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Exploration 2-3a: Limit of a Sum, Epsilon and Delta Proof

Date:

Objective: Prove by ε-δ techniques that the limit of a sum of two functions equals the sum of the two limits.

1. Write the definition of limit using ε and δ.

$|f(x)-L| \le \varepsilon$

$|x-c| \le \delta$

For Problems 2–5,

Let $y_1 = 5x + 1$.

Let $y_2 = x^2$.

Let $y_3 = y_1 + y_2$.

2. Find

$L_1 = \lim_{x\to 2} y_1 =$ 11 and $L_2 = \lim_{x\to 2} y_2 =$ 4

3. Let $\varepsilon = 0.1$. Find δ_1 such that y_1 is within $\varepsilon/2$ unit of L_1 on the positive side by substituting $(L_1 + 0.05)$ for y_1 and $(2 + \delta_1)$ for x.

$L_1+\frac{\varepsilon}{2} = 5(c+\delta)+1$

$11+\frac{\varepsilon}{2} = 5(2+\delta)+1$

$\frac{10+\frac{\varepsilon}{2}}{5} = 2+\delta$

$\delta = \frac{\varepsilon}{10} = 0.001$

4. Using $\varepsilon = 0.1$ as in Problem 3, find δ_2 such that y_2 is within $\varepsilon/2$ units of L_2 on the positive side by substituting $(L_2 + 0.05)$ for y_2 and $(2 + \delta_2)$ for x.

$|x^2-4| \le \frac{\varepsilon}{2}$

$4+\frac{\varepsilon}{2} = (2+\delta)^2$

$= \delta^2+2^2+4\delta$

$\frac{\varepsilon}{2} = \delta^2+4\delta$

$\delta = \frac{-4\pm\sqrt{16+2\varepsilon}}{2} = 2\sqrt{\text{''}}$ $\sqrt{4(4+\frac{\varepsilon}{2})}$

$= -2\pm\sqrt{4+\frac{\varepsilon}{2}} = \begin{matrix}0.0124\\ -4.0124\end{matrix}$

5. Add $L_1 + L_2$ to get $L = \lim_{x\to 2} y_3$.

Let $\delta = \min\{\delta_1, \delta_2\}$, which means that δ is the smaller of δ_1 and δ_2. Make a table of values of y_3 for several values of x between 2 and $(2 + \delta)$. Does the table show that y_3 is within $\varepsilon = 0.1$ unit of L whenever x is within δ units of 2 on the positive side?

$\delta = 0.001$ and it works

For Problems 6 and 7,

Let $f(x) = g(x) + h(x)$.

Let $L_1 = \lim_{x\to c} g(x)$.

Let $L_2 = \lim_{x\to c} h(x)$.

6. Suppose that someone has chosen a number $\varepsilon > 0$. Because L_1 is the limit of $g(x)$ as x approaches c, you can keep $g(x)$ as close as you like to L_1 just by keeping x close enough to c. Thus, there is a number δ_1 such that $g(x)$ can be kept within $\varepsilon/2$ units of L_1. That is,

$$L_1 + e/2 < g(x) < L_1 + \varepsilon/2$$

Similarly, you can keep $h(x)$ within $\varepsilon/2$ units of L_2 just by keeping x within δ_2 units of c. Write an inequality for $h(x)$ if x is within δ_2 units of c.

$L_2-\frac{\varepsilon}{2} < h(x) < L_2+\frac{\varepsilon}{2}$

7. Let $\delta = \min\{\delta_1, \delta_2\}$. Thus, both inequalities in Problem 6 will be true. By appropriate operations on these inequalities, show that $f(x)$ is within ε units of $(L_1 + L_2)$, and thus that $(L_1 + L_2)$ is the limit of $f(x)$ as x approaches c.

$L_1-\frac{\varepsilon}{2}+L_2-\frac{\varepsilon}{2} < g(x)+h(x) < L_1+\frac{\varepsilon}{2}+L_2+\frac{\varepsilon}{2}$

$L_1+L_2-\varepsilon < g+h(x) < L_1+L_2+\varepsilon$

8. State the property of the limit of a sum verbally.

$|g+h(x)-(L_1+L_2)| < \varepsilon$

limit of a sum of functions is the sum of their limits.

9. What did you learn as a result of doing this Exploration that you did not know before?

You can directly add inequalities