# 18.445 HOMEWORK 4 SOLUTIONS

**Exercise 1.** Let $X, Y$ be two random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{A} \subset \mathcal{F}$ be a sub-$\sigma$-algebra. The random variables $X$ and $Y$ are said to be independent conditionally on $\mathcal{A}$ is for every non-negative measurable functions $f, g$, we have

$$\mathbb{E}[f(X)g(Y) \,|\, \mathcal{A}] = \mathbb{E}[f(X) \,|\, \mathcal{A}] \times \mathbb{E}[g(Y) \,|\, \mathcal{A}] \quad a.s.$$

Show that $X, Y$ are independent conditionally on $\mathcal{A}$ is and only if for every non-negative $\mathcal{A}$-measurable random variable $Z$, and every non-negative measurable functions $f, g$, we have

$$\mathbb{E}[f(X)g(Y)Z] = \mathbb{E}[f(X)Z\mathbb{E}[g(Y) \,|\, \mathcal{A}]].$$

*Proof.* If $X$ and $Y$ are independent conditionally on $\mathcal{A}$ and $Z$ is $\mathcal{A}$-measurable, then

$$\begin{aligned}
\mathbb{E}[f(X)g(Y)Z] &= \mathbb{E}\big[\mathbb{E}[f(X)g(Y)Z \,|\, \mathcal{A}]\big] \\
&= \mathbb{E}\Big[\mathbb{E}[f(X)g(Y) \,|\, \mathcal{A}]Z\Big] \\
&= \mathbb{E}\Big[\mathbb{E}[f(X) \,|\, \mathcal{A}]\mathbb{E}[g(Y) \,|\, \mathcal{A}]Z\Big] \\
&= \mathbb{E}\Big[\mathbb{E}\big[f(X)\mathbb{E}[g(Y) \,|\, \mathcal{A}]Z \,|\, \mathcal{A}\big]\Big] \\
&= \mathbb{E}\Big[f(X)Z\mathbb{E}[g(Y) \,|\, \mathcal{A}]\Big].
\end{aligned}$$

Conversely, if this equality holds for every nonnegative $\mathcal{A}$-measurable $Z$, then in particular, for every $A \in \mathcal{A}$,

$$\mathbb{E}[f(X)g(Y)\mathbb{1}_A] = \mathbb{E}\Big[f(X)\mathbb{E}[g(Y) \,|\, \mathcal{A}]\mathbb{1}_A\Big].$$

It follows from the definition of conditional expectation that

$$\mathbb{E}[f(X)g(Y) \,|\, \mathcal{A}] = \mathbb{E}\Big[f(X)\mathbb{E}[g(Y) \,|\, \mathcal{A}] \,|\, \mathcal{A}\Big] = \mathbb{E}[f(Y) \,|\, \mathcal{A}]\mathbb{E}[g(Y) \,|\, \mathcal{A}],$$

so $X$ and $Y$ are independent conditionally on $\mathcal{A}$. $\square$

**Exercise 2.** Let $X = (X_n)_{n \geq 0}$ be a martingale.

*(1).* Suppose that $T$ is a stopping time, show that $X^T$ is also a martingale. In particular, $\mathbb{E}[X_{T \wedge n}] = \mathbb{E}[X_0]$.

*Proof.* Since $X$ is a martingale, first we have

$$\mathbb{E}[|X_n^T|] \leq \mathbb{E}[\max_{i \leq n} |X_i|] \leq \sum_{i=1}^{n} \mathbb{E}[|X_i|] < \infty.$$

Moreover, for every $n \geq m$,

$$\begin{aligned}
\mathbb{E}[X_n^T \,|\, \mathcal{F}_{n-1}] &= \mathbb{E}[X_{n-1}^T + (X_n - X_{n-1})\mathbb{1}_{T > n-1} \,|\, \mathcal{F}_{n-1}] \\
&= \mathbb{E}[X_{n-1}^T] + \mathbb{1}_{T > n-1}\mathbb{E}[X_n - X_{n-1} \,|\, \mathcal{F}_{n-1}] \\
&= \mathbb{E}[X_{n-1}^T].
\end{aligned}$$

We conclude that $X^T$ is a martingale. $\square$

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*Date*: April 16, 2015. Prepared by Cheng Mao.

*(2).* Suppose that $S \leq T$ are bounded stopping times, show that $\mathbb{E}[X_T \,|\, \mathcal{F}_S] = X_S, a.s.$ In particular, $\mathbb{E}[X_T] = \mathbb{E}[X_S]$.

*Proof.* Suppose $S$ and $T$ are bounded by a constant $N \in \mathbb{N}$. For $A \in \mathcal{F}_S$,

$$\begin{aligned}
\mathbb{E}[X_N \mathbb{1}_A] &= \sum_{i=1}^{N} \mathbb{E}[X_N \mathbb{1}_A \mathbb{1}_{S=i}] \\
&= \sum_{i=1}^{N} \mathbb{E}\Big[\mathbb{E}[X_N \,|\, \mathcal{F}_S] \mathbb{1}_A \mathbb{1}_{S=i}\Big] \\
&= \sum_{i=1}^{N} \mathbb{E}\Big[\mathbb{E}[X_N \,|\, \mathcal{F}_i] \mathbb{1}_A \mathbb{1}_{S=i}\Big] \\
&= \sum_{i=1}^{N} \mathbb{E}\Big[X_i \mathbb{1}_A \mathbb{1}_{S=i}\Big] \\
&= \mathbb{E}[X_S \mathbb{1}_A],
\end{aligned}$$

so $\mathbb{E}[X_N \,|\, \mathcal{F}_S] = X_S$. Similarly, $\mathbb{E}[X_N \,|\, \mathcal{F}_T] = X_T$. We conclude that

$$\mathbb{E}[X_T \,|\, \mathcal{F}_S] = \mathbb{E}\Big[\mathbb{E}[X_N \,|\, \mathcal{F}_T] \,|\, \mathcal{F}_S\Big] = \mathbb{E}[X_N \,|\, \mathcal{F}_S] = X_S.$$

□

*(3).* Suppose that there exists an integrable random variable $Y$ such that $|X_n| \leq Y$ for all $n$, and $T$ is a stopping time which is finite a.s., show that $\mathbb{E}[X_T] = \mathbb{E}[X_0]$.

*Proof.* Since $|X_n| \leq Y$ for all $n$ and $T$ is finite a.s., $|X_{n\wedge T}| \leq Y$. Then the dominated convergence theorem implies that

$$\lim_{n\to\infty} \mathbb{E}[X_{n\wedge T}] = \mathbb{E}[\lim_{n\to\infty} X_{n\wedge T}] = \mathbb{E}[X_T].$$

As $n \wedge T$ is a bounded stopping time, Part (2) implies that $\mathbb{E}[X_{n\wedge T}] = \mathbb{E}[X_0]$. Hence we conclude that $\mathbb{E}[X_T] = \mathbb{E}[X_0]$. □

*(4).* Suppose that $X$ has bounded increments, i.e. $\exists M > 0$ such that $|X_{n+1} - X_n| \leq M$ for all $n$, and $T$ is a stopping time with $\mathbb{E}[T] < \infty$, show that $\mathbb{E}[X_T] = \mathbb{E}[X_0]$.

*Proof.* We can write $\mathbb{E}[X_T] = \mathbb{E}[X_0] + \mathbb{E}[\sum_{i=1}^{T}(X_i - X_{i-1})]$, so it suffices to show that the last term is zero. Note that

$$\mathbb{E}[|\sum_{i=1}^{T}(X_i - X_{i-1})|] \leq \mathbb{E}[\sum_{i=1}^{T} |X_i - X_{i-1}|] \leq M\mathbb{E}[T] < \infty.$$

Then the dominated convergence theorem implies that

$$\begin{aligned}
\mathbb{E}[\sum_{i=1}^{T}(X_i - X_{i-1})] &= \mathbb{E}[\sum_{i=1}^{\infty}(X_i - X_{i-1})\mathbb{1}_{T\geq i}] \\
&= \sum_{i=1}^{\infty} \mathbb{E}[(X_i - X_{i-1})\mathbb{1}_{T\geq i}] \\
&= \sum_{i=1}^{\infty} \mathbb{E}[X_i - X_{i-1}]\mathbb{P}[T \geq i] \\
&= 0,
\end{aligned}$$

where we used that $X_i - X_{i-1}$ is independent of $\{T \geq i\} = \{T < i - 1\}$ as $T$ is a stopping time of the martingale $X$. □

**Exercise 3.** Let $X = (X_n)_{n\geq 0}$ be Gambler's ruin with state space $\Omega = \{0, 1, 2, ..., N\}$:

$$X_0 = k, \quad \mathbb{P}[X_{n+1} = X_n + 1 \,|\, X_n] = \mathbb{P}[X_{n+1} = X_n - 1 \,|\, X_n] = 1/2, \quad \tau = \min\{n : X_n = 0 \text{ or } N\}.$$

*(1).* Show that $Y = (Y_n := X_n^2 - n)_{n\geq 0}$ is a martingale.

*Proof.* By the definition of $X$,

$$\begin{aligned}
\mathbb{E}[Y_n \,|\, \mathcal{F}_{n-1}] &= \mathbb{E}[X_n^2 - n \,|\, \mathcal{F}_{n-1}] \\
&= \mathbb{E}[(X_n - X_{n-1})^2 + 2(X_n - X_{n-1})X_{n-1} + X_{n-1}^2 - n \,|\, \mathcal{F}_{n-1}] \\
&= \mathbb{E}[(X_n - X_{n-1})^2 \,|\, X_{n-1}] + 2\mathbb{E}[X_n - X_{n-1} \,|\, X_{n-1}]X_{n-1} + X_{n-1}^2 - n \\
&= 1 + 0 + X_{n-1}^2 - n = Y_{n-1},
\end{aligned}$$

so $Y$ is a martingale. □

*(2).* Show that $Y$ has bounded increments.

*Proof.* It is clear that

$$\begin{aligned}
|Y_n - Y_{n-1}| &= |X_n^2 - X_{n-1}^2 - 1| \\
&\leq |X_n + X_{n-1}||X_n - X_{n-1}| + 1 \\
&\leq |X_{n-1}| + 1 + |X_{n-1}| + 1 \\
&\leq 2N + 2,
\end{aligned}$$

so $Y$ has bounded increments. □

*(3).* Show that $\mathbb{E}[\tau] < \infty$.

*Proof.* First, let $\alpha$ be the probability that the chain increases for $N$ consecutive steps, i.e.

$$\alpha = \mathbb{P}[X_{i+1} - X_i = 1, X_{i+2} - X_{i+1} = 1, \ldots, X_{i+N} - X_{i+N-1} = 1]$$

which is positive and does not depend on $i$. If $\tau > mN$, then the chain never increases $N$ times consecutively in the first $mN$ steps. In particular,

$$\{\tau > mN\} \subset \bigcap_{i=0}^{m-1} \{X_{iN+1} - X_{iN} = 1, X_{iN+2} - X_{iN+1} = 1, \ldots, X_{iN+N} - X_{iN+N-1} = 1\}^c.$$

Since the events on the right-hand side are independent and each have probability $1 - \alpha < 1$,

$$\mathbb{P}[\tau > mN] \leq (1-\alpha)^m.$$

For $mN \leq l < (m+1)N$, $\mathbb{P}[\tau > l] \leq \mathbb{P}[\tau > mN]$, so

$$\mathbb{E}[\tau] = \sum_{l=0}^{\infty} \mathbb{P}[\tau > l] \leq \sum_{m=0}^{\infty} N\mathbb{P}[\tau > mN] \leq N \sum_{m=0}^{\infty} (1-\alpha)^m < \infty.$$

□

*(4).* Show that $\mathbb{E}[\tau] = k(N-k)$.

*Proof.* Since $\mathbb{E}[X_{n+1} - X_n \,|\, \mathcal{F}_n] = 0$ and $|X_{n+1} - X_n| = 1$, $X$ is a martingale with bounded increments. We also showed that $Y$ is a martingale with bounded increments. As $\mathbb{E}[\tau] < \infty$, Exercise 2 Part (4) implies that

$$k = \mathbb{E}[X_0] = \mathbb{E}[X_\tau] = \mathbb{P}[X_\tau = 0] \cdot 0 + \mathbb{P}[X_\tau = N] \cdot N \tag{1}$$

$$\text{and} \quad k^2 = \mathbb{E}[Y_0] = \mathbb{E}[Y_\tau] = \mathbb{E}[X_\tau^2] - \mathbb{E}[\tau]. \tag{2}$$

Then (1) gives, $\mathbb{P}[X_\tau = N] = k/N$. Hence it follows from (2) that

$$\mathbb{E}[\tau] = \mathbb{E}[X_\tau^2] - k^2 = \mathbb{P}[X_\tau = 0] \cdot 0 + \mathbb{P}[X_\tau = N] \cdot N^2 - k^2 = kN - k^2 = k(N-k).$$

□

**Exercise 4.** Let $X = (X_n)_{n\geq 0}$ be the simple random walk on $\mathbb{Z}$.

*(1).* Show that $(Y_n := X_n^3 - 3nX_n)_{n\geq 0}$ is a martingale.

*Proof.* We have

$$\begin{aligned}
&\mathbb{E}[Y_n - Y_{n-1} \,|\, \mathcal{F}_{n-1}] \\
&= \mathbb{E}[X_n^3 - 3nX_n - X_{n-1}^3 + 3(n-1)X_{n-1} \,|\, \mathcal{F}_{n-1}] \\
&= \mathbb{E}[(X_n - X_{n-1})^3 + 3(X_n - X_{n-1})^2 X_{n-1} + 3(X_n - X_{n-1})X_{n-1}^2 - 3n(X_n - X_{n-1}) - 3X_{n-1} \,|\, \mathcal{F}_{n-1}] \\
&= \mathbb{E}[(X_n - X_{n-1})^3] + 3\mathbb{E}[(X_n - X_{n-1})^2]X_{n-1} + 3\mathbb{E}[X_n - X_{n-1}]X_{n-1}^2 - 3n\mathbb{E}[X_n - X_{n-1}] - 3X_{n-1} \\
&= 0 + 3X_{n-1} + 0 - 0 - 3X_{n-1} \\
&= 0,
\end{aligned}$$

so $Y$ is a martingale. ☐

*(2).* Let $\tau$ be the first time that the walker hits either 0 or $N$. Show that, for $0 \leq k \leq N$, we have

$$\mathbb{E}_k[\tau \,|\, X_\tau = N] = \frac{N^2 - k^2}{3}.$$

*Proof.* Since $0 \leq X_n^\tau \leq N$, the martingale $Y^\tau$ is bounded and thus has bounded increments. The stopping time $\tau$ is the same as in Exercise 3, so the same argument implies that

$$k^3 = \mathbb{E}[Y_0] = \mathbb{E}[Y_\tau] = \mathbb{E}[X_\tau^3] - 3\mathbb{E}[\tau X_\tau].$$

We compute that $\mathbb{E}[X_\tau^3] = \mathbb{P}[X_\tau = 0] \cdot 0 + \mathbb{P}[X_\tau = N] \cdot N^3 = kN^2$. Hence

$$\frac{kN^2 - k^3}{3} = \mathbb{E}[\tau X_\tau] = \mathbb{P}[X_\tau = 0] \cdot 0 + \mathbb{P}[X_\tau = N] \cdot \mathbb{E}[\tau N \,|\, X_\tau = N] = k\mathbb{E}[\tau \,|\, X_\tau = N].$$

We conclude that

$$\mathbb{E}[\tau \,|\, X_\tau = N] = \frac{N^2 - k^2}{3}.$$

☐

**Exercise 5.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with filtration $(\mathcal{F}_n)_{n\geq 0}$.

*(1).* For any $m, m' \geq n$ and $A \in \mathcal{F}_n$, show that $T = m1_A + m'1_{A^c}$ is a stopping time.

*Proof.* Assume without loss of generality that $m \leq m'$ (since we can flip the roles of $A$ and $A^c$). If $l < m$, then $\{T \leq l\} = \varnothing \in \mathcal{F}_l$. If $m \leq l < m'$, then $\{T \leq l\} = A \in \mathcal{F}_n \subset \mathcal{F}_l$ as $n \leq m \leq l$. If $l \geq m'$, then $\{T \leq l\} = \Omega \in \mathcal{F}_l$. Hence $T$ is a stopping time. ☐

*(2).* Show that an adapted process $(X_n)_{n\geq 0}$ is a martingale if and only if it is integrable, and for every bounded stopping time $T$, we have $\mathbb{E}[X_T] = \mathbb{E}[X_0]$.

*Proof.* The "only if" part was proved in Exercise 2 Part (2) with $S \equiv 0$.

Conversely, suppose for every bounded stopping time $T$, we have $\mathbb{E}[X_T] = E[X_0]$. In particular, $\mathbb{E}[X_m] = \mathbb{E}[X_0]$ for every $m \in \mathbb{N}$. Moreover, for $n \leq m$ and $A \in \mathcal{F}_n$, Part (1) implies that $T = n\mathbb{1}_A + m\mathbb{1}_{A^c}$ is a bounded stopping time. Thus

$$\mathbb{E}[X_m] = \mathbb{E}[X_0] = \mathbb{E}[X_T] = \mathbb{E}[X_n\mathbb{1}_A + X_m\mathbb{1}_{A^c}],$$

so $\mathbb{E}[X_m\mathbb{1}_A] = \mathbb{E}[X_n\mathbb{1}_A]$. By definition, this means $\mathbb{E}[X_m \,|\, \mathcal{F}_n] = X_n$, so $X$ is a martingale. ☐

**Exercise 6.** Let $X = (X_n)_{n\geq 0}$ be a martingale in $L^2$.

*(1).* Show that its increments $(X_{n+1} - X_n)_{n\geq 0}$ are pairwise orthogonal, i.e. for all $n \neq m$, we have

$$\mathbb{E}[(X_{n+1} - X_n)(X_{m+1} - X_m)] = 0.$$

*Proof.* First, note that for any $n \leq m$,

$$\mathbb{E}[X_n X_m] = \mathbb{E}\Big[\mathbb{E}[X_n X_m \,|\, \mathcal{F}_n]\Big] = \mathbb{E}\Big[X_n \mathbb{E}[X_m \,|\, \mathcal{F}_n]\Big] = \mathbb{E}[X_n^2].$$

Now assume without loss of generality that $n < m$. Then

$$\begin{aligned}\mathbb{E}[(X_{n+1} - X_n)(X_{m+1} - X_m)] &= \mathbb{E}[X_{n+1}X_{m+1}] - \mathbb{E}[X_n X_{m+1}] - \mathbb{E}[X_{n+1}X_m] + \mathbb{E}[X_n X_m] \\ &= \mathbb{E}[X_{n+1}^2] - \mathbb{E}[X_n^2] - \mathbb{E}[X_{n+1}^2] + \mathbb{E}[X_n^2] = 0.\end{aligned}$$

□

*(2).* Show that $X$ is bounded in $L^2$ if and only if

$$\sum_{n\geq 0} \mathbb{E}[(X_{n+1} - X_n)^2] < \infty.$$

*Proof.* Note that

$$\mathbb{E}[X_0(X_{n+1} - X_n)] = \mathbb{E}[X_0^2] - \mathbb{E}[X_0^2] = 0$$

by the computation in Part (1). Thus for any $m$, we have

$$\mathbb{E}[X_m^2] = \mathbb{E}\Big[\big(X_0 + \sum_{n=0}^{m-1}(X_{n+1} - X_n)\big)^2\Big] = \mathbb{E}[X_0^2] + \sum_{n=0}^{m-1} \mathbb{E}[(X_{n+1} - X_n)^2]$$

where the cross terms disappear by Part (1). Therefore,

$$\sup_{m\geq 0} \mathbb{E}[X_m^2] = \mathbb{E}[X_0^2] + \sum_{n\geq 0} \mathbb{E}[(X_{n+1} - X_n)^2]. \tag{3}$$

If $X$ is bounded in $L^2$, i.e. the left-hand side in (3) is bounded, then the sum on the right-hand side is bounded. Conversely, if the sum is bounded, since $X_0$ is in $L^2$, the left-hand side is also bounded. □