Paris Dauphine University

Monte Carlo and Finite Difference Methods

MIDO, MASEF's program

Lookback Options

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1 Introduction

In financial markets, Options are divided into two main groups: European Options and American Options. The former gives the investors the right to exercise the option only at maturity, while the latter gives the investor the right to exercise the option anytime until maturity. In this project we are working on a particular case of European Option which are Lookback Options. They constitute an example of path-dependent options based on the extremal values of their underlying. We will focus on the pricing of the Lookback Put Option which gives its owner the right to sell an underlying asset for the maximal price it has reached before the maturity. The payoff, therefore, is the amount by which the underlying's maximal price has exceeded its terminal price, making them highly sought after by investors for their reward potential, but also very expensive [2]. This project will be based on the comparative study of three different ways to compute the price of the Lookback Put, first with naive approach, then with a Brownian bridge approach and finally by using a finite difference method for partial differential equations.

2 Pricing with Monte Carlo

2.1 Mathematical Framework

We let $T > 0$ the maturity and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ a filtered probability space. We will place ourselves within the Black-Scholes model. This model is a classic framework used for modeling the value of a risky asset. It is based on several assumptions:

1. **Underlying Asset Price (X_t)**: The price of the underlying asset, X_t, follows a Geometric Brownian Motion with constant volatility σ and constant drift μ, driven by a standard Brownian Motion $(W_t)_t$. It satisfies the differential equation:

$$dX_t = \mu X_t \, dt + \sigma X_t \, dW_t$$

2. **No Arbitrage Opportunity**: There are no risk-free financial strategies that allow for the creation of certain wealth at a future date from an initial cost of zero.

3. **Continuous Time**: Time is treated as a continuous function.

4. **Short Selling**: It is possible to engage in short selling.

5. **No Transaction Costs**: There are no costs associated with transactions.

6. **Risk-Free Interest Rate**: There exists a risk-free interest rate r, which is known in advance and remains constant.

7. **Perfectly Divisible Underlying**: The underlying asset is perfectly divisible, meaning it's possible to purchase fractional parts of the option (e.g., 1/100 of an option).

8. **No Dividends**: In the case of a stock, it is assumed that there are no dividends paid between the time of the option's valuation and its expiration. For the purposes of the model, dividends are considered to be zero.

Under these assumptions, arbitrage-free pricing theory posits that every payoff is attainable : the market is complete, and there exists a unique risk-neutral probability measure $\mathbb{Q}$, under which the dynamic of the risky asset is

$$\begin{cases} dX_t = rX_t dt + \sigma X_t dB_t, \\ X_0 = x_0, \end{cases} \tag{1}$$

where $(B_t)_{t\in[0,T]}$ is a $\mathbb{Q}$-brownian motion. We are considering a Lookback Put Option which has the following payoff

$$G := \left(\max_{t\in[0,T]} \{X_t\} - X_T \right)^+ = \max\left(\max_{t\in[0,T]} \{X_t\} - X_T, 0 \right). \tag{2}$$

Let us denote by $p(0, x_0)$ the price at time $t = 0$ of the Lookback Put if its underlying asset's spot price is x_0. For convenience, we will generally write $p := p(0, x_0)$. According to the fundamental principles of arbitrage-free pricing

$$p = e^{-rT}\mathbb{E}^{\mathbb{Q}}[G].$$

In our numerical simulations, we will take the following values for the parameters :

$$x_0 = 100, r = 0, \sigma = 0.4, T = 1.$$

2.2 Naive Approach

A first simple idea is to use the approximation

$$\max_{t\in[0,T]} X_t \approx \max_{k\in[\![0,m]\!]} X_{t_k},$$

with $0 = t_0 < t_1 < ... < t_m = T$ with $t_{i+1} - t_i = \frac{T}{m}$ $\forall i \in [\![0, m-1]\!]$. Then the Law of Large Numbers ensures that

$$\begin{aligned} p = e^{-rT}\mathbb{E}^{\mathbb{Q}}[G] &= \lim_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{n}\left[e^{-rT}\left(\max_{t\in[0,T]}\{X_t^i\} - X_T^i\right)^+\right] \\ &\approx \lim_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{n}\left[e^{-rT}\left(\max_{k\in[\![0,m]\!]}\{X_{t_k}^i\} - X_T^i\right)^+\right] \end{aligned} \tag{3}$$

where $\left((X_t^i)_{t\in[0,T]}\right)_{i\in[\![1,n]\!]}$ are n independent and identically distributed trajectories of the underlying X over $[0, T]$. The naive Monte Carlo estimator is intuitively defined as

$$\overline{p}_n^{MC} = \frac{1}{n}\sum_{i=1}^{n}\left[e^{-rT}\left(\max_{k\in[\![0,m]\!]}\{X_{t_k}^i\} - X_T^i\right)^+\right]$$

With the parameters specified above, a number of simulations $n = 10,000$ and a number of time steps $m = 100$, we find the following estimated price and confidence interval at level 95% :

$$\overline{p}_n^{\mathrm{MC}} = 30.6855, \;\; \mathrm{CI}^{\mathrm{MC}} = [30.2845, 31.0866].$$

This method faces a major issue, it leads to a weak error that is suggested to be of order $O(1/\sqrt{m})$ (see [4]), in the time step m because the maximum of the continuous underlying path is taken to be the maximum on the discretized grid rather than simulated explicitly which leads to an underestimation of the price of the Lookback Put.

Intuitively :

$$\max_{k=0,\ldots,m} X_{t_k} << \max_{t\in[0,T]} X_t,$$

therefore

$$\overline{p}_n^{MC} << p$$

when m is small.

We will therefore resort to the Brownian Bridge approach to circumvent this limitation.

2.3 Brownian Bridge Approach

In this section we will use the Brownian bridge approach to compute the price of the Lookback option.

This approach consists in simulating directly $\max_{t\in[0,T]} X_t$ the maximum of the asset path, conditionally to its discretised sample path $(X_{t_k})_{k=0,\ldots,m}$. It is conjectured that this method leads to a convergence rate of order $O(\frac{1}{m})$ (see [3, 8.1]).

For the following theoretical explanations in this subsection, credit goes to [1, 2.5].

After having simulated a discrete sample path $(X_{t_k})_{k=0,\dots,m}$, we simulate the maximum of the continuous sample path

$$\max_{t\in[0;T]} X_t = \max_{k=0,\dots,m-1} \max_{t\in[t_k;t_{k+1}]} X_t$$

conditionally to $(X_{t_k})_{k=0,\dots,m}$. By the Markov property of diffusions, we have for $k \in [\![0, m-1]\!]$

$$\mathcal{L}\big(\max_{t\in[t_k;t_{k+1}]} X_t|(X_{t_l})_{l=0,\dots,m}\big) = \mathcal{L}\big(\max_{t\in[t_k;t_{k+1}]} X_t|X_{t_k}, X_{t_{k+1}}\big).$$

We know that the cumulative density function of this distribution is given by :

$$F_m(M; x_k, x_{k+1}) := \mathbb{P}\big(\max_{t\in[t_k;t_{k+1}]} X_t \leq M|X_{t_k} = x_k, X_{t_{k+1}} = x_{k+1}\big) = 1 - \exp\Big(-2\frac{m}{T}\frac{(M-x_k)(M-x_{k+1})}{\sigma^2(x_k)}\Big)$$

for $M \geq \max\{x_k, x_{k+1}\}$. Besides, $F_m(\cdot; x_k, x_{k+1})$ is invertible on $[\max\{x_k, x_{k+1}\}; +\infty)$, and we get for $u \in (0; 1)$

$$\begin{aligned}
& F_m(M; x_k, x_{k+1}) = u \\
\iff & \log(1-u) = -2\frac{m}{T}\frac{(M-x_k)(M-x_{k+1})}{\sigma^2(x_k)} \\
\iff & -\frac{T}{2m}\sigma^2(x_k)\log(1-u) = M^2 - M(x_k + x_{k+1}) + x_k x_{k+1} \\
\iff & -\frac{T}{2m}\sigma^2(x_k)\log(1-u) = \big(M - \frac{x_k + x_{k+1}}{2}\big)^2 + x_k x_{k+1} - \frac{x_k + x_{k+1}}{4} \\
\iff & -\frac{T}{2m}\sigma^2(x_k)\log(1-u) = \big(M - \frac{x_k + x_{k+1}}{2}\big)^2 - \frac{(x_{k+1} - x_k)^2}{4} \\
\iff & M = \sqrt{\frac{1}{4}(x_{k+1} - x_i)^2 - \frac{T}{2m}\sigma^2(x_i)\log(1-u)} + \frac{x_{k+1} + x_k}{2}.
\end{aligned}$$

Therefore, we have

$$F_m^{-1}(u; x_k, x_{k+1}) := \sqrt{\frac{1}{4}(x_{k+1} - x_i)^2 - \frac{T}{2m}\sigma^2(x_i)\log(1-u)} + \frac{x_{k+1} + x_k}{2}.$$

We can hence simulate $\big(\max_{t\in[t_k;t_{k+1}]} X_t\big)_{k=0}^{m-1}$ conditionally given $(X_{t_k})_{k=0}^m = (x_k)_{k=0}^m$ by

$$\big(\max_{t\in[t_k;t_{k+1}]} X_t\big)_{k=0}^{m-1} \text{ knowing } \{(X_{t_k})_{k=0}^m = (x_k)_{k=0}^m\} \overset{\mathcal{L}}{=} (F_m^{-1}(U_k; x_k, x_{k+1}))_{k=0}^{m-1},$$

where $(U_k)_{k=0,\dots m-1}$ is a sequence off iid uniform variables on $[0; 1]$.

We have effectively simulated the maximum of the continuous asset path :

$$\max_{t\in[0;T]} X_t = \max_{k=0,\dots,m-1} \max_{t\in[t_k;t_{k+1}]} X_t.$$

The Brownian Bridge estimator is then

$$\overline{p}^{\mathrm{BB}} = \lim_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{n}\left[e^{-rT}\left(\max_{k\in[\![0,m-1]\!]}\{F_m^{-1}(U_k^i; X_{t_k}^i, X_{t_{k+1}}^i)\} - X_{t_m}^i\right)^+\right].$$

We test the Brownian Bridge method with the same parameters and the same underlying paths. Numerical simulation with $n = 10.000$ gives for the price and the confidence interval at level 95%

$$\overline{p}_n^{\mathrm{BB}} = 30.6863,\ \ \mathrm{CI}^{\mathrm{BB}} = [30.2852, 31.0873].$$

As expected the Brownian Bridge approach is slightly above the naive method.

2.4 Comparison between Naive and Brownian Bridge Methods

Here is a comparison of the Naive and Brownian Bridge confidence intervals for different values of the time steps m. Every time with the same number of simulations $n = 10.000$ and the same parameters for the model:

m	Naive Method CI	Brownian Bridge CI
10	[23.779, 24.562]	[23.782, 24.564]
50	[28.905, 29.707]	[28.906, 29.708]
100	[30.234, 31.041]	[30.235, 31.042]
500	[31.800, 32.606]	[31.800, 32.607]
1000	[32.106, 32.913]	[32.106, 32.913]

The Naive method always underestimates the price of the option in comparison to the Brownian Bridge method. The edge conferred by using the Brownian Bridge appears thin at glance, and does not motivate the use of this much more computationally expensive approach ($n(m-1)$ supplementary random variables need to be simulated because we are simulating the maximum of the underlying on every time step of $[0, T]$). The full potential of the Brownian Bridge method is appreciated in high-volatility scenarios, in which the underlying asset can fluctuate sharply during small-time steps and reach extreme values that may be missed by coarser sampling intervals.

2.5 Study of the Lookback Put

In this part, we will analyse the behaviour of the Lookback Put. We will use the Brownian Bridge approach to approximate better the price of the option. We will plot the estimated price against the spot price of the underlying and against the volatility, all other parameters kept constant.

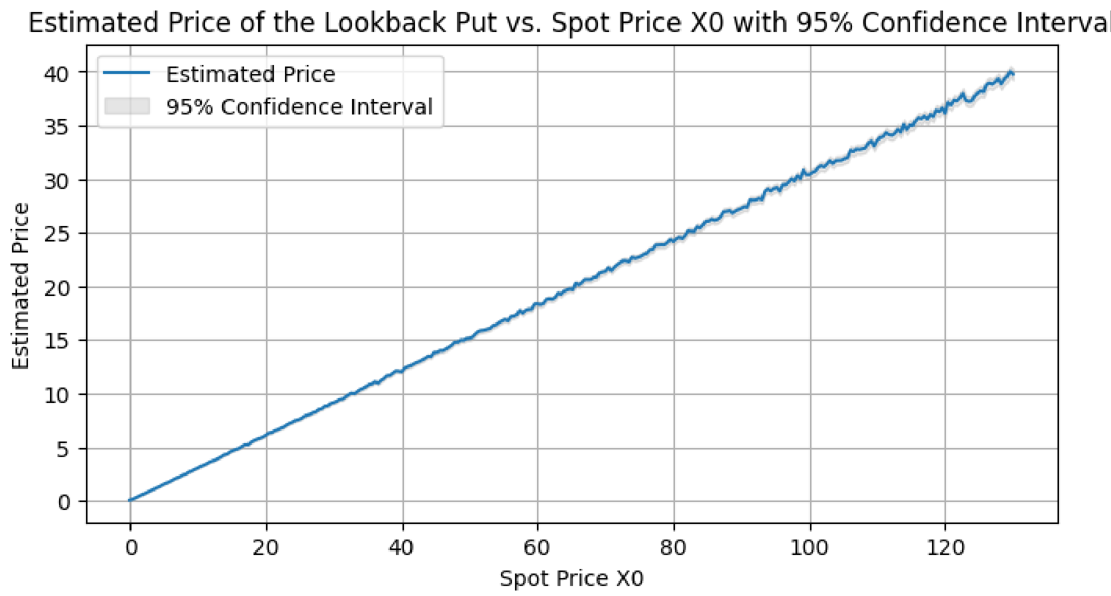

Figure 1: Graph of $\overline{p_n}^{BB}$ w.r.t. X_0 for $r = 0, \sigma = 0.4, T = 1, n = 10^4, m = 100$

The price of the Lookback Put exhibits a notable linear (of directing coefficient ≈ 0.3 which we will confirm thereafter) and non-decreasing tendency against the spot price. This makes sense because the Lookback Put's payoff is proportional to the spot price, and the higher the spot price, the higher the range of values the asset can explore during the option's lifetime.

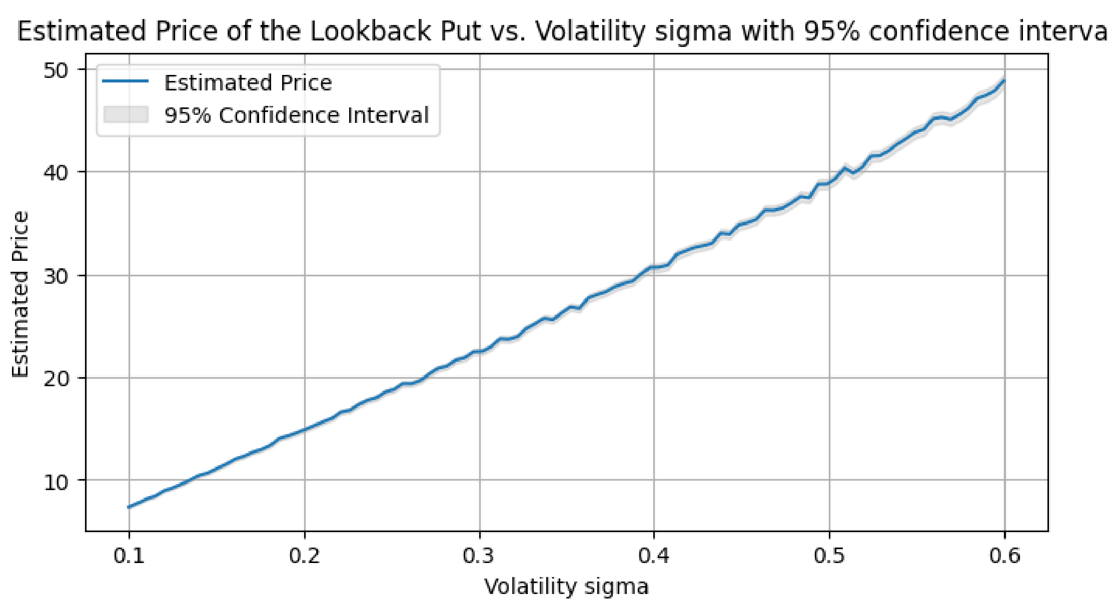

Figure 2: Graph of $\overline{p_n}^{BB}$ w.r.t. σ for $X_0 = 100, r = 0, T = 1, n = 10^4, m = 100$

Again, we observe a clear linear and non-decreasing pattern for the price w.r.t. the

Volatility. The non-decreasing pattern is very intuitive because the higher the volatility, the more fluctuating the asset and the more its maximum price will exceed its terminal price.

To conclude this section, let us estimate the delta of the Lookback Put. The Delta (Δ) refers to the sensitivity of the price of an option with respect to the price of its underlying :

$$\Delta = \frac{\partial p}{\partial x}.$$

It is fundamental in Finance as it enables to devise hedging strategies, a concept known as *Delta-Hedging*. To estimate the delta, a simple strategy consists in using the finite difference approximation of the derivative and plugging into it the corresponding prices estimated via Monte Carlo method :

$$\Delta \approx \frac{p(0, x_0 + \epsilon) - p(0, x_0 - \epsilon)}{2\epsilon} \approx \frac{\overline{p}_n^{BB}(0, x_0 + \epsilon) - \overline{p}_n^{BB}(0, x_0 - \epsilon)}{2\epsilon},$$

for a small $\epsilon > 0$. After numerical simulation, we get the following graph

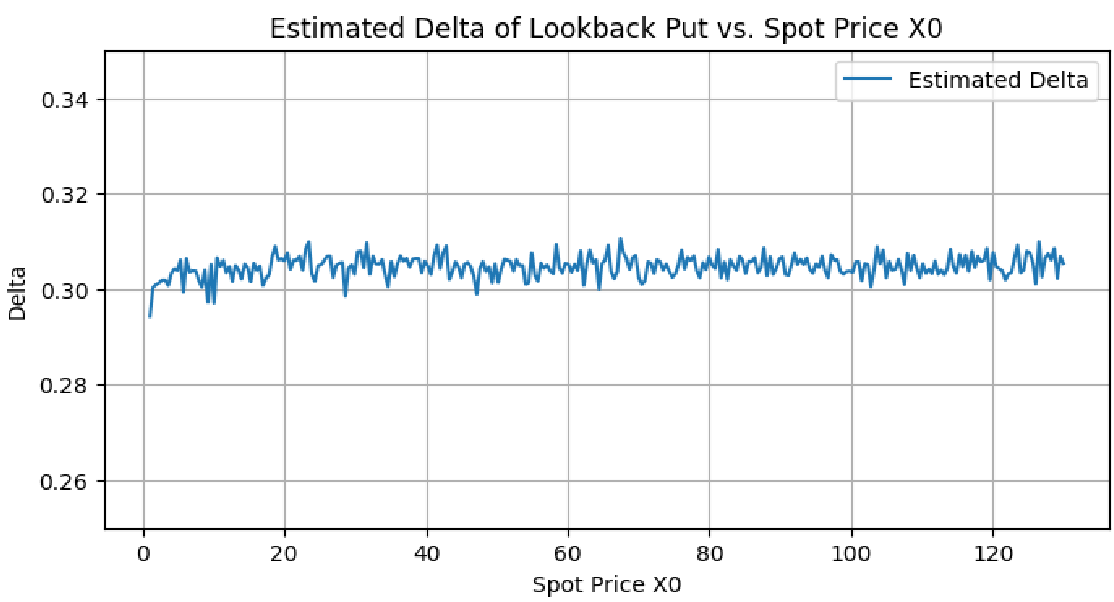

Figure 3: Delta (Δ) w.r.t. the spot price X_0

As we could have guessed from the linear nature of 1, the delta of the Lookback Put is a constant function of the spot price of values approximately equal to 0.305.

3 PDE approach

3.1 Framework

Let's denote $J(T) := \max_{t\in[0,T]}\{X_t\}$, then if we denote P as the the price of the option, then the payoff 2 becomes

$$P(X, J, T) := J(T)\max\left(1 - \frac{X_T}{J(T)}, 0\right) = J(T)W(\xi, T)$$

and we may have a solution of the form

$$P(X, J, t) := J(t)W(\xi, t)$$

where $W(\xi, T) = \max(1 - \xi, 0)$ and $\xi = \frac{X}{J}$. With these definitions, we find that W follows the partial differential equation given by [5, p. 215]

$$\frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2\xi^2\frac{\partial^2 W}{\partial \xi^2} + r\xi\frac{\partial W}{\partial \xi} - rW = 0 \text{ in } [0, T) \times (0, 1), \tag{4}$$

which is the Black-Scholes partial differential equation. The Dirichlet conditions are given by [5, p. 215]

$$\begin{cases} W(0, t) = e^{-r(T-t)}, \\ W(1, t) = \dfrac{\partial W}{\partial \xi}(1, t). \end{cases} \tag{5}$$

and the terminal condition is

$$W(\xi, T) = \max(1 - \xi, 0).$$

3.2 Discretization

Let us introduce the mesh

$$\Big\{(x_i, t_n) := (i\delta, nh);\ 0 \le i \le l+1,\ 0 \le n \le m\Big\},$$

where $h = \frac{T}{m}$ and $\delta = \frac{1}{l+1}$ to approximate $(W(x,t))_{(x,t)\in[0,1]\times[0,T]}$ by a finite dimensional matrix $(W_i^n)_{0\le i\le l+1,\, 0\le n\le m}$ such that $W_i^n \approx W(x_i, t_n)$. Because of 5, we let for all $n \in [\![0, m]\!]$ and for all $i \in [\![0, l+1]\!]$

$$W_0^n = e^{-r(T-t_n)},\ W_{l+1}^n = \frac{\partial W_{l+1}^n}{\partial \xi} \text{ and } W_i^m = \max(1 - x_i, 0).$$

We approximate the derivatives by finite differences. Namely, let us fix $n \in [\![0, m-1]\!]$ and $i \in [\![1, l]\!]$, we have

$$\frac{\partial W}{\partial \xi}(x_i, t_n) \approx \frac{W(x_i + \delta, t_n) - W(x_i - \delta, t_n)}{2\delta} \approx \frac{W_{i+1}^n - W_{i-1}^n}{2\delta},$$

and

$$\frac{\partial^2 W}{\partial \xi^2}(x_i,t_n) \approx \frac{W(x_i+\delta,t_n)+W(x_i-\delta,t_n)-2W(t_n,x_i)}{\delta^2}$$
$$\approx \frac{W^n_{i+1}+W^n_{i-1}-2W^n_i}{\delta^2}.$$

Due to the boundaries denoted in 5, we must have for $n \in [\![0,m-1]\!]$

$$\frac{\partial W^n_{l+1}}{\partial \xi} \approx \partial_\xi W(1,t_n) \approx \frac{W(1,t_n)-W(1-\delta,t_n)}{\delta}$$
$$\approx \frac{W^n_{l+1}-W^n_l}{\delta} \approx W^n_{l+1}$$
$$\Rightarrow W^n_{l+1} \approx \frac{W^n_l}{1-\delta}.$$

Therefore, thanks to 4, we have with the implicit scheme for $(i,n) \in [\![1,l]\!] \times [\![0,m-1]\!]$

$$\frac{W^{n+1}_i - W^n_i}{dt} + rx_i\frac{W^n_{i+1}-W^n_{i-1}}{2\delta} + \frac{\sigma^2 x_i^2}{2}\frac{W^n_{i+1}+W^n_{i-1}-2W^n_i}{\delta^2} - rW^n_i = 0,$$

which after being rewritten gives us

$$W^{n+1}_i = W^n_{i-1}\left(\frac{rx_ih}{2\delta} - \frac{\sigma^2x_i^2h}{2\delta^2}\right) + W^n_i\left(1+rh+\frac{\sigma^2x_i^2h}{\delta^2}\right) + W^n_{i+1}\left(-\frac{rx_ih}{2\delta} - \frac{\sigma^2x_i^2h}{2\delta^2}\right).$$

Therefore, we fix the tridiagonal matrix $\mathcal{A}_\delta \in \mathbb{R}^{l\times l}$ with $\mathcal{A}_\delta = (a_{i,j})_{(i,j)\in[\![1,l]\!]^2}$ given by

$$\begin{pmatrix}
(1+rh+\frac{\sigma^2x_1^2h}{\delta^2}) & (-\frac{rx_1h}{2\delta}-\frac{\sigma^2x_1^2h}{2\delta^2}) & \cdots & \cdots & \cdots \\
(\frac{rx_2h}{2\delta}-\frac{\sigma^2x_2^2h}{2\delta^2}) & (1+rh+\frac{\sigma^2x_2^2h}{\delta^2}) & (-\frac{rx_2h}{2\delta}-\frac{\sigma^2x_2^2h}{2\delta^2}) & \cdots & \cdots \\
\cdots & \ddots & \ddots & \ddots & \cdots \\
\cdots & \cdots & (\frac{rx_{l-1}h}{2\delta}-\frac{\sigma^2x_{l-1}^2h}{2\delta^2}) & (1+rh+\frac{\sigma^2x_{l-1}^2h}{\delta^2}) & (-\frac{rx_{l-1}h}{2\delta}-\frac{\sigma^2x_{l-1}^2h}{2\delta^2}) \\
\cdots & \cdots & \cdots & (\frac{rx_lh}{2\delta}-\frac{\sigma^2x_l^2h}{2\delta^2}) & (1+rh+\frac{\sigma^2x_l^2h}{\delta^2}+B)
\end{pmatrix}$$

where $B = (-\frac{rx_lh}{2\delta}-\frac{\sigma^2x_l^2h}{2\delta^2})\frac{1}{1-\delta}$ because of the Dirichlet conditions (5), hence if we denote $W^n = (W^n_i)_{i\in[\![1,l]\!]}$ we must have

$$W(\cdot,t_{n+1}) \approx W^{n+1} = \mathcal{A}_\delta W^n + f^n_\delta,$$

where we define for $n \in [\![0,m-1]\!]$ f^n_δ as follow: $f^n_\delta = (f(x_i,t_n))_{i\in[\![1,l]\!]}$ where f is null everywhere except for $i=1$,

$$f^n_{\delta,1} = \left(\frac{rx_1h}{2\delta} - \frac{\sigma^2x_1^2h}{2\delta^2}\right)e^{-r(T-t_n)}.$$

Therefore, we have the following relationship

$$W^n = \mathcal{A}_\delta^{-1}(W^{n+1} - f^n_\delta),$$

and we can solve recursively for $n = m-1, ..., 0$.

3.3 Convergence

The convergence of the finite difference method relies on two properties. The first one is the consistency. In our case, this condition is fulfilled as the implicit scheme is always consistent in order 2 in space and 1 in time. The second property is the stability of the scheme. To establish it, we have to check that there exists $C > 0$ independent of $\delta, h > 0$ such that

$$|W^n|_\infty \leq C|W^m|_\infty \ \forall n \in [\![0, m-1]\!],$$

to ensure the stability for the L^∞-norm of the scheme. In fact, we can easily check that $g(x) = \sigma x$ is uniformly elliptic which gives us the stability of the implicit scheme.

3.4 Analytical interpretation

For numerical simulation, we took the following parameters $r = 0$, $\sigma = 0.4$, $T = 1.0$ and $m, l = 200, 1000$. At time 0, we get the following graph which denote the value of $W(\cdot, 0)$

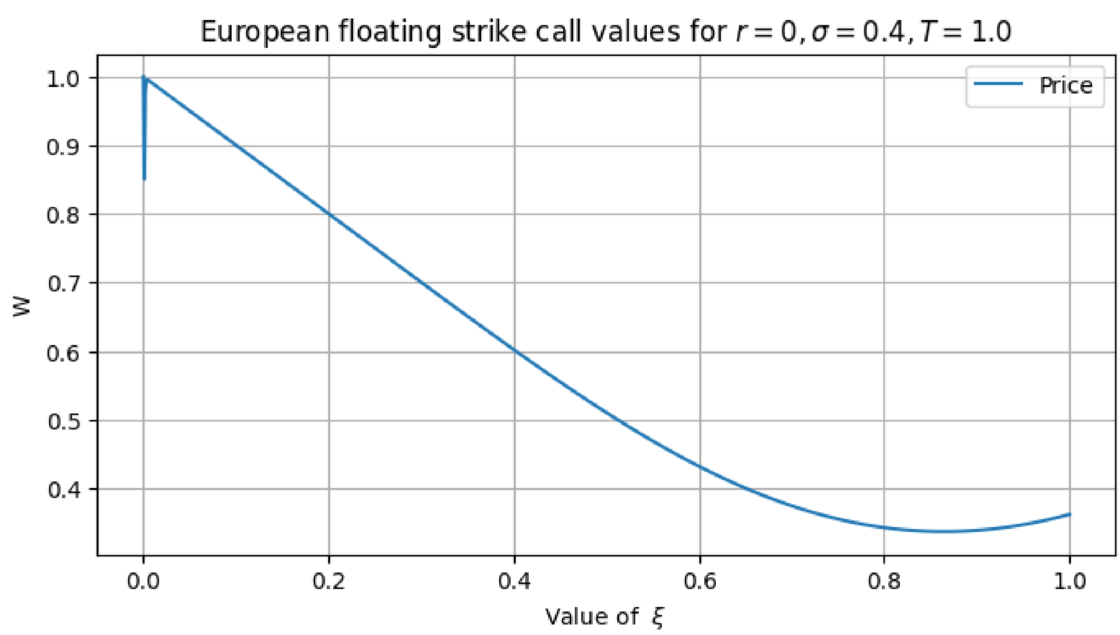

Figure 4: Lookback price with respect to ξ

To get the price of the European Lookback put with floating strike at time 0, we must have $\xi = 1$ because $\xi = \frac{X_0}{J(0)} = 1$, and we find $W(1, 0) \approx 0.3617$, then the price of the option has to be for $X_0 = 100$,

$$P(X, J, 0) = J(0)W(1, 0) \approx 36.17.$$

To get the price of the option at time t, we need the current price X_t and the current maximum price $J(t)$, then we must have $\xi = \frac{X_t}{J(t)}$, thus the price has to be

$$J(t)W(\xi, t).$$

The values of $W(\xi, t)$ are plotted as a surface thanks to the finite difference method and we get the following graph

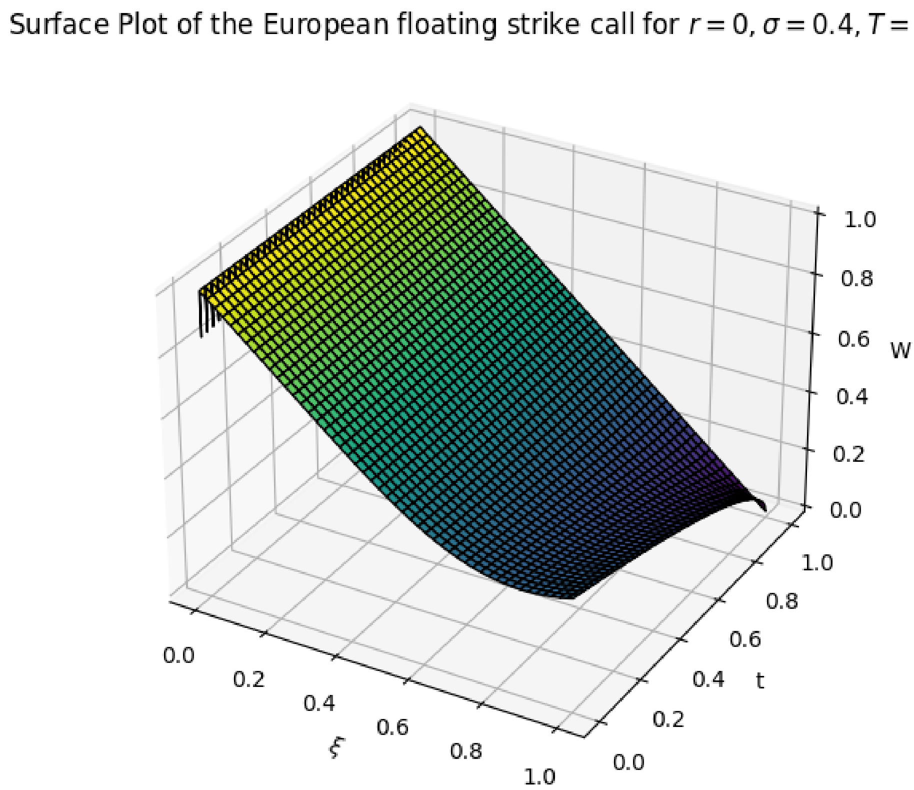

Figure 5: Surface Lookback option price

4 Conclusion

In conclusion, the pricing of European lookback put options with a floating strike using various computational methods was the focus of this project. The review thought about the Guileless methodology, which discretizes the basic resource's way, with the Brownian Scaffold approach, which recreates the greatest resource cost expressly. Using an implicit finite difference scheme, a partial differential equation method was also investigated.

Because the naive method always underestimates the cost, the Monte Carlo simulations demonstrated that the Brownian Bridge method typically provides estimates that are more accurate than the Naive method. The straight relationship saw in the Lookback Put choice cost concerning the spot cost and unpredictability was made sense of naturally inside the Black Scholes system.

Moreover, the PDE approach gave a numerical structure to evaluating Lookback Put choices, and its mathematical execution exhibited the development of choice costs over the long run and hidden resource costs. The finite difference method's consistency and stability were confirmed by the analytical interpretation of the results.

In outline, this task introduced a complete report on Lookback Put Choice evaluating, offering bits of knowledge into various computational techniques and their applications. The examinations and investigations gave important data to financial backers and monetary experts in understanding the intricacies and subtleties related with evaluating way subordinate choices. An aspect that could be explored is the pricing thanks to the tree method, with binomial or trinomial trees.

Aspect to improve

1. Implement Variance reduction methods (Importance sampling, Control Variates, Antithetic Control,...).

2. Use a faster programming language such as C, or use Cython because Python as an interpreted language is usually slower than low-level language such as C or C++.

3. Because of the implementation of the implicit scheme, we have to solve a linear system with a tridiagonal matrix $\mathcal{A}_\delta \in \mathbb{R}^{l\times l}$ with a lot of zeros. We could use sparse matrix for this case.

5 Bibliography

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