Source:

1 | **Problem**

Prove or give a counterexample:

$$
\begin{aligned}
&\text{dim}(U_1 + U_2 + U_3) \\
&\qquad = \text{dim } U_1 + \text{dim } U_2 + \text{dim } U_3 \\
&\qquad\quad - \text{dim}(U_1 \cap U_2) - \text{dim}(U_1 \cap U_3) - \text{dim}(U_2 \cap U_3) \\
&\qquad\quad + \text{dim}(U_1 \cap U_2 \cap U_3)
\end{aligned}
$$

2 | **Reasoning**

By Axler2.41 we know that

$$\text{dim}(U_1 + U_2) = \text{dim } U_1 + \text{dim } U_2 - \text{dim}(U_1 \cap U_2)$$

By applying this formula to itself, we find that

$$
\begin{aligned}
&\text{dim}(U_1 + U_2 + U_3) \\
&\qquad = \text{dim}((U_1 + U_2) + U_3) \\
&\qquad = \text{dim}(U_1 + U_2) + \text{dim } U_3 - \text{dim}((U_1 + U_2) \cap U_3) \\
&\qquad = \text{dim } U_1 + \text{dim } U_2 - \text{dim}(U_1 \cap U_2) + \text{dim } U_3 - \text{dim}((U_1 + U_2) \cap U_3)
\end{aligned}
$$

To show that the lemma is true, we would have to show that

$$
\begin{aligned}
&\cancel{\text{dim } U_1 + \text{dim } U_2 + \text{dim } U_3 - \text{dim}(U_1 \cap U_2)} - \text{dim}(U_1 \cap U_3) - \text{dim}(U_2 \cap U_3) + \text{dim}(U_1 \cap U_2 \cap U_3) \\
= &\cancel{\text{dim } U_1 + \text{dim } U_2 - \text{dim}(U_1 \cap U_2)} + \text{dim } U_3 - \text{dim}((U_1 + U_2) \cap U_3)
\end{aligned}
$$

and to provide a counterexample, we just need to find some U_1, U_2, U_3 such that

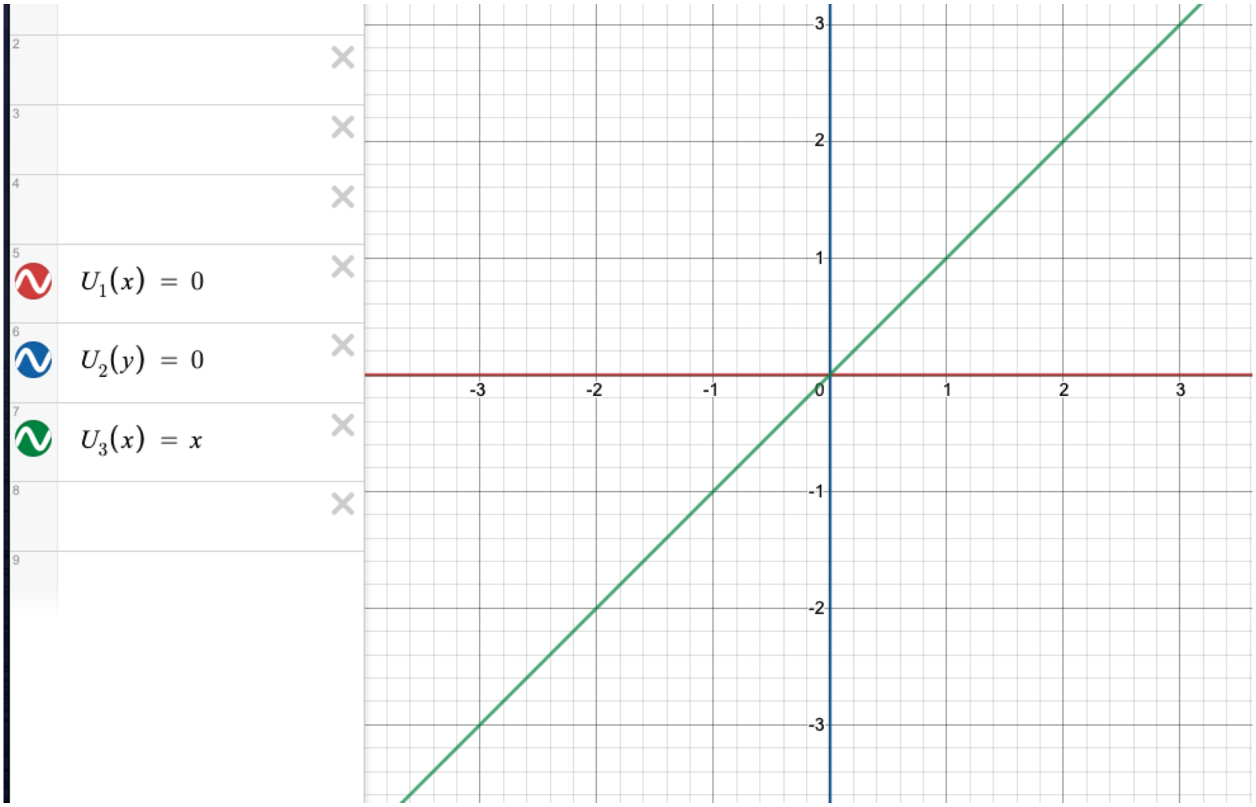
$$\text{dim}(U_1 \cap U_3) + \text{dim}(U_2 \cap U_3) - \text{dim}(U_1 \cap U_2 \cap U_3) \neq \text{dim}((U_1 + U_2) \cap U_3)$$

3 | **Counterexample**

If we choose

$$
\begin{aligned}
U_1 &= \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} : x \in \mathbb{R} \right\} \\
U_2 &= \left\{ \begin{pmatrix} 0 \\ x \end{pmatrix} : x \in \mathbb{R} \right\} \\
U_3 &= \left\{ \begin{pmatrix} x \\ x \end{pmatrix} : x \in \mathbb{R} \right\}
\end{aligned}
$$

then the graph of the subspaces looks like this:



and the dimesion of each intersection is 0 while the dimension of $(U_1 + U_2) \cap U_3 = 2$. Thus, we have

$$\begin{aligned} \text{dim}\cancelto{0}{(U_1 \cap U_3)} + \text{dim}\cancelto{0}{(U_2 \cap U_3)} - \text{dim}\cancelto{0}{(U_1 \cap U_2 \cap U_3)} &\neq \text{dim}((U_1 + U_2) \cap U_3) \\ &\implies 0 \neq 2 \end{aligned}$$

In summary, the sum of these subpsaces is $\mathbb{R}^2$ and the dimension of the sum is 2, but

$$\text{dim}(U_1 + U_2 + U_3) = 2 \neq 3 = 1 + 1 + 1 - 0 - 0 - 0 + 0$$