1 | upper triangular matrix def

A matrix in which all entries below the diagonal are zero

$$\begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

1.1 | results

1.1.1 | Axler5.26 Conditions for upper-triangular matrix

Suppose $T; i\mathcal{L}(V)$ and $v_1, \ldots, v_n$ is a basis of V. The following are equivalent:

- the matrix of T with respect to $v_1, \ldots, v_n$ is upper triangular
- $Tv_j \in$ span$(v_1, \ldots, v_j)$ for each $j = 1, \ldots, n$
- The span of each prefix of the basis is invariant under T.

1.1.2 | Axler5.27 Over $\mathbb{C}$, every operator has an upper-triangular matrix

Suppose V is a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$. Then T has an upper-triangular matrix wrt some basis of V.

1. intuition

   There are n eigenvalues (fundamental theorem of linear algebra) and each one should have a corresponding eigenvector that can sweep out a column? What happens when an eigenvalue has higher multiplicity?

2. proof

   (a) induction on the dimension of V. use the fact that the first column can be found, then use the remaining basis vectors as a smaller subspace and do the same thing?

1.1.3 | Axler5.30 Determination of invertibility from upper-triangular matrix

Suppose $T \in \mathcal{L}(V)$ has an upper-tringular matrix wrt some basis of V. Then, T is invertible iff all the entries on the diagonal of the upper-triangular matrix are nonzero.

1. intuition

   (a) if one of the diagonal vectors is zero, then there is an injectivity/surjectivity problem and the operator is singular

   (b) proof is by assuming all are nonzero and showing surjective, then by contradiction.

1.1.4 | Axler 5.32 Determination of eigenvalues from upper-triangular matrix

Suppose $T \in \mathcal{L}(V)$ has an upper-triangular matrix wrt some basis of V. Then the eigenvalues of T are precisely the entries on the diagonal of that upper-triangular matrix.

1. proof

$$\mathcal{M}(T) = \begin{pmatrix} \lambda_1 & & & * \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$
$$\mathcal{M}(T - \lambda I) = \begin{pmatrix} \lambda_1 - \lambda & & & * \\ & \lambda_2 - \lambda & & \\ & & \ddots & \\ 0 & & & \lambda_n - \lambda \end{pmatrix}$$

   And that second matrix is only singular when $\lambda \in \lambda_1, \ldots, \lambda_n$