5 Stationarity and linear filters

Problem 5.1. Let λ be a non-zero constant. Does a stochastic process with the covariance function $K(t,s) = \sin(\lambda(t-s))$ exist? **Options:**

- Yes;
- No;

Solution. Suggested functions is not symmetric, hence it is not a covariance function of any stochastic process.

Problem 5.2. Let Y_n be a stochastic process which is defined as follows: $Y_{n+1} = \alpha Y_n + X_n$, $n = 0, 1, \ldots$. Assume $Y_0 = 0$, $|\alpha| < 1$ and X_n is a sequence of i.i.d. standard normal random variables for $n = 0, 1, 2, \ldots$ Determine whether Y_n is stationary and find its mean and variance. **Options:**

- Y_n is stationary, $\mathbb{E}\, Y_n = 0$, $\operatorname{Var} Y_n = \alpha^2 + 1$;
- Y_n is non-stationary, $\mathbb{E}\, Y_n = 0$, $\operatorname{Var} Y_n = \alpha^2 + 1$;
- Y_n is non-stationary, $\mathbb{E}\, Y_n = 0$, $\operatorname{Var} Y_n = \frac{1-\alpha^{2n}}{1-\alpha^2}$;
- Y_n is stationary, $\mathbb{E}\, Y_n = 0$, $\operatorname{Var} Y_n = \frac{1-\alpha^{2n}}{1-\alpha^2}$;
- none of these.

Solution. Clearly $\mathbb{E}\, Y_n = 0$. This can be show either by induction on n, or via using the explicit formula for Y_n: $Y_n = X_n + \alpha X_{n-1} + \ldots + \alpha^n X_0$. Further,

$$\begin{aligned}
K(n_1, n_2) = \operatorname{cov}(Y_{n_1}, Y_{n_2}) &= \\
&= \operatorname{cov}\left(\sum_{k_1=0}^{n_1-1} \alpha^{k_1} X_{n_1-1-k_1}, \sum_{k_2=0}^{n_2-1} \alpha^{k_2} X_{n_2-1-k_2}\right) = \\
&= \sum_{k_1=0}^{n_1-1}\sum_{k_2=0}^{n_2-1} \operatorname{cov}\left(\alpha^{k_1} X_{n_1-1-k_1}, \alpha^{k_2} X_{n_2-1-k_2}\right) = \\
&= \sum_{k_1=0}^{n_1-1}\sum_{k_2=0}^{n_2-1} \alpha^{k_1+k_2} \operatorname{cov}\left(X_{n_1-1-k_1}, X_{n_2-1-k_2}\right) = \\
&= \sum_{k_1=0}^{n_1-1}\sum_{k_2=0}^{n_2-1} \alpha^{k_1+k_2} \mathbf{1}\{n_1 - k_1 = n_2 - k_2\} = \\
&= \sum_{k_1=0}^{n_1-1}\sum_{k_2=0}^{n_2-1} \alpha^{k_1+k_2} \mathbf{1}\{n_1 - n_2 = k_1 - k_2\},
\end{aligned}$$

i.e. can be represented as a function of $n_1 - n_2$ only. Hence Y_n is weakly stationary. Finally,

$$\begin{aligned}
\operatorname{Var} Y_n = \operatorname{Var}\left(\sum_{k=0}^{n-1} \alpha^k X_{n-1-k}\right) &= \\
&= \sum_{k=0}^{n-1} \operatorname{Var}(\alpha^k X_{n-1-k}) = \\
&= \sum_{k=0}^{n-1} \alpha^{2k} \operatorname{Var} X_{n-1-k} = \\
&= \sum_{k=0}^{n-1} \alpha^{2k} = \frac{1-\alpha^{2n}}{1-\alpha^2}.
\end{aligned}$$

Problem 5.3. Let W_t be a Brownian Motion and define $X_t = (1-t)W_{t/(1-t)}$ for $t \in (0,1)$. Choose all correct statements. **Options:**

- X_t is strictly stationary process;

- X_t is weakly stationary process;
- none of these.

Solution. Clearly $\mathbb{E}\, X_t = (1-t)\, \mathbb{E}\, W_{t/(1-t)} = (1-t)\cdot 0 = 0$. However,

$$\begin{aligned}\operatorname{cov}(X_t, X_s) &= \operatorname{cov}\left((1-t)\cdot W_{t/(1-t)}, (1-s)\cdot W_{s/(1-s)}\right) = \\ &= (1-t)\cdot(1-s)\cdot \operatorname{cov}\left(W_{t/(1-t)}, W_{s/(1-s)}\right) = \\ &= (1-t)\cdot(1-s)\cdot \min\left\{\frac{t}{1-t}, \frac{s}{1-s}\right\}.\end{aligned}$$

This function cannot be represented as a function of $t-s$ only. One way to show this explicitly is the following:

$$K(t,t) = (1-t)\cdot(1-t)\cdot \min\left\{\frac{t}{1-t}, \frac{t}{1-t}\right\} = (1-t)^2\cdot\frac{t}{1-t} = (1-t)\cdot t,$$

which is clearly not a constant.

Problem 5.4. Let W_t be a Brownian Motion and $h > 0$ be a fixed number. Find a covariance function of the process $X_t = W_{t+h} - W_t$. **Options:**

- $K(t,s) = \begin{cases} h - |t-s|, & |t-s| \le h, \\ 0, & |t-s| > h; \end{cases}$
- $K(t,s) = \begin{cases} \min(t,s), & |t-s| \le h, \\ 0, & |t-s| > h; \end{cases}$
- $K(t,s) = 0$, $\forall t, s$;
- none of above.

Solution. To begin with,

$$\begin{aligned}K(t,s) &= \operatorname{cov}(W_{t+h} - W_t, W_{s+h} - W_s) = \\ &= \operatorname{cov}(W_{t+h}, W_{s+h}) - \operatorname{cov}(W_{t+h}, W_s) - \operatorname{cov}(W_t, W_{s+h}) + \operatorname{cov}(W_t, W_s) = \\ &= \min\{t+h, s+h\} - \min\{t+h, s\} - \min\{t, s+h\} + \min\{t,s\}.\end{aligned}$$

If $|t-s| > h$ then either $\min\{t+h, s+h\} = \min\{t, s+h\} = s+h$ and $\min\{t+h, s\} = \min\{t,s\} = s$, or $\min\{t+h, s+h\} = \min\{t+h, s\} = t+h$ and $\min\{t, s+h\} = \min\{t,s\} = t$. In any case, the entire sum will be equal to 0, as all the terms cancel each other.

However, if $|t-s| \le h$, and, without losing the generality, $t > s$, then

$$\min\{t+h, s+h\} - \min\{t+h, s\} - \min\{t, s+h\} + \min\{t,s\} = (s+h) - (s) - (t) + (s) = h - (t-s).$$

By using the same logic for the case $s > t$ one can get that $K(t,s) = h - |t-s|$ if $|t-s| \le h$.

Problem 5.5. Let X_t is a process with independent and stationary increments and h is a positive constant. Moreover, $\mathbb{E}\, X_t = 0$ and $\mathbb{E}\, X_t^2 < \infty$. Is $Y_t = X_{t+h} - X_t$ a wide-sense stationary process? **Options:**

- Yes;
- No;
- Additional information on X_t is required.

Solution. $X_t \mapsto X_{t+h} - X_t$ is a linear filter, hence Y_t is weakly stationary by the theorem from lecture.

Problem 5.6. Let X_t be a wide-sense stationary process with autocovariance function γ, such that $\gamma(0) = 2$, $\gamma(1) = \gamma(-1) = 1$ and $\gamma(n) = 0$ for all other n. Find the spectral density $g_X(u)$ of this process. **Options:**

- $g_X(u) = \frac{1+2\cos u}{2\pi}$;
- $g_X(u) = \frac{1+\cos u}{2\pi}$;

- $g_X(u) = \frac{1+2\cos u}{\pi}$;
- $g_X(u) = \frac{1+\cos u}{\pi}$;
- None of above.

Solution. Use the definition:

$$g_X(u) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{-inu}\gamma(n) = \frac{1}{2\pi}\left(e^{iu} + 2 + e^{-iu}\right) = \frac{1}{\pi}(1+\cos u) = \frac{1+\cos u}{\pi}.$$

Problem 5.7. Let the autocovariance function of some stochastic process X_t be

$$\gamma_X(u) = \begin{cases} 3, & u = 0, \\ 1, & u = \pm 2, \\ 0, & \text{otherwise.} \end{cases}$$

Find the spectral density of $Y_t = 3X_t + 2X_{t-1} + X_{t-2}$. **Options:**

- $g_Y(u) = \frac{1}{2\pi}(3 + 2\cos(2u))(14 + 6\cos(2u) + 16\cos(u))$;
- $g_Y(u) = \frac{3}{2\pi}(1 + \cos(2u))(1 + \cos(2u) + 8\cos(u))$;
- $g_Y(u) = \frac{1}{2\pi}(1 + \cos(2u))(14 + 8\cos(2u) + 8\cos(u))$.

Solution. We first find $g_X(u)$ by definition:

$$g_X(u) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{-inu}\gamma(n) = \frac{1}{2\pi}\left(e^{2iu} + 3 + e^{-2iu}\right) = \frac{1}{2\pi}(3 + 2\cos 2u) = \frac{3+2\cos 2u}{2\pi}.$$

Then we employ the theorem from lecture, namely

$$g_Y(u) = g_X(u) \cdot \left|\mathscr{F}[\rho](u)\right|^2,$$

where

$$\mathscr{F}[\rho](u) = \int_{\mathbb{R}} e^{iux}\rho(x)\,\mathrm{d}x.$$

We now use the "modulo squared via complex conjugate" trick from the lecture:

$$\begin{aligned}
\left|\mathscr{F}[\rho](u)\right|^2 &= \int_{\mathbb{R}} e^{iux}\rho(x)\,\mathrm{d}x \cdot \int_{\mathbb{R}} e^{-iux}\rho(x)\,\mathrm{d}x = \\
&= (3 + 2e^{iu} + e^{2iu}) \cdot (3 + 2e^{-iu} + e^{-2iu}) = \\
&= 14 + 6(e^{iu} + e^{-iu}) + 3(e^{2iu} + e^{-2iu}) + 2(e^{-iu} + e^{iu}) = \\
&= 14 + 12\cos u + 6\cos 2u + 4\cos u = \\
&= 14 + 16\cos u + 6\cos 2u.
\end{aligned}$$