1 | Problem

Suppose $T \in \mathcal{L}(V)$. Prove that $T/(\text{null}\, T)$ is injective if and only if $(\text{null}\, T) \cap (\text{range}\, T) = \{0\}$

2 | Proof

2.1 | Condition Manipulation

First, we will rewrite the problem as logical statements for easier manipulation.

2.1.1 | Left Condition

The left-hand side "$T/(\text{null}\, T)$ is injective" is equivalent to:

$$\begin{aligned}
(T/U\,(v+U)=0) &\implies (v+U=0) && \text{(alternate definition of injective)}\\
Tv+U=\text{null}\, T &\implies v+U=\text{null}\, T && (T/U(v+U) \text{ is defined as } Tv+U)\\
Tv+(\text{null}\, T)=\text{null}\, T &\implies v+(\text{null}\, T)=\text{null}\, T && (U=\text{null}\, T)\\
Tv \in \text{null}\, T &\implies v \in \text{null}\, T\\
T^2v=0 &\implies v \in \text{null}\, T
\end{aligned}$$

2.1.2 | Right Condition

We can also rewrite the right-hand condition for easier manipulation. The intersection of the null space and the range being 0 is the same as (assuming $w \neq 0$) "if $w \in \text{null}\, T$ then $w \notin \text{range}\, T$" and "if $w \in \text{range}\, T$ then $w \notin \text{null}\, T$". Note that these are contrapositives of eachother, so we just need to work with the second statement.

Thus, assuming $w \neq 0$, these statements are equivalent:

$$\begin{aligned}
(\exists v : Tv=w) &\implies (Tw \neq 0) && \text{(definitions of null space and range)}\\
v \notin \text{null}\, T &\implies T^2v \neq 0 && (w \neq 0)\\
T^2v=0 &\implies v \in \text{null}\, T && \text{(contrapositive)}
\end{aligned}$$

2.2 | Proof

The statements are equivalent. ■