1 | Problem

Suppose $T \in \mathcal{L}(V)$. Prove that $T/(\text{null}\, T)$ is injective if and only if $(\text{null}\, T) \cap (\text{range}\, T) = \{0\}$

2 | Proof

2.1 | Condition Manipulation

First, we will rewrite the problem as logical statements for easier manipulation.

2.1.1 | Left Condition

The left-hand side "$T/(\text{null}\, T)$ is injective" is equivalent to:

$$\begin{aligned}
(T/U\,(v+U) = 0) &\implies (v+U=0) \\
Tv + U = \text{null}\, T &\implies v + U = \text{null}\, T \\
Tv + (\text{null}\, T) = \text{null}\, T &\implies v + (\text{null}\, T) = \text{null}\, T \\
Tv \in \text{null}\, T &\implies v \in \text{null}\, T \\
T^2 v = 0 &\implies Tv = 0
\end{aligned}$$

2.1.2 | Right Condition

We can also rewrite the right-hand condition for easier manipulation. The intersection of the null space and the range being 0 is the same as (assuming $w \neq 0$) "if $w \in \text{null}\, T$ then $w \notin \text{range}\, T$" and "if $w \in \text{range}\, T$ then $w \notin \text{null}\, T$". Note that these are contrapositives of eachother, so we just need to work with the second statement.

Thus, assuming $w \neq 0$, these statements are equivalent:

$$\begin{aligned}
(\exists v : Tv = w) &\implies && (Tw \neq 0) \\
T^2 v \neq 0 & && \forall v \notin \text{null}\, T \\
v \notin \text{null}\, T &\implies && T^2 v \neq 0 \\
Tv \neq 0 &\implies && T^2 v \neq 0 \\
T^2 v = 0 &\implies && Tv = 0
\end{aligned}$$

Note that the last two statements imply the original $(\text{null}\, T) \cap (\text{range}\, T) = \{0\}$.