

1 | Problem

What happens if the Gram–Schmidt Procedure is applied to a list of vectors that is not linearly independent?

2 | Answer

Suppose the list v_1, \dots, v_n is linearly dependent. Then, there exists some v_j s.t. v_1, \dots, v_{j-1} is linearly independent while v_1, \dots, v_j is not. Then, $v_j \in \text{span}(v_1, \dots, v_{j-1})$

Because the Gram-Schmidt procedure preserves prefix spans,

$$v_j \in \text{span}(e_1, \dots, e_{j-1})$$

Thus, the denominator is equivalent to

$$\langle v_j, v_j \rangle = \langle e_1, e_1 \rangle \langle v_j, v_j \rangle = \langle e_j, e_j \rangle$$