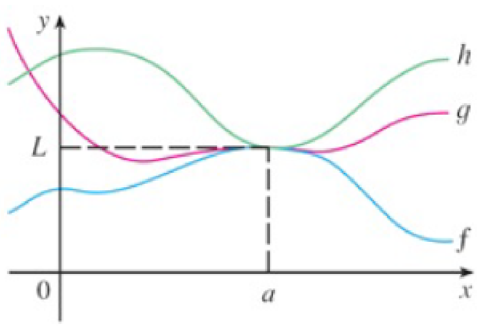
Calculus
2019-2020
Handout #4: Squeeze Theorem and Trigonometric Limits

Squeeze Theorem

3 The Squeeze Theorem If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then $$\lim_{x \to a} g(x) = L$$



1. Evaluate the following limits
   a. $\lim_{x\to\infty} \frac{\sin(x)}{x}$

$-1 \leq \sin(x) \leq 1$

$\frac{-1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$

$\lim_{x\to\infty} \frac{-1}{x} = \lim_{x\to\infty} \frac{1}{x} = 0$

$\therefore \lim_{x\to\infty} \frac{\sin x}{x} = \boxed{0}$

   b. $\lim_{x\to\infty} \frac{1-\cos(x)}{x}$

$1 \geq -\cos x \geq -1$

$-1 \leq \cos x \leq 1$

$\lim_{x\to\infty} 0 = 0$

$2 \geq 1-\cos x \geq 0$

$\lim_{x\to\infty} \frac{2}{x} = 0$

$\therefore \lim_{x\to\infty} \frac{1-\cos x}{x} = \boxed{0}$

??

$\frac{0}{x} \leq \frac{1-\cos x}{x} \leq \frac{2}{x}$

why can't you just divide by x? it doesn't work… 1-cosx)/x is not greater than 0

!! ask in class!

c. $\lim_{x\to\infty} x^2\cos\left(\frac{1}{x^2}\right)$

$-1 \le \cos\frac{1}{x^2} \le 1$

$-x^2 \le x^2\cos\frac{1}{x^2} \le x^2$

$\lim_{x\to\infty} -x^2 = -\infty$

$\lim_{x\to\infty} x^2 = \infty$

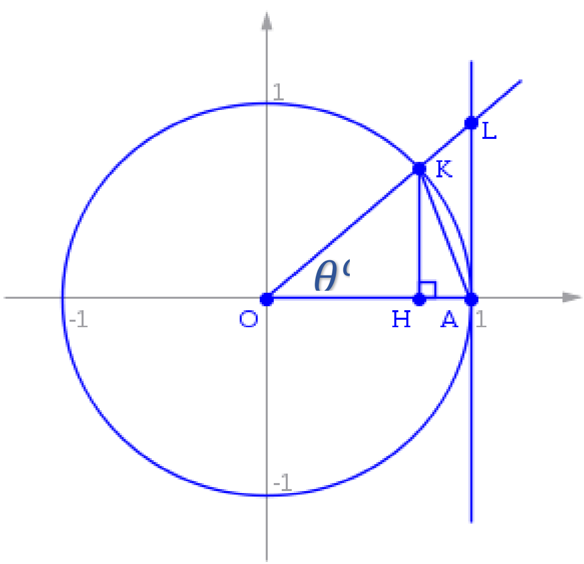
if it was $\lim_{x\to 0}$ then

$\lim_{x\to 0} x^2 = 0 = \lim_{x\to 0} -x^2$

$\therefore \lim_{x\to 0} x^2\cos\frac{1}{x^2} = \boxed{0}$

$\therefore$ squeeze thm is inconclusive

2. Prove $\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta} = 1$ using steps below.



The figure to the left is a sketch of a unit circle, where the angle θ is in radians. K is a point on the unit circle.

a. Please write the coordinates of point K in terms of θ.

$\cos\theta, \sin\theta$

b. What is the slope of line OK in terms of θ.

$\frac{\sin\theta}{\cos\theta} = \tan\theta$

c. Find the equation of line OL in terms of θ.
.

$y = x\tan\theta$

d. Find the coordinates of point A

$1, 0$

e. Find the coordinates of point L in terms of θ.

$1, \tan\theta$

f. Find the area of $\triangle$ OAK in terms of θ.

$\frac{\sin\theta}{2}$

g. Find the area of sector $\widehat{OAK}$ in terms of θ.

$\frac{\theta}{2\cancel{\pi}}(\cancel{\pi} r^2) = \frac{\theta}{2}r^2 = \frac{\theta}{2}$

h. Find the area of $\triangle$ OAL in terms of θ.

$\frac{\tan\theta}{2}$

i. Using the figure, we see *Area of* $\triangle$ OAK $<$ *Area of sector OAK* $<$ *Area of* $\triangle$ OAL. Write an inequality for your case in terms of θ.

$\frac{\sin\theta}{\cancel{2}} \le \frac{\theta}{\cancel{2}} \le \frac{\tan\theta}{\cancel{2}}$ $\quad \frac{\sin\theta}{\sin\theta} \le \frac{\theta}{\sin\theta} \le \frac{\tan\theta}{\sin\theta} = \frac{1}{\cos\theta}$

j. Find $\lim_{\theta\to 0}(\textit{Area of sector OAK})$.

$\lim_{\theta\to 0}(\textit{Area of } \triangle \text{OAK}) < \lim_{\theta\to 0}(\textit{Area of sector OAK}) < \lim_{\theta\to 0}(\textit{Area of } \triangle \text{OAL})$

$\lim_{\theta\to 0}\frac{1}{\cos\theta} = 1 \quad \therefore \quad 1 \le \frac{\theta}{\sin\theta} \le 1 \quad \therefore \quad \frac{\theta}{\sin\theta} = 1 \quad \therefore \quad \frac{1}{\frac{\theta}{\sin\theta}} = \frac{1}{1} = \boxed{1}$

3. Using $\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta} = 1$, evaluate the following limits using trigonometric identities.

a. $\lim_{\theta\to 0}\frac{\tan(\theta)}{\theta}$

$\frac{\sin\theta}{\theta\cos\theta} = 1\cos\theta = 1$

b. $\lim_{\theta\to 0}\left(\frac{1-\cos(\theta)}{\theta}\right)$ $\quad \tan\frac{x}{2} = \frac{1-\cos\theta}{\sin\theta} = \frac{1-\cos\theta}{\theta}\,\frac{\sin\theta}{\sin\theta}$

$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\lim_{\theta\to 0}\frac{1-\cos\theta}{\sin\theta} = 1\lim_{\theta\to 0}\tan\frac{\theta}{2} = \tan 0 = \boxed{0}$

c. $\lim_{\theta\to 0}\left(\frac{1-\cos(\theta)}{\theta^2}\right)$

$= \lim_{\theta\to 0}\frac{\tan\frac{\theta}{2}}{\theta} = \lim_{\theta\to 0}\frac{\tan\frac{\theta}{2}}{\sin\theta} = \frac{\overset{\cos}{\cancel{\tan}}\frac{\theta}{2}}{2\cancel{\sin\frac{\theta}{2}}\cos\frac{\theta}{2}} = \frac{1}{2}\;\frac{\cos\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \boxed{\frac{1}{2}}$

d. $\lim_{\theta\to\frac{\pi}{2}^+}(\sec(\theta) - \tan(\theta))$

$= \lim_{\theta\to\frac{\pi}{2}^+}\frac{1-\sin\theta}{\cos\theta} \Rightarrow \frac{1-\sin\theta}{\sqrt{1-\sin^2\theta}}$

I'm not sure how to do this, and have spent a while now (1.5h)…

e. $\lim_{\theta\to 0}(\csc(\theta) - \cot(\theta))$

f. $\lim_{\theta\to 0}\left(\frac{\sin(2\theta)}{\theta}\right)$

g. $\lim_{\theta \to 0} \left(\frac{\sec(\theta) - 1}{\theta} \right)$

h. $\lim_{\theta \to 0} \left(\frac{\sin(3\theta)}{\sin(2\theta)} \right)$