1 | Axler6.56 Minimizing the distance to a subspace

Suppose U is a finite-dimensional subspace of V, $v \in V$, and $u \in U$. Then,

$$\|v - P_U v\| \leq \|v - u\|$$

Because we often end up having to find the minimal $v - u$ where $u \in U$, this result makes linear algebra applicable to numerous real-world applications.

1.1 | Proof

$$\begin{aligned}
\|v - P_U v\|^2 &\leq \|v - P_U v\|^2 + \|P_U v - u\|^2 && \text{by } 0 \leq \|P_U v - u\|^2 \\
&= \|(v - P_U v) + (P_U v - u)\|^2 && \text{by the Pythagorean Theorem} \\
&= \|v - u\|^2
\end{aligned}$$

Inequality is an equality only when $u = P_U v$.

1.2 | An example

First define an inner product that will be our cost function. In this case, they use the integral of $f(x)g(x)$ on the range $[-\pi, \pi]$. Then, orthonormalize a basis of the polynomials up to degree 6 (using the Gram-Schmidt procedure) and take the orthonormal projection using the same inner product. This ends up with roughly

$$u(x) = 0 : 987862x - 0 : 155271x^3 + 0 : 00564312x^5$$