1 | general trends

1.1 | split fractions

Always consider splitting sums of fractions

$$\int \frac{a+b}{c} dx = \int \frac{a}{c} dx + \int \frac{b}{c} dx$$

1.2 | pull out constant factors

$$\int af(x)dx = a \int f(x)dx$$

2 | additive

$$\int f(x) + g(x)dx = \int f(x)dx + \int g(x)dx$$

3 | change of lower bound

$$\begin{aligned} \int_a^b f(x)dx + \int_b^c f(x)dx &= \int_a^c f(x)dx \\ \implies \int_a^x f(t)dt &= \int_b^x f(t)dt - \int_a^b f(t)dt \end{aligned}$$

4 | fundamental theorem of calculus

$$\begin{aligned} \int f'(x)dx &= f(x) + C \\ \frac{d}{dx} \int f(x)dx &= f(x) \\ \frac{d}{dx} \int_a^x f(t)dt &= f(x) \end{aligned}$$

(second one doesn't have a $+C$ because the derivative sends that to zero)

5 | net change theorem

$$\begin{aligned} \int_a^b f'(x)dx &= f(b) - f(a) \\ \frac{d}{dx} \int_a^x f(x)dx &= f(x) \\ \int_a^x f'(x)dx &= f(x) - f(a) \end{aligned}$$

6 | variable bounds

$$\frac{d}{dx}\int_a^{p(x)} f(x)dx = f(p(x))p'(x)$$
$$\int_{p(x)}^{q(x)} f(t)dt = \int_0^{q(x)} f(t)dt - \int_0^{p(x)} f(t)dt$$

An example

$$k(x) = \int_{x^2}^{e^x} \sqrt{\sin t}dt = \int_0^{e^x} \sqrt{\sin t}dt - \int_0^{x^2} \sqrt{\sin t}dt$$
$$k'(x) = \sqrt{\sin e^x}e^x - 2x\sqrt{\sin x^2}$$

7 | mean value theorem (for integrals)

There exists some point c over an integrable interval $[a, b]$ s.t.

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

8 | integration rules

8.1 | power rule for integrals

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

8.2 | k-angle formulas for sinusoids

$$\int \sin kx dx = -\frac{\cos kx}{k}$$
$$\int \cos kx dx = \frac{\sin kx}{k}$$

8.2.1 | for $\sin^2 x$ or $\cos^2 x$,

Try to get a double angle using the $\cos 2x$ identities.

8.2.2 | for a product of $\sin x \cos x$ or similar,

Use the double angle identity for $\sin 2x = 2\sin x \cos x$.

8.2.3 | for $\sec x$ and $\tan x$,

We know their derivatives contain themselves, so we can look for something cyclic.
For example,

$$\begin{aligned}\int \sec x dx &= \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} dx \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx\end{aligned}$$

Let $u = \sec x + \tan x$, $du = \sec x \tan x + \sec^2 x$

$$\begin{aligned}\int \sec x dx &= \int \frac{du}{u} dx \\ &= \int \frac{1}{u} du \\ &= \ln |u| \\ &= \ln |\sec x + \tan x|\end{aligned}$$

8.3 | u-substitution (on products, chain rule)

if it happens to work:

$$\int f'(g(x))g'(x)dx = f(g(x)) + C$$

an example:

$$\int 2x \sin(x^2) dx$$

Let $u = x^2$ and $du = fracdudxdx = 2xdx$

$$\begin{aligned}\int \sin x^2 2x dx &= \int \sin u du \\ &= -\cos(u) + C \\ &= -\cos(x^2) + C\end{aligned}$$

8.4 | integration by parts (on products, product rule)

$$\begin{aligned}\frac{d}{dx} f(x)g(x) &= f'(x)g(x) + g'(x)f(x) \\ \int \frac{d}{dx} f(x)g(x)dx &= \int f'(x)g(x) + g'(x)f(x)dx \\ f(x)g(x) + C &= \int f'(x)g(x)dx + \int g'(x)f(x)dx \\ \implies \int f'(x)g(x)dx &= f(x)g(x) - \int g'(x)f(x)dx + C\end{aligned}$$