Source:

1 | **Prompt**

Which of the following systems have a unique solution? You do NOT have to solve the 3 variable system by hand; you can graph it or use other resources. What does this have to do with linearly dependent/independent vectors??

2 | **Ideas**

I first focused on the systems of 2 var 2 equs. I thought of the first set

$$2x - 3y = 1$$
$$x + 3y = 3$$

as asking

$$(1, 3) \overset{?}{\in} \text{span}\left((2, 1), (-, 31)\right)$$

but that didn't really get me anywhere.

Then, I tried writing it as a matrix equation:

$$\begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

I figured that because we wanted to know whether the system is linearly independent or not, which is a boolean value, I had to compress the matrix down to some number that can then be compared. The only way I know how to do that is by taking the determinant, so I tried to find some connection between the determinant of a 2x2 matrix and whether it's component rows interpreted as vectors of $\mathbb{F}^2$ are linearly dependant.

3 | **Lemma**

A pair of vectors u, v in a vector space V over $\mathbb{F}^2$ are linearly dependent iff $\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} = 0$.

4 | **Proof**

In the forwards direction

Showing that if u, v are linearly dependent, then

$$\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} = 0$$

Suppose u, v are linearly dependent. Then, we can write v as $au : a \in \mathbb{F}$. Then the target determinant can be written as

$$\begin{vmatrix} u_1 & u_2 \\ au_1 & au_2 \end{vmatrix} = u_1 au_2 + -u_2 au_1$$

Because $u_1 au_2 = -u_2 au_1$, their sum is clearly 0.

In the reverse direction

Showing that if $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0$, then the vectors $(a,b), (c,d)$ are linearly dependent.

Two vectors $u, v \in \mathbb{F}^2 : u = (a,b), v = (c,d)$ and $a, b, c, d \in \mathbb{F}$ are linearly dependent if one is a linear combination of the other, or $(a,b) = u = av = (ac, ad)$ wlog.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = 0$$
$$\Rightarrow ad = bc$$
$$\Rightarrow \frac{a}{c} = \frac{b}{d}$$

If we define \$$k = \frac{a}{c} = \frac{b}{d}$