Delta-Hedging

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1 Introduction

This project is personal, and it has been made in parallel of the MASEF (ex DEA-MASE) program at the Paris Dauphine University. The objective is to simulate the delta-hedging strategy on at first a simple portfolio with European Call Option. The objective is to use the knowledge acquired during the Monte Carlo course at Paris Dauphine and in structured product, stochastic calculus. The first part is on the delta hedging algorithm on the European option, and the second part is about this strategy over a family of stochastic differential equation.

2 European option case

2.1 Black-Merton-Scholes model

Let $T > 0$, a time horizon and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ a filtered space. We consider the stochastic differential equation

$$\begin{cases} dS_t = \mu S_t dt + \sigma S_t dB_t \\ S_0 = s_0, \end{cases} \tag{1}$$

where S denote our risky asset in the Black-Merton-Scholes model, $B = \{B_t, t \geq 0\}$ is a brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$. We let μ and $\sigma > 0$ two real constants where μ is the instant return of the asset and σ its volatility. We consider also a riskless asset denoted $(S_t^0)_{t\geq 0}$ where

$$S_t^0 = S_0^0 e^{rt},$$

$r \geq 0$ a constant which represent the interest rate of the model. We'll consider also for more simplicity that $S_0^0 = 1$, and that during all the project, no arbitrage opportunity exists.

Remark Because $b : (t,x) \mapsto \mu x$ and $\theta : (t,x) \mapsto \sigma x$ are clearly homogeneous and lipchitz, hence the stochastic differential equation (1) admits a unique solution and it exists.

2.1.1 Equivalent martingale measure

To evaluate the price of an European option, let say an European Call option, we have to build an equivalent martingale measure such that the discounted process $(\tilde{S}_t)_{t\geq 0} = (\frac{S_t}{S_t^0})_{t\geq 0}$ is a martingale under this new measure. First, let recall that the discounted process verifies the stochastic differential equation

$$\begin{cases} d\tilde{S}_t = (\mu - r)\tilde{S}_t dt + \sigma \tilde{S}_t dB_t \\ \tilde{S}_0 = 1. \end{cases} \tag{2}$$

Let consider $\lambda^* = \frac{r-\mu}{\sigma}$, then the exponential process $(Z_t^{\lambda^*})_{t\geq 0} = \left(\exp\left(B_t \lambda^* - t\frac{(\lambda^*)^2}{2}\right)\right)_{t\geq 0}$ is a martingale under $\mathbb{P}$ because of the Novikov criterion. Thanks to the Girsanov's Theorem, we can define $\forall A \in \mathcal{F}_T$ a new equivalent probability to $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$, $\mathbb{Q}$ such that $\mathbb{Q}(A) = \mathbb{E}[Z_T^{\lambda^*} 1_A]$, under which the process $B^* = (B_t^*)_{0\leq t\leq T}$ defined by

$$B_t^* = B_t - \lambda^* t$$

is a $(\mathcal{F}_t)_{0\leq t\leq T}$-Brownian motion under $\mathbb{Q}$. Therefore, under this new measure, it's easy to see that the stochastic differential equation (1) becomes

$$\begin{cases} dS_t = rS_t dt + \sigma S_t dB_t^* \\ S_0 = s_0, \end{cases} \tag{3}$$

and that the stochastic differential equation (2) becomes

$$\begin{cases} d\tilde{S}_t = \sigma \tilde{S}_t dB_t^* \\ \tilde{S}_0 = 1. \end{cases} \tag{4}$$

Therefore $(\tilde{S}_t)_{0\leq t\leq T}$ is clearly a martingale under $\mathbb{Q}$ since the drift is null and we have

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \sigma^2 \tilde{S}_t^{\,2} dt\right] = \sigma^2 \int_0^T \mathbb{E}^{\mathbb{Q}}\left[\tilde{S}_t^{\,2}\right] < \infty$$

as $\tilde{S}_t = \exp(-\frac{\sigma^2}{2}t + \sigma B_t^*)$.

2.1.2 Strategy and portfolio framework

We consider a financial strategy where you can chose to buy in the risk-less asset or the risky one. We define the financial strategies by two stochastic processes

$$\Theta := \{\Theta_t,\ t \in [0,T]\}$$
$$\theta := \{\theta_t,\ t \in [0,T]\},$$

where Θ and θ denote respectively the number of risk-less and risky asset you have in your portfolio at time t. Therefore, the portfolio's value can be defined as the process

$$V_t = \Theta_t S_t^0 + \theta_t S_t,\ \tilde{V}_t = \Theta_t + \theta_t \tilde{S}_t$$
$$dV_t = \Theta_t dS_t^0 + \theta_t dS_t,\ d\tilde{V}_t = \theta_t d\tilde{S}_t.$$

Thus, under $\mathbb{Q}$, we have

$$\begin{cases} d\tilde{V}_t = \theta_t \sigma \tilde{S}_t dB_t^* \\ \tilde{V}_0 = \dfrac{x}{S_0^0}, \end{cases} \tag{5}$$

we denote $(V_t^{x,\theta})_{0\leq t\leq T}$ the solution of this stochastic differential equation. We define also $\mathcal{O}$ the set of the acceptable strategies such that

$$\mathcal{O} := \left\{(\theta_t)_{0\leq t\leq T} \text{ progressively measurable} \middle| \mathbb{E}^{\mathbb{Q}}\left[\int_0^T (\theta_t \tilde{S}_t)^2 dt\right] < \infty\right\}.$$

Let fix $\theta \in \mathcal{O}$, therefore from (5), $(\tilde{V}_t)_{0\leq t\leq T}$ is a $\mathbb{Q}$-martingale.

2.2 European option pricing

Let suppose that we're currently in a complete market such that for every payoff G, there exists $(x,\theta) \in \mathbb{R} \times \mathcal{O}$ such that $V_T^{x,\theta} = G$. Let recall that for an European Option, the payoff G is either $(S_T - K, 0)_+$ for a call or $(K - S_T, 0)_+$ for a put, where K is the strike of the option, T the maturity. Thanks to the call-put parity, we just have to compute the price of the European Call Option to get the put's price. For the computation, we consider the payoff $G = (S_T - K, 0)_+$ and we denote C_t^G the value of the European Call Option in the Black-Merton-Scholes model at time t. We denote also, because of the completeness of the market, (x^G, θ^G) the parameter for the replication such that $V_T^{x^G,\theta^G} = G$. We've seen that the discounted portfolio's process was a $\mathbb{Q}$-martingale, therefore we have

$$\tilde{V_T}^{x^G,\theta^G} = Ge^{-rT} = C_T^G e^{-rT},$$

thus

$$\begin{aligned}
e^{-rt}C_t^G &= \mathbb{E}^{\mathbb{Q}}[\tilde{V}_T^{x^G,\theta^G}|\mathcal{F}_t] \\
\Leftrightarrow C_t^G &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[(S_T-K,0)_+|\mathcal{F}_t].
\end{aligned}$$

But, $S_t = S_0 e^{(\mu-\frac{\sigma^2}{2})t+\sigma B_t}$, so $S_T = S_t e^{(\mu-\frac{\sigma^2}{2})(T-t)+\sigma(B_T-B_t)}$, therefore

$$C_t^G = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[(S_t e^{(\mu-\frac{\sigma^2}{2})(T-t)+\sigma(B_T-B_t)} - K)_+|\mathcal{F}_t].$$

We denote now $C_t^G = u(t, S_t)$, hence by independence

$$\begin{aligned}
u(t,x) &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[(S_t e^{(\mu-\frac{\sigma^2}{2})(T-t)+\sigma(B_T-B_t)} - K)_+|S_t = x] \\
&= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[(S_t e^{(r-\frac{\sigma^2}{2})(T-t)+\sigma(B_T^*-B_t^*)} - K)_+|S_t = x] \\
&= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[(x e^{(r-\frac{\sigma^2}{2})(T-t)+\sigma(B_T^*-B_t^*)} - K)_+] \\
&= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[(x e^{(r-\frac{\sigma^2}{2})(T-t)+\sigma\sqrt{T-t}Y} - K)_+]
\end{aligned}$$

where $Y \sim \mathcal{N}(0,1)$. So we get

$$u(t,x) = e^{-r(T-t)}\int_{\mathbb{R}}\Big(x e^{(r-\frac{\sigma^2}{2})(T-t)+\sigma\sqrt{T-t}y} - K\Big)_+ \frac{e^{\frac{-y^2}{2}}}{\sqrt{2\pi}}dy.$$

We have to find y such that $x e^{(r-\frac{\sigma^2}{2})(T-t)+\sigma\sqrt{T-t}y} \geq K$. Therefore

$$\begin{aligned}
& x e^{(r-\frac{\sigma^2}{2})(T-t)+\sigma\sqrt{T-t}y} \geq K \\
\Leftrightarrow\ & (r-\frac{\sigma^2}{2})(T-t) + \sigma\sqrt{T-t}y \geq \ln\big(\frac{K}{x}\big) \\
\Leftrightarrow\ & y \geq \frac{\ln\big(\frac{K}{x}\big) - (r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} = -d_2.
\end{aligned}$$

Hence

$$\begin{aligned}
u(t,x) &= e^{-r(T-t)}\int_{-d_2}^{\infty}\Big(x e^{(r-\frac{\sigma^2}{2})(T-t)+\sigma\sqrt{T-t}y} - K\Big)\frac{e^{\frac{-y^2}{2}}}{\sqrt{2\pi}}dy \\
&= e^{-r(T-t)}\int_{-\infty}^{d_2}\Big(x e^{(r-\frac{\sigma^2}{2})(T-t)-\sigma\sqrt{T-t}y} - K\Big)\frac{e^{\frac{-y^2}{2}}}{\sqrt{2\pi}}dy \\
&= e^{-r(T-t)} x e^{(r-\frac{\sigma^2}{2})(T-t)}\int_{-\infty}^{d_2} e^{-\sigma\sqrt{T-t}y}\frac{e^{\frac{-y^2}{2}}}{\sqrt{2\pi}}dy - e^{-r(T-t)}K\mathcal{N}(d_2) \\
&= x\int_{-\infty}^{d_2}\frac{1}{\sqrt{2\pi}}e^{\frac{-(y+\sigma\sqrt{T-t})^2}{2}}dy - e^{-r(T-t)}K\mathcal{N}(d_2) \\
&= x\int_{-\infty}^{d_2+\sigma\sqrt{T-t}}\frac{1}{\sqrt{2\pi}}e^{\frac{-z^2}{2}}dz - e^{-r(T-t)}K\mathcal{N}(d_2) \\
&= x\mathcal{N}(d_1) - e^{-r(T-t)}K\mathcal{N}(d_2)
\end{aligned}$$

where $d_1 = d_2 + \sigma\sqrt{T-t}$ and $\mathcal{N}(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}} dz$. Finally, we have

$$u(t, S_t) = S_t \mathcal{N}(d_1) - e^{-r(T-t)} K \mathcal{N}(d_2)$$

with

$$d_2 = \frac{\ln\left(\frac{S_t}{K}\right) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}. \tag{6}$$

We conclude that

$$C_t(S, T, K) = S_t \mathcal{N}(d_1) - e^{-r(T-t)} K \mathcal{N}(d_2),$$

and thanks to the call-put parity, we have for the put

$$P_t(S, T, K) = -S_t \mathcal{N}(-d_1) + K e^{-r(T-t)} \mathcal{N}(-d_2).$$

3 Delta-hedging idea

3.1 Delta hedged call option framework

The main idea of delta-hedging is to hedge your position against the **small** change of the underlying in the option in your portfolio. The delta (Δ) of an option is defined as the rate of change of the option price with respect to the price of the underlying. Mathematically speaking, it is the partial derivative of the price of your option with respect to the underlying. For example, let say the that the delta of your option is 0.5, it means that if the underlying change by an amount, the option's price will change by 50% of this amount. We denote

$$\Delta = \frac{\partial c}{\partial S},$$

where c is the price of the option and S the price of the underlying. So simplify, let say that the European Call option's price c depends only on the underlying S, and the time t, we denote $c(S_t, t)$ its price at time t. Thanks to the Taylor's expansion, we have

$$\Delta c(S_t, t) = \frac{\partial c}{\partial S}(S_t, t)\Delta S_t + \frac{\partial c}{\partial t}(S_t, t)\Delta t + \frac{1}{2}\Big(\frac{\partial^2 c}{\partial S^2}(S_t, t)(\Delta S_t)^2 + \frac{\partial^2 c}{\partial t^2}(S_t, t)(\Delta t)^2\Big) + \frac{\partial^2 c}{\partial S \partial t}(S_t, t)\Delta S_t \Delta t + ...$$

Now consider the hedged portfolio

$$P_t = c(S_t, t) + \alpha S_t$$

where α is a constant. Thanks to the Taylor's expansion, we have

$$\Delta P_t = \Big(\frac{\partial c}{\partial S}(S_t, t) + \alpha\Big)\Delta S_t + \frac{\partial c}{\partial t}(S_t, t)\Delta t + \frac{1}{2}\Big(\frac{\partial^2 c}{\partial S^2}(S_t, t)(\Delta S_t)^2 + \frac{\partial^2 c}{\partial t^2}(S_t, t)(\Delta t)^2\Big) + \frac{\partial^2 c}{\partial S \partial t}(S_t, t)\Delta S_t \Delta t + ...$$

The risk comes from S so we want to remove the high order risk, hence we chose $\alpha = -\frac{\partial c}{\partial S}(S_t, t)$. With this delta-hedging strategy, we hence have in the Black-Merton-Scholes model

$$dP_t = \left(\frac{\partial c}{\partial t}(S_t, t) + \mu S_t \frac{\partial c}{\partial S_t}(S_t, t) + \alpha\mu S_t + \frac{\sigma^2 S_t^2}{2}\frac{\partial^2 c}{\partial S_t^2}(S_t, t)\right)dt + \left(\frac{\partial c}{\partial S_t}(S_t, t) + \alpha\sigma S_t\right)dB_t,$$

with our delta-hedging constant, we have now

$$dP_t = \left(\frac{\partial c}{\partial t}(S_t,t) + \frac{\sigma^2 S_t^2}{2}\frac{\partial^2 c}{\partial S_t^2}(S_t,t)\right)dt.$$

Since we considered at the beginning that there weren't any arbitrage opportunities, we have to get

$$dP_t = rP_t dt.$$

Therefore, we have

$$dP_t = \left(rc(S_t,t) - r\frac{\partial c}{\partial S_t}(S_t,t)S_t\right)dt = \left(\frac{\partial c}{\partial t}(S_t,t) + \frac{\sigma^2 S_t^2}{2}\frac{\partial^2 c}{\partial S_t^2}(S_t,t)\right).$$

Hence, we must get

$$rc(S_t,t) = \frac{\partial c}{\partial t}(S_t,t) + \frac{\sigma^2 S_t^2}{2}\frac{\partial^2 c}{\partial S_t^2}(S_t,t) + r\frac{\partial c}{\partial S_t}(S_t,t)S_t.$$

It's not the goal of this project, but it's good to know that $\frac{\partial^2 c}{\partial S_t^2}(S_t,t)$ denote the Gamma of the option, which is very useful to hedge a portfolio against the big changes of the value of the underlying.

Informally, what we just proved is that, if you want your portfolio delta neutral to get hedged, if the delta of your portfolio is let say 60%, you have to sell 60% of the notional.

3.2 Delta of a call option

To apply the delta-hedging strategy on a portfolio with call option in our case, we have to compute the delta of this option. Thankfully, we have an explicit formula for this Greek and it is

$$\frac{\partial c}{\partial S_t}(S_t,t) = \mathcal{N}(d_1),$$

where d_1 has been defined at (6).

Proof.

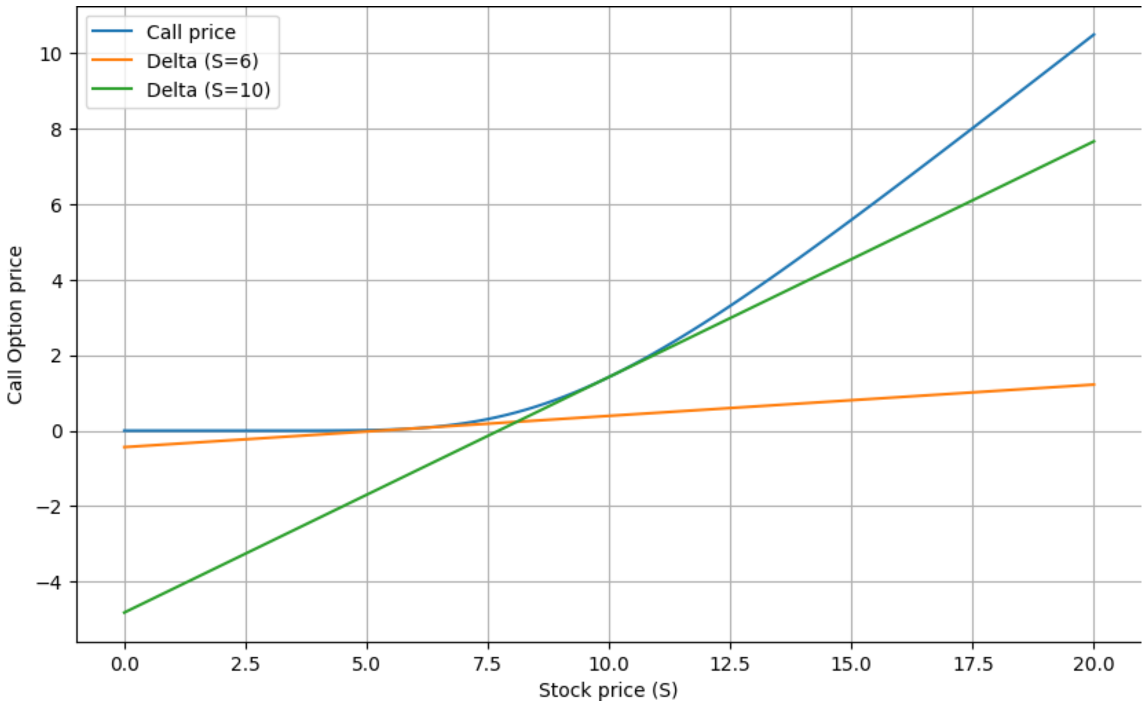
$$\begin{aligned}
\frac{\partial c}{\partial S_t}(S_t,t) &= \mathcal{N}(d_1) + S_t\frac{\partial \mathcal{N}(d_1)}{\partial S_t} - \frac{\partial \mathcal{N}(d_2)}{\partial S_t}Ke^{-r(T-t)} \\
&= \mathcal{N}(d_1) + S_t\frac{\partial \mathcal{N}(d_1)}{\partial d_1}\frac{\partial d_1}{\partial S_t} - \frac{\partial \mathcal{N}(d_2)}{\partial d_2}\frac{\partial d_2}{\partial S_t}Ke^{-r(T-t)} \\
&= \mathcal{N}(d_1) + S_t\frac{e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}} \times \frac{1}{S_t\sigma\sqrt{T-t}} - \frac{e^{-\frac{d_2^2}{2}}}{\sqrt{2\pi}} \times \frac{1}{S_t\sigma\sqrt{T-t}}Ke^{-r(T-t)} \\
&= \mathcal{N}(d_1) + \frac{1}{\sqrt{2\pi}S_t\sigma\sqrt{T-t}}\left[S_t e^{-\frac{d_1^2}{2}} - e^{-\frac{(d_1-\sigma\sqrt{T-t})^2}{2}}Ke^{-r(T-t)}\right] \\
&= \mathcal{N}(d_1) + \frac{e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}S_t\sigma\sqrt{T-t}}\left[S_t - Ke^{-\frac{\sigma^2(T-t)}{2}+d_1\sigma\sqrt{T-t}-r(T-t)}\right] \\
&= \mathcal{N}(d_1) + \frac{e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}S_t\sigma\sqrt{T-t}}\left[S_t - S_t e^{(r+\frac{\sigma^2}{2})(T-t)}e^{-\frac{\sigma^2(T-t)}{2}-r(T-t)}\right] \\
&= \mathcal{N}(d_1).
\end{aligned}$$

□

3.2.1 In practice

Due to transaction costs in reality we can't delta hedge continuously, therefore we have to choose when we want to delta hedge, every week or every day for instance. In practice, the delta is the slope of the Call Option price, and we can see it in figure 1 with stock prices $S = 6$ and $S = 10$.

Figure 1: Delta Hedging with parameters: $K = 10,\ \sigma = 30\%,\ r = 5\%,\ T = 1$.



For the numerical experiment, we chose $K = 100,\ \sigma = 20\%,\ r = 5\%,\ T = 20/52,\ S_0 = 95$ for the Black-Merton-Scholes model, also we consider that there are no transaction costs. We are the investor and we wrote 100 000 European Call Options with 20 weeks maturity that we want them delta hedged where the spot price is $S_0 = \$95$, we chose arbitrarily to delta hedge every week until the maturity. Thanks to the Black-Merton-Scholes formula, the price of a such option is $\$3,385$. In the case where the stock price closes below the strike at maturity, we have for a trajectory of the underlying in Figure 2, and the case where the stock close above the strike at maturity is in Figure 3.

Figure 2: Delta Hedging when to stock close below the strike at maturity.

Week	Stock price	Delta	Shares purchased	Shares held	Cost of shares purchased	Cumulative costs including interest cost	Interest costs
0	95.0	0.4221	42211.76	42211.76	4010117.62	4010117.62	3857.74
1	95.85	0.4446	2246.35	44458.11	215303.79	4229279.15	4064.86
2	94.66	0.3973	-4728.83	39729.28	-447636.35	3785707.66	3634.23
3	89.84	0.2306	-16672.72	23056.56	-1497843.78	2291498.11	2193.31
4	92.04	0.29	5939.42	28995.98	546666.0	2840357.42	2719.2
5	94.96	0.3846	9460.75	38456.73	898387.58	3741464.2	3583.45
6	93.88	0.3348	-4976.76	33479.97	-467229.99	3277817.66	3133.97
7	89.77	0.1829	-15192.26	18287.72	-1363773.12	1917178.51	1822.02
8	91.19	0.2142	3131.91	21419.63	285594.3	2204594.83	2096.76
9	90.46	0.1766	-3762.85	17656.78	-340401.77	1866289.82	1769.3
10	91.46	0.1936	1698.24	19355.02	155313.5	2023372.62	1918.71
11	93.85	0.2684	7487.83	26842.85	702712.68	2728004.0	2594.72
12	94.68	0.2877	1928.05	28770.9	182540.55	2913139.28	2770.32
13	93.23	0.2042	-8350.47	20420.43	-778525.85	2137383.75	2021.38
14	93.88	0.2086	442.27	20862.7	41519.34	2180924.46	2061.32
15	87.67	0.022	-18658.11	2204.59	-1635765.06	547220.73	487.71
16	88.65	0.019	-305.36	1899.23	-27069.79	520638.65	461.67
17	88.25	0.0059	-1310.25	588.98	-115625.66	405474.66	350.44
18	83.65	0.0	-588.61	0.37	-49237.72	356587.39	303.07
19	81.14	0.0	-0.37	0.0	-29.9	356860.56	303.05
20	77.87	0.0	-0.0	-0.0	-0.0	357163.61	0.0

Figure 3: Delta Hedging when to stock close above the strike at maturity.

Week	Stock price	Delta	Shares purchased	Shares held	Cost of shares purchased	Cumulative costs including interest cost	Interest costs
0	95.0	0.4221	42211.76	42211.76	4010117.62	4010117.62	3857.74
1	94.24	0.3899	-3218.01	38993.75	-303253.62	3710721.73	3566.01
2	91.72	0.2985	-9142.5	29851.25	-838541.06	2875746.68	2759.33
3	89.23	0.2129	-8557.86	21293.39	-763625.28	2114880.73	2024.72
4	88.13	0.1725	-4046.61	17246.78	-356642.37	1760263.08	1681.63
5	90.11	0.2172	4468.64	21715.42	402654.55	2164599.26	2068.98
6	88.0	0.1467	-7044.22	14671.2	-619864.47	1546803.77	1472.67
7	89.97	0.1888	4211.79	18882.99	378924.09	1927200.54	1837.2
8	86.76	0.0951	-9376.71	9506.28	-813515.2	1115522.54	1054.6
9	89.15	0.1383	4320.58	13826.86	385159.59	1501736.73	1425.12
10	91.0	0.1783	4000.26	17827.12	364018.66	1867180.51	1775.31
11	94.54	0.2983	12005.7	29832.81	1135016.98	3003972.79	2867.2
12	100.12	0.5608	26249.1	56081.91	2628157.87	5634997.86	5395.49
13	101.9	0.6498	8893.54	64975.45	906235.23	6546628.57	6267.28
14	101.9	0.6541	436.67	65412.12	44498.03	6597393.89	6310.09
15	104.21	0.7803	12621.71	78033.83	1315296.9	7919000.88	7575.41
16	110.49	0.971	19068.68	97102.51	2106974.27	10033550.56	9602.32
17	109.45	0.9752	414.72	97517.23	45389.25	10088542.12	9645.98
18	110.35	0.9951	1988.15	99505.38	219392.67	10317580.77	9857.04
19	112.4	1.0	493.61	99998.99	55479.54	10382917.35	9910.41
20	113.16	1.0	1.01	100000.0	114.62	10392942.38	0.0

In the case the option closes out of the money (Figure 2), the cost of the hedging strategy is $\$357,163$, so at maturity the option won't be exercised. Because we wrote 100 000 options at price $\$3,385$ and that the strategy costs $\$357,163$, we are totalling around $\$18,663$ of loses. In the case the option closes in the money (Figure 3), the cost of the hedging strategy is $\$10,392,942$, and it's clear that the option will be exercised and as delta approaches 1, we are covering our position. At the maturity, we give all the shares that we owe and we receive $\$K \times 100,000 = \$10,000,000$. Because we wrote $100,000$ options, we are totalling around $54500 of loses. We are always losing money using this hedging strategy, and it's because delta hedging a short position involves selling stock just after the price goes down, and buying stock just after the price goes up. To reduce this cost, there are two main possibilities, rebalance our position more frequently, but this imply more transaction cost, or we can predict if the stock will rather rise, or rather fall and pre delta hedge.

3.3 Remark, Euler Scheme

In the first part, we were considering the classic stochastic differential equation (3), under which we had an explicit formula for S;

$$S_t = s_0 e^{(r-\frac{\sigma^2}{2})t+\sigma B_t},$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion under $\mathbb{Q}$. But, in reality we can face some other stochastic differential equation where we can't compute the explicit formula. In this case, we have to use Monte Carlo methods to simulate the stochastic differential equation with Euler scheme. Let's consider a stochastic differential equation of the form

$$\begin{cases} dS_t = b(S_t)dt + \theta(S_t)dB_t \\ S_0 = s_0, \end{cases} \tag{7}$$

where $b : \mathbb{R} \to \mathbb{R}$ and $\theta : \mathbb{R} \to \mathbb{R}$ are lipchitz, so that there exists a unique solution to (7). We are also fixing

$$0 = t_0 < t_1 < ... < t_n = T$$

such that $t_{i+1} - t_i = \frac{T}{n}$. Hence, the Euler scheme gives us a construction to simulate (7) and we have the following construction

$$\begin{aligned} \bar{S}^n_{t_{i+1}} &= \bar{S}^n_{t_i} + b(\bar{S}^n_{t_i})\Delta t + \theta(\bar{S}^n_{t_i})\Delta B_i \\ \bar{S}^n_0 &= s_0, \end{aligned} \tag{8}$$

where $\Delta t = t_{i+1} - t_i = \frac{T}{n}$ and $\Delta B_i \sim \mathcal{N}(0, \Delta t)$. We also introduce the continuous Euler scheme as

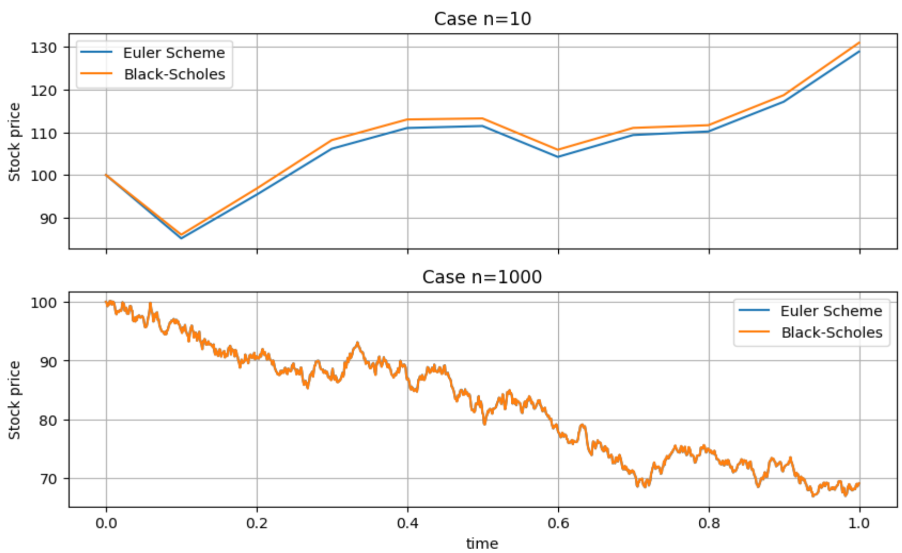
$$\bar{S}^n_t = s_0 + \int_0^t b(\bar{S}^n_{\phi(u)})du + \int_0^t \theta(\bar{S}^n_{\phi(u)})dB_u,$$

where $\phi(u) = \max\{t_i, t_i < t\}$. Let $p \geq 1$, [2, p. 340] give us an order for the strong error and it holds

$$\mathbb{E}\left[\sup_{t\in[0,T]} |S_t - \bar{S}^n_t|^p\right]^{\frac{1}{p}} \leq \frac{C}{\sqrt{n}},$$

C a constant, it ensures that our approximation is good. Under this construction, we can now simulate every risky asset following (7) and apply our delta hedging strategy on this asset. For example, if we apply this construction to simulate (3), we have $b : x \mapsto xr$ and $\theta : x \mapsto x\sigma$ that are clearly lipchitz. For numerical experiments, we chose $\sigma = 30\%,\ r = 5\%, s_0 = 100,\ m_1 = 10,\ m_2 = 1000$ and $T = 1$.

Figure 4: Difference from Euler scheme and explicit formula.

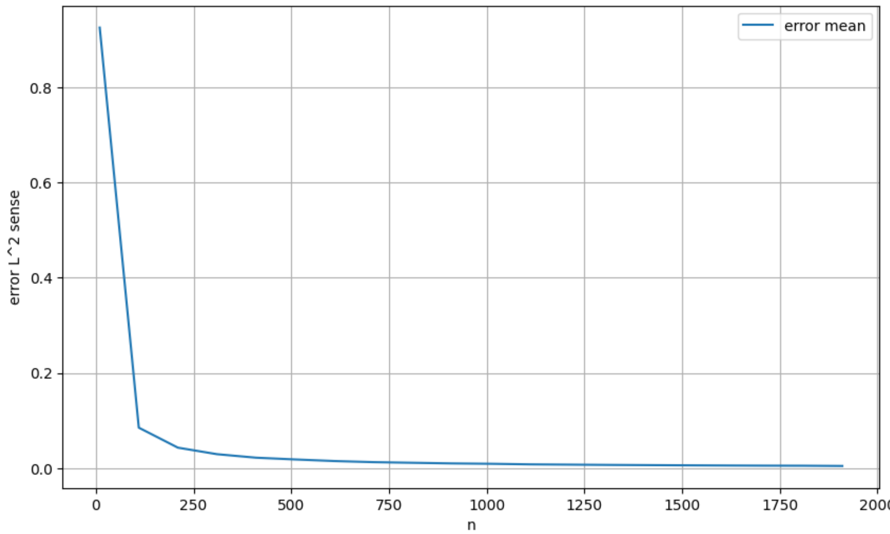


It's clear that as n increase, the difference between the two trajectories tends to 0. To model this result, we will compute with Monte Carlo methods the error in the L^2 sense which is

$$\mathbb{E}\left[\left(\bar{S}_T^n - S_T\right)^2\right] \approx \frac{1}{m}\sum_{i=1}^{m}\left(\bar{S}_T^{n,i} - S_T^i\right)^2$$

as n increase, and we fix $m = 10^4$, so we get

Figure 5: Error in L^2 sense with Monte Carlo approximation.



Numerically, we showed that

$$\lim_{n \to +\infty} \mathbb{E}\left[\left(\bar{S}_T^n - S_T\right)^2\right] = 0.$$

Then, we can just use the delta hedging strategy as before with the Euler scheme to model the hedging strategy over a portfolio with European Options.

4 Conclusion

Through this small project, we studied the delta hedging strategy over a simple portfolio with European Options and we briefly talked about the case where the underlying was following a different stochastic differential equation from the Black-Merton-Scholes model thanks to the Euler scheme.

Aspects to improve We could improve the hedging strategy in increasing the rebalancing frequency or in predicting in the future if the stock will rather rise of fall to pre delta hedge to prevent the cost as we are in a short position.

5 Bibliography

References

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