

**Source:**

## 1 | Problem

Suppose  $T \in \mathcal{L}(V, W)$  and  $U$  is a subspace of  $V$ . Let  $\pi$  denote the quotient map from  $V$  onto  $V/U$ . Prove that there exists  $S \in \mathcal{L}(V/U, W)$  such that  $T = S \circ \pi$  if and only if  $U \subseteq \text{null } T$ .

Intuitively, if we mod out part of the null  $T$ , then we should still be able to have a map that does what  $T$  would do, and if we are able to do what  $T$  would do, then we only removed part of null  $T$  and lost no information.

## 2 | Forward Direction by Contrapositive

Intuitively, if we lost information, then we can't reconstruct what  $T$  would do.

If  $U \not\subseteq \text{null } T$ , then there exists  $v \in U$  s.t.  $Tv \neq 0$ . This is some of the "information" that was "lost". Because  $v \in U$ ,

$$\pi v = U + v = U$$

Because  $U$  is the additive identity (0) in  $V/U$ , and because linear maps take zero to zero,  $S \in \mathcal{L}(V/U, W)$  must take  $\pi v = 0$  to zero. Thus, either  $S(\pi v) \neq Tv$  or  $S$  is not a linear map, both of which are contradictions.

## 3 | Reverse Direction