Source: [KBe2020math530refExr0nRetIndex]

1 | Prompt

Which of the following systems have a unique solution? You do NOT have to solve the 3 variable system by hand; you can graph it or use other resources. What does this have to do with linearly dependent/independent vectors??

2 | Ideas

I first focused on the systems of 2 var 2 equs. I thought of the first set

$$\begin{aligned} 2x - 3y &= 1 \\ x + 3y &= 3 \end{aligned}$$

as asking

$$(1,3) \overset{?}{\in} \text{span}\left((2,1),(-,31)\right)$$

but that didn't really get me anywhere.

Then, I tried writing it as a matrix equation:

$$\begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

I figured that because we wanted to know whether the system is linearly independent or not, which is a boolean value, I had to compress the matrix down to some number that can then be compared. The only way I know how to do that is by taking the determinant, so I tried to find some connection between the determinant of a 2x2 matrix and whether it's component rows interpreted as vectors of $\mathbb{F}^2$ are linearly dependant.

3 | Lemma

A pair of vectors u, v in a vector space V over $\mathbb{F}^2$ are linearly dependent iff $\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} = 0.$

4 | Proof

In the forwards direction

Showing that if u, v are linearly dependent, then

$$\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} = 0$$

Suppose u, v are linearly dependent. Then, we can write v as $au : a \in \mathbb{F}$. Then the target determinant can be written as

$$\begin{vmatrix} u_1 & u_2 \\ au_1 & au_2 \end{vmatrix} = u_1 a u_2 + -u_2 a u_1$$

Because $u_1 a u_2 = -u_2 a u_1$, their sum is clearly 0.

In the reverse direction

Showing that if $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0$, then the vectors $(a, b), (c, d)$ are linearly dependent.

Two vectors $u, v \in \mathbb{F}^2 : u = (a, b), v = (c, d)$ and $a, b, c, d \in \mathbb{F}$ are linearly dependent if one is a linear combination of the other, or $(a, b) = u = av = (ac, ad)$ wlog. Because

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$
