2 Poisson processes & Queueing theory

Problem 2.1. Compute the mathematical expectation of a Poisson process N_t with intensity λ. **Options:**

- λ;
- λx;
- λt;
- t.

Solution. $\mathbb{E}\, N_t = \lambda t$ by the third property of a homogeneous Poisson process (the number of events in any interval of length t is a Poisson random variable with parameter λt).

Problem 2.2. Find the probability generating function of a random variable with binomial distribution, $\mathbb{P}\{\xi = k\} = C_n^k p^k (1-p)^{n-k}$, $k = 0, 1, \ldots, n$, $p \in (0, 1)$. **Options:**

- $\varphi(u) = (up + (1-p))^k$;
- $\varphi(u) = \frac{p}{1-(1-p)u}$;
- $\varphi(u) = (up + (1-p))^n$;
- $\varphi(u) = \left(\frac{1-p}{1-pu}\right)^r$.

Solution. By the definition of the probability generating function:

$$\varphi(u) = \mathbb{E}(u^\xi) = \sum_{k=0}^{n} C_n^k p^k (1-p)^{n-k} u^k = \sum_{k=0}^{n} C_n^k (up)^k (1-p)^{n-k} = \Big((up) + (1-p)\Big)^n = (up + (1-p))^n.$$

Problem 2.3. Let N_t be a (homogeneous) Poisson process with intensity λ. Find the limit $\lim_{h\to 0} \mathbb{P}\{N_h = 1\}$. **Options:**

- $1 - \lambda h$;
- λh;
- 0;
- 1.

Solution. The third property in the alternative definition states that $\mathbb{P}\{N(t+h) - N(t) = 1\} = \lambda h + o(h)$. By setting $t = 0$, recalling that $N_0 = 0$ (almost surely) and letting $h \to 0$ one gets

$$\lim_{h\to 0} \mathbb{P}\{N_h = 1\} = \lim_{h\to 0} (\lambda h + o(h)) = 0.$$

Problem 2.4. 2 friends are chatting: one has a messaging speed equal to 3 messages per minute, another — 2 messages per minute. Assuming that for every person the process of writing the messages is modeled with Poisson process and these processes are independent, find the probability that there will be sent only 2 messages during the first minute. **Options:**

- $e^{-5t}\frac{25t}{k!}$;
- $e^{-3}\frac{6t^k}{k!}$;
- $e^{-2}\frac{6t^k}{k!}$;
- $e^{-5}\frac{25}{2}$.

Solution. By the total probability law, independency of N_1 and N_2, and some definitions one has:

$$\begin{aligned}
\mathbb{P}\{N_1 + N_2 = 2\} &= \sum_{k=0}^{2} \mathbb{P}\{N_1 = k\} \cdot \mathbb{P}\{N_2 = 2-k\} = \\
&= \sum_{k=0}^{2} \left(\frac{\lambda_1^k}{k!} e^{-\lambda_1}\right) \cdot \left(\frac{\lambda_2^{2-k}}{(2-k)!} e^{-\lambda_2}\right) = \\
&= e^{-(\lambda_1+\lambda_2)} \sum_{k=0}^{2} \frac{\lambda_1^k}{k!} \cdot \frac{\lambda_2^{2-k}}{(2-k)!} = \\
&= e^{-5} \sum_{k=0}^{2} \frac{2^k}{k!} \cdot \frac{3^{2-k}}{(2-k)!} = \\
&= e^{-5} \cdot \left(\frac{2^0}{0!} \cdot \frac{3^2}{2!} + \frac{2^1}{1!} \cdot \frac{3^1}{1!} + \frac{2^2}{2!} \cdot \frac{3^0}{0!}\right) = \\
&= e^{-5} \cdot \left(\frac{9}{2} + 2 \cdot 3 + 2\right) = \\
&= e^{-5} \frac{25}{2}.
\end{aligned}$$

Problem 2.5. Purchases in a shop are modeled by the homogeneous Poisson process: 30 purchases are made on average during an hour after the opening of the shop. Find the probability that the interval between k and $k+1$ purchases will be less than or equal to 4 minutes, given that the purchase number k was in the time moment s. **Options:**

- e^{-2s-2};
- $1 - e^{-2}$;
- $1 - e^{-s}$;
- e^{-2s}.

Solution. The answer is clearly equal to $1 - \mathbb{P}\{N(s+4) - N(s) = 0\}$, which, in its turn, is equal to $1 - \mathbb{P}\{N_4 = 0\}$, because the process is homogeneous. Further simple calculations show that

$$1 - \mathbb{P}\{N_4 = 0\} = 1 - e^{-(1/2)\cdot 4} = 1 - e^{-2}.$$

Problem 2.6. The amount of claims to an insurance company is modeled by the Poisson process, and the claim sizes are modeled by an exponential distribution. On average there are 100 claims per day, and the mean value of one claim is 5000 USD. Find the variance of the process X_t, which is equal to the total amount of claims till time t. **Options:**

- $t \cdot 10^9$;
- $5t \cdot 10^9$;
- 10^9;
- $5 \cdot 10^9$.

Solution. The process X_t is a compound Poisson process, hence its variance can be computed by the formula covered in lecture, namely

$$\operatorname{Var} X_t = \lambda t \, \mathbb{E}(\xi^2).$$

It should sound familiar to you from the course of probability theory that

$$\mathbb{E}(\xi^n) = n! \cdot \mathbb{E}\,\xi,$$

given that ξ has an exponential distribution. Therefore, one can easily compute

$$\lambda t \, \mathbb{E}(\xi^2) = 100 \cdot t \cdot 2! \cdot 5000^2 = 5t \cdot 10^9.$$

Problem 2.7. Purchases in a shop are modeled with non-homogeneous Poisson process: $30t^{5/4}$ purchases are made on average during t hours after the opening of the shop. Find the probability that the interval between k and $k+1$ purchases will be less than or equal to 2 minutes, given that the purchase number k was in the time moment s. **Options:**

- $1 - e^{-30(s+1/60)^{5/4}+30s^{5/4}} - e^{-30(s+1/20)^{5/4}+30s^{5/4}}$;
- $e^{-30(s+1/30)^{5/4}+30s^{5/4}}$;
- $e^{-30(s+1/60)^{5/4}+30s^{5/4}} - e^{-30(s+1/20)^{5/4}+30s^{5/4}}$;
- $1 - e^{-30(s+1/30)^{5/4}+30s^{5/4}}$.

Solution. The answer is clearly equal to $1 - \mathbb{P}\{N(s+2) - N(s) = 0\}$. One can recall that

$$N_{s+2} - N_s \sim \text{Pois}(\Lambda(s+2) - \Lambda(s)) = \text{Pois}(30(s+2/60)^{5/4} - 30s^{5/4}) = \text{Pois}(30(s+1/30)^{5/4} - 30s^{5/4}).$$

hence

$$1 - \mathbb{P}\{N_{s+2} - N_s = 0\} = 1 - e^{-(30(s+1/30)^{5/4}-30s^{5/4})} = 1 - e^{-30(s+1/30)^{5/4}+30s^{5/4}}.$$

Problem 2.8. The number of downloads of an app in Google-Play is modeled by a non-homogeneous Poisson process with intensity $\Lambda(t) = t^{13/5}$, where t is measured in hours after app's commencement time. Find the probability that the time between the 1000$^{\text{th}}$ and 1001$^{\text{st}}$ downloads is less than or equal to 36 seconds (0.01 hour) given 1000$^{\text{th}}$ download time being 14 hours after app's launch. **Options:**

- 0.83;
- 0.73;
- 0.6;
- 0.95.

Solution. The answer is clearly equal to $1 - \mathbb{P}\{N(14.01) - N(14.0) = 0\}$. One can recall that

$$N_{14.01} - N_{14} \sim \text{Pois}(\Lambda(14.01) - \Lambda(14)) = \text{Pois}(14.01^{13/5} - 14^{13/5}) \approx \text{Pois}(1.7743)$$

hence

$$1 - \mathbb{P}\{N_{14.01} - N_{14} = 0\} \approx 1 - e^{-1.7743} \approx 1 - 0.1696 \approx 0.83.$$

Problem 2.9. Find the probability generating function of the the random variable N_3 (where N_t is a homogeneous Poisson process). **Options:**

- $e^{-3\lambda(1-u)}$;
- $e^{3\lambda(1-u)}$;
- $e^{-3\lambda(u^2-1)}$.

Solution.

$$\begin{aligned}
\varphi_\xi(u) = \mathbb{E}(u^\xi) = \\
&= \sum_{n=0}^{\infty} u^n \cdot \mathbb{P}\{N_3 = n\} = \\
&= \sum_{n=0}^{\infty} u^n \cdot \frac{(3\lambda)^n}{n!} e^{-3\lambda} = \\
&= e^{-3\lambda} \cdot \sum_{n=0}^{\infty} u^n \cdot \frac{(3\lambda)^n}{n!} = \\
&= e^{-3\lambda} \cdot \sum_{n=0}^{\infty} \frac{(3\lambda u)^n}{n!} = \\
&= e^{-3\lambda} \cdot e^{3\lambda u} = \\
&= e^{3\lambda(u-1)}.
\end{aligned}$$

Problem 2.10. There is a speed limit on the street near the secondary school. To keep a lid on traffic violations, local administration decided to put a speed-register. If a car violates the speed limit, the register correctly identifies its ID number with probability 80%. Assume that the number of cars passing the school and violating the speed limit is modeled by the homogeneous Poisson process N_t with intensity equal to 20. Find the probability that during 2 hours after midday there will be 16 cars registered. **Options:**

- 7%;
- 10%;
- 40%;
- 5%.

Solution. According to the total probability law, the desired probability is equal to

$$\sum_{n=0}^{\infty} C_n^{16} \cdot 0.8^{16} \cdot (1-0.8)^{n-16} \cdot \frac{(2 \cdot 20)^n}{n!} \cdot e^{-(2 \cdot 20)} \approx 10\%,$$

where the last approximate equality was obtained via Wolfram Alpha.