18.445 Introduction to Stochastic Processes

Lecture 6: Lower bounds on mixing times

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Recall Suppose that P is irreducible with stationary measure π.

$$d(n) = \max_x ||P^n(x, \cdot) - \pi||_{TV}, \quad t_{mix} = \min\{n : d(n) \leq 1/4\}.$$

Today's Goal Find lower bounds for the mixing times.

- Bottleneck Ratio
- Distinguishing statistics
- Random walk on hypercube

Bottleneck ratio

Suppose that P is an irreducible transition matrix with stationary measure π. Define

$$Q(A, B) = \sum_{x \in A, y \in B} \pi(x) P(x, y).$$

$Q(A, B)$: the probability of moving from A to B within one step when starting from π.

Definition

For a subset $S \subset \Omega$, the bottleneck ratio of S is defined to be

$$\Phi(S) = Q(S, S^c)/\pi(S).$$

The bottleneck ratio of the whole chain is defined to be

$$\Phi_\star = \min\{\Phi(S) : \pi(S) \leq 1/2\}.$$

Bottleneck ratio

Consider simple random walk on a graph $G = (V, E)$.

$$P(x, y) = \frac{1}{deg(x)} 1_{[x \sim y]}, \quad \pi(x) = \frac{deg(x)}{2|E|}.$$

Then

$$Q(S, S^c) = \frac{|\partial S|}{2|E|}, \quad \Phi(S) = \frac{|\partial S|}{\sum_{x \in S} deg(x)}.$$

Bottleneck ratio

Theorem

Suppose that $\Phi_\star$ is the bottleneck ratio, then

$$t_{mix} \geq \frac{1}{4\Phi_\star}.$$

Lemma

For any subset $S \subset \Omega$, let μ_S be π conditioned on S :

$$\mu_S(A) = \frac{\pi(A \cap S)}{\pi(S)}.$$

Then

$$||\mu_S P - \mu_S||_{TV} = \Phi(S).$$

Distinguishing statistics

Goal : find a statistic f (a function on Ω) such that the distance between $f(X_n)$ and f can be bounded from below. Recall

$$\mu f = \sum_x \mu(x) f(x), \quad var_\mu(f) = \mu f^2 - (\mu f)^2.$$

Lemma

Let μ and ν be two probability distributions on Ω. Let f be a real-valued function on Ω. If

$$|\mu f - \nu f| \geq r\sigma, \quad \textit{where } \sigma^2 = \frac{1}{2}(var_\mu(f) + var_\nu(f)),$$

then

$$||\mu - \nu||_{TV} \geq \frac{r^2}{4 + r^2}.$$

Random walk on hypercube

$N-$dimensional hypercube is a graph with vertex set $\Omega = \{0, 1\}^N$; two vertices are connected by an edge when they differ in exactly one coordinate.

The simple random walk on hypercube moves from one vertex $(x^1, ..., x^N)$ by choosing a coordinate $j \in \{1, ..., N\}$ uniformly and setting the new state to $(x^1, ..., x^{j-1}, 1 - x^j, x^{j+1}, ..., x^N)$.

The lazy walk remains at its current position with probability $1/2$ and moves as above with probability $1/2$.

The lazy walk can be constructed using the following random mapping representation :
Uniformly select an element (j, B) in $\{1, ..., N\} \times \{0, 1\}$, and then update the coordinate j with B.
Let $(Z_n = (j_n, B_n))_{n \geq 1}$ be i.i.d. $\overset{d}{\sim}$ (j, B). At each step, the coordinate j_n of X_{n-1} is updated by B_n.

Random walk on hypercube

Theorem

For the lazy walk on hypercube, there exists a constant $c_0 > 0$ such that

$$t_{mix} \geq cN \log N.$$

Proof Suppose that that lazy walk starts from $X_0 = (1, ..., 1)$. Define

$$W(\overrightarrow{x}) = \sum_{j=1}^{N} x^j.$$

Lemma

If
$$|\mu f - \nu f| \geq r\sigma, \quad \text{where } \sigma^2 = \frac{1}{2}(var_\mu(f) + var_\nu(f)),$$
then
$$||\mu - \nu||_{TV} \geq \frac{r^2}{4 + r^2}.$$