1 | $\int_1^2 \frac{e^{\frac{1}{x}}}{x^2} dx$

$$\begin{aligned}\int -e^u du &= -e^u + C \\ &= -e^{\frac{1}{2}} + e^{\frac{1}{1}} \\ &= e - e^{\frac{1}{2}}\end{aligned}$$

2 | $\int_0^1 re^{\frac{r}{2}} dr$

$$\begin{aligned}\int_0^1 re^{\frac{r}{2}} dx &\implies r2e^{\frac{r}{2}} - \int 2e^{\frac{r}{2}} dr \\ &= 2re^{\frac{r}{2}} - \int 2e^{\frac{r}{2}} dr \\ &= 2re^{\frac{r}{2}} - 4e^{\frac{r}{2}} \\ &= 2re^{\frac{r}{2}} - 4e^{\frac{r}{2}} \\ &\implies 2e^{\frac{1}{2}} - 4e^{\frac{1}{2}} - (-4) \\ &= 4 - 2e^{\frac{1}{2}}\end{aligned}$$

3 | $\int_4^9 \frac{\ln y}{\sqrt{y}} dy$

$$\begin{aligned}\int \frac{\ln y}{\sqrt{y}} dy &= 2\ln y\sqrt{y} - \int 2\frac{1}{y}\sqrt{y} dy \\ &= 2\ln y\sqrt{y} - \int 2\frac{1}{\sqrt{y}} dy \\ &= 2\ln y\sqrt{y} - 2\int y^{-\frac{1}{2}} dy \\ &= 2\ln y\sqrt{y} - 4\sqrt{y} + C \\ &= 2\sqrt{y}(\ln y - 2) + C \\ \implies \quad & 6(\ln 9 - 2) - 4(\ln 4 - 2)\end{aligned}$$

4 | $\int_0^{\sqrt{\pi}} \cos\sqrt{x} dx$

Let $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}} dx$, $dx = 2udu$

$$\begin{aligned}\int_0^{\sqrt{\pi}} \cos\sqrt{x} dx &= 2\int u\cos u du \\ &= 2u\sin u - 2\int \sin u du \\ &= 2u\sin u + 2\cos u \\ &= 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} \\ \implies \quad & 2\pi^{\frac{1}{4}}\sin\pi^{\frac{1}{4}} + 2\cos\pi^{\frac{1}{4}} - 2\end{aligned}$$

5 | $\int_1^e \sin\ln x dx$

$$\begin{aligned}
\int_1^e \sin\ln x dx &= x\sin\ln x - \int x\frac{1}{x}\cos\ln x dx \\
&= x\sin\ln x - \int \cos\ln x dx \\
&= x\sin\ln x - \left(x\cos\ln x + \int \cancel{x}\frac{\cancel{1}}{x}\sin\ln x dx\right) \\
&= x\sin\ln x - x\cos\ln x - \int \sin\ln x dx \\
2\int \sin\ln x dx &= x\sin\ln x - x\cos\ln x \\
\int \sin\ln x dx &= \frac{1}{2}x(\sin\ln x - \cos\ln x) \\
\implies \quad & \frac{1}{2}e(\sin 1 - \cos 1) - \frac{1}{2}(\sin 0 - \cos 0) \\
&= \frac{e}{2}(\sin 1 - \cos 1) + \frac{1}{2}
\end{aligned}$$

6 | $\int_0^1 \frac{x^3}{\sqrt{4+x}}dx$

Let $u = 4 + x^2$, $du = 2xdx$

$$\begin{aligned}
\int \frac{x^3}{\sqrt{4+x}}dx &= \frac{1}{2}\int \frac{(u-4)}{\sqrt{u}}du \\
&= \frac{1}{2}\int \sqrt{u}dx - 2\int \frac{1}{\sqrt{u}}dx
\end{aligned}$$

7 | **(additional problems)**

8 | $\int \sin^2 xdx$

$$\begin{aligned}
\int \sin^2 xdx &= -\sin x\cos x + \int \cos^2 xdx \\
&= -\sin x\cos x + \int 1dx - \int \sin^2 xdx \\
2\int \sin^2 xdx &= -\sin x\cos x + x \\
\int \sin^2 xdx &= \frac{1}{2}(x - \sin x\cos x)
\end{aligned}$$

9 | $\int \cos^2 x dx$

$$\begin{aligned}
\int \cos^2 x dx &= \cos x \sin x + \int \sin^2 x dx \\
&= \cos x \sin x + \int 1 dx - \int \cos^2 x dx \\
2 \int \cos^2 x dx &= \cos x \sin x + x \\
\int \cos^2 x dx &= \frac{1}{2} \cos x \sin x + \frac{x}{2}
\end{aligned}$$

10 | $\int \sin^2 x \cos^2 x dx$

First, the integral of $\sin x \cos x$:

$$\begin{aligned}
\int \sin x \cos x dx &= \sin^2 x - \int \sin x \cos x dx \\
&= \frac{1}{2} \sin^2 x + C
\end{aligned}$$

And the derivative:

$$\begin{aligned}
\frac{d}{dx} \sin x \cos x &= -\sin^2 x + \cos^2 x \\
&= \cos^2 x - \sin^2 x
\end{aligned}$$

Now we can use differentiation by parts on $(\sin x \cos x)(\sin x \cos x)$:

$$\begin{aligned}
\int (\sin x \cos x)(\sin x \cos x) dx &= \frac{1}{2}(\sin x \cos x) \sin^2 x - \int \left(\cos^2 x - \sin^2 x\right) \left(\frac{1}{2} \sin^2 x\right) dx \\
&= \frac{1}{2}(\sin x \cos x) \sin^2 x - \frac{1}{2} \int \sin^2 x \cos^2 x dx - \frac{1}{2} \int \sin^4 x dx \\
\frac{3}{2} \int \sin^2 x \cos^2 x dx &= \frac{1}{2}(\sin x \cos x) \sin^2 x - \frac{1}{2} \int \sin^4 x dx
\end{aligned}$$

To evaluate the integral of $\sin^4 x$, we can use differentiation by parts after multiplying by one. We will need to use the integral of $\sin x \cos x$ from earlier in this problem, as well as the integral of $\sin^2 x$ from the previous problem.

$$\begin{aligned}
\int 1 \sin^4 x dx &= x \sin^4 x - 4 \int x \sin x \cos x dx \\
&= x \sin^4 x - 4 \left(x \left(\frac{1}{2} \sin^2 x \right) - \int \frac{1}{2} \sin^2 x dx \right) \\
&= x \sin^4 x - 4 \left(x \left(\frac{1}{2} \sin^2 x \right) - \frac{1}{2} \int \sin^2 x dx \right) \\
&= x \sin^4 x - 2x \sin^2 x + 4 \frac{1}{2} \left(x - \sin x \cos x \right) \\
&= x \sin^4 x - 2x \sin^2 x + 2x - 2 \sin x \cos x
\end{aligned}$$

Plugging back into the main problem:

$$
\begin{aligned}
\frac{3}{2}\int \sin^2 x\cos^2 x dx &= \frac{1}{2}(\sin x\cos x)\sin^2 x - \frac{1}{2}\int \sin^4 x dx \\
&= \frac{1}{2}(\sin x\cos x)\sin^2 x - \frac{1}{2}x\sin^4 x + x\sin^2 x - x + \sin x\cos x + C \\
\int \sin^2 x\cos^2 x dx &= \frac{1}{3}\sin^3 x\cos x - \frac{1}{3}x\sin^4 x + \frac{2}{3}x\sin^2 x - \frac{2}{3}x + \frac{3}{2}\sin x\cos x + C
\end{aligned}
$$

I think I did something wrong.

11 | $\int \sin^3 x dx$

$$
\begin{aligned}
\int \sin^3 x dx &= \int \sin x(1-\cos^2 x)dx \\
&= \int \sin x dx - \int \cos^2 x\sin x dx
\end{aligned}
$$

$$
\begin{aligned}
\int \sin^3 x dx &= \int 1\sin^3 x dx \\
&= x\sin^3 x - \int 3x\sin x\cos x dx \\
&= x\sin^3 x - 3x\sin x\cos x + 3\int \frac{1}{2}\sin^2 x dx \\
&= x\sin^3 x - 3x\sin x\cos x + \frac{3}{2}\int \sin^2 x dx \\
&= x\sin^3 x - 3x\sin x\cos x + \frac{3}{2}\frac{1}{2}(x - \sin x\cos x) + C \\
&= x\sin^3 x - 3x\sin x\cos x + \frac{3}{4}(x - \sin x\cos x) + C
\end{aligned}
$$