Introduction & Renewal processes

1. Let η be a random variable with distribution function $\mathbb{F}_\eta$. Define a stochastic process $X_t = \eta + t$. Compute the distribution function of a finite-dimensional distribution $(X_{t_1}, \ldots, X_{t_n})$, where $t_1, \ldots, t_n \in \mathbb{R}_+$:

   - $\mathbb{F}_\eta\{\min(t_1, \ldots, t_n)\}$;
   - $\mathbb{F}_\eta\{\min(x_1, \ldots, x_n)\}$;
   - $\mathbb{F}_\eta\{\min(x_1 - t_1, \ldots, x_n - t_n)\}$;
   - none of above.

   Solution:
$$\begin{aligned}\mathbb{F}_{(X_{t_1},\ldots,X_{t_n})}(x_1, \ldots, x_n) &= \mathbb{P}\{X_{t_1} \le x_1, \ldots, X_{t_n} \le x_n\} = \\ &= \mathbb{P}\{\eta + t_1 \le x_1, \ldots, \eta + t_n \le x_n\} = \\ &= \mathbb{P}\{\eta \le x_1 - t_1, \ldots, \eta \le x_n - t_n\} = \\ &= \mathbb{P}\{\eta \le \min(x_1 - t_1, \ldots, x_n - t_n)\} = \\ &= \mathbb{F}_\eta\{\min(x_1 - t_1, \ldots, x_n - t_n)\}.\end{aligned}$$

2. Let S_n be a renewal process such that $\xi_n = S_n - S_{n-1}$ takes the values 1 or 2 with equal probabilities $p = 1/2$. Find the mathematical expectation of the counting process N_t at $t = 3$:

   - 15/8;
   - 7/8;
   - 1/8;
   - 3;
   - none of above.

   Solution:
$$\begin{aligned}\mathbb{E}N_3 = &\ \mathbb{P}\{\xi_1 = 1, \xi_2 = 1, \xi_3 = 1\} \cdot 3 + \\ &+ \mathbb{P}\{\xi_1 = 1, \xi_2 = 1, \xi_3 = 2\} \cdot 2 + \\ &+ \mathbb{P}\{\xi_1 = 1, \xi_2 = 2\} \cdot 2 + \\ &+ \mathbb{P}\{\xi_1 = 2, \xi_2 = 1\} \cdot 2 + \\ &+ \mathbb{P}\{\xi_1 = 2, \xi_2 = 2\} \cdot 1 = \\ = &\ 3/8 + 2/8 + 2/4 + 2/4 + 1/4 = 15/8.\end{aligned}$$

3. Let $S_n = S_{n-1} + \xi_n$ be a renewal process and $p_\xi(x) = \lambda e^{-\lambda x}$. Find the mathematical expectation of the corresponding counting process N_t:

   - $1/\lambda$;
   - $1/\lambda^2$;
   - λ;
   - λ^2;
   - none of above.

   Solution:

   1. $p \to \mathcal{L}_p$:
$$\begin{aligned}\mathcal{L}_{p_\xi}(s) &= \int_0^\infty e^{-sx} p_\xi(x)\,\mathrm{d}x = \\ &= \int_0^\infty e^{-sx} \lambda e^{-\lambda x}\,\mathrm{d}x = \\ &= \lambda \int_0^\infty e^{-(s+\lambda)x}\,\mathrm{d}x = \\ &= \lambda \cdot -\frac{e^{-(s+\lambda)x}}{s+\lambda}\bigg|_0^\infty = \\ &= \frac{\lambda}{s+\lambda}.\end{aligned}$$

2. $\mathcal{L}_p \to \mathcal{L}_U$:

$$\begin{aligned}\mathcal{L}_U(s) &= \frac{\mathcal{L}_p(s)}{s(1-\mathcal{L}_p(s)} = \\ &= \frac{\frac{\lambda}{s+\lambda}}{s\left(1-\frac{\lambda}{s+\lambda}\right)} = \\ &= \frac{\frac{\lambda}{s+\lambda}}{s\cdot\frac{s}{s+\lambda}} = \\ &= \frac{\lambda}{s^2}.\end{aligned}$$

3. $\mathcal{L}_U \to U$: we **guess** that $U(t) = L_s^{-1}\left(\frac{\lambda}{s^2}\right)(t) = \lambda t$.

4. Let η be a random variable with distribution function $\mathbb{F}_\eta$. Define a stochastic process $X_t = e^\eta t^2$. What is the distribution function of $(X_{t_1}, \ldots, X_{t_n})$ for positive $t_1, \ldots, t_n$?

   - 0;
   - $\mathbb{F}_\eta\{\min(\ln(x_1/t_1^2), \ldots, \ln(x_n/t_n^2))\}$;
   - $\mathbb{F}_\eta\{\min(\ln(x_1/t_1), \ldots, \ln(x_n/t_n))\}$;
   - none of above.

   Solution:

$$\begin{aligned}\mathbb{F}_{(X_{t_1},\ldots,X_{t_n})}(x_1,\ldots,x_n) &= \mathbb{P}\left\{X_{t_1} \le x_1, \ldots, X_{t_n} \le x_n\right\} = \\ &= \mathbb{P}\{e^\eta t_1^2 \le x_1, \ldots, e^\eta t_n^2 \le x_n\} = \\ &= \mathbb{P}\{e^\eta \le x_1/t_1^2, \ldots, e^\eta \le x_n/t_n^2\} = \\ &= \mathbb{P}\{\eta \le \ln(x_1/t_1^2), \ldots, \eta \le \ln(x_n/t_n^2)\} = \\ &= \mathbb{P}\{\eta \le \min(\ln(x_1/t_1^2), \ldots, \ln(x_n/t_n^2))\} = \\ &= \mathbb{F}_\eta\{\min(\ln(x_1/t_1^2), \ldots, \ln(x_n/t_n^2))\}.\end{aligned}$$

5. Let N_t be a counting process of a renewal process $S_n = S_{n-1} + \xi_n$ such that the i.i.d. random variables $\xi_1, \xi_2, \ldots$ have a probability density function

$$p_\xi(x) = \begin{cases}\frac{1}{2}e^{-x}(x+1), & x \ge 0,\\ 0, & x < 0.\end{cases}$$

   Find the mean of N_t:

   - $-\frac{1}{9} + \frac{4}{3}t + \frac{1}{9}e^{-(3/2)t}$;
   - $-\frac{1}{9} + \frac{2}{3}t + \frac{1}{9}e^{-(3/2)t}$;
   - $-\frac{1}{9} + \frac{2}{3}t + \frac{1}{9}e^{3/2t}$;
   - none of above.

   Solution:

   1. $p \to \mathcal{L}_p$:

$$\begin{aligned}\mathcal{L}_{p_\xi}(s) &= \int_0^\infty e^{-sx} p_\xi(x)\,\mathrm{d}x = \\ &= \int_0^\infty e^{-sx}\frac{1}{2}e^{-x}(x+1)\,\mathrm{d}x = \\ &= \frac{1}{2}\int_0^\infty e^{-(s+1)x}(x+1)\,\mathrm{d}x = \\ &= \ldots = \\ &= \frac{s+2}{2(s+1)^2}.\end{aligned}$$

2. $\mathcal{L}_p \to \mathcal{L}_U$:

$$\begin{aligned}
\mathcal{L}_U(s) &= \frac{\mathcal{L}_p(s)}{s(1-\mathcal{L}_p(s))} = \\
&= \frac{\frac{s+2}{2(s+1)^2}}{s\left(1-\frac{s+2}{2(s+1)^2}\right)} = \\
&= \frac{\frac{s+2}{2(s+1)^2}}{s\cdot\frac{2s^2+3s}{2(s+1)^2}} = \\
&= \frac{s+2}{s^2(2s+3)}.
\end{aligned}$$

3. $\mathcal{L}_U \to U$: we first decompose $\mathcal{L}_U(s)$ into elementary fractions:

$$\frac{s+2}{s^2(2s+3)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{2s+3}.$$

One can check that $A = 2/3$, $B = -1/9$, $C = 2/9$.

We then **guess** that

$$\begin{aligned}
U(t) &= L_s^{-1}\left(\frac{s+2}{s^2(2s+3)}\right)(t) = \\
&= L_s^{-1}\left(\frac{2}{3s^2} - \frac{1}{9s} + \frac{2}{9(2s+3)}\right)(t) = \\
&= L_s^{-1}\left(\frac{2}{3s^2}\right)(t) - L_s^{-1}\left(\frac{1}{9s}\right)(t) + L_s^{-1}\left(\frac{2}{9(2s+3)}\right)(t) = \\
&= \frac{2}{3}t + \frac{1}{9}e^{-(3/2)t} - \frac{1}{9}.
\end{aligned}$$

6. Let ξ and η be 2 random variables. It is known that the distribution of η is symmetric, that is, $\mathbb{P}\{\eta > x\} = \mathbb{P}\{\eta < -x\}$ for any $x > 0$, and moreover $\mathbb{P}\{\eta = 0\} = 0$. Find the probability of the event that the trajectories of stochastic process $X_t = \xi^2 + t(\eta + t)$, $t \geq 0$ increase:

- 0;
- 1/2;
- 1/4;
- 1;
- none of above.

Solution: with ξ and η fixed, $X(t)$ increase iff $\dot{X}(t) > 0$ for all $t \in \mathbb{R}_+$. Simple calculations show that $\dot{X}(t) = \eta + 2t$. Hence, the inequality $\dot{X}(t) > 0$ for all $t \in \mathbb{R}_+$ is equivalent to $\eta + 2t > 0$ for all $t \in \mathbb{R}_+$. The last inequality, in its turn, is clearly equivalent to $\eta > 0$. Finally, it is well known that $\mathbb{P}\{\eta > 0\}$ is 1/2 for symmetric random variables with $\mathbb{P}\{\eta = 0\} = 0$. Indeed, $\mathbb{P}\{\eta > 0\} = \mathbb{P}\{\eta < 0\}$, and both sides of this equality have to be equal to 1/2, because their sum is $1 - \mathbb{P}\{\eta = 0\} = 1$.