15 Pigeonhole principle

Problem 15.1 (folklore)**.** Prove that in a group of at least 2 people there are always two people who have the same number of acquaintances in the group.

Problem 15.2. Show that among any $n+2$ integers, either there are two whose difference is a multiple of $2n$, or there are two whose sum is divisible by $2n$.

Problem 15.3 (Erdos–Szekeres theorem)**.** Any sequence of distinct real numbers with length at least $pq+1$ contains a monotonically increasing subsequence of length $p+1$ or a monotonically decreasing subsequence of length $q+1$.

Problem 15.4 (IMO'20 P4)**.** There is an integer $n > 1$. There are n^2 stations on a slope of a mountain, all at different altitudes. Each of two cable car companies, A and B, operates k cable cars; each cable car provides a transfer from one of the stations to a higher one (with no intermediate stops). The k cable cars of A have k different starting points and k different finishing points, and a cable car which starts higher also finishes higher. The same conditions hold for B. We say that two stations are linked by a company if one can start from the lower station and reach the higher one by using one or more cars of that company (no other movements between stations are allowed). Determine the smallest positive integer k for which one can guarantee that there are two stations that are linked by both companies.

Problem 15.5. Assume that 101 distinct points are placed in a square 10×10 such that no three of them lie on a line. Prove that we can choose three of the given points that form a triangle whose area is at most 1.

Problem 15.6. Let $n \geqslant 10$ and let S be a subset of $\{1, 2, \dots, n^2\}$ that has exactly n elements. Prove that there are two non-empty disjoint subsets A and B of the set S such that the sum of elements of A is equal to the sum of elements of B.

Problem 15.7. Given a set M of 1985 distinct positive integers, none of which has a prime divisor greater than 26, prove that M contains at least one subset of four distinct elements, whose product is the fourth power of an integer.

Problem 15.8. A set $S \subset \{2, 3, 4, 5, \dots\}$ is called *composite* if no two elements of S are relatively prime. Find the maximal $k \in \mathbb{N}$ for which the following proposition holds: Every composite set S with at least 3 elements has an element greater than or equal to $k \cdot |S|$. As usual, $|S|$ denotes the number of elements in S.

Problem 15.9 (ISL'11 N2)**.** Consider a polynomial $P(x) = \prod_{j=1}^{9}(x+d_j)$, where $d_1, d_2, \dots, d_9$ are nine distinct integers. Prove that there exists an integer N, such that for all integers $x \geqslant N$ the number $P(x)$ is divisible by a prime number greater than 20.

Problem 15.10. Assume that $P = (P_n)_{n=1}^{\infty}$ is a sequence of points from $\mathbb{R}^3$. Denote by $\alpha_m(P)$ the number of points with integer coordinates that are at a distance smaller than 2012 from the polygonal line $P_1P_2 \dots P_m$. Denote by $\ell_m(P)$ the length of the polygonal line $P_1P_2 \dots P_m$.

Prove that there exists numbers C and D such that for any sequence $P = (P_n)_{n=1}^{\infty}$ and any positive integer m the following inequality holds:

$$\alpha_m(P) \leqslant C \cdot \ell_m(P) + D.$$

Problem 15.11. Let $n \geqslant 3$, and let B be a set of more than $\frac{2^{n+1}}{n}$ distinct points with coordinates of the form $(\pm 1, \pm 1, \ldots, \pm 1)$ in n-dimensional space. Prove that there are three distinct points in B which are the vertices of an equilateral triangle.

Problem 15.12. Let $x_1, \ldots, x_{19}$ be positive integers each of which is less than or equal to 93. Let $y_1, \ldots, y_{93}$ be positive integers each of which is less than or equal to 19. Prove there exists a (nonempty) sum of some x_i's equal to a sum of some y_j's.

Problem 15.13. There are $n \geqslant 1$ boxes with infinitely many coins in each of them. Each box contains either regular or fake coins. All regular coins have equal weight w. All fake coins have the same weight that is different from w. We do not know whether they are heavier or lighter than the regular ones.

What is the smallest number of measurements on a digital scale that is necessary to find all boxes that have fake coins?

Problem 15.14 (China TST'12 P6)**.** Prove that there exists a positive real number C with the following property: for any integer $n \geqslant 2$ and any subset X of the set $\{1, 2, \ldots, n\}$ such that $|X| \geqslant 2$, there exist $x, y, z, w \in X$ (not necessarily distinct) such that

$$0 < |xy - zw| < C\alpha^{-4}$$

where $\alpha = \frac{|X|}{n}$.

Problem 15.15 (USA TST'11 P9)**.** Determine whether or not there exist two different sets A, B, each consisting of at most 2011^2 positive integers, such that every x with $0 < x < 1$ satisfies the following inequality:

$$\left| \sum_{a \in A} x^a - \sum_{b \in B} x^b \right| < (1 - x)^{2011}.$$