

1 | Problem

Suppose U and V are finite-dimensional vector spaces and $S \in \mathcal{L}(V, w)$ and $T \in \mathcal{L}(U, V)$. Prove that

$$\dim \text{null } ST \leq \dim \text{null } S + \dim \text{null } T.$$

2 | Proof

All vectors $v \in \text{null } ST$ must have been nulled by T or S , and therefore either it must be in $\text{null } T$ or Tv in $\text{range } T \cap \text{null } S$. Notationally,

$$\text{null } ST = \text{null } T \cup \{v : Tv \in (\text{range } T \cap \text{null } S)\}$$

Note that because this union is equal to $\text{null } ST$, it is a vector space. Because no vector can be in both $\text{null } T$ and $\{v : Tv \in (\text{range } T \cap \text{null } S)\}$, the dimension of the union is

$$\dim \text{null } ST = \dim \text{null } T + \dim (\{v : Tv \in (\text{range } T \cap \text{null } S)\})$$

Every value of w that satisfies $w \in (\text{range } T \cap \text{null } S)$ will be the output of Tv for some v , because the range is defined as all the outputs of T .

$$\dim \text{null } ST = \dim \text{null } T + \dim (\text{range } T \cap \text{null } S)$$

An intersection can only make the dimension of a set smaller, so $\dim (\text{range } T \cap \text{null } S) \leq \dim \text{null } S$ and

$$\dim \text{range } ST \leq \dim \text{null } S, \dim \text{null } T$$