# 18.445 Introduction to Stochastic Processes

## Lecture 19: Galton-Watson tree

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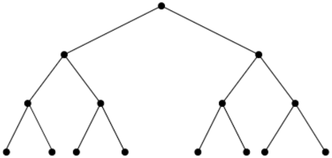
# Rooted trees

A **tree** is a connected graph with no cycles.
A **rooted tree** has a distinguished vertex $v_0$, called the root.
The **depth** of a vertex $v$ is its graph distance to the root.
A **leaf** is a vertex with degree one.
Consider a **regular rooted tree** :

- each vertex has a fixed number (say $m$) of offspring
- $Z_n$ : the number vertices in the $n$-th generation
- for regular tree : $Z_n = m^n$

A rooted binary tree



In real life, we often encounter trees where the number of offspring of a vertex is random.

# Galton-Watson tree

- It starts with one initial ancestor
- it produces a certain number of offspring according to some distribution $\mu$
- the new particles form the first generation
- each of the new particles produces offspring according to $\mu$, independently of each other
- the system regenerates
- $Z_n$ : the number of particles in $n$-th generation

**Observation :** If $Z_n = 0$ for some $n$, then $Z_m = 0$ for all $m \geq n$
$\rightarrow$ the family become extinct
**Question :** extinction probability $q = \mathbb{P}[Z_n = 0 \text{ eventuallly}]$ ?

# Extinction probability

**Notations :**

- $\mu$ : let $p_k$ be the probability that a particle has $k$ children, $k \geq 0$
- $\sum_0^\infty p_k = 1$
- $m := \mathbb{E}[Z_1] = \sum_0^\infty kp_k$
- Assume $p_0 + p_1 < 1$.
- Convention $0^0 = 1$
- extinction probability $q = \mathbb{P}[Z_n = 0 \text{ eventuallly}]$
- $f$ : the generating function of the reproduction law :

$$f(s) := \mathbb{E}[s^{Z_1}] = \sum_0^\infty s^k p_k.$$

  - $f(0) = p_0$, $f(1) = 1$, $f'(1) = m$.

**Theorem**

*The extinction probability $q$ is the smallest root of $f(s) = s$ for $s \in [0,1]$. In particular, $q = 1$ if $m \leq 1$, and $q < 1$ if $m > 1$.*

# Extinction probability

**Theorem**

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- In the subcritical case ($m < 1$), the GW tree dies out with probability 1
- In the critical case ($m = 1$), the GW tree dies out with probability 1
- In the supercritical case ($m > 1$), the GW tree survives with strictly positive probability $1 - q$.

**Question :** In the supercritical case $m > 1$, how fast the tree grows ? We know that $\mathbb{E}[Z_n] = m^n$, do we have $Z_n \sim m^n$ ?

# Growth rate

**Assumption :** $m \in (1, \infty)$. Define $W_n = Z_n / m^n$.

- $(W_n)_{n \geq 0}$ is a non-negative martingale
- $W_n \to W$ a.s.
- By Fatou's Lemma, we have $\mathbb{E}[W] \leq 1$

**Observation :** If $W > 0$, then $Z_n \sim m^n$; if $W = 0$, then $Z_n << m^n$.

Theorem (Kesten and Stigum)

$$\mathbb{E}[W] = 1 \Leftrightarrow \mathbb{P}[W > 0 \mid \textit{non-extinction}] \Leftrightarrow \mathbb{E}[Z_1 \log^+ Z_1] < \infty$$

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### Theorem

*If* $\mathbb{E}[Z_1^2] < \infty$*, then* $\mathbb{E}[W] = 1$ *and* $\mathbb{P}[W = 0] = q$*.*

### Lemma

$\mathbb{P}[W = 0]$ *is either* $q$ *or* $1$*.*