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Exploration 2-3b: Extension of the Limit Theorems by Mathematical Induction

Date:

Objective: Prove that the limit of a sum property is true for the sum of *any* finite number of terms.

Suppose that $f_n(x)$ is the sum of n other functions,

$$f_n(x) = g_1(x) + g_2(x) + g_3(x) + \cdots + g_n(x)$$

and that $g_1(x), g_2(x), g_3(x), \ldots, g_n(x)$ have limits $L_1, L_2, L_3, \ldots, L_n$, respectively, as x approaches c. Prove that

$$\lim_{x \to c} f_n(x) = L_1 + L_2 + L_3 + \cdots + L_n$$

for *all* integers $n \geq 2$.

Proof: (You supply the details!)

1. Explain how you know that the property is true for $n = 2$. (This fact is called the **anchor** of the proof.)

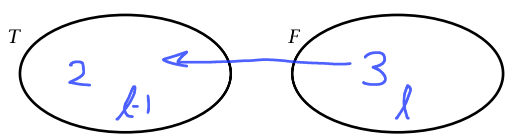
See 2-3A

KBerefLimitSumEpsDeltaProof

2. Assume that the property is false. What does this assumption tell you about values of n?

$n > 2$?

3. Let j be a value of n for which the property is not true. Let T be the set of values of n for which the property *is* true, and let F be the set of values of n for which the property is *false*. On the Venn diagram below, show a value of n that is in T and a value of n that is in F.



4. By a clever use of the associative property for addition, you can turn a sum of three terms into a sum of two terms. By doing this, show that the property is true for $n = 3$. You may write "lim" as an abbreviation for limit as $x \to c$. Write 3 in T.

$g_1(x) + g_2(x) + g_3(x)$

$= (g_1 + g_2)(x) + g_3(x)$

5. Suppose you assume that the property is true for $n = 5$. Show how you could prove that it is also true for $n = 6$.

$g_{1..6}(x) = g_{1..5}(x) + g_6(x)$

6. The **well-ordering axiom** states that any nonempty set of positive integers has a *least* element. How do you know that set F is a nonempty set of positive integers?

$n \in \mathbb{Z}_{\geq 2}$; we assume F to be non empty

this is what we are trying to contradict

7. Let l be the least element of F. How do you know that $l - 1$ is a *positive* integer? In which set is $l - 1$? Show l and $l - 1$ in the Venn diagram of Problem 3.

$l \in \mathbb{Z}_{\geq 2} \therefore l - 1 \in \mathbb{Z}_{\geq 1}$

l-1 must be in T becasue if it were in F, then l-1 would be the smallest in F but we said that l was the smallest in F

8. Write a statement about the limit of $f_{l-1}(x)$ and a statement about the limit of $f_l(x)$ as x approaches c.

$\lim f_{l-1}(x) = L_1 + L_2 + L_3 + \ldots L_{l-1}$

$\lim f_l(x) \neq L_1 + L_2 + \ldots + L_{l-1} + L_l$

9. By the definition of $f_n(x)$,

$$f_l(x) = g_1(x) + g_2(x) + g_3(x) + \cdots + g_{l-1}(x) + g_l(x)$$

By clever use of the associative property, as in Problem 5, associate the right side of this equation into a sum of *two* terms.

$f_l(x) = f_{l-1}(x) + g_l(x)$

(Over)

10. Take the limit of both sides of the equation in Problem 9. Use the anchor to write the right side of the equation as a sum of *two* limits.

$\lim f_l(x) = \lim f_{l-1}(x) + \lim g_l(x)$

11. Use the equation for the limit of $f_{l-1}(x)$ from Problem 8 to simplify the equation in Problem 10.

already did

12. Explain why the simplified equation in Problem 11 contradicts what you wrote about the limit of $f_l(x)$ in Problem 8.

$\lim f_l(x) = \left(\sum^{l-1} L\right) + L_l$

$= L_1 + L_2 + L_3 + \ldots + L_l$

13. The only place in the steps above that could account for the contradiction in Problem 12 is the assumption in Problem 2 that the property was *not* true for all integers $n \geq 2$. What can you conclude about this assumption? What can you conclude about the property?

thus, the assumption is false and the property is true for all integers n >= 2

The proof process in Problems 1 through 13 can be shortened if you realize that all you need to do is (a) find an anchor and (b) show that if the property is true for some integer $n = k$, then it is also true for the next integer, $n = k + 1$. This shortened process is called **mathematical induction.** Complete the following for the property of the limit of a sum.

Proof:

Anchor:

$\lim f_2(x) = \lim(g_1(x) + g_2(x))$

$= L_1 + L_2$

Induction hypothesis: Assume the property is true for $n = k > 2$.

Verification for $n = k + 1$:

$\lim f_{k+1}(x) = \lim f_k(x) + \lim g_{k+1}(x)$

$= (L_1 + L_2 + \ldots + L_k) + L_{k+1}$

Conclusion:

Q.E.D.

14. What did you learn as a result of doing this Exploration that you did not know before?