# 6 Ergodicity, differentiability, continuity

**Problem 6.1.** Let $W_t$ be a Brownian Motion considered at integer time points $t = 0, 1, 2, \ldots$. Choose the ergodic processes. **Options:**

- $X_t = Ct + W_t$, where $C$ is a non-zero constant;
- $X_t = \xi t + W_t$, where $\xi \sim \mathcal{N}(0, 1)$ and $\xi$ is independent of $W_t$;
- none of above.

**Solution.** Recall that discrete-time process is said to be ergodic if

$$\frac{1}{T}\sum_{t=1}^{T} X_t \xrightarrow[T\to\infty]{\mathbb{P}} c.$$

We now check this property for both processes:

1. $X_t = Ct + W_t$:

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T} X_t &= \frac{1}{T}\sum_{t=1}^{T}(Ct + W_t) = \\ &= \frac{1}{T}\sum_{t=1}^{T} Ct + \frac{1}{T}\sum_{t=1}^{T} W_t = \\ &= \frac{C}{T}\sum_{t=1}^{T} t + \frac{1}{T}\sum_{t=1}^{T} W_t = \\ &= \frac{C}{T}\frac{(T+1)T}{2} + \frac{1}{T}\sum_{t=1}^{T} W_t = \\ &= \frac{C(T+1)}{2} + \frac{1}{T}\sum_{t=1}^{T} W_t,\end{aligned}$$

   clearly does not converge to a constant.

2. By the same argument for $X_t = \xi t + W_t$ one gets

$$\frac{1}{T}\sum_{t=1}^{T} X_t = \frac{\xi(T+1)}{2} + \frac{1}{T}\sum_{t=1}^{T} W_t,$$

   which also does not converge to a constant.

**Problem 6.2.** Let $X_t = \cos(\omega t + \theta)$ be a stochastic process and $\theta \sim U([0, 2\pi])$, $\omega = \pi/10$. Is this process ergodic? Is it stationary? **Options:**

- It is ergodic and weakly stationary;
- It is non-ergodic and weakly stationary;
- It is ergodic and non-stationary;
- none of above.

**Solution.** Firstly, $\mathbb{E}\, X_t = \mathbb{E}\cos(\omega t + \theta) = 0$, because $\omega t$ is just a constant for fixed $t$, and hence $\theta \sim U([0, 2\pi])$ implies that $(\omega t + \theta) \pmod{2\pi}$ also has uniform distribution.

By the same argument, one can show that $K_X(t, s) = 0$, and therefore, $X_t$ is weakly stationary.

Moreover, by the same argument as in the lecture, one can get that $X_t$ is ergodic because the sum of any 20 consecutive $X_t$'s vanishes.

**Problem 6.3.** Let $X_t = \varepsilon_t + \xi \cos \frac{\pi t}{12}$, $t = 1, 2, \ldots$, where $\xi, \varepsilon_1, \varepsilon_2, \ldots$ are i.i.d. standard normal random variables. Choose the correct statement. **Options:**

- $X_t$ is weakly stationary and ergodic;
- $X_t$ is weakly stationary and non-ergodic;
- $X_t$ is not weakly stationary, but it is ergodic;
- $X_t$ is not weakly stationary and is not ergodic

**Solution.** Clearly $\mathbb{E}\, X_t = \mathbb{E}\, \varepsilon_t + \cos \frac{\pi t}{12} \cdot \mathbb{E}\, \xi = 0 + \cos \frac{\pi t}{12} \cdot 0 = 0$. However,

$$\begin{aligned}
K_X(t,s) = \operatorname{cov}(X_t, X_s) &= \operatorname{cov}\left(\varepsilon_t + \xi \cos \frac{\pi t}{12}, \varepsilon_s + \xi \cos \frac{\pi s}{12}\right) = \\
&= \operatorname{cov}(\varepsilon_t, \varepsilon_s) + \operatorname{cov}\left(\xi \cos \frac{\pi t}{12}, \varepsilon_s\right) + \operatorname{cov}\left(\varepsilon_t, \xi \cos \frac{\pi s}{12}\right) + \operatorname{cov}\left(\xi \cos \frac{\pi t}{12}, \xi \cos \frac{\pi s}{12}\right) = \\
&= \mathbf{1}\{t = s\} + 0 + 0 + \operatorname{cov}\left(\xi \cos \frac{\pi t}{12}, \xi \cos \frac{\pi s}{12}\right) = \\
&= \mathbf{1}\{t = s\} + \mathbb{E}\left[\left(\xi \cos \frac{\pi t}{12} - 0\right) \cdot \left(\xi \cos \frac{\pi s}{12} - 0\right)\right] = \\
&= \mathbf{1}\{t = s\} + \cos \frac{\pi t}{12} \cdot \cos \frac{\pi s}{12} \cdot \mathbb{E}\, \xi^2,
\end{aligned}$$

clearly not a function of $t - s$ only, because $K_X(t,t) = 1 + \cos^2 \frac{\pi t}{12}$, clearly not a constant. Therefore, $X_t$ is not weakly stationary.

However,

$$\frac{1}{T}\sum_{t=1}^{T} X_t = \frac{1}{T}\sum_{t=1}^{T}\left(\varepsilon_t + \xi \cos \frac{\pi t}{12}\right) = \frac{1}{T}\sum_{t=1}^{T}\varepsilon_t + \frac{1}{T}\sum_{t=1}^{T}\xi \cos \frac{\pi t}{12} = \frac{1}{T}\sum_{t=1}^{T}\varepsilon_t + \frac{\xi}{T}\sum_{t=1}^{T}\cos \frac{\pi t}{12}.$$

First sum converges to 0 by the classical law of large numbers, and the second sum converges to 0 by the same argument as in lecture (the sum of any 24 consecutive $X_t$'s vanishes). Hence, $X_t$ is ergodic.

**Problem 6.4.** Assume that for a process $X_t$ it is known that $\mathbb{E}\, X_t = \alpha + \beta t$, $\operatorname{cov}(X_t, X_{t+h}) = e^{-h\lambda}$ for all $h \geq 0$, $t > 0$, and some constants $\lambda > 0$, $\alpha$, $\beta$. Is the process $Y_t = X_{t+1} - X_t$ stationary and ergodic? **Options:**

- $Y_t$ is weakly stationary and ergodic;
- $Y_t$ is weakly stationary and non-ergodic;
- $Y_t$ is non-stationary and ergodic;
- none of above.

**Solution.** Firstly,

$$\mathbb{E}\, Y_t = \mathbb{E}[X_{t+1} - X_t] = \mathbb{E}\, X_{t+1} - \mathbb{E}\, X_t = (\alpha + \beta(t+1)) - (\alpha + \beta t) = \beta.$$

Secondly,

$$\begin{aligned}
K_Y(t,s) = \operatorname{cov}(Y_t, Y_s) &= \operatorname{cov}(X_{t+1} - X_t, X_{s+1} - X_s) = \\
&= \operatorname{cov}(X_{t+1}, X_{s+1}) - \operatorname{cov}(X_{t+1}, X_s) - \operatorname{cov}(X_t, X_{s+1}) + \operatorname{cov}(X_t, X_s).
\end{aligned}$$

Each of these 4 summands depends only on $t - s$ (it may look like some of them also depend on $t + 1 - s$ and $t - s - 1$, but these dependencies can be represented in terms of dependency on $t - s$). Therefore, $Y_t$ is weakly stationary.

Moreover,

$$\frac{1}{T}\sum_{t=0}^{T-1} Y_t = \frac{1}{T}\sum_{t=0}^{T-1}(X_{t+1} - X_t) = \frac{X_T - X_0}{T}.$$

This expression converges to $\beta$, because its expectation is clearly equal to $\beta$, and because

$$\operatorname{Var} X_t = \operatorname{cov}(X_t, X_t) = e^{-0\lambda} = 1$$

meaning that

$$\operatorname{Var} \frac{X_T}{T} = \frac{1}{\sqrt{T}} \xrightarrow[T\to\infty]{} 0$$

$T \to \infty$. Therefore, $Y_t$ is ergodic.

**Problem 6.5.** Let $X_t = \sigma W_t + ct$, where $W_t$ is Brownian motion, $\sigma, c > 0$. Choose the correct statements about this process. **Options:**

- $X_t$ is differentiable;
- $X_t$ has continuous trajectories;
- $X_t$ is weakly stationary;
- $X_t$ is strictly stationary;
- none of above.

**Solution.** $X_t$ is not differentiable because $\sigma W_t$ is not and $ct$ is, $X_t$ has continuous trajectories because both $\sigma W_t$ and $ct$ have. $X_t$ is neither weakly nor strictly stationary, because

$$\mathbb{E}\, X_t = \mathbb{E}(\sigma W_t + ct) = \mathbb{E}\,\sigma W_t + \mathbb{E}\, ct = \sigma\, \mathbb{E}\, W_t + ct = \sigma \cdot 0 + ct = ct,$$

clearly not a constant.

**Problem 6.6.** Let the process $X_t$ have an autocovariance function $\gamma(r) = e^{-\alpha|r|}$. Is $Y_t = X_t + w$ an ergodic process? **Options:**

- No, for any $w$.
- Yes, if $w$ has the same distribution as $X_t$.
- Yes, for any $w$.
- Yes, if $w$ is a constant.

**Solution.** To begin with,

$$\frac{1}{T}\sum_{t=1}^{T} Y_t = \frac{1}{T}\sum_{t=1}^{T} X_t + w = \frac{1}{T}\sum_{t=1}^{T} X_t + \frac{w}{T} \xrightarrow[T\to\infty]{} \frac{1}{T}\sum_{t=1}^{T} X_t.$$

Furthermore, we know that $X_t$ is ergodic by the theorem from lecture ($\gamma(r) \to 0$ as $r \to \infty$), hence so is $Y_t$.