11 | cubic and a line

11.1 | show tangency

$$\frac{dy}{dx} = \frac{d}{dx}(4x^2 - x^3) = 8x - 3x^2\Big|_3$$
$$= 24 - 27 = -3$$
$$\frac{dy}{dx} = \frac{d}{dx}(18 - 3x) = -3$$

11.2 | area between curves

$$\int_3^6 18 - 3x - 4x^2 + x^3 dx \to \frac{1}{4}x^4 - \frac{1}{3}4x^3 - \frac{1}{2}3x^2 + 18x + C$$
$$= \frac{1}{4}(6)^4 - \frac{1}{3}4(6)^3 - \frac{1}{2}3(6)^2 + 18(6) - \frac{1}{4}(3)^4 + \frac{1}{3}4(3)^3 + \frac{1}{2}3(3)^2 - 18(3)$$
$$= \boxed{\frac{261}{4}}$$

12 | estimate area

Right handed Riemann Sum:

$$0.5 + 4 + 10 + 13 + 10 + 0 = 37.5$$

13 | estimate area again

$$4(200 + 2700 + 1100 + 4000 + 200) = 32800$$

14 | area between curves

$$\int_0^{10} 2200e^{0.024t}dx - \int_0^{10} 1360e^{0.018t}dx = \frac{1}{0.024}2200e^{0.024t} - \frac{1}{0.018}1360e^{0.018t}$$
$$\implies \quad \frac{1}{0.024}2200e^{0.24} - \frac{1}{0.018}1360e^{0.18} - \frac{1}{0.024}2200 + \frac{1}{0.018}1360 \quad \approx 9964$$

15 | meaning of area

The shaded region represents the profit made between producing 50 units and 100 units.

16 | TODO slicing pizza into three using parallel cuts

The problem of placing slices is the same if we only worry about the top half of the pizza. Thus, we can choose some x for the first slice s.t.

$$2\int_{-7}^{x}\sqrt{7^2-t^2}dt = \int_{x}^{7}\sqrt{7^2-t^2}dt$$
$$2\int_{-7}^{x}\sqrt{7^2-t^2}dt - \int_{x}^{7}\sqrt{7^2-t^2}dt = 0$$
$$2\int_{-7}^{x}\sqrt{7^2-t^2}dt + \int_{7}^{x}\sqrt{7^2-t^2}dt = 0$$
$$2\left(\int_{0}^{x}\sqrt{7^2-t^2}dt - \int_{0}^{-7}\sqrt{7^2-t^2}dt\right) + \left(\int_{0}^{x}\sqrt{7^2-t^2}dt - \int_{0}^{7}\sqrt{7^2-t^2}dt\right) = 0$$
$$2\left(\int_{0}^{x}\sqrt{7^2-t^2}dt + \int_{-7}^{0}\sqrt{7^2-t^2}dt\right) + \left(\int_{0}^{x}\sqrt{7^2-t^2}dt - \int_{0}^{7}\sqrt{7^2-t^2}dt\right) = 0$$
$$2\int_{0}^{x}\sqrt{7^2-t^2}dt + 2\int_{-7}^{0}\sqrt{7^2-t^2}dt + \int_{0}^{x}\sqrt{7^2-t^2}dt - \int_{0}^{7}\sqrt{7^2-t^2}dt = 0$$
$$3\int_{0}^{x}\sqrt{7^2-t^2}dt + 2\int_{-7}^{0}\sqrt{7^2-t^2}dt - \int_{0}^{7}\sqrt{7^2-t^2}dt = 0$$
$$3\int_{0}^{x}\sqrt{7^2-t^2}dt + \int_{-7}^{0}\sqrt{7^2-t^2}dt = 0$$
$$3\int_{0}^{x}\sqrt{7^2-t^2}dt + \frac{49\pi}{4} = 0$$

Now, we need to use trigonometric substitution, apparently.

$$x = a\sin\theta, dx = a\cos\theta d\theta$$

Or, you could use David's method, which is just better (cut horizontally instead of vertically)

$$\int_{-7}^{7}\sqrt{49-x^2} - adx = \frac{49\pi}{3}$$
$$\int_{-7}^{7}\sqrt{49-x^2}dx - \int_{-7}^{7}adx = \frac{49\pi}{3}$$
$$\frac{49\pi}{2} - \int_{-7}^{7}adx = \frac{49\pi}{3}$$
$$\frac{49\pi}{2} - (7a - -7a) = \frac{49\pi}{3}$$
$$\frac{49\pi}{6} = 14a$$
$$a = \frac{49\pi}{84} = \frac{7\pi}{12}$$

Since a is the upper half of the center portion, the width of each slice is $2a = \frac{7\pi}{6}$

17 | **tractrix**

17.1 | **derivative**

At any moment, if the boat is at (x, y) and the puller is at $(0, h)$, then the velocity of the boat is in the direction

$$\frac{\Delta y}{\Delta x}$$

Where $\Delta x = -x$ and Δy can be found using the Pythagorean Theorem

$$\begin{aligned} L^2 &= \Delta y^2 + x^2 \\ \implies \quad \Delta y &= \sqrt{L^2 - x^2} \end{aligned}$$

Thus, the boat is moving in the direction

$$\frac{\sqrt{L^2 - x^2}}{-x}$$

17.2 | **integral**

$$\int \frac{\sqrt{L^2 - x^2}}{-x} dx = -\int \frac{1}{x} \sqrt{L^2 - x^2} dx$$

Let $x = L\sin\theta, dx = L\cos\theta d\theta$

$$\begin{aligned}
&= -\int \frac{1}{L\sin\theta} \sqrt{L^2 - L^2\sin^2\theta} dx \\
&= -\int \frac{1}{L\sin\theta} L\sqrt{1 - \sin^2\theta} dx \\
&= -\int \frac{1}{L\sin\theta} L\sqrt{\cos^2\theta} dx \\
&= -\int \frac{L\cos\theta}{L\sin\theta} L\cos\theta d\theta \\
&= -\int L\frac{\cos^2\theta}{\sin\theta} d\theta \\
&= -L\int \frac{1}{\sin\theta} d\theta + L\int \sin\theta d\theta \\
&= L\ln|\csc\theta + \cot\theta| - L\cos\theta + C \\
&= L\ln\left|\frac{L^2}{x^2} + \frac{\sqrt{L^2 - x^2}}{x^2}\right| - L\frac{\sqrt{L^2 - x^2}}{L^2} + C
\end{aligned}$$

18 | **TODO water displacement**

Plan: find a function $f(r)$ which represents the amount of water displaced for any radius, then take the derivative and find roots.