1 | Problem

Suppose $T \in \mathcal{L}(V)$. Prove that $T/(\text{null}\, T)$ is injective if and only if $(\text{null}\, T) \cap (\text{range}\, T) = \{0\}$

2 | Proof

First, we will rewrite the problem as logical statements for easier manipulation. The left-hand side "$T/(\text{null}\, T)$ is injective" is equivalent to:

$$\begin{aligned}
(T/U\,(v+U)=0) &\iff (v+U=0)\\
Tv+U=\text{null}\, T &\iff v+U=\text{null}\, T\\
Tv+(\text{null}\, T)=\text{null}\, T &\iff v+(\text{null}\, T)=\text{null}\, T\\
Tv\in\text{null}\, T &\iff v\in\text{null}\, T\\
T^2v=0 &\iff Tv=0
\end{aligned}$$

We can also rewrite the second condition for easier manipulation. The intersection of the null space and the range being 0 is the same as (assuming $w \neq 0$) "if $w \in \text{null}\, T$ then $w \notin \text{range}\, T$" and "if $w \in \text{range}\, T$ then $w \notin \text{null}\, T$". Note that these are contrapositives of eachother, so we just need to work with the second statement.

Assuming $w \neq 0$, these statements are equivalent:

$$\begin{aligned}
(\exists v : Tv = w) &\implies \quad (Tw \neq 0)\\
T^2v \neq 0 & \qquad \forall v \notin (\text{null}\, T)\\
v \notin \text{null}\, T &\implies \quad T^2v \neq 0
\end{aligned}$$

Note that this statement, along with its contrapositive, implies the original $(\text{null}\, T) \cap (\text{range}\, T) = \{0\}$ as desired.

Furthermore, keeping in mind that $w = Tv$ and $w \neq 0$,

$$\begin{aligned}
T^2v \neq 0 &\implies T(Tv) \neq 0\\
&\implies Tw \neq 0\\
&\implies w \notin \text{null}\, T
\end{aligned}$$

which shows the previous relation is an if-and-only-if relation:

$$v \notin \text{null}\, T \iff T^2v \neq 0$$