Source:

1 | Problem: Axler 3.E exercise 18

> Suppose $T \in \mathcal{L}(V, W)$ and U is a subspace of V. Let π denote the quotient map from V onto V/U. Prove that there exists $S \in \mathcal{L}(V/U, W)$ such that $T = S \circ \pi$ if and only if $U \subseteq$ null T.

Intuitively, if we mod out part of the null T, then we should still be able to have a map that does what T would do. If we are able to do what T would do, then when modding out U we only removed part of null T and lost no information.

2 | Forward Direction

Intuitively, we can treat $S \circ \pi$ as a single map and take a basis of V to the same place that T would, and the maps would be equal.

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~If $V$ is finite dimensional, suppose $v_1, \ldots, v_n$ is a basis of $V$ and $v_1, \ldots,
v_k$ is a basis of $U$ ($k = \text{dim } U$ and $n = \text{dim }V$). For each $k < j \le n$,
$\pi v_j \neq 0$, and we can control where $S$ should send it. Let $S$ be defined by: \ S(
\pi v_j) = T v_j \ Then, $S \circ \pi$ will send each vector in $U$ to 0 and each other vector
where $T$ would send it. Because $U \subseteq \text{null }T$, $S \circ \pi = T$. ~~ This argument
does not work for infinite dimensional vector spaces. Instead, perhaps we can send anything
not in $U$ to where $T$ would send it and show that the resulting $S$ is linear? I'm not convinced
by the following argument: ~~ Let $S : V/U \to W$ s.t. $S(\pi v) = Tv$. Then, $S \circ \pi
= T$. ~~ For $S$ to be linear, it needs to be additive and homogenous. For $u, v \in V$ and
$\lambda \in \mathbb F$, \ S\pi u + S\pi v = Tu + Tv = T(u+v) = S(\pi u + \pi v) \\ \lambda
S \pi u = \lambda T u = T(\lambda u) = S (\lambda \pi u) \
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In other words, $T$ is linear thus $S \circ \pi$ is also linear.
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Let $S : V/U \to W$ be defined by

$$S(U + v) = Tv$$

If S is well defined (every element in V/U is mapped to exactly one place), then S will inherit additivity and homogeneity from T and $S \circ \pi$ will equal T.

Let $v \in V$ and $v' \in V/U$ s.t. $v' = \pi v$ (v' is where π takes v).

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For some $u, v \in \text{range }T$,
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2.1 | define S(U + v) = T v

2.1.1 | check that it is well defined

1. every element is sent to exactly one place

2.1.2 | check that linearity is inhereted from T

3 | Reverse Direction by Contrapositive

Intuitively, if we lost information, then we can't reconstruct what T would do.

Assume $U \nsubseteq$ null T. There exists $v \in U$ s.t. $Tv \neq 0$. This is some of the ”information” that was ”lost”. Because $v \in U$,

$$\pi v = U + v = U$$

Because U is the additive identity (0) in V/U, and because linear maps take zero to zero, $S \in \mathcal{L}(V/U, W)$ must take $\pi v = 0$ to zero. Thus, either $S(\pi v) \neq Tv$ or S is not a linear map, both of which are contradictions.

This shows that if $U \nsubseteq$ null T, then $S \notin \mathcal{L}(V/U, W)$ or $T \neq S \circ \pi$. Thus, if $S \in \mathcal{L}(V/U, W)$ and $T = S \circ \pi$, then $U \subseteq$ null T.