Source:

1 | Problem: Axler 3.E exercise 18

> Suppose $T \in \mathcal{L}(V, W)$ and U is a subspace of V. Let π denote the quotient map from V onto V/U. Prove that there exists $S \in \mathcal{L}(V/U, W)$ such that $T = S \circ \pi$ if and only if $U \subseteq \text{null } T$.

Intuitively, if we mod out part of the null T, then we should still be able to have a map that does what T would do. If we are able to do what T would do, then when modding out U we only removed part of null T and lost no information.

2 | Forward Direction

Intuitively, we can treat $S \circ \pi$ as a single map and take a basis of V to the same place that T would, and the maps would be equal.

~~If V is finite dimensional, suppose $v_1, \ldots, v_n$ is a basis of V and $v_1, \ldots, v_k$ is a basis of U ($k = \dim U$ and $n = \dim V$). For each $k < j \leq n$, $\pi v_j \neq 0$, and we can control where S should send it. Let S be defined by:~~

$$S(\pi v_j) = T v_j$$

~~Then, $S \circ \pi$ will send each vector in U to 0 and each other vector where T would send it. Because $U \subseteq \text{null } T$, $S \circ \pi = T$.~~

~~This argument does not work for infinite dimensional vector spaces. Instead, perhaps we can send anything not in U to where T would send it and show that the resulting S is linear? I'm not convinced by the following argument:~~

~~Let $S : V/U \to W$ s.t. $S(\pi v) = Tv$. Then, $S \circ \pi = T$.~~

~~For S to be linear, it needs to be additive and homogenous. For $u, v \in V$ and $\lambda \in \mathbb{F}$,~~ +

$$S\pi u + S\pi v = Tu + Tv = T(u + v) = S(\pi u + \pi v)$$

+ +

$$\lambda S\pi u = \lambda Tu = T(\lambda u) = S(\lambda \pi u)$$

+

~~In other words, T is linear thus $S \circ \pi$ is also linear.~~

Let S be a relation between V/U and W defined by

$$S(U + v) = Tv$$

If S is well defined (every element in V/U is mapped to exactly one place), then S will inherit additivity and homogeneity from T and $S \circ \pi$ will equal T.

Let $v \in V$ and $v' \in V/U$ s.t. $v' = \pi v$ (v' is where π takes v). Then, to show that S is well defined, we must show that v has atleast one and at most one image through $S \circ \pi$.

Because πv is well defined, and $U + v$ was arbitrary in the definition of S, each v must have atleast one image in W.

Take S to be an arbitrary linear map. The only restriction on S that could cause $S(U + v) \neq Tv$ is $S(0) = 0$ (this statement is not watertight). Thus, S is defined if $\forall U + v = U = 0$, $Tv = 0$. Equivalently, S is defined if $U \subseteq \text{null } T$, which is given in the problem.

Thus, S is well defined. To show that it inherits additivity and homogeneity:

$$S(U+u)+S(U+v)=Tu+Tv=T(u+v)=S(U+u+U+v)=S(U+(u+v))$$

$$\lambda\left(S(U+v)\right)=\lambda Tv=T(\lambda v)=S(U+(\lambda v))$$

~~For some $u, v \in$ range T,~~

2.1 | define S(U + v) = T v

2.1.1 | check that it is well defined

1. every element is sent to exactly one place

2.1.2 | check that linearity is inhereted from T

3 | Reverse Direction by Contrapositive

Intuitively, if we lost information, then we can't reconstruct what T would do.

Assume $U \nsubseteq$ null T. There exists $v \in U$ s.t. $Tv \neq 0$. This is some of the "information" that was "lost". Because $v \in U$,

$$\pi v = U + v = U$$

Because U is the additive identity (0) in V/U, and because linear maps take zero to zero, $S \in \mathcal{L}(V/U, W)$ must take $\pi v = 0$ to zero. Thus, either $S(\pi v) \neq Tv$ or S is not a linear map, both of which are contradictions.

This shows that if $U \nsubseteq$ null T, then $S \notin \mathcal{L}(V/U, W)$ or $T \neq S \circ \pi$. Thus, if $S \in \mathcal{L}(V/U, W)$ and $T = S \circ \pi$, then $U \subseteq$ null T.