1 | **Problem**

Suppose $T \in \mathcal{L}(V)$. Prove that $T/(\text{null } T)$ is injective if and only if $(\text{null } T) \cap (\text{range } T) = \{0\}$

2 | **Proof**

2.1 | **Condition Manipulation**

First, we will rewrite the problem as logical statements for easier manipulation.

2.1.1 | **Left Condition**

The left-hand side "$T/(\text{null } T)$ is injective" is equivalent to:

$$
\begin{aligned}
(T/U \, (v + U) = 0) &\implies (v + U = 0) \\
Tv + U = \text{null } T &\implies v + U = \text{null } T \\
Tv + (\text{null } T) = \text{null } T &\implies v + (\text{null } T) = \text{null } T \\
Tv \in \text{null } T &\implies v \in \text{null } T \\
T^2 v = 0 &\implies Tv = 0
\end{aligned}
$$

2.1.2 | **Right Condition**

We can also rewrite the right-hand condition for easier manipulation. The intersection of the null space and the range being 0 is the same as (assuming $w \neq 0$) "if $w \in \text{null } T$ then $w \notin \text{range } T$" and "if $w \in \text{range } T$ then $w \notin \text{null } T$". Note that these are contrapositives of eachother, so we just need to work with the first statement.

Thus, assuming $w \neq 0$, these statements are equivalent:

$$
\begin{aligned}
(\exists v : Tv = w) &\implies \quad (Tw \neq 0) \\
T^2 v \neq 0 & \qquad \forall v \notin \text{null } T \\
v \notin \text{null } T &\implies \quad T^2 v \neq 0
\end{aligned}
$$

Note that this statement, along with its contrapositive, implies the original $(\text{null } T) \cap (\text{range } T) = \{0\}$.

Furthermore, keeping in mind that $w = Tv$ and $w \neq 0$,

$$
\begin{aligned}
T^2 v \neq 0 &\implies T(Tv) \neq 0 \\
&\implies Tw \neq 0
\end{aligned}
$$

which shows the previous relation is an if-and-only-if relation:

$$
v \notin \text{null } T \iff T^2 v \neq 0
$$