

$$1 \mid \int_1^2 \frac{e^{\frac{1}{x}}}{x^2} dx$$

$$\begin{aligned} \int -e^u du &= -e^u + C \\ &= -e^{\frac{1}{2}} + e^{\frac{1}{1}} \\ &= e - e^{\frac{1}{2}} \end{aligned}$$

$$2 \mid \int_0^1 r e^{\frac{r}{2}} dr$$

$$\begin{aligned} \int_0^1 r e^{\frac{r}{2}} dx &\Rightarrow r 2e^{\frac{r}{2}} - \int 2e^{\frac{r}{2}} dr \\ &= 2r e^{\frac{r}{2}} - \int 2e^{\frac{r}{2}} dr \\ &= 2r e^{\frac{r}{2}} - 4e^{\frac{r}{2}} \\ &= 2r e^{\frac{r}{2}} - 4e^{\frac{r}{2}} \\ &\Rightarrow 2e^{\frac{1}{2}} - 4e^{\frac{1}{2}} - (-4) \\ &= 4 - 2e^{\frac{1}{2}} \end{aligned}$$

$$3 \mid \int_4^9 \frac{\ln y}{\sqrt{y}} dy$$

$$\begin{aligned} \int \frac{\ln y}{\sqrt{y}} dy &= 2 \ln y \sqrt{y} - \int 2 \frac{1}{y} \sqrt{y} dy \\ &= 2 \ln y \sqrt{y} - \int 2 \frac{1}{\sqrt{y}} dy \\ &= 2 \ln y \sqrt{y} - 2 \int y^{-\frac{1}{2}} dy \\ &= 2 \ln y \sqrt{y} - 4\sqrt{y} + C \\ &= 2\sqrt{y}(\ln y - 2) + C \\ \Rightarrow &6(\ln 9 - 2) - 4(\ln 4 - 2) \end{aligned}$$

$$4 \mid \int_0^{\sqrt{\pi}} \cos \sqrt{x} dx$$

$$\text{Let } u = \sqrt{x}, du = \frac{1}{2\sqrt{x}} dx, dx = 2u du$$

$$\begin{aligned} \int_0^{\sqrt{\pi}} \cos \sqrt{x} dx &= 2 \int u \cos u du \\ &= 2u \sin u - 2 \int \sin u du \\ &= 2u \sin u + 2 \cos u \\ &= 2\sqrt{x} \sin \sqrt{x} + 2 \cos \sqrt{x} \\ \Rightarrow &2\pi^{\frac{1}{4}} \sin \pi^{\frac{1}{4}} + 2 \cos \pi^{\frac{1}{4}} - 2 \end{aligned}$$

5 | $\int_1^e \sin \ln x dx$

$$\begin{aligned}
 \int_1^e \sin \ln x dx &= x \sin \ln x - \int x \frac{1}{x} \cos \ln x dx \\
 &= x \sin \ln x - \int \cos \ln x dx \\
 &= x \sin \ln x - \left(x \cos \ln x + \int x \cancel{\frac{1}{x}} \sin \ln x dx \right) \\
 &= x \sin \ln x - x \cos \ln x - \int \sin \ln x dx \\
 2 \int \sin \ln x dx &= x \sin \ln x - x \cos \ln x \\
 \int \sin \ln x dx &= \frac{1}{2} x (\sin \ln x - \cos \ln x) \\
 \implies \frac{1}{2} e (\sin 1 - \cos 1) - \frac{1}{2} (\sin 0 - \cos 0) \\
 &= \frac{e}{2} (\sin 1 - \cos 1) + \frac{1}{2}
 \end{aligned}$$

6 | $\int_0^1 \frac{x^3}{\sqrt{4+x}} dx$

Let $u = 4 + x^2$, $du = 2x dx$

$$\begin{aligned}
 \int \frac{x^3}{\sqrt{4+x}} dx &= \frac{1}{2} \int \frac{(u-4)}{\sqrt{u}} du \\
 &= \frac{1}{2} \int \sqrt{u} dx - 2 \int \frac{1}{\sqrt{u}} dx
 \end{aligned}$$

7 | (additional problems)

8 | $\int \sin^2 x dx$

$$\begin{aligned}
 \int \sin^2 x dx &= -\sin x \cos x + \int \cos^2 x dx \\
 &= -\sin x \cos x + \int 1 dx - \int \sin^2 x dx \\
 2 \int \sin^2 x dx &= -\sin x \cos x + x \\
 \int \sin^2 x dx &= \frac{1}{2} (x - \sin x \cos x)
 \end{aligned}$$

9 | $\int \cos^2 x dx$

$$\begin{aligned}\int \cos^2 x dx &= \cos x \sin x + \int \sin^2 x dx \\ &= \cos x \sin x + \int 1 dx - \int \cos^2 x dx \\ 2 \int \cos^2 x dx &= \cos x \sin x + x \\ \int \cos^2 x dx &= \frac{1}{2} \cos x \sin x + \frac{x}{2}\end{aligned}$$

10 | $\int \sin^2 x \cos^2 x dx$

First, the integral of $\sin x \cos x$:

$$\begin{aligned}\int \sin x \cos x dx &= \sin^2 x - \int \sin x \cos x dx \\ &= \frac{1}{2} \sin^2 x + C\end{aligned}$$

And the derivative:

$$\begin{aligned}\frac{d}{dx} \sin x \cos x &= -\sin^2 x + \cos^2 x \\ &= \cos^2 x - \sin^2 x\end{aligned}$$

Now we can use differentiation by parts on $(\sin x \cos x)(\sin x \cos x)$:

$$\begin{aligned}\int (\sin x \cos x)(\sin x \cos x) dx &= \frac{1}{2} (\sin x \cos x) \sin^2 x - \int (\cos^2 x - \sin^2 x) \left(\frac{1}{2} \sin^2 x \right) dx \\ &= \frac{1}{2} (\sin x \cos x) \sin^2 x - \frac{1}{2} \int \sin^2 x \cos^2 x dx - \frac{1}{2} \int \sin^4 x dx \\ \frac{3}{2} \int \sin^2 x \cos^2 x dx &= \frac{1}{2} (\sin x \cos x) \sin^2 x - \frac{1}{2} \int \sin^4 x dx\end{aligned}$$

To evaluate the integral of $\sin^4 x$, we can use differentiation by parts after multiplying by one. We will need to use the integral of $\sin x \cos x$ from earlier in this problem, as well as the integral of $\sin^2 x$ from the previous problem.

$$\begin{aligned}\int 1 \sin^4 x dx &= x \sin^4 x - 4 \int x \sin x \cos x dx \\ &= x \sin^4 x - 4 \left(x \left(\frac{1}{2} \sin^2 x \right) - \int \frac{1}{2} \sin^2 x dx \right) \\ &= x \sin^4 x - 4 \left(x \left(\frac{1}{2} \sin^2 x \right) - \frac{1}{2} \int \sin^2 x dx \right) \\ &= x \sin^4 x - 2x \sin^2 x + 4 \frac{1}{2} (x - \sin x \cos x) \\ &= x \sin^4 x - 2x \sin^2 x + 2x - 2 \sin x \cos x\end{aligned}$$

Plugging back into the main problem:

$$\begin{aligned}
\frac{3}{2} \int \sin^2 x \cos^2 x dx &= \frac{1}{2} (\sin x \cos x) \sin^2 x - \frac{1}{2} \int \sin^4 x dx \\
&= \frac{1}{2} (\sin x \cos x) \sin^2 x - \frac{1}{2} x \sin^4 x + x \sin^2 x - x + \sin x \cos x + C \\
\int \sin^2 x \cos^2 x dx &= \frac{1}{3} \sin^3 x \cos x - \frac{1}{3} x \sin^4 x + \frac{2}{3} x \sin^2 x - \frac{2}{3} x + \frac{3}{2} \sin x \cos x + C
\end{aligned}$$

I think I did something wrong.

11 | $\int \sin^3 x dx$

$$\begin{aligned}
\int \sin^3 x dx &= \int \sin x (1 - \cos^2 x) dx \\
&= \int \sin x dx - \int \cos^2 x \sin x dx \\
\text{Let } u &= \cos x, du = -\sin x dx \\
&= \int \sin x dx + \int u^2 du \\
&= -\cos x + \frac{1}{3} u^3 + C \\
&= \frac{1}{3} \cos^3 x - \cos x + C
\end{aligned}$$

~~$$\begin{aligned}
\int \sin^3 x dx &= \int 1 \sin^3 x dx \\
&= x \sin^3 x - \int 3x \sin x \cos x dx \\
&= x \sin^3 x - 3x \sin x \cos x + 3 \int \frac{1}{2} \sin^2 x dx \\
&= x \sin^3 x - 3x \sin x \cos x + \frac{3}{2} \int \sin^2 x dx \\
&= x \sin^3 x - 3x \sin x \cos x + \frac{3}{2} \frac{1}{2} (x - \sin x \cos x) + C \\
&= x \sin^3 x - 3x \sin x \cos x + \frac{3}{4} (x - \sin x \cos x) + C
\end{aligned}$$~~