<u>MA1521 Cheat Sheet</u>

1 Pre-calculus

Real Numbers & Functions

$$a^2 - b^2 = (a+b)(a-b)$$
$$|x+y| \leq |x| + |y|$$
$$\log_a x = \frac{\ln x}{\ln a}$$
$$f : A \longrightarrow B,\ \ g \circ f = g(f(x)),\ \ g \circ f \neq f \circ g$$

A: domain, B: codomain, range: $f = \{f(x) \in B | x \in A\}$
Injective: $f(x) = f(y) \Rightarrow x = y$, surjective: $\forall z \in B, \exists x \in A, f(x) = z$
If f^{-1} exists, then f is bijective

Linear Equations

Slope-intercept: $y = mx + b$
Point-slope: $y - y_1 = m(x + x_1)$
Intercept: $\frac{x}{a} + \frac{y}{b} = 1$
b: y-intercept, a: x-intercept, (x_1, y_1) is a point on the line
The gradient of the normal of a line is $\frac{1}{m}$

Trigonometric Identities

$$\csc x = \frac{1}{\sin x}$$
$$\sec x = \frac{1}{\cos x}$$
$$\cot x = \frac{1}{\tan x}$$
$$\sin^2\theta + \cos^2\theta = 1$$
$$\tan^2\theta + 1 = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$
$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$
$$\sin 2A = 2 \sin A \cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$
$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$
$$\sin P \pm \sin Q = 2 \sin(\frac{P \pm Q}{2}) \cos(\frac{P \mp Q}{2})$$
$$\cos P + \cos Q = 2 \cos(\frac{P + Q}{2}) \cos(\frac{P - Q}{2})$$
$$\cos P - \cos Q = -2 \sin(\frac{P + Q}{2}) \sin(\frac{P - Q}{2})$$

Values of Trigonometric Functions

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	DNE

Range of a Function

First find the maximal domain of f
To find the range of f, let $y = f(x)$, solve for x, deduce the range (exclude values where $f(x)$ is undefined)

2 Limits

Continuity

c	$\lim_{x\to c^-} f(x)$	$\lim_{x\to c^+} f(x)$	$\lim_{x\to c} f(x)$	$f(x)$

Continuous if $\lim_{x\to c} f(x)$ exists (only if left = right limit) AND
Interior point: $\lim_{x\to c} f(x) = f(c)$
Left/right end-point: left/right limit equals $f(c)$
Polynomials, trigonometric/exponential/logarithmic functions and their combinations are continuous

Evaluation of Limits

$$\lim_{x\to\pm\infty} \frac{k_1 x^a}{k_2 x^b} = 0 (a < b), \frac{A}{B}(a = b), \pm\infty(a > b)$$

$$\lim_{x\to c} \frac{\sin(g(x))}{g(x)} = \lim_{x\to c} \frac{g(x)}{\sin(g(x))} = 1$$

$$\lim_{x\to c} \frac{\tan(g(x))}{g(x)} = \lim_{x\to c} \frac{g(x)}{\tan(g(x))} = 1$$

In particular, when $c = 0$ and $g(x) = x$

Squeeze Theorem

If $g(x) \leq f(x) \leq h(x)$ and

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L \Rightarrow \lim_{x\to c} f(x) = L$$

If $\lim_{x\to c} g(x) = 0$,

$$\lim_{x\to c} g(x) sin(h(x)) = 0, \lim_{x\to c} g(x) cos(h(x)) = 0$$

Intermediate Value Theorem

To show an equation $f(x) = c$ has a root between a and b, $f(x)$ must be continuous and $f(a) < c < f(b)$

3 Derivatives

Differentiability implies continuity (converse is not true in general)

Function	Derivative
$(f(x))^n$	$nf'(x)(f(x))^{n-1}$
$\cos(f(x))$	$-f'(x)\sin(f(x))$
$\sin(f(x))$	$f'(x)\cos(f(x))$
$\tan(f(x))$	$f'(x)\sec^2(f(x))$
$\sec(f(x))$	$f'(x)\sec(f(x))\tan(f(x))$
$\csc(f(x))$	$-f'(x)\csc(f(x))\cot(f(x))$
$\cot(f(x))$	$-f'(x)\csc^2(f(x))$
$e^{f(x)}$	$f'(x)e^{f(x)}$
$\ln(f(x))$	$\frac{f'(x)}{f(x)}$
$\sin^{-1}(f(x))$	$\frac{f'(x)}{\sqrt{1-f(x)^2}}$
$\cos^{-1}(f(x))$	$-\frac{f'(x)}{\sqrt{1-f(x)^2}}$
$\tan^{-1}(f(x))$	$\frac{f'(x)}{1+f(x)^2}$
$\cot^{-1}(f(x))$	$-\frac{f'(x)}{1+f(x)^2}$
$\sec^{-1}(f(x))$	$\frac{f'(x)}{\lvert f(x)\rvert\sqrt{f(x)^2-1}}$
$\csc^{-1}(f(x))$	$-\frac{f'(x)}{\lvert f(x)\rvert\sqrt{f(x)^2-1}}$

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$$
$$\frac{d}{dx}(\frac{u}{v}) = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2}$$
$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$$

Implicit Differentiation

Differentiate all terms w.r.t. x, chain rule on terms only in y e.g. $\frac{d}{dx}y^3 = 3y^2\frac{dy}{dx}$
For equations of the form $f(x, y) = 0$,

$$\frac{dy}{dx} = -\frac{\frac{d}{dx}f(x,y)}{\frac{d}{dy}f(x,y)}$$

Derivatives of Inverse Functions

For bijective function f,

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

$f'(f^{-1}(a))$ is the derivative of f evaluated at $f^{-1}(a)$

Parametric Equations

For curves defined by the equations $x = f(t), y = g(t)$

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$
$$\frac{d^2y}{dx^2} = \frac{d}{dt}(\frac{dy}{dx}) \div \frac{dx}{dt}$$

Concavity, Extremas

$f''(c) > 0 \Rightarrow$ concave upward / local minima, $f''(c) < 0 \Rightarrow$ concave downward / local maxima, $f''(c) = 0 \Rightarrow$ point of inflection
End-points are not considered to be local extremas
Critical point: not an end-point and $f'(c) = 0$ or DNE
Absolute extremum: occurs at end-point or critical point

L'Hôpital's Rule

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$$

4 Integrals

Function	Integral
$\int (ax+b)^n dx$	$\frac{(ax+b)^{n+1}}{(n+1)a} + C, (n \neq 1)$
$\int \frac{1}{ax+b} dx$	$\frac{1}{a}\ln\lvert ax+b\rvert + C$
$\int e^{ax+b} dx$	$\frac{1}{a}e^{ax+b} + C$
$\int \sin(ax+b)dx$	$-\frac{1}{a}\cos(ax+b) + C$
$\int \cos(ax+b)dx$	$\frac{1}{a}\sin(ax+b) + C$
$\int \tan(ax+b)dx$	$\frac{1}{a}\ln\lvert\sec(ax+b)\rvert + C$
$\int \sec(ax+b)dx$	$\frac{1}{a}\ln\lvert\sec(ax+b)+\tan(ax+b)\rvert + C$
$\int \csc(ax+b)dx$	$-\frac{1}{a}\ln\lvert\csc(ax+b)+\cot(ax+b)\rvert + C$
$\int \cot(ax+b)dx$	$-\frac{1}{a}\ln\lvert\csc(ax+b)\rvert + C$
$\int \sec^2(ax+b)dx$	$\frac{1}{a}\tan(ax+b) + C$
$\int \csc^2(ax+b)dx$	$-\frac{1}{a}\cot(ax+b) + C$
$\int \sec(ax+b)\tan(ax+b)dx$	$\frac{1}{a}\sec(ax+b) + C$
$\int \csc(ax+b)\cot(ax+b)dx$	$-\frac{1}{a}\csc(ax+b) + C$
$\int \frac{1}{a^2+(x+b)^2}dx$	$\frac{1}{a}\tan^{-1}(\frac{x+b}{a}) + C$
$\int \frac{1}{\sqrt{a^2-(x+b)^2}}dx$	$\sin^{-1}(\frac{x+b}{a}) + C$
$\int \frac{-1}{\sqrt{a^2-(x+b)^2}}dx$	$\cos^{-1}(\frac{x+b}{a}) + C$
$\int \frac{1}{a^2-(x+b)^2}dx$	$\frac{1}{2a}\ln\lvert\frac{x+b+a}{x+b-a}\rvert + C$
$\int \frac{1}{(x+b)^2-a^2}dx$	$\frac{1}{2a}\ln\lvert\frac{x+b-a}{x+b+a}\rvert + C$
$\int \frac{1}{\sqrt{(x+b)^2+a^2}}dx$	$\ln\lvert(x+b)+\sqrt{(x+b)^2+a^2}\rvert + C$
$\int \frac{1}{\sqrt{(x+b)^2-a^2}}dx$	$\ln\lvert(x+b)+\sqrt{(x+b)^2-a^2}\rvert + C$
$\int \sqrt{a^2-x^2}dx$	$\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$
$\int \sqrt{x^2-a^2}dx$	$\frac{x}{2}\sqrt{x^2-a^2} + \frac{a^2}{2}\ln\lvert x+\sqrt{x^2-a^2}\rvert + C$

Useful Identities

$$\cos^2 A = \frac{1}{2}(1+\cos 2A)$$
$$\sin^2 A = \frac{1}{2}(1-\cos 2A)$$
$$\sin A\cos B = \frac{1}{2}(\sin(A+B)+\sin(A-B))$$
$$\cos A\sin B = \frac{1}{2}(\sin(A+B)-\sin(A-B))$$
$$\cos A\cos B = \frac{1}{2}(\cos(A+B)+\cos(A-B))$$
$$\sin A\sin B = -\frac{1}{2}(\cos(A+B)-\cos(A-B))$$

Partial Fractions

Denominator Factors	Partial Fractions
$ax+b$	$\frac{A}{ax+b}$
$(ax+b)^2$	$\frac{A}{ax+b} + \frac{B}{(ax+b)^2}$
$ax^2+bx+c, b^2-4ac<0$	$\frac{Ax+B}{ax^2+bx+c}$

Improper fraction $\rightarrow$ proper fraction: long division

Integration by Substitution

$$\int f(g(x))g'(x)dx = \int f(u)du$$

Expression	Trigonometric Substitution
$\sqrt{a^2-(x+b)^2}$	$x+b = a\sin\theta, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
$\sqrt{a^2+(x+b)^2}$	$x+b = a\tan\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$
$\sqrt{(x+b)^2-a^2}$	$x+b = a\sec\theta, 0 < \theta < \frac{\pi}{2}$ *or* $\pi \leq \theta < \frac{3\pi}{2}$

Manipulate the expression to fit the forms

Integration by Parts

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx$$

Choice of integration:
log, inversion trigo, algebraic functions $\rightarrow$ differentiate
exponential functions $\rightarrow$ integrate
trigo functions $\rightarrow$ either

Fundamental Theorem of Calculus

$$\frac{d}{dx}\int_a^{u(x)} f(t)dt = f(u(x))u'(x)$$

$\int_{-a}^{a} f(x)dx = 0$ if $f(-x) = -f(x)$ (odd function)
$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$ if $f(-x) = f(x)$ (even function)

Improper Integrals

Type 1: integrals with infinite limits of integration

$$\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$$
$$\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx$$
$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx$$

Type 2: integrals of functions that become infinite at a point within the interval of integration

$$\int_a^b f(x)dx = \lim_{c\to a^+}\int_c^b f(x)dx$$

$f(x)$ is continuous on $(a, b]$ and is discontinuous at a

Area Between Curves

$$A = \int_a^b f(x) - g(x)dx$$

$f(x)$ is the curve above $g(x)$, if the two curves alternate between being top and bottom, split the regions into their respective integrals and intervals
For curves in terms of y, $f(y)$ is further to the right of the y-axis than $g(y)$,

$$A = \int_c^d f(y) - g(y)dy$$

Volume of Solid of Revolution

Disk method (revolve about the x-axis):

$$V = \pi\int_a^b f(x)^2dx - \pi\int_a^b g(x)^2dx$$

Use $f(y)$, $g(y)$ and differentiate w.r.t. y for revolution about the y-axis
Cylindrical shell method (use when difficult/impossible to express $y = f(x)$ as $x = f(y)$):

$$V = 2\pi\int_a^b x\lvert f(x) - g(x)\rvert dx$$

The above is for rotation about the **y-axis**, for rotation about the x-axis, use $y\lvert f(y) - g(y)\rvert dy$

Arc Length of a Curve

$$\int_a^b \sqrt{1+f'(x)^2}dx$$

5 Series

Common Infinite Series

Geometric Series

$$\sum_{n=1}^{\infty} ar^{n-1},\ (a \neq 0)$$
$$\sum_{i=1}^{n} ar^{i-1} = \frac{a(1-r^n)}{1-r}$$

Convergent to $\frac{a}{1-r}$ when $\lvert r\rvert < 1$

Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ is divergent}$$

Alternating Harmonic Series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \text{ converges to } ln2$$

p-Series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ is convergent} \iff p > 1$$

Convergence & divergence

n^{th} Term Test

$$\text{If } \sum_{n=1}^{\infty} a_n \text{ is convergent, then } \lim_{n\to\infty} a_n = 0$$

Therefore if $\lim_{n\to\infty} a_n$ does not exist or $\lim_{n\to\infty} a_n \neq 0$ then the series is divergent, inconclusive if $\lim_{n\to\infty} a_n = 0$

Integral Test

Use when integral is simple/known

$$\sum_{n=1}^{\infty} a_n \text{ is convergent} \iff \int_1^{\infty} f(x)dx \text{ is convergent}$$

Comparison Test

Use when able to establish an inequality to compare to a known series

$$\sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} b_n, \; a_n \leq b_n \text{ for all } n$$

$$\sum_{n=1}^{\infty} b_n \text{ is convergent} \rightarrow \sum_{n=1}^{\infty} a_n \text{ is convergent}$$

$$\sum_{n=1}^{\infty} a_n \text{ is divergent} \rightarrow \sum_{n=1}^{\infty} b_n \text{ is divergent}$$

Ratio Test and Root Test

Use root test on series with power n by taking n^{th} root

$$\sum_{n=1}^{\infty} a_n, \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L \textbf{ OR } \lim_{n\to\infty} \sqrt[n]{|a_n|} = L$$

$0 \leq L < 1 \rightarrow$ absolutely convergent
$L > 1 \rightarrow$ divergent
$L = 1 \rightarrow$ inconclusive

Alternating Series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n$$

If b_n is decreasing and $\lim_{n\to\infty} b_n = 0$ then the series is convergent

Power Series

Power series centered at a

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + ...$$

Exactly one of the following:

1. Converges at $x = a$ only (R = 0)
2. Converges for all x (R = ∞)
3. There exists a positive number R such that the series converges absolutely if $|x-a| < R$ and diverges if $|x-a| > R$

R is the radius of convergence, end points of the interval of convergence can converge or diverge
To compute R:

$$\lim_{n\to\infty} |\frac{c_{n+1}}{c_n}| = L \textbf{ OR } \lim_{n\to\infty} \sqrt[n]{|c_n|} = L, \; L \in \mathbb{R} \vee L = \infty, \; R = \frac{1}{L}$$

Power Series Representation

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1} + C, |x-a| < R$$

$$\int f(x)dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C, |x-a| < R$$

To find the power series representation, manipulate the expression into one of the known series to derive the summation

Taylor Series

Coefficients of a power series is given as $c_n = \frac{f^{(n)}(a)}{n!}$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Power series representation of f at a, extension of power series

Maclaurin Series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Power series representation of f at 0, special case of Taylor series

6 Vectors & Geometry of Space

Equation of a sphere with center $C(x_1, y_1, z_1)$ and radius r

$$(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2 = r^2$$

Distance formula: $|P_1P_2| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$
Length of a vector: $||v|| = \sqrt{(v_1)^2 + (v_2)^2 + (v_3)^2}$
Unit vector: $u = v/||v||$
Dot product: $a \cdot b = a_1b_1 + a_2b_2 + a_3b_3$, $a \cdot b = ||a||||b|| \cos\theta$, $a \cdot a = ||a||^2$
θ is the angle between a and b
a & b are perpendicular to each other if $a \cdot b = 0$

Projections

Component of b along a: $\text{comp}_a b = ||b|| \cos\theta = \frac{a \cdot b}{||a||}$
Vector projection of b onto a: $\text{proj}_a b = \text{comp}_a b \times \frac{a}{||a||} = \frac{a \cdot b}{||a||^2} a$

Distance from a point to a plane

Shortest distance from point $P(x_0, y_0, z_0)$ to the plane $ax + by + cz = d$

$$\frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

Cross Product

$$a \times b = (a_2b_3 - a_3b_2)i + (a_3b_1 - a_1b_3)j + (a_1b_2 = a_2b_1)k$$

$a \times b$ is perpendicular to both a and b
Area of a parallelogram: $||a \times b|| = ||a||||b|| \sin\theta$
Distance from Q to the line through P & R:

$$||\vec{PQ}|| \sin\theta = \frac{||\vec{PQ} \times \vec{PR}||}{||\vec{PR}||}$$

Lines

r_0: known point on the line, t: scalar multiple, v: direction vector of the line
Vector equation: $r = r_0 + tv$ or $\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$
Parametric equation: $x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$
If two lines in 3D space are not parallel and does not intersect, then they are skew

Planes

n: normal vector to the plane, r_0: point on the plane Vector equation: $n \cdot r = n \cdot r_0$
Linear equation: $ax + by + cz + d = 0, d = -(ax_0 + by_0 + cz_0)$
Two planes are parallel if their normal vectors are parallel
If two planes are not parallel, they intersect in a line and the angle between two planes is the acute angle between their normal vectors

7 Functions of Several Variables

Vector-valued Function

$r(t) = \langle f(t), g(t), h(t) \rangle = f(t)i + g(t)j + h(t)k$
$r'(t) = \langle f'(t), g'(t), h'(t) \rangle$

Derivative Rules

$f(t)$: differentiable scalar function, $r(t)$ & $s(t)$: differentiable vector-valued function
$\frac{d}{dt} f(t)r(t) = f'(t)r(t) + f(t)r'(t)$
$\frac{d}{dt} r(t) \cdot s(t) = r'(t) \cdot s(t) + r(t) \cdot s'(t)$
$\frac{d}{dt} r(t) \times s(t) = r'(t) \times s(t) + r(t) \times s'(t)$

Tangent Vector and Tangent Line to a Curve

$r'(t)$ is the tangent vector of $r(t)$, equation of tangent line can be found using a point on the curve and the corresponding tangent vector

Arc Length of a Space Curve

$$s = \int_a^b ||r'(t)||dt$$

Functions of Two Variables

Elliptic paraboloid symmetric about z-axis

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$

Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Partial Derivatives

Clairaut's Theorem

$f_{xy}(a,b) = f_{yx}(a,b)$, $f_{xyy}(a,b) = f_{yxy}(a,b) = f_{yyx}(a,b)$

Equation of Tangent Planes

$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

Chain Rule

One independent variable: $z = f(x,y), x = g(t), y = h(t)$

$$\frac{dz}{dt} = \frac{\delta f}{\delta x}\frac{dx}{dt} + \frac{\delta f}{\delta y}\frac{dy}{dt}$$

Two independent variables: $z = f(x,y), x = g(s,t), y = h(s,t)$

$$\frac{dz}{ds} = \frac{\delta f}{\delta x}\frac{\delta x}{\delta s} + \frac{\delta f}{\delta y}\frac{\delta y}{\delta s}$$
$$\frac{dz}{dt} = \frac{\delta f}{\delta x}\frac{\delta x}{\delta t} + \frac{\delta f}{\delta y}\frac{\delta y}{\delta t}$$

Implicit Differentiation

Two independent variables: $F(x,y,z) = 0, z = f(x,y)$

$$\frac{\delta z}{\delta x} = -\frac{F_x}{F_z}, \frac{\delta z}{\delta y} = -\frac{F_y}{F_z}$$

Approximation of Increment

Good approximation provided Δx & Δy are small
$\Delta z \approx dz = f_x(a,b)\Delta x + f_y(a,b)\Delta y$

Directional Derivative

Gradient of $f(x,y)$

$$\nabla f(x,y) = \langle f_x, f_y \rangle = f_x i + f_y j$$

Directional derivative of $f(x,y)$ in the direction of unit vector $u = \langle a, b \rangle$

$$D_u f(x,y) = \nabla f(x,y) \cdot u = f_x(x,y)a + f_y(x,y)b$$

Similarly for $f(x,y,z)$

$$D_u f(x,y,z) = \nabla f(x,y,z) \cdot u = f_x(x,y,z)a + f_y(x,y,z)b + f_z(x,y,z)c$$

Tangent Plane to Level Surface

∇f is normal to the level curve/surface f

$$\nabla f(x_0,y_0,z_0) \cdot \langle x-x_0, y-y_0, z-z_0 \rangle = 0$$

Maximum Rate of Increase/Decrease

Maximum value of $D_u f(P)$: $||\nabla f(P)||$, minimum value: $-||\nabla f(P)||$

Local Extrema

If $f(x,y)$ has a local maximum or minimum at (a,b), then
$f_x(a,b) = f_y(a,b) = 0$
(a,b) is a critical point if the above is true or one of the partial derivatives does not exist

Second Derivative Test

$$D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - (f_{xy}(a,b))^2$$

f_{yy} can be substituted as f_{xx} below

- Local maximum: $D > 0$ and $f_{xx}(a,b) > 0$
- Local minimum: $D > 0$ and $f_{xx}(a,b) < 0$
- Saddle point: $D < 0$
- No conclusion: $D = 0$

8 Double Integrals

Iterated Integral

$$V = \int\int_R f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$$

If $f(x,y)$ can be factored into $g(x)h(y)$

$$\int\int_R g(x)h(y)dA = (\int_a^b g(x)dx)(\int_c^d h(y)dy)$$

Double Integral Over a General Region

Type I domain, region between two functions of x

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx, y = g_1(x), y = g_2(x)$$

Type II domain, region between two functions of y

$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy, x = h_1(y), x = h_2(y)$$

Double Integrals in Polar Coordinates

$r^2 = x^2 + y^2, x = r\cos\theta, y = r\cos\theta$

$$\int\int_R f(x,y)dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)rdrd\theta$$

$0 \le a \le r \le b, \alpha \le \theta \le \beta$, sketch R to determine limits, use polar coordinates when region is circular

Surface Area

$$\int\int_D \sqrt{(f_x)^2 + (f_y)^2 + 1}\, dA$$

9 Ordinary Differential Equations

Separable ODE

$$\frac{dy}{dx} = f(x)g(y)$$

Separate the variables

$$\frac{1}{g(y)}dy = f(x)dx$$

Integrate both sides

$$\int \frac{1}{g(y)}dy = \int f(x)dx + C$$

Reduction to Separable Form

Radical

$$y' = g(\frac{y}{x})$$

Let $v = \frac{y}{x}$, then $y = vx$ and $y' = v + xv'$, and the equation becomes $v + xv' = g(v)$ which is separable

Linear

$$y' = f(ax + by + c)$$

Set $u = ax + by + c$

Linear First Order ODE

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Integrating factor: $I(x) = e^{\int P(x)dx}$
Multiply both sides by $I(x)$ and integrate both sides, solve for y

$$y \cdot I(x) = \int Q(x) \cdot I(x)dx$$

Bernoulli Equation

$$y' + p(x)y = q(x)y^n$$

Let $u = y^{1-n}$, then $u' = (1-n)y^{1-n}y'$, so

$$u' + (1-n)p(x)u = (1-n)q(x)$$

Which is a linear first order ODE

Improved Malthus Model of Population

$$N = \frac{N_\infty}{1 + (\frac{N_\infty}{N} - 1)e^{-Bt}}$$

$D = sN$, $N_\infty = \frac{B}{s}$ (carrying capacity), $N = \frac{B}{2s}$ (point of inflection)

Common ODE

Half-life of x

Let $x(t)$ be the amount of x at time t, $\frac{dx}{dt} = -k_x x(t) \rightarrow$
$x(t) = x_0 e^{-k_x t} \rightarrow k_x = \frac{ln2}{\text{half-life of x}}$

Newton's Law of Cooling

$$\frac{dT}{dt} = -k(T - T_{\text{amb}})$$

Newton's Second Law

$$\text{force} - \text{resistance} = m\frac{dV}{dt}$$

Mixture

$$\frac{dQ}{dt} = \text{input} - \text{output}$$