

**Source:**

## 1 | Problem

Suppose  $U$  and  $V$  are finite-dimensional vector spaces and  $S \in \mathcal{L}(V, w)$  and  $T \in \mathcal{L}(U, V)$ . Prove that

$$\dim \text{null } ST \leq \dim \text{null } S + \dim \text{null } T.$$

## 2 | Proof

All vectors  $v \in \text{null } ST$  must have been nulled by  $T$  or  $S$ , and therefore either it must be in  $\text{null } T$  or  $Tv$  in  $\text{range } T \cap \text{null } S$ . Notationally,

$$\text{null } ST = \text{null } T \cup \{v : Tv \in (\text{range } T \cap \text{null } S)\}$$

Note that because this union is equal to  $\text{null } ST$ , it is a vector space. Because no vector can be in both  $\text{null } T$  and  $\{v : Tv \in (\text{range } T \cap \text{null } S)\}$ , the dimension of the union is

$$\dim \text{null } ST = \dim \text{null } T + \dim (\{v : Tv \in (\text{range } T \cap \text{null } S)\})$$

Every value of  $w$  that satisfies  $w \in (\text{range } T \cap \text{null } S)$  will be the output of  $Tv$  for some  $v$ , because the range is defined as all the outputs of  $Tv$ .

$$\dim \text{null } ST = \dim \text{null } T + \dim (\text{range } T \cap \text{null } S)$$

An intersection can only make the dimension of a set smaller, so  $\dim (\text{range } T \cap \text{null } S) \leq \dim \text{null } S$  and

$$\dim \text{range } ST \leq \dim \text{null } S, \dim \text{null } T$$