8 Lévy processes

Problem 8.1. Let $X_t = bt + \sigma W_t + cN_t$, where W_t is a Brownian Motion, N_t is a Poisson process with intensity λ, and W_t, N_t are independent; $b, c \in \mathbb{R}$, $\sigma \geq 0$. Find the characteristic function of this process. **Options:**

- $iub + \frac{\lambda(e^{icu}-1)}{t} - \frac{(\sigma u)^2}{2}$;
- $\exp\left\{iub + \frac{\lambda(e^{icu}-1)}{t} - \frac{(\sigma u)^2}{2}\right\}$;
- $\exp\left\{iubt + \lambda t(e^{icu}-1) - \frac{t(\sigma u)^2}{2}\right\}$;
- none of above.

Solution. Straightforward computations:

$$\begin{aligned}
\Phi_X(u,t) &= \mathbb{E}\left[e^{iuX_t}\right] = \\
&= \mathbb{E}\left[e^{iu(bt+\sigma W_t + cN_t)}\right] = \\
&= e^{iubt} \cdot \mathbb{E}\left[e^{iu\sigma W_t}\right] \cdot \mathbb{E}\left[e^{iucN_t}\right] = \\
&= e^{iubt} \cdot \Phi_{\sigma W_t}(u) \cdot \Phi_{cN_t}(u) = \\
&= e^{iubt} \cdot \Phi_{\mathcal{N}(0,\sigma^2 t)}(u) \cdot \Phi_{c\,\mathrm{Pois}(\lambda t)}(u) = \\
&= e^{iubt} \cdot \exp\left\{-\tfrac{1}{2}t\sigma^2 u^2\right\} \cdot \Phi_{\mathrm{Pois}(\lambda t)}(cu) = \\
&= e^{iubt} \cdot \exp\left\{-\tfrac{t(\sigma u)^2}{2}\right\} \cdot \exp\left\{\lambda t(e^{icu}-1)\right\} = \\
&= \exp\left\{iubt - \tfrac{t(\sigma u)^2}{2} + \lambda t(e^{icu}-1)\right\}.
\end{aligned}$$

Problem 8.2. Consider the previous process $X_t = bt + \sigma W_t + cN_t$, where W_t is a Brownian Motion, N_t is a Poisson process with intensity λ, and W_t, N_t are independent; $b, c \in \mathbb{R}$, $\sigma \geq 0$. What are the mean, variance and covariance function of this process? **Options:**

- $\mathbb{E}\, X_t = t(b + c\lambda)$, $\operatorname{Var} X_t = t(\sigma^2 + c\lambda)$, $K(t,s) = 0$;
- $\mathbb{E}\, X_t = tb$, $\operatorname{Var} X_t = t(\sigma^2 + c^2\lambda)$, $K(t,s) = \lambda \min\{t,s\}$;
- $\mathbb{E}\, X_t = t(b + c\lambda)$, $\operatorname{Var} X_t = t(\sigma^2 + c^2\lambda)$, $K(t,s) = (c^2\lambda + \sigma^2)\min\{t,s\}$;
- none of above.

Solution. To begin with,

$$\mathbb{E}\, X_t = \mathbb{E}[bt + \sigma W_t + cN_t] = \mathbb{E}[bt] + \mathbb{E}[\sigma W_t] + \mathbb{E}[cN_t] = bt + 0 + c\lambda t = t(b + c\lambda t).$$

Going further,

$$\begin{aligned}
K_X(t,s) &= \operatorname{cov}(bt + \sigma W_t + cN_t, bs + \sigma W_s + cN_s) = \\
&= \operatorname{cov}(\sigma W_t, \sigma W_s) + \operatorname{cov}(cN_t, cN_t) = \\
&= \sigma^2 \min\{t,s\} + c^2\lambda \min\{t,s\} = \\
&= (\sigma^2 + c^2\lambda)\min\{t,s\}.
\end{aligned}$$

From here it immidiately follows that $\operatorname{Var} X_t = t(\sigma^2 + c^2\lambda)$.

Problem 8.3. Consider the previous process $X_t = bt + \sigma W_t + cN_t$, where W_t is a Brownian Motion, N_t is a Poisson process with intensity λ and W_t, N_t are independent; $b, c \in \mathbb{R}$, $\sigma \geq 0$. Denote the Lévy measure of this process by ν. What is measure ν of a Borel set B? **Options:**

- $\nu(B) = bt + \lambda$, if $1 \in B$ and 0 otherwise;
- $\nu(B) = \lambda \mathbb{P}\{1 \in B\}$, if $1 \in B$ and 0 otherwise;
- $\nu(B) = \lambda$, if $c \in B$ and 0 otherwise;
- none of above.

Solution. $\nu(B) = \mathbb{E}[\#t \in [0,1] : \Delta X_t \in B]$. $bt + \sigma W_t$ is a continuous part, and for N_t one has $\nu(B) = \lambda \mathbf{1}\{1 \in B\}$. Therefore, for cN_t one can write $\nu(B) = \lambda \mathbf{1}\{c \in B\}$.

Problem 8.4. Let X_t be a Lévy process. What is the correct expression for $\operatorname{Var} X_t$ in terms of characteristic exponent ψ? **Options:**

- $\operatorname{Var} X_t = -t\psi''(0)$;
- $\operatorname{Var} X_t = -t\psi''(0) - t^2(\psi'(0))^2$;
- $\operatorname{Var} X_t = -t\psi''(0) - t^2\psi'(0)$;
- none of above.

Solution. We first recall that $\operatorname{Var} X_t = t \operatorname{Var} X_1$. Then we recall how to compute variance from moments: $\operatorname{Var} X_1 = \mathbb{E}[X_1^2] - (\mathbb{E} X_1)^2$. Now we need to compute moments from characteristic function:

$$\mathbb{E} X_1 = i^{-1}\Phi'_{X_1}(0),$$
$$\mathbb{E}[X_1^2] = i^{-2}\Phi''_{X_1}(0).$$

We then use $\Phi_{X_1}(u) = e^{\psi(u)}$ to compute these derivatives:

$$\Phi'_{X_1}(u) = (e^{\psi(u)})' = \psi'(u)e^{\psi(u)},$$
$$\Phi''_{X_1}(u) = (e^{\psi(u)})'' = (\psi'(u)e^{\psi(u)})' = (\psi''(u) + (\psi'(u))^2)e^{\psi(u)}.$$

Evaluating these at $u = 0$ gives

$$\mathbb{E} X_1 = i^{-1}\psi'(0),$$
$$\mathbb{E}[X_1^2] = i^{-2}(\psi''(0) + (\psi'(0))^2).$$

By substituting these back into the formula for variance one gets

$$\operatorname{Var} X_1 = \mathbb{E}[X_1^2] - (\mathbb{E} X_1)^2 = -(\psi''(0) + (\psi'(u))^2) + (\psi'(0))^2 = -\psi''(0),$$

meaning that $\operatorname{Var} X_t = -t\psi''(0)$.

Problem 8.5. Let X_t be a Lévy process. Assuming that $X_1 \sim \mathcal{N}(0,1)$, find the mean and the variance of X_t. **Options:**

- $\mathbb{E} X_t = t$, $\operatorname{Var} X_t = t^2$;
- $\mathbb{E} X_t = 0$, $\operatorname{Var} X_t = t$;
- $\mathbb{E} X_t = 0$, $\operatorname{Var} X_t = 2t$;
- none of above.

Solution. By the theorem from lecture, $\mathbb{E} L_t = t\,\mathbb{E} L_1 = t \cdot 0 = 0$, and $\operatorname{Var} X_t = t \operatorname{Var} L_1 = t \cdot 1 = t$.

Problem 8.6. Let $X_t = bt + N_t$, where N_t is a Poisson process with intensity λ and $b \in \mathbb{R}$. Find the Lévy triplet of this process. **Options:**

- $(b, \lambda^2, 0)$;
- (b, λ, ν), where $\nu(B) = \mathbf{1}\{0 \in B\}$ for any Borel set B;
- $(b + \lambda, 0, \nu)$, where $\nu(B) = \lambda\mathbf{1}\{1 \in B\}$ for any Borel set B;
- $(\lambda, \lambda^2, \nu)$, where $\nu(B) = \lambda\mathbf{1}\{1 \in B\}$ for any Borel set B;
- none of above.

Solution. $\nu(B) = \mathbb{E}[\#t \in [0,1] : \Delta X_t \in B]$. bt is a continuous part, and for N_t one has $\nu(B) = \lambda\mathbf{1}\{1 \in B\}$.

Going further, $\mu = \frac{1}{t}\mathbb{E} X_t = \frac{1}{t}\mathbb{E}[bt + N_t] = b + \lambda$.

Finally, X_t has no Brownian Motion part, meaning that $\sigma = 0$.

Problem 8.7. Let L_t be a Lévy process. Choose the equality, which can serve as a proof of its infinite divisibility. **Options:**

- $L_t = L_t/n + (L_{2t} - L_t)/n + \ldots + (L_{nt} - L_{(n-1)t})/n$, $\forall t > 0$, $\forall n \in \mathbb{N}$;
- $L_t = L_{t/n} + (L_{2t/n} - L_{t/n}) + \ldots + (L_t - L_{(n-1)t/n})$, $\forall t > 0$, $\forall n \in \mathbb{N}$;
- $L_t = L_t/n + L_t/n + \ldots + L_t/n$, $\forall t > 0$, $\forall n \in \mathbb{N}$;
- $L_t = L_{t/n} + L_{t/n} + \ldots + L_{t/n}$, $\forall t > 0$, $\forall n \in \mathbb{N}$.

Solution. Indeed,

$$L_t = L_{t/n} + (L_{2t/n} - L_{t/n}) + \ldots + (L_t - L_{(n-1)t/n}) \stackrel{\mathrm{d}}{=} \underbrace{L_{t/n} + \ldots + L_{t/n}}_{n \text{ times}},$$

i.e. is the sum of n independent identically distributed random variables, for any n.

Problem 8.8. Let $T_a = \min\{s : B_s \geq a\}$, where B_s is a Brownian motion. Find the distribution function of the process T_a. Hint:

$$\mathbb{P}\left\{B_t - B_{T_a} > 0 | T_a < t\right\} = \mathbb{P}\left\{B_t - B_{T_a} > 0\right\}.$$

It follows from the fact that for the Brownian motion and all other Lévy processes the increments are independent. **Options:**

- $1 - \Phi\left(\frac{a}{\sqrt{t}}\right)$;
- $\Phi\left(\frac{a}{\sqrt{t}}\right)$;
- $2\Phi\left(\frac{a}{\sqrt{t}}\right)$;
- $1 - 2\Phi\left(\frac{a}{\sqrt{t}}\right)$;
- $2\left(1 - \Phi\left(\frac{a}{\sqrt{t}}\right)\right)$.