

4 Gaussian processes

Problem 4.1. Consider the condition from the Kolmogorov continuity theorem:

$$\mathbb{E} \left[|X_t - X_s|^\alpha \right] \leq K |t - s|^{1+\beta}, \quad \forall t, s > 0.$$

For which parameters α , K and β this condition holds, if X_t is a Brownian motion? **Options:**

- $\alpha = 3$, $K = 3$ and $\beta = 2$;
- $\alpha = 4$, $K = 2$ and $\beta = 3$;
- $\alpha = 4$, $K = 3$ and $\beta = 1$;
- none of the above.

Solution. Recall the lecture material:

$$\mathbb{E} \left[|X_t - X_s|^4 \right] = (t - s)^2 \mathbb{E} [\xi^4]$$

where $\xi \sim N(0, 1)$. Then one should recall that $\mathbb{E} [\xi^4] = 3$ for $\xi \sim N(0, 1)$, hence the answer.

Problem 4.2. Choose the right statements about the Brownian motion W_t . **Options:**

- W_t has continuous trajectories;
- $W_t - W_s \sim N(0, t - s)$;
- $W_0 = 0$ almost surely;
- W_t has symmetric distribution for any $t > 0$;
- W_t has independent increments.

Solution. Recall the lecture material.

Problem 4.3. Let $X_t = e^{W_t}$, where W_t is a Brownian motion. Find mathematical expectation $\mathbb{E}[X_t]$, variance $\text{Var}(X_t)$ and covariance function $K(t, s) = \text{cov}(X_t, X_s)$ (in the answers below it is assumed that $t > s \geq 0$). **Options:**

- $\mathbb{E}[X_t] = e^{t/2}$, $\text{Var}[X_t] = e^{2t} - e^t$, $\text{cov}(X_t, X_s) = e^{\frac{3s+t}{2}} - e^{\frac{s+t}{2}}$;
- $\mathbb{E}[X_t] = e^{2t} - e^t$, $\text{Var}[X_t] = e^{t/2}$, $\text{cov}(X_t, X_s) = e^{\frac{3s+2t}{2}} - e^{\frac{s-t}{2}}$;
- $\mathbb{E}[X_t] = e^{t/2}$, $\text{Var}[X_t] = e^{2t} - e^t$, $\text{cov}(X_t, X_s) = e^{\frac{3s+2t}{2}} - e^{\frac{s+t}{2}}$;
- none of the above.

Solution. Easy straightforward calculations show that

$$\mathbb{E} X_t = \mathbb{E} e^{W_t} = \int_{-\infty}^{\infty} \frac{e^x}{\sqrt{2\pi t}} \exp \left\{ -\frac{x^2}{2t} \right\} dx = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \exp \left\{ x - \frac{x^2}{2t} \right\} dx = \frac{1}{\sqrt{2\pi t}} \sqrt{2\pi t} e^{t/2} = e^{t/2}.$$

Still straightforward yet not so easy calculations show that

$$\begin{aligned} \text{Var}[X_t] &= \text{Var} \left[e^{W_t} \right] = \\ &= \mathbb{E} \left[\left(e^{W_t} - e^{t/2} \right)^2 \right] = \\ &= \mathbb{E} \left[e^{2W_t} \right] - 2e^t + e^t = \\ &= \mathbb{E} \left[e^{2W_t} \right] - e^t = \\ &= \int_{-\infty}^{\infty} \frac{e^{2x}}{\sqrt{2\pi t}} \exp \left\{ -\frac{x^2}{2t} \right\} dx - e^t = \\ &= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \exp \left\{ 2x - \frac{x^2}{2t} \right\} dx - e^t = \\ &= \frac{1}{\sqrt{2\pi t}} \sqrt{2\pi t} e^{2t} - e^t = \\ &= e^{2t} - e^t. \end{aligned}$$

Finally, somewhat harder calculations show that

$$\begin{aligned}
 \text{cov}(X_t, X_s) &= \text{cov}(e^{W_t}, e^{W_s}) = \\
 &= \mathbb{E}[e^{W_t+W_s}] - \mathbb{E}[e^{W_t}] \mathbb{E}[e^{W_s}] = \\
 &= \mathbb{E}[e^{W_t+W_s}] - e^{\frac{t+s}{2}} = \\
 &= \mathbb{E}[e^{2W_s+\xi(t-s)}] - e^{\frac{t+s}{2}} = \\
 &= \mathbb{E}[e^{2W_s}] \mathbb{E}[e^{\xi(t-s)}] - e^{\frac{t+s}{2}} = \\
 &= e^{2s} \mathbb{E}[e^{\xi(t-s)}] - e^{\frac{t+s}{2}} = \\
 &= e^{2s} e^{\frac{t-s}{2}} - e^{\frac{t+s}{2}} = \\
 &= e^{\frac{3s+t}{2}} - e^{\frac{t+s}{2}},
 \end{aligned}$$

where $\xi \sim N(0, 1)$.

Problem 4.4. Let W_t be the Brownian motion. Calculate $P\{W_1 + W_2 > 2\}$. In the possible answers below Φ is the distribution function of the standard normal distribution. **Options:**

- $1 - \Phi(2/\sqrt{5})$, where Φ is a normal distribution function;
- $\Phi(1/2)$, where Φ is a normal distribution function;
- $1/2$;
- $\Phi(2/\sqrt{5})$, where Φ is a normal distribution function
- none of above.

Solution. $W_2 = W_1 + \eta$, with $\eta \perp W_1$, where $\eta \sim N(0, 1)$, and $W_1 = \xi$, where $\xi \sim N(0, 1)$ hence we can write

$$P\{W_1 + W_2 > 2\} = P\{2\xi + \eta > 2\}.$$

It is known that $2N(0, 1) \sim N(0, 2)$ and that for independent random variables $N(0, 2) + N(0, 1) = N(0, 5)$. Finally,

$$P\{N(0, 5) > 2\} = P\{N(0, 1) > 2/\sqrt{5}\} = 1 - \Phi(2/\sqrt{5}).$$

Problem 4.5. Which properties hold for the covariance function $K(t, s)$ of a stochastic process? **Options:**

- K is symmetric, that is, $K(t, s) = K(s, t)$, $\forall t, s \in \mathbb{R}^+$;
- K is positive semidefinite, that is,

$$\sum_{j,k} u_j u_k K(t_j, t_k) \geq 0, \quad \forall t_1, \dots, t_n \in \mathbb{R}^+, \quad \forall u_1, \dots, u_n \in \mathbb{R}^+;$$

- $K(0, 0) = 1$;
- K is positive definite, that is,

$$\sum_{j,k} u_j u_k K(t_j, t_k) > 0, \quad \forall t_1, \dots, t_n \in \mathbb{R}^+, \quad \forall u_1, \dots, u_n \in \mathbb{R}^+;$$

- none of above.

Solution. Recall the lecture material.

Problem 4.6. Let W_t be the Brownian motion. Choose the processes which are also the Brownian motions. **Options:**

- aW_{t/a^2} with some fixed $a \neq 0$;
- $-W_t$;

- $W_{t+s} - W_s$ with some fixed $s > 0$;
- $tW_{1/t}$, $t > 0$, and $W_0 = 0$.

Solution. For the first three one can easily check the properties from the definition of a Brownian motion. For the last one, it is also a Brownian motion, even though it is somewhat harder to check: first we need to check that $\mathbb{E}(X_t X_s) = \min\{t, s\}$. Indeed:

$$\mathbb{E}[X_t X_s] = \mathbb{E}[tW_{1/t}sW_{1/s}] = ts \mathbb{E}[W_{1/t}W_{1/s}].$$

Given $t > s$ one gets $1/t < 1/s$ and hence

$$\mathbb{E}[W_{1/t}W_{1/s}] = \min\{1/t, 1/s\} = 1/t.$$

Substituting into the last equations gives $ts(1/t) = s = \min\{t, s\}$, as desired.

Now, we have that $\mathbb{E}[X_t - X_s]^2 = t - s$ for all $s < t$, and

$$\mathbb{E}[(X_t - X_s)(X_s - X_u)] = s - u - s - u = 0$$

for $u < s < t$. From this we have that the increments are independent and $N(0, t - s)$.

Finally, the trajectories are continuous since

$$\lim_{t \rightarrow 0} X_t = \lim_{s \rightarrow \infty} \frac{W_s}{s} = 0$$

as stated in the lecture.

Problem 4.7. Let $Y_{n+1} = aY_n + X_n$, where $n = 0, 1, 2, \dots$ $Y_0 = 0$, $|a| < 1$, $X_0, X_1, X_2, \dots \sim N(0, 1)$ i.i.d. Find $\text{cov}(Y_4, Y_3)$. **Options:**

- $a^5 + a^3 + a$;
- $a^5 + a^4 + a^3 + a^2 + a + 1$;
- $a^5 + a^4 + a^3 + a^2 + a$;
- $a^4 + a^2 + 1$.

Solution. First note that $\mathbb{E}(Y_0) = 0$ and

$$\mathbb{E}Y_{n+1} = \mathbb{E}(aY_n + X_n) = \mathbb{E}(aY_n) + \mathbb{E}X_n = \mathbb{E}(aY_n) = a \mathbb{E}Y_n = 0$$

by induction on n .

Next note that $\mathbb{E}(Y_0^2) = 0$ and

$$\begin{aligned} \mathbb{E}(Y_{n+1}^2) &= \mathbb{E}((aY_n + X_n)^2) = \\ &= \mathbb{E}(a^2Y_n^2 + 2aX_nY_n + X_n^2) = \\ &= a^2 \mathbb{E}(Y_n^2) + 2a \mathbb{E}(X_nY_n) + \mathbb{E}(X_n^2) \stackrel{Y_n \perp X_n}{=} \\ &= a^2 \mathbb{E}(Y_n^2) + 2a \mathbb{E}(X_n) \mathbb{E}(Y_n) + \mathbb{E}(X_n^2) = \\ &= a^2 \mathbb{E}(Y_n^2) + \mathbb{E}(X_n^2) = \\ &= a^2 \mathbb{E}(Y_n^2) + 1. \end{aligned}$$

From here one can derive the identity $\mathbb{E}(Y_n^2) = a^{2n} + a^{2(n-1)} + \dots + a^2 + 1$ by induction on n .

Finally, we see that

$$\begin{aligned} \text{cov}(Y_4, Y_3) &= \mathbb{E}(Y_4Y_3) - \mathbb{E}(Y_4) \mathbb{E}(Y_3) = \\ &= \mathbb{E}((aY_3 + X_3)Y_3) - \mathbb{E}(aY_3 + X_3) \mathbb{E}(Y_3) = \\ &= \mathbb{E}(aY_3^2 + X_3Y_3) - \mathbb{E}(aY_3 + X_3) \mathbb{E}(Y_3) = \\ &= \mathbb{E}(aY_3^2) - \mathbb{E}(aY_3) \mathbb{E}(Y_3) = \\ &= a \mathbb{E}(Y_3^2) - a(\mathbb{E}(Y_3))^2 = \\ &= a(a^4 + a^2 + 1) = a^5 + a^3 + a. \end{aligned}$$

Problem 4.8. Let X_t be a Brownian motion. Find

$$K(t, s) - \text{Var}\left(X_{\min\{t, s\}}\right).$$

Options:

- $\min\{t, s\}$;
- 0;
- $2(t + s)$;
- $\max\{t, s\}$.

Solution. First

$$K(t, s) = \min\{t, s\}$$

Second

$$\text{Var}\left(X_{\min\{t, s\}}\right) = \min\{t, s\}.$$

Therefore the answer is 0.