Source:

1 | Problem

Suppose $T \in \mathcal{L}(V, W)$ and U is a subspace of V. Let π denote the quotient map from V onto V/U. Prove that there exists $S \in \mathcal{L}(V/U, W)$ such that $T = S \circ \pi$ if and only if $U \subseteq$ null T.

Intuitively, if we mod out part of the null T, then we should still be able to have a map that does what T would do. If we are able to do what T would do, then when modding out U we only removed part of null T and lost no information.

2 | Forward Direction

Intuitively, we can treat $S \circ \pi$ as a single map and take a basis of V to the same place that T would, and the maps would be equal.

If V is finite dimensional, suppose $v_1, \ldots, v_n$ is a basis of V and $v_1, \ldots, v_k$ is a basis of U ($k =$ dim U and $n =$ dim V). For each $k < j \leq n$, $\pi v_j \neq 0$, and we can control where S should send it. Let S be defined by:

$$S(\pi v_j) = T v_j$$

Then, $S \circ \pi$ will send each vector in U to 0 and each other vector where T would send it. Because $U \subseteq$ null T, $S \circ \pi = T$.

This argument does not work for infinite dimensional vector spaces. Instead, perhaps we can send anything not in U to where T would send it and show that the resulting S is linear? I'm not convinced by the following argument:

Let $S : V/U \to W$ s.t. $S(\pi v) = Tv$. Then, $S \circ \pi = T$. Maybe one can say "T is linear so $S \circ \pi$ must also be linear", but I am unsure.

For S to be linear, it needs to be additive and homogenous. For $u, v \in V$ and $\lambda \in \mathbb{F}$,

$$S\pi u + S\pi v = Tu + Tv = T(u + v) = S(\pi u + \pi v)$$

$$\lambda S\pi u = \lambda T u = T(\lambda u) = S(\lambda \pi u)$$

3 | Reverse Direction by Contrapositive

Intuitively, if we lost information, then we can't reconstruct what T would do.

Assume $U \not\subseteq$ null T. There exists $v \in U$ s.t. $Tv \neq 0$. This is some of the "information" that was "lost". Because $v \in U$,

$$\pi v = U + v = U$$

Because U is the additive identity (0) in V/U, and because linear maps take zero to zero, $S \in \mathcal{L}(V/U, W)$ must take $\pi v = 0$ to zero. Thus, either $S(\pi v) \neq Tv$ or S is not a linear map, both of which are contradictions.

This shows that if $U \not\subseteq$ null T, then $S \notin \mathcal{L}(V/U, W)$ or $T \neq S \circ \pi$. Thus, if $S \in \mathcal{L}(V/U, W)$ and $T = S \circ \pi$, then $U \subseteq$ null T.