J. Math. Anal. Appl. 398 (2013) 278–295

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Bifurcation analysis of modified Leslie–Gower predator–prey model with Michaelis–Menten type prey harvesting

R.P. Gupta *, Peeyush Chandra

Department of Mathematics and Statistics, Indian Institute of Technology Kanpur, Kanpur-208016, India

ARTICLE INFO

Article history:
Received 1 June 2012
Available online 4 September 2012
Submitted by Junping Shi

Keywords:
Prey–predator model
Permanence
Stability
Bifurcation

ABSTRACT

In the present paper we discuss bifurcation analysis of a modified Leslie–Gower prey–predator model in the presence of nonlinear harvesting in prey. We give a detailed mathematical analysis of the model to describe some significant results that may arise from the interaction of biological resources. The model displays a complex dynamics in the prey–predator plane. The permanence, stability and bifurcation (saddle–node bifurcation, transcritical, Hopf–Andronov and Bogdanov–Takens) of this model are discussed. We have analyzed the effect of prey harvesting and growth rate of predator on the proposed model by considering them as bifurcation parameters as they are important from the ecological point of view. The local existence and stability of the limit cycle emerging through Hopf bifurcation is given. The emergence of homoclinic loops has been shown through simulation when the limit cycle arising though Hopf bifurcation collides with a saddle point. This work reflects that the feasible upper bound of the rate of harvesting for the coexistence of the species can be guaranteed. Numerical simulations using MATLAB are carried out to demonstrate the results obtained.

1. Introduction

Deterministic nonlinear mathematical models (ODE models) are widely used to understand the dynamics of interacting populations. They usually display similar dynamical behaviors such as coexistence of an equilibrium, a limit cycle or a chaotic attractor. The growing need for more food and resources has led to an increased exploitation of several biological resources. On the other hand there is a global concern to protect the ecosystem at large. In the face of these two contrasting scenarios, we look for a sustainable development policy in almost every sphere of human activity. This has necessitated a scientific management of commercial exploitation of the biological resources like fisheries and forestries [1,2]. It may be noted that species compete, evolve and disperse simply for the purpose of seeking resources to sustain their existence. Depending on specific settings of applications, the interacting populations can take the forms of resource–consumer, plant–herbivore, parasite–host, etc.

Predator–prey models are the building blocks of the ecosystems as biomasses are grown out of their resource masses. The simplest predator–prey dynamic model is the Lotka–Volterra model [3], which has been modified in many ways since its original formulation in the 1920s. In particular, Rosenzweig and MacArthur [4] improved the realism of the Lotka–Volterra model by introducing the density dependent growth of the prey population and a nonlinear saturating uptake of prey by the predator (functional response). Nowadays models are largely based on the Rosenzweig–MacArthur framework but are typically amended by emphasizing specific factors, such as inducible defenses in the prey [5] or adaptive foraging by the predator [6]. In population dynamics, the functional response of predator to prey density refers to the change in the density of prey attacked per unit time per predator as the prey density changes. Depending upon type of species and the environment

* Corresponding author.
E-mail addresses: rravi@iitk.ac.in (R.P. Gupta), peeyush@iitk.ac.in (P. Chandra).

0022-247X/$ – see front matter © 2012 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2012.08.057

where they are living, the predation terms are of different forms where the amount of food consumed by predator is a function of prey density only (e.g. Holling types I, II, and III) [7]. In most prey–predator models considered in the literature, the predator functional response to prey density is assumed to be monotonic increasing, the inherent assumption being that as there is more prey in the environment, it is better for the predator [8,9].

Dai and Tang [10] considered a predator–prey model in which two ecologically interacting species are harvested independently with constant rates. They proved that the harvested predator–prey system exhibits very complicated dynamics such as a spontaneous appearance of a homoclinic orbit and multiple limit cycles. An exhaustive study for bifurcation and pattern for a modified Lotka–Volterra model has been carried out by McGehee et al. [11]. Das et al. [12] studied a prey–predator model where both species grow logistically and are harvested with a nonlinear harvesting. Xiao and Ruan [13] discussed the Bogdanov–Takens bifurcation in detail for a prey–predator model with Holling-type II functional response and constant rate of harvesting in predator. The Beddington–DeAngelis type prey–predator model has been studied by Haque [14] for permanence and bifurcation. Leard et al. [15] and Lenzini and Rebaza [16] studied a ratio-dependent prey–predator model by considering a nonconstant harvesting in prey and predator respectively in which they reported several bifurcations and connecting orbits. A prey–predator system with Holling-type II functional response and constant rate of harvesting has been studied in detail by Peng et al. [17]. A detailed bifurcation analysis of a ratio-dependent prey–predator model with the Allee effect has been discussed by Sen et al. [18] in different parametric regions.

Li and Xiao [19] proposed a Leslie–Gower prey–predator model with Holling-type III functional response for its bifurcation analysis. Lin and Ho [20] studied the local and global stability for a predator–prey model of modified Leslie–Gower and Holling-type II with time-delay. Zhang et al. [21] proposed a Leslie–Gower prey–predator model with proportional harvesting in both prey and predator to study the persistence and global stability. The global stability of the unique interior equilibrium of the system was shown by defining a suitable Lyapunov function, which means that suitable harvesting has no influence on the persistent property of the harvesting system. Zhu and Lan [22] considered a Leslie–Gower model with constant harvesting in prey where they studied phase portraits near the interior equilibria. They also proved that the predator free equilibria can be saddle–nodes, saddles or unstable nodes depending on the choices of the parameters involved while the interior positive equilibria in the first quadrant are saddles, stable or unstable nodes, foci, centers, saddle–nodes or cusps. The dynamics of the Leslie–Gower model subjected to the Allee effect with proportionate harvesting has been studied by Mena-Lorca et al. [23]. Song and Li [24] proposed and analyzed the periodic prey–predator model with a modified Leslie–Gower Holling-type II scheme and impulsive effect for its dynamical behavior.

The aim of this paper is to give a detailed analysis of a two-dimensional autonomous differential equations model for a predator–prey system with nonlinear harvesting in prey. This model incorporates a modified version of the Leslie–Gower prey–predator model with Holling-type II functional response in the presence of nonlinear harvesting in prey.

2. Mathematical model

2.1. Basic model

Aziz-Alaoui and Daher Okiye [25] had proposed the following two-dimensional system of autonomous differential equation model for a prey–predator system which incorporates a modified version of Leslie–Gower and Holling-type II functional response:

$$\begin{cases} \dfrac{dx_1}{d\tau} = rx_1\left(1-\dfrac{x_1}{K}\right) - \dfrac{a_1x_1x_2}{n_1+x_1}, \\ \dfrac{dx_2}{d\tau} = sx_2\left(1-\dfrac{a_2x_2}{n_2+x_1}\right), \end{cases} \tag{1}$$

subjected to positive initial conditions $x_1(0) > 0, x_2(0) > 0$. Here, $x_1(t)$ and $x_2(t)$ are the prey and predator population densities respectively. r and K are intrinsic growth rate and environmental carrying capacity for the prey species respectively. a_1 is the maximum value of the per capita reduction rate of prey, $n_i, i = 1, 2$ measures the extent to which the environment provides protection to prey and predator respectively, s measures the growth rate of the predator species and sa_2 is the maximum value of the per capita reduction rate of predator.

This model has already been studied by several researchers. In particular, the boundedness of solutions and global stability of the positive equilibrium of this model has been investigated by Aziz-Alaoui and Daher Okiye [25]. Sufficient criteria for the permanence of systems and globally asymptotic stability of solutions were discussed by Du et al. [26]. Some sufficient conditions for the existence and global attractivity of positive periodic solutions of this model have been given by Zhu and Wang [27]. The long time behavior and persistent condition has been established for this model with stochastic perturbation by Ji et al. [28,29] under the assumption that the extent to which the environment provides protection to both the predator and prey is the same (i.e. $n_1 = n_2$).

2.2. Model with prey harvesting

Biological resources in the prey–predator system are most likely to be harvested and sold with the purpose of achieving the economic interest which motivates the introduction of harvesting in the prey–predator model. Amongst the several types of harvesting Michaelis–Menten type harvesting is more realistic from biological and economic points of view. For the details of this kind of harvesting one can see the Refs. [30,12,31]. The model (1) with nonlinear harvesting (Michaelis–Menten type) under the assumption that the extent to which the environment provides protection to both the predator and prey is the same [28,29] (i.e. $n_1 = n_2 = n$) is given by

$$\begin{cases} \dfrac{dx_1}{d\tau} = rx_1\left(1 - \dfrac{x_1}{K}\right) - \dfrac{a_1x_1x_2}{n + x_1} - \dfrac{qEx_1}{m_1E + m_2x_1}, \\ \dfrac{dx_2}{d\tau} = sx_2\left(1 - \dfrac{a_2x_2}{n + x_1}\right), \end{cases} \tag{2}$$

where q is the catchability coefficient, E is the effort applied to harvest the prey species, and m_1, and m_2 are suitable constants. The rest of the parameters have similar meanings as for the model (1). All the parameters are assumed to be positive due to biological considerations.

To investigate the dynamics of system (2), we shall consider the following non-dimensional scheme:

$$\begin{aligned} &x_1 = Kx, \qquad a_1x_2 = Ky, \qquad r\tau = t, \\ &\alpha = \frac{1}{r}, \qquad \beta = \frac{a_2}{a_1}, \qquad m = \frac{n}{K}, \qquad h = \frac{qE}{rm_2K}, \qquad c = \frac{m_1E}{m_2K}, \qquad \rho = \frac{s}{r}. \end{aligned} \tag{3}$$

Using the above non-dimensional scheme we obtain the following system of differential equations:

$$\begin{cases} \dfrac{dx}{dt} = x\left(1 - x - \dfrac{\alpha y}{m + x} - \dfrac{h}{c + x}\right) \equiv xf^{(1)}(x, y), \\ \dfrac{dy}{dt} = \rho y\left(1 - \dfrac{\beta y}{m + x}\right) \equiv yf^{(2)}(x, y) \end{cases} \tag{4}$$

with the initial conditions

$$x(0) = x_0 > 0, \qquad y(0) = y_0 > 0. \tag{5}$$

Here

$$f^{(1)}(x, y) = 1 - x - \frac{\alpha y}{m + x} - \frac{h}{c + x} \quad \text{and} \quad f^{(2)}(x, y) = \rho\left(1 - \frac{\beta y}{m + x}\right) \tag{6}$$

and α, β, m, h, c and ρ are all positive. System (4) is defined on the set:

$$\mathbb{R}_0^+ \times \mathbb{R}_0^+ = \{(x, y) \in \mathbb{R}^2 | x \geq 0, y \geq 0\}. \tag{7}$$

In the following we prove the positivity and boundedness of solutions as well as the permanence of system (4). Note that to prove the boundedness and permanence of the system (4), we use the following results [32]:

Lemma 1 (*Chen*)**.** *If* $a, b > 0$ *and* $\frac{dX}{dt} \leq (\geq) X(t)(a - bX(t))$ *with* $X(0) > 0$*, then*

$$\limsup_{t\to+\infty} X(t) \leq \frac{a}{b}\left(\liminf_{t\to+\infty} X(t) \geq \frac{a}{b}\right).$$

In fact, the above lemma is quantitatively equivalent to the following lemma.

Lemma 2. *If* $a, b > 0$ *and* $\frac{dX}{dt} \leq X(t)(a - bX(t))$ *with* $X(0) > 0$*, then for all* $t \geq 0$

$$X(t) \leq \frac{a}{b - Ce^{-at}}, \quad \text{with } C = b - \frac{a}{X(0)}.$$

In particular $X(t) \leq \max\left\{X(0), \frac{a}{b}\right\}$ *for all* $t \geq 0$.

2.2.1. Positivity and boundedness of solutions

Proposition 1. *(a) All solutions* $(x(t), y(t))$ *of the system* (4) *with the initial conditions* (5) *are positive, i.e.,* $x(t) > 0$, $y(t) > 0$, *for all* $t \geq 0$.

(b) All solutions $(x(t), y(t))$ *of the system* (4) *with the initial conditions* (5) *are bounded, for all* $t \geq 0$.

Proof. (a) From the prey equation of system (4) it follows that $x = 0$ is an invariant set. This implies that $x(t) > 0$, for all t if $x(0) > 0$. A similar argument, using the predator equation of the system (4), shows that $y = 0$ is also an invariant set, so $y(t) > 0$, for all t if $y(0) > 0$. Thus, any trajectory starting in R_+^2 cannot cross the coordinate axes. Hence the theorem follows.

(b) Using the positivity of variables x, y, from (4), we can write

$$\frac{dx}{dt} = x\left(1 - x - \frac{\alpha y}{m+x} - \frac{h}{c+x}\right) \le x(1-x). \tag{8}$$

From Lemma 2, we have

$$x(t) \le \max\{x(0), 1\} \equiv M_1 \quad \text{for all } t \ge 0.$$

Further, from (4) we have

$$\frac{dy}{dt} = \rho y\left(1 - \frac{\beta y}{m+x}\right) \le \rho y\left(1 - \frac{\beta y}{m+M_1}\right). \tag{9}$$

Again from the same Lemma 2 we have

$$y(t) \le \max\left\{y(0), \frac{m+M_1}{\beta}\right\} \equiv M_2 \quad \text{for all } t \ge 0.$$

This completes the proof of the boundedness of solutions and hence the system under consideration is dissipative. □

2.2.2. Permanence

We recall here the definition of permanence [33]:

Definition 1. System (4) is said to be permanent if there exist positive constants ζ_1 and ζ_2 $(0 < \zeta_1 < \zeta_2)$ such that each positive solution $(x(t, x_0, y_0),\ y(t, x_0, y_0))$ of system (4) with initial condition $(x_0, y_0) \in Int(R_+^2)$ satisfies,

$$\min\left\{\liminf_{t\to+\infty} x(t, x_0, y_0),\ \liminf_{t\to+\infty} y(t, x_0, y_0)\right\} \ge \zeta_1$$

$$\max\left\{\limsup_{t\to+\infty} x(t, x_0, y_0),\ \limsup_{t\to+\infty} y(t, x_0, y_0)\right\} \le \zeta_2.$$

Proposition 2. *The system (4) with the initial conditions (5) is permanent if* $\frac{\alpha(m+1)}{\beta m} + \frac{h}{c} < 1$.

Proof. From Eq. (8) and Lemma 1 it is clear that $0 < x(t) < 1$ for sufficiently large t. Also from Eq. (9) and Lemma 1 we get $y(t) \le \frac{m+1}{\beta}$ for sufficiently large t.

Hence, from the prey equation of system (4), we can write

$$\frac{dx}{dt} = x\left(1 - x - \frac{\alpha y}{m+x} - \frac{h}{c+x}\right) \ge x\left(1 - x - \frac{\alpha(m+1)}{\beta m} - \frac{h}{c}\right) = x(\omega_1 - x), \quad \text{for sufficiently large } t,$$

where $\omega_1 = 1 - \frac{\alpha(m+1)}{\beta m} - \frac{h}{c}$.

If $\omega_1 > 0$ $\left(i.e., \frac{\alpha(m+1)}{\beta m} + \frac{h}{c} < 1\right)$ then from Lemma 1, we have

$$\liminf_{t\to+\infty} x(t) \ge \omega_1.$$

Now using positivity of x, from the predator equation of system (4), we can write

$$\frac{dy}{dt} \ge \rho y\left(1 - \frac{\beta y}{m}\right),$$

which on using Lemma 1 gives the following result

$$\liminf_{t\to+\infty} y(t) \ge \frac{m}{\beta} \equiv \omega_2.$$

Also from inequalities (8) and (9), together with Lemma 1, we have

$$\limsup_{t\to\infty} x(t) \le 1 \quad \text{and} \quad \limsup_{t\to\infty} y(t) \le \frac{m+M_1}{\beta}.$$

Now choosing $\zeta_1 = \min(\omega_1, \omega_2)$ and $\zeta_2 = \max\left(1, \frac{m+M_1}{\beta}\right)$, we get the permanence of the system (4). □

3. Existence and stability of trivial equilibria

In order to find the equilibrium points of the system (4), we consider the zero growth isoclines of the system, which are given by:

$$xf^{(1)}(x,y)=0 \quad \text{and} \quad yf^{(2)}(x,y)=0. \tag{10}$$

The equilibrium points are now the points of intersection of these zero growth isoclines. The trivial equilibrium points of the system (4) are the following:

(i) The origin $E_0=(0,0)$.

(ii) The predator free equilibrium points at $E_L=(x_L,0)$ and $E_H=(x_H,0)$ where x_L and x_H are the roots of the quadratic equation

$$x^2-(1-c)x+h-c=0 \tag{11}$$

$$i.e.\ x_L=\frac{1}{2}\left(1-c-\sqrt{(1-c)^2-4(h-c)}\right) \quad \text{and} \quad x_H=\frac{1}{2}\left(1-c+\sqrt{(1-c)^2-4(h-c)}\right).$$

If $h>c$ then $E_L=(x_L,0)$ and $E_H=(x_H,0)$ both the equilibrium points exist whenever $c<1$ and $(1-c)^2>4(h-c)$ while if $h<c$ then $E_H=(x_H,0)$ only exists.

(iii) The prey extinction equilibrium point is $E_1=(0,\frac{m}{\beta})$.

Theorem 1. (i) *The origin $E_0=(0,0)$ is a saddle point if $h>c$ and unstable if $h<c$.*
(ii) *The axial equilibrium point $E_L=(x_L,0)$ is always unstable.*
(iii) *The equilibrium point $E_H=(x_H,0)$ is always a saddle point.*
(iv) *The axial equilibrium point $E_1=(0,\frac{m}{\beta})$ is stable if $\frac{\alpha}{\beta}+\frac{h}{c}>1$ and a saddle point if $\frac{\alpha}{\beta}+\frac{h}{c}<1$.*
(v) *System (4) undergoes a transcritical bifurcation around $E_1=(0,\frac{m}{\beta})$ if $\frac{\alpha}{\beta}+\frac{h}{c}=1$.*

Proof. (i) The Jacobian matrix of system (4) evaluated at the equilibrium point $E_0=(0,0)$ is given by

$$P=\begin{pmatrix} -\dfrac{h-c}{c} & 0 \\ 0 & \rho \end{pmatrix}.$$

The eigenvalues of P are $\lambda_1=-\frac{h-c}{c}$ and $\lambda_2=\rho>0$. Therefore the result follows.

(ii) The Jacobian matrix of system (4) evaluated at the point $(x,0)$ is given by

$$Q=\begin{pmatrix} x\left(-1+\dfrac{h}{(c+x)^2}\right) & -\dfrac{\alpha x}{m+x} \\ 0 & \rho \end{pmatrix}.$$

The eigenvalues of Q at $E_L=(x_L,0)$ are $\lambda_1=x_L\sqrt{(1-c)^2-4(h-c)}>0$ and $\lambda_2=\rho>0$. Therefore $E_L=(x_L,0)$ is always unstable.

(iii) The eigenvalues of Q at $E_H=(x_H,0)$ are $\lambda_1=-x_H\sqrt{(1-c)^2-4(h-c)}<0$ and $\lambda_2=\rho>0$. Therefore $E_L=(x_L,0)$ is always a saddle point.

(iv) The Jacobian matrix of system (4) evaluated at the equilibrium point $E_1=(0,\frac{m}{\beta})$ is given by

$$R=\begin{pmatrix} 1-\dfrac{\alpha}{\beta}-\dfrac{h}{c} & 0 \\ \dfrac{\rho}{\beta} & -\rho \end{pmatrix}.$$

The eigenvalues of R are $\lambda_1=1-\frac{\alpha}{\beta}-\frac{h}{c}$ and $\lambda_2=-\rho<0$. Therefore the result follows.

(v) Proof of this result is given in Section 4.2.1 (Theorem 6). $\square$

4. Existence, stability and bifurcation of interior equilibria

The interior equilibria are $E_{1*}=(x_{1*},y_{1*})$ and $E_{2*}=(x_{2*},y_{2*})$ where x_{1*} and x_{2*} are the roots of the quadratic equation

$$x^2+\left(\frac{\alpha}{\beta}+c-1\right)x+c\left(\frac{\alpha}{\beta}+\frac{h}{c}-1\right)=0. \tag{12}$$

$$i.e.\ x_{1*}=\frac{1}{2}\left(1-c-\frac{\alpha}{\beta}-\sqrt{\left(\frac{\alpha}{\beta}+c-1\right)^2-4c\left(\frac{\alpha}{\beta}+\frac{h}{c}-1\right)}\right) \tag{13}$$

and

$$x_{2*} = \frac{1}{2}\left(1 - c - \frac{\alpha}{\beta} + \sqrt{\left(\frac{\alpha}{\beta} + c - 1\right)^2 - 4c\left(\frac{\alpha}{\beta} + \frac{h}{c} - 1\right)}\right) \tag{14}$$

together with

$$y_{1*} = \frac{m + x_{1*}}{\beta} \quad \text{and} \quad y_{2*} = \frac{m + x_{2*}}{\beta}.$$

Since the number of equilibrium points depend upon the quantity $\frac{\alpha}{\beta} + \frac{h}{c} - 1$, we therefore consider the following case:

4.1. *Case I:* $\frac{\alpha}{\beta} + \frac{h}{c} > 1$.

(1) The two distinct interior equilibrium points $E_{1*} = (x_{1*}, y_{1*})$ and $E_{2*} = (x_{2*}, y_{2*})$ exist whenever $\frac{\alpha}{\beta} + c < 1$ and $\left(\frac{\alpha}{\beta} + c - 1\right)^2 > 4c\left(\frac{\alpha}{\beta} + \frac{h}{c} - 1\right)$ (see Fig. 1 (a)). Note that none of the two interior equilibrium points $E_{1*} = (x_{1*}, y_{1*})$ and $E_{2*} = (x_{2*}, y_{2*})$ exists for $\frac{\alpha}{\beta} + c > 1$.

(2) If $\frac{\alpha}{\beta} + c < 1$ and $\left(\frac{\alpha}{\beta} + c - 1\right)^2 = 4c\left(\frac{\alpha}{\beta} + \frac{h}{c} - 1\right)$, then the two interior equilibrium points $E_{1*} = (x_{1*}, y_{1*})$ and $E_{2*} = (x_{2*}, y_{2*})$ collide with each other and are denoted by the instantaneous equilibrium (saddle–node equilibrium) $\bar{E} = (\bar{x}, \bar{y})$ where $\bar{x} = \frac{1}{2}\left(1 - c - \frac{\alpha}{\beta}\right)$ (see Fig. 1(b)).

(3) If $\left(\frac{\alpha}{\beta} + c - 1\right)^2 < 4c\left(\frac{\alpha}{\beta} + \frac{h}{c} - 1\right)$, then no interior equilibrium point exists (see Fig. 1(c)).

4.1.1. *Stability and Hopf bifurcation of interior equilibria*

Theorem 2. (i) *The equilibrium point $E_{1*} = (x_{1*}, y_{1*})$ is always a saddle point.*

(ii) *The equilibrium point $E_{2*} = (x_{2*}, y_{2*})$ is stable if $x_{2*}\left(\frac{\alpha}{\beta(m+x_{2*})} + \frac{h}{(c+x_{2*})^2} - 1\right) < \rho$.*

(iii) *The system undergoes a Hopf-bifurcation with respect to bifurcation parameter ρ around the equilibrium point $E_{2*} = (x_{2*}, y_{2*})$ if $x_{2*}\left(\frac{\alpha}{\beta(m+x_{2*})} + \frac{h}{(c+x_{2*})^2} - 1\right) = \rho$.*

Proof. The Jacobian matrix of system (4) evaluated at the point (x, y) is given by

$$S = \begin{pmatrix} x\left(-1 + \dfrac{\alpha y}{(m+x)^2} + \dfrac{h}{(c+x)^2}\right) & -\dfrac{\alpha x}{m+x} \\ \dfrac{\rho\beta y^2}{(m+x)^2} & -\dfrac{\rho\beta y}{m+x} \end{pmatrix}.$$

(i) The *det* of matrix S evaluated at $E_{1*} = (x_{1*}, y_{1*})$ is

$$det\ S|_{E_{1*}} = -\frac{\rho\beta x_{1*} y_{1*}}{m + x_{1*}}\left(-1 + \frac{h}{(c + x_{1*})^2}\right).$$

Now using the value of x_{1*} from Eq. (13) we get

$$det\ S|_{E_{1*}} = -\frac{\rho x_{1*} y_{1*}}{(m + x_{1*})(c + x_{1*})}\sqrt{\left(\frac{\alpha}{\beta} + c - 1\right)^2 - 4c\left(\frac{\alpha}{\beta} + \frac{h}{c} - 1\right)} < 0.$$

Therefore the eigenvalues of the matrix S evaluated at $E_{1*} = (x_{1*}, y_{1*})$ have the real parts with the opposite signs. Hence the equilibrium point $E_{1*} = (x_{1*}, y_{1*})$ is always a saddle point.

(ii) Similarly the *det* of matrix S evaluated at $E_{2*} = (x_{2*}, y_{2*})$ is

$$det\ S|_{E_{2*}} = \frac{\rho x_{2*} y_{2*}}{(m + x_{2*})(c + x_{2*})}\sqrt{\left(\frac{\alpha}{\beta} + c - 1\right)^2 - 4c\left(\frac{\alpha}{\beta} + \frac{h}{c} - 1\right)} > 0.$$

Also *trace* of matrix S evaluated at $E_{2*} = (x_{2*}, y_{2*})$ is given by

$$tr\ S|_{E_{2*}} = x_{2*}\left(-1 + \frac{\alpha}{\beta(m + x_{2*})} + \frac{h}{(c + x_{2*})^2}\right) - \rho.$$

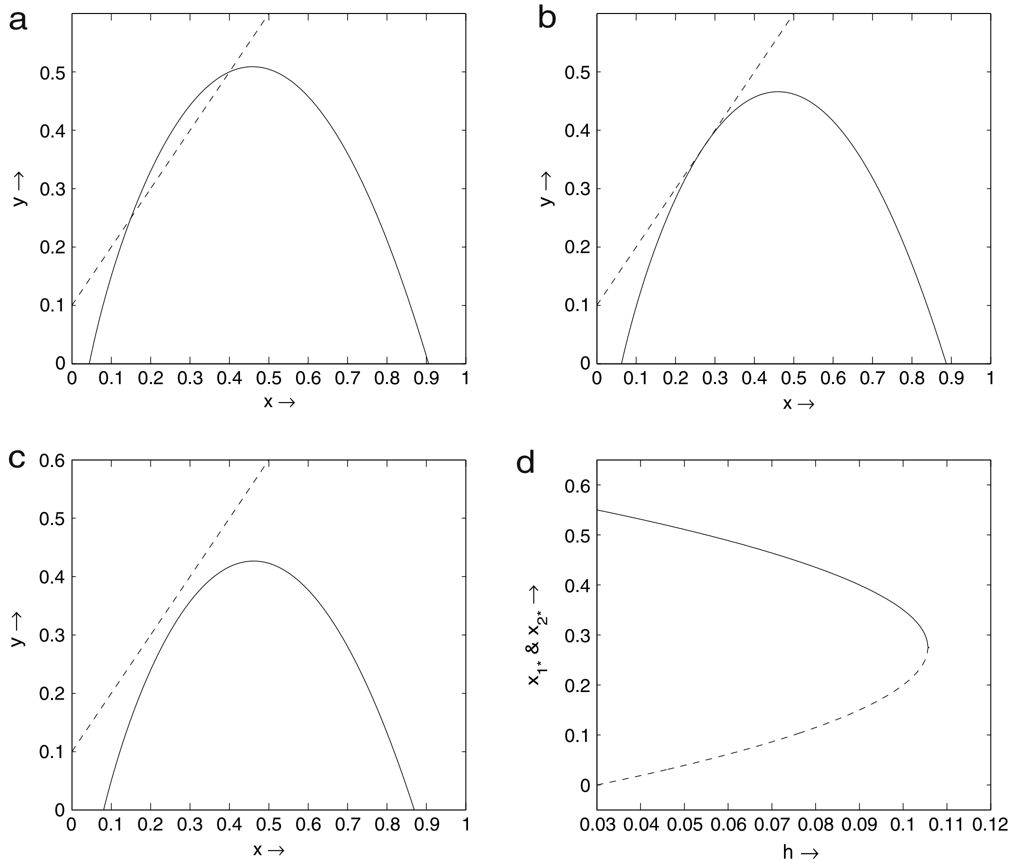

Fig. 1. In (a) to (c) the solid parabola is prey nullcline and dashed line is predator nullcline. This diagram shows how the number of interior equilibrium points changes with h while keeping other parameters fixed. For $\alpha = 0.1$, $\beta = 1$, $c = 0.05$ and (a) $h = 0.09$, there are two interior equilibria, (b) $h = h^{[sn]} = 0.105625$, the two interior equilibrium collide with each other, (c) $h = 0.12$, no interior equilibrium exists. (d) The solid curve stands for the stable equilibrium and the dotted curve stands for unstable equilibrium.

From the Routh–Hurwitz criterion the result follows.

(iii) We know that if the $tr\ S|_{E_{2*}} = 0$, then both the eigenvalues will be purely imaginary provided det $S|_{E_{2*}} > 0$. Therefore, from the implicit function theorem a Hopf bifurcation occurs where a periodic orbit is created as the stability of the equilibrium point $E_{2*}(x_{2*}, y_{2*})$ changes. The critical value for the Hopf bifurcation parameter is $\rho^{[hf]} = x_{2*}\left(\frac{\alpha}{\beta(m+x_{2*})} + \frac{h}{(c+x_{2*})^2} - 1\right)$. Further from part (ii) it is clear that under the given condition

(i) $tr\ S|_{E_{2*}} = 0$, (ii) det $S|_{E_{2*}} > 0$ and (iii) $\frac{d}{d\rho} tr\ S|_{E_{2*}} = -1 \neq 0$ at $\rho = \rho^{[hf]}$.

This guarantees the existence of Hopf bifurcation around $E_{2*}(u_{2*}, u_{2*})$. The stability of the limit cycle is discussed in Section 4.1.2. □

4.1.2. Stability of limit cycle

In order to discuss the stability (direction) of the limit cycle we now compute the Lyapunov coefficient [34,35] σ at the point $E_{2*}(x_{2*}, y_{2*})$ of the system (4).

We first translate the equilibrium $E_{2*}(x_{2*}, y_{2*})$ of system (4) to the origin by using the transformation $x = \hat{x} - x_{2*}$ and $y = \hat{y} - y_{2*}$. Then, using equilibrium equation (10) the system (4) in a neighborhood of the origin can be written as

$$\begin{cases} \dfrac{d\hat{x}}{dt} = a_{10}\hat{x} + a_{01}\hat{y} + a_{20}\hat{x}^2 + a_{11}\hat{x}\hat{y} + a_{02}\hat{y}^2 + a_{30}\hat{x}^3 + a_{21}\hat{x}^2\hat{y} + a_{12}\hat{x}\hat{y}^2 + a_{03}\hat{y}^3 + F_1(\hat{x}, \hat{y}), \\ \dfrac{d\hat{y}}{dt} = b_{10}\hat{x} + b_{01}\hat{y} + b_{20}\hat{x}^2 + b_{11}\hat{x}\hat{y} + b_{02}\hat{y}^2 + b_{30}\hat{x}^3 + b_{21}\hat{x}^2\hat{y} + b_{12}\hat{x}\hat{y}^2 + b_{03}\hat{y}^3 + F_2(\hat{x}, \hat{y}), \end{cases} \tag{15}$$

where,

$$a_{10} = -x_{2*} + \frac{hx_{2*}}{(c + x_{2*})^2} + \frac{\alpha\, q x_{2*}}{(m + x_{2*})^2}, \qquad a_{01} = -\frac{\alpha\, x_{2*}}{m + x_{2*}},$$

$$a_{20} = -\frac{hx_{2*}}{(c+x_{2*})^3} + \frac{h}{(c+x_{2*})^2} - \frac{\alpha\, x_{2*}y_{2*}}{(m+x_{2*})^3} + \frac{\alpha\, y_{2*}}{(m+x_{2*})^2} - 1,$$

$$a_{11} = \frac{\alpha\, x_{2*}}{(m+x_{2*})^2} - \frac{\alpha}{m+x_{2*}}, \qquad a_{02}=0, \qquad a_{30} = \frac{hx_{2*}}{(c+x_{2*})^4} - \frac{h}{(c+x_{2*})^3} + \frac{\alpha\, x_{2*}y_{2*}}{(m+x_{2*})^4} - \frac{\alpha\, y_{2*}}{(m+x_{2*})^3},$$

$$a_{21} = -\frac{\alpha\, x_{2*}}{(m+x_{2*})^3} + \frac{\alpha}{(m+x_{2*})^2}, \qquad a_{12}=0, \qquad a_{03}=0,$$

$$b_{10} = \frac{\rho\beta\, {y_{2*}}^2}{(m+x_{2*})^2}, \qquad b_{01} = -\frac{\rho\beta\, y_{2*}}{m+x_{2*}}, \qquad b_{20} = -\frac{\rho\beta\, {y_{2*}}^2}{(m+x_{2*})^3}, \qquad b_{11} = 2\,\frac{\rho\beta\, y_{2*}}{(m+x_{2*})^2},$$

$$b_{02} = -\frac{\rho\,\beta}{m+x_{2*}}, \qquad b_{30} = \frac{\rho\beta {y_{2*}}^2}{(m+x_{2*})^4},$$

$$b_{21} = -2\,\frac{\rho\beta\, y_{2*}}{(m+x_{2*})^3}, \qquad b_{12} = \frac{\rho\,\beta}{(m+x_{2*})^2}, \qquad b_{03}=0$$

and $F_k(\hat{u},\hat{v})$ (for $k=1,2$) are power series in powers of $\hat{u}^i\hat{v}^j$ satisfying $i+j\geq 4$, i.e.

$F_1(\hat{x},\hat{y}) = \Sigma_{i+j=4}^{\infty} a_{ij}\hat{x}^i\hat{y}^j$ and $F_2(\hat{x},\hat{y}) = \Sigma_{i+j=4}^{\infty} b_{ij}\hat{x}^i\hat{y}^j$.

Hence the first Lyapunov coefficient σ for a planar system (as defined in [34]) is given by

$$\begin{aligned}\sigma = -\frac{3\pi}{2a_{02}\Delta^{3/2}}\{&[a_{10}b_{10}({a_{11}}^2 + a_{11}b_{02} + a_{02}b_{11}) + a_{10}a_{01}({b_{11}}^2 + a_{20}b_{11} + a_{11}b_{02})\\ &+ {b_{10}}^2(a_{11}a_{02} + 2a_{02}b_{02}) - 2a_{10}b_{10}({b_{02}}^2 - a_{20}a_{02}) - 2a_{10}a_{01}({a_{20}}^2 - b_{20}b_{02})\\ &- {a_{01}}^2(2a_{20}b_{20} + b_{11}b_{20}) + (a_{01}b_{10} - 2{a_{10}}^2)(b_{11}b_{02} - a_{11}a_{20})]\\ &- ({a_{10}}^2 + a_{01}b_{10})[3(b_{10}b_{03} - a_{01}a_{30}) + 2a_{10}(a_{21}+b_{12}) + (b_{10}a_{12} - a_{01}b_{21})]\},\end{aligned}$$

where $\Delta = \frac{\rho x_{2*}y_{2*}}{(m+x_{2*})(c+x_{2*})}\sqrt{\left(\frac{\alpha}{\beta}+c-1\right)^2 - 4c\left(\frac{\alpha}{\beta}+\frac{h}{c}-1\right)}$.

Since the expression for Lyapunov number σ is complex enough we cannot say anything about the sign of σ and therefore we have given the following numerical example.

4.1.3. Numerical example

For $\alpha = 0.40, \beta = 1.00, m = 0.10, h = 0.10, c = 0.05$ we obtain $\rho^{[hf]} = 0.1798611111$ and Lyapunov number $\sigma = 949.3370580\pi > 0$. This implies that an unstable limit cycle is created around $E_{2*} = (x_{2*}, y_{2*}) = (0.35, 0.45)$ while $E_{1*} = (x_{1*}, y_{1*}) = (0.20, 0.30)$ is a saddle point and $E_1 = (0, \frac{m}{\beta})$ an attractor. Also for $0.2 = \rho > \rho^{[hf]}$, equilibrium point $E_{2*} = (x_{2*}, y_{2*})$ is stable and for $0.15 = \rho < \rho^{[hf]}$ it is unstable. For $\rho = 0.204985$ the limit cycle collides with the saddle point E_{1*} and gives a homoclinic orbit. Note that for $\rho = 0.204985$ the Lyapunov number $\sigma = 684.5999226\pi > 0$. This implies that the limit cycle remains unstable. These results are shown in Fig. 2.

4.1.4. Saddle–node bifurcation

As mentioned earlier, the two interior equilibrium points $E_{1*} = (x_{1*}, y_{1*})$ and $E_{2*} = (x_{2*}, y_{2*})$ collide with each other and system (4) has the unique instantaneous interior equilibrium (saddle–node interior equilibrium) $\bar{E} = (\bar{x}, \bar{y})$ for $\frac{\alpha}{\beta} + c < 1$ and $\left(\frac{\alpha}{\beta}+c-1\right)^2 = 4c\left(\frac{\alpha}{\beta}+\frac{h}{c}-1\right)$. Also one of the eigenvalues of the Jacobian evaluated at the point $\bar{E} = (\bar{x}, \bar{y})$ is zero so the point $\bar{E} = (\bar{x}, \bar{y})$ becomes non-hyperbolic and its stability cannot be studied by the linearization technique. Thus there is a chance of bifurcation around the instantaneous interior equilibrium $\bar{E} = (\bar{x}, \bar{y})$. Keeping all parameters fixed and varying the harvesting parameter h we can see the coexisting equilibrium points $E_{1*} = (x_{1*}, y_{1*})$ and $E_{2*} = (x_{2*}, y_{2*})$ collide with each other through saddle–node bifurcation as h crosses the critical magnitude $h^{[sn]} = \frac{1}{4}\left(\frac{\alpha}{\beta}+c-1\right)^2 - c\left(\frac{\alpha}{\beta}-1\right)$ and then mutually annihilated. The parametric surface

$$S_1 = \left\{(\alpha,\beta,h,c)\in\mathbb{R}_+^4 : \frac{\alpha}{\beta}+c<1,\ \left(\frac{\alpha}{\beta}+c-1\right)^2 = 4c\left(\frac{\alpha}{\beta}+\frac{h}{c}-1\right)\right\}$$

is known as the saddle–node bifurcation surface.

Theorem 3. *System (4) undergoes a saddle–node bifurcation around $\bar{E}(\bar{x},\bar{y})$ with respect to bifurcation parameter h if $\frac{\alpha}{\beta}+c<1$, $\left(\frac{\alpha}{\beta}+c-1\right)^2 = 4c\left(\frac{\alpha}{\beta}+\frac{h}{c}-1\right)$ and $\bar{x}\left(\frac{\alpha}{\beta(m+\bar{x})}+\frac{h}{(c+\bar{x})^2}-1\right)<\rho$.*

Proof. To prove that the model (4) undergoes a saddle–node bifurcation, we use Sotomayor's theorem [36,34] by considering h as the bifurcation parameter. According to Sotomayor's theorem one of the eigenvalues of the Jacobian at

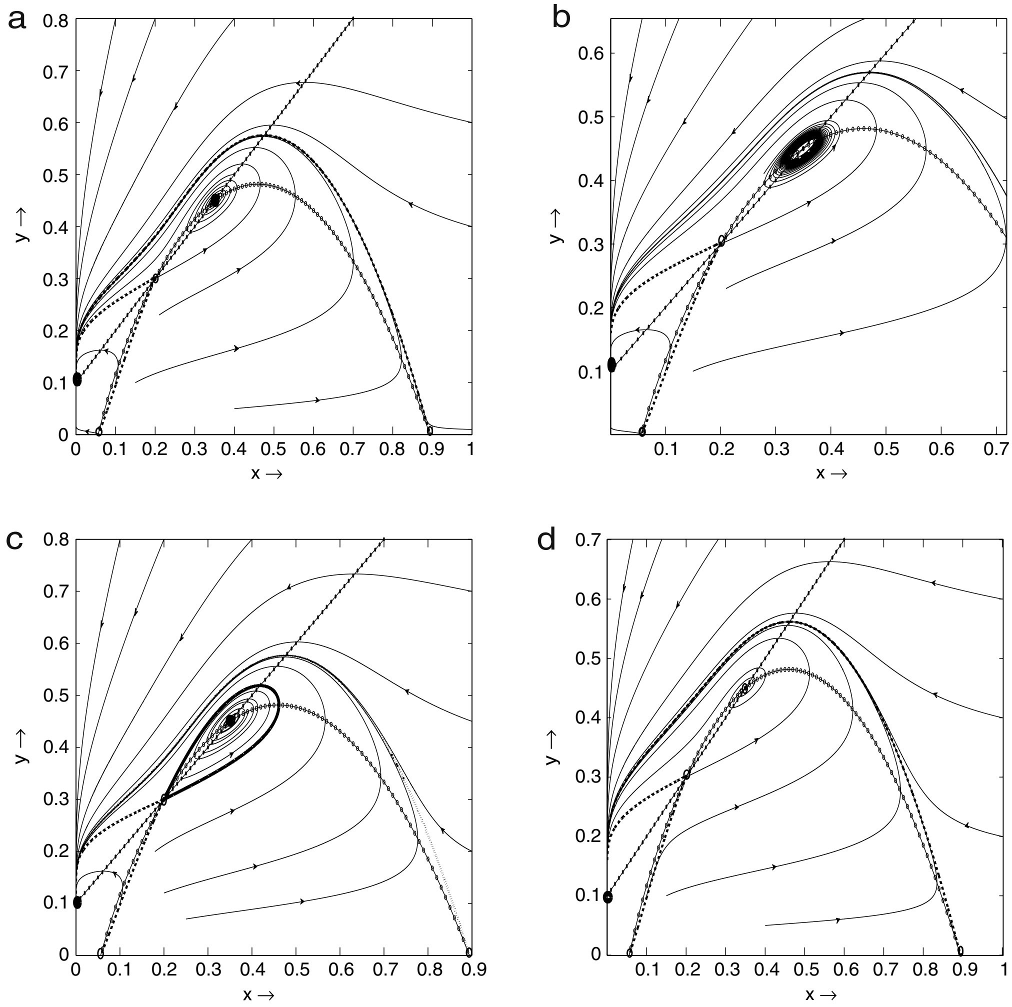

Fig. 2. In (a) to (d) the prey and predator nullcline are given by circle marked parabola and star marked line; stable and unstable manifolds of various equilibrium points are given by dashed and dotted curves. The interior equilibrium point $E_{1*} = (x_{1*}, y_{1*})$ is always a saddle point and the axial equilibrium $E_1 = \left(0, \frac{m}{\beta}\right)$ is always stable. (a) $E_{2*} = (x_{2*}, y_{2*})$ is locally asymptotically stable. (b) An unstable limit cycle bifurcates through Hopf-bifurcation around $E_{2*} = (x_{2*}, y_{2*})$. (c) This diagram shows that the limit cycle collides with the saddle point $E_{2*} = (x_{2*}, y_{2*})$ to give a homoclinic loop and (d) $E_{2*} = (x_{2*}, y_{2*})$ is unstable.

the saddle–node equilibrium point must be zero and the other eigenvalue must have negative real part, so we need to take the condition $\bar{x}\left(\frac{\alpha}{\beta(m+\bar{x})} + \frac{h}{(c+\bar{x})^2} - 1\right) < \rho$. Let $g = (g^{(1)}, g^{(2)})^T$ with $g^{(1)} \equiv xf^{(1)}$ and $g^{(2)} \equiv yf^{(2)}$ where $f^{(1)}$ and $f^{(2)}$ are already defined in Section 2. The Jacobian $\bar{J}$ at the equilibrium point $\bar{E}(\bar{u}, \bar{u})$ is given by

$$\bar{J} \equiv Dg(\bar{x}, \bar{y}) = \begin{pmatrix} \bar{x}\left(-1 + \dfrac{\alpha\bar{y}}{(m+\bar{x})^2} + \dfrac{h}{(c+\bar{x})^2}\right) & -\dfrac{\alpha\bar{x}}{m+\bar{x}} \\ \dfrac{\rho\beta\bar{y}^2}{(m+\bar{x})^2} & -\dfrac{\rho\beta\bar{y}}{m+\bar{x}} \end{pmatrix}.$$

Let $h^{[sn]}$ be the value of h such that the matrix $\bar{J}$ has a simple zero eigenvalue at $h = h^{[sn]}$. This demands $\det\bar{J} = 0$ at $h = h^{[sn]}$ giving $\frac{h}{(c+\bar{x})^2} = 1$.

Now, let $V = (v_1, v_2)^T$ and $W = (w_1, w_2)^T$ be the eigenvectors of $\bar{J}$ and $\bar{J}^T$ corresponding to zero eigenvalue respectively, a simple calculation yields $V = (\beta, 1)^T$ and $W = (1, -\frac{\alpha\bar{x}}{\rho(m+\bar{x})})^T$. Therefore, $\Omega_1 = W^T g_h(\bar{E}, h^{[sn]}) = -\frac{\bar{x}}{c+\bar{x}} < 0$ at $h = h^{[sn]}$, since

$$g_h(\bar{E}, h^{[sn]}) \equiv \frac{\partial g}{\partial h}(\bar{E}, h^{[sn]}) = \begin{pmatrix} -\dfrac{\bar{x}}{c+\bar{x}} \\ 0 \end{pmatrix} \quad \text{at } h = h^{[sn]}.$$

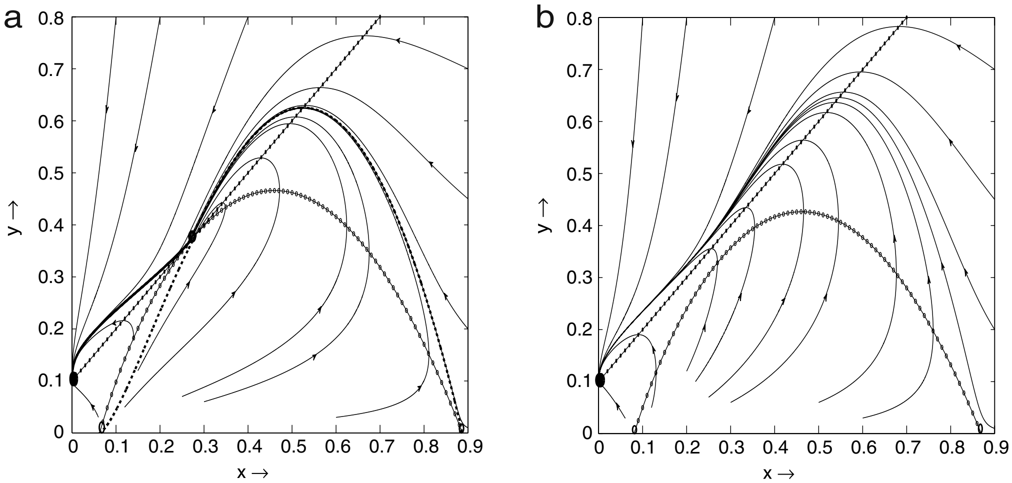

Fig. 3. Here circle marked parabola is prey nullcline, and star marked line is predator nullcline. (a) The two interior equilibrium collide with each other and a unique instantaneous equilibrium appears which is stable from one side and unstable from the other side and the axial equilibrium point $E_1 = \left(0, \frac{m}{\beta}\right)$ is locally asymptotically stable. The dashed curve is a separatrix which arises from stable and unstable manifolds of $\bar{E}(\bar{x}, \bar{y})$ and the dark solid curve is stable manifold of $E_1 = \left(0, \frac{m}{\beta}\right)$. (b) No interior equilibrium exists and all the solution trajectories approach the axial equilibrium point $E_1 = \left(0, \frac{m}{\beta}\right)$ which is locally asymptotically stable.

Now, $\Omega_2 = W^T[D^2 g(\bar{E}, h^{[sn]})(V, V)]$, where

$$D^2 g\left(X = (x, y)^T, h\right) = \begin{pmatrix} \nabla \dfrac{\partial g^{(1)}}{\partial x} & \nabla \dfrac{\partial g^{(1)}}{\partial y} \\ \nabla \dfrac{\partial g^{(2)}}{\partial x} & \nabla \dfrac{\partial g^{(2)}}{\partial y} \end{pmatrix},$$

$$\nabla \frac{\partial g^{(i)}}{\partial x} = \left(\frac{\partial^2 g^{(i)}}{\partial x^2}, \frac{\partial^2 g^{(i)}}{\partial x \partial y}\right)^T \quad \text{and} \quad \nabla \frac{\partial g^{(i)}}{\partial y} = \left(\frac{\partial^2 g^{(i)}}{\partial x \partial y}, \frac{\partial^2 g^{(i)}}{\partial x^2}\right)^T \quad \text{for } i = 1, 2.$$

After simplification, we get

$$\Omega_2 = -\frac{2\beta^2 \bar{x}}{c + \bar{x}} < 0.$$

Thus from Sotomayor's theorem the system undergoes a saddle–node bifurcation around $\bar{E}(\bar{x}, \bar{y})$ at $h = h^{[sn]}$. Hence, we can conclude that when the parameter h passes from one side of $h = h^{[sn]}$ to the other side, the number of interior equilibria of system (4) changes from zero to two. □

From the biological point of view of the optimal management of renewable resources, we would like to determine the harvesting rate h_{MSY} for the maximum sustainable yield (M.S.Y.) to ensure that both the populations can sustain itself. Hence the biological interpretation of the saddle–node bifurcation is that $h_{MSY} = h^{[sn]}$. The prey species are driven to extinction for $h > h^{[sn]}$ but do not go to extinction for a wide range of initial data when $0 < h < h^{[sn]}$, i.e., coexistence for model (4) is possible in the form of a positive equilibrium for certain choices of initial values.

4.1.5. Numerical example

For $\alpha = 0.40$, $\beta = 1.00$, $m = 0.10$, $c = 0.05$, we get $h^{[sn]} = 0.105625$. For $0 < h < h^{[sn]}$, the system (4) has two distinct interior equilibrium points $E_{1*} = (x_{1*}, y_{1*})$ and $E_{2*} = (x_{2*}, y_{2*})$ which collide with each other for $h = h^{[sn]}$ and no interior equilibrium for $h > h^{[sn]}$. For $\rho = 0.5$ together with the above parameter values an instantaneous equilibrium point is stable from the right side of the separatrix and unstable from the left of the separatrix (see Figs. 1 and 2(a) and 3).

4.1.6. Bogdanov–Takens bifurcation

The Jacobian matrix evaluated at $\bar{E} = (\bar{x}, \bar{y})$ is given by

$$\bar{J} = \begin{pmatrix} \dfrac{\alpha \bar{x}}{\beta(m + \bar{x})} & -\dfrac{\alpha\, \bar{x}}{m + \bar{x}} \\ \dfrac{\rho}{\beta} & -\rho \end{pmatrix} \triangleq \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

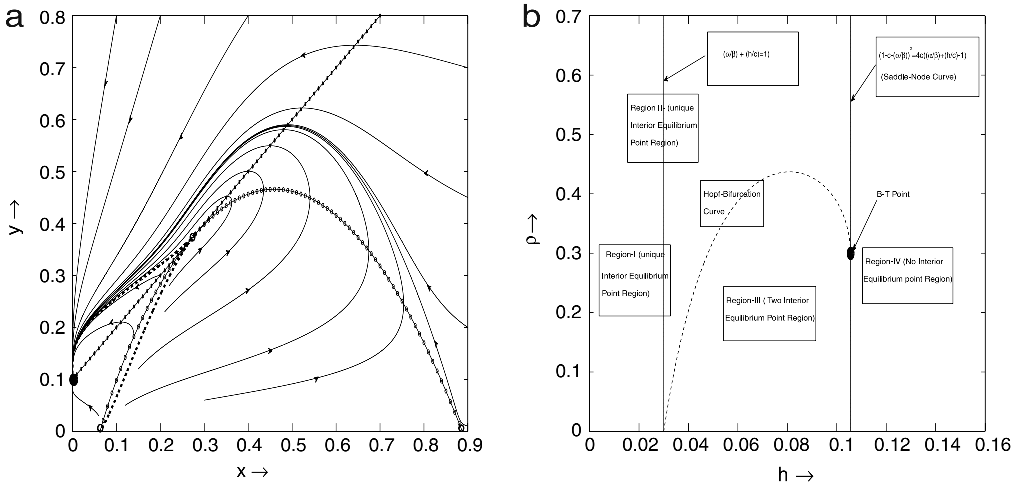

Fig. 4. (a) The prey and predator nullcline are given by circle marked parabola and star marked line. Dashed curves are stable and unstable manifolds of the equilibrium points E_1, E_L and $\bar{E}$. The unique instantaneous equilibrium point $\bar{E} = (\bar{x}, \bar{y})$ is a cusp (B–T point) of codimension two. (b) Parametric region is plotted with respect to various values of h by taking the parameter values $\alpha = 0.4$, $\beta = 1$, $c = 0.05$ and $m = 0.1$. The dotted curve is Hopf bifurcation curve for $E_{2*} = (x_{2*}, y_{2*})$, it is always unstable below this curve and is always stable above this curve. The colliding point of the saddle–node curve and Hopf bifurcation curve is a B–T point.

As $det\bar{J} = 0$, we now consider the case for which $tr\bar{J} = 0$ which gives double zero eigenvalues of $\bar{J}$. Clearly, $tr\bar{J} = 0$ if $\frac{\alpha}{\beta(m+\bar{x})} = \frac{\rho}{\bar{x}}$. In such a situation the matrix $\bar{J}$ is similar to the Jordan block of the form $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. According to the bifurcation theory [37], we know that under certain non-degeneracy conditions, the equilibrium point $\bar{E} = (\bar{x}, \bar{y})$ becomes a cusp of codimension 2. If we choose suitable bifurcation parameters, then the system (4) undergoes the Bogdanov–Takens bifurcation. For this purpose we concentrate on the following parametric region

$$S_2 = \left\{ (\alpha, \beta, h, c, \rho) \in \mathbb{R}^5_+ : \frac{\alpha}{\beta} + c < 1,\ \frac{\alpha}{\beta(m+\bar{x})} = \frac{\rho}{\bar{x}} \right\}.$$

Mathematically the surface represented by S_2 is called a B–T surface.

4.1.7. Numerical example for Bogdanov–Takens bifurcation

For $\alpha = 0.40$, $\beta = 1.00$, $m = 0.10$, $c = 0.05$ we obtain $h^{[BT]} = 0.105625$ and $\rho^{[BT]} = 0.2933333334$ together with $\bar{E} = (\bar{x}, \bar{y}) = (0.275, 0.375)$. For this set of parameter value both saddle–node and Hopf bifurcation collide with each other and the instantaneous equilibrium point becomes a cusp of co-dimension two (see Fig. 4).

Theorem 4. *If we choose h and ρ as two bifurcation parameters then the system (4) undergoes a Bogdanov–Takens bifurcation around the instantaneous equilibrium point $\bar{E}(\bar{x}, \bar{y})$ whenever $\frac{\alpha}{\beta} + c < 1$ and $\frac{\alpha}{\beta(m+\bar{x})} = \frac{\rho}{\bar{x}}$.*

Proof. Let $(h, \rho) = (h^{[BT]} + \lambda_1, \rho^{[BT]} + \lambda_2)$ be a neighboring point of the B–T point $(h^{[BT]}, \rho^{[BT]})$ with λ_i, $i = 1, 2$ sufficiently small. Then the system (4) at $(h, \rho) = (h^{[BT]} + \lambda_1, \rho^{[BT]} + \lambda_2)$ is given by

$$\begin{cases} \dfrac{dx}{dt} = x - x^2 - \dfrac{\alpha xy}{m+x} - \dfrac{h^{[BT]}x}{c+x} - \dfrac{\lambda_1 x}{c+x} \equiv F(x, y, \lambda_1) \\ \dfrac{dy}{dt} = \rho^{[BT]}y + \lambda_2 y - \dfrac{\rho^{[BT]}\beta y^2}{m+x} - \dfrac{\lambda_2 \beta y^2}{m+x} \equiv G(x, y, \lambda_2). \end{cases} \tag{16}$$

In the following we first reduce the system (16) in the canonical form of a Bogdanov–Takens bifurcation by employing a series of C^∞ change of coordinates [38,13] in a small neighborhood of (0, 0). Then we prove the non-degeneracy condition for the Bogdanov–Takens bifurcation with the help of the result from [37].

Making the origin (0, 0) as the bifurcation point by using the transformation $u = x - \bar{x}$, $v = y - \bar{y}$, we get

$$\begin{cases} \dfrac{du}{dt} = \alpha_{00} + \alpha_{10}u + \alpha_{01}v + \alpha_{20}u^2 + \alpha_{11}uv + \alpha_{02}v^2 + P_1(u, v), \\ \dfrac{dv}{dt} = \beta_{00} + \beta_{10}u + \beta_{01}v + \beta_{20}u^2 + \beta_{11}uv + \beta_{02}v^2 + P_2(u, v), \end{cases} \tag{17}$$

where,

$$\alpha_{00} = -\frac{\lambda_1\bar{x}}{c+\bar{x}}, \qquad \alpha_{10} = \frac{\alpha\bar{x}}{\beta(m+\bar{x})} - \frac{\lambda_1 c}{(c+\bar{x})^2}, \qquad \alpha_{01} = -\frac{\alpha\,\bar{x}}{m+\bar{x}},$$
$$\alpha_{20} = \frac{\alpha\,\bar{y}}{(m+\bar{x})^2} - 1 + \frac{h^{[BT]}}{(c+\bar{x})^2} - \frac{h\bar{x}}{(c+\bar{x})^3} - \frac{\alpha\,\bar{x}\bar{y}}{(m+\bar{x})^3} + \frac{c\lambda_1\bar{x}}{(c+\bar{x})^3}, \qquad \alpha_{11} = -\frac{\alpha}{m+\bar{x}} + \frac{\alpha\,\bar{x}}{(m+\bar{x})^2},$$
$$\alpha_{02} = 0,$$
$$\beta_{00} = \rho^{[BT]}\bar{y} + \lambda_2\bar{y} - \frac{\lambda_2\beta\,\bar{y}^2}{m+\bar{x}} - \frac{\rho^{[BT]}\beta\,\bar{y}^2}{m+\bar{x}}, \qquad \beta_{10} = \frac{\rho^{[BT]}\beta\,\bar{y}^2}{(m+\bar{x})^2} + \frac{\lambda_2\beta\,\bar{y}^2}{(m+\bar{x})^2},$$
$$\beta_{01} = \rho^{[BT]} + \lambda_2 - 2\,\frac{\rho^{[BT]}\beta\,\bar{y}}{m+\bar{x}} - 2\,\frac{\lambda_2\beta\,\bar{y}}{m+\bar{x}},$$
$$\beta_{20} = -\frac{\rho^{[BT]}\beta\,\bar{y}^2}{(m+\bar{x})^3} - \frac{\lambda_2\beta\,\bar{y}^2}{(m+\bar{x})^3}, \qquad \beta_{11} = 2\,\frac{\rho^{[BT]}\beta\,\bar{y}}{(m+\bar{x})^2} + 2\,\frac{\lambda_2\beta\,\bar{y}}{(m+\bar{x})^2}, \qquad \beta_{02} = -\frac{\rho^{[BT]}\beta}{m+\bar{x}} - \frac{\lambda_2\beta}{m+\bar{x}}$$

and $P_k(u, v), k = 1, 2$ are power series in (u, v) with powers $u^i v^j$ satisfying $i + j \geq 3$.

Now we introduce the affine transformation $\hat{u} = u$ and $\hat{v} = Au + Bv$ $\left(\text{where } A = \frac{\alpha\bar{x}}{\beta(m+\bar{x})}, B = -\frac{\alpha\,\bar{x}}{m+\bar{x}}\right)$, into the above system to get

$$\begin{cases} \dfrac{d\hat{u}}{dt} = \xi_{00}(\lambda) + \xi_{10}(\lambda)\hat{u} + \xi_{01}(\lambda)\hat{v} + \dfrac{1}{2}\xi_{20}(\lambda)\hat{u}^2 + \xi_{11}(\lambda)\hat{u}\hat{v} + \dfrac{1}{2}\xi_{02}(\lambda)\hat{v}^2 + Q_1(\hat{u}, \hat{v}), \\ \dfrac{d\hat{v}}{dt} = \eta_{10}(\lambda) + \eta_{10}(\lambda)\hat{u} + \eta_{01}(\lambda)\hat{v} + \dfrac{1}{2}\eta_{20}(\lambda)\hat{u}^2 + \eta_{11}(\lambda)\hat{u}\hat{v} + \dfrac{1}{2}\eta_{02}(\lambda)\hat{v}^2 + Q_2(\hat{u}, \hat{v}), \end{cases} \tag{18}$$

where,

$$\xi_{00}(\lambda) = -\frac{\lambda_1\bar{x}}{c+\bar{x}}, \qquad \xi_{10}(\lambda) = \frac{\alpha\,\bar{x}A}{B\,(m+\bar{x})} + \frac{h^{[BT]}\bar{x}}{(c+\bar{x})^2} + \frac{\alpha\,\bar{x}\bar{y}}{(m+\bar{x})^2} - \frac{\lambda_1}{c+\bar{x}} - \bar{x} + \frac{\lambda_1\bar{x}}{(c+\bar{x})^2},$$
$$\xi_{01}(\lambda) = -\frac{\alpha\,\bar{x}}{B\,(m+\bar{x})},$$
$$\begin{aligned}\xi_{20}(\lambda) \;=\; 2\Bigg(&\frac{\alpha\,A}{B\,(m+\bar{x})} - \frac{\alpha\,\bar{x}A}{B\,(m+\bar{x})^2} + \frac{\alpha\,\bar{y}}{(m+\bar{x})^2} - 1 + \frac{\lambda_1}{(c+\bar{x})^2} + \frac{h^{[BT]}}{(c+\bar{x})^2} - \frac{h^{[BT]}\bar{x}}{(c+\bar{x})^3} \\ &- \frac{\alpha\,\bar{x}\bar{y}}{(m+\bar{x})^3} - \frac{\lambda_1\bar{x}}{(c+\bar{x})^3}\Bigg),\end{aligned}$$
$$\xi_{11}(\lambda) = -\frac{\alpha}{B\,(m+\bar{x})} + \frac{\alpha\,\bar{x}}{B\,(m+\bar{x})^2}, \; \xi_{02}(\lambda) = 0,$$
$$\eta_{00}(\lambda) = \rho^{[BT]}\bar{y} + \lambda_2\bar{y} - \frac{\rho^{[BT]}\beta\,\bar{y}^2}{m+\bar{x}} - \frac{\lambda_2\beta\,\bar{y}^2}{m+\bar{x}},$$
$$\eta_{10}(\lambda) = -\frac{\rho^{[BT]}A}{B} - \frac{\lambda_2 A}{B} + 2\,\frac{\rho^{[BT]}\beta A\bar{y}}{B\,(m+\bar{x})} + 2\,\frac{\lambda_2\beta A\bar{y}}{B\,(m+\bar{x})} + \frac{\rho^{[BT]}\beta\,\bar{y}^2}{(m+\bar{x})^2} + \frac{\lambda_2\beta\,\bar{y}^2}{(m+\bar{x})^2},$$
$$\eta_{01}(\lambda) = \frac{\rho^{[BT]}}{B} + \frac{\lambda_2}{B} - 2\,\frac{\rho^{[BT]}\beta\,\bar{y}}{B\,(m+\bar{x})} - 2\,\frac{\lambda_2\beta\,\bar{y}}{B\,(m+\bar{x})},$$
$$\eta_{20}(\lambda) = 2\left(-\frac{\rho^{[BT]}\beta\,A^2}{B^2\,(m+\bar{x})} - \frac{\lambda_2\beta\,A^2}{B^2\,(m+\bar{x})} - 2\,\frac{\rho^{[BT]}\beta A\bar{y}}{B\,(m+\bar{x})^2} - 2\,\frac{\lambda_2\beta A\bar{y}}{B\,(m+\bar{x})^2} - 2\,\frac{\rho^{[BT]}\beta\,\bar{y}^2}{(m+\bar{x})^3} - 2\,\frac{\lambda_2\beta\,\bar{y}^2}{(m+\bar{x})^3}\right),$$
$$\eta_{11}(\lambda) = 2\,\frac{\rho^{[BT]}\beta\,A}{B^2\,(m+\bar{x})} + 2\,\frac{\lambda_2\beta\,A}{B^2\,(m+\bar{x})} + 2\,\frac{\rho^{[BT]}\beta\,\bar{y}}{B\,(m+\bar{x})^2} + 2\,\frac{\lambda_2\beta\,\bar{y}}{B\,(m+\bar{x})^2},$$
$$\eta_{02}(\lambda) = 2\left(-\frac{\rho^{[BT]}\beta}{B^2\,(m+\bar{x})} - \frac{\lambda_2\beta}{B^2\,(m+\bar{x})}\right)$$

and $Q_k(\hat{u}, \hat{v}), k = 1, 2$ are power series in $(\hat{u}, \hat{v})$ with powers $\hat{u}^i\hat{v}^j$ satisfying $i + j \geq 3$.

By using the equilibrium equations and expressions for A and B the system (18) is given by:

$$\begin{cases} \dfrac{d\hat{u}}{dt} = \bar{\xi}_{00} + \bar{\xi}_{10}\hat{u} + \hat{v} + \bar{\xi}_{20}\hat{u}^2 + \bar{\xi}_{11}\hat{u}\hat{v} + Q_1(\hat{u}, \hat{v}), \\ \dfrac{d\hat{v}}{dt} = \bar{\eta}_{01}\hat{v} + \bar{\eta}_{20}\hat{u}^2 + \bar{\eta}_{02}\hat{v}^2 + Q_2(\hat{u}, \hat{v}), \end{cases} \tag{19}$$

where,

$$\bar{\xi}_{00} = -\frac{\lambda_1\bar{x}}{c+\bar{x}}, \qquad \bar{\xi}_{10} = -\frac{\lambda_1 c}{(c+\bar{x})^2}, \qquad \bar{\xi}_{20} = -\frac{\bar{x}}{c+\bar{x}} + \frac{\lambda_1 c}{(c+\bar{x})^3}, \qquad \bar{\xi}_{11} = -\frac{m}{\bar{x}(m+\bar{x})}$$

$$\bar{\eta}_{01} = (\rho^{[BT]} + \lambda_2)\left(\frac{m+\bar{x}}{\alpha\bar{x}}\right), \qquad \bar{\eta}_{20} = -\frac{\rho^{[BT]} + \lambda_2}{\beta(m+\bar{x})}, \qquad \bar{\eta}_{02} = -(\rho^{[BT]} + \lambda_2)\left(\frac{\beta(m+\bar{x})}{\alpha^2\bar{x}^2}\right).$$

Now using the following C^∞ change of coordinates in a small neighborhood of $(0, 0)$:

$$y_1 = \hat{u} - \frac{1}{2}\left(\bar{\xi}_{11} + \bar{\eta}_{02}\right)\hat{u}^2 \quad \text{and} \quad y_2 = \hat{v} + \bar{\xi}_{20}\hat{u}^2 - \bar{\eta}_{02}\hat{u}\hat{v},$$

$$i.e.\ \hat{u} = y_1 + \frac{1}{2}\left(\bar{\xi}_{11} + \bar{\eta}_{02}\right){y_1}^2 \quad \text{and} \quad \hat{v} = y_2 - \bar{\xi}_{20}{y_1}^2 + \bar{\eta}_{02}y_1y_2$$

the system (19) can be transformed into

$$\begin{cases} \dfrac{dy_1}{dt} = r_{00} + r_{10}y_1 + y_2 + r_{20}{y_1}^2 + R_1(y_1, y_2), \\ \dfrac{dy_2}{dt} = s_{10}y_1 + s_{01}y_2 + s_{20}{y_1}^2 + s_{11}y_1y_2 + R_2(y_1, y_2), \end{cases} \tag{20}$$

where,

$$r_{00} = \bar{\xi}_{00}, \qquad r_{10} = \bar{\xi}_{10} - \bar{\xi}_{00}(\bar{\xi}_{11} + \bar{\eta}_{02}), \qquad r_{20} = -\frac{1}{2}(\bar{\xi}_{11} + \bar{\eta}_{02})\left(\bar{\xi}_{10} + \bar{\xi}_{00}(\bar{\xi}_{11} + \bar{\eta}_{02})\right),$$

$$s_{10} = 2\bar{\xi}_{00}\bar{\xi}_{20}, \qquad s_{01} = \bar{\eta}_{01} - \bar{\xi}_{00}\bar{\eta}_{02}, \qquad s_{20} = \bar{\eta}_{20} + 2\bar{\xi}_{10}\bar{\xi}_{20} + \bar{\xi}_{00}\bar{\xi}_{20}(\bar{\xi}_{11} + \bar{\eta}_{02}) - \bar{\xi}_{20}(\bar{\eta}_{01} - \bar{\xi}_{00}\bar{\eta}_{02}),$$

$$s_{11} = 2\bar{\xi}_{20} - \bar{\xi}_{00}\bar{\eta}_{02}^2 - \bar{\xi}_{10}\bar{\eta}_{02}$$

and $R_k(y_1, y_2)$, $k = 1, 2$ are power series in (y_1, y_2) with powers $y_1^i y_2^j$ satisfying $i + j \geq 3$.

Next, under the following C^∞ change of coordinates in a small neighborhood of $(0, 0)$

$$z_1 = y_1 \quad \text{and} \quad z_2 = r_{00} + r_{10}y_1 + y_2 + r_{20}{y_1}^2,$$

the system (20) becomes

$$\begin{cases} \dfrac{dz_1}{dt} = z_2 + \bar{R}_1(z_1, z_2), \\ \dfrac{dz_2}{dt} = \theta_{00} + \theta_{10}z_1 + \theta_{01}z_2 + \theta_{20}{z_1}^2 + \theta_{11}z_1z_2 + \bar{R}_2(z_1, z_2), \end{cases} \tag{21}$$

where,

$$\theta_{00} = -r_{00}s_{01}, \qquad \theta_{10} = s_{10} - r_{00}s_{11} - r_{10}s_{01}, \qquad \theta_{01} = r_{10} + s_{01}$$

$$\theta_{20} = s_{20} - r_{20}s_{01} - r_{10}s_{11}, \qquad \theta_{11} = s_{11} + 2r_{20}$$

and $\bar{R}_k(z_1, z_2)$, $k = 1, 2$ are power series in (z_1, z_2) with powers $z_1^i z_2^j$ satisfying $i + j \geq 3$.

Further we consider the following nonsingular change of coordinates in a small neighborhood of $(0, 0)$:

$$w_1 = z_1 \quad \text{and} \quad w_2 = z_2 + \bar{R}_1(z_1, z_2)$$

to reduce the system (21) into the following form

$$\begin{cases} \dfrac{dw_1}{dt} = w_2, \\ \dfrac{dw_2}{dt} = \theta_{00} + \theta_{10}w_1 + \theta_{01}w_2 + \theta_{20}{w_1}^2 + \theta_{11}w_1w_2 + G_1(w_1) + w_2G_2(w_1) + {w_2}^2G_3(w_1, w_2). \end{cases} \tag{22}$$

Here G_1, G_2 are power series in w_1 with powers ${w_1}^{k_1}$, ${w_1}^{k_2}$ satisfying $k_1 \geq 3$, $k_2 \geq 2$ respectively and G_3 is a power series in (w_1, w_2) with powers $w_1^i w_2^j$ satisfying $i + j \geq 1$.

Applying the Malgrange Preparation theorem [39], we have

$$\theta_{00}+\theta_{10}w_1+\theta_{01}w_2+\theta_{20}{w_1}^2+G_1(w_1)=\left({w_1}^2+\frac{\theta_{10}}{\theta_{20}}w_1+\frac{\theta_{00}}{\theta_{20}}\right)B_1(w_1,\lambda),$$

where $B_1(0,\lambda)=\theta_{20}$ and B_1 is a power series in w_1 whose coefficients depend on parameters $\lambda=(\lambda_1,\lambda_2)$.

Now let $u_1=w_1,\ u_2=\frac{w_2}{\sqrt{\theta_{20}}}$ and $T=\sqrt{\theta_{20}}dt$, then the system (22) becomes

$$\begin{cases}\dfrac{du_1}{dT}=u_2,\\ \dfrac{du_2}{dT}=\dfrac{\theta_{00}}{\theta_{20}}+\dfrac{\theta_{10}}{\theta_{20}}u_1+\dfrac{\theta_{01}}{\theta_{20}}u_2+{u_1}^2+\dfrac{\theta_{11}}{\sqrt{\theta_{20}}}u_1u_2+S_1(u_1,u_2,\lambda),\end{cases}\tag{23}$$

where $S_1(u_1,u_2,0)$ is a power series in (u_1,u_2) with powers $u_1^iu_2^j$ satisfying $i+j\geq 3$ and $j\geq 2$.

Again we consider $v_1=u_1+\frac{\theta_{10}}{2\theta_{20}},\ v_2=u_2$ and use a Taylor series expansion to transform the system (23) into the following

$$\begin{cases}\dfrac{dv_1}{dT}=v_2,\\ \dfrac{dv_2}{dT}=\mu_1(\lambda_1,\lambda_2)+\mu_1(\lambda_1,\lambda_2)v_2+{v_1}^2+(\varepsilon+\alpha(\lambda))v_1v_2+S_2(v_1,v_2,\mu),\end{cases}\tag{24}$$

where,

$$\mu_1(\lambda_1,\lambda_2)=\frac{\theta_{00}}{\theta_{20}}-\left(\frac{\theta_{00}}{2\theta_{20}}\right)^2,\qquad \mu_2(\lambda_1,\lambda_2)=\frac{\theta_{01}}{\theta_{20}}-\frac{\theta_{00}}{2(\theta_{20})^{\frac{3}{2}}},\qquad \varepsilon+\alpha(\lambda)=\frac{\theta_{11}}{\sqrt{\theta_{20}}}$$

with $\alpha(0)=0$ and $S_2(v_1,v_2,0)$ is a power series in (v_1,v_2) with powers $v_1^iv_2^j$ satisfying $i+j\geq 3$ and $j\geq 2$.

Thus the system (24) can be written as

$$\begin{cases}\dfrac{dv_1}{dT}=v_2,\\ \dfrac{dv_2}{dT}=\mu_1(\lambda_1,\lambda_2)+\mu_1(\lambda_1,\lambda_2)v_2+{v_1}^2+\varepsilon v_1v_2+S_3(v_1,v_2,\mu),\end{cases}\tag{25}$$

where $S_3(v_1,v_2,\mu)$ is a power series in (v_1,v_2,μ_1,μ_2) with powers $v_1^iv_2^j{\mu_1}^k{\mu_2}^l$ satisfying $i+j+k+l\geq 4$ and $i+j\geq 3$.

The system (25) is topologically equivalent to the normal form of the Bogdanov–Takens bifurcation given by

$$\begin{cases}\dfrac{dY_1}{dt}=Y_2,\\ \dfrac{dY_2}{dt}=\mu_1+\mu_2Y_2+{Y_1}^2+Y_1Y_2.\end{cases}\tag{26}$$

Now, we prove the non-degeneracy condition for the Bogdanov–Takens bifurcation [37]. Using $\frac{A}{B}=-\frac{1}{\beta}$, together with equilibrium equations from Eq. (13), we have

$$\xi_{00}(0)=0,\qquad \xi_{10}(0)=0,\qquad \xi_{01}(0)=1,\qquad \xi_{20}(0)=-\frac{2\bar{x}}{c+\bar{x}},\qquad \xi_{11}(0)=-\frac{m}{\bar{x}(m+\bar{x})},\qquad \xi_{02}(0)=0$$

and

$$\eta_{00}(0)=0,\qquad \eta_{10}(0)=0,\qquad \eta_{01}(0)=-\frac{\rho^{[BT]}}{B},\qquad \eta_{20}(0)=-\frac{2\rho^{[BT]}}{\beta(m+\bar{x})},\qquad \eta_{11}(0)=0,$$

$$\eta_{02}(0)=-\frac{2\beta\rho^{[BT]}}{B^2(m+\bar{x})}.$$

We see that the non-degeneracy condition of the Bogdanov–Takens bifurcation that $\xi_{20}(0)+\eta_{11}(0)\neq 0$ and $\eta_{20}(0)\neq 0$ are satisfied. We also see that

$$sign(\eta_{20}(0)(\xi_{20}(0)+\eta_{11}(0)))=+1.\quad\square$$

4.2. *Case II:* $\frac{\alpha}{\beta}+\frac{h}{c}\leq 1$

In this case only one interior equilibrium point exists and is denoted by $E_*=(x_*,y_*)\equiv(x_{2*},y_{2*})$.

Theorem 5. (i) *The equilibrium point* $E_* = (x_*, y_*)$ *is stable if* $x_*\left(\frac{\alpha}{\beta(m+x_*)} + \frac{h}{(c+x_*)^2} - 1\right) < \rho$.

(ii) *The system undergoes a Hopf bifurcation with respect to bifurcation parameter* ρ *around the equilibrium point* $E_* = (x_*, y_*)$ *if* $x_*\left(\frac{\alpha}{\beta(m+x_*)} + \frac{h}{(c+x_*)^2} - 1\right) = \rho$.

This result can be proved in a similar way as given in Section 4.1.1.

4.2.1. Transcritical bifurcation

We have seen that the axial equilibrium $E_1 = (0, \frac{m}{\beta})$ is stable and $E_{2*} = (x_{2*}, y_{2*})$ is unstable in the parametric region S_3 given by

$$S_3 = \left\{(\alpha, \beta, h, c, \rho) \in \mathbb{R}^5_+ : \frac{\alpha}{\beta} + \frac{h}{c} > 1,\ x_{2*}\left(\frac{\alpha}{\beta(m+x_{2*})} + \frac{h}{(c+x_{2*})^2} - 1\right) > \rho\right\}.$$

On the other hand the axial equilibrium $E_1 = (0, \frac{m}{\beta})$ is unstable and $E_{2*} = (x_{2*}, y_{2*})$ is stable in the region

$$S_4 = \left\{(\alpha, \beta, h, c, \rho) \in \mathbb{R}^5_+ : \frac{\alpha}{\beta} + \frac{h}{c} < 1,\ x_{2*}\left(\frac{\alpha}{\beta(m+x_{2*})} + \frac{h}{(c+x_{2*})^2} - 1\right) < \rho\right\},$$

therefore there is an exchange of stability between the two equilibrium points $E_1 = (0, \frac{m}{\beta})$ and $E_{2*} = (x_{2*}, y_{2*})$ in the two regions S_3 and S_4 as h crosses the critical magnitude $h^{[tc]} = c(1 - \frac{\alpha}{\beta})$.

The axial equilibrium $E_1 = (0, \frac{m}{\beta})$ loses its stability at $\frac{\alpha}{\beta} + \frac{h}{c} = 1$ and one of the eigenvalues of the Jacobian evaluated at the point $E_1 = (0, \frac{m}{\beta})$ is zero so the point $E_1 = (0, \frac{m}{\beta})$ becomes non-hyperbolic. Thus there is a chance of bifurcation around the axial equilibrium $E_1 = (0, \frac{m}{\beta})$. Similar to the case of saddle–node bifurcation, keeping all the parameters fixed and varying h, we can study the existence of transcritical bifurcation with the help of Sotomayor's theorem. This is shown in Fig. 5(d).

4.2.2. Numerical example

We take the numerical example is $\alpha = 0.80$, $\beta = 2.00$, $m = 0.10$, $c = 0.30$, $h = 0.17$ satisfying $\frac{\alpha}{\beta} + \frac{h}{c} < 1$ which gives $\rho^{[hf]} = 0.1180989313$ and the Lyapunov number $\sigma = -142.7814668\pi$. For this set of parameters the unique interior equilibrium is obtained as (0.330, 0.215). This equilibrium point is stable for $\rho = 0.2 > \rho^{[hf]}$ and the limit cycle collides with saddle point E_1 for $\rho = 0.09 < \rho^{[hf]}$ to give a homoclinic orbit (see Fig. 5(a), (b) and (c)).

Theorem 6. *The system* (4) *undergoes a transcritical bifurcation between* $E_1 = (0, \frac{m}{\beta})$ *and* $E_{2*} = (x_{2*}, y_{2*})$ *with respect to bifurcation parameter* h *if* $\frac{\alpha}{\beta} + \frac{h}{c} = 1$.

Proof. The Jacobian matrix R of the system (4) around the axial equilibrium point $E_1 = (0, \frac{m}{\beta})$ is given by

$$R = \begin{pmatrix} 1 - \dfrac{\alpha}{\beta} - \dfrac{h}{c} & 0 \\ \dfrac{\rho}{\beta} & -\rho \end{pmatrix}.$$

Clearly, one of the eigenvalues of R is negative and the other eigenvalue will be zero if and only if $\frac{\alpha}{\beta} + \frac{h}{c} = 1$ which gives $h = h^{[tc]} = c(1 - \frac{\alpha}{\beta})$. If $\Phi = (\phi_1, \phi_2)^T$ and $\Psi = (\psi_1, \psi_2)^T$ denote the eigenvectors corresponding to the eigenvalue zero of the matrices R and R^T, respectively, then we obtain $\Phi = (1, \frac{1}{\beta})^T$ and $\Psi = (1, 0)^T$. Now $\Delta_1 = \Psi^T[g_h(E_1, h^{[tc]})] = 0$, where $E_1 = (0, \frac{m}{\beta})$. Hence the system (4) does not attain any saddle–node bifurcation [34] around E_1. Again, $\Delta_2 = \Psi^T[Dg_h(E_1, h^{[tc]})\Phi] = -\frac{1}{c}$, where

$$Dg_h(E_1, h^{[tc]}) = \begin{pmatrix} -\dfrac{1}{c} & 0 \\ 0 & 0 \end{pmatrix}.$$

Similar to the calculation for saddle–node bifurcation

$$\Delta_3 = \Psi^T[D^2 f(E_1, h^{[tc]})(\Phi, \Phi)] = -\frac{2}{c}\left(\frac{\alpha}{\beta} + c - 1\right) \neq 0,$$

under the existence of the interior equilibrium point $E_{2*} = (x_{2*}, y_{2*})$. □

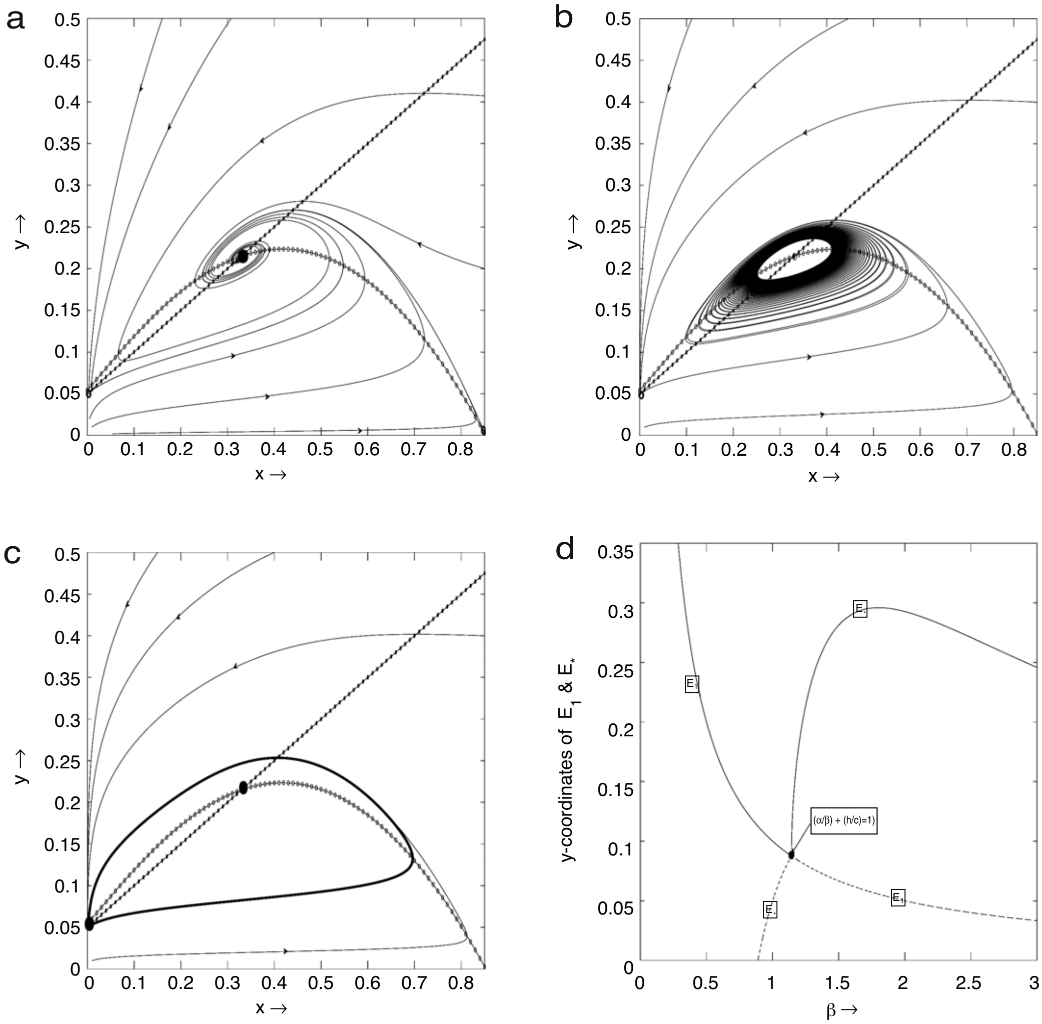

Fig. 5. In (a) to (c) the prey and predator nullclines are given by circle marked parabola and star marked line. (a) The unique interior equilibrium point $E_* = (x_*, y_*)$ is stable and the axial equilibrium point $E_1 = (0, \frac{m}{\beta})$ is a saddle point. (b) A stable limit cycle bifurcates from the interior equilibrium through Hopf bifurcation. (c) The limit cycle collides with the saddle point $E_1 = (0, \frac{m}{\beta})$ and a homoclinic curve appears. (d) This diagram shows the exchange of stability between $E_1 = (0, \frac{m}{\beta})$ and $E_{2*} = (x_{2*}, y_{2*})$. The y-coordinates of $E_1 = (0, \frac{m}{\beta})$ and $E_{2*} = (x_{2*}, y_{2*})$ are plotted on the y-axis, where the solid curve shows the stable branch and the dashed curve stands for the unstable branch.

5. Conclusion

In this paper, we have considered a modified Leslie–Gower predator–prey model with a nonlinear harvesting in prey where the protection provided by the environment for both the prey and predator is the same. The model shows rich and varied dynamics. We have discussed the permanence condition for the model which gives the condition for coexistence of the species for all future time with continuous harvesting in prey. The local stability of different steady states have been discussed. Under certain parametric conditions we have obtained a situation where solutions are highly dependent on the initial values i.e., the solutions of the system converge to the prey extinction equilibrium point for a large number of initial values while they converge to the coexisting equilibrium point if the initial conditions lie in the other region. This is known as a bistable situation in which the whole domain of prey–predator is divided into two regions through a separatrix. We observe that the system cannot collapse for any value of parameters as the origin is never stable.

The model exhibits several local bifurcations such as saddle–node, Hopf–Andronov, transcritical, homoclinic and Bogdanov–Takens. These bifurcations are ecologically important and the saddle–node and homoclinic bifurcations especially can lead to potentially dramatic shifts to the system dynamics [40]. A transcritical bifurcation transforms a prey extinction equilibrium point into a unstable equilibrium point and at the unstable coexisting equilibrium to a stable one. The system can have zero, one or two interior equilibria through saddle–node bifurcation as the bifurcation parameter crosses its critical value. We have found a parametric domain where one of the co-existing equilibria is saddle and other is stable which gives the existence of a saddle–node bifurcation. The ecological significance of saddle–node and transcritical bifurcations gives the maximum threshold for continuous harvesting without the extinction risk of the prey species. It has also been seen that a small unstable limit cycle bifurcates from the co-existing equilibrium whenever there are two interior equilibria. We have found a situation where the saddle–node bifurcation and Hopf bifurcation curves collide and the instantaneous equilibrium point becomes a cusp of co-dimension 2 (Bogdanov–Takens bifurcation). The stable and unstable manifolds are also plotted to separate the different region.

There is a unique interior equilibrium point which is locally stable for certain parametric restrictions and in this case all the axial equilibria become unstable. We have also seen that there is an exchange of stability between the prey free equilibrium and the co-existing equilibrium as the bifurcation parameter passes though the critical value. That means there exists a parametric region for the co-existence of the species. A stable limit cycle bifurcates around the unique interior equilibrium point. The local existence of limit cycles in different cases has been observed through Hopf bifurcation and the stability of limit cycles has been examined and validated through numerical simulations by calculating the first Lyapunov number. We have also seen that the size of the limit cycle increases with the change in bifurcation parameter and a situation arises where the limit cycle collides with the axial equilibrium point which is a saddle point. In this case a stable homoclinic loop enclosing the interior equilibrium point appears through homoclinic bifurcation. Exhaustive numerical simulations are carried out to ensure the number of equilibrium points and their stability properties.

Acknowledgments

The work of the first author (RPG) is supported by the Council of Scientific and Industrial Research, India. The authors are grateful to the anonymous reviewers for their valuable suggestions.

References

[1] C.W. Clark, Mathematical Bioeconomics: the Optimal Management of Renewable Resources, Wiley, New York, 1976.
[2] P.D.N. Srinivasu, Bioeconomics of a renewable resource in presence of a predator, Nonlinear Anal. RWA 2 (2001) 497–506.
[3] W. Murdoch, C. Briggs, R. Nisbet, Consumer-Resource Dynamics, Princeton University Press, New York, 2003.
[4] M.L. Rosenzweig, R.H. MacArthur, Graphical representation and stability conditions of predator–prey interactions, Am. Naturalist 97 (1963) 209.
[5] M. Vos, B.W. Kooi, D.L. DeAngelis, W.M. Mooij, Inducible defences and the paradox of enrichment, Oikos 105 (2004) 471–480.
[6] M. Kondoh, Foraging adaptation and the relationship between food-web complexity and stability, Science 299 (2003) 1388–1391.
[7] P. Turchin, Complex Population Dynamics. A Theoretical/Empirical Synthesis, in: Monogr. Population Biol., vol. 35, Princeton University Press, Princeton, NJ, 2003.
[8] H.I. Freedman, Deterministic Mathematical Models in Population Ecology, Marcel Dekker, 1980.
[9] H.I. Freedman, G.S.K. Wolkowicz, Predator–prey systems with group defence: The paradox of enrichment revisited, Bull. Math. Biol. 8 (1986) 493–508.
[10] G. Dai, M. Tang, Coexistence region and global dynamics of a harvested predator–prey system, SIAM J. Appl. Math. 58 (1) (1998) 193–210.
[11] E.A. McGehee, N. Schutt, D.A. Vasquez, E. Peacock-Lopez, Bifurcations and temporal and spatial patterns of a modified Lotka–Voltera model, Int. J. Bifurcation Chaos Appl. Sci. Eng. 18 (2008) 2223–2248.
[12] T. Das, R.N. Mukherjee, K.S. Chaudhari, Bioeconomic harvesting of a prey–predator fishery, J. Biol. Dyn. 3 (2009) 447–462.
[13] D. Xiao, S. Ruan, Bogdanov–Takens bifurcations in predator–prey systems with constant rate harvesting, Fields Inst. Commun. 21 (1999) 493–506.
[14] M. Haque, A detailed study of the Beddington–DeAngelis predator–prey model, Math. Biosci. 234 (2011) 116.
[15] B. Leard, C. Lewis, J. Rebaza, Dynamics of ratio-dependent predator–prey models with non-constant harvesting, Discrete Contin. Dyn. Syst. Ser. S 1 (2) (2008) 303–315.
[16] P. Lenzini, J. Rebaza, Non-constant predator harvesting on ratio-dependent predator–prey models, Appl. Math. Sci. 4 (16) (2010) 791–803.
[17] G.J. Peng, Y.L. Jiang, C.P. Li, Bifurcations of a Holling-type II predator–prey system with constant rate harvesting, Int. J. Bifurcation Chaos Appl. Sci. Eng. 19 (2009) 2499–2514.
[18] M. Sen, M. Banerjee, A. Morozov, Bifurcation analysis of a ratio-dependent prey–predator model with the Allee effect, Ecol. Complex. (2012) http://dx.doi.org/10.1016/j.ecocom.2012.01.002.
[19] Y. Li, D. Xiao, Bifurcations of a predator–prey system of Holling and Leslie types, Chaos Solitons Fractals 34 (2007) 606–620.
[20] C.M. Lin, C.P. Ho, Local and global stability for a predator–prey model of modified Leslie-Gower and Holling-type II with time-delay, Tunghai, Science 8 (2006) 33–61.
[21] N. Zhang, F. Chen, Q. Su, T. Wu, Dynamic behaviors of a harvesting Leslie-Gower predator–prey model, Discrete Dyn. Nat. Soc. 2011 (2011) http://dx.doi.org/10.1155/2011/473-949.
[22] C.R. Zhu, K.Q. Lan, Phase portraits, Hopf-bifurcations and limit cycles of Leslie-Gower predator–prey systems with harvesting rates, Discrete Contin. Dyn. Syst. Ser. B 14 (2010) 289–306.
[23] J. Mena-Lorca, E. González-Olivares, B. González-Yañez, The Leslie-Gower predator–prey model with Allee effect on prey: a simple model with a rich and interesting dynamics, in: R. Mondaini (Ed.), Proceedings of the International Symposium on Mathematical and Computational Biology BIOMAT 2006 (2007), E-papers Servios Editoriais Ltda., R' io de Janeiro, 105–132.
[24] X.Y. Song, Y.F. Li, Dynamic behaviors of the periodic predator–prey model with modified Leslie-Gower Holling-type II schemes and impulsive effect, Nonlinear Anal. RWA 9 (2008) 64–79.
[25] M.A. Aziz-Alaoui, M. Daher Okiye, Boundedness and global stability for a predator–prey model with modified Leslie-Gower and Holling-type II schemes, Appl. Math. Lett. 16 (7) (2003) 1069–1075.
[26] Y. Du, R. Peng, M. Wang, Effect of a protection zone in the diffusive Leslie predator–prey model, J. Differential Equations 246 (10) (2009) 3932–3956.
[27] Y. Zhu, K. Wang, Existence and global attractivity of positive periodic solutions for a predator–prey model with modified Leslie-Gower Holling-type II schemes, J. Math. Anal. Appl. 384 (2011) 400–408.
[28] C. Ji, D. Jiang, N. Shi, Analysis of a predator–prey model with modified Leslie-Gower and Holling-type II schemes with stochastic perturbation, J. Math. Anal. Appl. 359 (2009) 482–498.
[29] C. Ji, D. Jiang, N. Shi, A note on a predator–prey model with modified Leslie-Gower and Holling-type II schemes with stochastic perturbation, J. Math. Anal. Appl. 377 (1) (2011) 435–440.
[30] C.W. Clark, Mathematical models in the economics of renewable resources, SIAM Rev. 21 (1979) 81–99.
[31] S.V. Krishna, P.D.N. Srinivasu, B. Kaymackcalan, Conservation of an ecosystem through optimal taxation, Bull. Math. Biol. 60 (1998) 569–584.
[32] F.D. Chen, On a nonlinear non-autonomous predator–prey model with diffusion and distributed delay, J. Comput. Appl. Math. 180 (1) (2005) 33–49.

[33] P.J. Pal, S. Sarwardi, T. Saha, P.K. Mandal, Mean square stability in a modified Leslie-Gower and holling-typeII predator–prey model, J. Appl. Math. Inform. 29 (3–4) (2011) 781–802.
[34] L. Perko, Differential Equations and Dynamical Systems, Springer, New York, 1996.
[35] D. Xiao, L. Jennings, Bifurcations of a ratio-dependent predatorprey system with constant rate harvesting, SIAM J. Appl. Math. 65 (2005) 737–753.
[36] M. Haque, Ratio-dependent predator–prey models of interacting populations, Bull. Math. Biol. 71 (2009) 430–452.
[37] A.Y. Kuznetsov, Elements of Applied Bifurcation Theory, Appl. Math. Sci, Springer-Verlag, New York, 2004, p. 112.
[38] X. Lai, S. Liu, R. Lin, Rich dynamical behaviours for predator–prey model with weak Allee effect, Appl. Anal. 8 (89) (2010) 1271–1292.
[39] S.N. Chow, J.K. Hale, Methods of Bifurcation Theory, Tim Clutton-Brock Grundlehren Mathematischen Wissenschaften, Springer-Verlag, New York, Berlin, 1983, p. 251.
[40] B. Barnes, H. Sidhu, Plant-herbivore models where more grass means fewer grazers, Bull. Math. Biol. 67 (2005) 33–55.