1 | Problem

Suppose $T \in \mathcal{L}(V)$. Prove that $T/(\text{null}\, T)$ is injective if and only if $(\text{null}\, T) \cap (\text{range}\, T) = \{0\}$

2 | Proof

2.1 | Condition Manipulation

First, we will rewrite the problem as logical statements for easier manipulation.

2.1.1 | Left condition

The left-hand side "$T/(\text{null}\, T)$ is injective" is equivalent to:

$$\begin{aligned}
(T/U\,(v+U) = 0) &\iff (v+U=0) \\
Tv + U = \text{null}\, T &\iff v + U = \text{null}\, T \\
Tv + (\text{null}\, T) = \text{null}\, T &\iff v + (\text{null}\, T) = \text{null}\, T \\
Tv \in \text{null}\, T &\iff v \in \text{null}\, T \\
T^2 v = 0 &\iff Tv = 0
\end{aligned}$$

We can also rewrite the second condition for easier manipulation. The intersection of the null space and the range being 0 is the same as (assuming $w \neq 0$) "if $w \in \text{null}\, T$ then $w \notin \text{range}\, T$" and "if $w \in \text{range}\, T$ then $w \notin \text{null}\, T$". Note that these are contrapositives of eachother, so we just need to work with the second statement.

Assuming $w \neq 0$, these statements are equivalent:

$$\begin{aligned}
(\exists v : Tv = w) &\implies \quad (Tw \neq 0) \\
T^2 v \neq 0 & \qquad \forall v \notin (\text{null}\, T) \\
v \notin \text{null}\, T &\implies \quad T^2 v \neq 0
\end{aligned}$$

Note that this statement, along with its contrapositive, implies the original $(\text{null}\, T) \cap (\text{range}\, T) = \{0\}$ as desired.

Furthermore, keeping in mind that $w = Tv$ and $w \neq 0$,

$$\begin{aligned}
T^2 v \neq 0 &\implies T(Tv) \neq 0 \\
&\implies Tw \neq 0 \\
&\implies w \notin \text{null}\, T
\end{aligned}$$

which shows the previous relation is an if-and-only-if relation:

$$v \notin \text{null}\, T \iff T^2 v \neq 0$$