18.445 Introduction to Stochastic Processes

Lecture 16: Optional stopping theorem

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Recall

- $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space
- A filtration $(\mathcal{F}_n)_{n\geq 0}$
- $X = (X_n)_{n\geq 0}$ is adapted to $(\mathcal{F}_n)_{n\geq 0}$ and is integrable
- X is a martingale if $\mathbb{E}[X_n \,|\, \mathcal{F}_m] = X_m$ *a.s.* for all $n \geq m$.
- X is a supermartingale if $\mathbb{E}[X_n \,|\, \mathcal{F}_m] \leq X_m$ *a.s.* for all $n \geq m$.
- X is a submartingale if $\mathbb{E}[X_n \,|\, \mathcal{F}_m] \geq X_m$ *a.s.* for all $n \geq m$.

Today's Goal

- stopping time
- Optional stopping theorem : $\mathbb{E}[X_T] = \mathbb{E}[X_0]$?

Examples

Example 1 Let $(\xi_i)_{i\geq 1}$ be i.i.d with $\mathbb{E}[\xi_1] = 0$. Then $X_n = \sum_1^n \xi_i$ is a martingale.

Example 2 Let $(\xi_i)_{i\geq 1}$ be i.i.d with $\mathbb{E}[\xi_1] = 1$. Then $X_n = \Pi_1^n \xi_i$ is a martingale.

Example 3 Consider biased gambler's ruin : at each step, the gambler gains one dollar with probability p and losses one dollar with probability $(1-p)$. Let X_n be the money in purse at time n.

- If $p = 1/2$, then (X_n) is a martingale.
- If $p < 1/2$, then (X_n) is a supermartingale.
- If $p > 1/2$, then (X_n) is a submartingale.

Basic properties

About the expectations

- If $(X_n)_{n\geq 0}$ is a martingale, then $\mathbb{E}[X_n] = \mathbb{E}[X_0]$ for all n.
- If $(X_n)_{n\geq 0}$ is a supermartingale, then $\mathbb{E}[X_n]$ is decreasing in n.
- If $(X_n)_{n\geq 0}$ is a submartingale, then $\mathbb{E}[X_n]$ is increasing in n.

More

- If $(X_n)_{n\geq 0}$ is a supermartingale, then $(-X_n)_{n\geq 0}$ is a submartingale.
- If X is both supermartingale and submartingale, then it is a martingale.
- If $X = (X_n)_{n\geq 0}$ is a martingale, then $(|X_n|)_{n\geq 0}$ is a non-negative submartingale.

Lemma

If $(X_n)_{n\geq 0}$ is a martingale, and φ is a convex function,then $(\varphi(X_n))_{n\geq 0}$ is a submartingale.

Examples

Suppose that (Y_n) is a biased random walk on $\mathbb{Z}$: $p \neq 1/2$,

$$Y_{n+1} - Y_n = \begin{cases} 1 & \text{with probability } p \\ -1 & \text{with probability } 1 - p. \end{cases}$$

Lemma

Set $\mu = 2p - 1$, and set

$$X_n = Y_n - \mu n.$$

Then (X_n) is a martingale.

Stopping time

Suppose that $(\mathcal{F}_n)_{n\geq 0}$ is a filtration.

Definition

A stopping time $T : \Omega \to \mathbb{N} = 0, 1, 2, ..., \infty$ is a random variable such that

$$[T = n] \in \mathcal{F}_n, \quad \forall n.$$

Lemma

The following are equivalent.

- $[T = n] \in \mathcal{F}_n$ *for all* n.
- $[T \leq n] \in \mathcal{F}_n$ *for all* n.
- $[T > n] \in \mathcal{F}_n$ *for all* n.
- $[T \geq n] \in \mathcal{F}_{n-1}$ *for all* n.

Lemma

If S, T, T_j *are stopping times. The following are also stopping times.*

- $S \vee T$ *and* $S \wedge T$
- $\inf_j T_j$ *and* $\sup_j T_j$
- $\liminf_j T_j$ *and* $\limsup_j T_j$

Stopping time

$(\Omega, \mathcal{F}, \mathbb{P})$: a probability space with a filtration $(\mathcal{F}_n)_{n\geq 0}$.
$X = (X_n)_{n\geq 0}$: a process adapted to $(\mathcal{F}_n)_{n\geq 0}$

Definition

Let T be a stopping time. Define the σ-algebra $\mathcal{F}_T$ by

$$\mathcal{F}_T = \sigma\{A \in \mathcal{F} : A \cap [T \leq n] \in \mathcal{F}_n, \forall n\}.$$

- Intuitively, $\mathcal{F}_T$ is the information available at time T.
- If $T = n_0$, then $\mathcal{F}_T = \mathcal{F}_{n_0}$.
- $X_T 1_{[T<\infty]}$ is measurable with respect to $\mathcal{F}_T$.
- Let S and T be stopping times, if $S \leq T$, then $\mathcal{F}_S \subset \mathcal{F}_T$.

If $X = (X_n)_{n\geq 0}$ is a process, define $X^T = (X_n^T)_{n\geq 0}$ by $X_n^T = X_{T\wedge n}$.

- X^T is adapted.
- If X is integrable, then X^T is also integrable.

Optional Stopping Theorem

Goal : $\mathbb{E}[X_T] = \mathbb{E}[X_0]$?

Theorem

Let $X = (X_n)_{n \geq 0}$ *be a martingale.*

1. *If* T *is a stopping time, then* X^T *is also a martingale. In particular,* $\mathbb{E}[X_{T \wedge n}] = \mathbb{E}[X_0]$.
2. *If* $S \leq T$ *are bounded stopping times, then* $\mathbb{E}[X_T \mid \mathcal{F}_S] = X_S$, *a.s. In particular,* $\mathbb{E}[X_T] = \mathbb{E}[X_S]$.
3. *If there exists an integrable random variable* Y *such that* $|X_n| \leq Y$ *for all n, and* T *is a stopping time which is finite a.s., then* $\mathbb{E}[X_T] = \mathbb{E}[X_0]$.
4. *If* X *has bounded increments, i.e.* $\exists M > 0$ *such that* $|X_{n+1} - X_n| \leq M$ *for all n, and* T *is a stopping time with* $\mathbb{E}[T] < \infty$, *then* $\mathbb{E}[X_T] = \mathbb{E}[X_0]$.

Gambler's ruin

The gambler's situation can be modeled by a Markov chain on the state space $\{0, 1, ..., N\}$:

- X_0 : initial money in purse
- X_n : the gambler's fortune at time n
- $\mathbb{P}[X_{n+1} = X_n + 1 \mid X_n] = 1/2$,
- $\mathbb{P}[X_{n+1} = X_n - 1 \mid X_n] = 1/2$.
- The states 0 and N are absorbing.
- τ : the time that the gambler stops.

Theorem

Assume that $X_0 = k$ *for some* $0 \leq k \leq N$. *Then*

$$\mathbb{P}[X_\tau = N] = \frac{k}{N}, \quad \mathbb{E}[\tau] = k(N - k).$$

Optional Stopping Theorem

Theorem

Let $X = (X_n)_{n \geq 0}$ be a martingale.

1. *If T is a stopping time, then X^T is also a martingale. In particular, $\mathbb{E}[X_{T \wedge n}] = \mathbb{E}[X_0]$.*
2. *If $S \leq T$ are bounded stopping times, then $\mathbb{E}[X_T \mid \mathcal{F}_S] = X_S$, a.s. In particular, $\mathbb{E}[X_T] = \mathbb{E}[X_S]$.*
3. *If there exists an integrable random variable Y such that $|X_n| \leq Y$ for all n, and T is a stopping time which is finite a.s., then $\mathbb{E}[X_T] = \mathbb{E}[X_0]$.*
4. *If X has bounded increments, i.e. $\exists M > 0$ such that $|X_{n+1} - X_n| \leq M$ for all n, and T is a stopping time with $\mathbb{E}[T] < \infty$, then $\mathbb{E}[X_T] = \mathbb{E}[X_0]$.*

Optional Stopping Theorem

Theorem

Let $X = (X_n)_{n\geq 0}$ be a supermartingale.

1. *If T is a stopping time, then X^T is also a supermartingale. In particular, $\mathbb{E}[X_{T\wedge n}] \leq \mathbb{E}[X_0]$.*
2. *If $S \leq T$ are bounded stopping times, then $\mathbb{E}[X_T \mid \mathcal{F}_S] \leq X_S$, a.s. In particular, $\mathbb{E}[X_T] \leq \mathbb{E}[X_S]$.*
3. *If there exists an integrable random variable Y such that $|X_n| \leq Y$ for all n, and T is a stopping time which is finite a.s., then $\mathbb{E}[X_T] \leq \mathbb{E}[X_0]$.*
4. *If X has bounded increments, i.e. $\exists M > 0$ such that $|X_{n+1} - X_n| \leq M$ for all n, and T is a stopping time with $\mathbb{E}[T] < \infty$, then $\mathbb{E}[X_T] \leq \mathbb{E}[X_0]$.*

Optional Stopping Theorem

Theorem

Let $X = (X_n)_{n \geq 0}$ be a supermartingale.

1. *If T is a stopping time, then X^T is also a supermartingale. In particular, $\mathbb{E}[X_{T \wedge n}] \leq \mathbb{E}[X_0]$.*
2. *If $S \leq T$ are bounded stopping times, then $\mathbb{E}[X_T \mid \mathcal{F}_S] \leq X_S$, a.s. In particular, $\mathbb{E}[X_T] \leq \mathbb{E}[X_S]$.*
3. *If there exists an integrable random variable Y such that $|X_n| \leq Y$ for all n, and T is a stopping time which is finite a.s., then $\mathbb{E}[X_T] \leq \mathbb{E}[X_0]$.*
4. *If X has bounded increments, i.e. $\exists M > 0$ such that $|X_{n+1} - X_n| \leq M$ for all n, and T is a stopping time with $\mathbb{E}[T] < \infty$, then $\mathbb{E}[X_T] \leq \mathbb{E}[X_0]$.*
5. *Suppose that X is a non-negative supermartingale. Then for any stopping time T which is finite a.s., we have $\mathbb{E}[X_T] \leq \mathbb{E}[X_0]$.*