Source:

1 | find taylor series

1.1 | $y = \cos(x)$

$$\begin{aligned}
P_n(x) &= f(0) & &+\frac{d}{dx}f(0)x & &+\frac{\frac{d^2}{d^2x}f(0)}{2!}x^2 & &+\frac{\frac{d^3}{d^3x}f(0)}{3!}x^3 & &+\cdots \\
&= \cos(0) & &-\sin(0)x & &+\frac{-\cos(0)}{2!}x^2 & &+\frac{\sin(0)}{3!}x^3 & &+\cdots \\
&= 1 & &-0x & &-\frac{1}{2!}x^2 & &+\frac{0}{3!}x^3 & &+\cdots \\
&= 1 & & & &-\frac{1}{2!}x^2 & & & &+\frac{1}{4!}x^4+\cdots \\
&= \sum_{k=0}\frac{(-1)^k}{(2k)!}x^{2k}
\end{aligned}$$

1.2 | $y = e^x$

$$\begin{aligned}
P_n(x) &= f(0) & &+\frac{d}{dx}f(0)x & &+\frac{\frac{d^2}{d^2x}f(0)}{2!}x^2 & &+\frac{\frac{d^3}{d^3x}f(0)}{3!}x^3 & &+\cdots \\
&= e^0 & &+e^0x & &+\frac{e^0}{2!}x^2 & &+\frac{e^0}{3!}x^3 & &+\cdots \\
&= 1 & &+x & &+\frac{1}{2!}x^2 & &+\frac{1}{3!}x^3 & &+\cdots \\
&= \sum_{k=0}\frac{x^k}{k!}
\end{aligned}$$

1.3 | $y = \sqrt{x}$ centered at $x = 1$

$$\begin{aligned}
P_n(x) &= f(1) & &+\frac{d}{dx}f(1)(x-1) & &+\frac{\frac{d^2}{d^2x}f(1)}{2!}(x-1)^2 & &+\frac{\frac{d^3}{d^3x}f(1)}{3!}(x-1)^3 & &+\cdots \\
&= 1 & &+\frac{1}{2}(x-1) & &+\frac{\frac{1}{2}\frac{-1}{2}}{2!}(x-1)^2 & &+\frac{\frac{1}{2}\frac{-1}{2}\frac{-3}{2}}{3!}(x-1)^3 & &+\cdots
\end{aligned}$$

I don't know how to write it using summation notation though..

2 | prove approximations

2.1 | $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$

Proof by geometric series

2.2 | $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$

Plug $-x$ for x in the previous equation.

2.3 | $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$

Plug x^2 for x in the previous equation.

3 | more finding of polynomial

3.1 | TODO $ y = ln (1+x)$

3.2 | TODO $y = \tan^{-} x$

3.3 | $y = (1+x)^k$

$$
\begin{aligned}
P_n(x) &= f(0) && + \frac{d}{dx} f(0) x && + \frac{\frac{d^2}{d^2x} f(0)}{2!} x^2 && + \frac{\frac{d^3}{d^3x} f(0)}{3!} x^3 && + \cdots \\
&= 1 + k(1)^k && + k(k-1)(1)^{k-1} x && + \frac{k(k-1)(k-2)(1)^{k-2}}{2!} x^2 && + \frac{k(k-1)(k-2)(k-3)(1)^{k-3}}{3!} x^3 && + \cdots \\
&= 1 + k && + k(k-1) x && + \frac{k(k-1)(k-2)}{2!} x^2 && + \frac{k(k-1)(k-2)(k-3)}{3!} x^3 && + \cdots \\
&= 1 + k && + \frac{k!}{(k-1)!} x && + \frac{\frac{k!}{(k-2)!}}{2!} x^2 && + \frac{\frac{k!}{(k-3)!}}{3!} x^3 && + \cdots \\
&= 1 + k && + \frac{k! x}{(k-1)!} && + \frac{k!}{(k-2)!2!} x^2 && + \frac{k!}{(k-3)!3!} x^3 && + \cdots \\
&= \binom{k}{0} + \binom{k}{1} x && + \binom{k}{2} x^2 && + \binom{k}{3} x^3 && + \cdots \\
&= \sum_{i=0} \binom{k}{i} x^i
\end{aligned}
$$

4 | find sum of series by recognizing Taylor Series approximations of some functions

4.1 | $3 + \frac{9}{2!} + \frac{27}{3!} + \frac{81}{4!} + \cdots$

$$
3 + \quad \frac{9}{2!} + \quad \frac{27}{3!} + \quad \frac{81}{4!} = e^3 - 1
$$

4.2 | $1 - \ln 2 + \frac{\ln^2 2}{2!} + \frac{\ln^3 2}{3!} + \cdots$

$$
1 - \ln 2 + \frac{\ln^2 2}{2!} + \frac{\ln^3 2}{3!} + \cdots = e^{-\ln 2} = \frac{1}{2}
$$

4.3 | $\sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k+1}}{4^{2k+1}(2k+1)!}$

$$
\sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k+1}}{4^{2k+1}(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{\pi}{4}^{2k+1}\right)}{(2k+1)!} = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}
$$

5 | **evaluate limits using taylor series approx**

5.1 | $\lim\limits_{x\to 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5}$

$$
\begin{aligned}
\lim_{x\to 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5} = & \quad \lim_{x\to 0} \frac{\cancel{x} - \frac{x^3}{6} + \frac{x^5}{5!} + \cdots \cancel{-x} + \frac{x^3}{6}}{x^5} \\
&= \lim_{x\to 0} \frac{\cancel{-\frac{x^3}{6}} + \frac{x^5}{5!} + \cdots \cancel{+\frac{x^3}{6}}}{x^5} \\
&= \lim_{x\to 0} \frac{\frac{\cancel{x^5}}{5!} + \cdots}{x^5} \\
&= \lim_{x\to 0} \frac{1}{5!} + \frac{x^7}{x^5 7!} + \frac{x^9}{x^5 9!} \cdots \\
&= \frac{1}{5!}
\end{aligned}
$$

5.2 | **TODO** $\lim\limits_{x\to 0} \frac{x - \tan^{-} x}{x^3}$

6 | **find taylor series approximations**

6.1 | $y = e^x + e^{-x}$

$$
\begin{aligned}
& e^x + e^{-x} \\
& \quad = 1 + x + \frac{x^2}{2!}\frac{x^3}{3!} + \cdots + 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} \\
& \quad = 1 + 1 + x - x + \frac{x^2}{2!} + \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^3}{3^1} + \cdots \\
& \quad = 2\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) \\
& \quad = 2\sum_{k=0} \frac{x^{2k}}{(2k)!}
\end{aligned}
$$

6.2 | $y = \sin(\pi x)$

$$\sin(\pi x) = \sum_{k=0} \frac{(-1)^k (\pi x)^{2k+1}}{(2k+1)!}$$

(just plug it in)

6.3 | **TODO** $y = \sin^2 x$