18.445 Introduction to Stochastic Processes

Lecture 17: Martingle: a.s convergence and L^p-convergence

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Recall

- Martingale : $\mathbb{E}[X_n \mid \mathcal{F}_m] = X_m$ for $n \geq m$.
- Optional Stopping Theorem : $\mathbb{E}[X_T] = \mathbb{E}[X_0]$?

Today's goal

- a.s.martingale convergence
- Doob's maximal inequality
- convergence in L^p for $p > 1$

Various convergences

Spaces

- L^1 space : $\mathbb{E}[|X|] < \infty$.
  - L^1-norm : $||X||_1 = \mathbb{E}[|X|]$.
  - triangle inequality : $||X+Y||_1 \leq ||X||_1 + ||Y||_1$.
- L^p space for $p > 1$: $\mathbb{E}[|X|^p] < \infty$
  - L^p-norm : $||X||_p = \mathbb{E}[|X|^p]^{1/p}$.
  - triangle inequality : $||X+Y||_p \leq ||X||_p + ||Y||_p$.

Lemma

For $p > 1$, L^p is contained in L^1.

different notions of convergence

- almost sure convergence : $X_n \to X_\infty$ a.s.
- convergence in L^p : $X_n \to X_\infty$ in L^p.
- convergence in L^1 : $X_n \to X_\infty$ in L^1.

A.S. Martingale Convergence

Theorem

Let $X = (X_n)_{n\geq 0}$ *be a* *supermartingale* *which is bounded in* L^1*, i.e.* $\sup_n \mathbb{E}[|X_n|] < \infty$*. Then*

$$X_n \to X_\infty, \quad \text{almost surely, as} \quad n \to \infty,$$

for some $X_\infty \in L^1$*.*

Proof Attached on the website.

Corollary

Let $X = (X_n)_{n\geq 0}$ *be a* *non-negative supermartingale*. *Then* X_n *converges a.s. to some a.s. finite limit.*

Examples

Example 1 Let $(\xi_j)_{j\geq 1}$ be independent random variables with mean zero such that $\sum_{j=1}^{\infty} \mathbb{E}[|\xi_j|] < \infty$. Set

$$X_0 = 0, \quad X_n = \sum_{j=1}^{n} \xi_j.$$

- $(X_n)_{n\geq 0}$ is a martingale bounded in L^1.
- X_n converges a.s. to $X_\infty = \sum_{j=1}^{\infty} \xi_j$.
- In fact, X_n also converges to X_∞ in L^1.

Example 2 Let $(\xi_j)_{j\geq 1}$ be non-negative independent random variables with mean one. Set

$$X_0 = 1, \quad X_n = \Pi_{j=1}^{n} \xi_j.$$

- $(X_n)_{n\geq 0}$ is a non-negative martingale.
- X_n converges a.s. to some limit $X_\infty \in L^1$.

Question

Suppose that a martingale X is bounded in L^1, then we have the a.s. convergence.

Question : Do we have $\mathbb{E}[X_\infty] = \mathbb{E}[X_0]$?

Answer : It is true when we have convergence in L^1 .

- Convergence in L^p for $p > 1$ implies convergence in L^1. (Today)
- Convergence in L^1. (Next lecture)

Doob's maximal inequality

Theorem

Let $X = (X_n)_{n\geq 0}$ be a non-negative submartingale. Define $X_n^* = \max_{0\leq k\leq n} X_k$. Then

$$\lambda\mathbb{P}[X_n^* \geq \lambda] \leq \mathbb{E}[X_n 1_{[X_n^* \geq \lambda]}] \leq \mathbb{E}[X_n].$$

Theorem

Let $X = (X_n)_{n\geq 0}$ be a non-negative submartingale. Define $X_n^* = \max_{0\leq k\leq n} X_k$. Then, for all $p > 1$, we have

$$||X_n^*||_p \leq \frac{p}{p-1}||X_n||_p.$$

Recall Hölder inequality : $p > 1, q > 1$ and $1/p + 1/q = 1$, then

$$\mathbb{E}[|XY|] \leq \mathbb{E}[|X|^p]^{1/p} \times \mathbb{E}[|Y|^q]^{1/q}.$$

L^p Convergence for $p > 1$

Theorem

Let $X = (X_n)_{n \geq 0}$ be a martingale and $p > 1$, then the following statements are equivalent.

1. *X is bounded in L^p : $\sup_{n \geq 0} ||X_n||_p < \infty$*
2. *X converges a.s and in L^p to a random variable X_∞.*
3. *There exists a random variable $Z \in L^p$ such that*

$$X_n = \mathbb{E}[Z \,|\, \mathcal{F}_n] \quad a.s.$$

Corollary

Let $Z \in L^p$. Then

$$\mathbb{E}[Z \,|\, \mathcal{F}_n] \to \mathbb{E}[Z \,|\, \mathcal{F}_\infty], \quad a.s. \text{and in } L^p.$$

Example

Let $(\xi_j)_{j\geq 1}$ be independent random variables with mean zero such that $\sum_{j=1}^{\infty} \mathbb{E}[\xi_j^2] < \infty$. Set

$$X_0 = 0, \quad X_n = \sum_{j=1}^{n} \xi_j.$$

- $(X_n)_{n\geq 0}$ is a martingale bounded in L^2.
- X_n converges to $X_\infty = \sum_{j=1}^{\infty} \xi_j$ a.s. and in L^2.
- $\mathbb{E}[X_\infty^2] = \sum_{j=1}^{\infty} \mathbb{E}[\xi_j^2]$.

Example

Let $(\xi_j)_{j\geq 1}$ be non-negative independent random variables with mean one. Set

$$X_0 = 1, \quad X_n = \Pi_{j=1}^n \xi_j.$$

1. $(X_n)_{n\geq 0}$ is a non-negative martingale.
2. X_n converges a.s. to some limit $X_\infty \in L^1$.

Question :

1. Do we have $\mathbb{E}[X_\infty] = 1$?

Answer : Set $a_j = \mathbb{E}[\sqrt{\xi_j}] \in (0, 1]$.

1. If $\Pi_j a_j > 0$, then X converges in L^1 and $\mathbb{E}[X_\infty] = 1$. (Next lecture)
2. If $\Pi_j a_j = 0$, then $X_\infty = 0$ a.s.