1 | **Problem**

Suppose $T \in \mathcal{L}(V)$. Prove that $T/(\operatorname{null} T)$ is injective if and only if $(\operatorname{null} T) \cap (\operatorname{range} T) = \{0\}$

2 | **Proof**

2.1 | **Condition Manipulation**

First, we will rewrite the problem as logical statements for easier manipulation.

2.1.1 | **Left Condition**

The left-hand side "$T/(\operatorname{null} T)$ is injective" is equivalent to:

$$
\begin{aligned}
(T/U\,(v+U) = 0) &\iff (v+U = 0) \\
Tv + U = \operatorname{null} T &\iff v + U = \operatorname{null} T \\
Tv + (\operatorname{null} T) = \operatorname{null} T &\iff v + (\operatorname{null} T) = \operatorname{null} T \\
Tv \in \operatorname{null} T &\iff v \in \operatorname{null} T \\
T^2 v = 0 &\iff Tv = 0
\end{aligned}
$$

2.1.2 | **Right Condition**

We can also rewrite the right-hand condition for easier manipulation. The intersection of the null space and the range being 0 is the same as (assuming $w \neq 0$) "if $w \in \operatorname{null} T$ then $w \notin \operatorname{range} T$" and "if $w \in \operatorname{range} T$ then $w \notin \operatorname{null} T$". Note that these are contrapositives of eachother, so we just need to work with the second statement.

Thus, assuming $w \neq 0$, these statements are equivalent:

$$
\begin{aligned}
(\exists v : Tv = w) &\implies (Tw \neq 0) \\
T^2 v \neq 0 \quad\quad & \quad\quad \forall v \notin (\operatorname{null} T) \\
v \notin \operatorname{null} T &\implies T^2 v \neq 0
\end{aligned}
$$

Note that this statement, along with its contrapositive, implies the original $(\operatorname{null} T) \cap (\operatorname{range} T) = \{0\}$ as desired.

Furthermore, keeping in mind that $w = Tv$ and $w \neq 0$,

$$
\begin{aligned}
T^2 v \neq 0 &\implies T(Tv) \neq 0 \\
&\implies Tw \neq 0 \\
&\implies w \notin \operatorname{null} T
\end{aligned}
$$

which shows the previous relation is an if-and-only-if relation:

$$
v \notin \operatorname{null} T \iff T^2 v \neq 0
$$