Source:

1 | Problem: Axler 3.E exercise 18

> Suppose $T \in \mathcal{L}(V, W)$ and U is a subspace of V. Let π denote the quotient map from V onto V/U. Prove that there exists $S \in \mathcal{L}(V/U, W)$ such that $T = S \circ \pi$ if and only if $U \subseteq \text{null } T$.

Intuitively, if we mod out part of the null T, then we should still be able to have a map that does what T would do. If we are able to do what T would do, then when modding out U we only removed part of null T and lost no information.

2 | Forward Direction

Intuitively, we can treat $S \circ \pi$ as a single map and take a basis of V to the same place that T would, and the maps would be equal.

Let S be a relation between V/U and W defined by

$$S(U + v) = Tv$$

If S is well defined (every element in V/U is mapped to exactly one place), then S will inherit additivity and homogeneity from T and $S \circ \pi$ will equal T.

Let $v \in V$ and $v' \in V/U$ s.t. $v' = \pi v$ (v' is where π takes v). Then, to show that S is well defined, we must show that v has atleast one and at most one image through $S \circ \pi$.

Because πv is well defined, and $U + v$ was arbitrary in the definition of S, each v must have atleast one image in W.

Take S to be an arbitrary linear map. The only restriction on S that could cause $S(U + v) \neq Tv$ is $S(0) = 0$ (this statement is not watertight). Thus, S is defined if $\forall U + v = U = 0$, $Tv = 0$. Equivalently, S is defined if $U \subseteq \text{null } T$, which is given in the problem.

Thus, S is well defined. To show that it inherits additivity and homogeneity:

$$S(U + u) + S(U + v) = Tu + Tv = T(u + v) = S(U + u + U + v) = S(U + (u + v))$$

$$\lambda \left(S(U + v)\right) = \lambda Tv = T(\lambda v) = S(U + (\lambda v))$$

Thus, S is linear, and $S \circ \pi = T$ if $U \subseteq \text{null } T$.

3 | Reverse Direction by Contrapositive

Intuitively, if we lost information, then we can't reconstruct what T would do.

Assume $U \nsubseteq \text{null } T$. There exists $v \in U$ s.t. $Tv \neq 0$. This is some of the "information" that was "lost". Because $v \in U$,

$$\pi v = U + v = U$$

Because U is the additive identity (0) in V/U, and because linear maps take zero to zero, $S \in \mathcal{L}(V/U, W)$ must take $\pi v = 0$ to zero. Thus, either $S(\pi v) \neq Tv$ or S is not a linear map, both of which are contradictions.

This shows that if $U \nsubseteq \text{null } T$, then $S \notin \mathcal{L}(V/U, W)$ or $T \neq S \circ \pi$. Thus, if $S \in \mathcal{L}(V/U, W)$ and $T = S \circ \pi$, then $U \subseteq \text{null } T$.