

Suppose  $T \in \mathcal{L}(V)$  and  $U$  is a subspace of  $V$ . Prove that  $U$  is invariant under  $T$  iff  $U^\perp$  is invariant under  $T^*$ .

For all  $u \in U$ ,  $Tu = u' \in U$ . Let  $w \in U^\perp$ . Then,  $\langle T^*w, u \rangle = \langle w, Tu \rangle = \langle w, u' \rangle = 0$

$$\langle u, T^*w \rangle = \langle Tu, w \rangle = \langle u', w \rangle$$

For all pairs  $u \in U$  and  $w \in U^\perp$ ,

$$\langle Tu, w \rangle = 0$$

$$\langle u, T^*w \rangle = 0$$

This implies both directions, since  $U = U^{\perp\perp}$  and  $T = (T^*)^*$ .