Numerical Algorithms

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# New algorithms and convergence theorems for solving variational inequalities with non-Lipschitz mappings

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**Abstract** We propose and study new projection-type algorithms for solving pseudomonotone variational inequality problems in real Hilbert spaces without assuming Lipschitz continuity of the cost operators. We prove weak and strong convergence theorems for the sequences generated by these new methods. The numerical behavior of the proposed algorithms when applied to several test problems is compared with several previously known algorithms.

**Keywords** Projection-type method · pseudomonotone operator · viscosity method · variational inequality · weak convergence · strong convergence

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## 1 Introduction

We consider the variational inequality problem (VI) [10, 11] of finding a point $x^* \in C$ such that

$$\langle Fx^*, x - x^* \rangle \geq 0 \quad \forall x \in C, \tag{1}$$

where $C$ is a nonempty, closed and convex subset of a real Hilbert space $H$, $F : H \to H$ is a single-valued mapping, and $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ are the inner product and the induced norm on $H$,

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respectively. We denote by $Sol(C,F)$ the solution set of problem (1). Variational inequality problems are fundamental in a broad range of mathematical and applied sciences; the theoretical and algorithmic foundations, as well as the applications of variational inequalities have been extensively studied in the literature and continue to attract intensive research. For a detailed exposition of the field in the finite-dimensional setting, see, for instance, [9] and the extensive list of references therein.

Many authors have proposed and analyzed several iterative methods for solving the variational inequality (1). The simplest one is the following projection method, which can be considered an extension of the projected gradient method for optimization problems:

$$x_{n+1} = P_C(x_n - \lambda F x_n), \tag{2}$$

for each $n \geq 1$, where $P_C$ denotes by the metric projection from $H$ onto $C$. Convergence results for this method require some monotonicity properties of $F$. This method converges under quite strong hypotheses. If $F$ is Lipschitz continuous with Lipschitz constant $L$ and $\alpha$-strongly monotone, then the sequence generated by (2) converges to an element of $Sol(C,F)$ for $\lambda \in (0, \frac{2\alpha}{L^2})$.

In order to find an element of $Sol(C,F)$ under weaker hypotheses, Korpelevich [20] (also independently by Antipin [1]) proposed to replace method (2) by the extragradient method in the finite-dimensional Euclidean space $\mathbb{R}^m$ for a monotone and $L$-Lipschitz continuous operator $F : \mathbb{R}^m \to \mathbb{R}^m$. Her algorithm is of the form

$$x_0 \in C, \;\; y_n = P_C(x_n - \lambda F x_n), \;\; x_{n+1} = P_C(x_n - \lambda F y_n), \tag{3}$$

where $\lambda \in (0, \frac{1}{L})$. The sequence $\{x_n\}$ generated by (3) converges to an element of $Sol(C,F)$ provided that $Sol(C,F)$ is nonempty.

In recent years the extragradient method has been extended to infinite-dimensional spaces in various ways; see, for example, [3,4,5,6,21,24,25,29,30,31] and the references cited therein.

We may observe that, when $F$ is not Lipschitz continuous or the constant $L$ is very difficult to compute, Korpelevich's method is not so applicable because we cannot determine the step-size $\tau_n$. To overcome this difficulty, Iusem [15] proposed in the Euclidean space $\mathbb{R}^m$ the following iterative algorithm for solving $Sol(C,F)$:

$$y_n = P_C(x_n - \gamma_n F x_n), \;\; x_{n+1} = P_C(x_n - \lambda_n F y_n), \tag{4}$$

where $\gamma_n > 0$ is computed through an Armijo-type search and $\lambda_n = \frac{\langle F y_n, x_n - y_n \rangle}{\|F y_n\|^2}$. This modification has allowed the author to establish convergence without assuming Lipschitz continuity of the operator $F$.

In order to determine the step size $\gamma_n$ in (4), we need to use a line search procedure which contains one projection. So at iteration $n$, if this procedure requires $m_n$ steps to arrive at the appropriate $\gamma_n$, then we need to evaluate $m_n$ projections.

To overcome this difficulty, Iusem and Svaiter [18] proposed a modified extragradient method for solving monotone variational inequalities which only requires two projections onto $C$ at each iteration. A few years later, this method was improved by Solodov and Svaiter [29]. They introduced an algorithm for solving (1) in finite-dimensional spaces. As a matter of fact, their method applies to a more general case where $F$ is only continuous and satisfies the following condition:

$$\langle F x, x - x^* \rangle \geq 0 \;\; \forall x \in C \text{ and } x^* \in Sol(C,F). \tag{5}$$

Property (5) holds if $F$ is monotone or, more generally, pseudomonotone on $C$ in the sense of Karamardian [19]. More precisely, Solodov and Svaiter proposed the following algorithm:

**Algorithm 1.1**

***Initialization:*** *Given* $l \in (0,1), \mu \in (0,1)$*, let* $x_1 \in \mathbb{R}^m$ *be arbitrary*

***Iterative Steps:*** *Given the current iterate* $x_n$*, calculate* $x_{n+1}$ *as follows:*

***Step 1.*** *Compute*

$$z_n = P_C(x_n - Fx_n)$$

*and* $r(x_n) := x_n - z_n$*. If* $r(x_n) = 0$*, then stop;* $x_n$ *belongs to* $Sol(C,F)$*. Otherwise,*
***Step 2.*** *Compute*

$$y_n = x_n - \tau_n r(x_n),$$

*where* $\tau_n := l^{j_n}$ *and* $j_n$ *is the smallest non-negative integer* $j$ *satisfying*

$$\langle F(x_n - l^j r(x_n)), r(x_n)\rangle \geq \mu \|r(x_n)\|^2. \tag{6}$$

***Step 3.*** *Compute*

$$x_{n+1} = P_{C\cap H_n}(x_n),$$

*where*

$$H_n := \{x \in \mathbb{R}^m : \langle Fy_n, x - y_n\rangle \leq 0\}.$$

*Set* $n := n+1$ *and go to* ***Step 1****.*

Vuong and Shehu [34] have recently modified the result of Solodov and Svaiter in the spirit of Halpern [13], and obtained strong convergence in infinite-dimensional real Hilbert spaces. Their algorithm is of the following form:

**Algorithm 1.2**

***Initialization:*** *Given* $\{\alpha_n\} \subset (0,1), l \in (0,1), \mu \in (0,1)$*, let* $x_1 \in C$ *be arbitrary*

***Iterative Steps:*** *Given the current iterate* $x_n$*, calculate* $x_{n+1}$ *as follows:*

***Step 1.*** *Compute*

$$z_n = P_C(x_n - Fx_n)$$

*and* $r(x_n) := x_n - z_n$*. If* $r(x_n) = 0$*, then stop;* $x_n$ *belongs to* $Sol(C,F)$*. Otherwise,*
***Step 2.*** *Compute*

$$y_n = x_n - \tau_n r(x_n),$$

*where* $\tau_n := l^{j_n}$ *and* $j_n$ *is the smallest non-negative integer* $j$ *satisfying*

$$\langle F(x_n - l^j r(x_n)), r(x_n)\rangle \geq \frac{\mu}{2} \|r(x_n)\|^2. \tag{7}$$

***Step 3.*** *Compute*

$$x_{n+1} = \alpha_n x_1 + (1-\alpha_n)P_{C_n}(x_n),$$

*where*

$$C_n := \{x \in C : h_n(x_n) \le 0\}$$

*and*

$$h_n(x) = \langle Fy_n, x - y_n \rangle.$$

*Set* $n := n+1$ *and go to* ***Step 1***.

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Vuong and Shehu proved that if $F : H \to H$ is a pseudomonotone, uniformly continuous and weakly sequentially continuous on bounded subsets of $C$, and the sequence $\{\alpha_n\}$ satisfies the conditions $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, then the sequence $\{x_n\}$ generated by Algorithm 1.2 converges strongly to $p \in Sol(C,F)$, where $p = P_C x_1$.

Motivated and inspired by [29, 34], and by the ongoing research in these directions, in the present paper we introduce new algorithms for solving variational inequalities with uniformly continuous pseudomonotone operators. In particular, we use a different Armijo-type line search in order to obtain a hyperplane which strictly separates the current iterate from the solutions of the variational inequality under consideration.

Our paper is organized as follows. We first recall in Section 2 some basic definitions and results. Our algorithms are presented and analyzed in Section 3. In Section 4 we present several numerical experiments which illustrate the performance of the algorithms. They also provide a preliminary computational overview by comparing it with the performance of several related algorithms.

## 2 Preliminaries

Let $H$ be a real Hilbert space and $C$ be a nonempty, closed and convex subset of $H$. The weak convergence of a sequence $\{x_n\}_{n=1}^{\infty}$ to $x$ as $n \to \infty$ is denoted by $x_n \rightharpoonup x$ while the strong convergence of $\{x_n\}_{n=1}^{\infty}$ to $x$ as $n \to \infty$ is denoted by $x_n \to x$. For each $x, y \in H$, we have

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y \rangle.$$

**Definition 2.1** Let $F : H \to H$ be an operator. Then

1. the operator $F$ is called $L$-`Lipschitz continuous` with Lipschitz constant $L > 0$ if

$$\|Fx - Fy\| \le L\|x - y\| \quad \forall x, y \in H.$$

   If $L = 1$, then the operator $F$ is called `nonexpansive` and if $L \in (0,1)$, then $F$ is called a `strict contraction`.
2. $F$ is called `monotone` if

$$\langle Fx - Fy, x - y \rangle \ge 0 \quad \forall x, y \in H.$$

3. $F$ is called `pseudomonotone` if

$$\langle Fx, y - x \rangle \ge 0 \Longrightarrow \langle Fy, y - x \rangle \ge 0 \quad \forall x, y \in H.$$

4. $F$ is called $\alpha$ strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Fx - Fy, x - y \rangle \ge \alpha\|x - y\|^2 \quad \forall x, y \in H.$$

5. The operator $F$ is called `sequentially weakly continuous` if the weak convergence of a sequence $\{x_n\}$ to $x$ implies that the sequence $\{Fx_n\}$ converges weakly to $Fx$.

It is easy to see that every monotone operator is pseudomonotone, but the converse is not true.

For each point $x \in H$, there exists a unique nearest point in $C$, denoted by $P_C x$ which satisfies $\|x - P_C x\| \leq \|x - y\|$ $\forall y \in C$. The mapping $P_C$ is called the *metric projection* of $H$ onto $C$. It is known that $P_C$ is nonexpansive.

**Lemma 2.1** ([12]) *Let $C$ be a nonempty, closed and convex subset of a real Hilbert space $H$. if $x \in H$ and $z \in C$, then $z = P_C x \Longleftrightarrow \langle x - z, z - y\rangle \geq 0$ $\forall y \in C$.*

**Lemma 2.2** ([12]) *Let $C$ be a closed and convex subset of a real Hilbert space $H$ and let $x \in H$. Then*
*i) $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle$ $\forall y \in H$;*
*ii) $\|P_C x - y\|^2 \leq \|x - y\|^2 - \|x - P_C x\|^2$ $\forall y \in C$.*

More properties of the metric projection can be found in Section 3 in [12].

The following lemmas are useful in the convergence analysis of our proposed methods.

**Lemma 2.3** ([16,17]) *Let $H_1$ and $H_2$ be two real Hilbert spaces. Suppose $F : H_1 \to H_2$ is uniformly continuous on bounded subsets of $H_1$ and $M$ is a bounded subset of $H_1$. Then $F(M)$ is bounded.*

**Lemma 2.4** ([7, Lemma 2.1]) *Let $C$ be a nonempty, closed and convex subset of a real Hilbert space $H$, and let $F : C \to H$ be pseudomonotone and continuous. Then $x^*$ belongs to $Sol(C,F)$ if and only if*

$$\langle Fx, x - x^*\rangle \geq 0 \ \ \forall x \in C.$$

The following lemma can be found in [14].

**Lemma 2.5** *Let $C$ be a nonempty, closed and convex subset of a real Hilbert space $H$. Let $h$ be a real-valued function on $H$ and define $K := \{x \in C : h(x) \leq 0\}$. If $K$ is nonempty and $h$ is Lipschitz continuous on $C$ with modulus $\theta > 0$, then*

$$dist(x,K) \geq \theta^{-1} \max\{h(x), 0\} \ \ \forall x \in C,$$

*where $dist(x,K)$ denotes the distance of $x$ to $K$.*

**Lemma 2.6** ([27]) *Let $C$ be a nonempty set of $H$ and let $\{x_n\}$ be a squence in $H$ such that the following two conditions hold:*
*i) for every $x \in C$, $\lim_{n\to\infty} \|x_n - x\|$ exists;*
*ii) every sequential weak cluster point of $\{x_n\}$ is in $C$.*
*Then $\{x_n\}$ converges weakly to a point in $C$.*

The next technical lemma is very useful and has been used by many authors; see, for example, Liu [22] and Xu [33]. A variant of this lemma has already been used by Reich in [28].

**Lemma 2.7** *Let $\{a_n\}$ be sequence of nonnegative real numbers such that:*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n b_n,$$

*where $\{\alpha_n\} \subset (0,1)$ and $\{b_n\}$ is a sequence such that*
*a) $\sum_{n=0}^{\infty} \alpha_n = \infty$;*
*b) $\limsup_{n\to\infty} b_n \leq 0$.*
*Then $\lim_{n\to\infty} a_n = 0$.*

## 3 Main results

In this section we introduce two new methods for solving (1). In the convergence analysis of these algorithms the following three conditions are assumed.

**Condition 3.1** *The feasible set $C$ is a nonempty, closed, and convex subset of the real Hilbert space $H$.*

**Condition 3.2** *The operator $F: C \to H$ associated with the VI (1) is pseudomonotone and uniformly continuous on $C$.*

**Condition 3.3** *The mapping $F: H \to H$ satisfies the following property*

$$\text{whenever } \{x_n\} \subset C, x_n \rightharpoonup z, \text{ one has } \|F(z)\| \leq \liminf_{n\to\infty} \|Fx_n\|.$$

**Condition 3.4** *The solution set of the VI (1) is nonempty, that is, $Sol(C,F) \neq \emptyset$.*

### 3.1 Weak convergence

We begin by introducing a new projection-type algorithm.

**Algorithm 3.3**

***Initialization:*** *Given $\mu > 0, l \in (0,1), \lambda \in (0, \frac{1}{\mu})$, let $x_1 \in C$ be arbitrary*

***Iterative Steps:*** *Given the current iterate $x_n$, calculate $x_{n+1}$ as follows:*

***Step 1.*** *Compute*

$$z_n = P_C(x_n - \lambda F x_n)$$

*and $r_\lambda(x_n) := x_n - z_n$. If $r_\lambda(x_n) = 0$, then stop; $x_n$ belongs to $Sol(C,F)$. Otherwise,*
***Step 2.*** *Compute*

$$y_n = x_n - \tau_n r_\lambda(x_n),$$

*where $\tau_n := l^{j_n}$ and $j_n$ is the smallest non-negative integer $j$ satisfying*

$$\langle Fx_n - F(x_n - l^j r_\lambda(x_n)), r_\lambda(x_n)\rangle \leq \frac{\mu}{2}\|r_\lambda(x_n)\|^2. \tag{8}$$

***Step 3.*** *Compute*

$$x_{n+1} = P_{C_n}(x_n),$$

*where*

$$C_n := \{x \in C : h_n(x_n) \leq 0\}$$

*and*

$$h_n(x) = \langle Fy_n, x - x_n\rangle + \frac{\tau_n}{2\lambda}\|r(x_n)\|^2. \tag{9}$$

*Set $n := n+1$ and go to **Step 1**.*

**Lemma 3.8** *Assume that Conditions 3.1–3.4 hold. Then the Armijo-type search rule (8) is well defined.*

*Proof* Since $l \in (0,1)$ and $F$ is continuous on $C$, the sequence $\{\langle Fx_n - F(x_n - l^j r_\lambda(x_n)), r_\lambda(x_n)\rangle\}$ converges to zero as $j$ tends to infinity. On the other hand, as a consequence of Step 1, $\|r_\lambda(x_n)\| > 0$ (otherwise, the procedure would have stopped). Therefore there exists a non-negative integer $j_n$ satisfying (8).

**Lemma 3.9** *Assume that the sequence $\{x_n\}$ is generated by Algorithm 3.3. Then we have*

$$\langle Fx_n, r_\lambda(x_n)\rangle \geq \lambda^{-1}\|r_\lambda(x_n)\|^2.$$

*Proof* Since $P_C$ is the metric projection, we know that $\|x - P_C y\|^2 \leq \langle x - y, x - P_C y\rangle$ for all $x \in C$ and $y \in H$. Let $y = x_n - \lambda Fx_n, x = x_n$. Then

$$\|x_n - P_C(x_n - \lambda Fx_n)\|^2 \leq \lambda\langle Fx_n, x_n - P_C(x_n - \lambda Fx_n)\rangle$$

and so

$$\langle Fx_n, r_\lambda(x_n)\rangle \geq \lambda^{-1}\|r_\lambda(x_n)\|^2.$$

**Lemma 3.10** *Assume that Conditions 3.1–3.4 hold. Let $x^*$ be a solution of problem (1) and let the function $h_n$ be defined by (9). Then $h_n(x_n) = \frac{\tau_n}{2\lambda}\|r_\lambda(x_n)\|^2$ and $h_n(x^*) \leq 0$. In particular, if $r_\lambda(x_n) \neq 0$, then $h_n(x_n) > 0$.*

*Proof* The first claim of Lemma 3.10 is obvious. In order to prove the second claim, Let $x^*$ be a solution of problem (1) Then by Lemma 2.4 we have $h_n(x^*) = \langle Fy_n, y_n - x^*\rangle \geq 0$. We also have

$$\begin{aligned} h_n(x^*) &= \langle Fy_n, x^* - x_n\rangle + \frac{\tau_n}{2\lambda}\|r(x_n)\|^2 \\ &= -\langle Fy_n, x_n - y_n\rangle - \langle Fy_n, y_n - x^*\rangle + \frac{\tau_n}{2\lambda}\|r(x_n)\|^2 \\ &\leq -\tau_n\langle Fy_n, r_\lambda(x_n)\rangle + \frac{\tau_n}{2\lambda}\|r(x_n)\|^2. \end{aligned} \tag{10}$$

On the other hand, by (8) we have

$$\langle Fx_n - Fy_n, r_\lambda(x_n)\rangle \leq \frac{\mu}{2}\|r_\lambda(x_n)\|^2.$$

Thus

$$\langle Fy_n, r_\lambda(x_n)\rangle \geq \langle Fx_n, r_\lambda(x_n)\rangle - \frac{\mu}{2}\|r_\lambda(x_n)\|^2.$$

Using Lemma 3.9, we get

$$\langle Fy_n, r_\lambda(x_n)\rangle \geq (\frac{1}{\lambda} - \frac{\mu}{2})\|r_\lambda(x_n)\|^2. \tag{11}$$

Combining (10) and (11), we now see that

$$h_n(x^*) \leq -\frac{\tau_n}{2}(\frac{1}{\lambda} - \mu)\|r_\lambda(x_n)\|^2.$$

Thus $h_n(x^*) \leq 0$, as asserted.

**Lemma 3.11** *Assume that Conditions 3.1–3.4 hold. Let $\{x_n\}$ be a sequence is generated by Algorithm 3.3. If there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $z \in C$ and $\lim_{k\to\infty}\|x_{n_k} - z_{n_k}\| = 0$ then $z \in Sol(C,F)$.*

*Proof* Since $z_{n_k} = P_C(x_{n_k} - \lambda F_{n_k})$, we have

$$\langle x_{n_k} - \lambda F x_{n_k} - z_{n_k}, x - z_{n_k} \rangle \le 0 \ \ \forall x \in C$$

or equivalently,

$$\langle x_{n_k} - z_{n_k}, x - z_{n_k} \rangle \le \langle \lambda F x_{n_k}, x - z_{n_k} \rangle \ \ \forall x \in C.$$

This implies that

$$\langle \frac{x_{n_k} - z_{n_k}}{\lambda}, x - z_{n_k} \rangle + \langle F x_{n_k}, z_{n_k} - x_{n_k} \rangle \le \langle F x_{n_k}, x - x_{n_k} \rangle \ \ \forall x \in C. \tag{12}$$

Since $\|x_{n_k} - z_{n_k}\| \to 0$ as $k \to \infty$ and since the sequence $\{F x_{n_k}\}$ is bounded, taking $k \to \infty$ in (12), we get

$$\liminf_{k\to\infty} \langle F x_{n_k}, x - x_{n_k} \rangle \ge 0. \tag{13}$$

Next, to show that $z \in Sol(C,F)$, we first choose a decreasing sequence $\{\varepsilon_k\}$ of positive numbers which tends to 0. For each $k$, we denote by $N_k$ the smallest positive integer such that

$$\langle F x_{n_j}, x - x_{n_j} \rangle + \varepsilon_k \ge 0 \ \ \forall j \ge N_k, \tag{14}$$

where the existence of $N_k$ follows from (13). Since the sequence $\{\varepsilon_k\}$ is decreasing, it is easy to see that the sequence $\{N_k\}$ is increasing. Furthermore, for each $k$, since $\{x_{N_k}\} \subset C$, we have $F x_{N_k} \ne 0$ and setting

$$v_{N_k} = \frac{F x_{N_k}}{\|F x_{N_k}\|^2},$$

we have $\langle F x_{N_k}, x_{N_k} \rangle = 1$ for each $k$. Now, we can deduce from (14) that for each $k$,

$$\langle F x_{N_k}, x + \varepsilon_k v_{N_k} - x_{N_k} \rangle \ge 0.$$

Since the $F$ is pseudomonotone, it follows that

$$\langle F(x + \varepsilon_k v_{N_k}), x + \varepsilon_k v_{N_k} - x_{N_k} \rangle \ge 0.$$

This implies that

$$\langle Fx, x - x_{N_k} \rangle \ge \langle Fx - F(x + \varepsilon_k v_{N_k}), x + \varepsilon_k v_{N_k} - x_{N_k} \rangle - \varepsilon_k \langle Fx, v_{N_k} \rangle. \tag{15}$$

Next, we show that $\lim_{k\to\infty} \varepsilon_k v_{N_k} = 0$. Indeed, we have $x_{n_k} \rightharpoonup z \in C$ as $k \to \infty$. Since $F$ satisfies Condition 3.3, we have

$$0 < \|Fz\| \le \liminf_{k\to\infty} \|F x_{n_k}\|.$$

Since $\{x_{N_k}\} \subset \{x_{n_k}\}$ and $\varepsilon_k \to 0$ as $k \to \infty$, we obtain

$$0 \le \limsup_{k\to\infty} \|\varepsilon_k v_{N_k}\| = \limsup_{k\to\infty} \left( \frac{\varepsilon_k}{\|F x_{n_k}\|} \right) \le \frac{\limsup_{k\to\infty} \varepsilon_k}{\liminf_{k\to\infty} \|F x_{n_k}\|} = 0,$$

which implies that $\lim_{k\to\infty} \varepsilon_k v_{N_k} = 0$.

Now, letting $k \to \infty$, we see the right-hand side of (15) tends to zero because $F$ is uniformly continuous, the sequences $\{x_{N_k}\}$ and $\{v_{N_k}\}$ are bounded, and $\lim_{k\to\infty} \varepsilon_k v_{N_k} = 0$. Thus we get

$$\liminf_{k\to\infty} \langle Fx, x - x_{N_k} \rangle \ge 0.$$

Hence, for all $x \in C$, we have

$$\langle Fx, x - z \rangle = \lim_{k\to\infty} \langle Fx, x - x_{N_k} \rangle = \liminf_{k\to\infty} \langle Fx, x - x_{N_k} \rangle \ge 0.$$

Appealing to Lemma 2.4, we obtain $z \in Sol(C,F)$ and the proof is complete.

**Lemma 3.12** *Assume that Conditions 3.1–3.3 hold. Let $\{x_n\}$ be a sequence is generated by Algorithm 3.3. If $\lim_{n\to\infty}\tau_n\|r_\lambda(x_n)\|^2=0$, then $\lim_{n\to\infty}\|x_n-z_n\|=0$.*

*Proof* Ffirst we consider the case where $\liminf_{n\to\infty}\tau_n>0$. In this case, there is a constant $\tau>0$ such that $\tau_n\geq\tau>0$ for all $n\in\mathbb{N}$. We then have

$$\|x_n-z_n\|^2=\frac{1}{\tau_n}\tau_n\|x_n-z_n\|^2\leq\frac{1}{\tau}.\tau_n\|x_n-z_n\|^2=\frac{1}{\tau}\tau_n\|r(x_n)\|^2. \tag{16}$$

Combining the assumption and (16), we see that

$$\lim_{n\to\infty}\|x_n-z_n\|=0.$$

Second, we consider the case where $\liminf_{n\to}\tau_n=0$. In this case, we take a subsequence $\{n_k\}$ of $\{n\}$ such that

$$\lim_{k\to\infty}\tau_{n_k}=0$$

and

$$\lim_{k\to\infty}\|x_{n_k}-z_{n_k}\|=a>0. \tag{17}$$

Let $y_{n_k}=\frac{1}{l}\tau_{n_k}z_{n_k}+(1-\frac{1}{l}\tau_{n_k})x_{n_k}$. Since $\lim_{n\to\infty}\tau_n\|r_\lambda(x_n)\|^2=0$, we have

$$\lim_{k\to\infty}\|y_{n_k}-x_{n_k}\|^2=\lim_{k\to\infty}\frac{1}{l^2}\tau_{n_k}.\tau_{n_k}\|x_{n_k}-z_{n_k}\|^2=0. \tag{18}$$

From the step size rule (8) and the definition of $y_k$ it follows that

$$\langle Fx_{n_k}-Fy_{n_k},x_{n_k}-z_{n_k}\rangle>\frac{\mu}{2}\|x_{n_k}-z_{n_k}\|^2. \tag{19}$$

Since $F$ is uniformly continuous on bounded subsets of $C$, (18) implies that

$$\lim_{k\to\infty}\|Fx_{n_k}-Fy_{n_k}\|=0. \tag{20}$$

Combining now (19) and (20), we obtain

$$\lim_{k\to\infty}\|x_{n_k}-z_{n_k}\|=0.$$

This, however, is a contradiction to (17). It follows that $\lim_{n\to\infty}\|x_n-z_n\|=0$ and this completes the proof of the lemma.

**Theorem 3.5** *Assume that Conditions 3.1–3.4 hold. Then any sequence $\{x_n\}$ is generated by Algorithm 3.3 converges weakly to an element of $Sol(C,F)$.*

*Proof* **Claim 1.** We first prove that $\{x_n\}$ is a bounded sequence. Indeed, for $p\in Sol(C,F)$ we have

$$\begin{aligned}\|x_{n+1}-p\|^2=\|P_{C_n}x_n-p\|^2&\leq\|x_n-p\|^2-\|P_{C_n}x_n-x_n\|^2\\&=\|x_n-p\|^2-dist^2(x_n,C_n).\end{aligned} \tag{21}$$

This implies that

$$\|x_{n+1}-p\|\leq\|x_n-p\|.$$

and so $\lim_{n\to\infty}\|x_n - p\|$ exists. Thus, the sequence $\{x_n\}$ is bounded, and it also follows that the sequences $\{y_n\}$ and $\{Fy_n\}$ are bounded too.

**Claim 2.** We claim that

$$\left[\frac{\tau_n}{2\lambda L}\|r_\lambda(x_n)\|^2\right]^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \tag{22}$$

for some $L > 0$. Indeed, since the sequence $\{Fy_n\}$ is bounded, there exists $L > 0$ such that $\|Fy_n\| \le L$ for all $n$. Using this fact, we see that for all $u, v \in C_n$,

$$\|h_n(u) - h_n(v)\| = \|\langle Fy_n, u - v\rangle\| \le \|Fy_n\|\|u - v\| \le L\|u - v\|.$$

This implies that $h_n(\cdot)$ is $L$-Lipschitz continuous on $C_n$. By Lemma 2.5, we obtain

$$dist(x_n, C_n) \ge \frac{1}{L}h_n(x_n),$$

which, when combined with Lemma 3.10, yields the inequality

$$dist(x_n, C_n) \ge \frac{\tau_n}{2\lambda L}\|r_\lambda(x_n)\|^2. \tag{23}$$

Combining the proof of claim 1 with (23), we obtain

$$\|x_{n+1} - p\|^2 \le \|x_n - z\|^2 - \left[\frac{\tau_n}{2\lambda L}\|r_\lambda(x_n)\|^2\right]^2,$$

which implies, in its turn, Claim 2.

**Claim 3.** We claim that $\{x_n\}$ converges weakly to an element of $Sol(C,F)$. Indeed, since $\{x_n\}$ is a bounded sequence, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $z \in C$.

Appealing to Claim 2, we find that

$$\lim_{n\to\infty}\frac{\tau_n}{2\lambda L}\|r_\lambda(x_n)\|^2 = 0, \text{ that is }, \lim_{n\to\infty}\tau_n\|r_\lambda(x_n)\|^2 = 0.$$

Thanks to Lemma 3.12 we also get

$$\lim_{n\to\infty}\|x_n - z_n\| = 0. \tag{24}$$

Using Lemma 3.11 and (24), we may infer that $z \in Sol(C,F)$.
Thus, we have proved that

i) for every $p \in Sol(C,F)$, the limit $\lim_{n\to\infty}\|x_n - p\|$ exists;

ii) every sequential weak cluster point of the sequence $\{x_n\}$ is in $Sol(C,F)$.

Lemma 2.6 now implies that the sequence $\{x_n\}$ converges weakly to an element of $Sol(C,F)$.

*Remark 3.1*
1. When the operator $F$ is monotone, it is not necessary to assume its Condition 3.3 (see, [8, 35]).
2. It is worth noting that in our work, we use Condition 3.3 is weaker than the sequentially weakly continuity of $F$ which used in recent articles [32,34,35].

## 3.2 Strong convergence

In this section, we introduce an algorithm for strong convergence which it constructs based on viscosity method [26] and Algorithm 3.3 for solving VIs. In addition, we assume that $f : C \to C$ is a contractive mapping with a coefficient $\rho \in [0,1)$, and we add the following condition

**Condition 3.6** *Let $\{\alpha_n\}$ be a real sequences in $(0,1)$ such that*

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty.$$

**Algorithm 3.4**

***Initialization:*** *Given $\mu > 0, l \in (0,1), \lambda \in (0, \frac{1}{\mu})$. Let $x_1 \in C$ be arbitrary*

***Iterative Steps:*** *Given the current iterate $x_n$, calculate $x_{n+1}$ as follows:*

***Step 1.*** *Compute*

$$z_n = P_C(x_n - \lambda F x_n)$$

*and $r_\lambda(x_n) := x_n - z_n$ if $r_\lambda(x_n) = 0$ then stop and $x_n$ is a solution of $Sol(C,F)$. Otherwise,*
***Step 2.*** *Compute*

$$y_n = x_n - \tau_n r_\lambda(x_n),$$

*where $\tau_n := l^{j_n}$ and $j_n$ is the smallest non-negative integer $j$ satisfying*

$$\langle F x_n - F(x_n - l^j r_\lambda(x_n)), r_\lambda(x_n)\rangle \le \frac{\mu}{2}\|r_\lambda(x_n)\|^2.$$

***Step 3.*** *Compute*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) P_{C_n}(x_n),$$

*where*

$$C_n := \{x \in C : h_n(x_n) \le 0\}$$

*and*

$$h_n(x) = \langle F y_n, x - x_n\rangle + \frac{\tau_n}{2}\|r(x_n)\|^2.$$

*Set $n := n+1$ and go to **Step 1**.*

**Theorem 3.7** *Assume that Conditions 3.1–3.4 and 3.6 hold. Then any sequence $\{x_n\}$ is generated by Algorithm 3.4 converges strongly to $p \in Sol(C,F)$, where $p = P_{Sol(C,F)} \circ f(p)$.*

*Proof* **Claim 1.** We prove that $\{x_n\}$ is bounded. Indeed, let $w_n = P_{C_n} x_n$. Since (21) we have

$$\|w_n - p\|^2 \le \|x_n - p\|^2 - dist^2(x_n, C_n).$$

This implies that

$$\|w_n - p\| \le \|x_n - p\|. \tag{25}$$

Using (25) we have

$$\begin{aligned}
\|x_{n+1}-p\| &= \|\alpha_n f(x_n)+(1-\alpha_n)w_n-p\| \\
&= \|\alpha_n(f(x_n)-p)+(1-\alpha_n)(w_n-p)\| \\
&\le \alpha_n\|f(x_n)-p\|+(1-\alpha_n)\|w_n-p\| \\
&\le \alpha_n\|f(x_n)-f(p)\|+\alpha_n\|f(p)-p\|+(1-\alpha_n)\|w_n-p\| \\
&\le \alpha_n\rho\|x_n-p\|+\alpha_n\|f(p)-p\|+(1-\alpha_n)\|x_n-p\| \\
&\le [1-\alpha_n(1-\rho)]\|x_n-p\|+\alpha_n(1-\rho)\frac{\|f(p)-p\|}{1-\rho} \\
&\le \max\{\|x_n-p\|,\frac{\|f(p)-p\|}{1-\rho}\} \\
&\le \ldots \le \max\{\|x_1-p\|,\frac{\|f(p)-p\|}{1-\rho}\}.
\end{aligned}$$

Thus, the sequence $\{x_n\}$ is bounded. Consequently, the sequences $\{y_n\},\{f(x_n)\},\{Fy_n\}$ are bounded.

**Claim 2**. We prove that

$$\|w_n-x_n\|^2 \le \|x_n-p\|^2-\|x_{n+1}-p\|^2+2\alpha_n\langle f(x_n)-p,x_{n+1}-p\rangle.$$

Indeed, we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\alpha_n(f(x_n)-p)+(1-\alpha_n)(w_n-p)\|^2 \\
&\le (1-\alpha_n)\|w_n-p\|^2+2\alpha_n\langle f(x_n)-p,x_{n+1}-p\rangle \\
&\le \|w_n-p\|^2+2\alpha_n\langle f(x_n)-p,x_{n+1}-p\rangle. 
\end{aligned} \tag{26}$$

On the other hand, we have

$$\|w_n-p\|^2=\|P_{C_n}x_n-p\|^2 \le \|x_n-p\|^2-\|w_n-x_n\|^2 \tag{27}$$

Substitution (27) into (26) we get

$$\|x_{n+1}-p\|^2 \le \|x_n-p\|^2-\|w_n-x_n\|^2+2\alpha_n\langle f(x_n)-p,x_{n+1}-p\rangle.$$

This implies that

$$\|w_n-x_n\|^2 \le \|x_n-p\|^2-\|x_{n+1}-p\|^2+2\alpha_n\langle f(x_n)-p,x_{n+1}-p\rangle.$$

**Claim 3.** We show that

$$\Big[\frac{\tau_n}{2\lambda L}\|r_\lambda(x_n)\|^2\Big]^2 \le \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n\|f(x_n)-p\|^2.$$

According to (22) we get

$$\|w_n-p\|^2 \le \|x_n-p\|^2-\Big[\frac{\tau_n}{2\lambda L}\|r_\lambda(x_n)\|^2\Big]^2. \tag{28}$$

From the definition of the sequence $\{x_n\}$ and (28) we obtian

$$\begin{aligned}\|x_{n+1}-p\|^2 =&\|\alpha_n(f(x_n)-p)+(1-\alpha_n)(w_n-p)\|^2\\ =&\alpha_n\|f(x_n)-p\|^2+(1-\alpha_n)\|w_n-p\|^2-\alpha_n(1-\alpha_n)\|f(x_n)-w_n\|^2\\ \leq&\alpha_n\|f(x_n)-p\|^2+(1-\alpha_n)\|w_n-p\|^2\\ \leq&\alpha_n\|f(x_n)-p\|^2+(1-\alpha_n)\|x_n-p\|^2-(1-\alpha_n)\left[\frac{\tau_n}{2\lambda L}\|r_\lambda(x_n)\|^2\right]^2\\ \leq&\alpha_n\|f(x_n)-p\|^2+\|x_n-p\|^2-(1-\alpha_n)\left[\frac{\tau_n}{2\lambda L}\|r_\lambda(x_n)\|^2\right]^2.\end{aligned}$$

This implies that

$$(1-\alpha_n)\left[\frac{\tau_n}{2\lambda L}\|r_\lambda(x_n)\|^2\right]^2\leq\|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n\|f(x_n)-p\|^2.$$

**Claim 4.** We prove that

$$\|x_{n+1}-p\|^2\leq(1-(1-\rho)\alpha_n)\|x_n-p\|^2+(1-\rho)\alpha_n\frac{2}{1-\rho}\langle f(p)-p,x_{n+1}-p\rangle.$$

Indeed, we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\alpha_n f(x_n)+(1-\alpha_n)z_n-p\|^2\\ &= \|\alpha_n(f(x_n)-f(p))+(1-\alpha_n)(z_n-p)+\alpha_n(f(p)-p)\|^2\\ &\leq \|\alpha_n(f(x_n)-f(p))+(1-\alpha_n)(z_n-p)\|^2+2\alpha_n\langle f(p)-p,x_{n+1}-p\rangle\\ &\leq \alpha_n\|f(x_n)-f(p)\|^2+(1-\alpha_n)\|z_n-p\|^2+2\alpha_n\langle f(p)-p,x_{n+1}-p\rangle\\ &\leq \alpha_n\rho\|x_n-p\|^2+(1-\alpha_n)\|x_n-p\|^2+2\alpha_n\langle f(p)-p,x_{n+1}-p\rangle\\ &= (1-(1-\rho)\alpha_n)\|x_n-p\|^2+(1-\rho)\alpha_n\frac{2}{1-\rho}\langle f(p)-p,x_{n+1}-p\rangle.\end{aligned}$$

**Claim 5.** Now, we will show that the sequence $\{\|x_n-p\|^2\}$ converges to zero by considering two possible cases on the sequence $\{\|x_n-p\|^2\}$.

**Case 1:** There exists an $N\in\mathbb{N}$ such that $\|x_{n+1}-p\|^2\leq\|x_n-p\|^2$ for all $n\geq N$. This implies that $\lim_{n\to\infty}\|x_n-p\|^2$ exists. It implies from **Claim 2** that

$$\lim_{n\to\infty}\|x_n-w_n\|=0.$$

Since the sequence $\{x_n\}$ is bounded, it implies that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that weak convergence to some $z\in C$ such that

$$\limsup_{n\to\infty}\langle f(p)-p,x_n-p\rangle=\lim_{k\to\infty}\langle f(p)-p,x_{n_k}-p\rangle=\langle f(p)-p,z-p\rangle.$$

Now, according to **Claim 3**

$$\left[\frac{\tau_{n_k}}{2\lambda L}\|r_\lambda(x_{n_k})\|^2\right]^2=0.$$

This follows that

$$\lim_{k\to\infty}\tau_{n_k}\|r_\lambda(x_{n_k})\|^2=\lim_{k\to\infty}\tau_{n_k}\|x_{n_k}-z_{n_k}\|^2=0.$$

Thanks to Lemma 3.12 we obtain

$$\lim_{k\to\infty} \|x_{n_k} - z_{n_k}\| = 0. \tag{29}$$

Since $x_{n_k} \rightharpoonup z$ and (29), it implies that from Lemma 3.11 that $z \in Sol(C,F)$.
On the other hand

$$\|x_{n+1} - w_n\| = \alpha_n \|f(x_n) - w_n\| \to 0 \text{ as } n \to \infty.$$

Thus

$$\|x_{n+1} - x_n\| = \|x_{n+1} - w_n\| + \|x_n - w_n\| \to 0 \text{ as } n \to \infty.$$

Since $p = P_{Sol(C,F)} f(p)$ and $x_{n_k} \rightharpoonup z \in Sol(C,F)$ we get

$$\limsup_{n\to\infty} \langle f(p) - p, x_n - p\rangle = \langle f(p) - p, z - p\rangle \leq 0.$$

This implies that

$$\limsup_{n\to\infty} \langle f(p) - p, x_{n+1} - p\rangle \leq \limsup_{n\to\infty} \langle f(p) - p, x_{n+1} - x_n\rangle + \limsup_{n\to\infty} \langle f(p) - p, x_n - p\rangle \leq 0,$$

which, together with **Claim 4**, it implies from Lemma 2.7 that

$$x_n \to p \text{ as } n \to \infty.$$

**Case 2:** Assume that there is no $n_0 \in \mathbb{N}$ such that $\{\|x_n - p\|\}_{n=n_0}^{\infty}$ is monotonically decreasing. The technique of proof used here is adapted from [23]. Set $\Gamma_n = \|x_n - p\|^2$ for all $n \geq 1$ and let $\eta : \mathbb{N} \to \mathbb{N}$ be a mapping defined for all $n \geq n_0$ (for some $n_0$ large enough) by

$$\eta(n) := \max\{k \in \mathbb{N} : k \leq n, \Gamma_k \leq \Gamma_{k+1}\},$$

i.e. $\eta(n)$ is the largest number $k$ in $\{1,...,n\}$ such that $\Gamma_k$ increases at $k = \eta(n)$; note that, in view of Case 2, this $\eta(n)$ is well-defined for all sufficiently large $n$. Clearly, $\eta$ is a non-decreasing sequence such that $\eta(n) \to \infty$ as $n \to \infty$ and

$$0 \leq \Gamma_{\eta(n)} \leq \Gamma_{\eta(n)+1} \quad \forall n \geq n_0.$$

According to **Claim 2** we have

$$\begin{aligned}
\|w_{\eta(n)} - x_{\eta(n)}\|^2 &\leq \|x_{\eta(n)} - p\|^2 - \|x_{\eta(n)+1} - p\|^2 + 2\alpha_{\eta(n)} \langle f(x_{\eta(n)}) - p, x_{\eta(n)+1} - p\rangle. \\
&\leq \alpha_{\eta(n)} \langle f(x_{\eta(n)}) - p, x_{\eta(n)+1} - p\rangle \\
&\leq \alpha_{\eta(n)} \|f(x_{\eta(n)}) - p\| \|x_{\eta(n)+1} - p\| \to 0 \text{ as } n \to \infty.
\end{aligned}$$

From **Claim 3** we have

$$\begin{aligned}
(1 - \alpha_{\eta(n)}) \left[\frac{\tau_{\eta(n)}}{2\lambda L} \|r_\lambda(x_{\eta(n)})\|^2\right]^2 &\leq \|x_{\eta(n)} - p\|^2 - \|x_{\eta(n)+1} - p\|^2 + \alpha_{\eta(n)} \|f(x_{\eta(n)}) - p\|^2 \\
&\leq \alpha_{\eta(n)} \|f(x_{\eta(n)}) - p\|^2 \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Using the same arguments as in the proof of Case 1, we obtian

$$\lim_{k\to\infty} \|x_{\eta(n)} - z_{\eta(n)}\| = 0, \lim_{k\to\infty} \|x_{\eta(n)+1} - x_{\eta(n)}\| \to 0$$

and

$$\limsup_{n\to\infty}\langle f(p)-p,x_{\eta(n)+1}-p\rangle \le 0. \tag{30}$$

Thanks to **Claim 4** we get

$$\begin{aligned}\|x_{\eta(n)+1}-p\|^2 &\le (1-\alpha_{\eta(n)}(1-\rho))\|x_{\eta(n)}-p\|^2+2\alpha_{\eta(n)}\langle f(p)-p,x_{\eta(n)+1}-p\rangle\\ &\le (1-\alpha_{\eta(n)}(1-\rho))\|x_{\eta(n)+1}-p\|^2+2\alpha_{\eta(n)}\langle f(p)-p,x_{\eta(n)+1}-p\rangle.\end{aligned}$$

Thus

$$(1-\rho)\|x_{\eta(n)+1}-p\|^2 \le 2\langle f(p)-p,x_{\eta(n)+1}-p\rangle.$$

which, together with (30) implies that $\limsup_{n\to\infty}\|x_{\eta(n)+1}-p\|^2 \le 0$, that is $\lim_{n\to\infty}\|x_{\eta(n)+1}-p\| = 0$.
Now, we show that for all sufficiently large $n$ we have

$$0 \le \Gamma_n \le \Gamma_{\eta(n)+1}. \tag{31}$$

Indeed, for $n \ge n_0$, it is easy to observe that $\eta(n) \le n$ for $n \ge n_0$ and consider the three cases: $\eta(n) = n, \eta(n) = n-1$ and $\eta(n) < n-1$. For the first and second cases, it is obvious that $\Gamma_n \le \Gamma_{\eta(n)+1}$, for $n \ge n_0$. For the third case $\eta(n) \le n-2$, we have from the definition of $\eta(n)$ and for any integer $n \ge n_0$ that $\Gamma_j \ge \Gamma_{j+1}$ for $\eta(n)+1 \le j \le n-1$. Thus, $\Gamma_{(n)+1} \ge \Gamma_{(n)+2} \ge ... \ge \Gamma_{n-1} \ge \Gamma_n$. As a consequence, we obtain the inequality (31). Using (31) and $\lim_{n\to\infty}\|x_{\eta(n)+1}-p\| = 0$ we get $x_n \to p$ as $n \to \infty$.

Applying Algorithm 3.3 with $f(x) := x_1$ for all $x \in C$, we obtain the following corollary.

**Corollary 3.1** *Given $\mu > 0, l \in (0,1), \lambda \in (0,\frac{1}{\mu})$. Let $x_1 \in C$ be arbitrary. Compute*

$$z_n = P_C(x_n - \lambda F x_n)$$

*and $r_\lambda(x_n) := x_n - z_n$ if $r_\lambda(x_n) = 0$ then stop and $x_n$ is a solution of $Sol(C,F)$. Otherwise Compute*

$$y_n = x_n - \tau_n r_\lambda(x_n),$$

*where $\tau_n := l^{j_n}$ and $j_n$ is the smallest non-negative integer $j$ satisfying*

$$\langle Fx_n - F(x_n - l^j r_\lambda(x_n)), r_\lambda(x_n)\rangle \le \frac{\mu}{2}\|r_\lambda(x_n)\|^2.$$

*Compute*

$$x_{n+1} = \alpha_n x_1 + (1-\alpha_n)P_{C_n}(x_n),$$

*where*

$$C_n := \{x \in C : h_n(x_n) \le 0\} \quad \text{and} \quad h_n(x) = \langle F y_n, x - x_n\rangle + \frac{\tau_n}{2\lambda}\|r(x_n)\|^2.$$

*Assume that Conditions 3.1–3.4 hold. Then the sequence $\{x_n\}$ converges strongly to $p \in Sol(C,F)$, where $p = P_{Sol(C,F)}x_1$.*

## 4 Numerical Illustrations

In this section we provide several numerical examples regarding our proposed algorithms. We compare Algorithm 3.3 (Proposed Alg. 3.3 or TD Agl) with the Algorithm 1.1 (Solodov and Svaiter Alg. 1.1) and Algorithm 1.2 (Vuong and Shehu Alg. 1.2) in Examples 1 and 2. In Example 3, we compare Algorithm 3.4 with Algorithm 1.2. All the numerical experiments are performed on a HP laptop with Intel(R) Core(TM)i5-6200U CPU 2.3GHz with 4 GB RAM. All the programs are written in Matlab2015a.

*Example 1* We are consider a classical example (see ) for which the usual gradient method does not converge to a solution of the variational inequality. The feasible set is $C := \mathbb{R}^m$ (for some positive even integer m) and $F := (a_{ij})_{1\le i,j\le m}$ is the square matrix $m\times m$ whose terms are given by

$$a_{ij} = \begin{cases} 1 & \text{if } j = m+1-i > i, \\ 1 & \text{if } j = m+1-i < i, \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that, zero vector $x^\star = (0,...,0)$ is the solution of this test example. In the first examples, we take $\alpha_n = \frac{1}{n}$ and the starting points are $x_1 = (1,1,...,1)^T \in \mathbb{R}^m$. We terminate the iterations if $\|x_n - x^\star\| \le \varepsilon$ with $\varepsilon = 10^{-4}$ or iteration $\ge 1000$. The results are listed in Table 4.1 and Figs. 1,2 below.

| Methods | m=100 | | m=200 | | m=500 | |
|---|---|---|---|---|---|---|
| | Sec. | Iter. | Sec. | Iter. | Sec. | Iter |
| Solodov and Svaiter Alg. 1.1 | 0.23991 | 105 | 0.97432 | 108 | 6.7691 | 112 |
| Vuong and Shehu Alg. 1.2 | 2.3978 | 1000 | 8.6678 | 1000 | 64.63 | 1000 |
| Proposed Alg. 3.3 | 0.20413 | 53 | 0.64506 | 55 | 3.8467 | 57 |

Table 4.1: Numerical results obtained by other algorithms with $\lambda = 1.8, \mu = 0.5, l = 0.5$

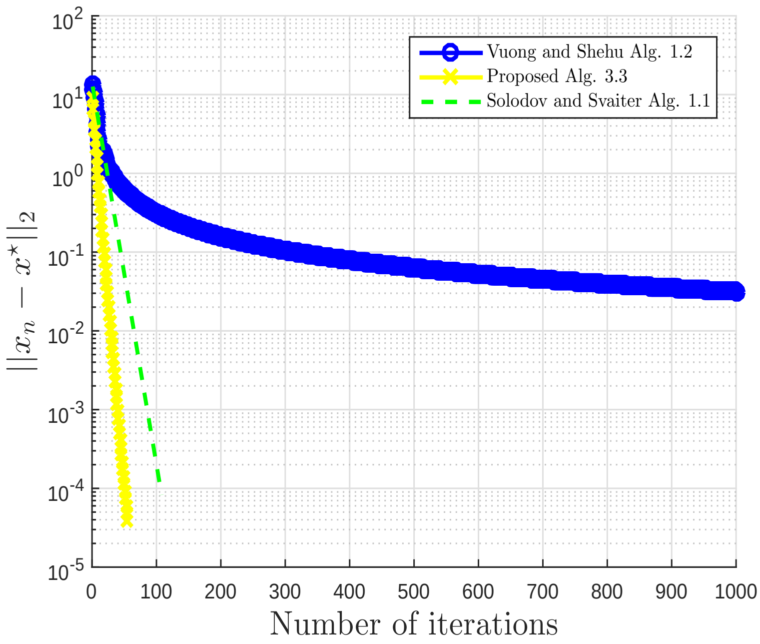


Fig. 1: Comparison of all algorithms with $m = 200$

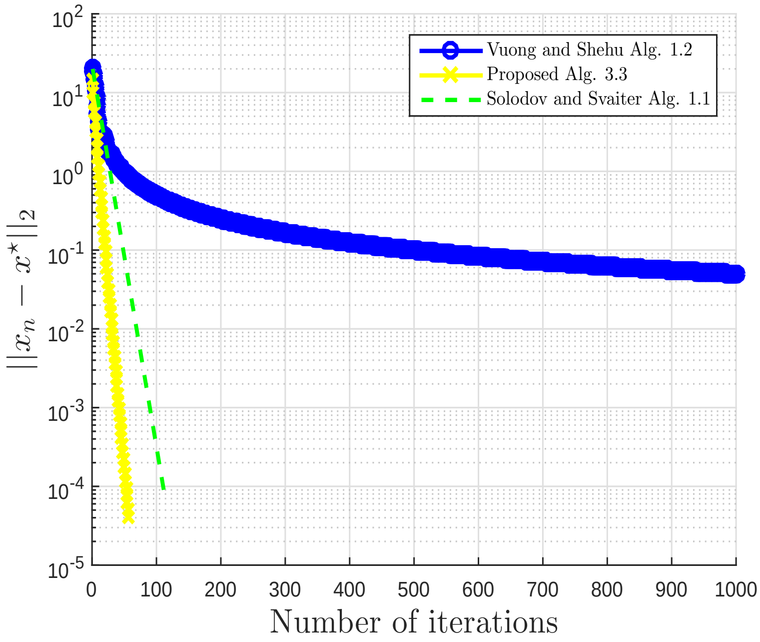


Fig. 2: Comparison of all algorithms with $m = 500$

*Example 2* Assume that $F : \mathbb{R}^m \to \mathbb{R}^m$ is defined by $F(x) = Mx + q$ with $M = NN^T + S + D$, $N$ is an $m \times m$ matrix, $S$ is an $m \times m$ skew-symmetric matrix, $D$ is an $m \times m$ diagonal matrix, whose diagonal entries are positive (so $M$ is positive definite), $q$ is a vector in $\mathbb{R}^m$, and

$$C := \{x \in \mathbb{R}^m : -5 \leq x_i \leq 5, i = 1, \cdots, m\}.$$

It is clear that $F$ is monotone and Lipschitz continuous with the Lipschitz constant $L = \|M\|$ so $F$ is uniformly continuous pseudomonotone operator. For $q = 0$, the unique solution of the corresponding variational inequality is $\{0\}$.

For experiment, all entries of $N$, $S$ and $D$ are generated randomly in $(-2, 2)$ and of $D$ are in (0,1). The start points are $x_1 = (1, 1, ..., 1)^T \in \mathbb{R}^m$ and $\alpha_n = \frac{1}{\sqrt{n}}$. We use stopping rule $\|x_n - x^\star \leq 10^{-4}\|$ or iteration $\geq 1000$ for all algorithms. The numerical results are described in Table 4.2 and Figs. 3 - 4.

| Methods | m=10 | | m=50 | | m=100 | |
|---|---|---|---|---|---|---|
| | Sec. | Iter. | Sec. | Iter. | Sec. | Iter |
| Solodov and Svaiter Alg. 1.1 | 0.16551 | 506 | 0.419 | 1000 | 0.50847 | 1000 |
| Vuong and Shehu Alg. 1.2 | 0.17286 | 1000 | 0.2 | 1000 | 0.24625 | 1000 |
| Proposed Alg. 3.3 | 0.012714 | 64 | 0.035282 | 125 | 0.062133 | 152 |

Table 4.2: Numerical results obtained by other algorithms with $\lambda = 1.8, \mu = 0.5, l = 0.5$

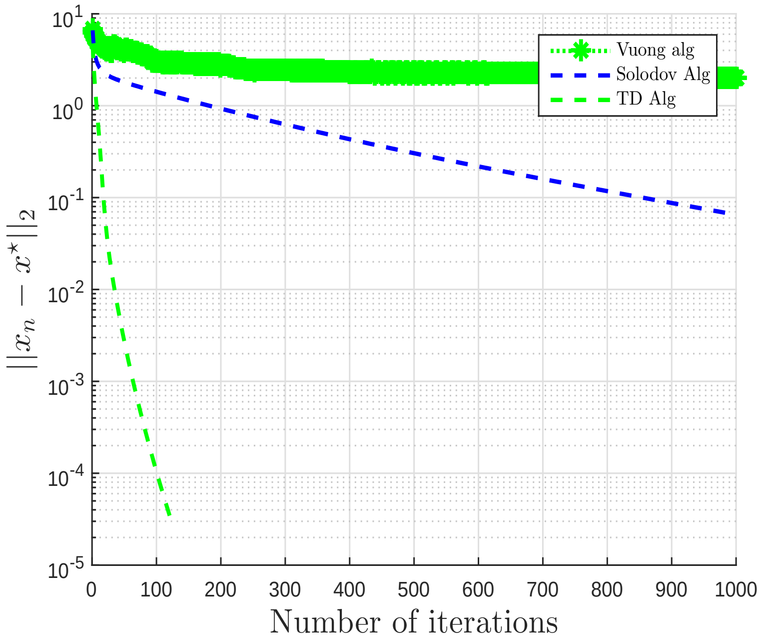


Fig. 3: Comparison of all algorithms with $m = 50$

*Example 3* Consider $C := \{x \in H : \|x\| \leq 2\}$. Let $g : C \to \mathbb{R}$ be defined by $g(u) := \frac{1}{1+\|u\|^2}$. Observe that $g$ is $L_g$-Lipchitz continuous with $L_g = \frac{16}{25}$ and $\frac{1}{5} \leq g(u) \leq 1, \ \ \forall u \in C$. Define the Volterra integral operator $A : L^2([0,1]) \to L^2([0,1])$ by

$$A(u)(t) := \int_0^t u(s)ds, \ \ \forall u \in L^2([0,1]), t \in [0,1].$$

Then $A$ is bounded linear monotone (see Exercise 20.12 of [2]) and $\|A\| = \frac{2}{\pi}$. Now, define $F : C \to L^2([0,1])$ by $F(u)(t) := g(u)A(u)(t), \ \ \forall u \in C, t \in [0,1]$. Then $F$ is pseudomonotone and $L_F$-Lipschitz-continuous with $L_F = \frac{82}{\pi}$.

Take $\mu = 0.3$, $l = 0.9$ and $\alpha_n = \frac{1}{n}$ in Algorithm 3.4 and Algorithm 1.2. Choose $\lambda = \frac{0.9}{\mu}$ and $f(x) := x_1$ in Algorithm 3.4. The initial point $x_0 = \sin(2\pi t^2)$.

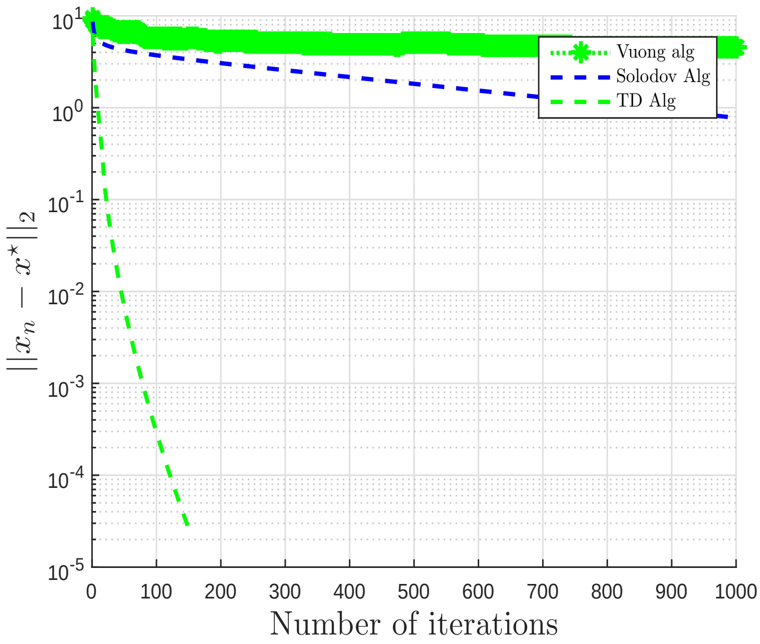


Fig. 4: Comparison of all algorithms with $m = 100$

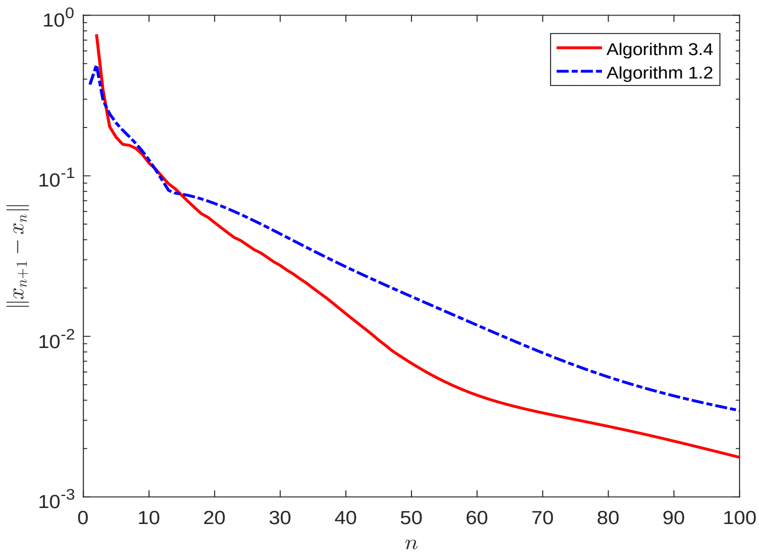


Fig. 5: Comparison of Algorithm 3.4 and Algorithm 1.2 in Example 3

We compared Algorithm 3.4 with Algorithm 1.2. The numerical results are described in Fig. 5. It shows that the performance of Algorithm 3.4 is better than that of Algorithm 1.2.

## 5 Conclusions

In this paper we have proposed new projection-type algorithms for solving variational inequalities in real Hilbert spaces. We have established weak and strong convergence theorems for these algorithms under a pseudomonotonicity assumtion imposed on the cost operator, which is not assumed to be Lipschitz continuous. Moreover, our algorithms require the calculation of only two projections onto the feasible set per each iteration. These two properties bring out the advantages of our proposed algorithms over several existing algorithms which have recently been proposed in the literature. Numerical experiments in both finite and infinite dimensional spaces illustrate the good performance of our new schemes.

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