18.445 Introduction to Stochastic Processes

Lecture 12: Countable state space chains 1

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30 March 2015

Recall

Ω : **finite** state space, P : transition matrix. A Markov chain $(X_n)_{n\geq 0}$ is a random process such that

$$\mathbb{P}[X_{n+1} = y \,|\, X_1 = x_1, ..., X_{n-1} = x_{n-1}, X_n = x]$$
$$= \mathbb{P}[X_{n+1} = y \,|\, X_n = x] = P(x, y).$$

Today

Ω : **countable** state space, P : transition matrix. A Markov chain $(X_n)_{n\geq 0}$ is a random process such that

$$\mathbb{P}[X_{n+1} = y \,|\, X_1 = x_1, ..., X_{n-1} = x_{n-1}, X_n = x]$$
$$= \mathbb{P}[X_{n+1} = y \,|\, X_n = x] = P(x, y).$$

Related notions

- stationary distribution : $\pi = \pi P$ and π has unit total mass
- irreducible : for any $x, y \in \Omega$, there exists n such that $P^n(x, y) > 0$
- first hitting time and first return time : for $x \in \Omega$

$$\tau_x = \min\{n \geq 0 : X_n = x\}, \quad \tau_x^+ = \min\{n \geq 1 : X_n = x\}.$$

Definition

We say a state $x \in \Omega$ is recurrent if

$$\mathbb{P}_x[\tau_x^+ < \infty] = 1.$$

Otherwise, we say x is transient.

Remark If Ω is finite and P is irreducible, every state is recurrent. However, when Ω is countable, we have two different cases : recurrent or transient.

Recurrence

A state $x \in \Omega$ is recurrent if $\mathbb{P}_x[\tau_x^+ < \infty] = 1$.

Lemma

Suppose that P is irreducible. Define Green's function

$$G(x, y) = \mathbb{E}_x[\sharp\textit{visits to } y] = \sum_0^\infty P^n(x, y).$$

The following four conditions are equivalent.

1. $G(x, x) = \infty$ *for some* $x \in \Omega$
2. $G(x, y) = \infty$ *for all* $x, y \in \Omega$
3. $\mathbb{P}_x[\tau_x^+ < \infty] = 1$ *for some* $x \in \Omega$
4. $\mathbb{P}_x[\tau_y^+ < \infty] = 1$ *for all* $x, y \in \Omega$

Recurrence

A state $x \in \Omega$ is recurrent if $\mathbb{P}_x[\tau_x^+ < \infty] = 1$.
Suppose that P is irreducible. The following two conditions are equivalent.

1. $\mathbb{P}_x[\tau_x^+ < \infty] = 1$ for some $x \in \Omega$
2. $\mathbb{P}_x[\tau_y^+ < \infty] = 1$ for all $x, y \in \Omega$

Therefore, for an irreducible chain, a single state is recurrent if and only if all states are recurrent. For this reason, an irreducible Markov chain can be classified as either recurrent or transient.

Examples

- simple random walk on $\mathbb{Z}^2$ is recurrent
- simple random walk on $\mathbb{Z}^3$ is transient

Infinite networks

For an infinite connected graph $G = (V, E)$ with edge conductances $\{c(e) : e \in E\}$. Fix the source $a \in V$.

- Let $(G_n = (V_n, E_n))$ be a sequence of finite connected subgraphs containing the source such that
  - $V_n \subset V_{n+1}$ for all n, and $\cup_n V_n = V$
  - E_n contains all edges in E that with both endpoints in V_n.
- For each n, construct a modified network G_n^* in which all the vertices in $V \setminus V_n$ are replaced by a single vertex z_n
  - z_n is adjacent to all vertices in V_n which are adjacent to $V \setminus V_n$
  - $c(x, z_n) = \sum_{z \in V \setminus V_n} c(x, z)$.
- Define

$$R(a \leftrightarrow \infty) = \lim_n R(a \leftrightarrow z_n).$$

Lemma

This is well-defined, i.e. the limit exists and does not depend on the choice of the sequence (G_n).

Effective Resistance and Escape Probability

For an infinite connected graph $G = (V, E)$ with edge conductances $\{c(e) : e \in E\}$. Fix the source $a \in V$.

$$R(a \leftrightarrow \infty) = \lim_n R(a \leftrightarrow z_n).$$

Theorem

$$\mathbb{P}_a[\tau_a^+ = \infty] = \frac{1}{c(a)R(a \leftrightarrow \infty)}.$$

Proof

$$\lim_n \mathbb{P}_a[\tau_{z_n} < \tau_a^+] = \lim_n \frac{1}{c(a)R(a \leftrightarrow z_n)}.$$

Effective Resistance and Energy of Flows

Definition

A flow θ on G from a to ∞ is an antisymmetric edge function such that

$$div\theta(a) \geq 0, \quad div\theta(x) = 0, \text{ for all } x \neq a.$$

- The strength $||\theta|| = div\theta(a)$.
- The energy $\mathcal{E}(\theta) = \sum_e \theta(e)^2 r(e)$.

Theorem

$$R(a \leftrightarrow \infty) = \inf\{\mathcal{E}(\theta) : \theta \textit{ unit flow from a to } \infty\}$$

Corollary

Suppose that (Π_k) *are disjoint edge-cut sets that separates a from* ∞*, then*

$$R(a \leftrightarrow \infty) \geq \sum_k \left(\sum_{e \in \Pi_k} c(e) \right)^{-1}.$$

Random walk on infinite network

Theorem

The following are equivalent.

1. *The random walk on the network is transient.*
2. *There exists $a \in V$ such that $R(a \leftrightarrow \infty) < \infty$.*
3. *There exists a flow θ from a to ∞ such that $||\theta|| > 0$ and $\mathcal{E}(\theta) < \infty$.*

Theorem

If there exists disjoint edge-cut sets (Π_k) that separates a from ∞ and that

$$\sum_k \left(\sum_{e \in \Pi_k} c(e) \right)^{-1} = \infty.$$

Then the random walk on the network is recurrent.

Simple random walk on $\mathbb{Z}^d$

Theorem

- *Simple random walk on $\mathbb{Z}^1$ is recurrent.*
- *Simple random walk on $\mathbb{Z}^2$ is recurrent.*

Theorem

- *Simple random walk on $\mathbb{Z}^3$ is transient.*
- *Simple random walk on $\mathbb{Z}^d$ is transient for $d \geq 3$.*

Positive recurrence

Definition

- A state x is recurrent if $\mathbb{P}_x[\tau_x^+ < \infty] = 1$.
- A state x is positive recurrent if $\mathbb{E}_x[\tau_x^+] < \infty$.

Lemma

Suppose that P is irreducible.

- *The following two conditions are equivalent.*
  - $\mathbb{P}_x[\tau_x^+ < \infty] = 1$ *for some* $x \in \Omega$
  - $\mathbb{P}_x[\tau_y^+ < \infty] = 1$ *for all* $x, y \in \Omega$
- *The following two conditions are equivalent.*
  - $\mathbb{E}_x[\tau_x^+] < \infty$ *for some* $x \in \Omega$
  - $\mathbb{E}_x[\tau_y^+] < \infty$ *for all* $x, y \in \Omega$

Positive recurrence

Definition

A state x is positive recurrent if $\mathbb{E}_x[\tau_x^+] < \infty$.

Lemma

Suppose that P is irreducible. The following two conditions are equivalent.

- $\mathbb{E}_x[\tau_x^+] < \infty$ *for some* $x \in \Omega$
- $\mathbb{E}_x[\tau_y^+] < \infty$ *for all* $x, y \in \Omega$

Therefore, for an irreducible chain, a single state is positive recurrent if and only if all states are positive recurrent. For this reason, an irreducible recurrent Markov chain can be classified as either positive recurrent or else which we call null recurrent.

Example Simple random walk on $\mathbb{Z}$ is null recurrent.