1 | Axler 6.A exercise 9

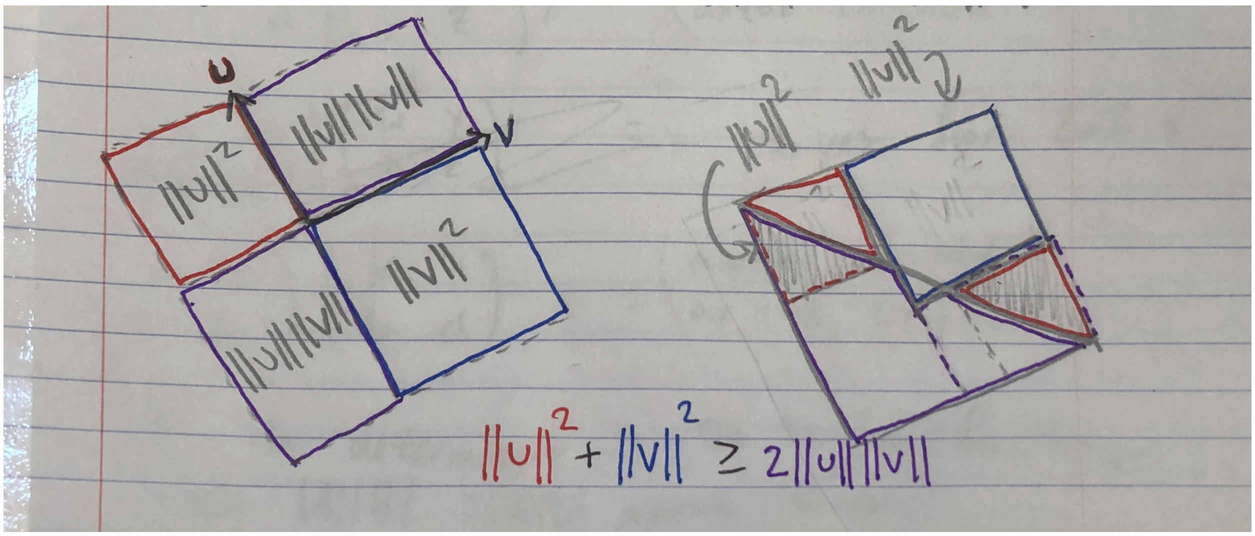
Suppose $u, v \in V$ and $\|u\| \leq 1$ and $\|v\| \leq 1$. Prove that

$$\sqrt{1 - \|u\|^2}\sqrt{1 - \|u\|^2} \leq 1 - |\langle u, v \rangle|$$

2 | Proof

2.1 | Useful Lemma

$$\|u\|^2 + \|v\|^2 \geq 2\|u\|\|v\|$$



This proof is only valid for inner product spaces over $\mathbb{F}^n$ and the Euclidean norm. An algebraic proof would be better.

2.2 | Cauchy-Schwarz Corollary

$$|\langle u, v \rangle| \leq \|u\|\|v\|$$
$$1 - \|u\|\|v\| \leq 1 - |\langle u, v \rangle|$$

This intermediate value is obtained using the Cauchy-Schwarz inequality.

2.3 | Main Proof

Now, to show that the square of the left hand side is less than or equal to the square of the right hand side,

$$\begin{aligned}
(1 - \|u\|^2)(1 - \|v\|^2) &= 1 - \|u\|^2 - \|v\|^2 + \|u\|^2\|v\|^2 \\
&= 1 - (\|u\|^2 + \|v\|^2) + \|u\|^2\|v\|^2 \\
&\leq 1 - 2\|u\|\|v\| + \|u\|^2\|v\|^2 && \text{by the earlier lemma} \\
&= (1 - \|u\|\|v\|)^2 \\
&\leq (1 - |\langle u, v \rangle|)^2 && \text{by the Cauchy-Schwarz corollary}
\end{aligned}$$

Taking square roots of both sides proves the desired result. □