Source:

1 | Lemma

The length of a linearly indpendent list is less than or equal to the length of a spanning list over some vector space V.

2 | Intermediate Result: Span of a linearly independent extension of a linearly independent list has more elements than the span of the original list.

2.1 | Lemma

Given a linearly independent list $v = v_1, \dots, v_k$ where each vector $v_1, \dots, v_k \in V$ and another vector v_{k+1} which is linearly independent with v, show that

$$\text{span}\,(v_1, \dots, v_k, v_{k+1})$$

contains elements that are not in

$$\text{span}\,(v_1, \dots, v_k)$$

2.2 | Proof

Because v_{k+1} is linearly independent with v, it cannot be written as a linear combination of elements in v. Thus,

$$v_{k+1} \notin \text{span}\,(v_1, \dots, v_k)$$

However, v_{k+1} must be in the span of the extended list, because we can write v_{k+1} as

$$0v_1 + 0v_2 + \dots + 0v_k + 1v_{k+1}$$

Thus, the extended list contains atleast one element that the original did not.

3 | Proof

Given a spanning list $u = u_1, \dots, u_j$ and a linearly independent list $v = v_1, \dots, v_k$, show that the $|u| \geq |v|$. The Linear Dependence Lemma states that while u is linearly dependent, it is possible to remove some vector u_i from u such that the span stays the same. Thus, there exists a linearly independent list b that has the same span as u, aka that also spans V. Because this list can be obtained by removing elements from u, $|b| \leq |u|$. The linearly independent list v must be shorter than or equal to b in length, because otherwise, span v would have more elements than span b by the intermediate result. Thus, $|v| \leq |b| \leq |u|$.