

## 1 | an example: semicircle revolved around the x-axis to create a sphere

We can make cuts perpendicular to the axis of rotation. In this case, you end up with a bunch of circular disks, where the height of each slice is your semicircle function.

Thus, the volume of the disk is

$$\pi f^2(x_i)\Delta x = (a^2 - x_i^2)\pi\Delta x$$

This is kinda like a Riemann Sum, but with more stuff added on. We can take the limit of the sum

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \pi(a^2 - x_i^2)\Delta x$$

Where  $\Delta x = \frac{1}{n}$  and  $x_i = -a + \frac{2ak}{n}$

Expressed as an integral:

$$\begin{aligned} \int_{-a}^a \pi(a^2 - x^2)dx &\rightarrow \int \pi a^2 dx - \int \pi x^2 dx \\ &= \pi a^2 x - \pi \frac{1}{3} x^3 \\ &\rightarrow \pi a^3 - \pi \frac{1}{3} a^3 + \pi a^3 + \pi \frac{1}{3} (-a)^3 \\ &= 2\pi a^3 - \pi \frac{2}{3} a^3 \\ &= \frac{4}{3} \pi a^3 \end{aligned}$$

## 2 | now lets try a cone

Rotate

$$y = -ax + b$$

Around the y-axis. Then, each circle (which is layed out flat) has thickness  $dy$  and radius  $x$  or  $\frac{y-b}{-a}$

The volume of the disk is then

$$\pi \left( \frac{y-b}{-a} \right)^2 dy$$

Or using  $r, h$  as the radius and height of the cone,

$$\pi \left( r - \frac{r}{h} y \right)^2 dy$$

And we can take the integral of that from 0 to  $h$

$$\begin{aligned} \int_0^h \pi \left( r - \frac{r}{h} y \right)^2 dy &\rightarrow \pi \int \left( r - \frac{r}{h} y \right)^2 dx \\ \text{Let } u &= r - \frac{r}{h} y, du = -\frac{r}{h} dy \\ &= \pi - \frac{h}{r} \int u^2 du \\ &= -\pi \frac{h}{r} \frac{1}{3} u^3 + C \\ &= -\pi \frac{h}{r} \frac{1}{3} \left( r - \frac{r}{h} y \right)^3 \end{aligned}$$

$$\begin{aligned}\int_0^h \pi \left( r - \frac{r}{h} y \right)^2 dy &\rightarrow \pi \int r^2 + \left( \frac{r}{h} y \right)^2 - 2r \left( \frac{r}{h} y \right) dx \\ &= \frac{1}{3} \pi r^3 + \frac{1}{3} \frac{r^2}{h^2} y^3\end{aligned}$$