Gaussian processes

1. Consider the condition from the Kolmogorov continuity theorem:

$$\mathsf{E}\Big[|X_t - X_s|^\alpha\Big] \le K|t-s|^{1+\beta}, \quad \forall t, s > 0.$$

For which parameters α, K and β this condition holds, if X_t is a Brownian motion?

- $\alpha = 3$, $K = 3$ and $\beta = 2$;
- $\alpha = 4$, $K = 2$ and $\beta = 3$;
- $\alpha = 4$, $K = 3$ and $\beta = 1$;
- none of the above.

Solution: recall the lecture material:

$$\mathsf{E}\Big[|X_t - X_s|^4\Big] = (t-s)^2\mathsf{E}\big[\xi^4\big]$$

where $\xi \sim N(0,1)$. Then one should recall that $\mathsf{E}\big[\xi^4\big] = 3$ for $\xi \sim N(0,1)$, hence the answer.

Note: it wasn't necessary to remember the exact value of K because pair (α, β) is different for all options.

2. Choose the right statements about the Brownian motion W_t:

- W_t has continuous trajectories;
- $W_t - W_s \sim N(0, t-s)$;
- $W_0 = 0$ almost surely;
- W_t has symmetric distribution for any $t > 0$;
- W_t has independent increments.

Solution: recall the lecture material.

3. Let $X_t = e^{W_t}$, where W_t is a Brownian motion. Find mathematical expectation $\mathsf{E}[X_t]$, variance $\mathrm{Var}(X_t)$ and covariance function $K(t,s) = \mathrm{cov}(X_t, X_s)$ (in the answers below it is assumed that $t > s \ge 0$).

- $\mathsf{E}[X_t] = e^{t/2}$, $\mathrm{Var}[X_t] = e^{2t} - e^t$, $\mathrm{cov}(X_t, X_s) = e^{\frac{3s+t}{2}} - e^{\frac{s+t}{2}}$;
- $\mathsf{E}[X_t] = e^{2t} - e^t$, $\mathrm{Var}[X_t] = e^{t/2}$, $\mathrm{cov}(X_t, X_s) = e^{\frac{3s+2t}{2}} - e^{\frac{s-t}{2}}$;
- $\mathsf{E}[X_t] = e^{t/2}$, $\mathrm{Var}[X_t] = e^{2t} - e^t$, $\mathrm{cov}(X_t, X_s) = e^{\frac{3s+2t}{2}} - e^{\frac{s+t}{2}}$;
- none of the above.

Solution: easy straightforward calculations show that

$$\begin{aligned}\mathsf{E}[X_t] = \mathsf{E}\Big[e^{W_t}\Big] &= \\ &= \int_{-\infty}^{\infty} \frac{e^x}{\sqrt{2\pi t}} \exp\left\{-\frac{x^2}{2t}\right\} \mathrm{d}x = \\ &= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \exp\left\{x - \frac{x^2}{2t}\right\} \mathrm{d}x = \\ &= \frac{1}{\sqrt{2\pi t}} \sqrt{2\pi t} e^{t/2} = e^{t/2}.\end{aligned}$$

Still straightforward yet not so easy calculations show that

$$\begin{aligned}
\mathrm{Var}[X_t] = \mathrm{Var}\left[e^{W_t}\right] &= \\
&= \mathsf{E}\left[\left(e^{W_t} - e^{t/2}\right)^2\right] = \\
&= \mathsf{E}\left[e^{2W_t}\right] - 2e^t + e^t = \\
&= \mathsf{E}\left[e^{2W_t}\right] - e^t = \\
&= \int_{-\infty}^{\infty} \frac{e^{2x}}{\sqrt{2\pi t}} \exp\left\{-\frac{x^2}{2t}\right\} \mathrm{d}x - e^t = \\
&= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \exp\left\{2x - \frac{x^2}{2t}\right\} \mathrm{d}x - e^t = \\
&= \frac{1}{\sqrt{2\pi t}} \sqrt{2\pi t} e^{2t} - e^t = \\
&= e^{2t} - e^t.
\end{aligned}$$

Finally, somewhat harder calculations show that

$$\begin{aligned}
\mathrm{cov}(X_t, X_s) = \mathrm{cov}\left(e^{W_t}, e^{W_s}\right) &= \\
&= \mathsf{E}\left[e^{W_t + W_s}\right] - \mathsf{E}\left[e^{W_t}\right]\mathsf{E}\left[e^{W_s}\right] = \\
&= \mathsf{E}\left[e^{W_t + W_s}\right] - e^{\frac{t+s}{2}} = \\
&= \mathsf{E}\left[e^{2W_s + \xi(t-s)}\right] - e^{\frac{t+s}{2}} = \\
&= \mathsf{E}\left[e^{2W_s}\right]\mathsf{E}\left[e^{\xi(t-s)}\right] - e^{\frac{t+s}{2}} = \\
&= e^{2s}\mathsf{E}\left[e^{\xi(t-s)}\right] - e^{\frac{t+s}{2}} = \\
&= e^{2s} e^{\frac{t-s}{2}} - e^{\frac{t+s}{2}} = \\
&= e^{\frac{3s+t}{2}} - e^{\frac{t+s}{2}},
\end{aligned}$$

where $\xi \sim N(0,1)$.

4. Let W_t be the Brownian motion. Calculate $\mathsf{P}\{W_1 + W_2 > 2\}$. In the possible answers below Φ is the distribution function of the standard normal distribution.

   - $1 - \Phi(2/\sqrt{5})$, where Φ is a normal distribution function;
   - $\Phi(1/2)$, where Φ is a normal distribution function;
   - $1/2$;
   - $\Phi(2/\sqrt{5})$, where Φ is a normal distribution function
   - none of above.

   Solution: $W_2 = W_1 + \eta$, with $\eta \perp\!\!\!\perp W_1$, where $\eta \sim N(0,1)$, and $W_1 = \xi$, where $\xi \sim N(0,1)$ hence we can write

   $$\mathsf{P}\{W_1 + W_2 > 2\} = \mathsf{P}\{2\xi + \eta > 2\}.$$

   It is known that $2N(0,1) \sim N(0,2)$ and that for independent random variables $N(0,2) + N(0,1) = N(0,5)$. Finally,

   $$\begin{aligned}
   \mathsf{P}\{N(0,5) > 2\} = \mathsf{P}\{N(0,1) > 2/\sqrt{5}\} &= \\
   &= 1 - \Phi(2/\sqrt{5}).
   \end{aligned}$$

   Note: it was clear from the beginning that out answer is less than $1/2$.

5. Which properties hold for the covariance function $K(t,s)$ of a stochastic process?

   - K is symmetric, that is, $K(t,s) = K(s,t)$, $\forall t, s \in \mathbb{R}^+$;
   - K is positive semidefinite, that is,

   $$\sum_{j,k} u_j u_k K(t_j, t_k) \geq 0, \quad \forall t_1, \dots, t_n \in \mathbb{R}^+, \quad \forall u_1, \dots, u_n \in \mathbb{R}^+;$$

   - $K(0,0) = 1$;
   - K is positive definite, that is,

   $$\sum_{j,k} u_j u_k K(t_j, t_k) > 0, \quad \forall t_1, \dots, t_n \in \mathbb{R}^+, \quad \forall u_1, \dots, u_n \in \mathbb{R}^+;$$

   - none of above.

   Solution: recall the lecture material.

6. Let W_t be the Brownian motion. Choose the processes which are also the Brownian motions.

   - aW_{t/a^2} with some fixed $a \neq 0.$;
   - $-W_t$;
   - $W_{t+s} - W_s$ with some fixed $s > 0$;
   - $tW_{1/t}$, $t > 0$, and $W_0 = 0$.

   Solution: for the first three one can easily check the properties from the definition of a Brownian motion. For the last one, it is also a Brownian motion, even though it is somewhat harder to check: first we need to check that $\mathsf{E}(X_t X_s) = \min\{t, s\}$. Indeed:

   $$\mathsf{E}[X_t X_s] = \mathsf{E}\left[tW_{1/t} s W_{1/s}\right] = ts\mathsf{E}[W_{1/t} W_{1/s}].$$

   Given $t > s$ one gets $1/t < 1/s$ and hence

   $$\mathsf{E}[W_{1/t} W_{1/s}] = \min\{1/t, 1/s\} = 1/t.$$

   Substituting into the lest equations gives $ts(1/t) = s = \min\{t, s\}$, as desired.

   Now, we have that $\mathsf{E}[X_t - X_s]^2 = t - s$ for all $s < t$, and

   $$\mathsf{E}\Big[(X_t - X_s)(X_s - X_u)\Big] = s - u - s - u = 0$$

   for $u < s < t$. From this we have that the increments are independent and $N(0, t-s)$.

   Finally, the trajectories are continuous since

   $$\lim_{t \to 0} X_t = \lim_{s \to \infty} \frac{W_s}{s} = 0$$

   as stated in the lecture.

7. Let $Y_{n+1} = aY_n + X_n$, where $n = 0, 1, 2, \dots$. $Y_0 = 0$, $|a| < 1$, $X_0, X_1, X_2, \dots \sim N(0,1)$ i.i.d. Find $\operatorname{cov}(Y_4, Y_3)$.

   - $a^5 + a^3 + a$;
   - $a^5 + a^4 + a^3 + a^2 + a + 1$;
   - $a^5 + a^4 + a^3 + a^2 + a$;
   - $a^4 + a^2 + 1$.

Solution: first note that $\mathsf{E}(Y_0) = 0$ and

$$\begin{aligned}\mathsf{E}(Y_{n+1}) &= \mathsf{E}(aY_n + X_n) = \\ &= \mathsf{E}(aY_n) + \mathsf{E}(X_n) = \\ &= \mathsf{E}(aY_n) = \\ &= a\mathsf{E}(Y_n) = \\ &= 0\end{aligned}$$

by induction on n.

Next note that $\mathsf{E}(Y_0^2) = 0$ and

$$\begin{aligned}\mathsf{E}(Y_{n+1}^2) &= \mathsf{E}((aY_n + X_n)^2) = \\ &= \mathsf{E}(a^2Y_n^2 + 2aX_nY_n + X_n^2) = \\ &= a^2\mathsf{E}(Y_n^2) + 2a\mathsf{E}(X_nY_n) + \mathsf{E}(X_n^2) \overset{Y_n \perp\!\!\!\perp X_n}{=} \\ &= a^2\mathsf{E}(Y_n^2) + 2a\mathsf{E}(X_n)\mathsf{E}(Y_n) + \mathsf{E}(X_n^2) = \\ &= a^2\mathsf{E}(Y_n^2) + \mathsf{E}(X_n^2) = \\ &= a^2\mathsf{E}(Y_n^2) + 1.\end{aligned}$$

From here one can derive the identity $\mathsf{E}(Y_n^2) = a^{2n} + a^{2(n-1)} + \ldots + a^2 + 1$ by induction on n.

Finally, we see that

$$\begin{aligned}\operatorname{cov}(Y_4, Y_3) &= \mathsf{E}(Y_4Y_3) - \mathsf{E}(Y_4)\mathsf{E}(Y_3) = \\ &= \mathsf{E}((aY_3 + X_3)Y_3) - \mathsf{E}(aY_3 + X_3)\mathsf{E}(Y_3) = \\ &= \mathsf{E}(aY_3^2 + X_3Y_3) - \mathsf{E}(aY_3 + X_3)\mathsf{E}(Y_3) = \\ &= \mathsf{E}(aY_3^2) - \mathsf{E}(aY_3)\mathsf{E}(Y_3) = \\ &= a\mathsf{E}(Y_3^2) - a(\mathsf{E}(Y_3))^2 = \\ &= a(a^4 + a^2 + 1) = a^5 + a^3 + a.\end{aligned}$$

8. Let X_t be a Brownian motion. Find

$$K(t, s) - \operatorname{Var}\Big(X_{\min\{t,s\}}\Big).$$

- $\min\{t, s\}$;
- 0;
- $2(t + s)$;
- $\max\{t, s\}$.

Solution: first

$$K(t, s) = \min\{t, s\}$$

Second

$$\operatorname{Var}\Big(X_{\min\{t,s\}}\Big) = \min\{t, s\}.$$

Therefore the answer is 0.