1 | Axler 5.B Exercise 13

> Suppose W is a complex vector space and $T \in \mathcal{L}(W)$ has no eigenvalues. Prove that every subspace of W invariant under T is either $\{0\}$ or infinite-dimensional.

2 | Proof

5.21 states

> Every operator on a finite-dimensional, nonzero, complex vector space has an eigenvalue.

W is given as a complex vector space, so to have no eigenvalues, it must be zero or infinite-dimensional. If W is zero, then all subspaces must also be zero. Thus, only the infinite-dimensional case remains to be shown.

By definition (5.14), for all subspaces V of W invariant under T, $T\big|_V$ exists in $\mathcal{L}(V)$.

Suppose for the sake of contradiction that V is nonzero and finite-dimensional. By 5.21, $T\big|_V$ has an eigenvalue. Then, there exists some $\lambda \in \mathbb{C}$ and some $v \neq 0 \in V$ s.t.

$$T\big|_V(v) = \lambda v$$

However, $T\big|_V$ is defined by $v \mapsto Tv$, which implies that

$$Tv = T\big|_U(v) = \lambda v$$

for $v \neq 0 \in W$, which makes λ an eigenvalue of T. This contradicts T having no eigenvalues, so there must be no subspaces V invariant under T that are nonzero and finite-dimensional. Thus, all such subspaces must be 0 on infinite-dimensional. $\square$