1 | Cauchy-Schwarz Inequality important

'One of the most important inequalities in mathematics'

Suppose $u, v \in V$ (where V is an inner product space). Then

$$|\langle u, v\rangle| \leq \|u\|\|v\|$$

The inequality is an equality iff one of u, v is a scalar multiple of the other.

1.1 | intuition

For the Euclidean inner product, this is true because $\langle u, v\rangle = \|u\|\|v\|\cos\theta$. However, the Cauchy-Schwarz inequality works for all inner product spaces, using the generalized Pythagorean theorem (instead of the law of cosines).

1.2 | proof is by the orthogonal decomposition

By homogeneity of norms,

$$\left\|\frac{\langle u, v\rangle}{\|v\|}v\right\|^2 = \left|\frac{\langle u, v\rangle}{\|v\|}\right|^2 \|v\|^2$$

1.3 | results

1.3.1 | triangle inequality

Suppose $u, v \in V$. Then

$$\|u+v\| \leq \|u\| + \|v\|$$

The inequality is an equality if and only if one of u, v is a non-negative multiple of the other (degenerate triangle)

This is proven by noticing that $\langle u, v\rangle + \langle v, u\rangle = \langle u, v\rangle\overline{\langle v, u\rangle} = 2Re\langle u, v\rangle \leq 2|\langle u, v\rangle| \leq 2\|u\|\|v\|$ by conjugate symmetry and Cauchy-Schwarz

1.3.2 | Parallelogram Equality

Suppose $u, v \in V$. Then

$$\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2)$$