

1 | A real valued matrix

Let $A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$

$$AA^T = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 8 \\ 8 & 13 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 13 & 8 \\ 8 & 5 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{pmatrix}$$

Then, $A^T A$ is the same thing, but with b, c swapped.

2 | For complex matrices

$$\begin{pmatrix} a+bi & c+di \\ f+gi & j+ki \end{pmatrix} \begin{pmatrix} a+bi & f+gi \\ c+di & j+ki \end{pmatrix} = \begin{pmatrix} a^2 - b^2 + 2abi + c^2 - d^2 + 2cdi & af + agi + bfi - bg \\ af + agi + bfi - bg & f^2 - g^2 + 2fgi + j^2 - k^2 + 2jki \end{pmatrix}$$

I'm not sure if I'm noticing anything different from the real ones, although maybe the variables are just too confusing.

3 | Complex conjugate ($A^* A$ vs AA^*)

$$\begin{pmatrix} a+bi & c+di \\ f+gi & j+ki \end{pmatrix} \begin{pmatrix} a-bi & f-gi \\ c-di & j-ki \end{pmatrix} = \begin{pmatrix} a^2 + b^2 + c^2 + d^2 & () \\ () & f^2 + g^2 + j^2 + k^2 \end{pmatrix}$$

$$\begin{pmatrix} a-bi & f-gi \\ c-di & j-ki \end{pmatrix} \begin{pmatrix} a+bi & c+di \\ f+gi & j+ki \end{pmatrix} = \begin{pmatrix} a^2 + b^2 + f^2 + g^2 & () \\ () & c^2 + d^2 + j^2 + k^2 \end{pmatrix}$$

The diagonals are real-valued, and the matrices are symmetric about the diagonal. I wonder if this means the matrices have identical eigenvalues... how do the diagonals of complex matrices change when they are upper-triangularized?

4 | Transpose distributivity with matrix multiplication

$$(AB)^T = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w & x \\ y & z \end{pmatrix} \right)^T = \begin{pmatrix} aw + by & cw + dy \\ ax + bz & cx + dz \end{pmatrix} = \begin{pmatrix} w & y \\ x & z \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = B^T A^T$$

I have no good proof of this for larger matrices or non-square matrices, but it makes sense because both scalar addition and scalar multiplication are commutative and transposing swaps rows for columns. Thus, when a matrix on the left is multiplied by a matrix on the right, it is the same as the left matrix becoming the right matrix but after a transpose, because both operations swap the rows and columns in some sense so they "cancel out".

5 | Determinant distributivity with matrix multiplication

$$\begin{aligned}
 & \left| \begin{pmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{pmatrix} \right| \\
 &= (aw + by)(cx + dz) - (ax + bz)(cw + dy) = \cancel{acwx} + adwz + bcxy + \cancel{bdyz} - (\cancel{acwx} + adxy + bcwz + \cancel{bdyz}) \\
 &= adwz - adxy - bcwz + bcxy = (ad - bc)(wz - xy) \\
 &= \left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right| \left| \begin{pmatrix} w & x \\ y & z \end{pmatrix} \right|
 \end{aligned}$$