

1 | Cauchy-Schwarz Inequality important

'One of the most important inequalities in mathematics'

Suppose $u, v \in V$ (where V is an inner product space). Then

$$|\langle u, v \rangle| \leq \|u\| \|v\|$$

The inequality is an equality iff one of u, v is a scalar multiple of the other.

1.1 | intuition

For the Euclidean inner product, this is true because $\langle u, v \rangle = \|u\| \|v\| \cos \theta$. However, the Cauchy-Schwarz inequality works for all inner product spaces, using the generalized Pythagorean theorem (instead of the law of cosines).

1.2 | proof is by the orthogonal decomposition

By homogeneity of norms,

$$\left\| \frac{\langle u, v \rangle}{\|v\|^2} v \right\|^2 = \left| \frac{\langle u, v \rangle}{\|v\|^2} \right|^2 \|v\|^2$$

1.3 | results

1.3.1 | triangle inequality

Suppose $u, v \in V$. Then

$$\|u + v\| \leq \|u\| + \|v\|$$

The inequality is an equality if and only if one of u, v is a non-negative multiple of the other (degenerate triangle)

1.3.2 | Parallelogram Equality

Suppose $u, v \in V$. Then

$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2)$$