18.445 HOMEWORK 5 SOLUTIONS

Exercise 1. If $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, show that the class

$$\{\mathbb{E}[X \mid \mathcal{A}] : \mathcal{A} \text{ sub } \sigma\text{-algebra of } \mathcal{F}\}$$

is Uniformly Integrable.

(1). Show that, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\mathbb{E}[|X|1_A] \leq \epsilon, \quad \text{whenever} \quad \mathbb{P}[A] \leq \delta.$$

Proof. Suppose the converse holds. Then for some $\varepsilon > 0$ there exists $A_k \subset \Omega$ for each k such that $\mathbb{P}[A_k] \leq 2^{-k}$ and $\mathbb{E}[|X|\mathbb{1}_{A_k}] \geq \varepsilon$. Since $\sum_k \mathbb{P}[A_k] < \infty$, the Borel-Cantelli lemma shows that $\mathbb{E}[\limsup_k \mathbb{1}_{A_k}] = 0$. Since $X \in L^1$, $\mathbb{E}[\limsup_k |X|\mathbb{1}_{A_k}] = 0$. Hence Fatou's lemma implies that

$$0 = \mathbb{E}[\limsup_k |X|\mathbb{1}_{A_k}] \geq \limsup_k \mathbb{E}[|X|\mathbb{1}_{A_k}] \geq \varepsilon$$

which is a contradiction. □

(2). Show the conclusion.

Proof. Fix $\varepsilon > 0$. Choose $\delta > 0$ which satisfies the condition in Part (1). For any $\mathcal{A} \subset \mathcal{F}$, Chebyshev's inequality and Jensen's inequality imply that

$$\mathbb{P}\Big[\Big|\mathbb{E}[X \mid \mathcal{A}]\Big| \geq C\Big] \leq \frac{1}{C}\mathbb{E}\Big[\Big|\mathbb{E}[X \mid \mathcal{A}]\Big|\Big] \leq \frac{1}{C}\mathbb{E}[|X|].$$

Since $X \in L^1$, we can choose C large enough so that $\mathbb{P}\Big[\Big|\mathbb{E}[X \mid \mathcal{A}]\Big| \geq C\Big] \leq \delta$. Then it follows from Part (1) and Jensen's inequality that

$$\mathbb{E}\Big[\Big|\mathbb{E}[X \mid \mathcal{A}]\Big|\mathbb{1}_{\{|\mathbb{E}[X \mid \mathcal{A}]| \geq C\}}\Big] \leq \mathbb{E}\Big[|X|\mathbb{1}_{\{|\mathbb{E}[X \mid \mathcal{A}]| \geq C\}}\Big] \leq \varepsilon.$$

Therefore $\mathbb{E}[X \mid \mathcal{A}]$ is uniformly integrable. □

Exercise 2. Customers arrive in a supermarket as a Poisson process with intensity N. There are N aisles in the supermarket and each customer selects one of them at random, independently of the other customers. Let X_t^N denote the proportion of aisles which remain empty by time t. Show that

$$X_t^N \to e^{-t}, \quad \text{in probability as } N \to \infty.$$

Proof. First, consider the simplified scenario where m people selects one of N aisles at random. Let $X = X(m)$ denote the number of empty aisles. Let $X_i = 1$ if the i-th aisle is empty and $X_i = 0$ otherwise. Note that the probability that the i-th aisle is empty is $\mathbb{E}[X_i] = (1 - 1/N)^m$. Since $X = \sum_{i=1}^N X_i$, we have

$$\mathbb{E}[X] = \sum_{i=1}^N \mathbb{E}[X_i] = N(1 - \frac{1}{N})^m. \tag{1}$$

Moreover, for $i \neq j$, the probability that both the i-th and the j-th aisles are empty is $\mathbb{E}[X_i X_j] = (1-2/N)^m$. Hence

$$\mathbb{E}[X^2] = \mathbb{E}[(\sum_{i=1}^N X_i)^2] = \sum_{i=1}^N \mathbb{E}[X_i^2] + \sum_{i \neq j} \mathbb{E}[X_i X_j] = N(1 - \frac{1}{N})^m + N(N-1)(1 - \frac{2}{N})^m. \tag{2}$$

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Next, if the customers arrive as a Poisson process with intensity N, then at time t the number of customers M has a Poisson distribution with intensity Nt, i.e.

$$\mathbb{P}[M = m] = \frac{e^{-Nt}(Nt)^m}{m!}.$$

Let Y denote the number of empty aisles at time t. Then $\mathbb{E}[Y \mid M = m] = X(m) = X$ and by (1),

$$\begin{aligned}
\mathbb{E}[Y] &= \sum_{m=0}^{\infty} \mathbb{P}[M = m]\mathbb{E}[Y \mid M = m] \\
&= \sum_{m=0}^{\infty} \frac{e^{-Nt}(Nt)^m}{m!} N(1 - \frac{1}{N})^m \\
&= e^{-Nt} N \sum_{m=0}^{\infty} \frac{t^m (N-1)^m}{m!} \\
&= e^{-Nt} N e^{t(N-1)} = Ne^{-t}.
\end{aligned}$$

If X_t^N denotes the proportion of empty aisles, then

$$\mathbb{E}[X_t^N] = \mathbb{E}[\frac{Y}{N}] = e^{-t}. \tag{3}$$

Moreover, $\mathbb{E}[Y^2 \mid M = m] = X(m)^2 = X^2$ and by (2),

$$\begin{aligned}
\mathbb{E}[Y^2] &= \sum_{m=0}^{\infty} \mathbb{P}[M = m]\mathbb{E}[Y^2 \mid M = m] \\
&= \sum_{m=0}^{\infty} \frac{e^{-Nt}(Nt)^m}{m!} [N(1 - \frac{1}{N})^m + N(N-1)(1 - \frac{2}{N})^m] \\
&= Ne^{-t} + e^{-Nt} N(N-1) \sum_{m=0}^{\infty} \frac{t^m (N-2)^m}{m!} \\
&= Ne^{-t} + e^{-Nt} N(N-1) e^{t(N-2)} \\
&= Ne^{-t} + N(N-1)e^{-2t}.
\end{aligned}$$

It follows that

$$\operatorname{Var}[Y] = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 = Ne^{-t} + N(N-1)e^{-2t} - N^2 e^{-2t} = Ne^{-t} - Ne^{-2t},$$

so

$$\operatorname{Var}[X_t^N] = \frac{1}{N^2} \operatorname{Var}[Y] = \frac{1}{N}(e^{-t} - e^{-2t}). \tag{4}$$

Finally, we deduce from Chebyshev's inequality and (4) that

$$\mathbb{P}[|X_t^N - \mathbb{E}[X_t^N]| > \varepsilon] \le \frac{1}{\varepsilon^2} \mathbb{E}\big[\big|X_t^N - \mathbb{E}[X_t^N]\big|^2\big] = \frac{1}{\varepsilon^2} \operatorname{Var}[X_t^N] \longrightarrow 0$$

as $N \to \infty$. This together with (3) implies that $X_t^N \to \mathbb{E}[X_t^N] = e^{-t}$ in probability. $\square$

Exercise 3. Let $T_1, T_2, ...$ be independent exponential random variables of parameter λ.

(1). For all $n \ge 1$, the sum $S = \sum_{i=1}^{n} T_i$ has the probability density function

$$f_S(x) = \frac{\lambda^n x^{n-1}}{(n-1)!} e^{-\lambda x}, \quad x > 0.$$

This is called the Gamma(n, λ) distribution.

Proof. We prove this by induction. The case $n = 1$ is obvious. Suppose $S = \sum_{i=1}^{n} T_i$ has the stated density function. Then the density of $S + T_{n+1}$ is the convolution

$$\int_{\substack{\{y\geq 0,\\ x-y\geq 0\}}} \frac{\lambda^n y^{n-1}}{(n-1)!} e^{-\lambda y} \cdot \lambda e^{-\lambda(x-y)}\, dy = \frac{\lambda^{n+1}}{(n-1)!} e^{-\lambda x} \int_0^x y^{n-1}\, dy = \frac{\lambda^{n+1}}{n!} e^{-\lambda x} x^n.$$

This completes the induction. □

(2). Let N be an independent geometric random variable with

$$\mathbb{P}[N = n] = \beta(1-\beta)^{n-1}, \quad n = 1, 2, \ldots.$$

Show that $T = \sum_{i=1}^{N} T_i$ has exponential distribution of parameter $\lambda\beta$.

Proof. To sample T, it is equivalent to sample $\sum_{i=1}^{n} T_i$ with probability $\beta(1-\beta)^{n-1}$. Since the density of $\sum_{i=1}^{n} T_i$ was established in Part (1), we can compute the density of T:

$$\begin{aligned}
\sum_{n=1}^{\infty} \beta(1-\beta)^{n-1} \frac{\lambda^n x^{n-1}}{(n-1)!} e^{-\lambda x} &= \lambda\beta e^{-\lambda x} \sum_{n=1}^{\infty} \frac{[\lambda(1-\beta)x]^{n-1}}{(n-1)!} \\
&= \lambda\beta e^{-\lambda x} e^{\lambda(1-\beta)x} \\
&= \lambda\beta e^{-\lambda\beta x}.
\end{aligned}$$

It follows that T has the exponential distribution with parameter $\lambda\beta$. □

Exercise 4. Let $(N^i)_{i\geq 1}$ be a family of independent Poisson processes with respect positive intensities $(\lambda_i)_{i\geq 1}$. Then

(1). Show that any two distinct Poisson processes in this family have no points in common.

Proof. First, for a fixed $t > 0$, $\mathbb{P}[N^i(t-\varepsilon, t] = 0] \to 1$ as $\varepsilon \to 0$ by the definition of a Poisson process, so a.s. N^i does not jump at time t.

Let T_n^i denote the n-th jump time of N^i. For $i = j$, N^i and N^j are independent. Hence conditional on N^j (and thus on T_n^j), N^i has the same law and a.s. does not jump on one T_n^j by the above argument. Since there are countably many T_n^j, a.s. N^i does not jump on any T_n^j. We conclude that two distinct Poisson process in this family have no simultaneous jumps (i.e. no points in common). □

(2). If $\sum_{i\geq 1} \lambda_i = \lambda < \infty$, then $\sum_{i\geq 1} N_t^i = N_t$ defines the counting process of a Poisson process with intensity λ.

Proof. Let X_i be independent Poisson random variables with mean λ_i. Using the discrete convolution formula, we have

$$\begin{aligned}
\mathbb{P}[X_1 + X_2 = k] &= \sum_{j=0}^{k} e^{-\lambda_1} \frac{\lambda_1^j}{j!} e^{-\lambda_2} \frac{\lambda_2^{k-j}}{(k-j)!} \\
&= e^{-(\lambda_1+\lambda_2)} \frac{1}{k!} \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} \lambda_1^j \lambda_2^{k-j} \\
&= e^{-(\lambda_1+\lambda_2)} \frac{(\lambda_1+\lambda_2)^k}{k!}.
\end{aligned}$$

Hence $X_1 + X_2$ is a Poisson random variable with mean $\lambda_1 + \lambda_2$. Inductively, we see that $\sum_{i=1}^{n} X_i$ is a Poisson random variable with mean $\sum_{i=1}^{n} \lambda_i$.

Define $X = \sum_{i\geq 1} X_i$ pointwise. By the monotone convergence theorem,

$$\mathbb{E}[X] = \sum_{i\geq 1} \mathbb{E}[X_i] = \sum_{i\geq 1} \lambda_i = \lambda,$$

so in particular, X is a.s. finite. Hence $\mathbb{1}_{\sum_{i=1}^n X_i = k} \to \mathbb{1}_{X=k}$ a.s. and by the dominated convergence theorem,

$$\mathbb{P}[X = k] = \lim_{n\to\infty} \mathbb{P}[\sum_{i=1}^n X_i = k] = \lim_{n\to\infty} \exp(-\sum_{i=1}^n \lambda_i) \frac{(\sum_{i=1}^n \lambda_i)^k}{k!} = e^{-\lambda} \frac{\lambda^k}{k!}.$$

Therefore X is a Poisson random variable with mean λ.

Next, since for each i and $0 < t_1 < \cdots < t_m$, $N_{t_1}^i, N^i(t_1, t_2], \ldots, N^i(t_{m-1}, t_m]$ are independent, it is easily seen that $N_{t_1}, N(t_1, t_2], \ldots, N(t_{m-1}, t_m]$ are independent. Moreover, for $(a, b] \subset \mathbb{R}_+$, $N^i(a, b]$ is a Poisson random variable with mean $\lambda_i(b-a)$, so our argument above implies that $N(a, b]$ is a Poisson random variable with mean $\lambda(b - a)$. This by definition shows that N is a Poisson process with intensity λ. $\square$

Exercise 5. (Optional, 3 bonus points)

(1). Let $(N_t)_{t\geq 0}$ be a Poisson process with intensity $\lambda > 0$ and let $(X_i)_{i\geq 0}$ be a sequence of i.i.d. random variables, independent of N. Show that if $g(s, x)$ is a function and T_j are jump times of N then

$$\mathbb{E}\left[\exp\left(\theta \sum_{j=1}^{N_t} g(T_j, X_j)\right)\right] = \exp\left(\lambda \int_0^t ds \mathbb{E}\left[e^{\theta g(s,X)} - 1\right]\right).$$

This is called Campbell's Theorem.

Proof. First we establish a uniform property of jump times of a Poisson process. Namely, claim that conditioned on $N_t = n$, the jump times T_j have the same joint distribution as $U_{(1)}, \ldots, U_{(n)}$, the order statistics of n i.i.d. uniform random variables on $[0, t]$, whose density is given by

$$f(t_1, \ldots, t_n) = \frac{n!}{t^n}, \quad 0 < t_1 < \cdots < t_n < t.$$

We use $f(X = t)$ to denote the density function of X at t. Let $T_0 = 0$ and for $1 \leq j \leq n + 1$, the inter-arrival times $E_j = T_j - T_{j-1}$ are independent exponential variables with parameter λ. Hence we have

$$\begin{aligned}
f(T_1 = t_1, \ldots, T_n = t_n \,|\, N_t = n) &= \frac{f(T_1 = t_1, \ldots, T_n = t_n, N_t = n)}{f(N_t = n)} \\
&= \frac{f(E_1 = t_1, E_2 = t_2 - t_1, \ldots, E_n = t_n - t_{n-1}, E_{n+1} > t - t_n)}{e^{-\lambda t}(\lambda t)^n/n!} \\
&= \frac{\lambda^n e^{-\lambda t_1} e^{\lambda(t_1-t_2)} \cdots e^{\lambda(t_{n-1}-t_n)} e^{\lambda(t_n-t)}}{e^{-\lambda t}(\lambda t)^n/n!} \\
&= \frac{n!}{t^n}
\end{aligned}$$

as claimed.

If $U_{(j)}$ are the order statistics of U_j, let σ be the permutation such that $\sigma((j)) = j$. Since X_j are i.i.d., $(X_{\sigma(j)})$ has the same joint distribution as (X_j). Hence

$$\sum_{j=1}^n g(U_{(j)}, X_j) = \sum_{j=1}^n g(U_j, X_{\sigma(j)}) \stackrel{d}{=} \sum_{j=1}^n g(U_j, X_j).$$

This fact and the claim above imply that

$$\begin{aligned}
\mathbb{E}\Big[\exp\Big(\theta\sum_{j=1}^{N_t} g(T_j, X_j)\Big)\Big] &= \sum_{n=0}^{\infty}\mathbb{P}[N_t = n]\cdot\mathbb{E}\Big[\exp\Big(\theta\sum_{j=1}^{n} g(T_j, X_j)\Big) \mid N_t = n\Big] \\
&= \sum_{n=0}^{\infty}\frac{e^{-\lambda t}(\lambda t)^n}{n!}\mathbb{E}\Big[\exp\Big(\theta\sum_{j=1}^{n} g(U_{(j)}, X_j)\Big)\Big] \\
&= \sum_{n=0}^{\infty}\frac{e^{-\lambda t}(\lambda t)^n}{n!}\mathbb{E}\Big[\prod_{j=1}^{n}\exp\big(\theta g(U_j, X_j)\big)\Big] \\
&= \sum_{n=0}^{\infty}\frac{e^{-\lambda t}(\lambda t)^n}{n!}\prod_{j=1}^{n}\mathbb{E}_{U_j}\mathbb{E}_{X_j}\exp\big(\theta g(U_j, X_j)\big) \\
&= e^{-\lambda t}\sum_{n=0}^{\infty}\frac{(\lambda t)^n}{n!}\Big(\frac{1}{t}\int_0^t \mathbb{E}\big[e^{\theta g(s,X)}\big]\,ds\Big)^n \\
&= e^{-\lambda t}\exp\Big(\lambda\int_0^t \mathbb{E}\big[e^{\theta g(s,X)}\big]\,ds\Big) \\
&= \exp\Big(\lambda\int_0^t \mathbb{E}\big[e^{\theta g(s,X)} - 1\big]\,ds\Big).
\end{aligned}$$

□

(2). Cars arrive at the beginning of a long road in a Poisson stream of intensity λ from time $t = 0$ onwards. A car has a fixed velocity V miles per hour, where $V > 0$ is a random variable. The velocities of cars are i.i.d. and are independent of the arrival process. Cars can overtake each other freely. Show that the number of cars on the first x miles of the road at time t has a Poisson distribution with mean $\lambda\mathbb{E}[\min\{t, x/V\}]$.

Proof. Let N_t be the number of cars that enter the road before time t. Suppose the j-th car enters the road at time T_j and has velocity V_j. Note that the j-th car is on the first x miles of the road at time t if and only if $t - T_j \le x/V_j$. Hence the number of cars on the first x miles of the road at time t is given by

$$\sum_{j=1}^{N_t}\mathbb{1}_{\{t-T_j\le x/V_j\}}.$$

Using Part (1), we compute its moment-generating function:

$$\begin{aligned}
\varphi(\theta) &= \mathbb{E}\Big[\exp\Big(\theta\sum_{j=1}^{N_t}\mathbb{1}_{\{t-T_j\le x/V_j\}}\Big)\Big] \\
&= \exp\Big(\lambda\int_0^t \mathbb{E}\big[\exp(\theta\mathbb{1}_{\{t-s\le x/V\}}) - 1\big]\,ds\Big) \\
&= \exp\Big(\lambda\mathbb{E}\Big[\int_0^t \exp(\theta\mathbb{1}_{\{t-s\le x/V\}}) - 1\,ds\Big]\Big).
\end{aligned}$$

If $t \le x/V$, then $\exp(\theta\mathbb{1}_{\{t-s\le x/V\}}) - 1 = e^\theta - 1$, so

$$\varphi(\theta) = \exp\big(\lambda t(e^\theta - 1)\big).$$

If $t > x/V$, for $s < t - x/V$, $\exp(\theta\mathbb{1}_{\{t-s\le x/V\}}) - 1 = e^0 - 1 = 0$; and for $s \ge t - x/V$, $\exp(\theta\mathbb{1}_{\{t-s\le x/V\}}) - 1 = e^\theta - 1$. Hence

$$\varphi(\theta) = \exp\big(\lambda\mathbb{E}\big[\frac{x}{V}(e^\theta - 1)\big]\big) = \exp\big(\lambda\mathbb{E}[x/V](e^\theta - 1)\big).$$

It follows that

$$\varphi(\theta) = \exp\big(\lambda\mathbb{E}[\min\{t, x/V\}](e^\theta - 1)\big)$$

which is exactly the moment-generating function of a Poisson distribution with mean $\lambda\mathbb{E}[\min\{t, x/V\}]$. This completes the proof. □

Exercise 6. (Optional, 3 bonus points) Customers enter a supermarket as a Poisson process with intensity 2. There are two salesmen near the door who offer passing customers samples of a new product. Each customer takes an exponential time of parameter 1 to think about the new product, and during this time occupies the full attention of one salesman. Having tried the product, customers proceed into the store and leave by another door. When both salesmen are occupied, customers walk straight in. Assuming that both salesmen are free at time 0, find the probability that both are busy at a later time t.

Proof. We can construct a continuous-time Markov chain as follows. The chain has three states $\{0, 1, 2\}$, namely, zero salesmen are occupied, one salesman is occupied and two salesmen are occupied. Since the inter-arrival times of the Poisson process are exponential variables with intensity 2, we see that the Q-matrix associated to the chain is

$$Q = \begin{bmatrix} q_{00} & q_{01} & q_{02} \\ q_{10} & q_{11} & q_{12} \\ q_{20} & q_{21} & q_{22} \end{bmatrix} = \begin{bmatrix} -2 & 2 & 0 \\ 1 & -3 & 2 \\ 0 & 2 & -2 \end{bmatrix}.$$

It is easy to get that the eigenvalues of Q are $-5, -2$ and 0, so the eigendecomposition of Q is

$$Q = U \begin{bmatrix} -5 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^T.$$

Hence the transition matrix is

$$P(t) = e^{tQ} = U \begin{bmatrix} e^{-5t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & 1 \end{bmatrix} U^T.$$

The probability that both salesmen are busy at time t is $p_{02}(t)$, which must have the form

$$p_{02}(t) = ae^{-5t} + be^{-2t} + c$$

for some constants a, b and c.

Since $P(0) = I$, $P'(0) = Q$ and $P''(0) = Q^2$, we see that $p_{02}(0) = 0, p'_{02}(0) = 0$, and $p''_{02}(0) = 4$. Hence

$$\begin{cases} a + b + c = 0 \\ -5a - 2b = 0 \\ 25a + 4b = 4 \end{cases} \implies \begin{cases} a = \frac{4}{15} \\ b = -\frac{2}{3} \\ c = \frac{2}{5} \end{cases}.$$

We conclude that the probability that both salesmen are busy at time t is $\frac{4}{15}e^{-5t} - \frac{2}{3}e^{-2t} + \frac{2}{5}$. $\square$