

Source:

1 | find taylor series

1.1 | $y = \cos(x)$

$$\begin{aligned}
 P_n(x) &= f(0) + \frac{d}{dx}f(0)x + \frac{\frac{d^2}{d^2x}f(0)}{2!}x^2 + \frac{\frac{d^3}{d^3x}f(0)}{3!}x^3 + \dots \\
 &= \cos(0) - \sin(0)x + \frac{-\cos(0)}{2!}x^2 + \frac{\sin(0)}{3!}x^3 + \dots \\
 &= 1 - 0x - \frac{1}{2!}x^2 + \frac{0}{3!}x^3 + \dots \\
 &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \dots \\
 &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}
 \end{aligned}$$

1.2 | $y = e^x$

$$\begin{aligned}
 P_n(x) &= f(0) + \frac{d}{dx}f(0)x + \frac{\frac{d^2}{d^2x}f(0)}{2!}x^2 + \frac{\frac{d^3}{d^3x}f(0)}{3!}x^3 + \dots \\
 &= e^0 + e^0x + \frac{e^0}{2!}x^2 + \frac{e^0}{3!}x^3 + \dots \\
 &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots \\
 &= \sum_{k=0}^{\infty} \frac{x^k}{k!}
 \end{aligned}$$

1.3 | $y = \sqrt{x}$ **centered at** $x = 1$

$$\begin{aligned}
 P_n(x) &= f(1) + \frac{d}{dx}f(1)(x-1) + \frac{\frac{d^2}{d^2x}f(1)}{2!}(x-1)^2 + \frac{\frac{d^3}{d^3x}f(1)}{3!}(x-1)^3 + \dots \\
 &= 1 + \frac{1}{2}(x-1) + \frac{\frac{1}{2} \cdot \frac{-1}{2}}{2!}(x-1)^2 + \frac{\frac{1}{2} \cdot \frac{-1}{2} \cdot \frac{-3}{2}}{3!}(x-1)^3 + \dots
 \end{aligned}$$

I don't know how to write it using summation notation though..

2 | prove approximations

2.1 | $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$

Proof by geometric series

2.2 | $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$

Plug $-x$ for x in the previous equation.

$$2.3 \mid \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots$$

Plug x^2 for x in the previous equation.

3 | more finding of polynomial

$$3.1 \mid \text{TODO } y = \ln(1+x)$$

$$3.2 \mid \text{TODO } y = \tan^{-1} x$$

$$3.3 \mid y = (1+x)^k$$

$$\begin{aligned}
 P_n(x) &= f(0) + \frac{d}{dx}f(0)x + \frac{\frac{d^2}{dx^2}f(0)}{2!}x^2 + \frac{\frac{d^3}{dx^3}f(0)}{3!}x^3 + \dots \\
 &= 1 + k(1)^k + k(k-1)(1)^{k-1}x + \frac{k(k-1)(k-2)(1)^{k-2}}{2!}x^2 + \frac{k(k-1)(k-2)(k-3)(1)^{k-3}}{3!}x^3 + \dots \\
 &= 1 + k + k(k-1)x + \frac{k(k-1)(k-2)}{2!}x^2 + \frac{k(k-1)(k-2)(k-3)}{3!}x^3 + \dots \\
 &= 1 + k + \frac{k!}{(k-1)!}x + \frac{\frac{k!}{(k-2)!}}{2!}x^2 + \frac{\frac{k!}{(k-3)!}}{3!}x^3 + \dots \\
 &= 1 + k + \frac{k!x}{(k-1)!} + \frac{k!}{(k-2)!2!}x^2 + \frac{k!}{(k-3)!3!}x^3 + \dots \\
 &= \binom{k}{0} + \binom{k}{1}x + \binom{k}{2}x^2 + \binom{k}{3}x^3 + \dots \\
 &= \sum_{i=0}^k \binom{k}{i} x^i
 \end{aligned}$$

4 | find sum of series by recognizing Taylor Series approximations of some functions

$$4.1 \mid 3 + \frac{9}{2!} + \frac{27}{3!} + \frac{81}{4!} + \dots$$

$$3 + \frac{9}{2!} + \frac{27}{3!} + \frac{81}{4!} = e^3 - 1$$

$$4.2 \mid 1 - \ln 2 + \frac{\ln^2 2}{2!} + \frac{\ln^3 2}{3!} + \dots$$

$$1 - \ln 2 + \frac{\ln^2 2}{2!} + \frac{\ln^3 2}{3!} + \dots = e^{-\ln 2} = \frac{1}{2}$$

$$4.3 \mid \sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k+1}}{4^{2k+1} (2k+1)!}$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k+1}}{4^{2k+1} (2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{\pi}{4}\right)^{2k+1}}{(2k+1)!} = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

5 | evaluate limits using taylor series approx

5.1 | $\lim_{x \rightarrow 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5}$

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5} &= \lim_{x \rightarrow 0} \frac{\cancel{x} - \frac{x^3}{6} + \frac{x^5}{5!} + \dots \cancel{x} + \frac{x^3}{6}}{x^5} \\
 &= \lim_{x \rightarrow 0} \frac{\cancel{-\frac{x^3}{6}} + \frac{x^5}{5!} + \dots \cancel{+\frac{x^3}{6}}}{x^5} \\
 &= \lim_{x \rightarrow 0} \frac{\cancel{\frac{x^5}{5!}} + \dots}{x^5} \\
 &= \lim_{x \rightarrow 0} \frac{1}{5!} + \frac{x^7}{x^5 7!} + \frac{x^9}{x^5 9!} \dots \\
 &= \frac{1}{5!}
 \end{aligned}$$

5.2 | **TODO** $\lim_{x \rightarrow 0} \frac{x - \tan^{-1} x}{x^3}$

6 | find taylor series approximations

6.1 | $y = e^x + e^{-x}$

$$\begin{aligned}
 e^x + e^{-x} &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots \\
 &= 1 + 1 + x - x + \frac{x^2}{2!} + \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^3}{3!} + \dots \\
 &= 2 \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \right) \\
 &= 2 \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}
 \end{aligned}$$

6.2 | $y = \sin(\pi x)$

$$\sin(\pi x) = \sum_{k=0}^{\infty} \frac{(-1)^k (\pi x)^{2k+1}}{(2k+1)!}$$

(just plug it in)

6.3 | **TODO** $y = \sin^2 x$