**Source**:

# 1 | Lemma

The length of a linearly indpendent list is less than or equal to the length of a spanning list over some vector space $V$.

# 2 | Intermediate Result: Span of a linearly independent extension of a linearly independent list has more elements than the span of the original list.

## 2.1 | Lemma

Given a linearly independent list $v = v_1, \ldots, v_k$ where each vector $v_1, \ldots, v_k \in V$ and another vector $v_{k+1}$ which is linearly independent with $v$, show that

$$\text{span}\left(v_1, \ldots, v_k, v_{k+1}\right)$$

contains elements that are not in

$$\text{span}\left(v_1, \ldots, v_k\right)$$

## 2.2 | Proof

Because $v_{k+1}$ is linearly independent with $v$, it cannot be written as a linear combination of elements in $v$. Thus,

$$v_{k+1} \notin \text{span}\left(v_1, \ldots, v_k\right)$$

However, $v_{k+1}$ must be in the span of the extended list, because we can write $v_{k+1}$ as

$$0v_1 + 0v_2 + \ldots + 0v_k + 1v_{k+1}$$

Thus, the extended list contains atleast one element that the original did not.

# 3 | Proof

Given a spanning list $u = u_1, \ldots, u_j$ and a linearly independent list $v = v_1, \ldots, v_k$, show that the $|u| \geq |v|$. Assume for the sake of contradiction that $|u| > |v|$.