

**Source:**

## 1 | Problem: Axler 3.E exercise 18

Suppose  $T \in \mathcal{L}(V, W)$  and  $U$  is a subspace of  $V$ . Let  $\pi$  denote the quotient map from  $V$  onto  $V/U$ . Prove that there exists  $S \in \mathcal{L}(V/U, W)$  such that  $T = S \circ \pi$  if and only if  $U \subseteq \text{null } T$ .

Intuitively, if we mod out part of the null  $T$ , then we should still be able to have a map that does what  $T$  would do. If we are able to do what  $T$  would do, then when modding out  $U$  we only removed part of null  $T$  and lost no information.

## 2 | Forward Direction

Intuitively, we can treat  $S \circ \pi$  as a single map and take a basis of  $V$  to the same place that  $T$  would, and the maps would be equal.

~If  $V$  is finite dimensional, suppose  $v_1, \dots, v_n$  is a basis of  $V$  and  $v_1, \dots, v_k$  is a basis of  $U$  ( $k = \dim U$  and  $n = \dim V$ ). For each  $k < j \leq n$ ,  $\pi v_j \neq 0$ , and we can control where  $S$  should send it. Let  $S$  be defined by:

$$S(\pi v_j) = T v_j$$

Then,  $S \circ \pi$  will send each vector in  $U$  to 0 and each other vector where  $T$  would send it. Because  $U \subseteq \text{null } T$ ,  $S \circ \pi = T$ .

This argument does not work for infinite dimensional vector spaces. Instead, perhaps we can send anything not in  $U$  to where  $T$  would send it and show that the resulting  $S$  is linear? I'm not convinced by the following argument:

Let  $S : V/U \rightarrow W$  s.t.  $S(\pi v) = T v$ . Then,  $S \circ \pi = T$ .

For  $S$  to be linear, it needs to be additive and homogenous. For  $u, v \in V$  and  $\lambda \in \mathbb{F}$ ,

$$S\pi u + S\pi v = T u + T v = T(u + v) = S(\pi u + \pi v)$$

$$\lambda S\pi u = \lambda T u = T(\lambda u) = S(\lambda \pi u)$$

In other words,  $T$  is linear thus  $S \circ \pi$  is also linear.~

Let  $S : V/U \rightarrow W$  be defined by

$$S(U + v) = T v$$

If  $S$  is well defined (every element in  $V/U$  is mapped to exactly one place), then  $S$  will inherit additivity and homogeneity from  $T$  and  $S \circ \pi$  will equal  $T$ .

For some  $u, v \in \text{range } T$

### 2.1 | define $S(U + v) = T v$

#### 2.1.1 | check that it is well defined

1. every element is sent to exactly one place

### 2.1.2 | check that linearity is inherited from T

## 3 | Reverse Direction by Contrapositive

Intuitively, if we lost information, then we can't reconstruct what  $T$  would do.

Assume  $U \not\subseteq \text{null } T$ . There exists  $v \in U$  s.t.  $Tv \neq 0$ . This is some of the "information" that was "lost". Because  $v \in U$ ,

$$\pi v = U + v = U$$

Because  $U$  is the additive identity (0) in  $V/U$ , and because linear maps take zero to zero,  $S \in \mathcal{L}(V/U, W)$  must take  $\pi v = 0$  to zero. Thus, either  $S(\pi v) \neq Tv$  or  $S$  is not a linear map, both of which are contradictions.

This shows that if  $U \not\subseteq \text{null } T$ , then  $S \notin \mathcal{L}(V/U, W)$  or  $T \neq S \circ \pi$ . Thus, if  $S \in \mathcal{L}(V/U, W)$  and  $T = S \circ \pi$ , then  $U \subseteq \text{null } T$ .