

1 | Problem

Suppose $T \in \mathcal{L}(V)$. Prove that $T/(\text{null } T)$ is injective if and only if $(\text{null } T) \cap (\text{range } T) = \{0\}$

2 | Proof

First, we will rewrite the problem as logical statements for easier manipulation. The left-hand side " $T/(\text{null } T)$ is injective" is equivalent to:

$$\begin{aligned} (T/U(v+U) = 0) &\iff (v+U = 0) \\ Tv + U = \text{null } T &\iff v + U = \text{null } T \\ Tv + (\text{null } T) = \text{null } T &\iff v + (\text{null } T) = \text{null } T \\ Tv \in \text{null } T &\iff v \in \text{null } T \\ T^2v = 0 &\iff Tv = 0 \end{aligned}$$

We can also rewrite the second condition for easier manipulation. The intersection of the null space and the range being 0 is the same as (assuming $w \neq 0$) "if $w \in \text{null } T$ then $w \notin \text{range } T$ " and "if $w \in \text{range } T$ then $w \notin \text{null } T$ ". Note that these are contrapositives of each other, so we just need to work with the second statement.

Assuming $w \neq 0$,

$$\begin{aligned} (\exists v : Tv = w) &\implies (Tw \neq 0) \\ T^2v \neq 0 &\quad \forall v \notin (\text{null } T) \\ v \notin \text{null } T &\implies T^2v \neq 0 \end{aligned}$$

Note that this statement along with its contrapositive implies $(\text{null } T) \cap (\text{range } T) = \{0\}$.

Furthermore, keeping in mind that $w = Tv$ and $w \neq 0$,

$$\begin{aligned} T^2v &\neq 0 \\ T(Tv) &\neq 0 \\ Tw &\neq 0 \\ w &\notin T \end{aligned}$$

which shows the previous relation is an if-and-only-if relation: