Source:

1 | Lemma

The size of a linearly indpendent set is less than or equal to the size of a spanning set over some vector space V.

2 | Intermediate Result: Span of a linearly independent extension of a linearly independent list has more elements than the span of the original list.

2.1 | Lemma

Given a linearly independent list $v = v_1, \ldots, v_k$ where each vector $v_1, \ldots, v_k \in V$ and another vector v_{k+1} which is linearly independent with v, show that

$$\text{span}\,(v_1, \ldots, v_k, v_{k+1})$$

contains elements that are not in

$$\text{span}\,(v_1, \ldots, v_k)$$

2.2 | Proof

Because v_{k+1} is linearly independent with v, it cannot be written as a linear combination of elements in v. Thus,

$$v_{k+1} \notin \text{span}\,(v_1, \ldots, v_k)$$