

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. Prove that λ is an eigenvalue of T iff $\bar{\lambda}$ is an eigenvalue of T^* .

Given λ is an eigenvalue of T , show that $\bar{\lambda}$ is an eigenvalue of T^* . This will imply both directions, since $\lambda = \overline{\bar{\lambda}}$ and $T = T^{**}$.

Suppose $\mathcal{M}(T)$ is the matrix of T wrt some orthonormal basis. Then, the matrix $\mathcal{M}(T^*)$ of T^* wrt the same orthonormal basis will equal the conjugate transpose of $\mathcal{M}(T)$.

Eigenvalues lie on the diagonal of a matrix, so the conjugate transpose will have the effect of conjugating each eigenvalue. Thus, the eigenvalues of $\mathcal{M}(T)$ are conjugates of the eigenvalues of $\mathcal{M}(T^*)$.

$$\langle T - \lambda I v, v \rangle = \langle v, (T - \lambda I)^* \rangle = \langle v, T^* - \bar{\lambda} I v \rangle$$