Source:

1 | TODO construct and write proof for Axler 2.C ex 17

2 | Problem

Prove or give a counterexample:

$$\begin{aligned}
&\dim(U_1 + U_2 + U_3) \\
&\qquad = \dim U_1 + \dim U_2 + \dim U_3 \\
&\qquad\quad - \dim(U_1 \cap U_2) - \dim(U_1 \cap U_3) - \dim(U_2 \cap U_3) \\
&\qquad\quad + \dim(U_1 \cap U_2 \cap U_3)
\end{aligned}$$

3 | Reasoning

By Axler2.41 we know that

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$$

By applying this formula to itself, we find that

$$\begin{aligned}
&\dim(U_1 + U_2 + U_3) \\
&\qquad = \dim((U_1 + U_2) + U_3) \\
&\qquad = \dim(U_1 + U_2) + \dim U_3 - \dim((U_1 + U_2) \cap U_3) \quad = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2) + \dim U_3 - \dim(
\end{aligned}$$

To show that the lemma is true, we would have to show that

$$\begin{aligned}
&\cancel{\dim U_1 + \dim U_2 + \dim U_3 - \dim(U_1 \cap U_2)} - \dim(U_1 \cap U_3) - \dim(U_2 \cap U_3) + \dim(U_1 \cap U_2 \cap U_3) \\
= &\cancel{\dim U_1 + \dim U_2 - \dim(U_1 \cap U_2) + \dim U_3} - \dim((U_1 + U_2) \cap U_3)
\end{aligned}$$

and to provide a counterexample, we just need to find some U_1, U_2, U_3 such that

$$\dim(U_1 \cap U_3) + \dim(U_2 \cap U_3) - \dim(U_1 \cap U_2 \cap U_3) \neq \dim((U_1 + U_2) \cap U_3)$$