1 | Axler 6.A exercise 9

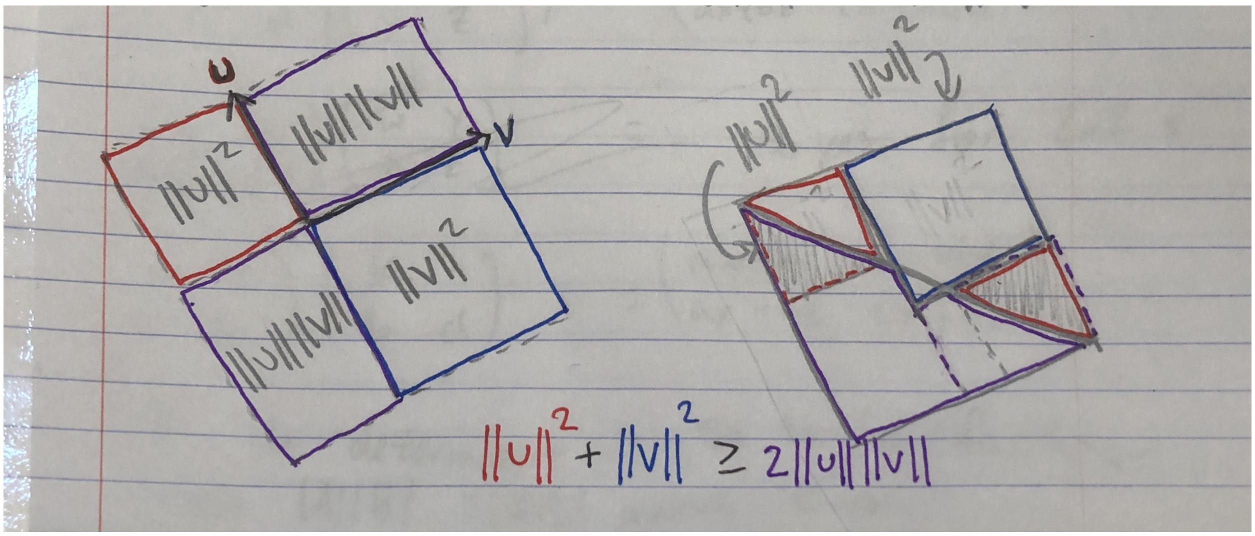
Suppose $u, v \in V$ and $\|u\| \leq 1$ and $\|v\| \leq 1$. Prove that

$$\sqrt{1-\|u\|^2}\sqrt{1-\|u\|^2} \leq 1 - |\langle u, v\rangle|$$

2 | Proof

2.1 | Useful Lemma

$$\|u\|^2 + \|v\|^2 \geq 2\|u\|\|v\|$$



2.2 | Cauchy-Schwarz Corollary

$$|\langle u, v\rangle| \leq \|u\|\|v\|$$
$$1 - \|u\|\|v\| \leq 1 - |\langle u, v\rangle|$$

This intermediate value is obtained using the Cauchy-Schwarz inequality.

2.3 | Main Proof

Now, to show that the square of the left hand side is less than or equal to the square of the right hand side,

$$\begin{aligned}
&=(1-\|u\|^2)(1-\|v\|^2) \\
&=1-\|u\|^2-\|v\|^2+\|u\|^2\|v\|^2 \\
&=1-(\|u\|^2+\|v\|^2)+\|u\|^2\|v\|^2 \\
&\leq 1-2\|u\|\|v\|+\|u\|^2\|v\|^2 \\
&=(1-\|u\|\|v\|)^2 \\
&\leq (1-|\langle u, v\rangle|)^2
\end{aligned}$$