Markov Chains

1. Find stationary distribution of Markov chain with the following 1-step transition matrix P:

$$P = \begin{pmatrix} 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 0 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 1/2 & 1/2 & 0 & 0 \end{pmatrix}$$

- $\begin{pmatrix} \frac{1}{12} & \frac{3}{12} & \frac{5}{12} & \frac{1}{12} & \frac{2}{12} \end{pmatrix}$;
- $\begin{pmatrix} \frac{2}{12} & \frac{3}{12} & \frac{5}{12} & \frac{1}{12} & \frac{1}{12} \end{pmatrix}$;
- $\begin{pmatrix} \frac{1}{12} & \frac{5}{12} & \frac{3}{12} & \frac{1}{12} & \frac{2}{12} \end{pmatrix}$;
- $\begin{pmatrix} \frac{1}{12} & \frac{1}{12} & \frac{5}{12} & \frac{3}{12} & \frac{2}{12} \end{pmatrix}$.

Solution: all we have to do is solve the following system of linear equations:

$$\vec{\pi} P = \vec{\pi},$$

which is the same as

$$\vec{\pi}(P - I) = 0.$$

Straightforward computations give that

$$\vec{\pi} = \begin{pmatrix} \frac{1}{12} & \frac{3}{12} & \frac{5}{12} & \frac{1}{12} & \frac{2}{12} \end{pmatrix}.$$

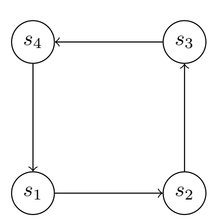
2. Choose all one-step transition matrices which correspond to ergodic Markov chains:

$$P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$P_2 = \begin{pmatrix} 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

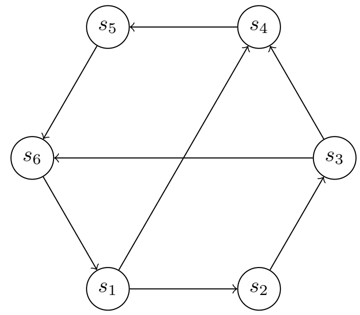
- P_1;
- P_2;
- P_1 and P_2;
- none of the above;

Solution: let us consider the graphs of these chains:



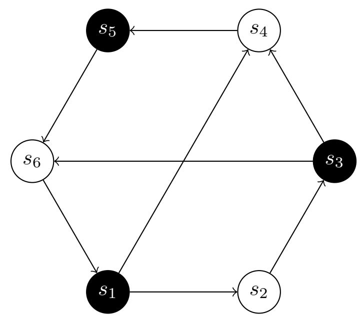
graph of P_1

Clearly all these states are periodic with period 4, hence P_1 is not ergodic. Let us now consider P_2:



graph of P_2

All these states are periodic with period 2 as shown by the following coloring:



colored graph of P_2

The color of the current states changes after every transition, hence P_2 is not ergodic as well.

3. Choose all periodic states of the Markov chain with the following 1-step transition matrix:

$$P = \begin{pmatrix} 1/3 & 1/3 & 1/3 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 0 & 1/2 \\ 0 & 1/2 & 0 & 1/2 \end{pmatrix}$$

- all states are aperiodic;
- 1 and 4;
- all states;
- 1, 2 and 3;
- 2, 3 and 4.

Solution: clearly states 1, 2 and 4 are aperiodic because they have loops, and states 3 is aperiodic because it is in the same equivalence class with, say, state 1, as there are edges $1 \to 3$ and $3to1$.

4. Let's consider once more the Markov chain from the previous task. How many equivalence classes has this chain?

- 0;
- 1;
- 2;
- 3;
- 4.

Solution: there is a path $1 \to 3 \to 4 \to 2 \to 1$, hence there is only one equivalence class.

5. Assume that there is a series of integer numbers, in which numbers $1, 2, \ldots, 9$ appear randomly and independently of each other with equal probabilities. Let x_n be a quantity of different numbers in n first elements of the series. Find a stationary distribution of this chain.

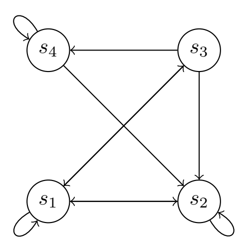
   - $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.9 & 0.1 \end{pmatrix}$;
   - $\begin{pmatrix} 0 & 0.1 & 0.9 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$;
   - $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$;
   - $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$.

   Solution: the only recurrent state is 9, hence it will accumulate all the probability distribution in itself, and the answer is clear.

6. Draw a graphic representation of a Markov chain with the following 1-step transition matrix:

$$P = \begin{pmatrix} 1/3 & 1/3 & 1/3 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 0 & 1/2 \\ 0 & 1/2 & 0 & 1/2 \end{pmatrix}$$

   Solution:



graph of P

7. At time moment $t = 0$ a six-dot side of a simple die faces upwards. Each time moment a die randomly flips on one of its sides. Find the $\vec{\pi}^2$ (the distribution of the Markov process after two flips). Note that the total number of dots on opposite sides of a simple die is equal to 7.

   - $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$;
   - $\begin{pmatrix} \frac{1}{2} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & 0 \end{pmatrix}$;
   - $\begin{pmatrix} \frac{1}{4} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{4} \end{pmatrix}$.

   Solution: One can notice, that on each turn there are 4 options to which side to flip for a die, hence the total number of "paths" is 16. It is easy to check that 4 of them return to the original position, 4 lead to "one dot"-side facing up, and the rest is distributed equally among other positions.

   Note: it is clear that any side of a die *can* face upwards after two moves, hence the only sane option is the last one.

8. At time moment $t = 0$ a six-dot side of a simple die faces upwards. Each time moment a die randomly flips on one of its sides. Find all stationary distributions. Note that the total number of dots on opposite sides of a simple die is equal to 7.

   - $\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{3} & 0 & \frac{1}{6} & 0 \end{pmatrix}$;
   - $\begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix}$;
   - $\begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \end{pmatrix}$.

Solution: the transition matrix of this chain is the following:

$$P = \begin{pmatrix} 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 \\ 1/4 & 0 & 1/4 & 1/4 & 0 & 1/4 \\ 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 \\ 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 \\ 1/4 & 0 & 1/4 & 1/4 & 0 & 1/4 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 \end{pmatrix}$$

One now can solve the system

$$\vec{\pi}P = \vec{\pi},$$

which is the same as

$$\vec{\pi}(P - I) = 0$$

to get the stationary distribution.

Note: it is clear that any stationary distribution should have at least 4 nonzero components, because for every state there are 4 states reachable from it, hence the only sane option is the second one.

9. Jane and Peter participate in a chess championship. For Jane, the probabilities of wining, draw, and losing a game number t are (w, d, l). Peter is slightly more emotional. If he wins the current game, the probabilities of the win, draw, and lose in the next game change from (w, d, l) to $(w + \epsilon, d, l - \epsilon)$; if the current game ends in a draw, then the corresponding probabilities remain (w, d, l); if Peter loses — the result of next game is distributed as $(w - \epsilon, d, l + \epsilon)$. Find the condition, which guarantees that the probability of wining in stationary distribution for Peter is larger then that for Jane. In other words, under which conditions is it better to be slightly emotional?

   - $2l < w$;
   - $l > w$;
   - $0.5l < w$;
   - $l < w$.

   Solution: the mathematical expectation of the change in the probability of winning for Peter is $w\epsilon - l\epsilon = (w - l)\epsilon$. This expression is greater than 0 iff $w > l$. This is, however, not a rigorous proof, rather an intuition for the correct answer.