

## 1 | Problem

Prove or give a counterexample:

$$\begin{aligned} \dim(U_1 + U_2 + U_3) \\ &= \dim U_1 + \dim U_2 + \dim U_3 \\ &\quad - \dim(U_1 \cap U_2) - \dim(U_1 \cap U_3) - \dim(U_2 \cap U_3) \\ &\quad + \dim(U_1 \cap U_2 \cap U_3) \end{aligned}$$

## 2 | Reasoning

By Axler 2.41 we know that

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$$

By applying this formula to itself, we find that

$$\begin{aligned} \dim(U_1 + U_2 + U_3) \\ &= \dim((U_1 + U_2) + U_3) \\ &= \dim(U_1 + U_2) + \dim U_3 - \dim((U_1 + U_2) \cap U_3) \\ &= \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2) + \dim U_3 - \dim((U_1 + U_2) \cap U_3) \end{aligned}$$

To show that the lemma is true, we would have to show that

$$\begin{aligned} &\dim U_1 + \dim U_2 + \dim U_3 - \dim(U_1 \cap U_2) - \dim(U_1 \cap U_3) - \dim(U_2 \cap U_3) + \dim(U_1 \cap U_2 \cap U_3) \\ &= \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2) + \dim U_3 - \dim((U_1 + U_2) \cap U_3) \end{aligned}$$

and to provide a counterexample, we just need to find some  $U_1, U_2, U_3$  such that

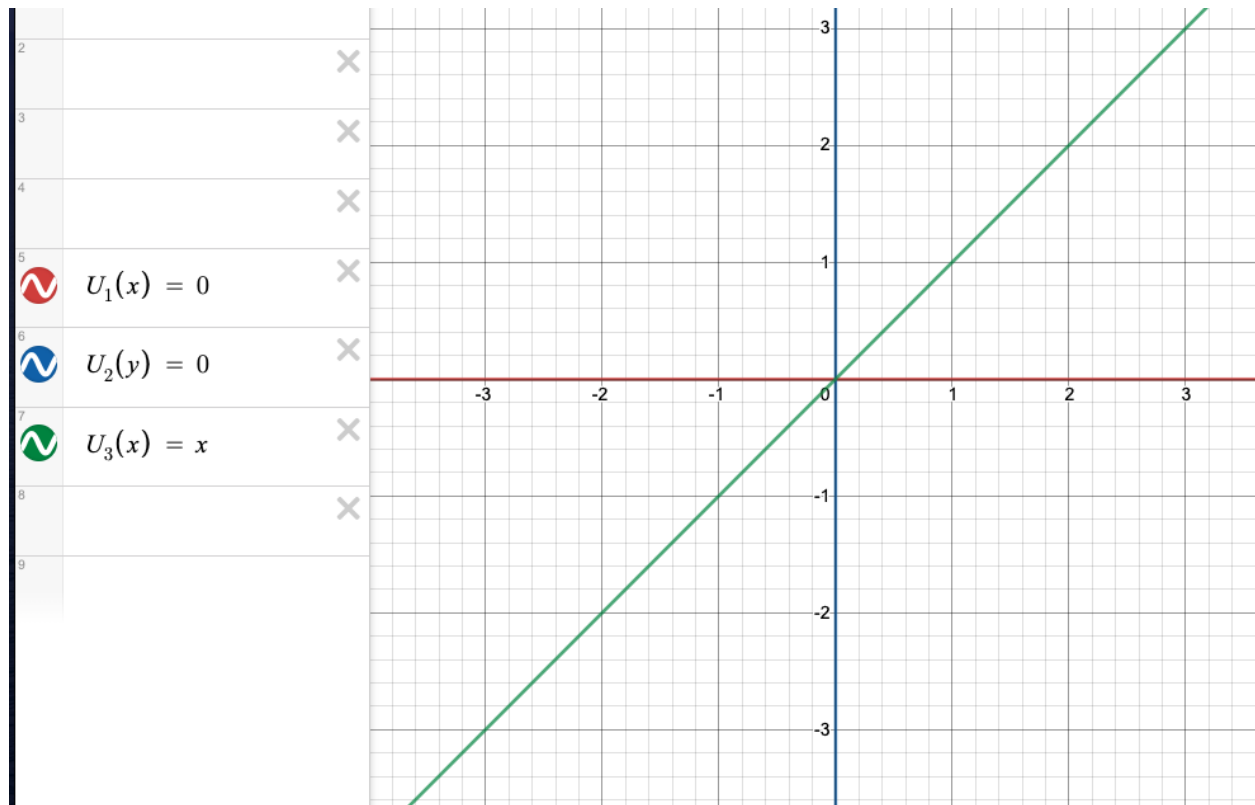
$$\dim(U_1 \cap U_3) + \dim(U_2 \cap U_3) - \dim(U_1 \cap U_2 \cap U_3) \neq \dim((U_1 + U_2) \cap U_3)$$

## 3 | Counterexample

If we choose

$$\begin{aligned} U_1 &= \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} : x \in \mathbb{R} \right\} \\ U_2 &= \left\{ \begin{pmatrix} 0 \\ x \end{pmatrix} : x \in \mathbb{R} \right\} \\ U_3 &= \left\{ \begin{pmatrix} x \\ x \end{pmatrix} : x \in \mathbb{R} \right\} \end{aligned}$$

then the graph of the subspaces looks like this:



and the dimension of each intersection is 0 while the dimension of  $(U_1 + U_2) \cap U_3 = 2$ . Thus, we have

$$\overset{0}{\dim(U_1 \cap U_3)} + \overset{0}{\dim(U_2 \cap U_3)} - \overset{0}{\dim(U_1 \cap U_2 \cap U_3)} \neq \dim((U_1 + U_2) \cap U_3) \\ \implies 0 \neq 2$$

In summary, the sum of these subspaces is  $\mathbb{R}^2$  and the dimension of the sum is 2, but

$$\dim(U_1 + U_2 + U_3) = 2 \neq 3 = 1 + 1 + 1 - 0 - 0 - 0 + 0$$