1 | Axler 6.A exercise 9

Suppose $u, v \in V$ and $\|u\| \leq 1$ and $\|v\| \leq 1$. Prove that

$$\sqrt{1 - \|u\|^2}\sqrt{1 - \|u\|^2} \leq 1 - |\langle u, v\rangle|$$

2 | Proof

We will prove this by showing that the left hand side is less than or equal to an intermediate term, which is less than or equal to the right hand side.

$$\begin{aligned} |\langle u, v\rangle| &\leq \|u\|\|v\| \\ \implies 1 - |\langle u, v\rangle| &\geq 1 - \|u\|\|v\| \end{aligned}$$

This intermediate value is obtained using the Cauchy-Schwarz inequality.

Now, to show that the left hand side is less than or equal to the right hand side,

$$\begin{aligned} &\sqrt{1 - \|u\|^2}\sqrt{1 - \|v\|^2} \\ =&\sqrt{(1 - \|u\|^2)(1 - \|v\|^2)} \\ =&\sqrt{1 - \|u\|^2 - \|v\|^2)} \end{aligned}$$