18.445 Introduction to Stochastic Processes

Lecture 7: Summary on mixing times

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Recall Suppose that P is irreducible with stationary measure π.

$$d(n) = \max_x ||P^n(x, \cdot) - \pi||_{TV}, \quad t_{mix} = \min\{n : d(n) \le 1/4\}.$$

Today's Goal Summary of the results on the mixing times.

- Upper bounds and lower bounds on mixing times
- Gambler's ruin, Coupon collecting
- Random walk on hypercube
- Random walk on $N-$cycle
- Top-to-random shuffle

Upper bounds

Suppose that P is irreducible with stationary distribution π.

Theorem (Coupling of two Markov chains)

Let $(X_n, Y_n)_{n\geq 0}$ be a coupling of Markov chains with transition matrix P for which $X_0 = x$, $Y_0 = y$. Define τ to be their first meet time : $\tau = \min\{n \geq 0 : X_n = Y_n\}$*. Then*

$$||P^n(x, \cdot) - P^n(y, \cdot)||_{TV} \leq \mathbb{P}_{x,y}[\tau > n]; \quad d(n) \leq \max_{x,y} \mathbb{P}_{x,y}[\tau > n].$$

Theorem (Strong stationary time)

Let $(X_n)_{n\geq 0}$ be a Markov chain with transition matrix P. If τ is a strong stationary time for (X_n), then

$$d(n) := \max_x ||P^n(x, \cdot) - \pi||_{TV} \leq \max_x \mathbb{P}[\tau > n].$$

Lower bounds

Suppose that P is irreducible with stationary measure π.

Theorem (Bottleneck ratio)

Define $Q(A,B) = \sum_{x \in A, y \in B} \pi(x) P(x,y)$, $\Phi(S) = Q(S, S^c)/\pi(S)$. *The bottleneck ratio of the chain is defined to be*

$$\Phi_\star = \min\{\Phi(S) : \pi(S) \le 1/2\}.$$

Then

$$t_{mix} \ge \frac{1}{4\Phi_\star}$$

Theorem (Distinguishing statistic)

Let μ *and* ν *be two probability distributions on* Ω. *Let* f *be a real-valued function on* Ω. *If*

$$|\mu f - \nu f| \ge r\sigma, \quad \textit{where } \sigma^2 = \frac{1}{2}(var_\mu(f) + var_\nu(f)),$$

then

$$||\mu - \nu||_{TV} \ge \frac{r^2}{4 + r^2}.$$

Gambler's ruin

Consider a gambler betting on the outcome of a sequence of independent fair coin tosses.
If head, he gains one dollar. If tail, he loses one dollar.
If he reaches a fortune of N dollars, he stops. If his purse is ever empty, he stops.
The gambler's situation can be modeled by a Markov chain on the state space $\{0, 1, ..., N\}$:

- X_0 : initial money in purse
- X_n : the gambler's fortune at time n
- τ : the time that the gambler stops.

Theorem

Assume that $X_0 = k$ for some $0 \leq k \leq N$. Then

$$\mathbb{P}[X_\tau = N] = \frac{k}{N}, \quad \mathbb{E}[\tau] = k(N-k).$$

Coupon collecting

A company issues N different types of coupons. A collector desires a complete set. The collector's situation can be modeled by a Markov chain on the state space $\{0, 1, ..., N\}$:

- $X_0 = 0$
- X_n : the number of different types among the collector's first n coupons.
- $\mathbb{P}[X_{n+1} = k + 1 \mid X_n = k] = (N - k)/N$,
- $\mathbb{P}[X_{n+1} = k \mid X_n = k] = k/N$.
- τ : the first time that the collector obtains all N types.

Theorem

$$\mathbb{E}[\tau] = N \sum_{k=1}^{N} \frac{1}{k} \approx N \log N.$$

For any $\alpha > 0$, we have that

$$\mathbb{P}[\tau > N \log N + \alpha N] \leq e^{-\alpha}.$$

Random walk on hypercube

The lazy walk on hypercube can be constructed using the following random mapping representation : Uniformly select an element (j, B) in $\{1, ..., N\} \times \{0, 1\}$, and then update the coordinate j with B.

Let $(Z_n = (j_n, B_n))_{n\geq 1}$ be i.i.d. $\overset{d}{\sim} (j, B)$. At each step, the coordinate j_n of X_{n-1} is updated by B_n. Define

$$\tau = \min\{n : \{j_1, ..., j_n\} = \{1, .., N\}\}.$$

This is the first time that all the coordinates have been selected at least once for updating.

Theorem

There exists constants $c > 0, C < \infty$ such that

$$CN \log N \geq t_{mix} \geq cN \log N.$$

Proof Upper bound : strong stationary time.
Lower bound : distinguishing statistic.

Random walk on $N-$cycle

Lazy walk : it remains in current position with probability $1/2$, moves left with probability $1/4$, right with probability $1/4$.

- It is irreducible.
- The stationary measure is the uniform measure.

Theorem

For the lazy walk on $N-$cycle, there exists some constant $c_0 > 0$ such that

$$c_0 N^2 \leq t_{mix} \leq N^2.$$

Proof

Upper bound : Coupling of two Markov chains.
Lower bound.

Top-to-random shuffle

Consider the following method of shuffling a deck of N cards :
Take the top card and insert it uniformly at random in the deck.
The successive arrangements of the deck are a random walk $(X_n)_{n\geq 0}$ on the group S_N starting from $X_0 = (123\cdots N)$.
The uniform measure is the stationary measure.
Let τ_{top} be the time one move after the first occasion when the original bottom card has moved to the top of the deck. The arrangements of cards at time τ_{top} is uniform in S_N.

Theorem

There exist constant $c_0 \in (0, \infty)$ such that

$$N \log N - c_0 N \leq t_{mix} \leq N \log N + c_0 N.$$

Proof
Upper bound : τ_{top} is strong stationary.
Lower bound.