Source:

1 | Definition

#definition Axler3.2 Linear Map #aka linear transformation A *linear map* from V to W is a function $T : V \to W$ with the following properties:

1.1 | Additivity

$$T(u+v) = Tu + Tv \forall u, v \in V$$

1.2 | Homogenity

$$T(\lambda v) = \lambda(Tv) \forall \lambda \in \mathbb{F}, v \in V$$

2 | Other Notation

2.1 | Set of Maps

#definition Axler3.3 $\mathcal{L}(V, W)$

The set of all linear maps from V to W is denoted $\mathcal{L}(V, W)$.

3 | Examples

3.1 | zero (0)

Zero is a function $0 : V \to W$ s.t. $0v = 0 \forall v \in V$. (It takes all vectors in V and maps them to the additive identity of W)

3.2 | identity (I)

The identity maps each from one vector space to itself (in the same vector space):

$$I \in \mathcal{L}(V, V), v \in V : Iv = v$$

3.3 | differentiation (D)

$$D \in \mathcal{L}\left(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R})\right) : Dp = p'$$

Basically stating that for two polynomials $a, b \in \mathcal{P}(\mathbb{R})$, $a' + b' = (a+b)'$ and with a constant $\lambda \in \mathcal{R}$ $(\lambda a)' = \lambda a'$.

3.4 | integration

3.5 | multiplication by x^2

$$T \in \mathcal{L}\left(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R})\right) : (Tp)(x) = x^2 p(x)$$

is a linear map

3.6 | **backward shift**

F^∞ is the vector space of all sequences of elements in $\mathbb{F}$.

$$T \in \mathcal{L}\left(\mathbb{F}^\infty, \mathbb{F}^\infty\right) : T(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots)$$

3.7 | $\mathbb{F}^n \to \mathbb{F}^m$

Given a "coefficent matrix" $A : A_{j,k} \in \mathbb{F} \forall j = 1, \ldots, m; \forall k = 1, \ldots, n$, define $T \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$:

$$T(x_1, \ldots, x_n) = (A_{1,1}x_1 + A_{1,2}x_2 + \cdots + A_{1,n}x_n,\ A_{2,1}x_1 + \cdots + A_{2,n}x_n,\ \ldots,\ A_{m,1}x_1 + \cdots + A_{m,n}x_n)$$

Notice that this is equivalent to taking A as a $m \times n$ matrix and dot producting it with the $n \times 1$ matrix $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.

4 | **Results**

4.1 | **Axler3.5 Linear maps and basis of domain**

If $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_n \in W$, then there exists a unique linear map $T : V \to W$ s.t.

$$Tv_j = w_j \forall j \in 1, \ldots, n$$

#aka given a basis v of V, there is a unique linear map that maps v to each $w \in W$.

4.1.1 | **#careful**

1. same dimension

   V and W are both of dimension n.
2. same field

   We defined V and W to both be vector spaces over the same field $\mathbb{F}$ which is either $\mathbb{R}$ or $\mathbb{C}$.
3. v is a basis

   $v_1, \ldots, v_n$ must be a basis of V (because that fact is used in the proof)

4.1.2 | **Questions**

1. DONE #question what does it mean that "T is uniquely determined on span$(v_1, \ldots, v_n)$? question

   There's no ambiguity and so we know exactly which map it's refering to, and thus it is uniquely determined.