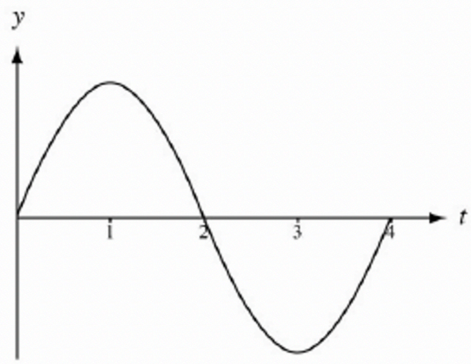
Calculus
2020-2021
Handout 19: Fundamental Theorem of Calculus

The graph of the function $y=f(t)$ is shown below. The function is defined for $0 \le t \le 4$ and has the following properties:

- The graph of f has odd symmetry around the point $(2,0)$.
- On the interval $[0,2]$, the graph of f is symmetric with respect to the line $t=1$.
- $\int_0^1 f(t)\,dt = \frac{4}{3}$.



Graph of $y=f(t)$

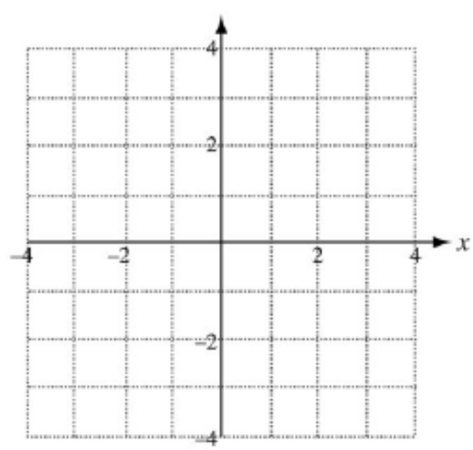
1. Let $F(x) = \int_0^x f(t)\,dt$.

a. Complete the following table of values.

x	0	1	2	3	4
F(x)					

b. Sketch your best estimate of the graph of F on the grid below.

$y = F(x)$

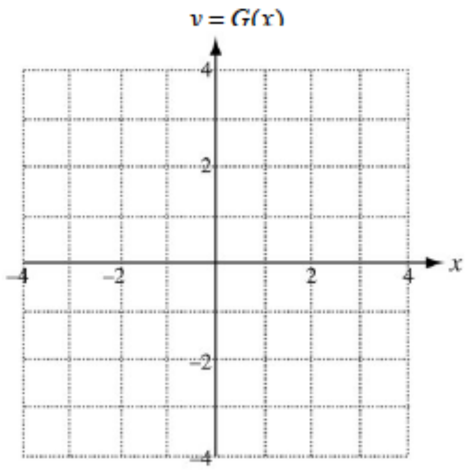


2. Let $G(x) = \int_2^x f(t)\,dt$.

a. Complete the following table of values.

x	0	1	2	3	4
$G(x)$					

b. Sketch the graph of G on the grid below.

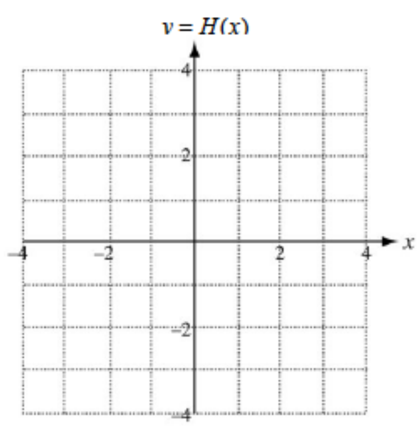


3. Let $H(x) = \int_4^x f(t)\,dt$.

a. Complete the following table of values.

x	0	1	2	3	4
$H(x)$					

b. Sketch the graph of H on the grid below.



4. Complete the following table.

	$F(x)$	$G(x)$	$H(x)$
The maximum value of the function occurs at what x-value(s)?			
The minimum value of the function occurs at what x-value(s)?			
The function increases on what interval(s)?			
The function decreases on what interval(s)?			

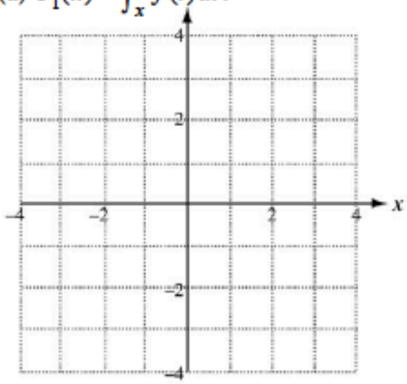
5. Although the tables in questions 1, 2, and 3 asked only for the three functions to be evaluated at integer values of x, those functions were all continuous on the domain of $0 \le x \le 4$. Refer back to the answers you gave for function F in the table above, and explain why you believe each of these answers is correct when one considers F on its entire domain. Write your arguments in the table below. Your explanations should not rely on the graphs you sketched.

	Justification of the answers above for $F(x)$
The maximum value of the function occurs at what x-value(s)?	
The minimum value of the function occurs at what x-value(s)?	
The function increases on what interval(s)?	
The function decreases on what interval(s)?	

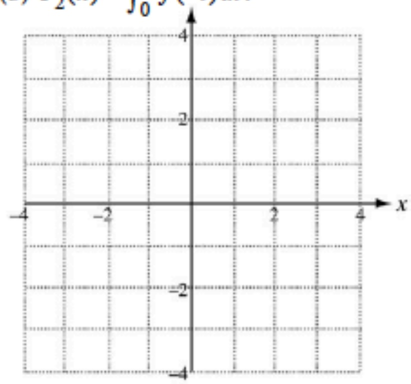
6. What conjectures would you make about the family of functions of the form $W(x) = \int_k^x f(t)\,dt$ for $0 \le k \le 4$, where f is the graph given at the beginning of this worksheet?

7. **Extend your understanding by sketching each of the following functions.**

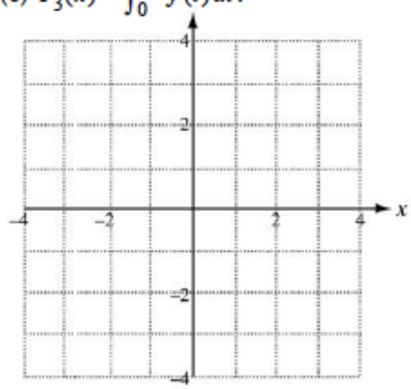
(a) $F_1(x) = \int_x^0 f(t)\,dt$.



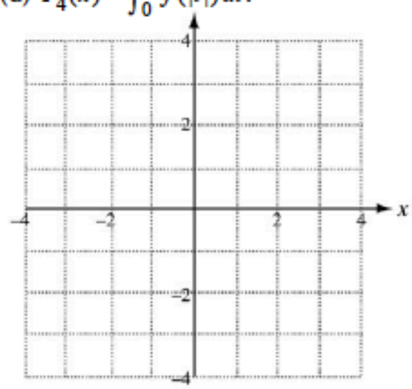
(b) $F_2(x) = \int_0^x f(-t)\,dt$.



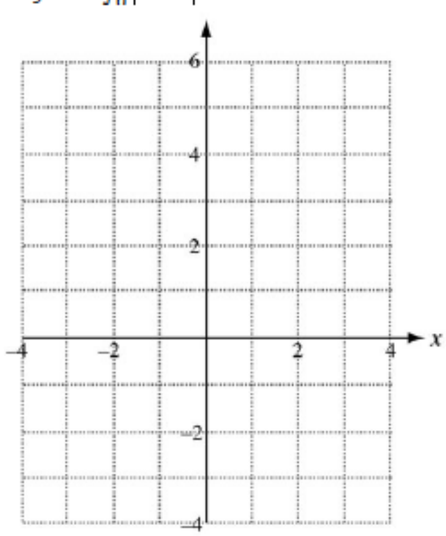
(c) $F_3(x) = \int_0^{2x} f(t)\,dt$.



(d) $F_4(x) = \int_0^x f(|t|)\,dt$.



(e) $F_5(x) = \int_0^x |f(t)|\,dt$.



The Mean Value Theorem for Integrals, Part 1

If $f(x)$ is continuous over an interval $[a, b]$, then there is at least one point $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx.$$

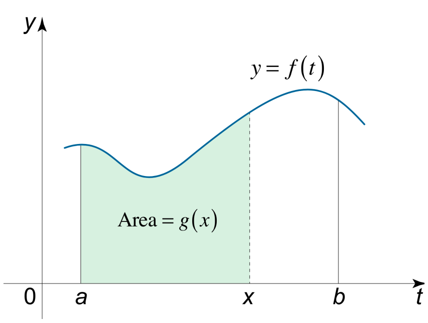
This formula can also be stated as

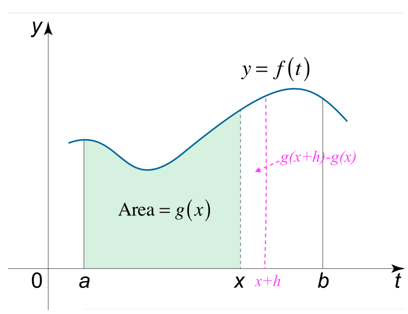
$$\int_a^b f(x)dx = f(c)(b-a).$$

The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\, dt \qquad a \leqslant x \leqslant b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.





Given $g(x) = \int_a^x f(t)dt$ then $g(x+h) = \int_a^{x+h} f(t)dt$

$$g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$g'(x) = \lim_{h\to 0}\frac{\int_a^{x+h} f(t)dt - \int_a^x f(t)dt}{h}$$

$$g'(x) = \lim_{h\to 0}\frac{\int_x^{x+h} f(t)dt}{h}$$

$g'(x) = \lim_{h\to 0}\frac{f(x)\cdot h}{h}$ since $\int_x^{x+h} f(t)dt \approx f(x)\cdot h \; using\ left\ hand\ approximation\ of\ the\ area$

$$g'(x) = f(x)$$

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The Fundamental Theorem of Calculus, Part 2 If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that $F' = f$.

$If\ m(x) = \int_a^{k(x)} f(t)dt\ \ then\ m'(x) = f\big(k(x)\big) \cdot k'(x)$

8. Please find the derivatives of the following integral functions
   a. $F(x) = \int_{-1}^{x^2} \sin(t^3 - 1)\,dt$

   b. $F(x) = \int_0^{2x} \ln(t-3)\,dt$

9. Use the following table to answer the questions below:

x	1	2	4	9
f(x)	-3	2	8	4
g(x)	7	11	15	12

   a. If $g(x) = \int_0^{x^2} f(t)dt$, find $g'(x)$.

   b. If $h(x) = g(x) + x^2$, use the table above to find $h'(1)$