1 | A real valued matrix

Let $A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$

$$AA^T = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 8 \\ 8 & 13 \end{pmatrix}$$
$$A^TA = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 13 & 8 \\ 8 & 5 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{pmatrix}$$

Then, A^TA is the same thing, but with b, c swapped.

2 | For complex matrices

$$\begin{pmatrix} a+bi & c+di \\ f+gi & j+ki \end{pmatrix}\begin{pmatrix} a+bi & f+gi \\ c+di & j+ki \end{pmatrix} = \begin{pmatrix} a^2-b^2+2abi+c^2-d^2+2cdi & af+agi+bfi-bg \\ af+agi+bfi-bg & f^2-g^2+2fgi+j^2-k^2+2jki \end{pmatrix}$$

I'm not sure if I'm noticing anything different from the real ones, although maybe the variables are just too confusing.

3 | Complex conjugate (A^*A vs AA^*)

$$\begin{pmatrix} a+bi & c+di \\ f+gi & j+ki \end{pmatrix}\begin{pmatrix} a-bi & f-gi \\ c-di & j-ki \end{pmatrix} = \begin{pmatrix} a^2+b^2+c^2+d^2 & () \\ () & f^2+g^2+j^2+k^2 \end{pmatrix}$$

$$\begin{pmatrix} a-bi & f-gi \\ c-di & j-ki \end{pmatrix}\begin{pmatrix} a+bi & c+di \\ f+gi & j+ki \end{pmatrix} = \begin{pmatrix} a^2+b^2+f^2+g^2 & () \\ () & c^2+d^2+j^2+k^2 \end{pmatrix}$$

The diagonals are real-valued, and the matrices are symmetric about the diagonal. I wonder if this means the matrices have identical eigenvalues… how do the diagonals of complex matricies change when they are upper-triangularized?

4 | Transpose distributivity with matrix multiplication

$$(AB)^\top = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} w & x \\ y & z \end{pmatrix}\right)^\top = \begin{pmatrix} aw+by & cw+dy \\ ax+bz & cx+dz \end{pmatrix} = \begin{pmatrix} w & y \\ x & z \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = B^\top A^\top$$

I have no good proof of this for larger matrices or non-square matrices, but it makes sense because both scalar addition and scalar multiplication are commutative and transposing swaps rows for columns. Thus, when a matrix on the left is multiplied by a matrix on the right, it is the same as the left matrix becoming the right matrix but after a transpose, because both operations swap the rows and columns in some sense so they "cancel out".

5 | Determinant distributivity with matrix multiplication

$$\begin{aligned}
&\left|\begin{pmatrix} aw+by & ax+bz \\ cw+dy & cx+dz \end{pmatrix}\right| \\
&= (aw+by)(cx+dz) - (ax+bz)(cw+dy) \\
&= \cancel{acwx} + adwz + bcxy + \cancel{bdyz} - (\cancel{acwx} + adxy + bcwz + \cancel{bdyz}) \\
&= adwz - adxy - bcwz + bcxy \\
&= (ad-bc)(wz-xy) \\
&= \left|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right| \left|\begin{pmatrix} w & x \\ y & z \end{pmatrix}\right|
\end{aligned}$$

This makes sense given that the determinant of a matrix can be considered a "scaling factor." We had talked about the determinant being the area of the parallelogram of