

## 1 | Exercises

### 1.1 | interpreting in terms of area



### 1.3 | subtracting integrals

I expect

$$\int_1^2 f(x) dx = \int_1^5 f(x) dx - \int_2^5 f(x) dx = -3 - 4 = -7$$

In fact, I expect

$$\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$$

### 1.4 | show $\int_a^b x^2 dx = \frac{b^3 - a^3}{3}$

(see attached pages)

Keep in mind

$$\sum_{k=1}^n af(x) = a \sum_{k=1}^n f(x)$$
$$\sum_{k=1}^n (a + f(x)) = an + \sum_{k=1}^n f(x)$$

$$\begin{aligned}
\int_a^b x^2 dx &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \left( \frac{b-a}{n} \left( a + k \frac{b-a}{n} \right)^2 \right) \\
&= \lim_{n \rightarrow \infty} \frac{b-a}{n} \sum_{k=1}^n \left( a + k \frac{b-a}{n} \right)^2 \\
&= \lim_{n \rightarrow \infty} \frac{b-a}{n} \sum_{k=1}^n \left( a^2 + \left( k \frac{b-a}{n} \right)^2 + 2ak \frac{b-a}{n} \right) \\
&= \lim_{n \rightarrow \infty} \frac{b-a}{n} \sum_{k=1}^n a^2 + \sum_{k=1}^n \left( k \frac{b-a}{n} \right)^2 + \sum_{k=1}^n 2ak \frac{b-a}{n} \\
&= \lim_{n \rightarrow \infty} \frac{b-a}{n} \left( a^2 n + \sum_{k=1}^n k^2 \left( \frac{b-a}{n} \right)^2 + 2a \frac{b-a}{n} \sum_{k=1}^n k \right) \\
&= \lim_{n \rightarrow \infty} \frac{b-a}{n} \left( a^2 n + \left( \frac{b-a}{n} \right)^2 \sum_{k=1}^n k^2 + 2a \frac{b-a}{n} \sum_{k=1}^n k \right) \\
&= \lim_{n \rightarrow \infty} \frac{b-a}{n} \left( a^2 n + \left( \frac{b-a}{n} \right)^2 \frac{n(n+1)(2n+1)}{6} + 2a \frac{b-a}{n} \frac{n(n+1)}{2} \right) \\
&= \lim_{n \rightarrow \infty} (b-a) \left( a^2 + \left( \frac{b-a}{n} \right)^2 \frac{(n+1)(2n+1)}{6} + 2a \frac{b-a}{n} \frac{(n+1)}{2} \right) \\
&= \lim_{n \rightarrow \infty} (b-a) \left( a^2 + \frac{(b-a)^2}{n^2} \left( n \frac{(2n+1)}{6} + \frac{(2n+1)}{6} \right) + a \frac{b-a}{n} (n+1) \right) \\
&= \lim_{n \rightarrow \infty} (b-a) \left( a^2 + \frac{(b-a)^2}{n^2} \left( n \frac{(2n+1)}{6} + \frac{(2n+1)}{6} \right) + a \cancel{n} \frac{b-a}{\cancel{n}} + a \frac{b-a}{n} \right) \\
&= \lim_{n \rightarrow \infty} (b-a) \left( a^2 + \frac{(b-a)^2}{n^2} \left( n \frac{(2n+1)}{6} + \frac{(2n+1)}{6} \right) + a(b-a) + a \frac{b-a}{n} \right) \\
&= \lim_{n \rightarrow \infty} (b-a) \left( a^2 + \frac{(b-a)^2}{n^2} \cancel{n} \frac{(2n+1)}{6} + \frac{(b-a)^2}{\cancel{n^2}} \frac{(2n+1)}{6} + a(b-a) \right) \\
&= \lim_{n \rightarrow \infty} (b-a) \left( a^2 + \frac{(b-a)^2}{n} \frac{(2n+1)}{6} + a(b-a) \right) \\
&= \lim_{n \rightarrow \infty} (b-a) \left( a^2 + \frac{\cancel{n} (b-a)^2}{3 \cancel{n}} + \frac{1}{6} \frac{(b-a)^2}{\cancel{n}} + a(b-a) \right) \\
&= \lim_{n \rightarrow \infty} (b-a) \left( a^2 + \frac{(b-a)^2}{3} + a(b-a) \right) \\
&= \lim_{n \rightarrow \infty} a^2(b-a) + \frac{b^2 + a^2 - 2ab}{3} (b-a) + a(b-a)^2 \\
&= \lim_{n \rightarrow \infty} \frac{1}{3} (3a^2(b-a) + b^2(b-a) + a^2(b-a) - 2ab(b-a) + 3a(b-a)^2) \\
&= \lim_{n \rightarrow \infty} \frac{1}{3} (3a^2b - 3a^3 + b^3 - ab^2 + a^2b - a^3 - 2ab^2 + 2a^2b + 3a(b^2 + a^2 - 2ab)) \\
&= \lim_{n \rightarrow \infty} \frac{1}{3} (3a^2b - 3a^3 + b^3 - ab^2 + a^2b - a^3 - 2ab^2 + 2a^2b + 3ab^2 + 3a^3 - 6a^2b) \\
&= \lim_{n \rightarrow \infty} \frac{1}{3} (3a^2b - 6a^2b - 3a^3 - a^3 + 3a^3 + b^3 + a^2b + 2a^2b - ab^2 - 2ab^2 + 3ab^2)
\end{aligned}$$