Riemann Sum Lesson Plan March

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1 Riemann Sum

Goal: The Area Under a Curve

Find the area of the region S that lies under the curve $y = f(x)$, above the x-axis and between the vertical lines $x = a$ and $x = b$.

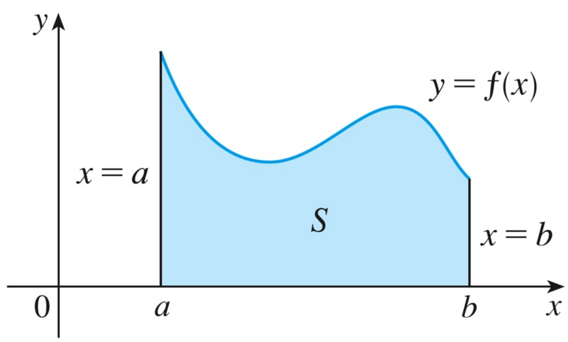

Figure 1: S=$\int_a^b f(x)dx$

The area under the curve f and above the x-axis, bounded between the vertical lines $x = a$ and $x = b$, is denoted by the integral expression

$$S = \int_a^b f(x)dx$$

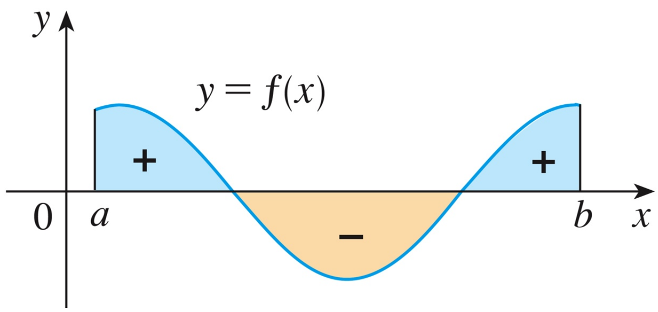

Figure 2: $\int_a^b f(x)dx$ is the net area

Definition 1.1. *A definite integral can be interpreted as the* ***net positive area*** *under the curve. That is, putting A_1 the area of the region above the x-axis and below the graph of f, and A_2 the area of the region below the x-axis and above the graph of f, the definite integral expresses the difference of areas*

$$A_1 - A_2 = \int_a^b f(x)dx.$$

Exercise 1.2. *The graph of f is shown. Evaluate each integral by interpreting it in terms of areas.*

1. $\int_0^2 f(x)dx$
2. $\int_0^5 f(x)dx$
3. $\int_5^7 f(x)dx$
4. $\int_0^9 f(x)dx$

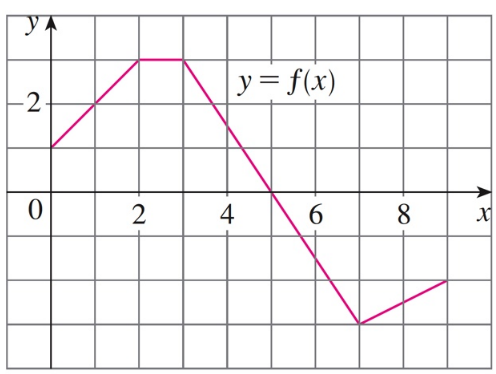

Figure 3: Area

Remark: Area is defined with respect to a rectangle, and we are familiar with extensions of the rectangular concept of area to various polygons.

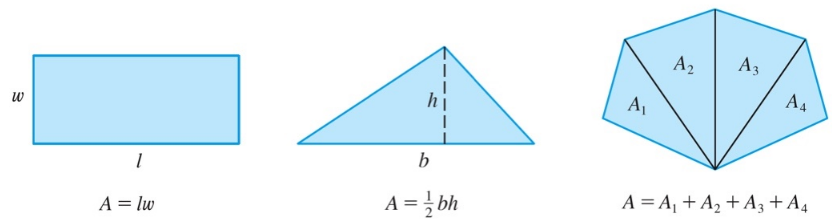

Figure 4: The area of Polygons

Hence for definite integrals we must formulate a **new** notion of area, consistently built upon the rectangular notion.

Exercise 1.3. *Approximate the region under the curve $x^3 - 6x$ and between $a = 0$, $b = 3$, with $n = 6$ rectangular regions. Then approximate $S = \int_a^b x^3 - 6x$ by the sum of the areas of the rectangles.*

The rectangles for the the last exercise could have been chosen in infinitely many ways, however there are several standardized ways of estimating the area under a curve. The most common method is:

Method 1: The Right Endpoint Estimate To compute a right endpoint estimate of the area under a curve we partition the considered domain into n equal sub-intervals and construct n rectangles whose bases are the lengths of the sub-intervals and whose heights are the values of the function at the right endpoints of the sub-intervals respectively. This is illustrated below for the curve $y = x^2$ on the domain $[a, b] = [0, 1]$ with $n = 4$.

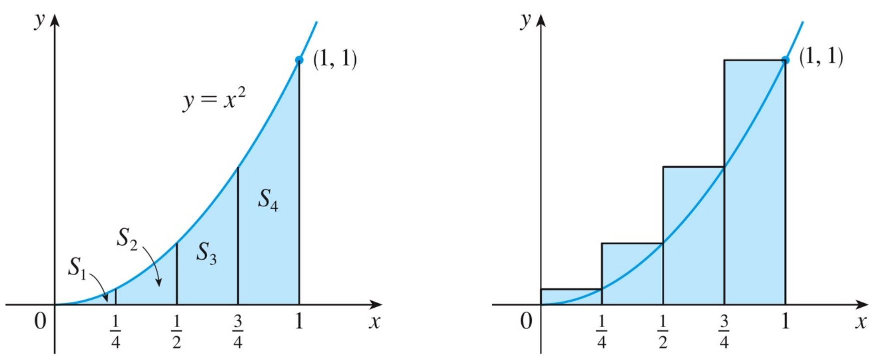

Figure 5: Right Endpoint Estimation

Analogously we may compute:

Method 2: The Left Endpoint Estimate This is similar to the right endpoint estimate, however the heights of the rectangles are the values of the function at the left endpoints of the sub-intervals respectively. The left endpoint estimate is illustrated below for the curve $y = x^2$ on the domain $[a, b] = [0, 1]$ with $n = 4$.

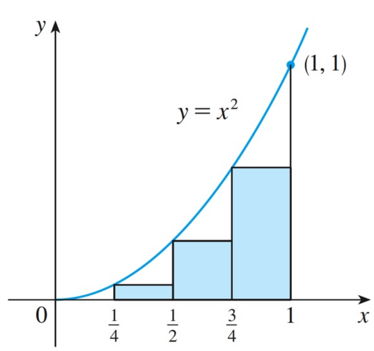

Figure 6: Left Endpoint Estimation

Remark: When the function is increasing the right endpoint estimate is an *over estimate* and the left endpoint estimate is an *under estimate*, however when the function is decreasing the right endpoint estimate is an *under estimate* while the left endpoint estimate is an *over estimate*.

There are two well known methods that attempt to balance the under and over estimates given by the left and right endpoint methods above. The first of these is:

Method 3: The Midpoint Estimate To compute the midpoint estimate of the area under a curve, we take as the heights of the rectangles the values of the function at the midpoints of the sub-intervals respectively. The midpoint estimate is illustrated below for the curve $y = x^2$ on the domain $[a, b] = [0, 1]$ with $n = 4$.

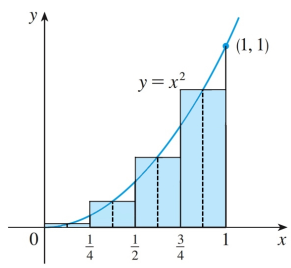

Figure 7: The Midpoint Estimate

Finally we have:

Method 4: The Trapezoid Estimate The trapezoid estimate is computed by summing the areas of n trapezoids. The trapezoids are constructed such that their left and right heights are the values of the function at the left and right endpoints of the sub-intervals respectively. It can be shown directly from the area formula for a trapezoid that the trapezoid estimate produces the average of the left endpoint and the right endpoint estimates.

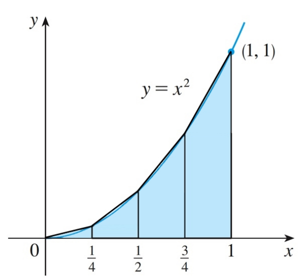

Figure 8: The Trapeziod Estimate

Exercise 1.4. *Use both the left endpoint and the right endpoint estimation methods to approximate the area under the parabola $y = x^2$ from $a = 0$ to $b = 1$.*

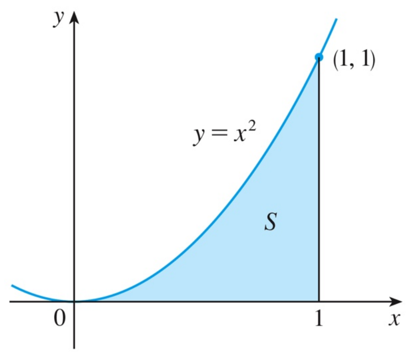

Figure 9: Estimate the area under curve

Exercise 1.5. *The speed of a runner increased steadily during the first three seconds of a race. Her speed at half-second intervals is given in the table. Compute left, right, and midpoint estimates for the distance that she traveled during these three seconds.*

t (s)	0	0.5	1.0	1.5	2.0	2.5	3.0
v (ft/s)	0	6.2	10.8	14.9	18.1	19.4	20.2

Figure 10: Table

Example 1.6. *Evaluate the following integrals by interpreting each in terms of areas.*

1. $\int_0^1 \sqrt{x^2+1}dx$

2. $\int_0^3 (x-1)dx$

Example 1.7. *Using summation notation write an expression for the right endpoint estimation of the area under the curve $y = x^2$ over the domain $[0, 1]$ partitioned into n equal pieces. Then write a similar expression for the general left endpoint estimation.*

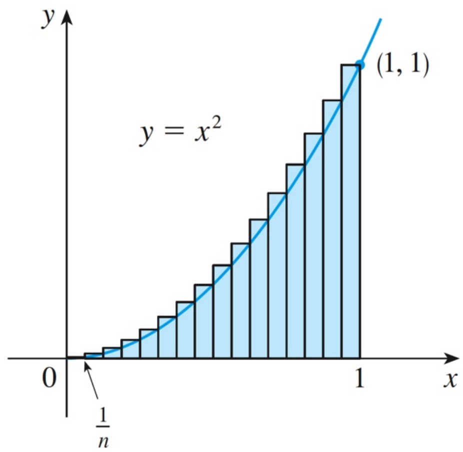

Figure 11: Estimate the area under curve

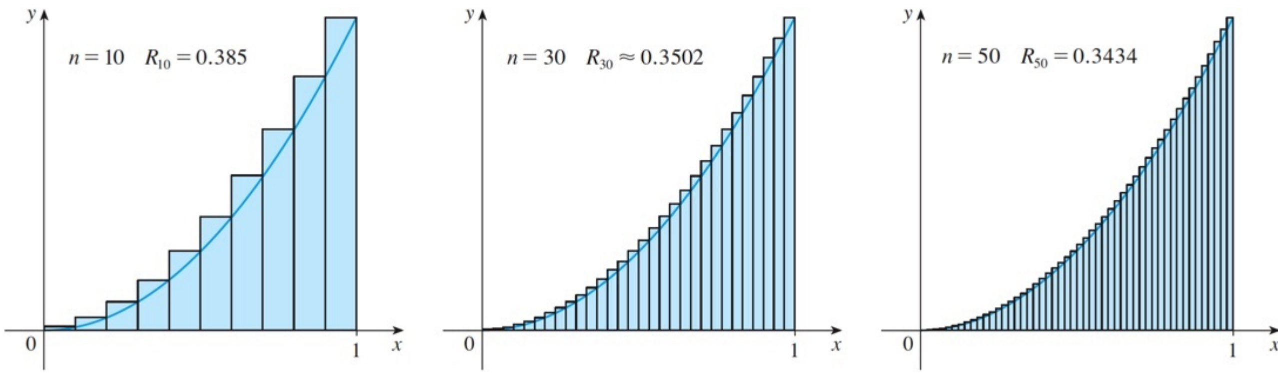

Figure 12: Right Endpoint Estimation

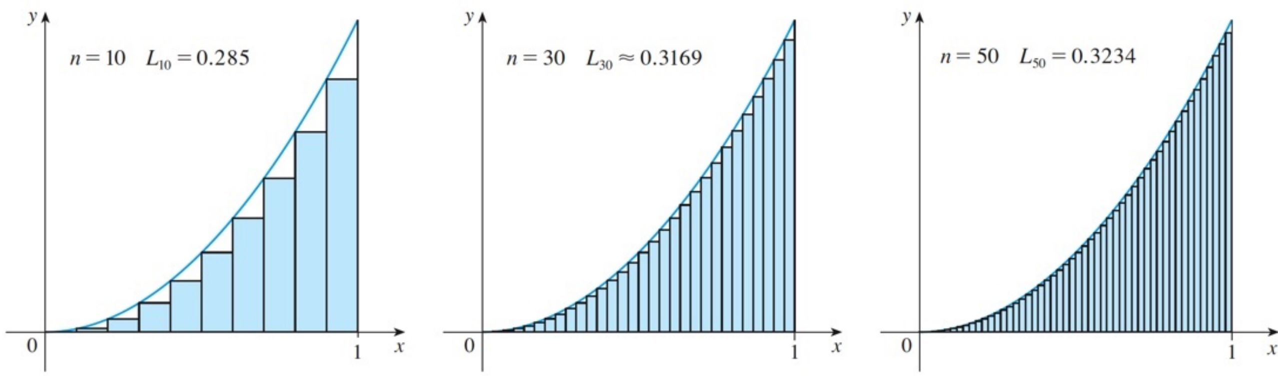

Figure 13: Left Endpoint Estimation

Definition 1.8. ***Definition of a Definite Integral***

If f is a continuous function on an closed interval $[a, b]$, *we divide the interval into* n *sub-intervals of equal width*

$$\Delta x = \frac{(b-a)}{n}.$$

We let

$$x_0 = a < x_1 < x_2 < ... < x_n = b$$

then the n *sub-intervals are*

$$[x_0, x_1], [x_1, x_2], ...[x_{n-1}, x_n].$$

We let x_1^*, x_2^*, . . . , x_n^* *be any sample points in these sub-intervals, (which means* x_i^* *lies in the ith sub-interval* $[x_{i-1}, x_i]$*).*

Then the definite integral of f from a to b is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

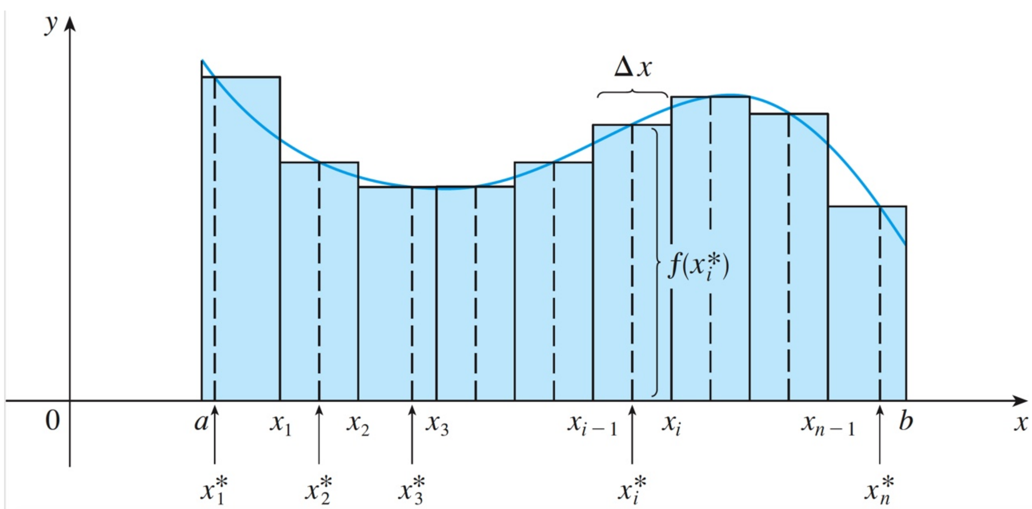

Figure 14: Sample Point Estimation

Note 1.9. *We form* ***lower*** *(and* ***upper****) sums by choosing the sample points* x_i^* *so that* $f(x_i^*)$ *is the minimum (and maximum) value of* f *on the ith sub-interval (*$[x_i, x_{i+1}]$ *)*

Theorem 1.10. *Let f is integrable on $[a, b]$, then*

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

Where

$$\Delta x = \frac{b-a}{n} \quad \& \quad x_i = a + i\Delta x$$

Reminder

$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

$\sum_{i=1}^{n} i^3 = (\frac{n(n+1)}{2})^2$

Exercise 1.11. *Set up an expression for $\int_{\pi}^{2\pi} \cos(x)dx$ as a limit of sums.*

Exercise 1.12. *Express the integral as a limit of Riemann sums. Do not evaluate the limit.*

1. $\int_1^3 \sqrt{4+x^2}dx$

2. $\int_2^5 (x^2+\frac{1}{x})dx$

Exercise 1.13. *Express the limit as a definite integral on the given interval.*

1. $\lim_{n\to\infty}\sum_{i=1}^n \dfrac{e^{x_i}}{1+x_i}\Delta x$ *on* $[0,1]$

2. $\lim_{n\to\infty}\sum_{i=1}^n x_i\sqrt{1+x_i^3}\Delta x$ *on* $[2,5]$

3. $\lim_{n\to\infty}\sum_{i=1}^n \dfrac{x_i}{x_i^2+4}\Delta x$ *on* $[1,3]$

Exercise 1.14. *Prove that*

$$\int_a^b x = \frac{b^2 - a^2}{2}$$