1 | Axler 6.A exercise 9

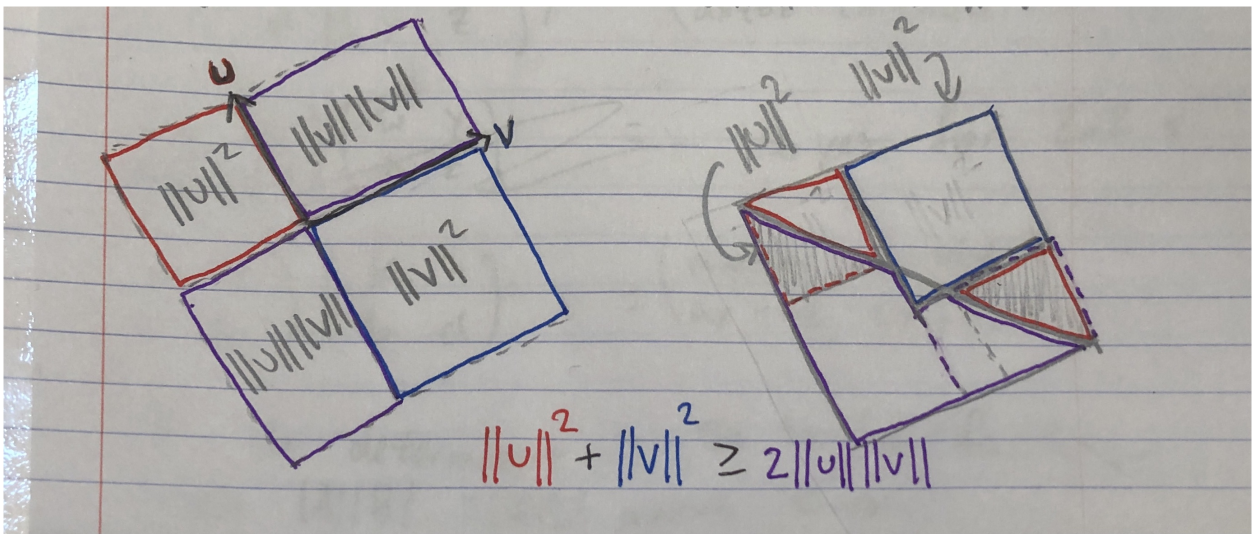
Suppose $u, v \in V$ and $\|u\| \leq 1$ and $\|v\| \leq 1$. Prove that

$$\sqrt{1-\|u\|^2}\sqrt{1-\|u\|^2} \leq 1 - |\langle u, v\rangle|$$

2 | Proof

2.1 | Useful Lemma

$$\|u\|^2 + \|v\|^2 \geq 2\|u\|\|v\|$$



This proof is only valid for inner product spaces over $\mathbb{F}^n$ and the Euclidean norm. An algebraic proof would be better.

2.2 | Cauchy-Schwarz Corollary

$$|\langle u, v\rangle| \leq \|u\|\|v\| \implies 1 - \|u\|\|v\| \quad \leq 1 - |\langle u, v\rangle|$$

2.3 | Main Proof

Now, to show that the square of the left hand side is less than or equal to the square of the right hand side,

$$\begin{aligned}
(1-\|u\|^2)(1-\|v\|^2) =& 1 - \|u\|^2 - \|v\|^2 + \|u\|^2\|v\|^2 \\
=& 1 - (\|u\|^2 + \|v\|^2) + \|u\|^2\|v\|^2 \\
\leq& 1 - 2\|u\|\|v\| + \|u\|^2\|v\|^2 && \text{by the earlier lemma} \\
=& (1 - \|u\|\|v\|)^2 \\
\leq& (1 - |\langle u, v\rangle|)^2 && \text{by the Cauchy-Schwarz corollary}
\end{aligned}$$

Taking square roots of both sides proves the desired result. $\square$