# Chapter 7

# Relations and Functions

## 7.1 Ordered pairs

The definition of a set explicitly disregards the order of the set elements, all that matters is who's in, not who's in first. However, sometimes the order is important. This leads to the notion of an *ordered pair* of two elements $x$ and $y$, denoted $(x, y)$. The crucial property is:

$$(x, y) = (u, v) \text{ if and only if } x = u \text{ and } y = v.$$

This notion can be extended naturally to define an *ordered $n$-tuple* as the ordered counterpart of a set with $n$ elements.

Give two sets $A$ and $B$, their *cartesian product* $A \times B$ is the set of all ordered pairs $(x, y)$, such that $x \in A$ and $y \in B$:

$$A \times B = \{(x, y) : x \in A, y \in B\}.$$

Here is a useful special case:

$$A^2 = A \times A = \{(x, y) : x, y \in A\}.$$

And here is a general definition: $A^1 = A$, and for $n \geq 2$,

$$A^n = \big\{(x_1, x_2, \ldots, x_n) : x_1, x_2, \ldots, x_n \in A\big\}.$$

For example, $\mathbb{R}^2$ is the familiar cartesian plane, and $\mathbb{R}^n$ is often referred to as the $n$-dimensional Euclidean space. If we omit the parentheses and the commas, $\{a, b\}^4$ is comprised of child babble and a 70s pop band:

$$\{aaaa, baba, abab, baaa, baab, aaab, aaba, abaa, abba, bbaa, bbba, bbbb, aabb, abbb, babb, bbab\}.$$

**Proposition 7.1.1.** $(A \cup B) \times C = (A \times C) \cup (B \times C)$

*Proof.* Recall that for two sets $X$ and $Y$, $X = Y$ if and only if $X \subseteq Y$ and $Y \subseteq X$.

Consider any element $(u, v) \in (A \cup B) \times C$. By definition, $u \in A \cup B$ and $v \in C$. Thus, $u \in A$ or $u \in B$. If $u \in A$ then $(u, v) \in A \times C$ and if $u \in B$ then $(u, v) \in B \times C$.

Thus $(u, v)$ is in $A \times C$ or in $B \times C$, and $(u, v) \in (A \times C) \cup (B \times C)$. This proves that $(A \cup B) \times C \subseteq (A \times C) \cup (B \times C)$.

Now consider any element $(u, v) \in (A \times C) \cup (B \times C)$. This implies that $(u, v) \in A \times C$ or $(u, v) \in B \times C$. In the first case $u \in A$ and $v \in C$ and in the second case $u \in B$ and $v \in C$. Thus $u \in A \cup B$ and $v \in C$, which implies $(u, v) \in (A \cup B) \times C$. □

## 7.2 Relations

Given a set $A$, a *relation on* $A$ is some property that is either true or false for any ordered pair $(x, y) \in A^2$. For example, "greater than" is a relation on $\mathbb{Z}$, denoted by $>$. It is true for the pair $(3, 2)$, but false for the pairs $(2, 2)$ and $(2, 3)$. In more generality,

**Definition 7.2.1.** *Given sets $A$ and $B$, a* relation between $A$ and $B$ *is a subset of $A \times B$.*

By this definition, a relation $R$ is simply a specification of which pairs are related by $R$, that is, which pairs the relation $R$ is true for. For the relation $>$ on the set $\{1, 2, 3\}$,

$$> = \{(2, 1), (3, 1), (3, 2)\}.$$

This notation might look weird because we do not often regard the symbol "$>$" as a meaningful entity in itself. It is, at least from the vantage point of the foundations of mathematics: This symbol is a particular relation.

The common usage of the symbol "$>$" (as in $3 > 2$) is an instance of a useful notational convention: For a relation $R$, $(a, b) \in R$ can also be specified as $aRb$. Thus, in the above example, $(2, 1) \in >$ can be written as $2 > 1$. How convenient!

Common mathematical relations that will concern us include $<$, $>$, $\leq$, $\geq$, $=$, $\neq$, $|$, $\equiv_n$, $\subset$, $\subseteq$, etc. For example, the relation $=$ on the set $\mathbb{Z}$ is precisely the set $\{(n, n) : n \in \mathbb{Z}\}$ and the relation $\leq$ on $\mathbb{R}$ is the set $\{(x, x + |y|) : x, y \in \mathbb{R}\}$.

The concept of a relation is as general as the concept of a set, and is not limited to strictly mathematical settings. For instance, we can define the relation *likes* between the set {Anna,Britney,Caitlyn} and the set {Austin,Brian,Carlos}, such that

likes = {(Britney,Austin),(Caitlyn,Austin),(Britney, Carlos),(Anna,Austin),(Caitlyn,Brian)}.

In this setting we can write *Britney likes Austin*.

## 7.3 Kinds of relations

A relation $R$ on a set $A$ is called

- *reflexive* if for all $a \in A$, $aRa$.
- *symmetric* if for all $a, b \in A$, $aRb$ implies $bRa$.
- *antisymmetric* if for all $a, b \in A$, $aRb$ and $bRa$ implies $a = b$.
- *transitive* if for all $a, b, c \in A$, $aRb$ and $bRc$ implies $aRc$.

**Equivalence relations.** A relation that is reflexive, symmetric, and transitive is called an *equivalence relation*. Clearly, the common relation $=$ on the set $\mathbb{R}$, say, is an equivalence relation. Also, we have seen earlier that the congruence relation $\equiv_n$ on the set $\mathbb{Z}$ is reflexive, symmetric, and transitive, thus it is also an equivalence relation. The similarity relation on the set of triangles in the plane is another example.

Equivalence relations are special in that they naturally partition the underlying set into *equivalence classes*. For example, the relation $\equiv_2$ partitions the integers into even and odd ones. These are, respectively, the integers that are related (by $\equiv_2$) to 0, and the ones related to 1. Let's formalize these concepts.

**Definition 7.3.1.** *A* partition *of a set $A$ is a set $\mathcal{X} \subseteq 2^A \setminus \{\emptyset\}$, such that*

*(a) Each $a \in A$ belongs to some $S \in \mathcal{X}$.*

*(b) If $S, T \in \mathcal{X}$, either $S = T$ or $S \cap T = \emptyset$.*

Stated differently, this definition says that the set $A$ is the union of the members of $\mathcal{X}$, and these members are disjoint. Now, given an equivalence relation $R$ on $A$, the *equivalence class* of $a \in A$ is defined as

$$R[a] = \{b \in A : aRb\}.$$

**Theorem 7.3.2.** *Let $R$ be an equivalence relation on a set $A$. Then $\{R[a] : a \in A\}$ is a partition of $A$.*

*Proof.* Consider an equivalence relation $R$ on $A$. Due to reflexivity, every element $a \in A$ belongs to $R[a]$, which implies (a). Now, consider two equivalence classes $R[a]$ and $R[b]$. If $aRb$, then for any $c \in R[a]$, by transitivity and symmetry, $bRc$ and $c \in R[b]$. This shows $R[a] \subseteq R[b]$. We can symmetrically argue that $R[b] \subseteq R[a]$, which together implies $R[a] = R[b]$.

Otherwise, if $a\not{R}b$ then consider some $c \in R[a]$. If $c \in R[b]$ then $aRc$ and $bRc$, which imply, by transitivity and reflexivity, $aRb$, leading to a contradiction. Thus no element of $R[a]$ belongs to $R[b]$ and $R[a] \cap R[b] = \emptyset$. This shows (b) and concludes the theorem. $\square$

**Order relations.** A relation that is reflexive, antisymmetric, and transitive is called a *partial order*. The relations $\leq$, $\geq$, and $\mid$ on the set $\mathbb{Z}$, as well as the relation $\subseteq$ on the powerset $2^A$ of any set $A$, are familiar partial orders. Note that a pair of elements can be *incomparable* with respect to a partial order. For example, $\mid$ is a partial order on $\mathbb{Z}$, but $2 \nmid 3$ and $3 \nmid 2$. A set $A$ with a partial order on $A$ is called a *partially ordered set*, or, more commonly, a *poset*.

A relation $R$ on a set $A$ is a *total order* if it is a partial order and satisfies the following additional condition:

- For all $a, b \in A$, either $aRb$ or $bRa$ (or both).

For example, the relations $\geq$ and $\leq$ are total orders on $\mathbb{R}$, but $\mid$ is not a total order on $\mathbb{Z}$. Finally, a *strict order* on $A$ is a relation $R$ that satisfies the following two conditions:

- For all $a, b, c \in A$, $aRb$ and $bRc$ implies $aRc$. (Transitivity.)
- Given $a, b \in A$, exactly one of the following holds (and not the other two): $aRb$, $bRa$, $a = b$.

The familiar $<$ and $>$ relations (on $\mathbb{R}$, say) are examples of strict orders.

## 7.4 Creating relations

There are a few ways to define new relations from existing ones, and we describe two important such ways below.

**Restrictions of relations.** Here is one notion that is sometimes useful: Given a relation $R$ on a set $A$, and a subset $S \subseteq A$, we can use $R$ to define a relation on $S$ called the *restriction* of $R$ to $S$. Denoted by $R|_S$, it is defined as

$$R|_S = \big\{(a, b) \in R : a, b \in S\big\}.$$

**Compositions of relations.** For three sets $A, B, C$, consider a relation $R$ between $A$ and $B$, and a relation $S$ between $B$ and $C$. The *composition of $R$ and $S$* is a relation $T$ between $A$ and $C$, defined as follows: $aTc$ if and only if there exists some $b \in B$, such that $aRb$ and $bSc$. The composition of $R$ and $S$ is commonly denoted by $R \circ S$.

Note that by this definition, the composition of relations on the same set $A$ is always well-defined. In particular, given a relation $R$ on $A$ we can recursively define $R^1 = R$ and $R^n = R^{n-1} \circ R$ for all $n \geq 2$. Now consider the infinite union

$$T = \bigcup_{i \in \mathbb{N}^+} R^i.$$

This relation $T$ is called the *transitive closure* of $R$.

**Proposition 7.4.1.** *Important properties of transitive closure:*

*(a) $T$ is transitive.*

*(b) $T$ is the smallest transitive relation that contains $R$. (That is, if $U$ is a transitive relation on $A$ and $R \subseteq U$, then $T \subseteq U$.)*

*(c) If $|A| = n$ then*

$$T = \bigcup_{i=1}^{n} R^i.$$

## 7.5 Functions

The general concept of a function in mathematics is defined very similarly to relations. In fact, as far as the definitions go, functions *are* relations, of a special type:

**Definition 7.5.1.** *Given two sets $A$ and $B$, a function $f : A \to B$ is a subset of $A \times B$ such that*

*(a) If $x \in A$, there exists $y \in B$ such that $(x, y) \in f$.*

*(b) If $(x, y) \in f$ and $(x, z) \in f$ then $y = z$.*

A function is sometimes called a *map* or *mapping*. The set $A$ in the above definition is the *domain* and $B$ is the *codomain* of $f$.

A function $f : A \to B$ is effectively a special kind of relation between $A$ and $B$, which relates every $x \in A$ to *exactly one* element of $B$. That element is denoted by $f(x)$.

If the above definition is followed rigidly, particular functions should be defined by specifying all the pairs $(x, f(x))$. This is often cumbersome and unnecessary, and we will mostly continue describing a function from $A$ to $B$ as we did before: as a rule for picking an element $f(x) \in B$ for every element $x \in A$. As in, "Consider a function $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = x^2$ for all $x \in \mathbb{R}$."

**Kinds of Functions.** For a function $f : A \to B$, the set $f(A) = \{f(x) : x \in A\}$ is called the *range* of $f$. The range is a subset of the codomain but may be different from it. If $f(A) = B$ then we say that $f$ is *onto*. More precisely, a function $f : A \to B$ is a *surjection* (or *surjective*), or *onto* if each element of $B$ is of the form $f(x)$ for at least one $x \in A$.

Today is the day of weird names, so: A function $f : A \to B$ is an *injection* (or *injective*), or *one-to-one* if for all $x, y \in A$, $f(x) = f(y)$ implies $x = y$. Put differently, $f : A \to B$ is one-to-one if each element of $B$ is of the form $f(x)$ for at most one $x \in A$.

As if this wasn't enough: A function $f : A \to B$ is a *bijection* (or *bijective*), or a *one-to-one correspondence* if it is both one-to-one and onto. Alternatively, $f : A \to B$ is a bijection if each element of $B$ is of the form $f(x)$ for exactly one $x \in A$.

**Compositions and Inverse Functions.** Given two functions $f : A \to B$ and $g : B \to C$ we can define a new function $g \circ f : A \to C$ by $(g \circ f)(x) = g(f(x))$ for all $x \in A$.

One useful function that can be defined for any set $A$ is the *identity function* $i_A : A \to A$, defined by $i_A(x) = x$ for all $x \in A$. We can use identity functions to define *inverse functions*. Specifically, if $f : A \to B$ is a bijection, then its inverse $f^{-1} : B \to A$ is defined so that $f^{-1} \circ f = i_A$ and $f \circ f^{-1} = i_B$. Of course, we haven't shown that $f^{-1}$ even exists or that it is unique, but these properties do hold, assuming that $f : A \to B$ is a bijection. (This assumption is necessary.)

Another result that is sometimes used is the following: If $f : A \to B$ and $g : B \to C$ are bijections then $g \circ f : A \to C$ is a bijection, and

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$

We omit the proof.

**Bijections and cardinality.** Bijections allow us to rigorously define when two sets are of the same cardinality:

**Definition 7.5.2.** *Two sets $A$ and $B$ have the same number of elements if and only if there exists a bijection $f : A \to B$.*

# Chapter 10

# Binomial Coefficients

## 10.1 Basic properties

Recall that $\binom{n}{k}$ is the number of $k$-element subsets of an $n$-element set, and

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{\prod_{i=0}^{k-1}(n-i)}{k!}.$$

The quantities $\binom{n}{k}$ are called *binomial coefficients* because of their role in the *Binomial Theorem*, apparently known to the 11th century Persian scholar Omar Khayyam. Before we state and prove the theorem let us consider some important identities that involve binomial coefficients. One that follows immediately from the algebraic definition is

$$\binom{n}{k} = \binom{n}{n-k}.$$

This also has a nice combinatorial interpretation: Choosing a $k$-element subset $B$ from an $n$-element set uniquely identifies the complement $A \setminus B$ of $B$ in $A$, which is an $(n-k)$-subset of $A$. This defines a bijection between $k$-element and $(n-k)$-element subsets of $A$, which implies the identity.

Another relation between binomial coefficients is called *Pascal's rule*, although it was known centuries before Pascal's time in the Middle East and India:

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}.$$

This can be easily proved algebraically:

$$
\begin{aligned}
\binom{n}{k-1}+\binom{n}{k} &= \frac{n!}{(k-1)!(n+1-k)!}+\frac{n!}{k!(n-k)!}\\
&= \frac{n!k}{k!(n+1-k)!}+\frac{n!(n+1-k)}{k!(n+1-k)!}\\
&= \frac{n!k+n!(n+1-k)}{k!(n+1-k)!}\\
&= \frac{(n+1)!}{k!(n+1-k)!}\\
&= \binom{n+1}{k}.
\end{aligned}
$$

Pascal's rule also has a combinatorial interpretation: $\binom{n+1}{k}$ is the number of $k$-element subsets of an $n$-element set $A$. Fix an element $a \in A$. A subset of $A$ either contains $a$ or it doesn't. $k$-element subsets of $A$ that do not contain $a$ are in fact $k$-element subsets of $A \setminus \{a\}$ and their number is $\binom{n}{k}$. $k$-element subsets of $A$ that do contain $a$ bijectively correspond to $(k-1)$-element subsets of $A \setminus \{a\}$, the number of which is $\binom{n}{k-1}$. The identity follows.

Another illuminating identity is the *Vandermonde convolution*:

$$\binom{m+n}{l}=\sum_{k=0}^{l}\binom{m}{k}\binom{n}{l-k}.$$

We only give a combinatorial argument for this one. We are counting the number of ways to choose an $l$-element subset of an $(m+n)$-element set $A$. Fix an $m$-element subset $B \subseteq A$. Any $l$-element subset $S$ of $A$ has $k$ elements from $B$ and $l-k$ elements from $A \setminus B$, for some $0 \leq k \leq l$. For a particular value of $k$, the number of $k$-element subsets of $B$ that can be part of $S$ is $\binom{m}{k}$ and the number of $(l-k)$-element subsets of $A \setminus B$ is $\binom{n}{l-k}$. We can now use the sum principle to sum over the possible values of $k$ and obtain the identity. An interesting special case is

$$\sum_{k=0}^{n}\binom{n}{k}^2=\binom{2n}{n}.$$

It follows from the Vandermonde convolution by taking $l=m=n$ and remembering that $\binom{n}{k}=\binom{n}{n-k}$.

## 10.2 Binomial theorem

**Theorem 10.2.1.** *For $n \in \mathbb{N}$ and $x, y \in \mathbb{R}$,*

$$(x+y)^n=\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}.$$

*Proof.* By induction on $n$. When $n = 0$ both sides evaluate to 1. Assume the claim holds for $n = m$ and consider the case $n = m + 1$.

$$
\begin{aligned}
(x+y)^{m+1} &= (x+y)\cdot(x+y)^m && (10.1)\\
&= (x+y)\cdot\sum_{k=0}^{m}\binom{m}{k}x^k y^{m-k} && (10.2)\\
&= x\cdot\sum_{k=0}^{m}\binom{m}{k}x^k y^{m-k} + y\cdot\sum_{k=0}^{m}\binom{m}{k}x^k y^{m-k} && (10.3)\\
&= \sum_{k=0}^{m}\binom{m}{k}x^{k+1} y^{m-k} + \sum_{k=0}^{m}\binom{m}{k}x^k y^{m+1-k} && (10.4)\\
&= \sum_{k=1}^{m+1}\binom{m}{k-1}x^{k} y^{m+1-k} + \sum_{k=0}^{m}\binom{m}{k}x^k y^{m+1-k} && (10.5)\\
&= \left(x^{m+1} + \sum_{k=1}^{m}\binom{m}{k-1}x^{k} y^{m+1-k}\right) + \left(y^{m+1} + \sum_{k=1}^{m}\binom{m}{k}x^k y^{m+1-k}\right) && (10.6)\\
&= x^{m+1} + y^{m+1} + \sum_{k=1}^{m}\left(\binom{m}{k-1} + \binom{m}{k}\right)x^k y^{m+1-k} && (10.7)\\
&= x^{m+1} + y^{m+1} + \sum_{k=1}^{m}\binom{m+1}{k}x^k y^{m+1-k} && (10.8)\\
&= \sum_{k=0}^{m+1}\binom{m+1}{k}x^k y^{m+1-k}. && (10.9)
\end{aligned}
$$

Here (5) follows from (4) by noting that

$$\sum_{k=0}^{m} f(k) = \sum_{k=1}^{m+1} f(k-1)$$

and (8) follows from (7) by Pascal's rule. The other steps are simple algebraic manipulation. This completes the proof by induction. $\square$

The binomial theorem can be used to immediately derive an identity we have seen before: By substituting $x = y = 1$ into the theorem we get

$$\sum_{k=0}^{n}\binom{n}{k} = 2^n.$$

Here is another interesting calculation: Putting $x = -1$ and $y = 1$ yields

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0.$$

This implies

$$\sum_{k \text{ odd}} \binom{n}{k} = \sum_{k \text{ even}} \binom{n}{k} = 2^{n-1}.$$

This means that the number of odd-size subsets of an $n$-element set $A$ is the same as the number of even-size subsets, and equals $2^{n-1}$. This can be proved by a combinatorial argument as follows: Fix an element $a \in A$ and note that the number of subsets of $A \setminus \{a\}$ is $2^{n-1}$. There is a bijective map between subsets of $A \setminus \{a\}$ and odd-size subsets of $A$, as follows: Map an odd-sized subset of $A \setminus \{a\}$ to itself, and map an even-sized subset $B \subseteq A \setminus \{a\}$ to $B \cup \{a\}$. Observe that this is a bijection and conclude that the number of odd-sized subsets of $A$ is $2^{n-1}$. Even-size subsets can be treated similarly, or by noting that their number is $2^n$ minus the number of odd-size ones.

Una **progresión geométrica** es una sucesión donde cada término se obtiene multiplicando un determinado número fijo al término anterior. Ese número fijo por el que vamos multiplicando para obtener el término siguiente se llama **razón** y se representa con la letra **r**.

Aquí tienes un ejemplo de progresión geométrica:

$$1,\ 3,\ 9,\ 27,\ 81, \ldots$$

$$a_n = a_1 . r^{n-1}$$

Vamos a ver un ejemplo:

Vamos a calcular el término general de la siguiente progresión geométrica:

$$1,\ 3,\ 9,\ 27,\ 81, \ldots$$

donde el primer término es 1 y la razón es 3:

$$a_1 = 1 \quad r = 3$$

Aplicamos la fórmula anterior:

$$a_n = a_1 . r^{n-1}$$

Sustituimos a1 y r por su valores:

$$a_n = 1.3^{n-1}$$

Una vez conocemos el término general, podemos calcular directamente cualquier término de la progresión.

Por ejemplo, vamos a obtener el valor del término que se encuentra en la posición 10. Para ello, sólo tenemos que sustituir n por 10 en la fórmula del termino general y calcular:

$$a_{10} = 1.3^{10-1} = 1.3^9 = 19683$$

$$a_n = a_k . r^{n-k}$$

donde k es la posición del término que conocemos.

Por ejemplo, en la progresión anterior:

$$1,\ 3,\ 9,\ 27,\ 81, \ldots$$

Vamos a suponer que sólo conocemos el cuarto termino y la razón:

$$a_4 = 27 \quad r = 3$$

Vamos a calcular el término general partiendo de esta fórmula:

$$a_n = a_k . r^{(n-k)}$$

Sustituimos k por 4, que es la posición que conocemos y r por 3:

$$a_n = a_4 . 3^{(n-4)}$$

Y finalmente sustituimos a4 por su valor:

$$a_n = 27 . 3^{(n-4)}$$

Puede parecer que el término general obtenido es distinto al del apartado anterior siendo la misma progresión, lo cual no es posible, ya que tiene que ser el mismo, pero realmente es el mismo término general.

Puedes comprobarlo obteniendo cualquier término con ambas fórmulas o también si aplicamos las propiedades de las potencias y operamos llegamos a la misma expresión del apartado anterior, es decir, expresando el 27 como 3 al cubo y luego multiplicando las potencias manteniendo la base y sumando los exponentes:

$$a_n = 3^3 . 3^{(n-4)} = 3^{3+n-4} = 3^{n-1}$$

### Cómo calcular la razón si conocemos dos o más términos seguidos

Si conocemos dos o más términos seguidos de una progresión geométrica, para calcular la razón, tan solo tenemos que dividir el término de una posición entre el término de la posición anterior:

$$r = \frac{a_n}{a_{n-1}}$$

Por ejemplo, en la progresión:

$$1, \ 3, \ 9, \ 27, \ 81, \ldots$$

Para calcular la razón dividimos el segundo término entre el primero:

$$r = \frac{a_2}{a_1} =$$

$$= \frac{3}{1} = 3$$

## Fórmula para calcular la suma de los n primeros términos de una progresión geométrica

Es posible sumar directamente los n primeros términos de una progresión geométrica sin necesidad de realizar la suma manualmente, cosa que puede resultar muy tediosa si la suma de términos es muy grande.

Esta suma se realiza de con la siguiente fórmula:

$$S_n = a_1 . \frac{r^n - 1}{r - 1}$$

donde n es la posición del término hasta el que queremos sumar, a1 es el primer término y r es la razón.

Por ejemplo, vamos a sumar los 7 primeros términos de la siguiente progresión geométrica:

$$1,\ 3,\ 9,\ 27,\ 81, \ldots$$

Aplicamos la fórmula anterior:

$$S_n = a_1 . \frac{r^n - 1}{r - 1}$$

donde en este caso, n=7, a1=1 y r=3:

$$S_7 = 1 . \frac{3^7 - 1}{3 - 1} = \frac{3^7 - 1}{2} = 1093$$

## Fórmula para calcular la suma de todos los términos de una progresión geométrica decreciente

Si la progresión geométrica es decreciente, es posible calcular la suma de todos los términos de la progresión, siempre y cuando el valor de la razón esté comprendido entre -1 y 1:

$$-1 < r < 1$$

En este caso, la suma de todos los términos de la progresión geométrica es:

$$S = \frac{a_1}{1 - r}$$

donde a1 es el primer término de la progresión y r es la razón,

Vamos a ver un ejemplo: Calcular la suma de todos los términos de la siguiente progresión

$$64,\ 32,\ 16,\ 8,\ 4, \dots$$

En primer lugar obtenemos la razón dividiendo el segundo término entre el primero:

$$r = \frac{32}{64} = 0{,}5$$

El valor de r está entre -1 y 1 por lo que se puede realizar la suma.

Sustituimos a1 y r por sus valores y operamos:

$$S = \frac{64}{1 - 0{,}5} = \frac{64}{0{,}5} = 128$$

Todos los términos de la progresión suman 128.

**Cómo calcular la razón si conocemos dos términos alternos**

¿Cómo calculamos la razón de una progresión geométrica si no conocemos dos terminos seguidos, si no que conocemos dos términos que estan alternos?

Vamos a verlo con un ejemplo.

Supongamos que solo conocemos el segundo y el cuarto término de la progresión anterior:

$$a_2 = 3 \qquad a_4 = 27$$

En este caso, para calcular la razón partimos de la fórmula del término general cuando no conocemos el primer término:

$$a_n = a_k . r^{(n-k)}$$

Donde siempre n debe ser mayor que k:

$$n > k$$

En nuestro caso:

$$n = 4 \qquad k = 2$$

Sustituimos n y k en la fórmula:

$$a_4 = a_2 . r^{(4-2)}$$

Y ahora sustituimos los valores de a2 y a4:

$$27 = 3 . r^2$$

Me queda una ecuación en la que tengo que despejar r.

Primero paso el 3 dividiendo al miembro contrario:

$$r^2 = \frac{27}{3}$$

Opero en el segundo miembro:

$$r^2 = 9$$

Y finalmente paso el cuadrado como raíz al miembro contrario y opero:

$$r = \sqrt{9} = 3$$

**Mathematical induction**, es una técnica para probar resultados o establecer declaraciones para números naturales. Esta parte ilustra el método a través de una variedad de ejemplos.

# Definición

**Mathematical Induction** es una técnica matemática que se utiliza para demostrar que un enunciado, una fórmula o un teorema es verdadero para cada número natural.

La técnica implica dos pasos para probar una declaración, como se indica a continuación:

**Step 1(Base step)** - Prueba que una afirmación es verdadera para el valor inicial.

**Step 2(Inductive step)**- Demuestra que si el enunciado es verdadero para la enésima iteración (o el número $n$ ), entonces también lo es para $(n + 1)$ *la* iteración (o el número $n + 1$ ).

# Cómo hacerlo

**Step 1**- Considere un valor inicial para el cual la afirmación es verdadera. Debe demostrarse que la afirmación es verdadera para n = valor inicial.

**Step 2**- Suponga que la afirmación es verdadera para cualquier valor de $n = k$ . Luego demuestre que el enunciado es verdadero para $n = k + 1$ . De hecho, dividimos $n = k + 1$ en dos partes, una parte es $n = k$ (que ya está probada) y tratamos de probar la otra parte.

# Problema 1

$3^n - 1$ es un múltiplo de 2 para n = 1, 2, ...

**Solution**

**Step 1** - Para $n = 1, 3^1 - 1 = 3 - 1 = 2$ que es múltiplo de 2

**Step 2** - Supongamos que $3^n - 1$ es cierto para $n = k$, por lo tanto, $3^k - 1$ es cierto (es una suposición)

Tenemos que demostrar que $3^{k+1} - 1$ también es múltiplo de 2

$3^{k+1} - 1 = 3\, times 3^k - 1 = (2\, times 3^k) + (3^k - 1)$

La primera parte $(2\, times 3k)$ seguramente será un múltiplo de 2 y la segunda parte $(3k - 1)$ también es cierta como nuestra suposición anterior.

Por tanto, $3^{k+1} - 1$ es un múltiplo de 2.

Entonces, se demuestra que $3^n - 1$ es un múltiplo de 2.

# Problema 2

$1 + 3 + 5 + \ldots + (2n - 1) = n^2$ para $n = 1, 2,\ dots$

**Solution**

**Step 1** - Para $n = 1, 1 = 1^2$, Por lo tanto, se cumple el paso 1.

**Step 2** - Supongamos que la afirmación es verdadera para $n = k$.

Por lo tanto, $1 + 3 + 5 +\ dots + (2k - 1) = k^2$ es cierto (es una suposición)

Tenemos que demostrar que $1 + 3 + 5 + \ldots + (2(k + 1) - 1) = (k + 1)^2$ también se cumple

$1 + 3 + 5 +\ puntos + (2(k + 1) - 1)$

$= 1 + 3 + 5 +\ puntos + (2k + 2 - 1)$

$= 1 + 3 + 5 +\ puntos + (2k + 1)$

$= 1 + 3 + 5 +\ puntos + (2k - 1) + (2k + 1)$

$= k^2 + (2k + 1)$

$= (k + 1)^2$

Entonces, $1 + 3 + 5 +\ dots + (2(k + 1) - 1) = (k + 1)^2$ hold que satisface el paso 2.

Por tanto, se demuestra $1 + 3 + 5 +\ dots + (2n - 1) = n^2$

# Problema 3

Demuestre que $(ab)^n = a^n b^n$ es cierto para cada número natural $n$

**Solution**

**Step 1** - Para $n = 1, (ab)^1 = a^1 b^1 = ab$. Por lo tanto, se cumple el paso 1.

**Step 2** - Supongamos que el enunciado es verdadero para $n = k$, por lo tanto, $(ab)^k = a^k b^k$ es verdadero (es una suposición).

Tenemos que demostrar que $(ab)^{k+1} = a^{k+1} b^{k+1}$ también se cumple

Dado, $(ab)^k = a^k b^k$

O $(ab)^k (ab) = (a^k b^k)(ab)$ [Multiplicar ambos lados por 'ab']

O $(ab)^{k+1} = (aa^k)(bb^k)$

O $(ab)^{k+1} = (a^{k+1} b^{k+1})$

Por tanto, se prueba el paso 2.

Entonces, $(ab)^n = a^n b^n$ es cierto para cada número natural n.

# Definición

Una relación de recurrencia es una ecuación que define de forma recursiva una secuencia donde el siguiente término es una función de los términos anteriores (Expresando $F_n$ como una combinación de $F_i$ con $i < n$).

**Example** - Serie de Fibonacci - $F_n = F_{n-1} + F_{n-2}$, Torre de Hanoi - $F_n = 2F_{n-1} + 1$

# Relaciones de recurrencia lineal

Una ecuación de recurrencia lineal de grado k u orden k es una ecuación de recurrencia que tiene el formato $x_n = A_1 x_{n-1} + A_2 x_{n-1} + A_3 x_{n-1} + \dots A_k x_{nk}$ ($A_n$ es una constante y $A_k \neq 0$) en una secuencia de números como polinomio de primer grado.

Estos son algunos ejemplos de ecuaciones de recurrencia lineal:

| Relaciones de recurrencia | Valores iniciales | Soluciones |
|---|---|---|
| $F_{norte} = F_{norte-1} + F_{norte-2}$ | $a_1 = a_2 = 1$ | Número de Fibonacci |
| $F_{norte} = F_{norte-1} + F_{norte-2}$ | una $_1 = 1$, una $_2 = 3$ | Número de Lucas |
| $F_{norte} = F_{norte-2} + F_{norte-3}$ | una $_1 =$ una $_2 =$ una $_3 = 1$ | Secuencia de Padovan |
| $F_n = 2F_{n-1} + F_{n-2}$ | una $_1 = 0$, una $_2 = 1$ | Número de Pell |

## Permutations

The calculator provided computes one of the most typical concepts of permutations where arrangements of a fixed number of elements **r**, are taken from a given set **n**. Essentially this can be referred to as **r-permutations of n or partial permutations**, denoted as $_nP_r$, $^nP_r$, $P_{(n,r)}$, or $P(n,r)$ among others. In the case of permutations without replacement, all possible ways that elements in a set can be listed in a particular order are considered, but the number of choices reduces each time an element is chosen, rather than a case such as the "combination" lock, where a value can occur multiple times, such as 3-3-3. For example, in trying to determine the number of ways that a team captain and goalkeeper of a soccer team can be picked from a team consisting of 11 members, the team captain and the goalkeeper cannot be the same person, and once chosen, must be removed from the set. The letters **A** through **K** will represent the 11 different members of the team:

A B C D E F G H I J K   11 members; A is chosen as captain

B C D E F G H I J K   10 members; B is chosen as keeper

As can be seen, the first choice was for **A** to be captain out of the 11 initial members, but since **A** cannot be the team captain as well as the goalkeeper, **A** was removed from the set before the second choice of the goalkeeper **B** could be made. The total possibilities if every single member of the team's position were specified would be 11 × 10 × 9 × 8 × 7 × ... × 2 × 1, or 11 factorial, written as 11!. However, since only the team captain and goalkeeper being chosen was important in this case, only the first two choices, 11 × 10 = 110 are relevant. As such, the equation for calculating permutations removes the rest of the elements, 9 × 8 × 7 × ... × 2 × 1, or 9!. Thus, the generalized equation for a permutation can be written as:

$$_nP_r = \frac{n!}{(n-r)!}$$

Or in this case specifically:

$$_{11}P_2 = \frac{11!}{(11-2)!} = \frac{11!}{9!} = 11 \times 10 = 110$$

Again, the calculator provided does not calculate permutations with replacement, but for the curious, the equation is provided below:

$$_nP_r = n^r$$

## Combinations

Combinations are related to permutations in that they are essentially permutations where all the redundancies are removed (as will be described below), since order in a combination is not important. Combinations, like permutations, are denoted in various ways, including $_nC_r$, $^nC_r$, $C_{(n,r)}$, or **C(n,r)**, or most commonly as simply

$\binom{n}{r}$. As with permutations, the calculator provided only considers the case of combinations without replacement, and the case of combinations with replacement will not be discussed. Using the example of a soccer team again, find the number of ways to choose 2 strikers from a team of 11. Unlike the case given in the permutation example, where the captain was chosen first, then the goalkeeper, the order in which the strikers are chosen does not matter, since they will both be strikers. Referring again to the soccer team as the letters **A** through **K**, it does not matter whether **A** and then **B** or **B** and then **A** are chosen to be strikers in those respective orders, only that they are chosen. The possible number of arrangements for all **n** people, is simply **n!**, as described in the permutations section. To determine the number of combinations, it is necessary to remove the redundancies from the total number of permutations (110 from the previous example in the permutations section) by dividing the redundancies, which in this case is 2!. Again, this is because order no longer matters, so the permutation equation needs to be reduced by the number of ways the players can be chosen, **A** then **B** or **B** then **A**, 2, or 2!. This yields the generalized equation for a combination as that for a permutation divided by the number of redundancies, and is typically known as the binomial coefficient:

$$_nC_r = \frac{n!}{r! \times (n - r)!}$$

Or in this case specifically:

$$_{11}C_2 = \frac{11!}{2! \times (11 - 2)!} = \frac{11!}{2! \times 9!} = 55$$

It makes sense that there are fewer choices for a combination than a permutation, since the redundancies are being removed. Again for the curious, the equation for combinations with replacement is provided below:

$$_nC_r = \frac{(r + n - 1)!}{r! \times (n - 1)!}$$

$$Probability\ of\ occurence\ of\ an\ event = \frac{Total\ number\ of\ favourable\ outcome}{Total\ number\ of\ Outcomes}$$

As the occurrence of any event varies between 0% and 100%, the probability varies between 0 and 1.

## Steps to find the probability

Step 1 – Calculate all possible outcomes of the experiment.

Step 2 – Calculate the number of favorable outcomes of the experiment.

Step 3 – Apply the corresponding probability formula.

## Tossing a Coin

If a coin is tossed, there are two possible outcomes – Heads $(H)$ or Tails $(T)$

So, Total number of outcomes = 2

Hence, the probability of getting a Head $(H)$ on top is 1/2 and the probability of getting a Tails $(T)$ on top is 1/2

## Throwing a Dice

When a dice is thrown, six possible outcomes can be on the top – $1, 2, 3, 4, 5, 6$ .

The probability of any one of the numbers is 1/6

The probability of getting even numbers is 3/6 = 1/2

The probability of getting odd numbers is 3/6 = 1/2

## Taking Cards From a Deck

From a deck of 52 cards, if one card is picked find the probability of an ace being drawn and also find the probability of a diamond being drawn.

Total number of possible outcomes – 52

Outcomes of being an ace – 4

Probability of being an ace = 4/52 = 1/13

Probability of being a diamond = 13/52 = 1/4

# Probability Axioms

- The probability of an event always varies from 0 to 1. $[0 \leq P(x) \leq 1]$
- For an impossible event the probability is 0 and for a certain event the probability is 1.
- If the occurrence of one event is not influenced by another event, they are called mutually exclusive or disjoint.

If $A_1, A_2 \ldots . A_n$ are mutually exclusive/disjoint events, then

$P(A_i \cap A_j) = \emptyset$ for $i \neq j$ and

$$P(A_1 \cup A_2 \cup \ldots . A_n) = P(A_1) + P(A_2) + \ldots . . P(A_n)$$

# Properties of Probability

- If there are two events $x$ and $\overline{x}$ which are complementary, then the probability of the complementary event is –

$$p(\overline{x}) = 1 - p(x)$$

- For two non-disjoint events A and B, the probability of the union of two events –

$$P(A \cup B) = P(A) + P(B)$$

- If an event A is a subset of another event B (i.e. $A \subset B$ ), then the probability of A is less than or equal to the probability of B. Hence, $A \subset B$ implies $P(A) \leq p(B)$

# Conditional Probability

The conditional probability of an event B is the probability that the event will occur given an event A has already occurred. This is written as $P(B|A)$ .

Mathematically – $P(B|A) = P(A \cap B)/P(A)$

If event A and B are mutually exclusive, then the conditional probability of event B after the event A will be the probability of event B that is $P(B)$ .

**Problem 1**

In a country 50% of all teenagers own a cycle and 30% of all teenagers own a bike and cycle. What is the probability that a teenager owns bike given that the teenager owns a cycle?

**Solution**

Let us assume A is the event of teenagers owning only a cycle and B is the event of teenagers owning only a bike.

So, $P(A) = 50/100 = 0.5$ and $P(A \cap B) = 30/100 = 0.3$ from the given problem.

$$P(B|A) = P(A \cap B)/P(A) = 0.3/0.5 = 0.6$$

Hence, the probability that a teenager owns bike given that the teenager owns a cycle is 60%.

## Bayes' Theorem

**Theorem** – If A and B are two mutually exclusive events, where $P(A)$ is the probability of A and $P(B)$ is the probability of B, $P(A|B)$ is the probability of A given that B is true. $P(B|A)$ is the probability of B given that A is true, then Bayes' Theorem states –

$$P(A|B) = \frac{P(B|A)P(A)}{\sum_{i=1}^{n} P(B|Ai)P(Ai)}$$

## Application of Bayes' Theorem

- In situations where all the events of sample space are mutually exclusive events.
- In situations where either $P(A_i \cap B)$ for each $A_i$ or $P(A_i)$ and $P(B|A_i)$ for each $A_i$ is known.

**Problem**

Consider three pen-stands. The first pen-stand contains 2 red pens and 3 blue pens; the second one has 3 red pens and 2 blue pens; and the third one has 4 red pens and 1 blue pen. There is equal probability of each pen-stand to be selected. If one pen is drawn at random, what is the probability that it is a red pen?

**Solution**

Let $A_i$ be the event that i$^{th}$ pen-stand is selected.

Here, i = 1,2,3.

Since probability for choosing a pen-stand is equal, $P(A_i) = 1/3$

Let B be the event that a red pen is drawn.

The probability that a red pen is chosen among the five pens of the first pen-stand,

$$P(B|A_1) = 2/5$$

The probability that a red pen is chosen among the five pens of the second pen-stand,

$$P(B|A_2) = 3/5$$

The probability that a red pen is chosen among the five pens of the third pen-stand,

$$P(B|A_3) = 4/5$$

According to Bayes' Theorem,

$$P(B) = P(A_1).P(B|A_1) + P(A_2).P(B|A_2) + P(A_3).P(B|A_3)$$

$$= 1/3.2/5 + 1/3.3/5 + 1/3.4/5$$

$$= 3/5$$