

1 | Exercises

1.1 | interpreting in terms of area



1.3 | subtracting integrals

I expect

$$\int_1^2 f(x) dx = \int_1^5 f(x) dx - \int_2^5 f(x) dx = -3 - 4 = -7$$

In fact, I expect

$$\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$$

1.4 | show $\int_a^b x^2 dx = \frac{b^3 - a^3}{3}$

(see attached pages)

Keep in mind

$$\sum_{k=1}^n af(x) = a \sum_{k=1}^n f(x)$$

$$\sum_{k=1}^n (a + f(x)) = an + \sum_{k=1}^n f(x)$$

$$\begin{aligned} \int_a^b x^2 dx &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\frac{b-a}{n} \left(a + k \frac{b-a}{n} \right)^2 \right) \\ &= \lim_{n \rightarrow \infty} \frac{b-a}{n} \sum_{k=1}^n \left(a + k \frac{b-a}{n} \right)^2 \\ &= \lim_{n \rightarrow \infty} \frac{b-a}{n} \sum_{k=1}^n \left(a^2 + \left(k \frac{b-a}{n} \right)^2 + 2ak \frac{b-a}{n} \right) \\ &= \lim_{n \rightarrow \infty} \frac{b-a}{n} \sum_{k=1}^n a^2 + \sum_{k=1}^n \left(k \frac{b-a}{n} \right)^2 + \sum_{k=1}^n 2ak \frac{b-a}{n} \\ &= \lim_{n \rightarrow \infty} \frac{b-a}{n} \left(a^2 n + \sum_{k=1}^n k^2 \left(\frac{b-a}{n} \right)^2 + 2a \frac{b-a}{n} \sum_{k=1}^n k \right) \\ &= \lim_{n \rightarrow \infty} \frac{b-a}{n} \left(a^2 n + \left(\frac{b-a}{n} \right)^2 \sum_{k=1}^n k^2 + 2a \frac{b-a}{n} \sum_{k=1}^n k \right) \\ &= \lim_{n \rightarrow \infty} \frac{b-a}{n} \left(a^2 n + \left(\frac{b-a}{n} \right)^2 \frac{n(n+1)(2n+1)}{6} + 2a \frac{b-a}{n} \frac{n(n+1)}{2} \right) \\ &= \lim_{n \rightarrow \infty} (b-a) \left(a^2 + \left(\frac{b-a}{n} \right)^2 \frac{(n+1)(2n+1)}{6} + 2a \frac{b-a}{n} \frac{(n+1)}{2} \right) \\ &= \lim_{n \rightarrow \infty} (b-a) \left(a^2 + \frac{(b-a)^2}{n^2} \left(n \frac{(2n+1)}{6} + \frac{(2n+1)}{6} \right) + a \frac{b-a}{n} (n+1) \right) \\ &= \lim_{n \rightarrow \infty} (b-a) \left(a^2 + \frac{(b-a)^2}{n^2} \left(n \frac{(2n+1)}{6} + \frac{(2n+1)}{6} \right) + a \cancel{\frac{b-a}{n}} + a \frac{b-a}{n} \right) \\ &= \lim_{n \rightarrow \infty} (b-a) \left(a^2 + \frac{(b-a)^2}{n^2} \left(n \frac{(2n+1)}{6} + \frac{(2n+1)}{6} \right) + a(b-a) + \cancel{a \frac{b-a}{n}} \right) \\ &= \lim_{n \rightarrow \infty} (b-a) \left(a^2 + \frac{(b-a)^2}{n^2} \cancel{\frac{(2n+1)}{6}} + \frac{(b-a)^2 \cancel{(2n+1)}}{n^2 \cancel{6}} + a(b-a) \right) \\ &= \lim_{n \rightarrow \infty} (b-a) \left(a^2 + \frac{(b-a)^2}{n} \frac{(2n+1)}{6} + a(b-a) \right) \\ &= \lim_{n \rightarrow \infty} (b-a) \left(a^2 + \frac{(b-a)^2}{3} + \cancel{\frac{1}{6}} \frac{(b-a)^2}{n} + a(b-a) \right) \\ &= \lim_{n \rightarrow \infty} (b-a) \left(a^2 + \frac{(b-a)^2}{3} + a(b-a) \right) \\ &= \lim_{n \rightarrow \infty} a^2(b-a) + \frac{b^2 + a^2 - 2ab}{3}(b-a) + a(b-a)^2 \end{aligned}$$