Calculus 2020-2021

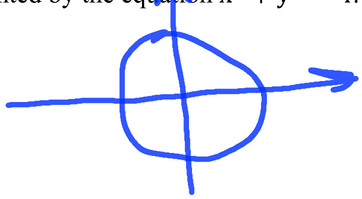
Handout #13: Worksheet on implicit differentiation, derivative of inverse functions and derivative of inverse trigonometric functions.

**Steps for Implicit Differentiation**

**1.** Differentiate both sides of the equation with respect to x (y is a function of x, so use chain rule).
**2.** Collect all $\frac{dy}{dx}$ terms on the left side of the equation and move all other terms to the right side.
**3.** Factor $\frac{dy}{dx}$ out of the left side of the equation if there is more than one $\frac{dy}{dx}$ term.
**4.** Solve for $\frac{dy}{dx}$ (It is okay to have both x's and y's in your answer).

**Note:** To find $\frac{dy}{dx}$ at a given point, take the derivative of both sides then immediately plug in the point and simplify.

1. Graph a circle represented by the equation $x^2 + y^2 = 4$.



a. Convert the implicit equation $x^2 + y^2 = 4$ to explicit function, $y = f(x)$. Find the derivative of the explicit function.

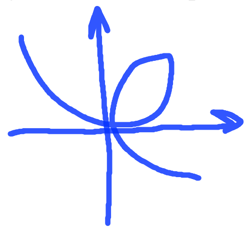
b. Find the derivative of the implicit equation $x^2 + y^2 = 4$ using implicit differentiation.

c. In which quadrants is the derivative positive and in which quadrants is the derivative negative?

d. Find the equation of the tangent lines at (2,0) and (0,-2).

e. Find the equation of the tangent line at $(\sqrt{3}, 1)$.

2. Given the curve, $x^3 + y^3 = 6xy$. Sketch this implicit curve on paper.



a. Find the derivative of the implicit equation $x^3 + y^3 = 6xy$

$$y' = \frac{x^2 - 2y}{2x - y^2}$$

b. Find the equation of the tangent and normal line to the curve $x^3 + y^3 = 6xy$ at $\left(\frac{4}{3}, \frac{8}{3}\right)$

$$y = \frac{4}{5}\left(x - \frac{4}{3}\right) + \frac{8}{3}$$

$y = \frac{-5}{4}$ 〃

c. Find the location on the curve where it has a vertical tangent.

$(0,0)$, $\left(\frac{16^{2/3}}{2}, \sqrt[3]{16}\right)$

$2x - y^2 = 0$, $x^3 + y^3 = 6xy$

d. Find the location on the curve where it has a horizontal tangent.

$x^2 - 2y = 0$, $x^3 + y^3 = 6xy$

$(0, 0)$ $\left(\frac{16^{2/3}}{2}, \sqrt[3]{16}\right)$

3. Please solve #6, 8, 10, 15, 18. Please ensure that you use product rule and chain rule when appropriate.

**5–20** Find $dy/dx$ by implicit differentiation.

**5.** $x^3 + y^3 = 1$

**6.** $2\sqrt{x} + \sqrt{y} = 3$

**7.** $x^2 + xy - y^2 = 4$

**8.** $2x^3 + x^2y - xy^3 = 2$

**9.** $x^4(x + y) = y^2(3x - y)$

**10.** $xe^y = x - y$

**11.** $y \cos x = x^2 + y^2$

**12.** $\cos(xy) = 1 + \sin y$

**13.** $4 \cos x \sin y = 1$

**14.** $e^y \sin x = x + xy$

**15.** $e^{x/y} = x - y$

**16.** $\sqrt{x + y} = 1 + x^2y^2$

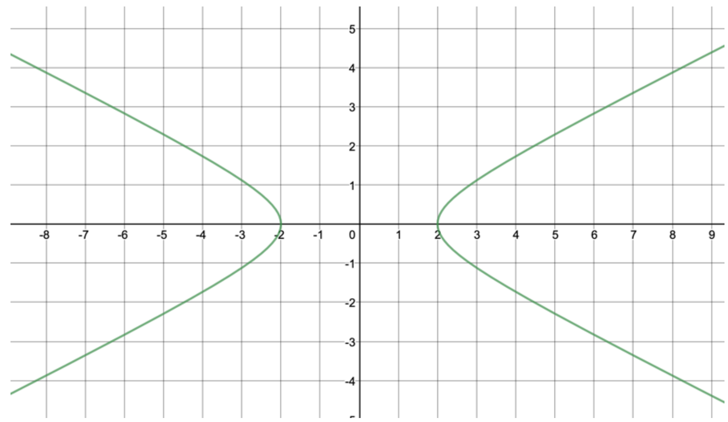
**17.** $\tan^{-1}(x^2y) = x + xy^2$

**18.** $x \sin y + y \sin x = 1$

**19.** $e^y \cos x = 1 + \sin(xy)$

**20.** $\tan(x - y) = \dfrac{y}{1 + x^2}$

4. Below is the graph of $\frac{x^2}{4} - y^2 = 1$.



$$\frac{2x}{4} - 2yy' = 0$$

$$\frac{x}{2} - 2yy' = 0$$

$$-2yy' = \frac{-x}{2}$$

a. Find $\frac{dy}{dx}$ using implicit differentiation

$$y' = \frac{x}{4y}$$

b. Graphically, where does this graph have vertical tangents? Does this graph have horizontal tangents? Explain.

vertical at $x = 2$, $x = -2$

no horiz. bc ~~$\lim_{x \to \infty}$, $\lim_{x \to -\infty} = \infty$ or $-\infty$~~

this is for asymtotes.. see derivative = 0 for tangents.

c. In which quadrants is this curve increasing? In which quadrants is this curve decreasing? Justify your answers.

increasing = I, III bc deriv positive $\left(\frac{\text{Positive}}{\text{Positive}}\right)$ or $\left(\frac{\text{negative}}{\text{negative}}\right)$

others are decreasing

d. In which quadrants is this curve concave down? In which quadrants is this curve concave up?

q1, q2 is down, q3, q4 is up

e. Prove that $\frac{d^2y}{dx^2} = \frac{-4}{16y^3}$. Explain how this equation supports your noticings about the concavity of the curve.

$$y'' = \frac{y - xy'}{4y^2}$$

$$= \frac{y - \frac{x^2}{4y}}{4y^2}$$

... (see scratch)

2nd deriv will be negative when y is positive

thus concave down when above x axis

$$\frac{x^2}{4} - y^2 = 1 \qquad y' = \frac{x}{4y}$$

$$y'' = \frac{y - xy'}{4y^2} = \frac{y - x^2/4y}{4y^2} = \frac{\frac{4y^2 - x^2}{4y}}{4y^2}$$

$$= \frac{4y^2 - x^2}{16y^3} = \frac{y^2 - \frac{x^2}{4}}{4y^3}$$

$$= \frac{y^2 - \frac{x^2}{4}}{4y^3} = \frac{-1}{4y^3}$$

$$= \frac{-\left(\frac{x^2}{4} - y^2\right)}{4y^3}$$

$$= \frac{-4}{16y^3}$$

that was a lot of simplification

5. Consider the curve in the xy-plane given by $x^2 - \frac{y^2}{5} = 1$. It is known that $\frac{dy}{dx} = \frac{5x}{y}$ and $\frac{d^2y}{dx^2} = \frac{-25}{y^3}$.
   a. Find the critical points of this curve. Justify your answer.

   b. Is the curve increasing or decreasing in Quadrant IV? Justify your answer.

   c. Is the curve concave up or concave down in Quadrant IV? Justify your answer.

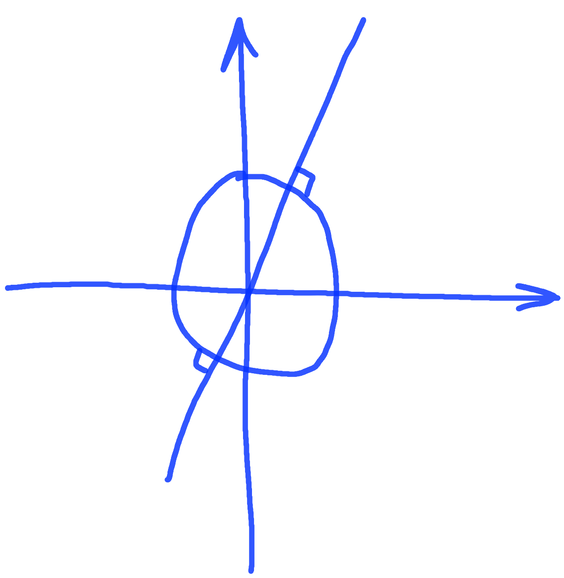
6. Two curves are orthogonal if their tangent lines are perpendicular at each point of intersection. Show that the given families of curves are orthogonal trajectories of each other, that is, every curve in one family is orthogonal to every curve in the other family. Sketch both families of curves on the same axes.

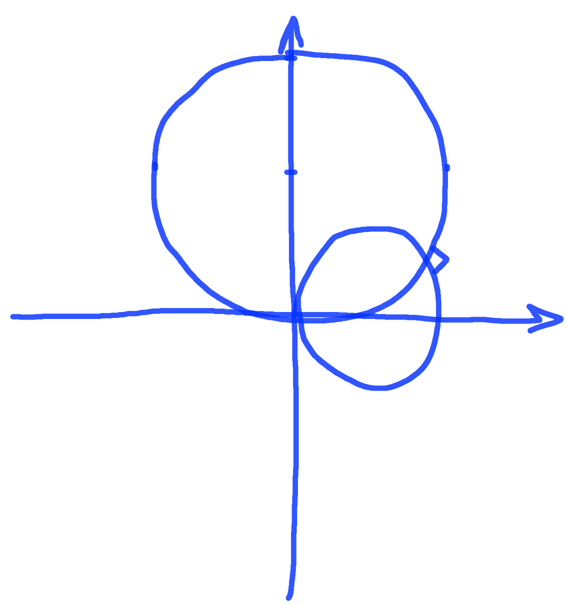
**59.** $x^2 + y^2 = r^2, \quad ax + by = 0$

**60.** $x^2 + y^2 = ax, \quad x^2 + y^2 = by$

confused

nvm, see other pdf.





7. Please define of the following concepts and then describe them with the help of diagrams if necessary.
   a. What is a function? Is $f(x) = x^2$ a function?

   b. What is a one to one (1:1) function? Is $f(x) = x^2$ a $1:1$ function?

   c. What is an inverse function? Give an example. Let us call g(x) as the inverse function of $f(x) = x^2$?

   d. How are the graphs of the inverse function and function related?

   e. What are some of the properties of inverse functions?

   f. Fill in the table with the three ordered pairs from your graph of f(x) and g(x). What relationships do you see between the ordered pairs?

| x | f(x) | f'(x) |
|---|---|---|
| 1 | 1 | 2 |
| 4 | 16 | 8 |
| 9 | 81 | 18 |
| 16 | 256 | 32 |

$\frac{1}{2\sqrt{x}}$

| x | g(x) | g'(x) |
|---|---|---|
| 1 | 1 | 1/2 |
| 4 | 2 | 1/4 |
| 9 | 3 | 1/6 |
| 16 | 4 | 1/8 |

   g. Compare the tables of $f'(x)$ and g'(x). What relationships do you see between the slopes?

reciprocal

also $\frac{d}{dx} f^{-}(x) = \frac{1}{f'(f^{-}(x))}$

h. If an inverse function is represented by $f^{-1}(x)$. A property of inverse functions is $f(f^{-1}(x)) = x$. Using this property of inverse function and chain rule to prove that the derivative of inverse function, $(f^{-1}(x))'$ is given by $(f^{-1}(x))' = \frac{1}{f'f^{-1}(x)}$.

8. Using the result $(f^{-1}(x))' = \frac{1}{f'f^{-1}(x)}$, find the derivative of $y = \ln(x)$

9. For $f(x) = 3x + 6$, find $(f^{-1})'(x)$.

$\boxed{10.}$ Let $f(x) = x^3 + x$. If $g(x) = f^{-1}(x)$ and $g(2) = 1$, what is the value of $g'(2)$?

$$f^{-1}(x) = \frac{1}{f'(f^{-}(x))}$$

$$f'(x) = 3x^2 + 1$$

$$g'(2) = \frac{1}{3(1)^2+1} = \frac{1}{4}$$

11. The table below gives selected values for a differentiable and decreasing function f and its derivative. If $f^{-1}(x)$ is the inverse function of f, what is the value of $(f^{-1}(2))'$?

| $x$ | $f(x)$ | $f'(x)$ |
|---|---|---|
| 0 | 49 | 0 |
| 1 | 2 | –8 |
| 2 | –1 | –80 |

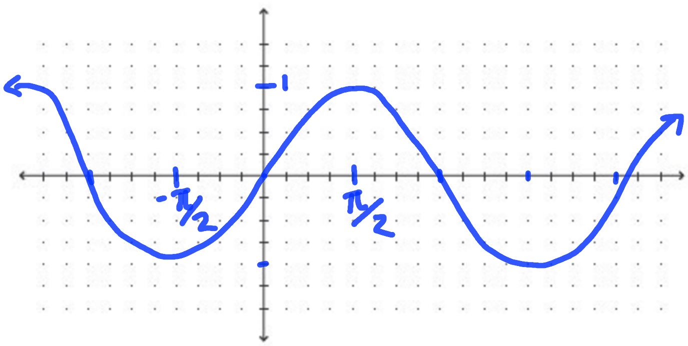
$= \frac{d}{dx} f^{-}(2)$ ? if so, then

$\frac{1}{f'(1)} = \frac{1}{-8} = \frac{-1}{8}$

12. Suppose that g is the inverse function of $f(x) = 3x^5 + 6x^3 + 4$. Find $g'(13)$.

13. Find the equation of the tangent line to the inverse of $f(x) = x^5 + 2x^3 + x - 4$ at the point $(-4,0)$.

14. Graph $y = \sin(x)$




a. Domain: $\mathbb{R}$

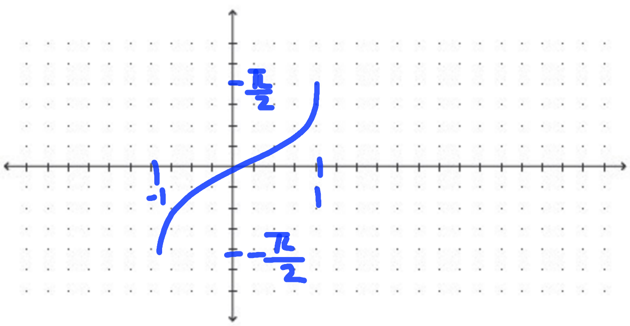
b. Range: $[-1, 1]$

In order to find the inverse function, you want $y = \sin(x)$ to be 1:1.

c. Restricted domain of 1:1 sin(x): $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

d. Range of this function 1:1 sin(x): $[-1, 1]$

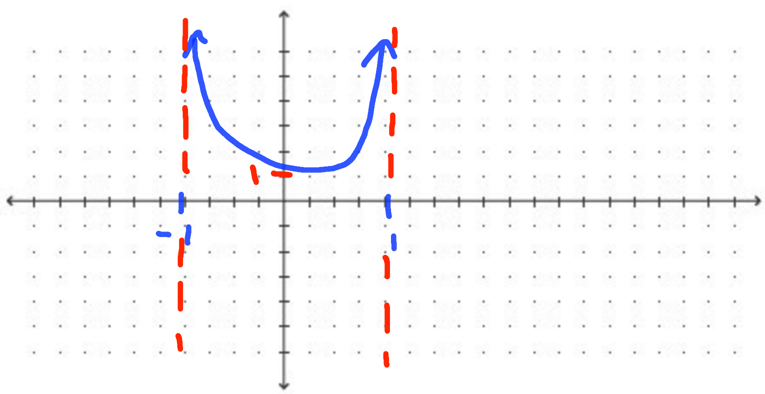
e. Graph $y = \sin^{-1}(x)$.




f. Domain of $y = \sin^{-1}(x)$ $[-1, 1]$

g. Range of $y = \sin^{-1}(x)$ $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

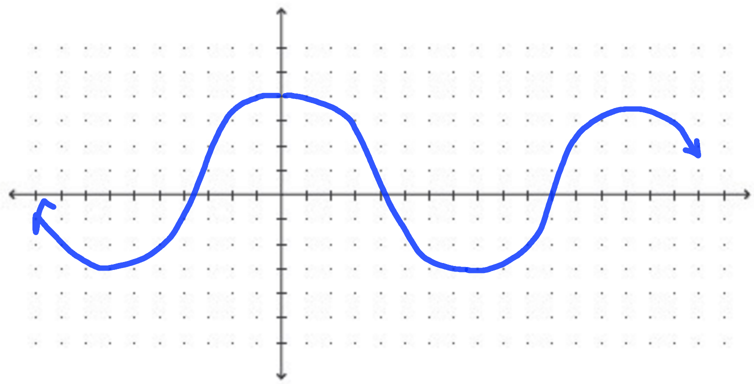
h. Graph of the derivative of $y = \sin^{-1}(x)$ using your graph of $y = \sin^{-1}(x)$ .




i. Use implicit differentiation to find the derivative of $y = \sin^{-1}(x)$.

$\sin^{-1}(x) = \dfrac{1}{\cos(\sin^{-}(x))}$ but how to simplify?

15. Graph $y = \cos(x)$



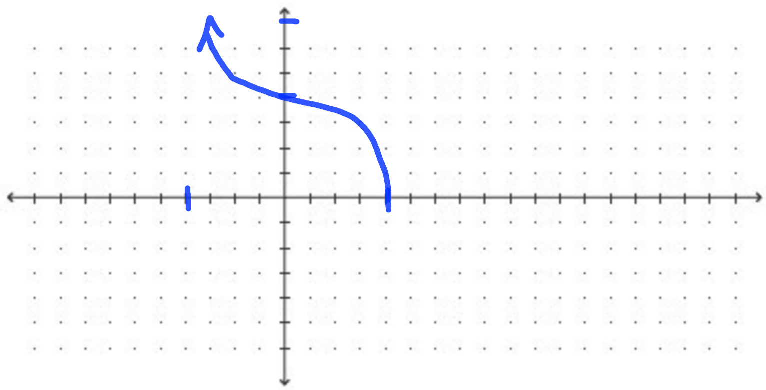
a. Domain: $\mathbb{R}$
b. Range: $[-1, 1]$

In order to find the inverse function, you want $y = \cos(x)$ to be 1:1.

c. Restricted domain of 1:1 cos(x): $0, \pi$

d. Range of this function 1:1 cos(x): $-1, 1$

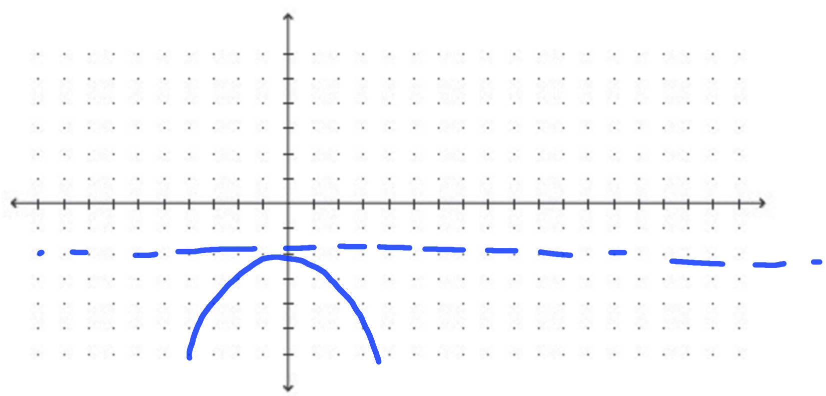
e. Graph $y = \cos^{-1}(x)$.



f. Domain of $y = \cos^{-1}(x)$

g. Range of $y = \cos^{-1}(x)$

h. Graph of the derivative of $y = \cos^{-1}(x)$ using your graph of $y = \cos^{-1}(x)$ .



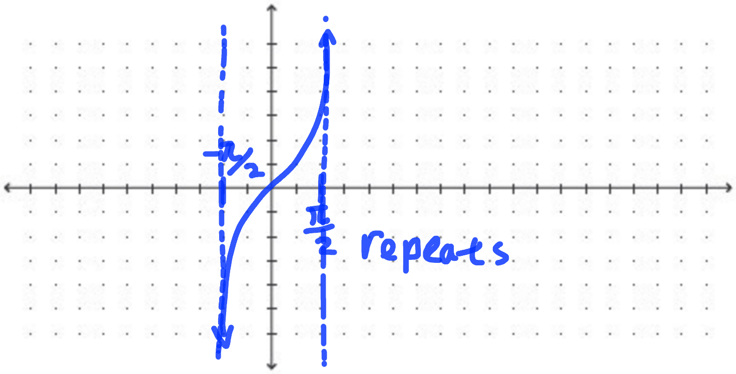
$< 0$

i. Use implicit differentiation to find the derivative of $y = \cos^{-1}(x)$.

~~Same strategy as above except -sin = cos, cancels with taking the - from ±~~

$y' = \frac{-1}{\sqrt{1-x^2}}$ take the negative bc look at the graph!

16. Graph $y = \tan(x)$




a. Domain: $\mathbb{R}$
b. Range: $\mathbb{R}$

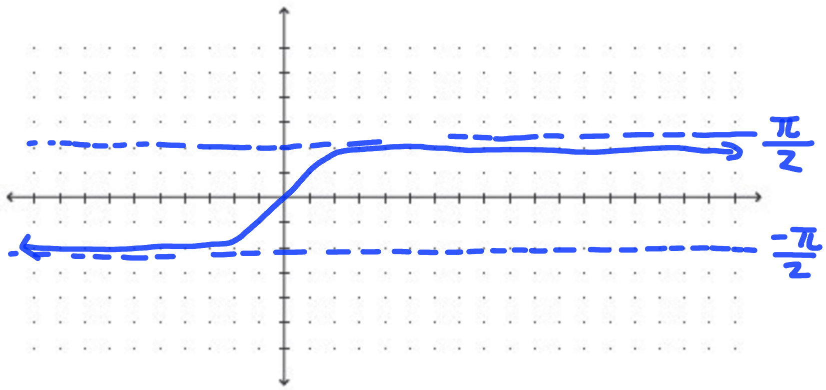
In order to find the inverse function, you want $y = \tan(x)$ to be 1:1.

c. Restricted domain of 1:1 $y = \tan(x)$: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

d. Range of this function 1:1 $y = \tan(x)$:

$\mathbb{R}$

e. Graph $y = \tan^{-1}(x)$.



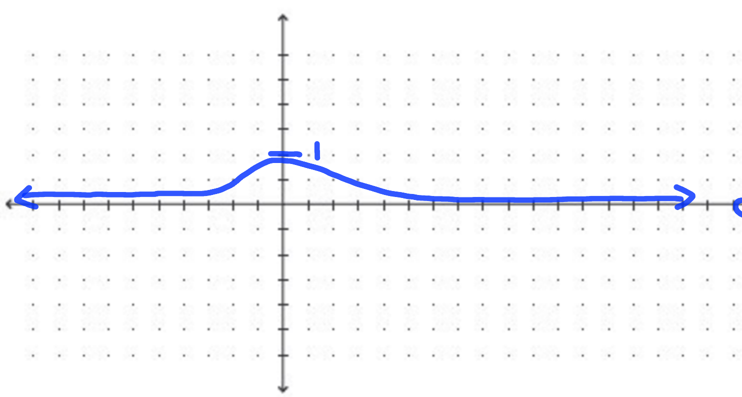

f. Domain of $y = \tan^{-1}(x)$

$\mathbb{R}$

g. Range of $y = \tan^{-1}(x)$

$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

h. Graph of the derivative of $y = \tan^{-1}(x)$ using your graph of $y = \tan^{-1}(x)$.



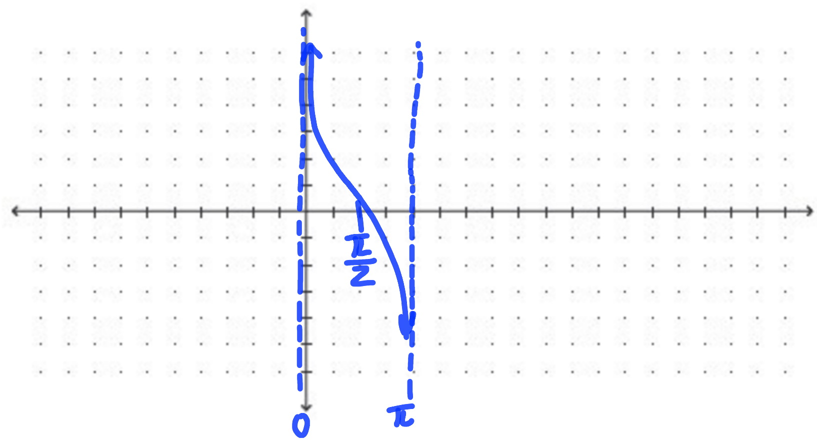

asymtote at $y=0$

i. Use implicit differentiation to find the derivative of $y = \tan^{-1}(x)$.

See external notes

17. Also, graph $y = \cot(x)$ and $y = \cot^{-1}(x)$. Also find the derivative of $y = \cot^{-1}(x)$. Also sketch the graph of the derivative of $y = \cot^{-1}(x)$ using the graph of the function, $y = \cot^{-1}(x)$.

$y = \cot(x)$  $\cot x = \tan\left(-x + \frac{\pi}{2}\right)$




a. Domain: $\mathbb{R}$

b. Range: $\mathbb{R}$

In order to find the inverse function, you want $y = \cot(x)$ to be 1:1.

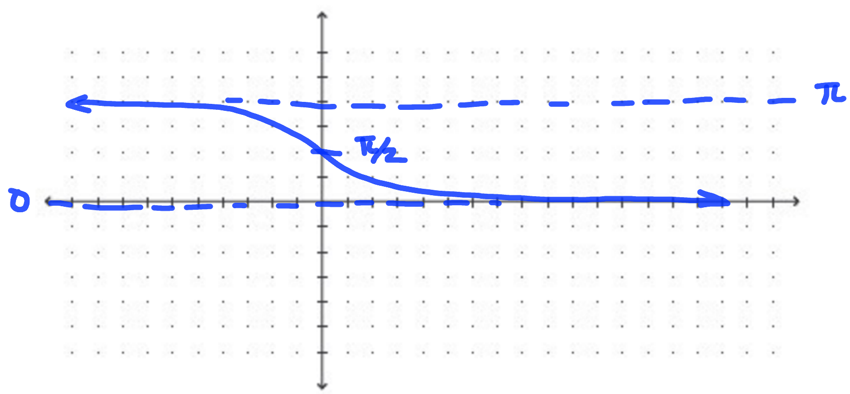
c. Restricted domain of 1:1 $y = \cot(x)$

$(0, \pi)$

d. Range of this function 1:1 $y = \cot(x)$

$\mathbb{R}$

e. Graph $y = \cot^{-1}(x)$.



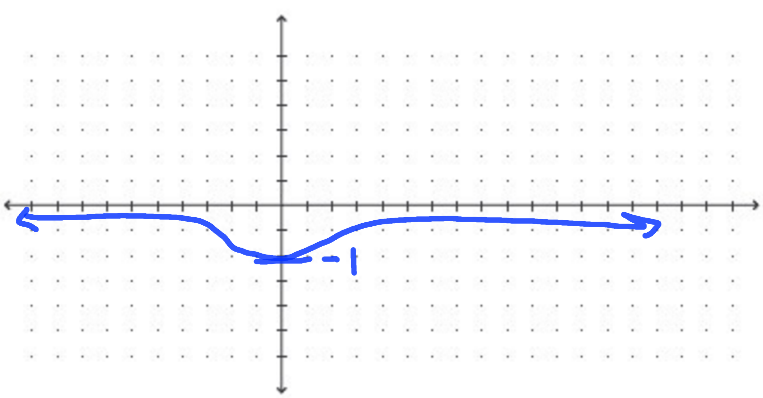

f. Domain of $y = \cot^{-1}(x)$.

$\mathbb{R}$

g. Range of $y = \cot^{-1}(x)$.

$(0, \pi)$

h. Graph of the derivative of $y = \cot^{-1}(x)$ using your graph of $y = \cot^{-1}(x)$




i. Use implicit differentiation to find the derivative of $y = \cot^{-1}(x)$.

tis flipped $\frac{-1}{x^2+1}$

18. Also, graph $y = \sec(x)$ and $y = \sec^{-1}(x)$. Also find the derivative of $y = \sec^{-1}(x)$. Also sketch the graph of the derivative of $y = \sec^{-1}(x)$ using the graph of the function, $y = \sec^{-1}(x)$.

19. Also, graph $y = \csc(x)$ and $y = \csc^{-1}(x)$. Also find the derivative of $y = \csc^{-1}(x)$. Also sketch the graph of the derivative of $y = \csc^{-1}(x)$ using the graph of the function, $y = \csc^{-1}(x)$.

Do you see any connections?

20. How are the graphs of $y = \cos^{-1}(x)$ and $y = \sin^{-1}(x)$ related? Could you have found the derivative of $y = \cos^{-1}(x)$ in a more efficient way using your knowledge of the derivative using the graph of the function, $y = \sin^{-1}(x)$.

21. How are the graphs of $y = \cot^{-1}(x)$ and $y = \tan^{-1}(x)$ related? Could you have found the derivative of $y = \cot^{-1}(x)$ in a more efficient way using your knowledge of the derivative using the graph of the function, $y = \tan^{-1}(x)$.

22. How are the graphs of $y = \csc^{-1}(x)$ and $y = \sec^{-1}(x)$ related? Could you have found the derivative of $y = \csc^{-1}(x)$ in a more efficient way using your knowledge of the derivative using the graph of the function, $y = \sec^{-1}(x)$.

Please make sure you enter the derivatives of all the 6 inverse trigonometric functions into your table of derivatives. Thank you!

23. Give an exact or approximate value of the expression in radians or degrees.
    a. $\cos^{-1}(\frac{\sqrt{2}}{2})$ \pi/4
    b. $\arcsin(-1)$ 3 pi / 2
    c. $\cos^{-1}(1)$ 0
    d. $\csc^{-1}(\sqrt{2})$ pi/4
    e. $\operatorname{arcsec}(2)$ pi/3

24. Let $y = \sin^{-1}(3x)$. Find $y'$.

3/(sqrt(1-9x^2)) (by plugging previous formula and chain rule)

25. For $y = \tan^{-1}(4x^2)$, find $\frac{dy}{dx}$.

$$\frac{8x}{1+16x^4}$$

26. If $\arcsin x = \ln y$, find $\frac{dy}{dx}$.

$$y' = \frac{y}{\sqrt{1-x^2}}$$

27. Find the equation of the line tangent to $y = \arcsin x$ at $x = \frac{\sqrt{3}}{2}$.

$$y = 2\left(x - \frac{\sqrt{3}}{2}\right) + \frac{\pi}{3}$$

28. For $y = \arccos(x^7)$, find $y'$.