Source:

1 | Definitions

1.1 | affine subset

An affine subset of a vector space V is of the form $U + v$ where $U \subseteq V$ and $v \in V$.

1.2 | product space

The product of some vector spaces $V_1 \times \cdots \times V_n$ is the set of lists of vectors with one from each respective space:

$$\{(v_1, \ldots, v_n) : v_1 \in V_1, \ldots, v_n \in V_n\}$$

1.3 | quotient space

A quotient space V/U is the set of affine subsets $\{U + v : v \in V\}$ (although some of those affine subsets are equivalent).

1.4 | equivalence relation

An equivalence relation is a set of elements that are considered equivalent (equal to eachother). For example, in a vector space U, $U + 0 = U + u \forall u \in U$.

2 | Why ”product” and ”quotient” are used to describe these operations

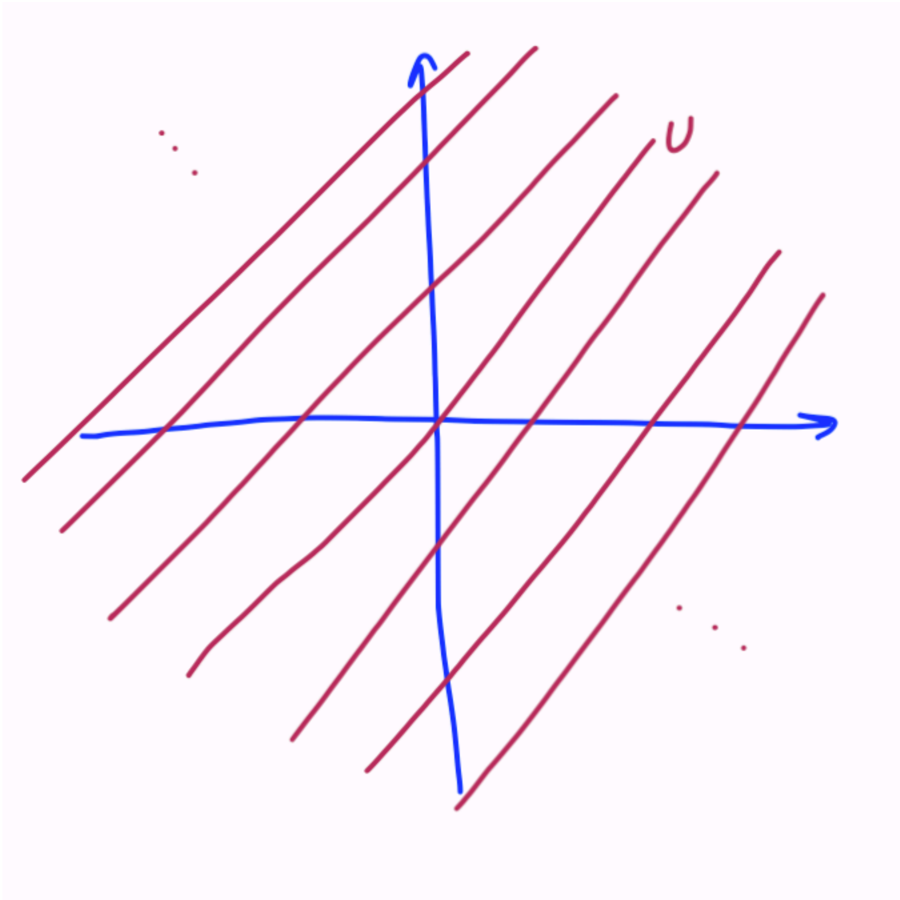
The product of vector spaces is essentially a cartesian product. With real numbers, the product is like stacking copies of an operand in a new direction (product of two numbers for area of a plane, product of three for the volume of a space). Here, we are doing essentially the same thing for vector spaces (each vector is combined within a list with other vector spaces, but they do not interact with eachother and are orthagonal, in a sense).

A quotient space is like taking (dividing) out part of a vector space. It's like taking a modulo because a subset (U) is collapsed to zero and some things become equivalent. It is like removing (”dividing”) a subset of the basis (those that form a basis of U), where the basis itself can be represented as the cartesian product $\mathbb{F}^n$ (where n is the dimension of V).

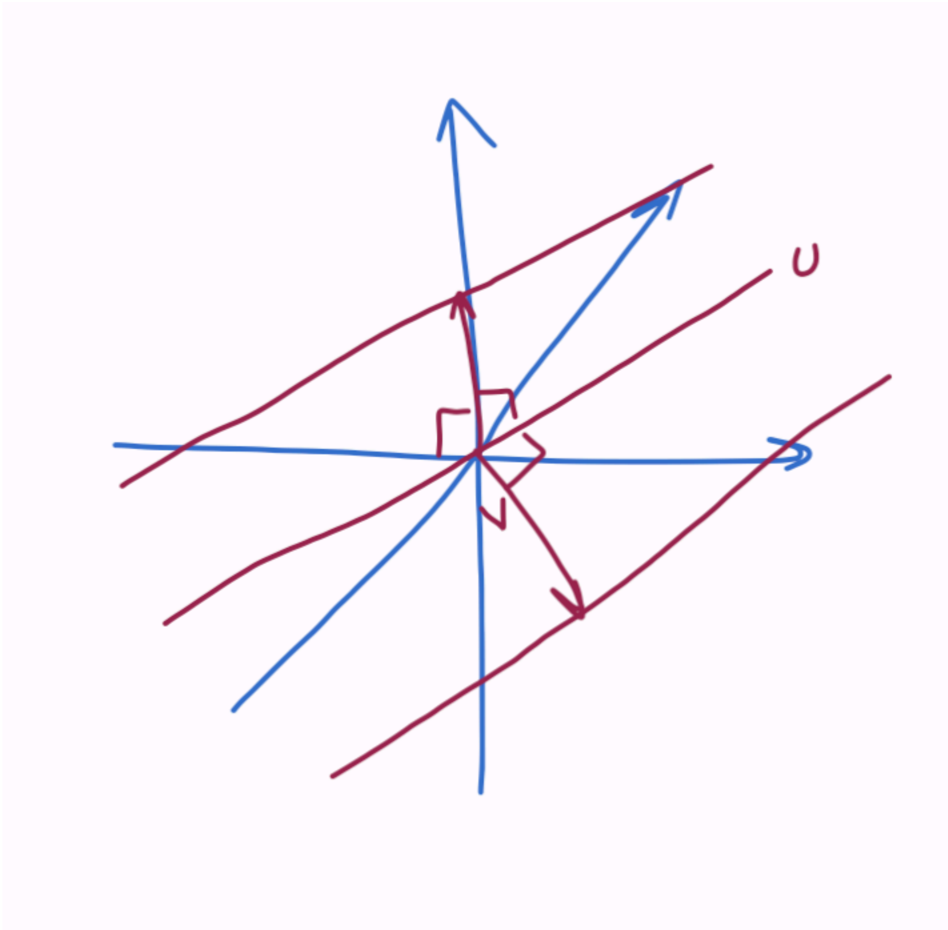
3 | Examples of quotient spaces

Let $U = \{(x, y) : y = 2x; x, y \in \mathbb{R}\}$.

In $\mathbb{R}^2/U$, vectors are of the form $U + v$ where $v \in V$ and any $v \in U$ is equivalent to $U + 0$ or the line $y = 2x$, which is the additive identity. Parallel spaces (”copies” that are shifted over) are the other elements in the space. The space is dimension one, since the copies can be shifted only in one orthoganal direction. A reasonable standard basis of the space is $(U + (-2, 1))$



In $\mathbb{R}^3/U$, the afine subsets extend out in two orthoganal directions. A reasonable standard basis is $(U + (-2,1,0), U + (0,0,1))$.



4 | Proove $\dim V/U = \dim V - \dim U$ when V is a finite dim vec space

Let $\pi \in \mathcal{L}(V, V/U)$ be defined by

$$\pi v = U + v$$

For each $v \in v_1, \ldots, v_n$ where $v_1, \ldots, v_n$ is a basis of V.

$\dim$ null $\pi = \dim U$ because U is 0 and $U + u = U$ iff $u \in U$. $\dim$ range $\pi = \dim V/U$ because v is arbitrary and every vector in V/U is of the form $U + v$.

Then, by the Fundamental Theorem of Linear Maps,

$$\begin{aligned} \dim V &= \dim V/U + \dim U \\ \dim V - \dim U &= \dim V/U \end{aligned}$$

5 | Suppose V is finite dimensional and $S, T \in \mathcal{L}(V)$. Prove that

5.1 | ST is invertible iff S and T are invertable

5.1.1 | if

Given that S and T are invertible, $\dim$ null $S = \dim$ null $T = 0$. ST is an operator on V, so if it is injective then it is invertible.

For ST to send some v to zero, either $v \in$ null T or $Tv \in$ null S. For some $s_1, \ldots, s_n$ is a basis of null S and $t_1, \ldots, t_m$ is a basis of null T, $v \in$ null ST iff v can be expressed as a linear combination of $s_1, \ldots, s_n, t_1, \ldots, t_m$. Let $U =$ span $s_1, \ldots, s_n, t_1, \ldots, t_m$. Then, $v \in$ null ST iff $v \in U$, aka null $ST = U$. Because $\dim U \leq n + m$, and $n = m = 0$, $\dim$ null $ST = 0$ and ST is injective, bijective, and invertible.

5.1.2 | **only if**

ST is invertible implies $\dim$ null $ST = 0$. S, T are operators, and if an operator is not injective then it is also not surjective and vise versa. Because linear maps send zero to zero (and using the logic from the previous part),

$$\dim \text{null } ST \geq \max\{\text{dim null } S, \text{dim null } T\}$$
$$\max\{\text{dim null } S, \text{dim null } T\} \leq \dim \text{null } ST = 0$$

Thus, dim null $S,$ dim null $T \leq 0$, S, T are injective, and invertible.

5.2 | $ST = I$ **iff** $TS = I$

Let $n = \dim V$. Given that S, T are operators, and $TS = I$, S, T are invertible (both injective and surjective) from the previous problem. Operators take bases to bases, because there exists two bases of V $s_1, \ldots, s_n$ and $t_1, \ldots, t_n$ s.t.

$$S(s_i) = t_i$$

and

$$T(t_i) = s_i$$

Then, for each $t_i \in t_1, \ldots, t_n$,

$$ST(t_i) = S(s_i) = t_i$$

Because the identity map also takes each $t_i \to t_i$, and linear maps are uniquely defined by where they take bases, TS must be the identity map. The argument is symetric for the only if direction.