1 | Axler 6.A exercise 9

Suppose $u, v \in V$ and $\|u\| \leq 1$ and $\|v\| \leq 1$. Prove that

$$\sqrt{1 - \|u\|^2}\sqrt{1 - \|u\|^2} \leq 1 - |\langle u, v \rangle|$$

2 | Proof

2.1 | Useful Lemma

$$2\|u\|\|v\| \leq \|u\|^2 + \|v\|^2$$

2.2 | Cauchy-Schwarz Result

$$\begin{gathered} |\langle u, v \rangle| \leq \|u\|\|v\| \\ 1 - \|u\|\|v\| \leq 1 - |\langle u, v \rangle| \end{gathered}$$

This intermediate value is obtained using the Cauchy-Schwarz inequality.

2.3 | Main Proof

Now, to show that the square of the left hand side is less than or equal to the square of the right hand side,

$$\begin{aligned} &=(1 - \|u\|^2)(1 - \|v\|^2) \\ &=1 - \|u\|^2 - \|v\|^2 + \|u\|^2\|v\|^2 \\ &=1 - (\|u\|^2 + \|v\|^2) + \|u\|^2\|v\|^2 \\ &=1 - 2\|u\|\|v\| + \|u\|^2\|v\|^2 \end{aligned}$$