Trapped Ion Quantum Computing

David Skocic

December 9 2025

Basics of Quantum Computing

Qubits

- Analog to the classical bit (Qubit = Quantum bit)
- Any two-state quantum system can theoretically be a qubit
- We use our familiar Dirac notation to represent states

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Qubits can have a superposition unlike a classical bit

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \alpha \, |0\rangle + \beta \, |1\rangle$$

Electron Spin

- Electron spin is a familiar two-state quantum system
- Recall our spin operators for the different dimensions

$$S_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad S_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad S_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$ are the z-basis eigenvectors
- Observe how these basis vectors can map to the Pauli spin matrices

$$|0\rangle\langle 0| + |1\rangle\langle 1| = I \qquad |0\rangle\langle 1| + |1\rangle\langle 0| = \sigma_x$$
$$i(|0\rangle\langle 1| - |1\rangle\langle 0|) = \sigma_y \qquad |1\rangle\langle 1| - |0\rangle\langle 0| = \sigma_z$$

Unitary Operators

- The evolution of a quantum system is described by unitary operators

$$|\psi_t\rangle = U(t, t_0)\,|\psi_0\rangle$$

- Unitary Operators are *mathematically reversible*:

$$UU^\dagger = I$$

- You can apply unitary operators to a qubit to alter its state in a desirable manner

Quantum Gates

- In quantum computing, unitary operators are usually called **quantum gates**
- Many of the basic quantum gates are look very familiar!

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- There are also less familiar gates such as the *Hadamard* Gate:

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Quantum Gates

- The X gate is also called the NOT gate:

$$NOT\,|0\rangle = |1\rangle \text{ and } NOT\,|1\rangle = |0\rangle$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \beta \\ \alpha \end{bmatrix}$$

- The Hadamard gate creates a superposition:

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

Quantum Circuits

- A series of quantum gates constitutes a **quantum circuit**
- It is standard to represent these as if they were on a wire

$$\Updownarrow$$

$$ZYH\left|0\right\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -i \\ -i \end{bmatrix}$$

Basics of Trapped Ion Quantum Systems

Why Use Trapped Ions?

- Qubits need to be isolated from the rest of the universe
- An ion is much easier to manipulate than a neutral atom
- They have exceptionally long coherence times
- Quantum gates have been implemented with exceptional fidelity
- Quantum states are easy to prepare and measure
- The physics behind them is well understood and relatively simple

Encoding a Trapped Ion with a Qubit

Take any two long-lived states of different energies $|g\rangle$ and $|e\rangle$

Just like spin, these states can map to the Pauli spin matrices:

$$|g\rangle\langle g| + |e\rangle\langle e| = I \qquad |g\rangle\langle e| + |e\rangle\langle g| = \sigma_x$$
$$i(|g\rangle\langle e| - |e\rangle\langle g|) = \sigma_y \qquad |e\rangle\langle e| - |g\rangle\langle g| = \sigma_z$$

We can think of $|g\rangle$ as $|0\rangle$ and $|e\rangle$ as $|1\rangle$

The two-level Hamiltonian is represented by

$$\hat{H} = \hbar(\omega_g |g\rangle\langle g| + \omega_e |e\rangle\langle e|) \rightarrow \hat{H} = \hbar\frac{\omega_0}{2}\sigma_z$$

Measuring a Qubit

- Detection of states involves photon scattering
- States are carefully chosen so one ”bright” state scatters incoming photons while the other has no effect
- The trajectories of the photons are detected, making is easy to tell which qubits scattered the photons

What Ions Make Good Qubits?

- There are a few features that are desirable in an ion:
  - Hydrogen-like atomic structure
  - Singly ionized species
  - Metastable states in the visible spectrum
- This makes heavier alkali earth metals a common choice

Rabi Frequencies

The Rabi frequency comes up when a two-state system is placed in an electric field

This frequency is the rate at which the the system 'flops' between states

Rabi frequencies are a way to measure the strength of the coupling of the states and the field applied defined by

$$\Omega = -\frac{\langle e|\,\vec{\mu}\cdot\vec{\epsilon}\,E_0\,|g\rangle}{\hbar}$$

Classical Ion Trapping

Classical Ion Trapping

- Static electric fields cannot trap an ion
  - Gauss's law $\nabla \cdot E = 0 \rightarrow$ electric potential has no local minima
  - This is overcome by varying the electric field with time
- These traps typically have potentials oscillating in the radio frequency range
- Construct a trap with a time-varying and time-independent part:

$$V(x,y,z,t) = U\frac{1}{2}(\alpha x^2+\beta y^2+\gamma z^2)+\tilde{U}\cos(\omega_{rf}t)\frac{1}{2}(\alpha' x^2+\beta' y^2+\gamma' z^2)$$

Classical Ion Trapping

$$V(x,y,z,t) = U\frac{1}{2}(\alpha x^2 + \beta y^2 + \gamma z^2) + \tilde{U}\cos(\omega_{rf} t)\frac{1}{2}(\alpha' x^2 + \beta' y^2 + \gamma' z^2)$$

Laplace's Equation ($\nabla^2 V = 0$) says:

$$\alpha + \beta + \gamma = 0$$
$$\alpha' + \beta' + \gamma' = 0$$

Which is underconstrained – there are many ways to place electrodes

Classical Ion Trapping

FILL IN WITH EXAMPLE TRAP WITH PARAMS AND IMAGE

This is a Linear Paul Trap, with

$$-(\alpha + \beta) = \gamma$$
$$\alpha' = -\beta'$$

Classical Ion Trapping

- The problem can be one dimensionalized:

$$\ddot{x} = -\frac{Ze}{m}\frac{\partial V}{\partial x} = -\frac{Ze}{m}[U\alpha + \tilde{U}\cos(\omega_{rf}t)\alpha']x$$

- And transformed into a Mathieu differential equation:

$$\frac{d^2x}{d\xi^2} + [a_x - 2q_x\cos(2\xi)]x = 0$$

with

$$\xi = \frac{\omega_{rf}t}{2}, \quad a_x = \frac{4ZeU\alpha}{m\omega_{rf}^2}, \quad q_x = \frac{2Ze\tilde{U}\alpha'}{m\omega_{rf}^2}$$

Classical Ion Trapping

Mathieu differential equations have solution

$$x(\xi) = Ae^{i\beta_x\xi}\sum_{n=-\infty}^{\infty} C_{2n}e^{i2n\xi} + B^{-i\beta_x\xi}\sum_{n=-\infty}^{\infty} C_{2n}e^{-i2n\xi}$$

where β_x and C_{2n} are functions of a_x and q_x.

Assuming $|a_x|, q_x^2 \ll 1$, $\beta \approx \sqrt{a_x + \frac{q_x^2}{2}}$, and (to first order):

$$x(t) \propto \cos(\beta_x \frac{\omega_{rf}}{2} t) \; [1 - \frac{q_x}{2}\cos(\omega_{rf} t)]$$

Quantum Ion Trapping

Quantum Ion Trapping

Begin with our same potential, but rewrite it:

$$V(t) = \frac{m}{2} W(t) \hat{x}^2 \quad \text{with} \quad W(t) = \frac{\omega_{rf}^2}{4} [a_x - 2q_x \cos(\omega_{rf} t)]$$

This looks familiar! We already know the solutions to

$$V(x) = \frac{1}{2} m \omega x^2$$

Maybe there are some raising and lowering operators...

Quantum Ion Trapping

Our Hamiltonian of motion can be written,

$$\hat{H}^{(m)} = \frac{\hat{p}^2}{2m} + \frac{m}{2}W(t)\hat{x}^2$$

First, we can write the operators in the Heisenberg picture:

$$\dot{\hat{x}}_H = \frac{1}{i\hbar}[\hat{x}, \hat{H}^{(m)}] = \frac{\hat{p}}{m}$$
$$\dot{\hat{p}}_H = \frac{1}{i\hbar}[\hat{p}, \hat{H}^{(m)}] = -mW(t)\hat{x}$$

which combine into a Mattheiu equation!

$$\ddot{\hat{x}}_H + W(t)\hat{x}_H = 0$$

$$\ddot{\hat{x}}_H + \frac{\omega_{rf}^2}{4}[a_x - 2q_x \cos(\omega_{rf} t)]\hat{x}_H = 0$$

Quantum Ion Trapping

Consider some other $u(t)$ that satisfies the same equation:

$$\ddot{u(t)} + W(t)\hat{u(t)} = 0$$

with boundary conditions $u(0) = 1, \ \dot{u}(0) = iv$

from before, with $A = 0$ and $B = 1$

$$u(t) = e^{i\beta_x \omega_{rf}/2} \sum_{n=-\infty}^{\infty} C_{2n} e^{in\omega_{rf}t} \equiv e^{i\beta_x \omega_{rf}/2} \Phi(t)$$

And applying the boundary conditions,

$$u(0) = \sum_{n=-\infty}^{\infty} C_{2n} = 1, \ \ v = \omega_{rf} \sum_{n=-\infty}^{\infty} C_{2n}(\beta_x/2 + n)$$

Quantum Ion Trapping

Since $u(t)$ and $\hat{x}$ satisfy the same differential equation, so

$$\hat{C}(t) = \sqrt{\frac{m}{2\hbar v}} i[u(t)\dot{\hat{x}} - \dot{u}(t)\hat{x}]$$

is proportional to their Wronskian identity, and is constant in time

$$\hat{C}(t) = \hat{C}(0) = \frac{1}{\sqrt{2m\hbar v}}(mv\hat{x}(0) + i\hat{p}(0)) = \hat{a}$$

We have recovered exactly the lowering operator!

Quantum Ion Trapping

Using this relation, we can write our operators in terms of $\hat{a}$ and $\hat{a}^\dagger$:

$$\hat{x}(t) = \sqrt{\frac{\hbar}{2mv}}[\hat{a}u^*(t) + \hat{a}^\dagger u(t)]$$

$$\hat{p}(t) = \sqrt{\frac{\hbar m}{2v}}[\hat{a}\dot{u}^*(t) + \hat{a}^\dagger \dot{u}(t)]$$

And just like we're used to, we have a state $\left|n=0\right\rangle_v$ such that

$$\hat{a}\left|n=0\right\rangle_v = 0$$

In the Schrödinger picture ($\hat{U}(t) = e^{-(i/\hbar)\hat{H}^{(m)}}$):

$$\hat{C}_S(t)\hat{U}(t)\left|n=0\right\rangle = \hat{C}_S(t)\left|n=0,t\right\rangle = 0$$

Quantum Ion Trapping

These states form a complete orthonormal basis via the creation operator:

$$|n, t\rangle = \frac{[\hat{C}_S^\dagger(t)]^n}{\sqrt{n!}} |n = 0, t\rangle$$

write ground state lowest order approx

Quantum Gate Implementation

Rotation Gates

We use lasers to change states

The total Hamiltonian can be written as a sum of the ion's internal electron state, its motional state, and the interaction with the laser:

$$\hat{H} = \hat{H}_E + \hat{H}_M + \hat{H}_L$$

The laser Hamiltonian is defined by Ω, ω, k, and ϕ

$$\hat{H}_L = \frac{\hbar\Omega}{2}(\hat{\sigma}^+ + \hat{\sigma}^-)[e^{i(k\hat{x}-\omega t+\phi)} + e^{-i(k\hat{x}-\omega t+\phi)}]$$

Rotation Gates

But this problem is much easier in the *interaction picture*

Let $\hat{H}_0 = \hat{H}_E + \hat{H}_M$ and $\hat{U}_0 = e^{-(i/\hbar)\hat{H}_0 t}$, so

$$\hat{H}_{int} = \hat{U}_0^\dagger \hat{H}_L \hat{U}_0.$$

After plugging in and much simplification and approximation

$$\hat{H}_{int}(t) = \frac{\hbar\Omega}{2}[e^{i\eta(\hat{a}^\dagger e^{ivt} + \hat{a}e^{-ivt})}\hat{\sigma}^+ e^{-i((\omega-\omega_0)t-\phi)} + h.c.]$$

We get resonance when $\omega - \omega_0$ is small and the ϕ term dominates

Rotation Gates

When the phase term dominates:

$$\hat{H}_{int} = \frac{\hbar\Omega}{2}(\hat{\sigma}^+ e^{i\phi} + \hat{\sigma}^- e^{-i\phi}) \equiv \frac{\hbar\Omega}{2}\hat{\sigma}_\phi$$

And this Hamiltonian describes rotation gates!

Let's put this back into the Schrödinger equation:

$$\frac{d}{dt}\left|\psi(t)\right\rangle = -\frac{i}{\hbar}\hat{H}\left|\psi(t)\right\rangle \to \left|\psi(t)\right\rangle = e^{-\frac{i\Omega}{2}t\hat{\sigma}_\phi}\left|\psi(0)\right\rangle \equiv \hat{U}(t)\left|\psi(0)\right\rangle$$

So we have a unitary operator evolving the system...

Rotation Gates

Letting $\Omega t \equiv \theta$:

$$\hat{U}(t) = \sum_{k=0}^{\infty} \frac{1}{k!}(-i\frac{\theta}{2}\hat{\sigma}_\phi)^k =$$

$$I \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\left(\frac{\theta}{2}\right)^{2k} - i\hat{\sigma}_\phi \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\left(\frac{\theta}{2}\right)^{2k+1} =$$

$$I \cos\left(\frac{\theta}{2}\right) - i\hat{\sigma}_\phi \sin\left(\frac{\theta}{2}\right) = \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2}e^{-i\phi} \\ -i\sin\frac{\theta}{2}e^{i\phi} & \cos\frac{\theta}{2} \end{pmatrix}$$

We recover the X, Y, and Z gates with the right choices of θ and ϕ!!