

## 1 | Problem

Suppose  $T \in \mathcal{L}(V)$ . Prove that  $T/(\text{null } T)$  is injective if and only if  $(\text{null } T) \cap (\text{range } T) = \{0\}$

## 2 | Proof

First, we will rewrite the problem as logical statements for easier manipulation. The left-hand side " $T/(\text{null } T)$  is injective" is equivalent to:

$$\begin{aligned} (T/U(v+U) = 0) &\iff (v+U = 0) \\ Tv + U = \text{null } T &\iff v + U = \text{null } T \\ Tv + (\text{null } T) = \text{null } T &\iff v + (\text{null } T) = \text{null } T \\ Tv \in \text{null } T &\iff v \in \text{null } T \\ T^2v = 0 &\iff Tv = 0 \end{aligned}$$

We can also rewrite the second condition for easier manipulation. The intersection of the null space and the range being 0 is the same as (assuming  $w \neq 0$ ) "if  $w \in \text{null } T$  then  $w \notin \text{range } T$ " and "if  $w \in \text{range } T$  then  $w \notin \text{null } T$ ". Note that these are contrapositives of each other, so we just need to work with the second statement.

Assuming  $w \neq 0$ , these statements are equivalent:

$$\begin{aligned} (\exists v : Tv = w) &\implies (Tw \neq 0) \\ T^2v \neq 0 &\quad \forall v \notin (\text{null } T) \\ v \notin \text{null } T &\implies T^2v \neq 0 \end{aligned}$$

Note that this statement along with its contrapositive implies  $(\text{null } T) \cap (\text{range } T) = \{0\}$ .

Furthermore, keeping in mind that  $w = Tv$  and  $w \neq 0$ ,

$$\begin{aligned} T^2v \neq 0 &\implies T(Tv) \neq 0 \\ &\implies Tw \neq 0 \\ &\implies w \notin \text{null } T \end{aligned}$$

which shows the previous relation is an if-and-only-if relation: