Source: [KBe2020math530refExr0nRetIndex]

Lemma

Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

Proof

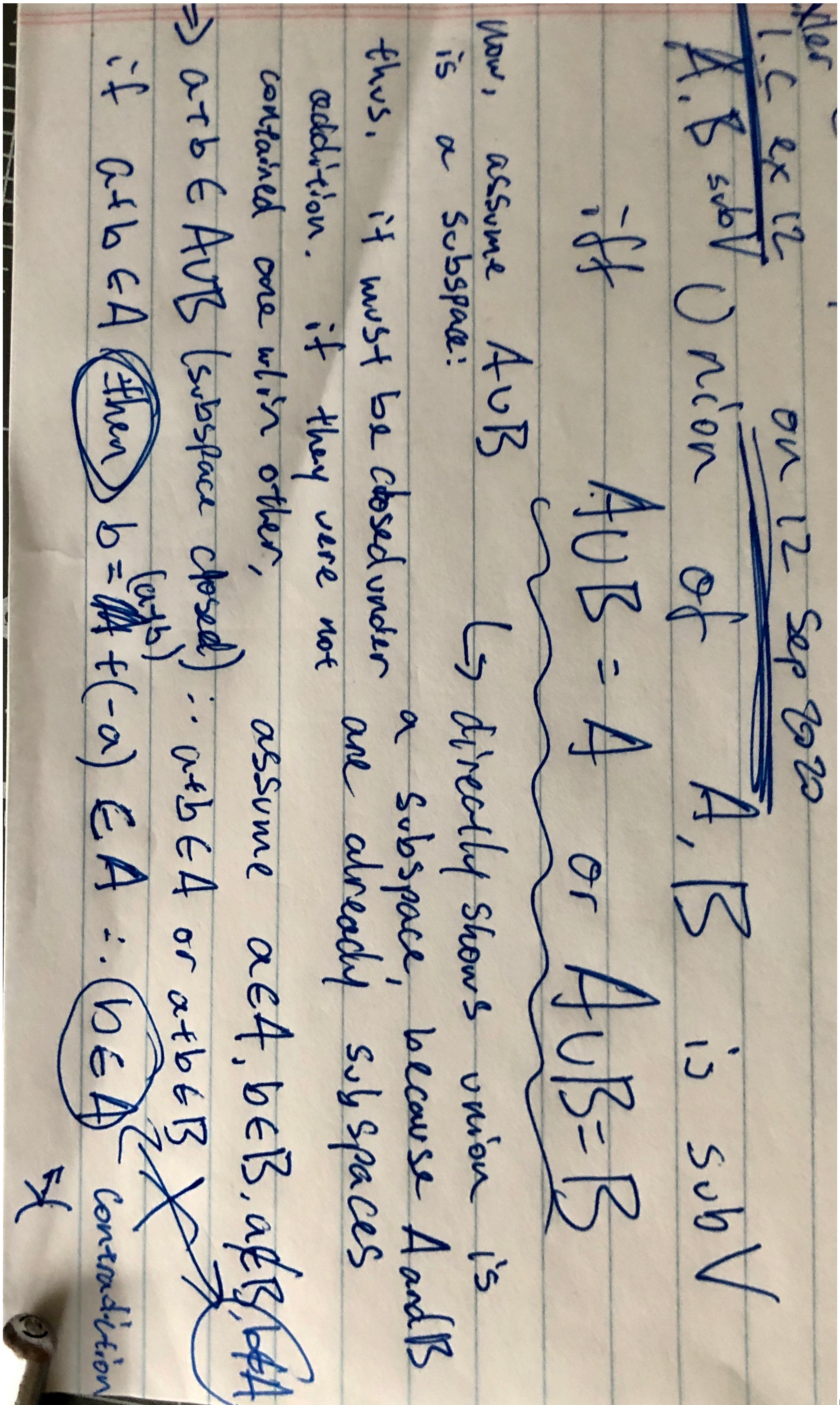
Given two subspaces A, B of vector space V, one is contained within the other if and only if $A \subseteq B$ or $B \subseteq A$, equivalently $A \cup B = B$ or $A \cup B = A$. Thus, we just need to show that $A \cup B$ is a subspace of V if and only if $A \cup B = A$ or $A \cup B = B$.

First, assume that $A \cup B = A$. $A \cup B$ must be a subspace of V because A is a subspace of V. The argument is symmetric for the case $A \cup B = B$.

Now, given that $A \cup B$ is a subspace of V, show that $A \cup B = A$ or $A \cup B = B$.

Assume for the sake of contradiction that $A \cup B \neq A$ and $A \cup B \neq B$. There must exist some elements $a \in A, b \in B$ such that $a \notin B$ and $b \notin A$. Additionally, because A, B are vector spaces, $-a \in A$ and $-b \in B$. Because $A \cup B$ is a subspace of V, it must also be a vector space and closed under addition. Thus, $a+b \in A \cup B$. Because $A \cup B$ is comprised exclusively of elements in A or in B, $a+b \in A$ or $a+b \in B$. Because A is closed under addition and $-a \in A$, if $a+b \in A$, the sum $(a+b)+(-a) = \cancel{a} + b \cancel{-a} = b \in A$. This contradicts the earlier definition of $b \notin A$ and completes the proof. The argument is symmetric for the case $a + b \in B$.

Appendix: Working it out



Axler 1.C ex 12 on 12 Sep 2020

A, B subV Union of A, B is subV

iff $A \cup B = A$ or $A \cup B = B$

↳ directly shows union is a subspace, because A and B are already subspaces

now, assume $A \cup B$ is a subspace:

thus, it must be closed under addition. if they were not contained one w/in other, assume $a \in A, b \in B, a \notin B, b \notin A$

$\Rightarrow a+b \in A \cup B$ (subspace closed) $\therefore a+b \in A$ or $a+b \in B$

if $a+b \in A$ then $b = (a+b) + (-a) \in A \therefore b \in A$ contradiction