1 | Escape Velocity and Gravitational Potential Energy

1.1 | Newton's Universal Gravitation Law

$$\vec{F_g} = -\frac{GM_1M_2}{r^2}\hat{r} \tag{1}$$

where, $\vec{F_g}$ is the force of gravity on M_2; M_1 and M_2 are two point masses; G the universal gravitation constant; r the magnitude of the vector $\vec{r}$ from M_1 to M_2 and $\hat{r}$ the unit vector in the $\vec{r}$ direction.

1.2 | Equation for Gravitational Potential Energy

1.2.1 | Needed Definitions

To begin, we need to modify the ***Newton's Universal Gravitation Law*** to fit the parameters of the scenario. Namely, we need to treat both Earth and our object as point masses, and assign M_1 to be Earth and M_2 to be our object.

Also, it is necessary to define the coordinate system: that our object, M_2, is defined to be to the left/more negative side of the coordinate compared to the location occupied by the Earth, $M_e = M_1$ (as, per the problem, the "zero" point is set at $r = \infty$.)

With this assumption, we could therefore claim $\vec{r}$ to be pointing from the origin to the ***negative*** side of the axis, rendering it represented by the value -1 for this system.

Hence, with the necessary variable substitutions as highlighted before, we arrive at the following equation:

$$\vec{F_{em}}(r) = \frac{GM_eM_2}{r^2} \tag{2}$$

1.2.2 | Deducing Gravitational Potential energy

The general equation for work is as follows:

$$W = F(x)dx \tag{3}$$

In this case, as we will be deducing the total gravitational potential energy as per the setup above, we need to be integrating upon $\vec{F_{em}}(r)dr$. Hence, the integral is therefore:

$$W = \int \frac{GM_eM_2}{r^2}dr \tag{4}$$

Determining the total gravitation potential energy will therefore involve evaluating the integral:

$$\begin{aligned} W &= \int \frac{GM_eM_2}{r^2}dr && (5)\\ W &= GM_eM_2\int \frac{1}{r^2}dr && (6)\\ W &= GM_eM_2\int r^{-2}dr && (7)\\ W &= \frac{-GM_eM_2}{r} && (8) \end{aligned}$$

1.3 | **Escape Velocity**

To deduct the escape velocity of the Earth, our object M_2 must do enough work such that the work Earth exerts upon it — ***as deducted above*** — is canceled out.

1.3.1 | **Calculation Setup**

Hence, to figure the escape velocity, we must deduct the kinetic energy needed to perform the exact work deducted above in the opposite direction. That,

$$-\frac{1}{2}M_2\vec{V}^2 = \frac{-GM_eM_2}{r} \tag{9}$$

Notably, the negative sign before the kinetic energy equation corresponds to the fact that the work is done in an opposite direction as that done by Earth's gravitational potential energy.

Also, as the object to escape Earth's gravitation is starting at the surface of Earth, $r = R_e$, the radius of the earth.

1.3.2 | **Deducing Escape Velocity**

The final calculations after substitution is therefore simply a matter of isolating $\vec{V}$, the velocity vector.

$$-\frac{1}{2}M_2\vec{V}^2 = \frac{-GM_eM_2}{R_e} \tag{10}$$

$$\vec{V}^2 = 2\frac{GM_e}{R_e} \tag{11}$$

$$\vec{V} = \sqrt{2\frac{GM_e}{R_e}} \tag{12}$$

1.3.3 | **Calculating Escape Velocity**

Taking $G = 6.674 \times 10^{-11} m^3 \frac{m^3}{kg\ s^2}$, $M_e = 5.97 \times 10^{24} kg$, and $R_e = 6.371 \times 10^6 km$...

$$|\vec{V}| \approx 1.119 \times 10^4 \frac{m}{s} = 2.503 \times 10^4 \frac{M}{h} \tag{13}$$

2 | **Center of Mass**

2.1 | **Finding an Expression for the Position of the Center of Mass**

The net force of a system is described as:

$$\sum_{i=1}^{n} F_{\vec{net,i}} = (\sum_{i=1}^{n} m_i) r_{\ddot{\vec{CM}}} \tag{14}$$

where, $F_{\vec{net,i}}$ is the net force on m_i, $r_{\vec{CM}}$ the position vector of the center of mass CM.

In order to isolate $r_{\vec{CM}}$, the equation above must be integrated twice w.r.t. time, namely:

$$\int\int\sum_{i=1}^{n}\vec{F_{net,i}}dtdt = \int\int(\sum_{i=1}^{n}m_i)\ddot{\vec{r_{CM}}}dtdt \tag{15}$$

For this evaluation, we will also set $\sum_{i=1}^{n} m_i = M$, define $\vec{a_i}$ as acceleration of m_i, $\vec{r_i}$ as position of m_i, and apply Newton's second law:

$$\int\int\sum_{i=1}^{n}\vec{F_{net,i}}dtdt = \int\int(\sum_{i=1}^{n}m_i)\ddot{\vec{r_{CM}}}dtdt \tag{16}$$

$$\int(\sum_{i=1}^{n}m_i\int\vec{a_i}dt)dt = \int(M\int\ddot{\vec{r_{CM}}}dt)dt \tag{17}$$

$$\int(\sum_{i=1}^{n}m_i\int\frac{d^2\vec{r_i}}{dt^2}dt)dt = \int(M\int\frac{d^2\vec{r_{CM}}}{dt^2}dt)dt \tag{18}$$

$$\int(\sum_{i=1}^{n}m_i\frac{d\vec{r_i}}{dt})dt = \int M\frac{d\vec{r_{CM}}}{dt}dt \tag{19}$$

$$\sum_{i=1}^{n}m_i\int\frac{d\vec{r_i}}{dt}dt = M\int\frac{d\vec{r_{CM}}}{dt}dt \tag{20}$$

$$\sum_{i=1}^{n}m_i\vec{r_i} = M\vec{r_{CM}} \tag{21}$$

$$\frac{1}{M}\sum_{i=1}^{n}m_i\vec{r_i} = \vec{r_{CM}} \tag{22}$$

As such, the position $\vec{r_{CM}}$ of the center of mass CM is therefore:

$$\vec{r_{CM}} = \frac{1}{M}\sum_{i=1}^{n}m_i\vec{r_i} \tag{23}$$

Where, M is the total mass of the system, m_i the mass of component i of the system, and $\vec{r_i}$ the position of m_i.

2.2 | **Simplifying to ignore internal forces**

Per the definition of internal force, it does ***not*** result in work performed on the system, meaning that the system as a whole would not have moved a distance.

Because of the fact that $\vec{r_{CM}} = \frac{1}{M}\sum_{i=1}^{n}m_i\vec{r_i}$ as derived ***above***, $\vec{r_{CM}}$ changes when the system as a whole moves. However, internal forces does not do this, meaning the existence of internal forces does not change $\vec{r_{CM}}$.

This fact allows for a simplification of the equation:

$$\sum_{i=1}^{n}\vec{F_{net,i}} = (\sum_{i=1}^{n}m_i)\ddot{\vec{r_{CM}}} \tag{24}$$

to:

$$\sum_{j=1}^{m}\vec{F_{ext,j}} = M\ddot{\vec{r_{CM}}} \tag{25}$$

by applying $\sum_{i=1}^{n} m_i = M$ as per aforementioned and the external forces argument above.