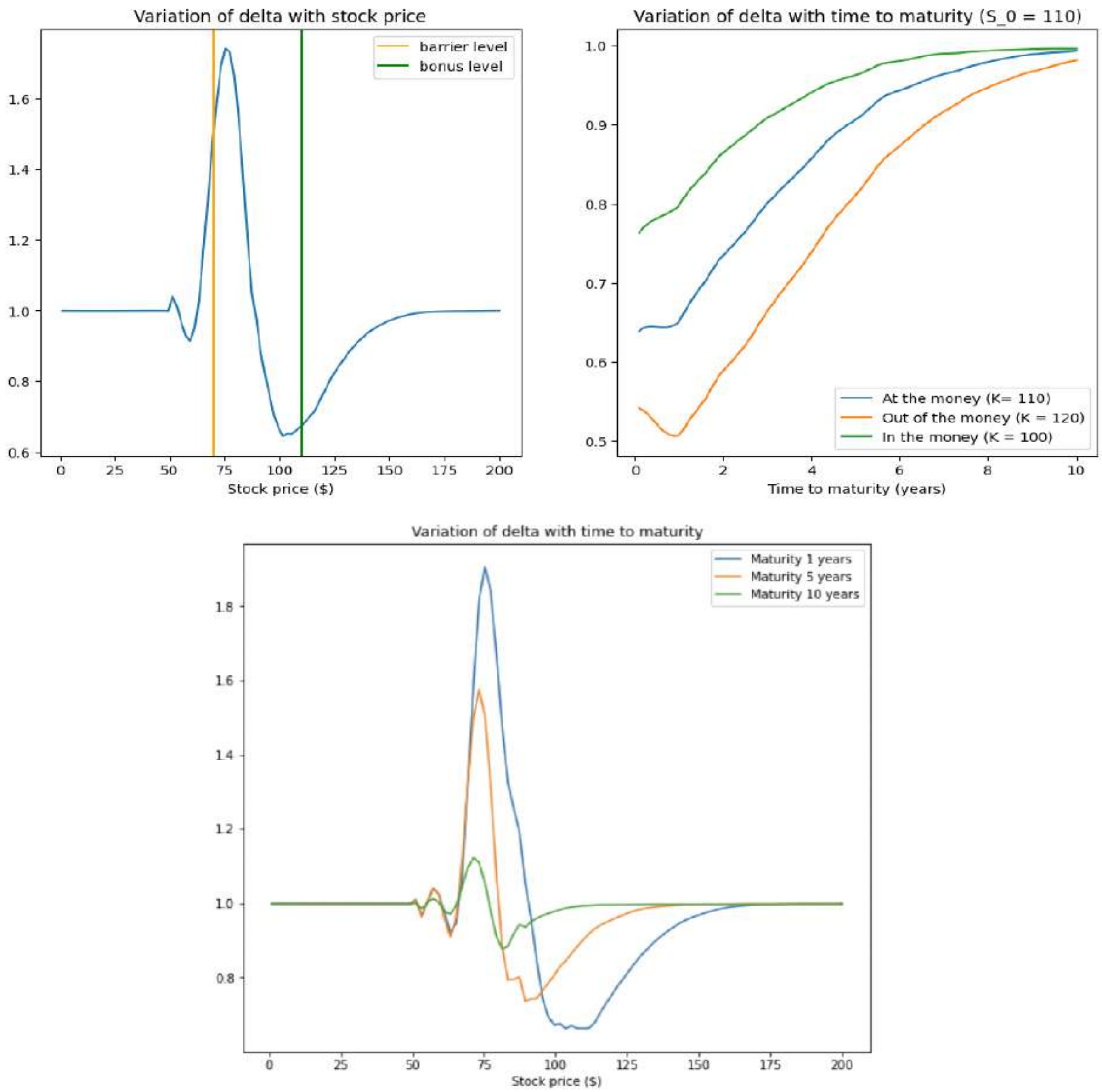


Figure 4: Typical pattern for Delta

As we can see on the graph on the left, the Delta of Bonus Certificate jumps and reaches its maximum level when the underlying price is close to its Barrier level. It reaches its minimum level when the underlying price is close to the strike. Otherwise the Delta is constant 1 because a variation of the underlying price has direct impact on the option.
On the graph on the right, we see that a highest strike coincide with less sensitivity for the Delta and no matter the Strike, the gamma is increasing with the maturity.
The third graph illustrate the fact that the greater maturity, the greater probability to cross the barrier for the option and otherwise the Delta is stabilized to 1. This is why we say that a Bonus Certificate is short term (2 to 4 years).

2.3.3 Gamma

The Gamma represents the rate of change between an option's Delta and the underlying asset's price. Higher Gamma values indicate that the Delta could change dramatically with even very small price changes in the underlying stock or fund.

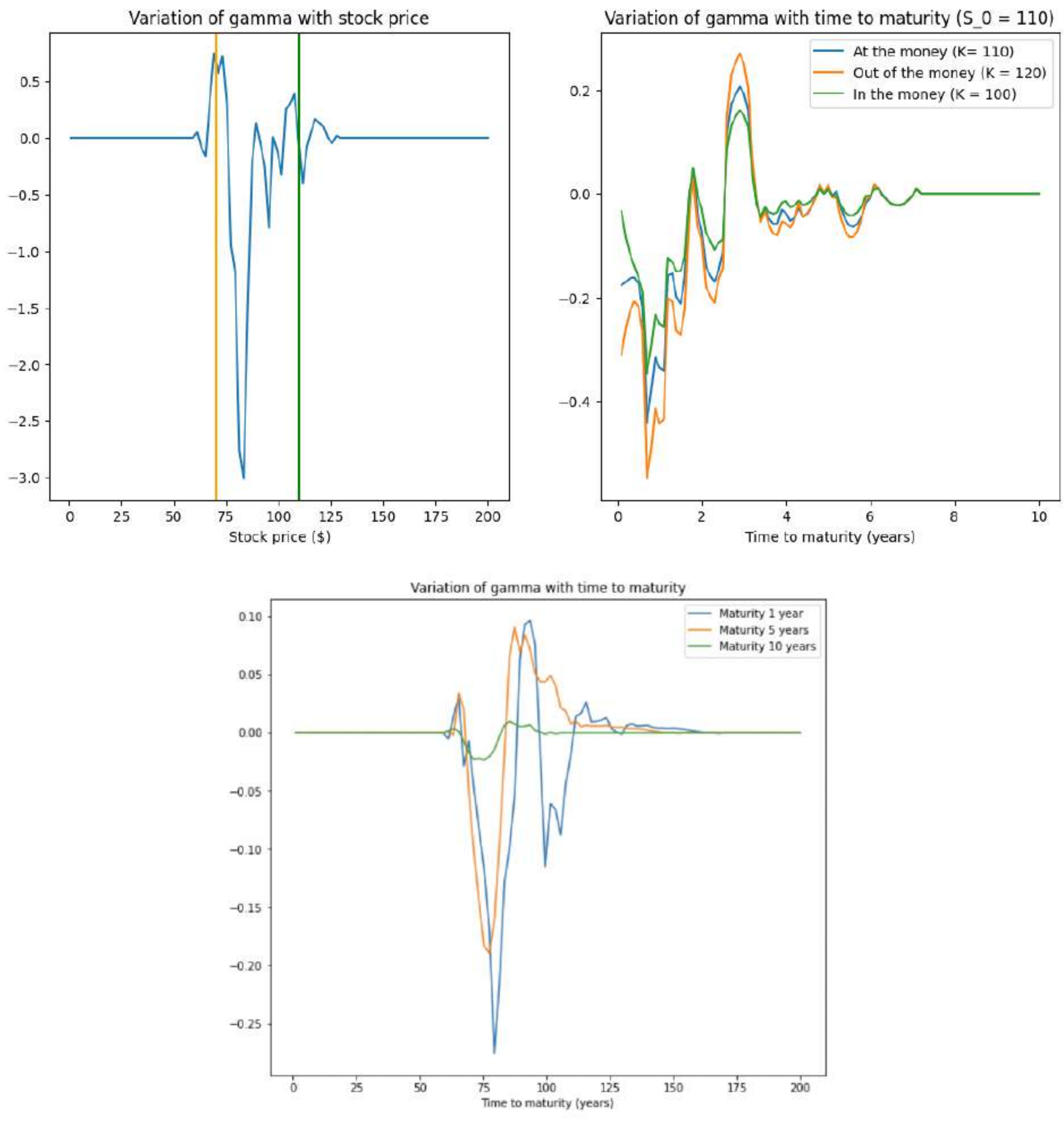


Figure 5: Typical pattern for Gamma

We can see on the first graph that between the barrier and the bonus level, the Gamma is very sensitive due to a high Delta. It tends to stabilize around the bonus level.
When the spot price is close to the barrier, a small change in the price can significantly impact the likelihood of the underlying asset moving above or below the barrier.
A longer maturity implies low impact of the volatility on the option as it is the same for the Delta.

2.3.4 Theta

The Theta measures how much value an option might lose each day as it approaches expiration.

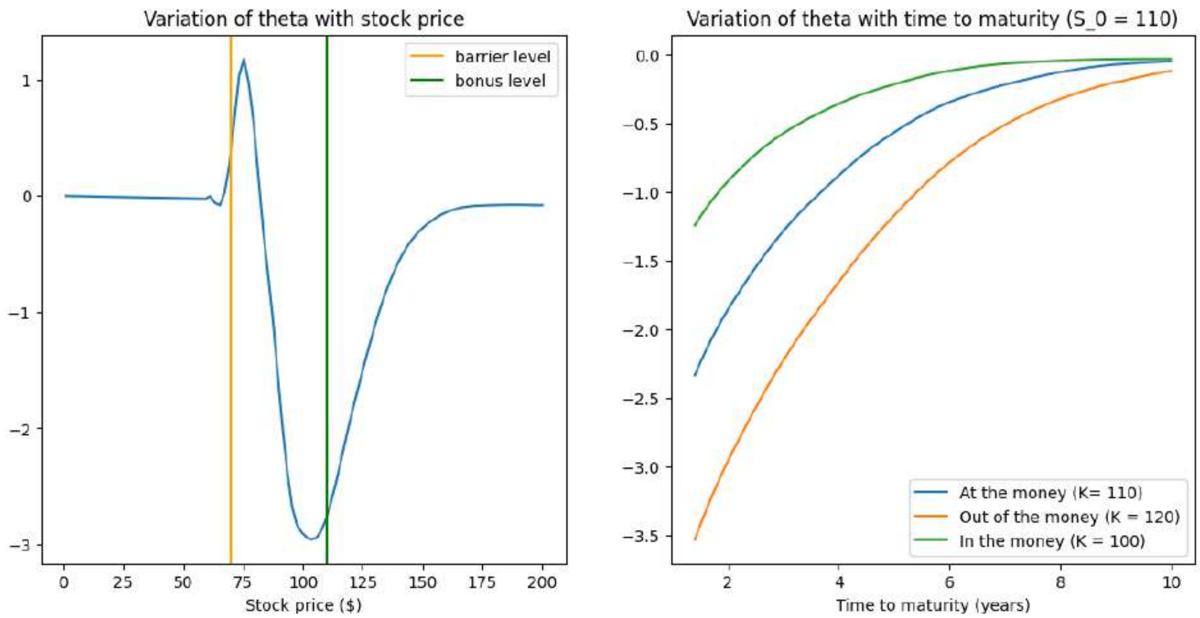


Figure 6: Typical pattern for Theta

On the left we can see that the Theta of Bonus Certificate jumps and reaches its maximum level when the underlying price is close to its Barrier level. It reaches its minimum level when the underlying price is close to the Strike. Otherwise the Theta is constant because a variation of the underlying price does not have much effect on the price of the Bonus Certificate. It is almost the same variation as the Delta except the fact that the Delta reaches much more to the Barrier level and the Theta go down stronger to the Strike.
Thus, when the price of the underlying is below the Barrier, one day less increase the probability of getting above the barrier that involve the increase of Bonus Certificate price, whereas, when prices are above the barrier, one day less increase the probability to stay under the barrier at maturity that decrease the option's price.

2.3.5 Vega

The Vega measures the sensitivity of the option price to a 1% change in the implied volatility.

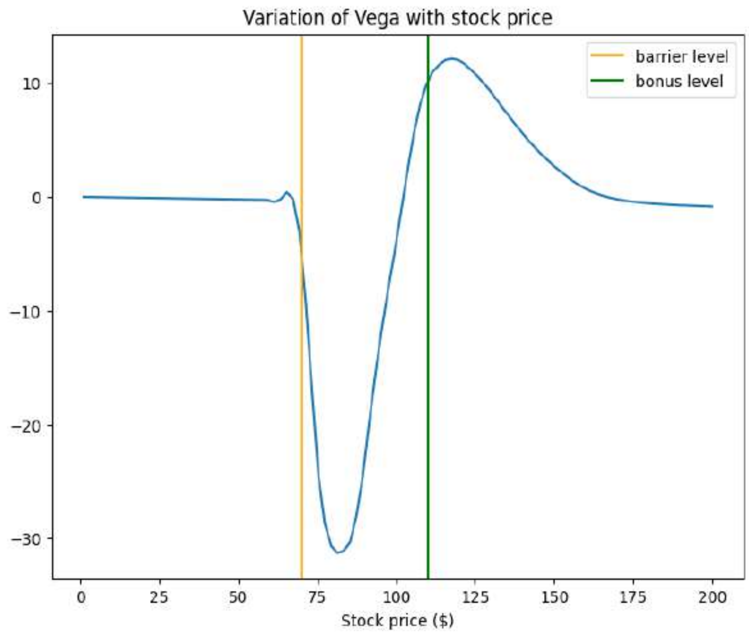


Figure 7: Typical pattern for Vega

The price has a high sensitivity to the changes in the volatility around the barrier. Unlike the Delta, the Vega decreases highly at the barrier level and jumps considerably at the bonus level. When the price of the underlying is below the Barrier, an increase of volatility doesn't impact the option. On the contrary, one time the barrier crossed an increase of volatility increases the probability of getting below the barrier and thus decreases the price of the option. Close to the strike, the Vega is the highest which means that an increase of the volatility implies an increasing price.

2.3.6 Volga

The Volga indicates the rate at which an option's Vega changes as volatility changes. Vega changes rapidly with volatility if Volga is high. Vega remains relatively steady despite volatility shifts if Volga is low.

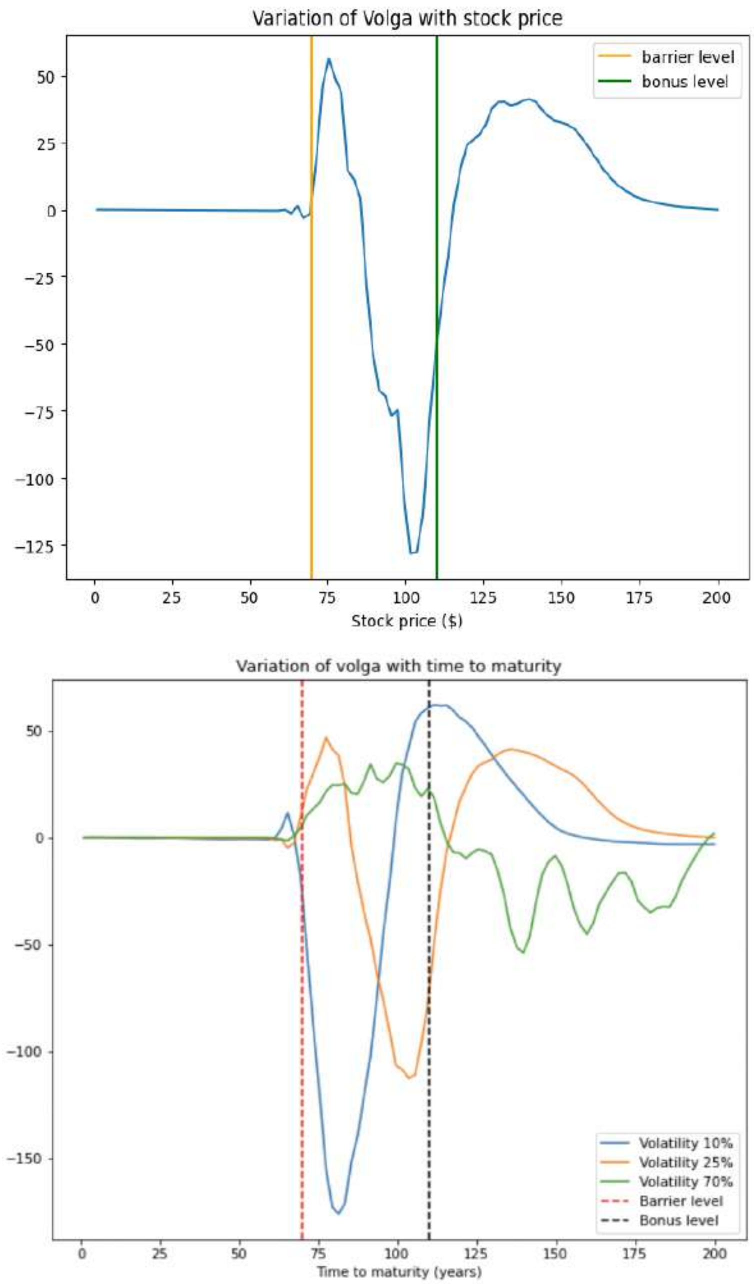


Figure 8: Typical pattern for Volga

This graph shows that the Vega of the product is very sensitive to a change in volatility between the barrier and the bonus level. We can see the higher the volatility the flatter the Volga curve. It involves that the Vega of the product is more sensitive to a change in volatility when the volatility is low and when the underlying price is near to the barrier.

3 Conclusion

In conclusion, our exploration of sensitivity analysis, focusing on the Greeks Delta, Gamma, Theta, Vega, and Volga, has provided valuable insights into the dynamics of a Bonus Certificate under the Heston model. These Greeks serve as indispensable tools in risk management, offering a nuanced understanding of how the option's value responds to changes in underlying parameters.

Numerical computations using Monte Carlo methods allowed us to analyze the price and sensitivities of a Bonus Certificate under specific parameters. The price exhibited interesting relationships based on the underlying asset's position relative to the barrier and bonus levels. The Delta, Gamma, Theta, Vega, and Volga displayed distinctive patterns, emphasizing the option's sensitivity to changes in stock price, time to maturity, and implied volatility.

To go further: We used a model with constant interest rate, we could apply a model with a variable one and we could compare our results with another model such as the Black-Merton-Scholes model or with the Dupire's approach.