1 Introduction & Renewal processes

Problem 1.1.

Let η be a random variable with distribution function $\mathbb{F}_\eta$. Define a stochastic process $X_t = \eta + t$. Compute the distribution function of a finite-dimensional distribution $(X_{t_1}, \dots, X_{t_n})$, where $t_1, \dots, t_n \in \mathbb{R}_+$. **Options:**

- $\mathbb{F}_\eta\{\min(t_1, \dots, t_n)\}$;
- $\mathbb{F}_\eta\{\min(x_1, \dots, x_n)\}$;
- $\mathbb{F}_\eta\{\min(x_1 - t_1, \dots, x_n - t_n)\}$;
- none of above.

Solution.

$$\begin{aligned}
\mathbb{F}_{(X_{t_1},\dots,X_{t_n})}(x_1, \dots, x_n) &= \mathbb{P}\{X_{t_1} \le x_1, \dots, X_{t_n} \le x_n\} = \\
&= \mathbb{P}\{\eta + t_1 \le x_1, \dots, \eta + t_n \le x_n\} = \\
&= \mathbb{P}\{\eta \le x_1 - t_1, \dots, \eta \le x_n - t_n\} = \\
&= \mathbb{P}\{\eta \le \min(x_1 - t_1, \dots, x_n - t_n)\} = \\
&= \mathbb{F}_\eta\{\min(x_1 - t_1, \dots, x_n - t_n)\}.
\end{aligned}$$

Problem 1.2. Let S_n be a renewal process such that $\xi_n = S_n - S_{n-1}$ takes the values 1 or 2 with equal probabilities $p = 1/2$. Find the mathematical expectation of the counting process N_t at $t = 3$. **Options:**

- 15/8;
- 7/8;
- 1/8;
- 3;
- none of above.

Solution.

$$\begin{aligned}
\mathbb{E}N_3 = &\ \mathbb{P}\{\xi_1 = 1, \xi_2 = 1, \xi_3 = 1\} \cdot 3 + \\
&+ \mathbb{P}\{\xi_1 = 1, \xi_2 = 1, \xi_3 = 2\} \cdot 2 + \\
&+ \mathbb{P}\{\xi_1 = 1, \xi_2 = 2\} \cdot 2 + \\
&+ \mathbb{P}\{\xi_1 = 2, \xi_2 = 1\} \cdot 2 + \\
&+ \mathbb{P}\{\xi_1 = 2, \xi_2 = 2\} \cdot 1 = \\
= &\ 3/8 + 2/8 + 2/4 + 2/4 + 1/4 = 15/8.
\end{aligned}$$

Problem 1.3. Let $S_n = S_{n-1} + \xi_n$ be a renewal process and $p_\xi(x) = \lambda e^{-\lambda x}$. Find the mathematical expectation of the corresponding counting process N_t. **Options:**

- $1/\lambda$;
- $1/\lambda^2$;
- λ;
- λ^2;
- none of above.

Solution.

1. $p \to \mathscr{L}_p$.

$$\mathscr{L}_{p_\xi}(s) = \int_0^\infty e^{-sx} p_\xi(x)\,\mathrm{d}x = \int_0^\infty e^{-sx} \lambda e^{-\lambda x}\,\mathrm{d}x = \lambda \int_0^\infty e^{-(s+\lambda)x}\,\mathrm{d}x = \lambda \cdot -\frac{e^{-(s+\lambda)x}}{s+\lambda}\bigg|_0^\infty = \frac{\lambda}{s+\lambda}.$$

2. $\mathscr{L}_p \to \mathscr{L}_U$.

$$\mathscr{L}_U(s) = \frac{\mathscr{L}_p(s)}{s(1 - \mathscr{L}_p(s))} = \frac{\frac{\lambda}{s+\lambda}}{s\left(1 - \frac{\lambda}{s+\lambda}\right)} = \frac{\frac{\lambda}{s+\lambda}}{s \cdot \frac{s}{s+\lambda}} = \frac{\lambda}{s^2}.$$

3. $\mathscr{L}_U \to U$. we **guess** that $U(t) = \mathscr{L}_s^{-1}\left(\frac{\lambda}{s^2}\right)(t) = \lambda t$.

Problem 1.4. Let η be a random variable with distribution function $\mathbb{F}_\eta$. Define a stochastic process $X_t = e^\eta t^2$. What is the distribution function of $(X_{t_1}, \dots, X_{t_n})$ for positive $t_1, \dots, t_n$? **Options:**

- 0;
- $\mathbb{F}_\eta\{\min(\ln(x_1/t_1^2), \dots, \ln(x_n/t_n^2))\}$;
- $\mathbb{F}_\eta\{\min(\ln(x_1/t_1), \dots, \ln(x_n/t_n))\}$;
- none of above.

Solution.

$$\begin{aligned}
\mathbb{F}_{\left(X_{t_1},\dots,X_{t_n}\right)}(x_1, \dots, x_n) &= \mathbb{P}\left\{X_{t_1} \le x_1, \dots, X_{t_n} \le x_n\right\} = \\
&= \mathbb{P}\{e^\eta t_1^2 \le x_1, \dots, e^\eta t_n^2 \le x_n\} = \\
&= \mathbb{P}\{e^\eta \le x_1/t_1^2, \dots, e^\eta \le x_n/t_n^2\} = \\
&= \mathbb{P}\{\eta \le \ln(x_1/t_1^2), \dots, \eta \le \ln(x_n/t_n^2)\} = \\
&= \mathbb{P}\{\eta \le \min(\ln(x_1/t_1^2), \dots, \ln(x_n/t_n^2))\} = \\
&= \mathbb{F}_\eta\{\min(\ln(x_1/t_1^2), \dots, \ln(x_n/t_n^2))\}.
\end{aligned}$$

Problem 1.5. Let N_t be a counting process of a renewal process $S_n = S_{n-1} + \xi_n$ such that the i.i.d. random variables $\xi_1, \xi_2, \dots$ have a probability density function

$$p_\xi(x) = \begin{cases} \frac{1}{2}e^{-x}(x+1), & x \ge 0, \\ 0, & x < 0. \end{cases}$$

Find the mean of N_t. **Options:**

- $-\frac{1}{9} + \frac{4}{3}t + \frac{1}{9}e^{-(3/2)t}$;
- $-\frac{1}{9} + \frac{2}{3}t + \frac{1}{9}e^{-(3/2)t}$;
- $-\frac{1}{9} + \frac{2}{3}t + \frac{1}{9}e^{3/2t}$;
- none of above.

Solution.

1. $p \to \mathscr{L}_p$.

$$\begin{aligned}
\mathscr{L}_{p_\xi}(s) &= \int_0^\infty e^{-sx} p_\xi(x)\,\mathrm{d}x = \\
&= \int_0^\infty e^{-sx}\tfrac{1}{2}e^{-x}(x+1)\,\mathrm{d}x = \\
&= \frac{1}{2}\int_0^\infty e^{-(s+1)x}(x+1)\,\mathrm{d}x = \\
&= \frac{s+2}{2(s+1)^2}.
\end{aligned}$$

2. $\mathscr{L}_p \to \mathscr{L}_U$.

$$\mathscr{L}_U(s) = \frac{\mathscr{L}_p(s)}{s(1-\mathscr{L}_p(s))} = \frac{\frac{s+2}{2(s+1)^2}}{s\left(1 - \frac{s+2}{2(s+1)^2}\right)} = \frac{\frac{s+2}{2(s+1)^2}}{s \cdot \frac{2s^2+3s}{2(s+1)^2}} = \frac{s+2}{s^2(2s+3)}.$$

3. $\mathscr{L}_U \to U$. we first decompose $\mathscr{L}_U(s)$ into elementary fractions.

$$\frac{s+2}{s^2(2s+3)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{2s+3}.$$

One can check that $A = 2/3$, $B = -1/9$, $C = 2/9$.

We then **guess** that

$$\begin{aligned}
U(t) &= \mathscr{L}_s^{-1}\left(\frac{s+2}{s^2(2s+3)}\right)(t) = \\
&= \mathscr{L}_s^{-1}\left(\frac{2}{3s^2} - \frac{1}{9s} + \frac{2}{9(2s+3)}\right)(t) = \\
&= \mathscr{L}_s^{-1}\left(\frac{2}{3s^2}\right)(t) - \mathscr{L}_s^{-1}\left(\frac{1}{9s}\right)(t) + \mathscr{L}_s^{-1}\left(\frac{2}{9(2s+3)}\right)(t) = \\
&= \frac{2}{3}t + \frac{1}{9}e^{-(3/2)t} - \frac{1}{9}.
\end{aligned}$$

Problem 1.6. Let ξ and η be 2 random variables. It is known that the distribution of η is symmetric, that is, $\mathbb{P}\{\eta > x\} = \mathbb{P}\{\eta < -x\}$ for any $x > 0$, and moreover $\mathbb{P}\{\eta = 0\} = 0$. Find the probability of the event that the trajectories of stochastic process $X_t = \xi^2 + t(\eta + t)$, $t \geq 0$ increase. **Options:**

- 0;
- 1/2;
- 1/4;
- 1;
- none of above.

Solution. With ξ and η fixed, $X(t)$ increase iff $\dot{X}(t) > 0$ for all $t \in \mathbb{R}_+$. Simple calculations show that $\dot{X}(t) = \eta + 2t$. Hence, the inequality $\dot{X}(t) > 0$ for all $t \in \mathbb{R}_+$ is equivalent to $\eta + 2t > 0$ for all $t \in \mathbb{R}_+$. The last inequality, in its turn, is clearly equivalent to $\eta > 0$. Finally, it is well known that $\mathbb{P}\{\eta > 0\}$ is 1/2 for symmetric random variables with $\mathbb{P}\{\eta = 0\} = 0$. Indeed, $\mathbb{P}\{\eta > 0\} = \mathbb{P}\{\eta < 0\}$, and both sides of this equality have to be equal to 1/2, because their sum is $1 - \mathbb{P}\{\eta = 0\} = 1$.