18.445 Introduction to Stochastic Processes

Lecture 2: Markov chains: stationary distribution

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17 February 2015

Announcement : The midterm is postponed to April 6th.

Recall

Consider a Markov chain with state space Ω and transition matrix P :

$$\mathbb{P}[X_{n+1} = y \mid X_n = x] = P(x, y).$$

- μ_0 : the distribution of X_0 ; μ_n : the distribution of X_n.
- We call a probability measure π is stationary if $\pi = \pi P$.
- If π is stationary and $\mu_0 = \pi$, then $\mu_n = \pi, \forall n$.

Today's goal

- Irreducible, aperiodic
- existence and uniqueness of stationary distribution.

Irreducible

Definition

A transition matrix P is called irreducible, if
for any $x, y \in \Omega$, there exists a number n (possibly depending on x, y) such that

$$P^n(x, y) > 0.$$

Definition

For any $x \in \Omega$, define $T(x) = \{n \geq 1 : P^n(x, x) > 0\}$. The period of state x is the greatest common divisor of $T(x)$, denoted by $gcd(T(x))$.

Lemma

If P is irreducible, then $gcd(T(x)) = gcd(T(y))$ for all $x, y \in \Omega$.

Aperiodic

Definition

For an irreducible chain, the period of the chain is defined to be the period which is common to all states.
The chain is aperiodic if all states have period 1.

Example Consider a simple random walk on $N-$cycle where N is odd. Then the walk is irreducible and aperiodic.

Theorem

If P is irreducible and aperiodic, then there exists an integer r such that

$$P^n(x, y) > 0, \quad \forall x, y \in \Omega, \forall n \geq r.$$

Existence of stationary distribution

Definition

For $x \in \Omega$, define

$$\tau_x = \inf\{n \geq 0 : X_n = x\}, \quad \tau_x^+ = \inf\{n \geq 1 : X_n = x\}.$$

τ_x : the hitting time for x. τ_x^+ : the first return time when $X_0 = x$.

Lemma

Suppose that P is irreducible. Then, for any $x, y \in \Omega$, we have

$$\mathbb{E}_x[\tau_y^+] < \infty.$$

Theorem

Suppose that P is irreducible, then there exists a probability measure π such that $\pi = \pi P$ and $\pi(x) > 0$ for all $x \in \Omega$.

Uniqueness of stationary distribution

Recall and Definition

- μ : a measure on Ω, μP is still a measure on Ω.
- f : a function on Ω, Pf is still a function on Ω.
- If $\mu = \mu P$, we say that μ is stationary.
- If $f = Pf$, we say that f is harmonic.

Lemma

Suppose that P is irreducible. Then any harmonic function f on Ω has to be constant.

Theorem

Suppose that P is irreducible. Then there exists a unique stationary distribution. Moreover,

$$\pi(x) = \frac{1}{\mathbb{E}_x[\tau_x^+]}, \quad \forall x \in \Omega.$$

Summary about stationary distribution

- In the proof of the existence and the uniqueness of stationary distribution, the crucial assumption is that P is irreducible.
- When P is irreducible, we can explicitly write out the stationary distribution
$$\pi(x) = \frac{1}{\mathbb{E}_x[\tau_x^+]} > 0, \forall x \in \Omega.$$
- In fact, the existence does not require irreducibility, but the uniqueness does.