

1 | Problem

Suppose $T \in \mathcal{L}(V)$. Prove that $T/(\text{null } T)$ is injective if and only if $(\text{null } T) \cap (\text{range } T) = \{0\}$

2 | Proof

2.1 | Condition Manipulation

First, we will rewrite the problem as logical statements for easier manipulation.

2.1.1 | Left Condition

The left-hand side " $T/(\text{null } T)$ is injective" is equivalent to:

$$\begin{aligned}
 (T/U(v+U) = 0) &\implies (v+U = 0) && \text{(alternate definition of injective)} \\
 Tv + U = \text{null } T &\implies v + U = \text{null } T \\
 Tv + (\text{null } T) = \text{null } T &\implies v + (\text{null } T) = \text{null } T \\
 Tv \in \text{null } T &\implies v \in \text{null } T \\
 T^2v = 0 &\implies v \in \text{null } T
 \end{aligned}$$

2.1.2 | Right Condition

We can also rewrite the right-hand condition for easier manipulation. The intersection of the null space and the range being 0 is the same as (assuming $w \neq 0$) "if $w \in \text{null } T$ then $w \notin \text{range } T$ " and "if $w \in \text{range } T$ then $w \notin \text{null } T$ ". Note that these are contrapositives of each other, so we just need to work with the second statement.

Thus, assuming $w \neq 0$, these statements are equivalent:

$$\begin{aligned}
 (\exists v : Tv = w) &\implies (Tw \neq 0) \\
 v \notin \text{null } T &\implies T^2v \neq 0 && (w \neq 0) \\
 T^2v = 0 &\implies v \in \text{null } T && \text{(contrapositive)}
 \end{aligned}$$

2.2 | Proof

The statements are equivalent.