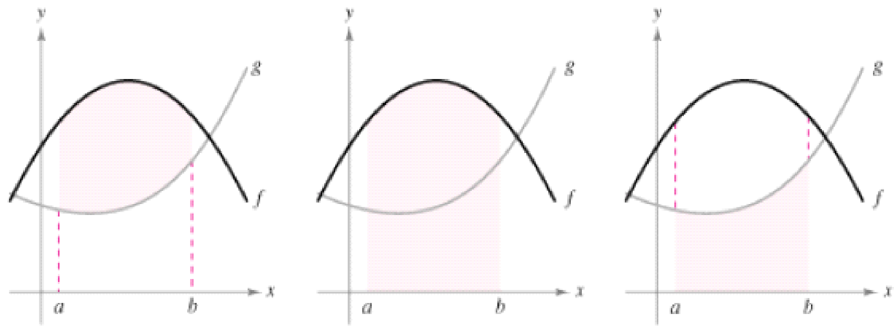
Area of a Region Between Two Curves

If f and g are continuous on $[a, b]$ and $g(x) \le f(x)$ for all x in $[a, b]$, then the area of the region bounded by the graphs of f and g and the vertical lines $x = a$ and $x = b$ is

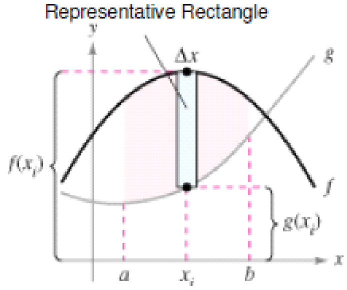
$$A = \int_a^b [f(x) - g(x)]\, dx$$



$$\int_a^b [f(x) - g(x)]dx \quad = \quad \int_a^b f(x)\, dx \quad - \quad \int_a^b g(x)\, dx$$

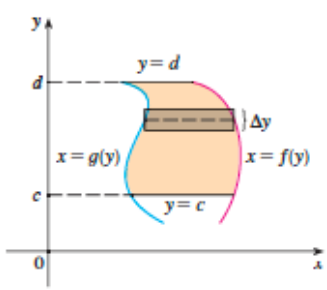
Area of region between f and g = Area of region under f - Area of region under g

Representative Rectangle



$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i) - g(x_i)]\Delta x$$

$$\text{Area} = \int_a^b [f(x) - g(x)]dx$$



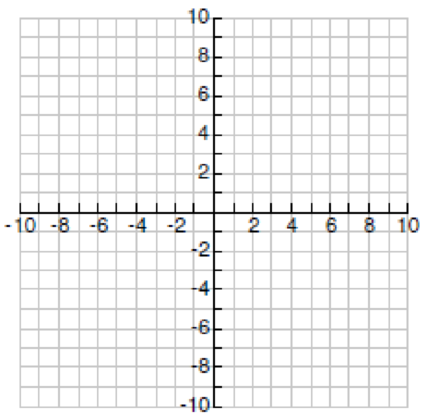
$$\text{Area} = \int_c^d (f(y) - g(y))dy$$

Calculus 2020-2021
Handout #25: Area Between Two Curves

Please sketch a graph of the functions and shade in the area you need to calculate.

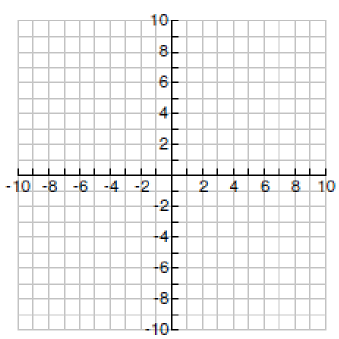
Example 1) Find the area of the region bounded by the graphs of:

$$y = x^2 + 2, \qquad y = -x, \qquad x = 0, \quad \textit{and} \quad x = 1$$



For two intersecting functions, we'll often look at the points of intersection to assign values to a and b.

Example 2) Sketch the region bounded by the graphs and find the area of the region.

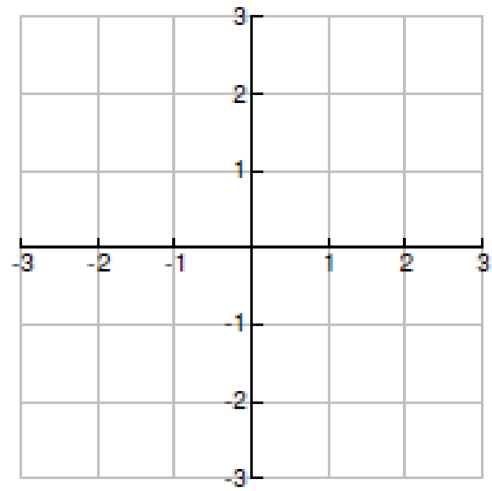


$f(x) = x^2 + 2x + 1$

$g(x) = 2x + 5$

3)

Find the area of the region bounded by the graph of
$x = 3 - y^2$ *and* $x = y + 1$



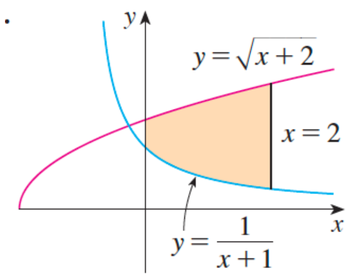
4)

1. Let f and g be the functions given by $f(x) = e^x$ and $g(x) = \ln x$.

   (a) Find the area of the region enclosed by the graphs of f and g between $x = \frac{1}{2}$ and $x = 1$.

Example 5-8: Set up and evaluate the integral to find the area of the shaded regions below.

5)



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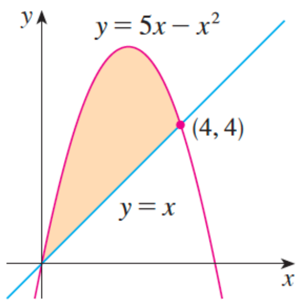

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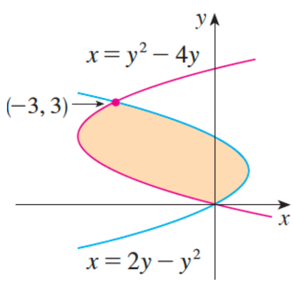

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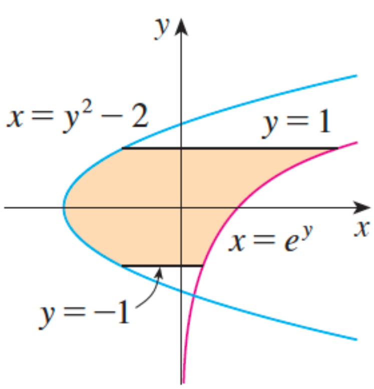

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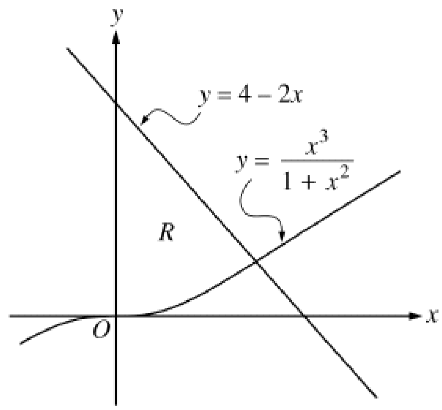
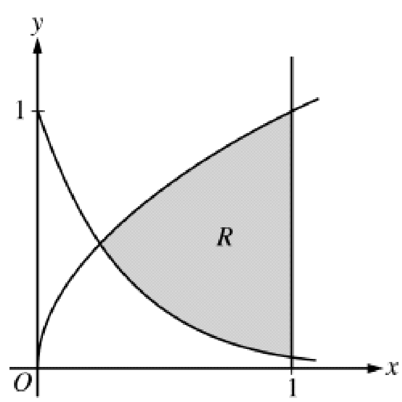


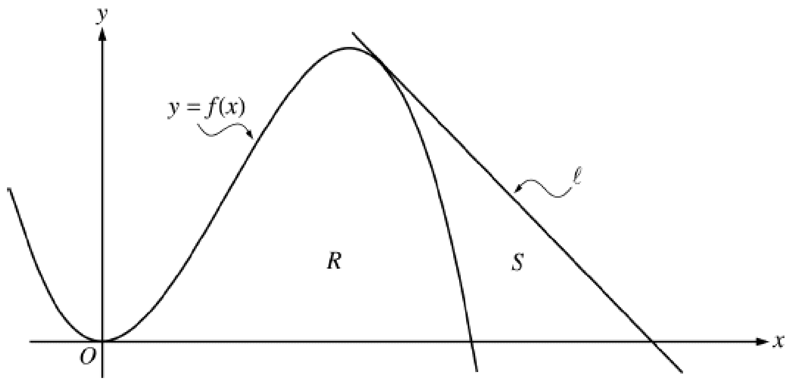
1. Let R be the region bounded by the y-axis and the graphs of $y = \dfrac{x^3}{1 + x^2}$ and $y = 4 - 2x$, as shown in the figure above.
   (a) Find the area of R.

10)



1. Let R be the shaded region bounded by the graphs of $y = \sqrt{x}$ and $y = e^{-3x}$ and the vertical line $x = 1$, as shown in the figure above.
   (a) Find the area of R.

11)



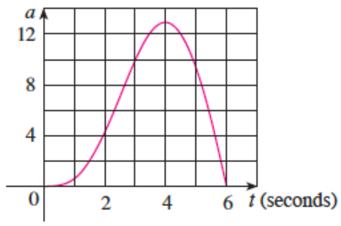
1. Let f be the function given by $f(x) = 4x^2 - x^3$, and let ℓ be the line $y = 18 - 3x$, where ℓ is tangent to the graph of f. Let R be the region bounded by the graph of f and the x-axis, and let S be the region bounded by the graph of f, the line ℓ, and the x-axis, as shown above.

   (a) Show that ℓ is tangent to the graph of $y = f(x)$ at the point $x = 3$.

   (b) Find the area of S.

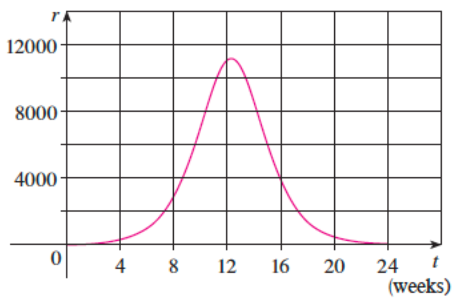
12)

The graph of the acceleration $a(t)$ of a car measured in ft/s^2 is shown. Use Simpson's Rule to estimate the increase in the velocity of the car during the 6-second time interval.



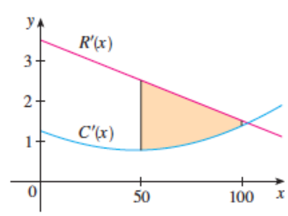
13)

A population of honeybees increased at a rate of $r(t)$ bees per week, where the graph of r is as shown. Use Simpson's Rule with six subintervals to estimate the increase in the bee population during the first 24 weeks.



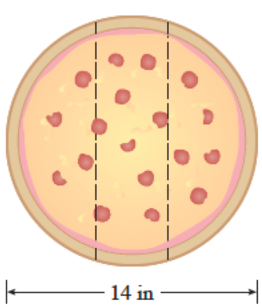
14) If the birth rate of a population is $b(t) = 2200e^{0.024t}$ people per year and the death rate is $d(t) = 1460e^{0.018t}$ people per year, find the area between these curves for $0 \le t \le 10$. What does this area represent?

15) The figure shows graphs of the marginal revenue function $R'(x)$ and the marginal cost function $C'(x)$ for a manufacturer. R(x) and C(x) represent revenue and cost in thousands of dollars when x unit are manufactured. What is the meaning of the area of the shaded region?



16)

1. Three mathematics students have ordered a 14-inch pizza. Instead of slicing it in the traditional way, they decide to slice it by parallel cuts, as shown in the figure. Being mathematics majors, they are able to determine where to slice so that each gets the same amount of pizza. Where are the cuts made?



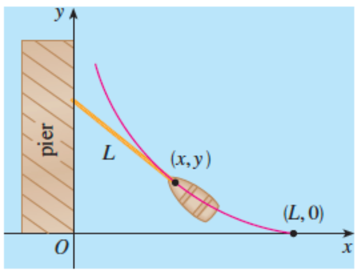
17)

6. A man initially standing at the point O walks along a pier pulling a rowboat by a rope of length L. The man keeps the rope straight and taut. The path followed by the boat is a curve called a *tractrix* and it has the property that the rope is always tangent to the curve (see the figure).

(a) Show that if the path followed by the boat is the graph of the function $y = f(x)$, then

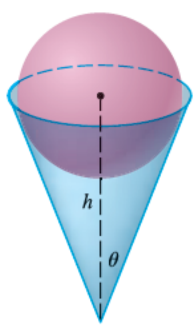
$$f'(x) = \frac{dy}{dx} = \frac{-\sqrt{L^2 - x^2}}{x}$$

(b) Determine the function $y = f(x)$.



18)

A paper drinking cup filled with water has the shape of a cone with height h and semivertical angle θ (see the figure). A ball is placed carefully in the cup, thereby displacing some of the water and making it overflow. What is the radius of the ball that causes the greatest volume of water to spill out of the cup?



19)

6. Archimedes' Principle states that the buoyant force on an object partially or fully submerged in a fluid is equal to the weight of the fluid that the object displaces. Thus, for an object of density ρ_0 floating partly submerged in a fluid of density ρ_f, the buoyant force is given by $F = \rho_f g \int_{-h}^{0} A(y)\,dy$, where g is the acceleration due to gravity and $A(y)$ is the area of a typical cross-section of the object. The weight of the object is given by

$$W = \rho_0 g \int_{-h}^{L-h} A(y)\,dy$$

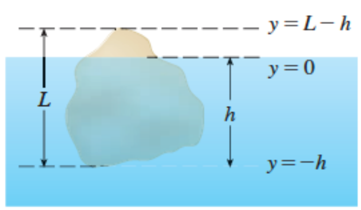
(a) Show that the percentage of the volume of the object above the surface of the liquid is

$$100\,\frac{\rho_f - \rho_0}{\rho_f}$$

(b) The density of ice is 917 kg/m³ and the density of seawater is 1030 kg/m³. What percentage of the volume of an iceberg is above water?

(c) An ice cube floats in a glass filled to the brim with water. Does the water overflow when the ice melts?

(d) A sphere of radius 0.4 m and having negligible weight is floating in a large freshwater lake. How much work is required to completely submerge the sphere? The density of the water is 1000 kg/m³.



20)

14. A rocket is fired straight up, burning fuel at the constant rate of b kilograms per second. Let $v = v(t)$ be the velocity of the rocket at time t and suppose that the velocity u of the exhaust gas is constant. Let $M = M(t)$ be the mass of the rocket at time t and note that M decreases as the fuel burns. If we neglect air resistance, it follows from Newton's Second Law that

$$F = M\frac{dv}{dt} - ub$$

where the force $F = -Mg$. Thus

$$\boxed{1} \qquad M\frac{dv}{dt} - ub = -Mg$$

Let M_1 be the mass of the rocket without fuel, M_2 the initial mass of the fuel, and $M_0 = M_1 + M_2$. Then, until the fuel runs out at time $t = M_2 b$, the mass is $M = M_0 - bt$.

(a) Substitute $M = M_0 - bt$ into Equation 1 and solve the resulting equation for v. Use the initial condition $v(0) = 0$ to evaluate the constant.
(b) Determine the velocity of the rocket at time $t = M_2/b$. This is called the *burnout velocity*.
(c) Determine the height of the rocket $y = y(t)$ at the burnout time.
(d) Find the height of the rocket at any time t.