18.445 Introduction to Stochastic Processes

Lecture 24: Stationary distribution

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Recall : Suppose that $X = (X_t)_{t \geq 0}$ is a continuous time Markov chain.

- the jump times $J_0, J_1, J_2, ...$: $J_0 = 0, J_{n+1} = \inf\{t > J_n : X_t \neq X_{J_n}\}$.
- the jump chain $Y_0, Y_1, Y_2, ...$: $Y_n = X_{J_n}$

Irreducible : X is irreducible if and only if Y is irreducible.
Recurrent : Suppose that X is an irreducible Markov chain. Then X is recurrent if and only if Y is recurrent.

Today's Goal :

- positive recurrent
- stationary distribution
- summary on topics in the final

Stationary

Suppose that X is a continuous time Markov chain with generator A, and that Y is its jump chain.

Definition

A measure π is said to be stationary for X if $\pi A = 0$.

Lemma

If π is stationary, then $\pi P_t = \pi$ for all $t \geq 0$.

Lemma

A measure π is stationary for X if and only if the measure μ, defined by $\mu(x) = q_x \pi(x)$, is stationary for Y.

Positive recurrence

Definition

Define $T_x^+ = \inf\{t \geq J_1 : X_t = x\}$. A state x is positive recurrent if

$$\mathbb{E}_x[T_x^+] < \infty.$$

Theorem

Let X be an irreducible continuous time Markov chain with generator A. The following statements are equivalent.

- *every state is positive recurrent.*
- *some state is positive recurrent.*
- *X has a stationary distribution.*

When one of the statement is true, the stationary distribution is

$$\pi(x) = \frac{1}{q_x \mathbb{E}_x[T_x^+]}.$$

Topics covered in the final

Martingale : Suppose that $X = (X_n)_{n \geq 0}$ is a martingale.

1. Optional Stopping Theorem
2. Martingale convergence theorems
   - If X is bounded in L^1, then $X_n \to X_\infty$ a.s.
   - If X is bounded in L^p for $p > 1$, then $X_n \to X_\infty$ a.s. and in L^p.
   - If X is UI, then $X_n \to X_\infty$ a.s. and in L^1.

Poisson process : Suppose that $(N_t)_{t \geq 0}$ is a Poisson process.

1. Definition, Markov property
2. Superposition
3. Characterization