Albert H 8 Sep 2020 1h 30min

Calculus
2019-2020

Handout #5: Formal Definition of Limits: An Introduction

*Please sketch the graph of the functions in each problem.*

1. Suppose a company produces a circular plate to repair the skull after brain surgery. The area of such a plate must be 16π square millimeters.
   a. What is the necessary radius for the plate? (recall the area of a circle)

      4

   b. Suppose the plate still works with an error in area of 0.04π square millimeters. What is the maximum error in radius that will guarantee the area is in error of at most 0.04π square millimeters? (Round to nearest 6th decimal place)

      $\cancel{y = \sqrt{\frac{x}{\pi}}}$

      $\cancel{|x-\delta| < 1}$

      $\cancel{\left|\sqrt{\frac{x+\delta}{\pi}}\right|}$

      $\sqrt{\frac{16.04\pi}{\pi}} = \sqrt{16.04} \simeq 4.004997$

      $3.994997$

      $+\text{error} = 0.004997$

      $\boxed{-\text{error} = 0.005003}$

2. Suppose we need to manufacture a ball bearing of volume 36π cubic inches (recall: the volume of a sphere is $V = (4/3)\pi r^3$. Note that the volume of the ball is a function of radius. If the volume of our ball bearing can be in error at most 0.4π, what is the maximum permissible error in radius?

   $V = \frac{4}{3}\cancel{\pi} r^3$

   $36.4\cancel{\pi}$

   $r = \sqrt[3]{\frac{3}{4}\ 35.6} \Rightarrow 0.01115$

3. Suppose a toilet paper company needs to make hollow cylinders (that is, without a top or bottom) with surface area 12π square centimeters. Suppose the radius must remain a constant 3 square centimeters but the surface area of the cylinder can be of error at most 0.2π square centimeters. What is the maximum permissible error in height?

   $A = C * L = 2\pi r * L$

   $6\cancel{\pi} L = 12.2\cancel{\pi}$

   $e = 0.0333$

Formal definition of a Limit

$$\lim_{x \to a} f(x) = L$$

**means** for all $\varepsilon > 0$, there exists $\delta > 0$ such that if 0<$|x - a| < \delta$ then $|f(x) - L| < \varepsilon$.

**Let's explore the above definition graphically:**

4. Consider the function $f(x) = 2x + 1$.
   a. Graph the function on Desmos. What do you the limit of the function is at $x = 3$?

   7

   b. Suppose $\varepsilon = 1$. Find $\delta > 0$ such that it satisfies the limit definition (that is, find the $\delta > 0$ such that if $|x - 3| < \delta$ then $|(2x + 1) - L| < \varepsilon$).

   $L = 7$

   $|2x+1-7| < \varepsilon$
   $|2x-6| < \varepsilon$
   $|2(x-3)| < \varepsilon$
   $2|x-3| < \varepsilon$
   $|x-3| < \frac{\varepsilon}{2}$ $\delta = \frac{\varepsilon}{2}$
   $|x-3| < \delta$

   c. Suppose $\varepsilon = 0.5$. Find $\delta > 0$ such that it satisfies the limit definition (up to 4 decimal places).

   $\delta = \frac{\varepsilon}{2} = \frac{0.05}{2} = 0.2500$

   d. Suppose $\varepsilon = 0.25$. Find $\delta > 0$ such that it satisfies the limit definition (up to 4 decimal places).

   $\delta = \frac{\varepsilon}{2} = \frac{0.25}{2} = 0.1250$

5. Consider the function $f(x) = x^2 - 2x + 1$.
   a. Graph the function on Desmos. What do you the limit of the function is at $x = 3$?

   4

   b. Suppose $\varepsilon = 1$. Find $\delta > 0$ such that it satisfies the limit definition (that is, find the $\delta > 0$ such that if $|x - 3| < \delta$ then $|(x^2 - 2x + 1) - L| < \varepsilon$).

   What the heck

   ~~$|x-1-3| \leq 1$~~

   ~~$|x-3| \leq 1$~~

   ~~$2 \leq x \leq 4$~~

   ~~$\delta = \min\{|x-3|, 1\}$~~

   $|(x^2 - 2x + 1) - 4| < \varepsilon$

   $|x^2 - 2x - 3| < \varepsilon$

   $|(x-3)(x+1)| < \varepsilon$

   $|x-3||x+1| < \varepsilon$

   $|x-3| \leq \frac{\varepsilon}{|x+1|}$

   $\therefore |x-3| \leq \frac{\varepsilon}{5}$ — take the upper bound

   $\delta = \min\{1, \frac{\varepsilon}{5}\}$

   $|x-3| \leq 1$

   $-1 \leq x-3 \leq 1$

   $2 \leq x \leq 4$

   $3 \leq x+1 \leq 5$

   $3 \leq |x+1| \leq 5$

   why not lower?

   $\boxed{\frac{1}{5}}$

   c. Suppose $\varepsilon = 0.5$. Find $\delta > 0$ such that it satisfies the limit definition (up to 4 decimal places).

   $$\underbrace{\delta = \min\left\{1, \frac{0.5}{5}\right\}}_{} = \frac{0.5}{5} = \frac{1}{10}$$

   d. Suppose $\varepsilon = 0.25$. Find $\delta > 0$ such that it satisfies the limit definition(up to 4 decimal places).

   $\boxed{\frac{1}{20}}$