18.445 Introduction to Stochastic Processes

Lecture 18: Martingale: Uniform integrable

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Announcement

- The drop date is April 23rd.
- Extra office hours today 1pm-3pm.

Recall Suppose that $X = (X_n)_{n\geq 0}$ is a martingale.

- If X is bounded in L^1, then $X_n \to X_\infty$ a.s.
- If X is bounded in L^p for $p > 1$, then $X_n \to X_\infty$ a.s. and in L^p.

Today's goal

- Do we have convergence in L^1 ?
- Uniform integrable
- Optional stopping theorem for UI martingales
- Backward martingale

Uniformly integrable

Definition

A collection $(X_i, i \in I)$ of random variables is uniformly integrable (UI) if

$$\sup_i \mathbb{E}[|X_i| 1_{[|X_i|>\alpha]}] \to 0, \quad \text{as } \alpha \to \infty.$$

1. A UI family is bounded in L^1, but the converse is not true.
2. If a family is bounded in L^p for some $p > 1$, then the family is UI.

Theorem

If $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, then the class

$$\{\mathbb{E}[X \mid \mathcal{H}] : \mathcal{H} \text{ sub } \sigma\text{-algebra of } \mathcal{F}\}$$

is UI.

L^1 convergence

A collection $(X_i, i \in I)$ of random variables is uniformly integrable (UI) if

$$\sup_i \mathbb{E}[|X_i| 1_{[|X_i| > \alpha]}] \to 0, \quad \text{as } \alpha \to \infty.$$

Theorem

Let $X = (X_n)_{n \geq 0}$ be a martingale. The following statements are equivalent.

1. *X is UI.*
2. *X_n converges to X_∞ a.s. and in L^1.*
3. *There exists $Z \in L^1$ such that $X_n = \mathbb{E}[Z \mid \mathcal{F}_n]$ a.s. for all $n \geq 0$.*

Lemma

Let $X \in L^1, X_n \in L^1$ and $X_n \to X$ a.s. Then

$$X_n \to X \text{ in } L^1 \quad \text{if and only if} \quad (X_n)_{n \geq 0} \text{ is UI.}$$

L^1 convergence

- If X is a UI martingale, then $X_n \to X_\infty$ a.s. and in L^1. Moreover, $X_n = \mathbb{E}[X_\infty \mid \mathcal{F}_n]$ a.s.
- If X is a UI supermartingale, then $X_n \to X_\infty$ a.s. and in L^1. Moreover, $X_n \geq \mathbb{E}[X_\infty \mid \mathcal{F}_n]$ a.s.
- If X is a UI submartingale, then $X_n \to X_\infty$ a.s. and in L^1. Moreover, $X_n \leq \mathbb{E}[X_\infty \mid \mathcal{F}_n]$ a.s.

Example

Let $(\xi_j)_{j\geq 1}$ be non-negative independent random variables with mean one. Set

$$X_0 = 1, \quad X_n = \Pi_{j=1}^n \xi_j.$$

1. $(X_n)_{n\geq 0}$ is a non-negative martingale.
2. X_n converges a.s. to some limit $X_\infty \in L^1$.

Question :

1. Do we have $\mathbb{E}[X_\infty] = 1$?

Answer : Set $a_j = \mathbb{E}[\sqrt{\xi_j}] \in (0, 1]$.

1. If $\Pi_j a_j > 0$, then X converges in L^1 and $\mathbb{E}[X_\infty] = 1$.
2. If $\Pi_j a_j = 0$, then $X_\infty = 0$ a.s.

Optional Stopping Theorem

Theorem

Let $X = (X_n)_{n \geq 0}$ be a martingale. If $S \leq T$ are bounded stopping times, then $\mathbb{E}[X_T \mid \mathcal{F}_S] = X_S$, a.s. In particular, $\mathbb{E}[X_T] = \mathbb{E}[X_S]$.

Theorem

Let $X = (X_n)_{n \geq 0}$ be a UI martingale. If $S \leq T$ are stopping times, then $\mathbb{E}[X_T \mid \mathcal{F}_S] = X_S$, a.s. In particular, $\mathbb{E}[X_T] = \mathbb{E}[X_S]$.

$$X_T = \sum_0^\infty X_n 1_{[T=n]} + X_\infty 1_{[T=\infty]}.$$

Summary

Suppose that $X = (X_n)_{n\geq 0}$ is a martingale.

- If X is bounded in L^1, then $X_n \to X_\infty$ a.s.
- If X is bounded in L^p for $p > 1$, then $X_n \to X_\infty$ a.s. and in L^p.
- If X is UI, then $X_n \to X_\infty$ a.s. and in L^1.

Suppose that $X = (X_n)_{n\geq 0}$ is a UI martingale.

- For any stopping times $S \leq T$, we have $\mathbb{E}[X_T \mid \mathcal{F}_S] = X_S$ a.s.
- In particular, $\mathbb{E}[X_\infty] = \mathbb{E}[X_0]$.

Applications

Theorem (Kolmogorov's 0-1 law)

Let $(X_n)_{n\geq 0}$ be i.i.d. Let $\mathcal{G}_n = \sigma(X_k, k \geq n)$ and $\mathcal{G}_\infty = \cap_{n\geq 0}\mathcal{G}_n$. Then $\mathcal{G}_\infty$ is trivial, i.e. every $A \in \mathcal{G}_\infty$ has probability $\mathbb{P}[A]$ is either 0 or 1.

Backwards martingale

Definition

- $(\Omega, \mathcal{G}, \mathbb{P})$ probability space
- A filtration indexed by $\mathbb{Z}_-$: $\cdots \subseteq \mathcal{G}_{-2} \subseteq \mathcal{G}_{-1} \subseteq \mathcal{G}_0$.
- A process $X = (X_n)_{n\leq 0}$ is called a backwards martingale, if it is adapted to the filtration, $X_0 \in L^1$ and for all $n \leq -1$, we have

$$\mathbb{E}[X_{n+1} \,|\, \mathcal{G}_n] = X_n, \, a.s.$$

Consequences

- For all $n \leq 0$, we have $\mathbb{E}[X_0 \,|\, \mathcal{G}_n] = X_n$.
- The process $X = (X_n)_{n\leq 0}$ is automatically UI.

Theorem

Suppose that $X = (X_n)_{n\geq 0}$ is a forwards martingale and $(\mathcal{F}_n)_{n\geq 0}$ is the filtration.

- *If X is bounded in L^p for $p > 1$, then*

$$X_n \to X_\infty \quad a.s. \text{and in } L^p; \quad X_n = \mathbb{E}[X_\infty \,|\, \mathcal{F}_n] \quad a.s.$$

- *If X is UI, then*

$$X_n \to X_\infty \quad a.s. \text{and in } L^1; \quad X_n = \mathbb{E}[X_\infty \,|\, \mathcal{F}_n] \quad a.s.$$

Theorem

Suppose that $X = (X_n)_{n\leq 0}$ is a backwards martingale and $(\mathcal{G}_n)_{n\leq 0}$ is the filtration. Recall that $\mathbb{E}[X_0 \,|\, \mathcal{G}_n] = X_n$.

- *If $X_0 \in L^p$ for $p \geq 1$, then*

$$X_n \to X_{-\infty} \quad a.s. \text{and in } L^p; \quad X_{-\infty} = \mathbb{E}[X_0 \,|\, \mathcal{G}_{-\infty}] \quad a.s.$$

where $\mathcal{G}_{-\infty} = \cap_{n\leq 0}\mathcal{G}_n$.

Applications

Theorem (Strong Law of Large Numbers)

Let $X = (X_n)_{n\geq 0}$ *be i.i.d. in* L^1 *with* $\mu = \mathbb{E}[X_1]$*. Define*

$$S_n = (X_1 + \cdots + X_n)/n.$$

Then

$$S_n/n \to \mu, \quad a.s. \text{and in } L^1.$$