

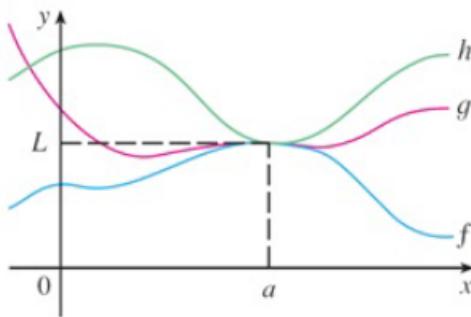
Squeeze Theorem

3 The Squeeze Theorem If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$$

then

$$\lim_{x \rightarrow a} g(x) = L$$



1. Evaluate the following limits

a. $\lim_{x \rightarrow \infty} \frac{\sin(x)}{x}$

$-1 \leq \sin(x) \leq 1$

$\frac{-1}{x} \leq \frac{\sin(x)}{x} \leq \frac{1}{x}$

$$\lim_{x \rightarrow \infty} \frac{-1}{x} = \lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

$$\therefore \lim_{x \rightarrow \infty} \frac{\sin x}{x} = \boxed{0}$$

b. $\lim_{x \rightarrow \infty} \frac{1-\cos(x)}{x}$

$|-1 \leq \cos x \leq 1|$

$-1 \leq \cos x \leq 1$

$$\lim_{x \rightarrow \infty} 0 = 0$$

$2 \geq 1 - \cos x \geq 0$

$$\lim_{x \rightarrow \infty} \frac{2}{x} = 0$$

$$\therefore \lim_{x \rightarrow \infty} \frac{1 - \cos x}{x} = \boxed{0}$$

??

$0 \leq \frac{1 - \cos x}{x} \leq 2$

why can't you just divide by x? it doesn't work... $1 - \cos x/x$ is not greater than 0

September 2019

!! ask in class !!

It actually does work,

NEAR where the limit is 1

Handout #4: Squeeze Theorem and Trigonometric Limits

c. $\lim_{x \rightarrow \infty} x^2 \cos\left(\frac{1}{x^2}\right)$

$$-1 \leq \cos\frac{1}{x^2} \leq 1 \quad \text{if it was } \lim_{x \rightarrow 0}$$

$$-x^2 \leq x^2 \cos\frac{1}{x^2} \leq x^2 \quad \lim_{x \rightarrow 0} x^2 = 0 = \lim_{x \rightarrow 0} -x^2$$

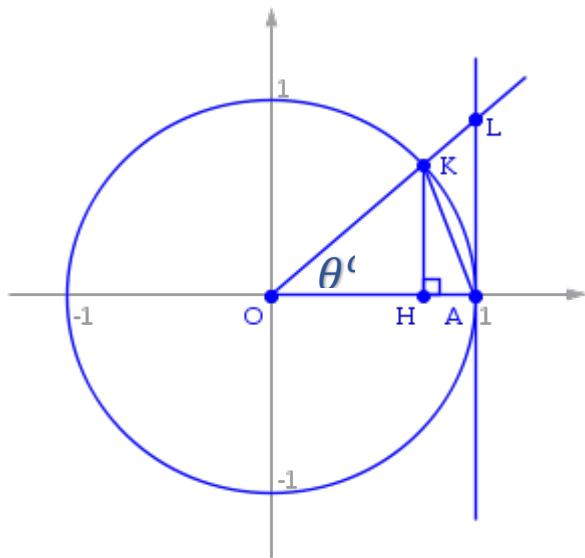
$$\lim_{x \rightarrow \infty} -x^2 = -\infty$$

$$\lim_{x \rightarrow \infty} x^2 = \infty$$

2. Prove $\lim_{\theta \rightarrow 0} \frac{\sin(\theta)}{\theta} = 1$ using steps below.

$$\therefore \lim_{x \rightarrow 0} x^2 \cos\frac{1}{x^2} \neq 0$$

\therefore Squeeze thm is inconclusive



The figure to the left is a sketch of a unit circle, where the angle θ is in radians. K is a point on the unit circle.

- a. Please write the coordinates of point K in terms of θ .

$$\cos \theta, \sin \theta$$

- b. What is the slope of line OK in terms of θ .

$$\frac{\sin \theta}{\cos \theta} = \tan \theta$$

- c. Find the equation of line OL in terms of θ .

$$y = x \tan \theta$$

- d. Find the coordinates of point A

$$1, 0$$

- e. Find the coordinates of point L in terms of θ .

$$1, \tan \theta$$

- f. Find the area of $\triangle OAK$ in terms of θ .

$$\frac{\sin \theta}{2}$$

- g. Find the area of sector \widehat{OAK} in terms of θ .

$$\frac{\theta}{2\pi} (\pi r^2) = \frac{\theta}{2} r^2 = \frac{\theta}{2}$$

- h. Find the area of $\triangle OAL$ in terms of θ .

$$\frac{\tan \theta}{2}$$

- i. Using the figure, we see $\text{Area of } \triangle OAK < \text{Area of sector } OAK < \text{Area of } \triangle OAL$. Write an inequality for your case in terms of θ .

$$\frac{\sin \theta}{\theta} \leq \frac{\theta}{2} \leq \frac{\tan \theta}{\theta} \quad \frac{\sin \theta}{\sin \theta} \leq \frac{\theta}{\sin \theta} \leq \frac{\tan \theta}{\sin \theta} = \frac{1}{\cos \theta}$$

- j. Find $\lim_{\theta \rightarrow 0} (\text{Area of sector } OAK)$.

$$\lim_{\theta \rightarrow 0} (\text{Area of } \triangle OAK) < \lim_{\theta \rightarrow 0} (\text{Area of sector } OAK) < \lim_{\theta \rightarrow 0} (\text{Area of } \triangle OAL)$$

$$\lim_{\theta \rightarrow 0} \frac{1}{\cos \theta} = 1 \quad \therefore 1 \leq \frac{\theta}{\sin \theta} \leq 1 \quad \therefore \frac{\theta}{\sin \theta} = 1 \quad \therefore \frac{1}{\frac{\theta}{\sin \theta}} = \frac{1}{1} = \boxed{1}$$

3. Using $\lim_{\theta \rightarrow 0} \frac{\sin(\theta)}{\theta} = 1$, evaluate the following limits using trigonometric identities.

a. $\lim_{\theta \rightarrow 0} \frac{\tan(\theta)}{\theta}$

$$\frac{\sin \theta}{\theta \cos \theta} = 1 \cdot \frac{1}{\cos \theta} = 1$$

b. $\lim_{\theta \rightarrow 0} \left(\frac{1 - \cos(\theta)}{\theta} \right) \tan \frac{\theta}{2} = \frac{1 - \cos \theta}{\sin \theta} = \frac{1 - \cos \theta}{\theta} \cdot \frac{\sin \theta}{\sin \theta}$

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\sin \theta} = \lim_{\theta \rightarrow 0} \tan \frac{\theta}{2} = -\tan 0 = \boxed{0}$$

c. $\lim_{\theta \rightarrow 0} \left(\frac{1 - \cos(\theta)}{\theta^2} \right)$

$$= \lim_{\theta \rightarrow 0} \frac{\tan \frac{\theta}{2}}{\theta} = \lim_{\theta \rightarrow 0} \frac{\tan \frac{\theta}{2}}{\sin \theta} = \frac{\tan \frac{\theta}{2}}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} = \frac{1}{2} \cdot \frac{\cos \frac{\theta}{2}}{\cos \frac{\theta}{2}} = \boxed{\frac{1}{2}}$$

d. $\lim_{\theta \rightarrow \frac{\pi}{2}^+} (\sec(\theta) - \tan(\theta))$

$$= \lim_{\theta \rightarrow \frac{\pi}{2}^+} \frac{1 - \sin \theta}{\cos \theta} \Rightarrow \frac{1 - \sin \theta}{\sqrt{1 - \sin^2 \theta}}$$

I'm not sure how to do this, and have spent a while now (1.5h)...

$$\frac{1 - \sin \theta}{\cos \theta} \cdot \frac{1 + \sin \theta}{1 + \sin \theta} = \frac{1 - \sin^2 \theta}{\cos \theta (1 + \sin \theta)} = \frac{\cos^2 \theta}{\cos \theta (1 + \sin \theta)} = \frac{0}{1 + 1} = 0$$

e. $\lim_{\theta \rightarrow 0} (\csc(\theta) - \cot(\theta))$

f. $\lim_{\theta \rightarrow 0} \left(\frac{\sin(2\theta)}{\theta} \right)$

Calculus
2019-2020
Handout #4: Squeeze Theorem and Trigonometric Limits

g. $\lim_{\theta \rightarrow 0} \left(\frac{\sec(\theta) - 1}{\theta} \right)$

h. $\lim_{\theta \rightarrow 0} \left(\frac{\sin(3\theta)}{\sin(2\theta)} \right)$