7 Stochastic integration & Itô formula

Problem 7.1. Let $I(f) = \int_0^1 t^2 \, \mathrm{d}W_t$. Find the mean of $I(f)$. **Options:**

- 0;
- 1/4;
- 1/2;
- 1;
- none of above.

Solution. By the theorem from lecture, $I(f) \sim \mathcal{N}\left(0, \int_0^t f^2(x)\, \mathrm{d}x\right)$. Hence the mean of $I(f)$ is 0.

Problem 7.2. Let $I(f) = \int_0^1 t^2 \, \mathrm{d}W_t$. Find the variance of $I(f)$. **Options:**

- 1/5;
- 1/2;
- 1;
- 5/4;
- none of above.

Solution. By the theorem from lecture, $I(f) \sim \mathcal{N}\left(0, \int_0^t f^2(x)\, \mathrm{d}x\right)$. One can easily compute $\int_0^t t^4 \, \mathrm{d}t = \frac{5}{4}$.

Problem 7.3. Let N_t be a Poisson process. Find the mean, covariance function and variance of $I(f) = \int_0^t N_s \, \mathrm{d}s$ (in the answers below $t > s \geq 0$). **Options:**

- $\mathbb{E}\, I(f) = \lambda t$, $\operatorname{Var} I(f) = (\lambda t)^2$, $K(t,s) = 0$;
- $\mathbb{E}\, I(f) = \frac{\lambda t^2}{2}$, $\operatorname{Var} I(f) = \frac{\lambda t^3}{3}$, $K(t,s) = \lambda\left(-\frac{t^3}{6} + \frac{st^2}{2}\right)$;
- $\mathbb{E}\, I(f) = \frac{\lambda t^2}{2}$, $\operatorname{Var} I(f) = \frac{\lambda t^2}{3}$, $K(t,s) = \lambda\left(-\frac{t^3}{6} + \frac{st^3}{2}\right)$;
- $\mathbb{E}\, I(f) = \lambda t$, $\operatorname{Var} I(f) = \lambda t$, $K(t,s) = 0$;
- none of above.

Solution. By the theorem from lecture,

$$\mathbb{E}\, I(f) = \int_0^t \mathbb{E}\, N_s \, \mathrm{d}s = \int_0^t \lambda s \, \mathrm{d}s = \frac{\lambda t^2}{2},$$

and

$$\begin{aligned}
K_I(t_1, t_2) &= \int_0^{t_1} \int_0^{t_2} K_N(s_1, s_2)\, \mathrm{d}s_2 \, \mathrm{d}s_1 = \\
&= \int_0^{t_1} \int_0^{t_2} (\lambda \min\{s_1, s_2\} + \lambda^2 s_1 s_2)\, \mathrm{d}s_2 \, \mathrm{d}s_1 = \\
&= \lambda \int_0^{t_1} \int_0^{t_2} \min\{s_1, s_2\}\, \mathrm{d}s_2 \, \mathrm{d}s_1 + \lambda^2 \int_0^{t_1} \int_0^{t_2} s_1 s_2 \, \mathrm{d}s_2 \, \mathrm{d}s_1 = \\
&= \lambda \int_0^{t_1} \int_0^{t_2} \min\{s_1, s_2\}\, \mathrm{d}s_2 \, \mathrm{d}s_1 + \frac{\lambda^2 t_1^2 t_2^2}{4} = \\
&= \lambda \left(\frac{t_1 t_2 \min\{t_1, t_2\}}{2} - \frac{\min\{t_1, t_2\}^3}{6}\right) + \frac{\lambda^2 t_1^2 t_2^2}{4}.
\end{aligned}$$

Finally

$$\operatorname{Var} I(f) = K_I(t,t) = \lambda\left(\frac{t^3}{2} - \frac{t^3}{6}\right) + \frac{\lambda^2 t^4}{4} = \frac{\lambda t^3}{3} + \frac{\lambda^2 t^4}{4}.$$

Problem 7.4. Let

$$X_t = \begin{cases} \xi_1, & 0 \le t \le 1, \\ \xi_2, & 1 \le t \le 2, \\ \xi_3, & 2 \le t, \end{cases}$$

where ξ_1, ξ_2, ξ_3 — i.i.d. random variables having exponential distribution with parameter λ.

Find the mean and the variance of $\int_0^T X_t \, \mathrm{d}t$. **Options:**

- $\mathbb{E}\left[\int_0^T X_t \, \mathrm{d}t\right] = \operatorname{Var}\left(\int_0^T X_t \, \mathrm{d}t\right) = \frac{T}{\lambda}$;
- $\mathbb{E}\left[\int_0^T X_t \, \mathrm{d}t\right] = \frac{T}{\lambda}$, $\operatorname{Var}\left(\int_0^T X_t \, \mathrm{d}t\right) = \begin{cases} \frac{T^2}{\lambda^2}, & 0 \le T < 1, \\ \frac{1}{\lambda^2} + \frac{(T-1)^2}{\lambda^2}, & 1 \le T < 2, \\ \frac{2}{\lambda^2} + \frac{(T-2)^2}{\lambda^2}, & 2 \le T; \end{cases}$
- $\mathbb{E}\left[\int_0^T X_t \, \mathrm{d}t\right] = \frac{T}{\lambda}$, $\operatorname{Var}\left(\int_0^T X_t \, \mathrm{d}t\right) = \frac{T^2}{\lambda^2}$;
- none of the above.

Solution. By the theorem from lecture,

$$\mathbb{E}\, I(f) = \int_0^T \mathbb{E}\, X_t \, \mathrm{d}t = \int_0^T \frac{\mathrm{d}t}{\lambda} = \frac{T}{\lambda},$$

and

$$\begin{aligned} \operatorname{Var} I(f) = K(T,T) = \\ &= \int_0^T \int_0^T K(t,s) \, \mathrm{d}s \, \mathrm{d}t = \\ &= \begin{cases} \int_0^T \int_0^T \operatorname{Var} \xi_1 \, \mathrm{d}s \, \mathrm{d}t, & 0 \le T \le 1 \\ \int_0^1 \int_0^1 \operatorname{Var} \xi_1 \, \mathrm{d}s \, \mathrm{d}t + \int_1^T \int_1^T \operatorname{Var} \xi_2 \, \mathrm{d}s \, \mathrm{d}t, & 1 \le T \le 2 \\ \int_0^1 \int_0^1 \operatorname{Var} \xi_1 \, \mathrm{d}s \, \mathrm{d}t + \int_1^2 \int_1^2 \operatorname{Var} \xi_2 \, \mathrm{d}s \, \mathrm{d}t + \int_2^T \int_2^T \operatorname{Var} \xi_3 \, \mathrm{d}s \, \mathrm{d}t, & 2 \le T; \end{cases} \\ &= \begin{cases} \frac{T^2}{\lambda^2}, & 0 \le T \le 1 \\ \frac{1}{\lambda^2} + \frac{(T-1)^2}{\lambda^2}, & 1 \le T \le 2 \\ \frac{2}{\lambda^2} + \frac{(T-2)^2}{\lambda^2}, & 2 \le T. \end{cases} \end{aligned}$$

Problem 7.5. Find the equivalent expression for the stochastic integral $\int_0^T W_t^2 \, \mathrm{d}W_t$, where W_t is a Brownian motion. **Options:**

- $\frac{1}{3}W_T^3 - \frac{1}{2}W_T^2 + \frac{1}{2}T$;
- $\frac{1}{3}W_T^3$;
- $\frac{1}{3}W_T^3 - \int_0^T W_s \, \mathrm{d}s$;
- none of above.

Solution. Let us use Itô's formula. We have $g(t,x) = x^2$, hence $f(t,x) = \frac{1}{3}x^3$. Therefore,

$$\int_0^T g(t, W_t) \, \mathrm{d}W_t = f(T, W_T) - f(0,0) - \int_0^T \frac{\partial f(t, W_t)}{\partial t} + \frac{1}{2} \frac{\partial g(t, W_t)}{\partial W_t} \, \mathrm{d}t = \frac{1}{3} W_T^3 - \int_0^T W_t \, \mathrm{d}t.$$

Problem 7.6. Compute the variance of the stochastic integral $\int_0^T W_t \, \mathrm{d}W_t$, where W_t is a Brownian motion. **Options:**

- $\frac{T^2}{2}$;
- W_T^2;
- T^2;
- none of above.

Solution. We first write the explicit definition:

$$\operatorname{Var}\left(\int_0^T W_t \, \mathrm{d}W_t\right) = \mathbb{E}\left[\left(\int_0^T W_t \, \mathrm{d}W_t - \mathbb{E}\left[\int_0^T W_t \, \mathrm{d}W_t\right]\right)^2\right].$$

One can show that, $\mathbb{E}\left[\int_0^T W_t \, \mathrm{d}W_t\right] = 0$. Therefore our variance is

$$\mathbb{E}\left[\left(\int_0^T W_t \, \mathrm{d}W_t\right)^2\right].$$

By Itô's isometry we get that our variance is

$$\int_0^T \mathbb{E}\left[W_t^2\right] \mathrm{d}t = \int_0^T t \, \mathrm{d}t = \frac{T^2}{2}.$$

Problem 7.7. Choose the process X_t which satisfies the following property:

$$X_t = X_0 + \int_0^t X_s \, \mathrm{d}W_s + \int_0^t e^{W_s + s/2} \, \mathrm{d}s.$$

Options:

- $X_t = e^{W_t - t/2} - e^{W_t + t/2}$;
- $X_t = e^{W_t + t/2} + e^{W_t - t/2}$;
- $X_t = e^{W_t + t/2}$;
- $X_t = e^{W_t + t/2} - e^{W_t - t/2}$.