**Source**:

# 1 | find taylor series

## 1.1 | $y = \cos(x)$

$$
\begin{aligned}
P_n(x) = f(0) && + \frac{d}{dx}f(0)x && + \frac{\frac{d^2}{d^2x}f(0)}{2!}x^2 && + \frac{\frac{d^3}{d^3x}f(0)}{3!}x^3 && + \cdots \\
= \cos(0) && -\sin(0)x && + \frac{-\cos(0)}{2!}x^2 && + \frac{\sin(0)}{3!}x^3 && + \cdots \\
= 1 && -0x && - \frac{1}{2!}x^2 && + \frac{0}{3!}x^3 && + \cdots \\
= 1 && && - \frac{1}{2!}x^2 && && + \frac{1}{4!}x^4 + \cdots \\
= \sum_{k=0} \frac{(-1)^k}{(2k)!}x^{2k}
\end{aligned}
$$

## 1.2 | $y = e^x$

$$
\begin{aligned}
P_n(x) = f(0) && + \frac{d}{dx}f(0)x && + \frac{\frac{d^2}{d^2x}f(0)}{2!}x^2 && + \frac{\frac{d^3}{d^3x}f(0)}{3!}x^3 && + \cdots \\
= e^0 && +e^0 x && + \frac{e^0}{2!}x^2 && + \frac{e^0}{3!}x^3 && + \cdots \\
= 1 && +x && + \frac{1}{2!}x^2 && + \frac{1}{3!}x^3 && + \cdots \\
= \sum_{k=0} \frac{x^k}{k!}
\end{aligned}
$$

## 1.3 | $y = \sqrt{x}$ centered at $x = 1$

$$
\begin{aligned}
P_n(x) = f(1) && + \frac{d}{dx}f(1)(x-1) && + \frac{\frac{d^2}{d^2x}f(1)}{2!}(x-1)^2 && + \frac{\frac{d^3}{d^3x}f(1)}{3!}(x-1)^3 && + \cdots \\
= 1 && + \frac{1}{2}(x-1) && + \frac{\frac{1}{2}\frac{-1}{2}}{2!}(x-1)^2 && + \frac{\frac{1}{2}\frac{-1}{2}\frac{-3}{2}}{3!}(x-1)^3 && + \cdots
\end{aligned}
$$

I don't know how to write it using summation notation though..

# 2 | prove approximations

## 2.1 | $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$

Proof by geometric series

## 2.2 | $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$

Plug $-x$ for $x$ in the previous equation.

2.3 | $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$

Plug $x^2$ for $x$ in the previous equation.

# 3 | more finding of polynomial

### 3.1 | **TODO $ y = ln (1+x)$**

### 3.2 | **TODO** $y = \tan^- x$

### 3.3 | $y = (1+x)^k$

$$
\begin{aligned}
P_n(x) = f(0) &\quad + \frac{d}{dx}f(0)x &\quad + \frac{\frac{d^2}{d^2 x}f(0)}{2!}x^2 &\quad + \frac{\frac{d^3}{d^3 x}f(0)}{3!}x^3 &\quad + \cdots \\
= 1 + k(1)^k &\quad + k(k-1)(1)^{k-1}x &\quad + \frac{k(k-1)(k-2)(1)^{k-2}}{2!}x^2 &\quad + \frac{k(k-1)(k-2)(k-3)(1)^{k-3}}{3!}x^3 &\quad + \cdots \\
= 1 + k &\quad + k(k-1)x &\quad + \frac{k(k-1)(k-2)}{2!}x^2 &\quad + \frac{k(k-1)(k-2)(k-3)}{3!}x^3 &\quad + \cdots \\
= 1 + k &\quad + \frac{k!}{(k-1)!}x &\quad + \frac{\frac{k!}{(k-2)!}}{2!}x^2 &\quad + \frac{\frac{k!}{(k-3)!}}{3!}x^3 &\quad + \cdots \\
= 1 + k &\quad + \frac{k!x}{(k-1)!} &\quad + \frac{k!}{(k-2)!2!}x^2 &\quad + \frac{k!}{(k-3)!3!}x^3 &\quad + \cdots \\
= \binom{k}{0} &\quad + \binom{k}{1}x &\quad + \binom{k}{2}x^2 &\quad + \binom{k}{3}x^3 &\quad + \cdots \\
= \sum_{i=0} \binom{k}{i}x^i
\end{aligned}
$$

# 4 | find sum of series by recognizing Taylor Series approximations of some functions

### 4.1 | $3 + \frac{9}{2!} + \frac{27}{3!} + \frac{81}{4!} + \cdots$

$$3 + \frac{9}{2!} + \frac{27}{3!} + \frac{81}{4!} = e^3 - 1$$

### 4.2 | $1 - \ln 2 + \frac{\ln^2 2}{2!} + \frac{\ln^3 2}{3!} + \cdots$

$$1 - \ln 2 + \frac{\ln^2 2}{2!} + \frac{\ln^3 2}{3!} + \cdots = e^{-\ln 2} = \frac{1}{2}$$

### 4.3 | $\sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k+1}}{4^{2k+1}(2k+1)!}$

$$\sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k+1}}{4^{2k+1}(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{\pi}{4}^{2k+1}\right)}{(2k+1)!} = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

## 5 | evaluate limits using taylor series approx

5.1 | $\lim\limits_{x\to 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5}$

$$\lim_{x\to 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5}$$

$$= \lim_{x\to 0} \frac{\cancel{x} - \frac{x^3}{6} + \frac{x^5}{5!} + \cdots - \cancel{x} + \frac{x^3}{6}}{x^5}$$

$$= \lim_{x\to 0} \frac{-\cancel{\frac{x^3}{6}} + \frac{x^5}{5!} + \cdots + \cancel{\frac{x^3}{6}}}{x^5}$$

$$= \lim_{x\to 0} \frac{\cancel{x^5}}{5!} + \cdots}{x^5}$$

$$= \lim_{x\to 0} \frac{1}{5!} + \frac{x^7}{x^5 7!} + \frac{x^9}{x^5 9!} \cdots$$

$$= \frac{1}{5!}$$

5.2 | **TODO** $\lim\limits_{x\to 0} \frac{x - \tan^- x}{x^3}$

## 6 | find taylor series approximations

6.1 | $y = e^x + e^{-x}$

$$e^x + e^{-x}$$

$$= 1 + x + \frac{x^2}{2!} \frac{x^3}{3!} + \cdots + 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!}$$

$$= 1 + 1 + x - x + \frac{x^2}{2!} + \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^3}{3^1} + \cdots$$

$$= 2\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)$$

$$= 2\sum_{k=0} \frac{x^{2k}}{(2k)!}$$

6.2 | $y = \sin(\pi x)$

$$\sin(\pi x) = \sum_{k=0} \frac{(-1)^k (\pi x)^{2k+1}}{(2k+1)!}$$

(just plug it in)

6.3 | **TODO** $y = \sin^2 x$