

## 1 | shoestring loop

$$\begin{aligned}x &= t^2 \\y &= t^3 - 3t \\ \frac{dx}{dt} &= 2t \\ \frac{dy}{dt} &= 3t^2 - 3 \\ \frac{dy}{dx} &= \frac{3t^2 - 3}{2t}\end{aligned}$$

### 1.1 | tangents are horizontal or vertical

#### 1.1.1 | horizontal

$$\begin{aligned}3t^2 - 3 &= 0 \\ 3t^2 &= 3 \\ t^2 &= 1 \\ t &= \pm 1\end{aligned}$$

#### 1.1.2 | vertical

$$\begin{aligned}2t &= 0 \\ t &= 0\end{aligned}$$

### 1.2 | concave up

$$\begin{aligned}\frac{d}{dx} \frac{dy}{dx} &= \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3} = \frac{2t(6t) - (6t^2 - 3)(2)}{8t^3} \\ &= \frac{6t^2 - 6t^2 + 3}{4t^3} = \frac{3}{4t^3} > 0 \\ &\therefore \text{concave up for } t > 0\end{aligned}$$

### 1.3 | concave down

Using similar logic, the curve is concave down for  $t \leq 0$ .

## 2 | polar curves + converting to cartesian

polar sketches

Also see the desmos.

## 3 | cardioid

3.1 | **sketch**

Oops I thought cosine was sine

3.2 | **crosses the origin**

Only happens when  $\theta = 0$ .

$$\begin{aligned}
 r &= 1 + 2 \cos \theta = 0 \\
 2 \cos \theta &= -1 \\
 \cos \theta &= -\frac{1}{2} \\
 \theta &= \cos^{-1}\left(-\frac{1}{2}\right) \\
 &= \frac{2\pi}{3}, -\frac{2\pi}{3}
 \end{aligned}$$

3.3 | **derivatives to verify crossing**

$$\begin{aligned}
 y &= r \sin \theta = (1 + 2 \cos \theta) \sin \theta = \sin \theta + 2 \cos \theta \sin \theta = \sin \theta + \sin 2\theta \\
 x &= r \cos \theta = (1 + 2 \cos \theta) \cos \theta = \cos \theta + 2 \cos^2 \theta \\
 \frac{dy}{d\theta} &= \cos \theta + 2 \cos 2\theta \implies -\frac{1}{2} + 2\left(-\frac{1}{2}\right) = -\frac{3}{2} \\
 \frac{dx}{d\theta} &= -\sin \theta - 2(2 \cos \theta) \sin \theta = -\sin \theta - 2 \sin 2\theta = -\frac{\sqrt{3}}{2} - 2\left(-\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2} \\
 \frac{d^2y}{d\theta^2} &= -\sin \theta - 4 \sin 2\theta = -\frac{\sqrt{3}}{2} - 4\left(-\frac{\sqrt{3}}{2}\right) = 3\frac{\sqrt{3}}{2} \\
 \frac{d^2x}{d\theta^2} &= -\cos \theta - 4 \cos 2\theta = \frac{5}{2} \\
 \frac{dy}{dx} &= \frac{\dot{y}}{\dot{x}} = \frac{\cos \theta + 2 \cos 2\theta}{\sin \theta + 2 \sin 2\theta} = -\frac{\frac{3}{2}}{\frac{\sqrt{3}}{2}} = -\sqrt{3} \\
 \frac{dy^2}{dx^2} &= \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\dot{x}^3} = \frac{\frac{\sqrt{3}}{2}3\frac{\sqrt{3}}{2} - \frac{5}{2}\frac{3}{2}}{\left(\frac{\sqrt{3}}{2}\right)^3} \\
 &= \frac{\frac{9}{4} - \frac{15}{4}}{\frac{3\sqrt{3}}{8}} = -\frac{3}{2} \frac{8}{3\sqrt{3}} = -\frac{4}{\sqrt{3}}
 \end{aligned}$$

3.4 | **points where tangent is horizontal or vertical**