Source:

1 | **Problem**

> Suppose $T \in \mathcal{L}(V, W)$ and U is a subspace of V. Let π denote the quotient map from V onto V/U. Prove that there exists $S \in \mathcal{L}(V/U, W)$ such that $T = S \circ \pi$ if and only if $U \subseteq \text{null } T$.

Intuitively, if we mod out part of the null T, then we should still be able to have a map that does what T would do. If we are able to do what T would do, then when modding out U we only removed part of null T and lost no information.

2 | **Forward Direction**

Intuitively, we can treat $S \circ \pi$ as a single map and take a basis of V to the same place that T would, and the maps would be equal.

If V is finite dimensional, suppose $v_1, \ldots, v_n$ is a basis of V and $v_1, \ldots, v_k$ is a basis of U ($k =$ dim U and $n =$ dim V). For each $k < j \leq n$, $\pi v_j \neq 0$, and we can control where S should send it.

3 | **Reverse Direction by Contrapositive**

Intuitively, if we lost information, then we can't reconstruct what T would do.

Assume $U \nsubseteq \text{null } T$. There exists $v \in U$ s.t. $Tv \neq 0$. This is some of the "information" that was "lost". Because $v \in U$,

$$\pi v = U + v = U$$

Because U is the additive identity (0) in V/U, and because linear maps take zero to zero, $S \in \mathcal{L}(V/U, W)$ must take $\pi v = 0$ to zero. Thus, either $S(\pi v) \neq Tv$ or S is not a linear map, both of which are contradictions.

This shows that if $U \nsubseteq \text{null } T$, then $S \notin \mathcal{L}(V/U, W)$ or $T \neq S \circ \pi$. Thus, if $S \in \mathcal{L}(V/U, W)$ and $T = S \circ \pi$, then $U \subseteq \text{null } T$.