18.445 Introduction to Stochastic Processes

Lecture 20: Poisson process

Hao Wu

MIT

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Random point process

A random point process is a countable random set of points of the real line which corresponds to the sequence of the times of occurrence of some event. For instance, the arrival times of customers.

Definition

A random point process on $\mathbb{R}_+$ is a sequence of random variables $(T_n)_{n\geq 0}$ such that

- $0 = T_0 < T_1 < T_2 < \cdots$
- $\lim_n T_n = \infty$

Definition

The interevent sequence : $S_n = T_n - T_{n-1}$ for $n \geq 1$.
The counting process : For $(a, b] \subset \mathbb{R}_+$, define

$$N(a, b] = \sum_{n\geq 1} 1_{(a,b]}(T_n)$$

Counting process

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In particular, set $N_t = N(0, t]$. Then

- $N_0 = 0$
- $N(a, b] = N_b - N_a$
- $t \mapsto N_t$ is right-continuous

Poisson process

Definition

A point process N on $\mathbb{R}_+$ is called a Poisson process with intensity $\lambda > 0$ if

- For any $k \geq 1$, any $0 \leq t_1 \leq t_2 \leq \cdots \leq t_k$, the random variables $N(t_i, t_{i+1}], i = 1, ..., k-1$ are independent.
- For any interval $(a, b] \subset \mathbb{R}_+$, the variable $N(a, b]$ is a Poisson random variable with mean $\lambda(b-a)$, i.e.

$$\mathbb{P}[N(a,b] = k] = e^{-\lambda(b-a)} \frac{(\lambda(b-a))^k}{k!}.$$

Theorem

The interevent sequence $(S_n)_{n \geq 1}$ of a Poisson process with intensity λ is i.i.d. with exponential distribution of parameter λ.

Poisson process — Markov property

Theorem (Markov property)

Let $(N_t)_{t\geq 0}$ be a Poisson process. Then, $\forall s \geq 0$,

- *the process $(N_{t+s} - N_s)_{t\geq 0}$ is also a Poisson process*
- *and it is independent of $(N_u)_{u\leq s}$*

Theorem (Strong Markov property)

Let $(N_t)_{t\geq 0}$ be a Poisson process. Suppose that T is a stopping time, then conditional on $[T < \infty]$,

- *the process $(N_{t+T} - N_T)_{t\geq 0}$ is also a Poisson process*
- *and it is independent of $(N_u)_{u\leq T}$*

Poisson process — Superposition

Theorem

Let $(N^i)_{i\geq 1}$ be a family of independent Poisson processes with respective positive intensities $(\lambda_i)_{i\geq 1}$. Then

- *two distinct Poisson processes in this family have no points in common*
- *if $\sum_{i\geq 1}\lambda_i = \lambda < \infty$, then $N_t = \sum_{i\geq 1} N^i_t$ defines the counting process of a Poisson process with intensity λ.*

Theorem

In this situation of the above theorem with $\sum \lambda_i = \lambda < \infty$. Denote by Z the first event time of $N = \sum N^i$ and by J the index of the Poisson process responsible for it. Then

$$\mathbb{P}[J = i, Z \geq a] = \mathbb{P}[J = i] \times \mathbb{P}[Z \geq a] = \frac{\lambda_i}{\lambda} e^{-\lambda a}.$$

Poisson process — Characterization

Theorem

Let $(X_t)_{t\geq 0}$ be an increasing right-continuous process taking values in $\{0, 1, 2, ...\}$ with $X_0 = 0$. Let $\lambda > 0$. Then the following statements are equivalent.

- *$(X_t)_{t\geq 0}$ is a Poisson process with intensity λ.*
- *X has independent increments, and as $\epsilon \downarrow 0$, uniformly in t, we have*

$$\mathbb{P}[X_{t+\epsilon} - X_t = 0] = 1 - \lambda\epsilon + o(\epsilon);$$

$$\mathbb{P}[X_{t+\epsilon} - X_t = 1] = \lambda\epsilon + o(\epsilon).$$

- *X has independent and stationary increments, and for all $t \geq 0$ we have $X_t \sim \text{Poisson}(\lambda t)$.*