

1 | upper triangular matrix def

A matrix in which all entries below the diagonal are zero

$$\begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

1.1 | results

1.1.1 | Axler 5.26 Conditions for upper-triangular matrix

Suppose $T \in \mathcal{L}(V)$ and v_1, \dots, v_n is a basis of V . The following are equivalent:

- the matrix of T with respect to v_1, \dots, v_n is upper triangular
- $Tv_j \in \text{span}(v_1, \dots, v_j)$ for each $j = 1, \dots, n$
- The span of each prefix of the basis is invariant under T .

1.1.2 | Axler 5.27 Over \mathbb{C} , every operator has an upper-triangular matrix

Suppose V is a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$. Then T has an upper-triangular matrix wrt some basis of V .

1. intuition

There are n eigenvalues (fundamental theorem of linear algebra) and each one should have a corresponding eigenvector that can sweep out a column? What happens when an eigenvalue has higher multiplicity?

2. proof

- induction on the dimension of V . use the fact that the first column can be found, then use the remaining basis vectors as a smaller subspace and do the same thing?

1.1.3 | Axler 5.30 Determination of invertibility from upper-triangular matrix

Suppose $T \in \mathcal{L}(V)$ has an upper-triangular matrix wrt some basis of V . Then, T is invertible iff all the entries on the diagonal of the upper-triangular matrix are nonzero.

1. intuition

- if one of the diagonal vectors is zero, then there is an injectivity/surjectivity problem and the operator is singular
- proof is by assuming all are nonzero and showing surjective, then by contradiction.

1.1.4 | Axler 5.32 Determination of eigenvalues from upper-triangular matrix

Suppose $T \in \mathcal{L}(V)$ has an upper-triangular matrix wrt some basis of V . Then the eigenvalues of T are precisely the entries on the diagonal of that upper-triangular matrix.

1. proof

$$\mathcal{M}(T) = \begin{pmatrix} \lambda_1 & & * \\ & \lambda_2 & \\ & & \ddots \\ 0 & & & \lambda_n \end{pmatrix}$$
$$\mathcal{M}(T - \lambda I) = \begin{pmatrix} \lambda_1 - \lambda & & * \\ & \lambda_2 - \lambda & \\ & & \ddots \\ 0 & & & \lambda_n - \lambda \end{pmatrix}$$