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Natali Hritonenko and
Yuri Yatsenko

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Modeling and Optimization of the Lifetime of Technologies

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PREFACE

Modern economic growth is characterized by structural changes based on the introduction of new technologies into economics. The replacement and renovation of technologies in industrial environments undergoing technical change is clearly one of the key aspects of economic development.

The mathematical modeling of evolutionary economics under technical change (TC) has been rigorously considered by many authors during last decades. There is a wide variety of economic approaches and models describing different aspects of technical change. Among these are the models of embodied technical progress [19], [35], [70], [129], endogenous growth models [94], [102], the models of technological innovations [31], [32], [41], and others. The perspective self-organization evolutionary approach is developed in [20], [38], [122], [123], [124], [126], which unites the aspects of diffusion of new technologies, technological and behavioral diversity of firms, learning mechanisms, age-dependent effects, and other important features of real-life economics . On the whole, an interest in evolutionary economics has brought considerable progress in the description and conceptualization of the sources, characteristics, direction and effects of technical change [125].

However, the modeling and control of technology lifetime under technical change has received rather little attention in mathematical economics in contrary to other aspects of technical progress. The lifetime of technologies has rarely been formally treated as a part of more general mathematical theory of economic dynamics. A problem which is still to be resolved consists in establishing the rational strategies of technologies' replacement under various assumptions on the behavior of technical change.

On the other hand, certain formal methods and constructions were developed in economic optimization theory, which enables us to study the problem of optimal technological renovation on the basis of integral economic models ([19], [45], [53], [74], [75], [96], [107], [129], etc.).

In this work, we shall study the nature and general properties of technological renovation in an evolutionary economics characterized by technological diversity and variable rate of technical change. In so doing, we shall focus on the modeling of the lifetime of technologies. This problem is posed and analyzed as the optimal control of specialized nonlinear integral equations.

Integral dynamical models (IDMs) under consideration fall into the class of macroeconomic models with embodied (materialized) technological change [19] and take into account the controlled liquidation of obsolete capacity involved into industrial production. They have been primarily developed by mathematicians and economists of former Soviet Union over the last two decades (see, e.g., [45], [48], [49], [50], [65], [66]. [67], [75], [76], [146], [147], [153], and the references therein).

This work is concerned mainly with optimal control problems for the mentioned economic IDMs. Its focus is on the qualitative investigation of optimal trajectories and its interpretation as the properties of technological renovation.

Major advantages of the proposed models are associated with their easy-to-use form for describing the lifetime of operating production means (capacities) as an endogenous dynamic variable depending on the dynamics of technical change and other economic factors. Our investigation is more concerned with general features of technological renovation than with specific implication of innovation process.

We will assume that at any time only one new technology prevails in an economy and is available for all agents. If a new technology is better, then it instantly replaces its predecessor. No account will be taken in this book for the uncertainty of innovation processes, firms' competitiveness, search and learning mechanisms in the replacement of technologies.

The technology is assumed to be capital embodied (i.e., embodied in given technical features of production inputs). To allow an analytic study of the models, other technology's characteristics different from its lifetime will be simplified in comparison with well-known modern models of technological innovation, e.g. self-organization endogenous growth models [122], [124], [125]. Namely, the following hypotheses are used:

- the pre-history dynamics and future course of technical progress are assumed to be completely known;
- the internal uncertainty of innovation processes is eliminated;

- an optimizing behavior of economic agents (enterprises, firms, etc.) is assumed;
- the time of technology diffusion is neglected.

Thus, we consider *a technology* as connected mainly with some amount of industrial equipment with given technological characteristics (specific productivity, price, operating cost, etc.) that vary in time due to TC. By *the lifetime of technology* is meant the service time (life-in-service) of corresponding piece of equipment, i.e. the period between putting into operation and scrapping this equipment.

The replacement of technologies is the replacement of one equipment unit by another unit. By the *technological renovation* is meant the replacement of a technology (obsolete vintage) by more efficient one (modern vintage) in virtue of the TC influence.

In contrast to classical renewal theory ([2], [77], [135], etc.) based on stochastic models, our arguments are deterministic and the emphasis is placed on the determination of unknown *variable aftereffect duration of dynamical system* (the lifetime of system elements). Our models do not assume the existence of any steady-state distributions of economic factors and cannot be termed as "equilibrium" models.

Under given simplified assumptions, the mentioned IDMs provide their analytical study, which leads to highly non-trivial results, in particular, to turnpike properties of optimal scrapping time for worse equipment. Such analysis is carried out in this book. It allows us to gain an insight into the character of rational scrapping of obsolete industrial equipment in evolutionary economics.

First, we will discuss some fundamental notions of mathematical economics to describe the essence and major achievements of the present book. Second, on this background, the extended review of the book's content will be exposed.

0.1 SOME BASIC NOTIONS

Aggregate models of economic dynamics describe the interaction between several generalized homogeneous economic indices. They usually reflect nonlinear interconnections in economic system and are based on the notion of *production function*.

The production function (PF) is termed the relation $y = \phi(x_1, ...x_n)$ between *productive output* y and *resources* (productive factors) x_i, $i = 1, .., n$. The resources may include labor force, capital, fixed assets, material expenditure, energy consumption, natural resources (land, water, minerals), etc. The output y and resources x_i are assumed to be homogeneous. For example, labor force is the quantity of workers which are identical in productive sense.

There are different types of the PF: Cobb-Douglas PF, the PF with constant elasticity of substitution (PF CES), the PF with fixed proportions (piece-wise linear PF), etc. The Cobb-Douglas PF and PF CES are usually used in aggregated economic-mathematical models, whereas the PF with fixed proportions is used in multi-sector "input-output" models.

The most widespread PFs are two-factor ones of the form:

$$Q = F(K, L), \tag{0.1}$$

where Q is output, K is the amount of capital (fixed assets), L is the amount of labor (manpower). If $F(K, L)$ is linearly homogeneous function (i.e., $F(kK, kL) = kF(K, L)$, $k > 0$), then the PF (0.1) can be written in another form:

$$\beta = f(\lambda), \tag{0.2}$$

where $\beta = Q/L$ is *productivity*, $\lambda = K/L$ is *capital labor ratio*, $f(\lambda) = F(\lambda, 1)$, $f(0) = 0$, $df/d\lambda > 0$, $d^2f/d\lambda^2 < 0$. This form of PF is more convenient for analysis and illustration. In particular, the *two-factor Cobb-Douglas PF* has the form:

$$F(K, L) = F_0 K^\alpha L^{1-\alpha} \quad \text{or} \quad \beta(\lambda) = \beta_0 \lambda^\alpha, \quad 0 < \alpha < 1. \tag{0.3}$$

The most known one-sector model of macroeconomic dynamics *(Solow-Shell model)* connects total output Q, the output C of consumption goods, the output S of accumulation product (investments), the amount K of capital, and the quantity L of labor resource. This is described by the following relations [120]:

$$Q = F(K, L), \qquad Q = C + S, \tag{0.4}$$

$$dK/dt = S - \mu K, \quad 0 < \mu < 1, \quad \mu = const \tag{0.5}$$

$$dL/dt = \nu L, \qquad \nu = const. \tag{0.6}$$

Thus, it is supposed that the μ-th part of capital K is removed from production during a time unit.

The endogenous (control) variables in the Solow-Shell model (0.4)-(0.6) are usually assumed to be the *accumulation norm* $s = Q/S$, $0 < s < 1$ (the part of investment in total output) and the *capital labor ratio* $\lambda = K/L$. Commonly considered *optimization problem (OP)* for the Solow-Shell model consists in finding the sought-for functions $s(t)$, $\lambda(t)$, $0 \leq s(t) \leq 1$, $\lambda_{min} \leq \lambda(t) \leq \lambda_{max}$, $t \in [t_0, T]$, which maximize the output C of consumption goods on a planned interval $[t_0, T]$:

$$\int_{t_0}^{T} e^{-qt} C(t) dt \longrightarrow \max_{s,\lambda}. \tag{0.7}$$

Here $q > 0$ is so-called *discounting multiplier* that reflects decreasing the utility of the output $C(t)$ produced at more distanced time t [19].

Note that the Solow-Shell model makes it possible to describe the extensive regime of economic growth only because it fails to account for the TC influence.

Modeling of Technological Change

Technological change (TC) in mathematical economics is understood to be a combination of the effects which lead to increasing an output when the amount of used resources does not increase. Such concept of TC also includes a progress in the methods of production management.

There are various directions of the TC description: *autonomous, embodied (materialized), exogenous, endogenous TC*, and rather different modeling results are obtained depending on these directions.

The autonomous TC is *exogenous* and is introduced into an economic system from outside. The efficiency of used resources is supposed to increase independently of capital investments and dynamics of labor force. It is described by means of the *dynamic PF*

$$Q = F(K, L, t), \tag{0.8}$$

depending explicitly on time t. So, the autonomous TC is equivalent to the PF increase in t: $\partial F/\partial t > 0$ [136]. The Cobb-Douglas PF with exponential autonomous TC $F(K, L, t) = e^{pt}K^{\alpha}L^{1-\alpha}$ is termed *Cobb-Douglas-Tinbergen PF.*

The Solow-Shell model (0.4)-(0.6) with autonomous TC is known as *Ramsey model.* The first equation (0.4) of the model (0.4)-(0.6) is replaced by $Q(t) = e^{pt}F(K, L)$, whereas other equations are the same. Analysis of the factors having influence on TC dynamics is not provided by similar models. Besides, the feedback mechanisms between TC and economic growth are not revealed, i.e. the TC is costless and uncontrolled process that is far from reality.

One of the main disadvantage of autonomous TC is following. In a static PF $Q = F(K, L)$ the variable K is a quantity of the equipment of the same type, and L is the number of identical workers. If $\partial F/\partial t > 0$, then the equipment created at different time cannot be considered as identical. To account for this fact, it is required to pass to the models of embodied TC.

The embodied (materialized) TC is introduced into an economic system with new more improved equipment or more qualified labor force. The models with embodied TC set up the connections between progress and capital investments. They take into account the heterogeneity of production elements undergoing TC and allow us to control the liquidation (reduction) of the obsolete elements with low technical-economic indices. This is a powerful means to raise the efficiency of economic systems functioning under the condition of material and human resource shortage. Such models are usually described by means of integral relations and represent the main subject of this monograph. They are discussed at greater length in Chapter 2.

The new feature of present study consists in the consideration of the endogenous lifetime L(t) of system elements in the integral models with embodied TC. The above-mentioned Solow-Shell and Ramsey models represent an aggregate analog of the problems studied below for the integral models. Namely, the sought-for variables $s(.)$ and $\lambda(.)$ of the Solow-Shell model (0.4)-(0.6) will be added by the lifetime $L(.)$ of production elements. Optimal choice of the lifetime $L(.)$ will be studied in the one-sector integral models with embodied TC in Chapters 4-6. The interaction between optimal $\lambda(.)$ and $L(.)$ will be considered in Section 6.2, and the interplay of optimal $s(.)$ and $L(.)$ will be studied by means of two-sector model in Chapters 8 and 9.

However, in classical models with embodied TC ([19], [129], etc.) the equipment improvement and raising the efficiency of elements are also given, after

all, from outside as a function of time (i.e. this is exogenous TC, too). Then the source of TC is still unclear. To explain the reasons and sources of TC, the models with *endogenous TC* are introduced.

The endogenous (induced) TC is assumed to be connected with previous development of an economy and is a result of such development. A simple ***aggregate model with endogenous TC*** can be constructed as a modification of the Solow-Shell model (0.4)-(0.6). Let us add to the characteristics of this model the new variable ***innovation*** I (the expenditure for raising the level of technical knowledge or for TC). The first two equations (0.4) of the model (0.4)-(0.6) are replaced by the following:

$$Q = A(t)F(K, L), \qquad Q = C + I + S, \tag{0.9a}$$

and the equations (0.5),(0.6) are the same. The dynamics of the index $A(t)$ of TC efficiency is described by some additional equation of the type:

$$A = \delta(A, I) \quad \text{or} \quad dA/dt = \tau I - \sigma A. \tag{0.9b}$$

Here τ is the coefficient of science efficiency, and σ characterizes the expenditure for science management.

More complicate models with endogenous TC are based on various hypotheses defining a relationship between TC rates and economic environment, for example, the amount of new types of capital equipment. As a result, the efficiency of the newest capital depends on investment and innovation. Hence, there is a need to distinguish different ages of capital according to the levels of investments at the time of its producing. For this purpose the age-dependent IDMs are also required.

The lifetime of optimal elements will be studied in the IDMs with endogenous embodied TC in Chapters 6 and 10, in so doing, three- sector integral model with separate TC branch is considered in Chapter 10.

The *TC as a separate branch of production* is the perspective approach for the description of economic development. Even when a middle-size economic system is considered, its TC is formed mainly within system itself, i.e. by politics of capital investments into R&D, design offices, experimental production, etc. Tak-

ing into account the importance of TC, corresponding economic-mathematical models must provide a separate TC block.

TC is realized through *innovation process.* Technological innovations have been investigated by means of substitution, diffusion, and evolution models, each of them dealing with different aspects of the innovation process [20], [38], [122], [124], [125]. Such models are briefly considered in Section 15.1.

In whole, evolutionary economics under TC relates to the complex systems with high dimension and uncertainty of inherent relationships. Nevertheless, deterministic models are widely used for simulation of general tendencies of economic evolution. In our opinion, it is due to unjustified complexity of mathematical description using stochastic factors, without substantial insight in the essence of modeled process. Strictly speaking, the deterministic models operate always with some averaging probabilistic performances of process (expected value of "manpower" instead of real "manpower", and so on). Once a stochastic model has been built and a trend of process is apparent, one can use a corresponding deterministic model to analyze the dynamics of the average values of stochastic variables. We concentrate here on some deterministic models of embodied TC.

Turnpike Approach

In the present monograph, qualitative study of the optimal lifetime of technologies is substantially based on ***turnpike approach. The turn-pike theorems*** establish that for a long planned interval the optimal trajectory of dynamical model is closed in certain sense to some curve which is called a ***turnpike*** (turnpike trajectory) and is independent of initial state and interval length [34], [84], [94], [97], [114].

There are various types of the turnpike theorems: in weak, normal, strong and the strongest forms. In this book we will use the following types of such theorems.

Weak turnpike theorems: for a large length $T - t_0 \gg 1$ of optimization horizon the optimal trajectory $x_T(t)$, $t \in [t_0, T]$ tends to one of effective (infinite-optimal) trajectories: $\lim_{T,t\to\infty} \rho(x_T(t), X_\infty(t)) = 0$, where $X_\infty(t)$ is a set of effective trajectories.

The turnpike theorems in normal form: at $T - t_0 \to \infty$ the optimal trajectory $x_T(t)$ can have a finite deviation from the turnpike set $X_\infty(t)$ only

on a finite subset Δ of optimization interval $[t_0, T]$, the total duration of Δ being independent of $T - t_0$.

The turnpike theorems in strong form: for any $\epsilon > 0$ the time $N(\epsilon)$ exists such that $\rho(x_T(t), X_\infty(t)) \leq \epsilon$ at $t \in [t_0 + N(\epsilon), T - N(\epsilon)]$, i.e. the optimal trajectory can deviate from a turnpike more than by value ϵ at initial and final instants only.

The turnpike theorem in the strongest form: for any $\epsilon > 0$ a time $N(\epsilon)$ exists such that the optimal trajectory $x_T(t)$ coincides with unique turnpike one $x_\infty(t)$ at $t \in [t_0 + N(\epsilon), T - N(\epsilon)]$.

The distance $\rho(.,.)$ is understood in the sense of angular deviation in Neumann-Gale models, and it can be a linear distance for other types of models.

The turnpike theorems were primarily proved for the optimization problems in linear multi-sector Neumann-Gale models [34], [84], [114]. The similar properties are also present in nonlinear aggregated economic-mathematical models, including the models considered above. In particular, the turnpike theorem (in the strongest form) holds true for optimal trajectories $\lambda(.)$ and $s(.)$ in the Solow-Shell model (0.4)-(0.6) with the objective functional (0.7). For the Ramsey model (0.4)-(0.8) with autonomous TC, turnpike properties are proved in special cases only, for example, for Cobb-Douglas-Tinbergen PF: $F(K, L, t) = e^{pt} K^\alpha L^{1-\alpha}$ $0 < \alpha < 1$. In the aggregated model (0.5), (0.6), (0.9) with endogenous TC, the sought-for controls are the accumulation norm $s_1(t) = Q/S$ and the part $s_2(t) = Q/I$ of the expenditure for science in total output, $s_1, s_2 \geq 0$, $s_1 + s_2 \leq 1$, $t \in [t_0, T]$. A turnpike theorem was proved for this model in the special case $s_1 + s_2 \equiv 1$, $\mu = 0$, $A = \delta(I)$, $F = F_1(K)F_2(L)$ [156].

It will be established in this work that the optimal lifetime of technologies in the models with embodied TC also possesses turnpike properties that is of interest in its own right.

The applied importance of turnpike properties consists in finding some "effective" trajectories (turnpikes) which are close to the solution of optimization problem and have simpler structure. Besides, it is easier to find the turnpikes and to search for their properties rather than to solve the optimization problem directly. In general, the turnpike theorems indicate the basic tendencies, the laws of economic development.

It should be noted that the turnpike properties are guaranteed only for a long optimization horizon and for discounted *infinite-horizon optimization*. For this reason the infinite-horizon optimization seems to be more appropriate from the practical point of view. In the latter case, the turnpike trajectories of the lifetime of technologies can be used as a basis of the decision-making rules for the rational scrapping of obsolete industrial equipment even for moderate time intervals.

0.2 THE REVIEW OF BOOK CONTENTS

The monograph consists of four parts and 15 Chapters and is organized as follows.

Part 1. INTEGRAL DYNAMICAL MODELS OF EVOLVING SYSTEMS

The first part of the monograph is devoted to different applications of the IDMs under consideration and mathematical problems arising in their study.

Chapter 1. Integral Dynamical Models in Control Theory

In this book emphasis is placed on the deterministic models of *dynamical systems (DS)* in continuous time. The traditional mathematical means for the description of such DS are *differential and integral equations*. The integral dynamical models of the form

$$\mathbf{x}(t) = \int_{t-T}^{t} \mathbf{K}(\tau, t, \mathbf{x}(\tau))d\tau, \quad \mathbf{x}, \mathbf{K} \in R^n \tag{0.10}$$

were investigated originally by Vito Volterra [140], [141] and have since been and still used extensively in the analysis of dynamical systems in various applications, primarily in automatic control theory [12], [39], [42], [73], [93]. The most thoroughly studied is the linear model (0.10), the so-called renewal equation [77], [101], [135]. The quantity T in (0.10) is usually called *the aftereffect duration* [140] or *the memory of the DS*. The widening of application field has

recently led to the necessity of studying the model (0.10) in which the memory is a function of time t: $T = T(t) > 0$, including the sought function. This monograph deals with the theory and application of such models called *the IDMs with controllable memory (IDM-CM).*

The use of IDMs becomes a must in the simulation of economic system renovation under assumption of heterogeneity of economic factors involved in production. But, as has been shown in [46], [47], [146], such IDMs also have a general system-theoretic interpretation as *the models of evolving systems (ES)*, which is discussed in Section 1.4. In so doing, the ES is defined as a developing self-organizing system with variable age-dependent structure. By the structure of ES we understand a distribution of the system elements in accordance with their functions (external and internal), times of creation, and the efficiency of function execution. The internal functions of ES ensure the improvement and development of the system itself, for example, the production of new elements, the rise of efficiency of produced elements, etc. The renovation of ES consists in putting new more efficient elements into the ES and in removing the oldest elements.

Such interpretation causes considerable extension of the sphere of IDMs application. It allows us to study the development of complex technical, scientific, ecological, and other systems whose self-improvement is of interest for investigation.

The statements of different mathematical problems in these models are briefly exposed in Section 1.6. They are the systems of Volterra-type integral equations, optimization and identification problems.

Chapter 2. Integral Dynamical Models of Economic Systems

Here we briefly describe the economic IDMs.

The integral economic models studied in this book fall into the class of macroeconomic models with embodied technological change [19]. Among them the most known is the Solow model [129]. According to this model, the technological change (technical progress) is incarnated in the productive capacity (fixed assets, production means): the recently created capacity units are more effective than those created in earlier time. This model is described by a system of Volterra-type equations (0.10) with infinite memory ($T = \infty$).

In contrast to the Solow model, the models given below allow the management of the liquidation of obsolete (worse) fixed assets (technologies, capacities, equipment) under the technological change. Leonid Kantorovich, the famous Russian mathematician and economist, Nobel prize winner, had introduced similar model for the first time into mathematical economics in 1959 [74]. However, this model was not investigated further, and, in 1973, he proposed another integral macroeconomic model [75] which is the improvement of the Solow model [129]. What is principally new in it is the introduction of the new endogenous (sought-for) time limit of the use of fixed assets: the assets created before this limit are not used at current time.

The essence of the IDMs with controllable lifetime of capacity can be explained in the following manner. We assume that economic system is open, i.e., that production units can be created (new production units enter the system) and closed down (units leave the system). Let the quantity $m(\tau)$ of new equipment units (EUs) put into operation at year τ, and the EU's characteristics (the productivity β and the capital labor ratio λ have been defined in Section 0.1) depend on this instant τ. Then the output Q (the amount of operated capacity) can be represented as an aggregation over EUs between current year t and the year $a(t)$ of commissioning the oldest unit:

$$Q(t) = \sum_{\tau=a(t)}^{t} \beta(\tau, \lambda(\tau))m(\tau)$$

or, in continuous time t, as

$$Q(t) = \int_{a(t)}^{t} \beta(\tau, \lambda(\tau))m(\tau)d\tau. \tag{0.11}$$

All other desired balance relations (in labor, production expenditures, and so on) can be constructed in analogous way and contain similar integral operators. In so doing, the interval $[a(t), t]$ represents *the lifetime of the oldest EU.* When the time $a(t)$ is treated as endogenous parameter (along with other model variables), we arrive at an IDM with the controllable liquidation of obsolete capacity.

Independently of Kantorovich, Victor Glushkov, the well-known Ukrainian cybernetician, had introduced in [45] the two-sector integral macroeconomic

model representing the relation between two production sectors: the production of production means (capacity) and the production of consumption goods. From this time, an intensive research of economic IDMs was started at the Cybernetics Institute of the Ukrainian Academy of Sciences, and, by present impressive results have been achieved (see [48], [49], [68], [65], [66], [67], [146], [148], [150], etc.). Such models are described by the systems of the Volterra-type equations with sought-for function in the lower limits of integration.

An "integral" production function, at first sight different from models mentioned above, was suggested in [107]. In this function the integral is evaluated for labor expenditure rates and the limit of integration is also sought for. It has been shown in [68], [146] that under natural assumptions a substitution of variables reduces this model to the form of previous models.

The fact that the various mentioned economic-mathematical models were introduced independently for describing similar economic processes (replacements and renewals of productive capacities) indicates that these models have become indispensable and, therefore, their investigation should be of high theoretic interest.

Chapter 3. Integral Dynamical Models in Mathematical Ecology

The IDMs find increasing use in mathematical simulation of complex biological systems, including ecological systems. Such models appear in mathematical ecology when the influence of age structure and the prehistory of process upon population dynamics *(the hereditary effects)* is considered. The best known integral models of mathematical ecology such as Volterra model of population self-intoxication, Volterra "predator-prey" model with aftereffect, Lotka age-dependent model are discussed in this Chapter, and their connection with corresponding partial differential models is considered. There is a deep analogy between biological and technological processes, which is useful for the development of the mathematical models of technological change as well as of corresponding biological systems.

The specific mathematical problems arising in the ecological IDMs are emphasized. They consist in the study of *the stationary solutions* of nonlinear IDMs and finding *the bifurcational values* of parameters at which new stationary solutions appear.

Part 2. ANALYSIS OF ONE-SECTOR INTEGRAL DYNAMICAL MODELS

This part is devoted to the study of various optimization problems in one-sector IDM of economic renovation. The asymptotic behavior and turnpike properties of the optimal lifetime of technologies are major focus of interest.

Chapter 4. Basic Optimization Problem in One-Sector Model

The optimization problems (OPs) evoke usually great interest among those engaged in economic-mathematical simulation, which has been displayed in the investigation of considered IDMs. The simplest optimization problem (OP1) in one-sector IDM is studied in the Chapter 4. It is the classical problem of total output maximization under given manpower. But the control influence on economic system is non-classical and is provided by means of change of the renovation intensity of the system technological structure.

The OP1 has only one sought-for control variable (the quantity of introduced EUs) and one phase variable (the EU lifetime) that substantially simplifies the OP investigation. This allows us to obtain essential results in the study of the behavior of optimal trajectories, among them the explicit solutions of nonlinear OPs and the turnpike theorems for the lifetime of EUs.

The OP1 is studied in detail and is the basic problem for other considered OPs in one-sector IDMs. In particular, the necessary and sufficient conditions of extremum are established for the OP1, and structure and asymptotical behavior of the solutions are studied for a large horizon as well as for the infinite horizon $[t_0, \infty)$.

Chapter 5. Asymptotical Behavior of Optimal Trajectories and Turnpike Theorems

This chapter is the continuation of Chapter 4 and completes the investigation of basic OP1 for the simplest one-sector IDM.

During investigation of the optimization problems for the IDMs, the results were taken [67], [117], [146], [147], which enable us to suppose the existence of so-called turnpike properties of optimal trajectories (see [34], [84], [94], [97],

[102], [114], and Section 0.1). Turnpike theorems are of great interest for mathematical economics. This monograph represents the study of turnpike properties of optimal technology lifetime in economic IDMs and is probably one of the first steps in the analysis of this problem.

In considered IDMs the analysis of turnpike properties leads to the nonlinear integral-functional equations (IFEs) of the form:

$$\int_t^{a^{-1}(t)} F(\tau, t, a(\tau))d\tau = f(t, a(t)), \qquad (F(\tau, t, t) \equiv 0) \tag{0.12}$$

with the sought-for variable a in upper integral limit (here $a^{-1}(.)$ is the inverse function of $a(.)$). The interval $[t, a^{-1}(t)]$ represents the sought-for future lifetime of the new EUs which are brought into operation at current time t, whereas the value $t - a(t)$ in (0.11) is the lifetime of the oldest EUs which are removed from service at the time t. The equation (0.12) links these two important characteristics of technological structure with the cost $\lambda(t)$ of new EUs that provides the analysis of the rational strategies of technological renovation.

The IFE (0.12) arises in the dual (adjoint) problem for the OP1, but it can be directly deduced from some evident economic assumptions, which is also fulfilled in this Chapter. The solvability and qualitative behavior of the solutions of the IFE (0.12) are studied in Section 5.1.

On the basis of study of IFE (0.12), the turnpike properties of the optimal trajectories of system renovation (the optimal lifetime of EUs) are established for basic OP1. The turnpike theorem in the strongest form is proved for the infinite-horizon discounted optimization, and the normal and strong theorems are proved for the finite-horizon case. These results have been obtained for the first time in mathematical economics.

Chapter 6. Other Optimization Problems in One-Sector Models

A number of other OPs are also analyzed in one-sector IDM: the minimization of production expenditure, the choice of the optimal cost of EU, the choice of the optimal productivity of new equipment, the account for the instantaneous and distributed expenditure for R&D. The basic structure of OP1 is unchanged

in considered OPs, i.e., the control influence is carried out by the change of the renovation intensity of technological structure.

A sketch for various investigated OPs in one-sector model is given in Chapter 6. The objective of this Chapter is to trace the change of qualitative behavior of the OP solutions if the various features of economic development are taken into account. Special emphasis is laid on generality and distinctions of turnpike properties as against those of the basic OP1.

The first mentioned problem is the inverse of the basic OP1 (expenditure minimization under given output). So, the behavior of their solutions turns to be similar to that of OP1 on a qualitative level. But the conditions of the existence of a turnpike EU lifetime are more strict than for output maximization.

The second OP implies the simultaneous optimization of EU lifetime and EU cost under embodied TC on the basis of the Kantorovich IDM (see Chapter 2). Here the output is supposed to depend on two factors (capital and manpower), which is classical in mathematical economics. The OP includes respectively two independent sought-for control variables (EU cost and the quantity of new EUs) and one phase variable (EU lifetime). The turnpike property is also proved for this OP. In case of the Cobb-Douglas production function and the exponential rates of embodied TC and discount, the turnpike trajectories are the constant lifetime of EUs and an exponential increase of the cost of new EUs.

Next, the rise of the productivity of new EUs (the rate of embodied TC) is supposed to be controlled by means of certain sought-for innovations, i.e., we consider the simplest IDM with endogenous embodied TC. Then turnpike properties are absent. Hence, under given assumptions the direct innovations into TC are more strong than other types of economic control.

The last two OPs take into account the expenditure (instantaneous or distributed in time) for the development of new vintages along with the cost of each new EU. There are no classical turnpike properties in these OPs. Nevertheless, the tendency present for the rational lifetime of EUs to strive to some "favorable" regime (if it exists). The last three OPs may have two local maxima: a trivial solution (the lack of R&D investment) and non-trivial one which corresponds to a regime of intensive technological renovation.

Obtained results allow us to study the structure and asymptotical properties of the optimal trajectories of technological renovation under different assumptions.

Part 3. ANALYSIS OF MULTI-SECTOR INTEGRAL DYNAMICAL MODELS

The Part 3 is the extension of previous one. It deals mainly with the theoretical analysis of OPs for more complex IDMs as well as with arising integral equations. The results of the optimization in two-, three-, and n-sector IDMs are exposed in this Part.

Chapter 7. The Volterra Integral Equations with Sought-For Lower Limits of Integration

As distinct from the one-sector model, more complex multi-sector IDMs contain certain systems of the Volterra integral equations (VIEs) with sought-for lower limits of integration. This Chapter is concerned with the solvability of the various VIEs arising in the IDMs under consideration. At first we study the solution existence and uniqueness for the system of two nonlinear VIEs of the second kind with one sought-for limit of integration. Next, the analogous results are established for similar VIE system of the first kind as well as for the system of $2n$ VIEs with n sought-for lower integral limits, $n > 1$.

The obtained results are the basis for further study of optimal economic renovation in multi-sector IDMs. They allow us to investigate the solvability of such models with respect to sought-for phase variables under given control variables, which is the first necessary step in the theoretical analysis of optimal control problems.

Chapter 8. Optimization in Two-Sector Models

Two-sector IDM (see Section 2.5) takes into account the liquidation of obsolete capacity and the production of new capacity inside an economic system. For this model the OP is investigated, which ensures simultaneous optimization of the EU renovation and distribution between production sectors. In this Chapter we use more general interpretation of the two-sector IDM as a basic *model of evolving system* (see also Sections 1.6 and 2.5.1).

The structure and asymptotic behavior of optimal model trajectories are examined, in particular, the turn-pike properties are established. In so doing, the same turnpike properties of technology lifetime are proved as for the basic OP1 in one-sector IDM. The obtained results develop those exposed above in Part

2 for the one-sector IDMs of economic renovation. They allow us to find some basic tendencies of economic self-development.

Namely, the turn-pike theorem in the strongest form is proved for infinite-horizon OP. This theorem implies that the optimal EU lifetime coincides with a turnpike trajectory except some initial finite interval. For analogous finite-horizon OP we show just less strong turn-pike properties (the turnpike theorems in normal and strong forms). The corresponding optimal distributions of EUs between production sectors are constructed and analyzed.

We also study the case of different dynamics of the technological renovation in production sectors. For this purpose the same OP is considered in the generalization of two-sector model that includes two independent sought-for lifetimes $a_1(t)$ and $a_2(t)$ of EUs for different sectors A and B (instead of one common control $a(t)$ for the entire system). The main result is that turnpike lifetime is less in such sector where the rate of embodied TC is larger.

Chapter 9. Optimization of Industry Conversion Rates

In this Chapter the considered IDMs are used for such applied problem as the optimization of the conversion processes in defense industries. Mathematical modeling of the conversion of defense plants and industries is a complex scientific problem, and here we consider only the issue regarding technical refitting rates of industrial systems undergoing conversion. Namely, we employ a two-sector IDM for describing an industrial system that manufactures two types of products (defense product and civilian product) in two types of plants using a common labor resource. The objective is output maximization and cost minimization over a certain planning horizon under a given labor force. The defense production target is also assumed as given.

The obtained results lead to approximate modeling of optimal conversion strategies in the presence of technical progress and enable us to study their properties without solving the optimization problem (by means of turnpike approach). Specifically, the interpretation of the proved turnpike features of the model produces different conversion strategies for different assumptions on TC rate.

Chapter 10. Optimization in Three-Sector Model with Endogenous Technical Change

In this Chapter, the IDM of three-sector macroeconomic system under endogenous embodied TC is constructed that implies the following three sectors of macroeconomics: the production of EUs (sector A), the production of consumption goods (sector B), and the production of technical progress (sector C). Sector C is understood to be the sector of technological innovation (the industrial R&D) whose function consists in raising the efficiency of the new EUs produced in the sector A. In contrast to the two-sector IDM (Chapter 8), this TC is assumed to be endogenous, i.e. the rate of EU efficiency depends on the investment into the sector C. This provides simultaneous optimization of the TC rate in EUs production and the renovation of EUs.

We reduce the OP for the three-sector IDM to the corresponding OP in one-sector model, which has been studied above in Section 6.3. The optimal regimes of technological renovation and the optimal distribution of output between production sectors are analyzed in this IDM. The case of different TC rates in the production sectors A, B, C is also briefly considered.

Chapter 11. Optimization in Multi-Sector Models

The use of multi-sector IDMs for the purpose of simulating the process of replacement and modernization of the elements of a manufacturing system is analyzed in this Chapter. We consider manufacturing processes whose peculiarity is the need to exequte operations at specific calendar periods (for example, numerous technological processes in the areas of agriculture, machine building, construction, etc.) according to a definite technological sequence for annual production cycle. The demand for equipment is determined on the basis of peak periods that require the maximal quantity of EUs.

Depending on the relationship between the number of unknowns and the number of equations in the model, we have one of two problems in the constructed model: the problem of predicting the development of a manufacturing system in a definite planning time interval, an optimization problem in the IDM.

The prediction problem with control of processes involved in replacement of elements of the manufacturing system, i.e., with sought-for limits of integration, represents an important real world problem. In this case, the IDM constitutes a system of VIEs of the first kind with unknown lower limit of integration. The-

oretical questions concerned with the existence and uniqueness of the solution are investigated for this system.

Analysis is fulfilled for the problem of expenditure minimization in the constructed multi-sector IDM. In some cases this OP is reduced to an OP in one-sector model, whose turnpike properties have been established in Chapter 6. The next considered OP generalizes the basic OP1 to the case of multi-sector IDM, and some results, proven above in Chapter 4, are extended to it. The last OP is the problem of restructuring a multi-branch economic system in minimal time. This problem with non-fixed time (for response speed) and secured right-end is investigated without taking into account the constraints $a'(t) \geq 0$ under assumption that the "prehistory" is specified ad infinitum on the interval $(-\infty, t_0]$.

Chapter 12. Optimization of Technological Renovation in Hierarchical Ecological-Economic System

In this Chapter we consider the two-level optimization of control of the development of hierarchical ecological-economic systems that takes into account embodied TC in manufacturing and environmental impact. It is assumed that a "Center", which is on the upper level of the hierarchy, controls "Plants" by imposing penalties for each unit by which the pollution standard is exceeded. Maximizing the product output while taking into account the capital expenditures for production and the expenditures for environmental pollution penalties is the internal problem of optimizing the "Plant" operation. Control is implemented at the "Plant" by the cut-over of new, more efficient EUs and the decomissioning of the oldest ones. In so doing, the one-sector IDM is used to describe the "Plant" operation.

The optimal strategies of the "Center" and the "Plants" are established. With sufficiently realistic assumptions, the optimal strategy of the "Center" is a maximum penalty for exceeding the environmental pollution standards, while the optimal strategy of the "Plant" is up-to-the-limit manufacturing modernization rates as long as the pollution level exceeds its admissible value. Such "Plant" strategy corresponds to a maximal fast liquidation of the most environmentally-damaging equipment, a rapid introduction of modern equipment and, as a rule, a maximal-fast reduction in total pollution level. When the pollution level does not exceed its admissible value, the "Plant" modernization rates are determined by TC rate and have turnpike properties.

Part 4. APPLIED PROBLEMS OF INTEGRAL DYNAMICAL MODELS

In Part 4 the applied problems, such as numerical algorithms and the examples of real economic simulation, are drawn for the IDMs.

Chapter 13. Numerical Algorithms for Integral Dynamical Models

The item of this Chapter is the construction and analysis of numerical algorithms for the considered IDMs. The numerical treatment of the IDMs is necessary for forecasting and optimization of economic renovation.

We will dwell first on linear VIEs in which all the lower integration limits are specified functions. As was already noted, the theory of such equations is analogous to that of classical VIEs, and the algorithms for solving them are also based on the familiar results for ordinary VIEs, but have specific distinctiveness, which makes it necessary to investigate them separately. Here, a numerical algorithm for solving a system of second-kind VIEs and regularizing algorithms for first-kind VIE are discussed. These algorithms are computationally-optimal in some sense (precision, order of convergence, etc.).

Next, algorithms for the numerical solution of the simplest non-linear equation with a sought-for integration limit (which corresponds to the problem of inverting a primitive) are proposed. Iteration algorithm for solving the system of $n = m+l$ VIEs with l sought-for lower limits z_j and m "second-kind" unknowns x_i is investigated. The general scheme of the algorithm is based on a sequence of states for solving a system of m linear second-kind VIE for x_i, $i = \overline{1,m}$, (for specified z_j), and states for solving (for fixed discrete values of time t) a system of l non-linear equations in z_j, $j = \overline{1,l}$.

The collocation-based algorithm for solving the second-kind VIEs with a single sought-for limit of integration is also constructed, studied in detail, and implemented in software. The spline collocation methods well developed for ordinary VIEs and VIEs with delay [21], [22], [23] offer a means of computational effectiveness increment for the VIEs with sought-for integral lower limits. Such integral equations have been earlier solved by the numerical methods of lower precision degree [146].

Some notes on numerical algorithms for the treatment of OPs are made. OPs are the most computationally-complex problems for IDMs. Presently known algorithms for solving such OPs (zeroth-order methods that reduce the OP to non-linear programming problems, and gradient methods) are described there.

The analysis of various OPs in the IDMs carried out in previous Chapters suggests that in many important cases the optimal lifetimes of technologies are approximately described by nonlinear integral-functional equations (IFEs) of the form (0.12) and similar. For effective solving such IFEs it is required to know given functions on sufficiently large time interval. In the opposite case it runs into great difficulties, and we can at most estimate a solution. The approximate algorithms for solving the IFE (0.12) are developed in Section 13.4 for the cases of minimally possible time interval and for large intervals.

A number of the algorithms for solving various IDM problems mentioned in the preceding have been implemented as software packages intended for the solution of applied problems.

Chapter 14. Application of Integral Models to Optimization of Technological Renovation

The key applied problem arising in the IDMs use is the identification of the IDMs. It primarily consists in the determination of unknown kernel of an IDM (functions of the type $\mathbf{K}$ in (0.10) or β in (0.11)). The IDM identification implies solving mathematical problems which are substantially complex from theoretical viewpoint as well as with respect to required input information. Such problems are usually incorrect and require involving a priori economic knowledge. A technique for identification of the considered IDMs on the basis of available economic information was proposed and developed in [4], [146], [150]. It is based on certain additional assumptions about the behavior of an economic system on its prehistory. Particularly, these assumptions take into account the key factors only that have influence on output (embodied TC, autonomous TC, and physical wear). This procedure has repeatedly modified and applied to different IDMs. We consider the essense of proposed approach for the simplest one-sector IDM.

The numerical study of the IDMs was carried out for test data as well as for real economic information. The simulation of the following economic systems is briefly described in this Chapter: the macroeconomy of the former USSR (FSU) in 1946-1990 years (on the basis of two-sector IDM from Chapter 8);

a separate plant (on the basis of one-sector IDMs from Chapters 4-6); a regional agricultural system (on the basis of results of Chapter 11); technological renovation with accounting for environmental impact (on the basis of two-level control problem from Chapter 12).

Of special interest is modeling of the optimal (turnpike) lifetime of technologies. The estimation of turnpike lifetime was performed for fixed assets of the FSU economy. It turns out that the non-efficiency of this economic system did not permit the existence of turnpike regime. Nevertheless, the turnpike regimes were possible for some separate plants and branches of FSU industry.

Chapter 15. Perspectives and Open Problems of Integral Models

Some open theoretical problems of the IDMs are discussed from the viewpoint of evolutionary self-organization approach. The results exposed in this monograph represent only the first step in the modelling of rational lifetime of technologies. They are valid for large planned intervals only, provided that future TC course is completely known. The main present-day trend of TC modeling is the endogenous self-organization models which describe the diffusion of new technologies under the conditions of uncertainty and endogeneity of the market as a disequilibrium stochastic process. In this connection, it is worthwhile to develop the evolutionary endogenous growth models with more detailed description of the control of scrapping the obsolete technologies (equipment) ubder the market conditions. The construction and investigation of the IDMs with stochastic effects, as well as the estimation of the future TC rate influenced by innovation strategy, are among the most interesting unsolved issues.

Some prospective applications of the proposed IDMs are also discussed in this Chapter. The problems of *optimization of the unknown aftereffect duration of dynamical systems* arise in various economic and technical applications. Those are the determination of optimal replacement (renewal, repair) time interval in *renewal and replacement models*, the determination of optimal cycle time in *inventory models*, the minimization of variable transient time in engineering systems, and so on.

* * * * *

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PART I

INTEGRAL DYNAMICAL MODELS OF EVOLVING SYSTEMS

1

INTEGRAL DYNAMICAL MODELS IN CONTROL THEORY

In this book, emphasis is placed on the deterministic models of *dynamical systems (DS)* in continuous time. We mean the notion of DC in a wide sense (as an inertial system, system with memory, etc.) rather than in a strong sense (as dynamic system). Traditional mathematical means for such DS description are ***differential and integral equations.*** Various forms of integral models of DS and their connection with differential models are considered in Sections 1.1-1.4.

Considered IDMs have a general interpretation as the models of evolving (developing) systems that is discussed in Section 1.5. Such interpretation causes the considerable extension of IDMs application sphere. It allows the study of the development of complex technical, scientific, ecological, and other systems whose self-improvement is of interest to investigation.

1.1 EXPLICIT INTEGRAL MODELS OF DYNAMICAL SYSTEMS

Denote by $u(t)$ and $x(t)$ the *input and output signals* of DS respectively, where t is time. Suppose that $u = (u_1, u_2, .., u_m)$ is m-dimensional vector and $x = (x_1, x_2, .., x_n)$ is n-dimensional one.

The integral dynamical models (IDMs) assigning the explicit connection between the input and output of the DS are traditionally used in automatic control theory ("black box" models). Thus, an arbitrary finite-dimensional linear DS is described by integral model of the following form:

$$x(t) = \int_{-\infty}^{t} K(\tau, t)u(\tau)d\tau \tag{1.1}$$

where $K(\tau, t)$ is termed *the unit impulse response (UIR)* of DS. If function $K(\tau, t) \neq 0$ for all $t - \tau > 0$, then DS is referred to as *the DS with infinite memory.*

Practically, the DS output $x(t)$ depends on the input signals $u(\tau)$ whose time instants τ are distant from t at no more than time $T > 0$, i.e. $K(\tau, t) \equiv 0$ for $t - \tau > T$. In this case we have the DS with finite memory, whose model is

$$x(t) = \int_{t-T}^{t} K(\tau, t)u(\tau)d\tau \tag{1.2}$$

The value T is called *the aftereffect duration* or *the DS memory.*

Consider also the DS non-excited at $t < 0$. Then $u(\tau) \equiv 0$ for $\tau \leq 0$, and the DS model is of the form:

$$x(t) = \int_{0}^{t} K(\tau, t)u(\tau)d\tau \tag{1.3}$$

If DS is stationary, then its output $x(t)$ depends only on the time $t - \tau$ after the input instant τ: $K(\tau, t) = K(t - \tau)$. The dynamics of stationary DS can be described efficiently in terms of so-called *transfer function* and *DS natural frequency response*, which represent respectively the Fourier and Laplace transforms of the UIR. In the case of nonstationary DS this way runs into severe difficulties, and the basic DS characteristic is the UIR.

1.1.1 Nonlinear Case

The case of nonlinear DS is more complex. Some nonlinear systems are described by the following explicit nonlinear IDM:

$$x(t) = \int_{t-T}^{t} F[\tau, t, u(\tau)]d\tau \tag{1.4}$$

The nonlinearity in DS is occasionally represented by a static nonlinear transformation of input u, and the DS dynamics remains to be linear. The integral model of such systems is called *the Hammerstein model*:

$$x(t) = \int_{t-T}^{t} K(\tau, t) f[u(\tau)] d\tau \tag{1.5}$$

which has some essential advantage for identification [39], [92]. Such model arises in a natural way in the description of nonlinear interconnection of economic factors in the models with embodied TC (see Sections 2.2 and 6.2).

But, in general, IDM (1.4) falls far short of being a conventional model for nonlinear systems. Many such DSs, for example, nonlinear feedback control systems, cannot be represented in the form (1.4). The failure of superposition principle in nonlinear case gives no way for the DS transformation to establish a input-output relation.

The mathematical description "input-output" for more wide class of nonlinear DSs (with inertialess nonlinear part) is given by *the Volterra series*:

$$x(t) = \sum_{n=1}^{\infty} \int_0^t \cdots \int_0^t K_n(\tau_1, ..., \tau_n) \prod_{i=1}^{n} u(t - \tau_i) d\tau_1 \cdots d\tau_n$$

which generalizes the IDM (1.3) of linear stationary DS and has a clear mathematical interpretation.

The IDMs (1.1)-(1.5) can describe univariate DSs as well as *multivariable* DSs. In the latter case $x(t)$ and $u(t)$ are vector- functions and $K(\tau, t)$ is corresponding matrix. The consideration of *multidimensional DSs* corresponds to the passage from one-dimensional integrals to many-dimensional ones in integral models (1.1)-(1.5).

1.2 CONNECTION WITH DIFFERENTIAL MODELS

The finite-dimensional linear DS is often described by the system of *ordinary differential equations (ODEs)* of the first order:

$$dx/dt = A(t)x + G(t)u \tag{1.6}$$

(where A and G are the matrices of corresponding dimension) or by one ODE of n-th order :

$$a_n(t)x^{(n)} + ... + a_1(t)x^{(1)} + a_0(t)x = b_m(t)u^{(m)} + ... + b_1(t)u^{(1)} + b_0(t)u. \tag{1.7}$$

If the parameters of DS are constant *(stationary DS)*, then the coefficients of the models (1.6) and (1.7) do not depend on t (particularly, $A(t) = A$, $G(t) = G$).

The exact analytical solution of ODE (1.6) is of the form:

$$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, \tau)G(\tau)u(\tau)d\tau, \tag{1.8}$$

where so-called *transition matrix* $\Phi(t, \tau)$ is defined by the differential equation: $d\Phi(t, \tau)/dt = A(t)\Phi(t, \tau)$, $\quad (\Phi(\tau, \tau) = E$ is identity matrix). The expression (1.8) represents a simple explicit IDM of the form (1.3).

A deterministic model of nonlinear DS is often described by the system of nonlinear ODEs:

$$dx/dt = F(t, x, u), \tag{1.9}$$

where F is a n-dimensional function of $n + m + 1$ variables x_1, $x_2, \ldots, x_n$, u_1, $u_2, \ldots, u_m$, t. In the most cases the exact solution of nonlinear ODEs is not found and their solving requires the use of approximate methods and computers. So, the connection between nonlinear differential and integral models is not so evident as for linear DS, and a separate investigation is needed for each concrete case.

It should be noted that, in contrast to the differential models of DS, the integral ones take into account the *aftereffect (persistence, contagion, hereditary effects)* when the continuous sequence of past DS states $u(\tau)$, $\tau < t$, makes influence on future evolution of DS. In mechanics the integral model of elastic persistence in

the form (1.1) (where x - deformation, u - stress) was introduced by Boltzman in XIX century. Vito Volterra developed the Boltzman theory and introduced the aftereffect concept in other applications, specifically, in ecology. The aftereffect is, in general, defined as an arbitrary nonlinear functional of $u(\tau)$, $\infty < \tau \leq t$ [140].

1.3 INTEGRAL MODELS OF SYSTEMS WITH CONTROLLABLE MEMORY

Denote by $a(t)$ the lower integral limit in IDMs (1.1)-(1.3). Then we obtain the following uniform mathematical notation of these IDMs:

$$x(t) = \int_{a(t)}^{t} K(\tau, t)u(\tau)d\tau, \quad a(t) < t, \tag{1.10}$$

where $a(t) = \{-\infty,\ t - T,\ 0\}$. It is evident that for a nonstationary DS the function $a(t)$ may be an arbitrary one such that $\infty < a(t) \leq t$ (*DS with variable memory*). Moreover, for some important applications this function can be an unknown control (see Chapter 2). Then the model (1.10) is referred to as *the IDM with controllable memory (IDM-CM).*

Such models are used below for the description of renovation processes in economic systems (see Sections 2.2-2.5).

1.4 IMPLICIT INTEGRAL MODELS

Let there be given the structure of DS and the characteristics of DS elements. It is common to be the case for economic and technical systems. By the *DS structure* we mean a set of the connections between DS elements and their intensities. It is reasonable to define the structural connections of linear DS by a linear relation of DS input u and output x:

$$u(t) = Y(t)x(t) + u_0(t). \tag{1.11}$$

If these connections are non-linear, then the equality (1.11) defines them in a first approximation. The coupling types and intensities are determined by matrix $Y(t) = \{y_{ij}(t)\}$. If $y_{ij} \neq 0$, then there is the feedback for output i (positive at $y_{ii} > 0$ and negative at $y_{ii} < 0$). If the DS has not active elements (for example, in economics), then $|y_{ij}| < 1$. Different types of DS connections can be described by choosing $Y(t)$, in particular, the parallel and sequential connections among DS elements [146].

Substituting the relation (1.11) into explicit IDM, for example into (1.10), we obtain the following *implicit IDM* of linear DS:

$$x(t) = \int_{a(t)}^{t} K(\tau, t)Y(\tau)x(\tau)d\tau + f(t), \tag{1.12}$$

where $f(t) = \int_{a(t)}^{t} K(\tau, t)u_0(\tau)d\tau$ is a given DS input.

Moreover, the function $a(t)$ may be different for different inputs:

$$A(t) = \{a_1(t), a_2(t), \ldots, a_n(t)\}.$$

In this case the IDM-CM can be written in the form:

$$x_i(t) = \sum_{j=1}^{n} \int_{a_j(t)}^{t} K_{ij}(\tau, t)y_{ij}(\tau)x_j(\tau)d\tau + f_i(t), \quad i, j = \overline{1, n}. \tag{1.13}$$

The IDM (1.13) describes the nonstationary linear DSs with variable structure ($A = A(t)$, $Y = Y(\tau)$). If some part of output $Y(t)$ is supposed to be used for the purpose of development of the DS under consideration, then we obtain *the models of evolving systems* (see next Section).

1.5 INTEGRAL MODELS OF EVOLVING SYSTEMS

Let us consider some abstract (natural or artificial) *evolving system (ES)* consisting of separate *system elements (SEs)* which execute certain *ES functions.*

Suppose that the ES elements differ in accordance with their functions, the times of their creation, and the efficiency of function execution.

System functions may be divided into external and internal ones. *The external functions* of ES can be interpreted as some aggregate production of system (external in relation to system). *The internal functions* ensure the functioning, improvement, and development of system itself (for example, the creation of new SEs, the rise of efficiency of produced SEs, etc.). The entire system may be divided into *subsystems* (sectors) whose elements execute uniform functions.

We shall consider the systems with variable structure. By *the structure of ES* we understand the distribution of SEs in accordance with their functions, times of their creation, and the efficiency of function execution. Suppose also that a certain age-dependent progress is connected with the ES, in accordance with which recently created SEs are more effective that those created at earlier time.

It is assumed that there is *the renovation of system*, which consists in putting new more efficient SEs into operation and the liquidation (shut down) of the oldest SEs. New elements can be entered into ES from outside or produced inside (in a subsystem of ES improvement).

Such ES can be described by different IDMs considered below.

One-sector IDMs (2.6)-(2.10) describe the ES which executes a uniform external function, and new elements enter into the ES from outside at the expense of the part of ES production output.

Two-sector IDM (2.12)-(2.14) describes the ES consisting of two subsystems: a self-improving subsystem A which ensures the creation of new SEs and the liquidation of obsolete SEs, and a subsystem B which ensures the execution of external uniform aggregate function. In relation to macroeconomics, this two-sector IDM represents two production sectors: the production of production means (A) and the production of consumption goods (B) (see below Section 2.3).

Three-sector IDM (10.2)-(10.5) studied in Chapter 10 contains another self-improving subsystem C along with the subsystems A and B. The function of the subsystem C consists in the raise of efficiency of new SEs which are created in the first self-improving subsystem A. The economic content of subsystem C may be a sector of technological innovation (the industrial R&D).

1.6 ARISING MATHEMATICAL PROBLEMS: IDENTIFICATION, INTEGRAL EQUATIONS, OPTIMAL CONTROL PROBLEMS

Most of the mathematical problems in IDMs under consideration may be classified into three chief groups: identification problems (in unknown UIR K); Volterra integral equations of the second and first kind (with respect to sought-for x_i, $y_{i,j}$, a_j; and optimal control problems.

1.6.1 Identification Problems

Classical problems of automatic control theory have been connected mainly with the sought-for UIR of studied IDMs [39], [42], [73], [92], [128] (in deterministic or stochastic approach). Most of the developed methods for the determination of UIR $K(\tau, t)$ can be used for the IDMs (1.10), (1.12), (1.13) with controllable memory (because these models are linear in input-output) provided that necessary input information is available. If a nonlinear IDM-CM (with nonlinear integrand of the form (1.4),(1.5)) arises, it is essential to use a priori knowledge about the character of nonlinearity. As pointed out in [92], the potential variety of nonlinear IDMs does not often allow us to construct a nonstructural model (1.4) of "black box" type.

In this book, no consideration is given to theoretical aspects of the identification of IDMs under study, and only some applied algorithms are discussed in Chapter 14 for the identification of economic IDMs.

1.6.2 Volterra-Type Integral Equations

The IDMs under study fit into the general scheme of equations with deviating arguments (see [37], [57], and other), but differ somewhat from the latter by arising mathematical issues. Whereas, in equations with deviating arguments (in particular, with lagging arguments) considerable attention is paid to the study of their functional structure, stability, periodicity of solutions, boundary problems, etc., the basic traditionally investigated problems of integral equations are the study of their solvability and the development of efficient

algorithms for their approximate solutions. Other questions are investigated, as a rule, for specific equations that arise in contextual applied problems.

The IDM (1.12) or (1.13) represents a ***system of Volterra integral equations (VIEs) of the second kind*** with respect to output $x(t)$. These equations are obtained from the problem of UIR's definition under the transfer from explicit linear IDM "input-output" to implicit model (by virtue of (1.11)). If certain part of outputs x_i is given and some elements y_{ij} of matrix Y are unknown, then it leads to the ***mixed systems of VIEs of the first and second kind.*** If the variable lower limits $a_i(t)$ of integration in (1.3) are specified functions, then the theory of such equations is similar to the well-developed theory of ordinary VIEs (see [23], [25], [54], [79], [80], [81], [100], etc.).

In this book, emphasis is placed on the equations (1.10), (1.12), (1.13) with sought-for functions $a_j(t)$ in the lower limits of integration. The presence of at least one unknown $a_j(t)$ in IDM-CM (1.13) adds a novelty to the investigation of the basic theoretical features (existence and uniqueness of solutions, stability, etc.) of these equations. These questions are studied in Chapter 7 of present book.

In all considered cases the sought-for functions $x_i(t)$, $y_{ij}(t)$ must be given on a prehistory of modeled process to provide the unique solvability of the equations under study.

The IDMs encompass the processes described by various mathematical means, in particular, by ODEs, differential equations with delay, and difference equations, and, consequently, allow for very complex pictures of qualitative behavior of the solutions. A number of examples of reducing IDM-CM to equations from the indicated classes and an investigation of oscillatory regimes in them are presented in [46], [64]. Problems of finding stationary solutions of a nonlinear IDM and the bifurcation values of the parameters for which new stationary solution appears (*"eigenvalue problems"*) are examined in [146] (see also Section 3.3). On a qualitative level, these results are equivalent to previously known results for partial differential model that describes the same biological community.

1.6.3 Optimal Control Problems

If the number of sought-for functions x_i, y_{ij}, a_j in the IDM (1.13) is greater than the number of equations (1.13), then the problem is usually closed by

means of introducing some extremal criterion, which means the transfer to the *optimal control problems (optimization problems)* for the IDM (1.13) [142].

Optimal control problems evoke great interest in applications (see Chapter 2), but they are the most complex from the theoretical point of view. In this book we will study the optimization problems for the IDM (1.13), which contain the sought-for functions $a_i(t)$ along with other endogenous (control) variables. The investigations focus on the qualitative analysis of the optimal trajectories and its contextual interpretation. Such *optimization of the unknown aftereffect duration of dynamical systems* is of interest for different applications:

- - the renovation of *macroeconomic systems* under technological change,
- - the optimal replacement (renewal, repair) time interval in *replacement and renewal models*,
- - the optimal cycle time in *inventory models*,
- - the minimization of *the average delay in digital networks*,
- - the *minimization of variable transient time* in engineering, and so on.

2

INTEGRAL DYNAMICAL MODELS OF ECONOMIC SYSTEMS

The necessity to apply integral models to mathematical economics comes while considering the heterogeneity of the economic factors related to industrial production, such as capital, productive capacity, fixed assets, industrial equipment, labor force, and so on (see [19], [122], [129], etc).

The integral macroeconomic models first allow us to investigate theoretically the rational rates of technological renovation and the optimal lifetime of capacity. These problems are considered in the correlation with other economic aspects (presence of resources, production efficiency criteria, technical progress rate, etc.). Universality of the IDMs provides their application for solving the problems on different levels of economic management.

The applied importance of such problems has caused the necessity of investigation of the integral models and their application to optimization of economic renovation. More than five hundred publications, primarily of the FSU authors, have been devoted to such models for the last twenty years.

The examined models belong to the class of the macroeconomic *models with embodied (materialized) technological change* [19]. The most known of them is Solow model [129].

2.1 SOLOW INTEGRAL MODEL (MODELING OF EMBODIED TECHNOLOGICAL CHANGE)

To describe the evolution of an economics, one has to determine the system, the elements, and their interactions relevant for the problem considered. Let us choose the *equipment unit (EU)* as a basic production unit. The EU is defined as the quantity of equipment (capital, capacity) per one worker and is assumed to be the smallest production unit possible.

According to the Solow model [129], technological change (technical progress) is incarnated in capital (productive capacity) by the following way. The EUs created recently are more effective than the EUs created at earlier time instants. The EUs created at the same instant have identical efficiency. This is described below. Introduce into consideration the following functions:

- $K(\tau, t)$ - the amount of productive capacity (measured in capital units) which is created at time τ and is still in service at current time t, $\tau < t$,
- $m(\tau, t)$ - the amount of labor for the capital $K(\tau, t)$.

In accordance with the above definition of EU, the value $m(\tau, t)$ is the quantity of the EUs created at instant τ and remaining in operation at time t. Then the output at time t produced by EUs created at instant τ can be described as

$$q(\tau, t) = F[\tau, K, m] = F[\tau, K(\tau, t), m(\tau, t)], \tag{2.1}$$

where $F(\tau, K, m)$ is the production function for capital $K(\tau, t)$ and labor $m(\tau, t)$. It is chosen in [129] as the Cobb-Douglas function with exponential technical progress (see Section 0.1):

$$F(\tau, K, m) = e^{p\tau} m^{\alpha} K^{1-\alpha}, \quad 0 < \alpha < 1, \tag{2.2}$$

where $p > 0$ is the rate of increasing new EUs productivity (the *rate of embodied technological change*).

Then the total production output $Q(t)$ at time t is the aggregation of the outputs $q(\tau, t)$ over EUs from the current year t ad infinitum:

$$Q(t) = \int_{-\infty}^{t} F[\tau, K(\tau, t), m(\tau, t)] d\tau, \tag{2.3}$$

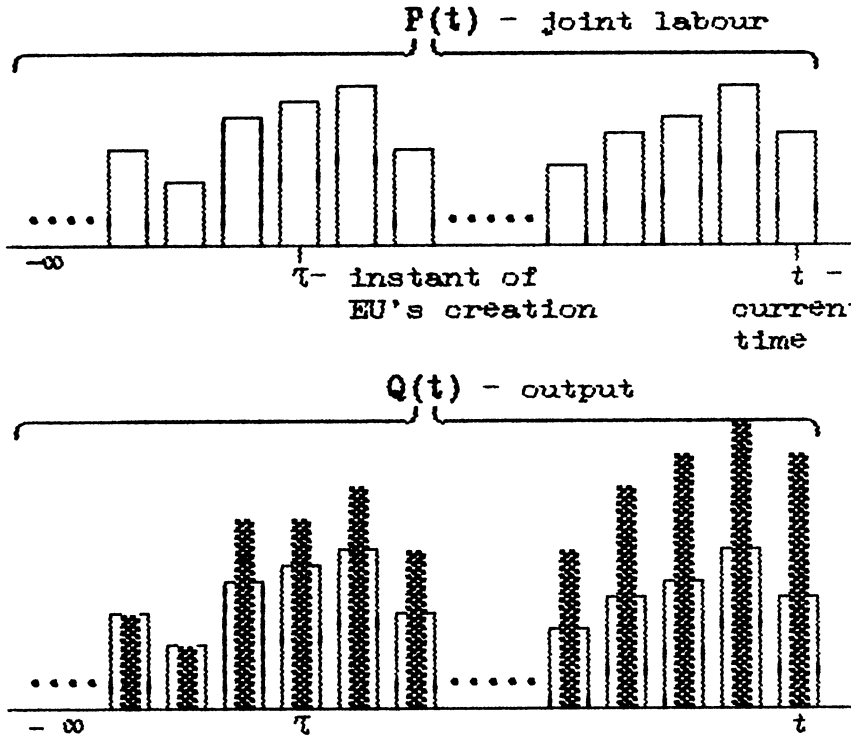

Figure 2.1 The scheme of the Solow integral model with embodied technological change

and total manpower $P(t)$ is

$$P(t) = \int_{-\infty}^{t} m(\tau, t) d\tau. \tag{2.4}$$

The equalities (2.3)-(2.4) describe the *integral Solow model.* It is supposed that the EUs are removed from operation due to their physical wear (decay, deterioration) with a constant wear rate $\delta > 0$:

$$K(\tau, t) = I(\tau) e^{-\delta(\tau - t)}, \tag{2.5}$$

where $I(\tau)$ is gross investment at instant τ.

Graphic scheme of the model (2.3)-(2.5) is given in Figure 2.1.

Henceforth we shall often exploit the following definition.

Definition. The value $r(t) = f'(t)/f(t)$ is called the *increase rate of function* $f(t)$. If $r(t) = const$ (the constant rate), then $f(t) = Ce^{rt}$.

As was shown in [19], at the constant rates of wear δ and EU productivity growth p the integral Solow model (2.2)-(2.5) leads to the same conclusions as the aggregate (non-integral) Solow model with exponential autonomous TC (i.e., the Ramsey model exposed in Section 0.1). But this result is of a particular character. In the general case of nonuniform economic development the integral models possess principally new features which are not inherent in other types of economic-mathematical models. Such features are the subject of the present book.

In contrast to Solow model, the integral models given below allow us to manage *the liquidation (scrapping, winding-up, retirement, elimination) of obsolete capacity (capital, fixed assets, equipment).*

2.2 KANTOROVICH INTEGRAL MODEL (MODELING OF OBSOLETE EQUIPMENT LIQUIDATION)

The integral macroeconomic models seem to have been used for the first in mathematical economics by Leonid Kantorovich in 1959 [74]. However, the model introduced in [74] was not investigated further, and, in 1973, Kantorovich proposed another clustered macroeconomic model [75] which was studied systematically in [76], etc.

The model [75] is the improvement of the Solow model (2.2)-(2.5). It is described by the following equations:

$$Q(t) = \int_{a(t)}^{t} F[\tau, K(\tau), m(\tau)]d\tau, \tag{2.6}$$

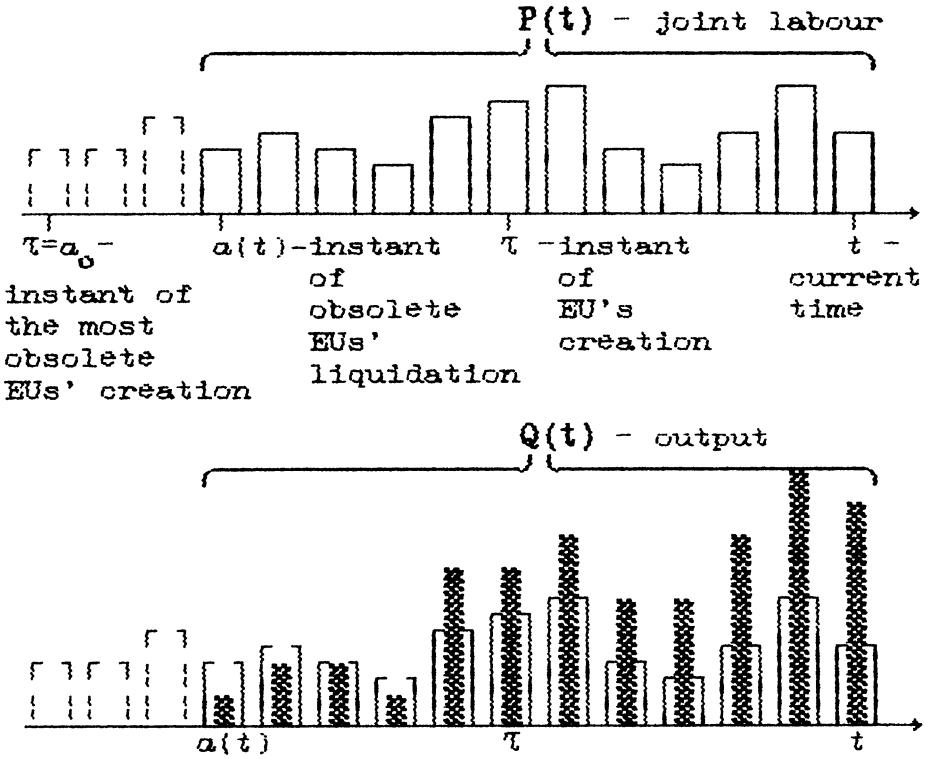

Figure 2.2 The scheme of the Kantorovich integral model with the liquidation of obsolete equipment

$$P(t) = \int_{a(t)}^{t} m(\tau)d\tau, \tag{2.7}$$

where, as in the above model, $m(\tau)$ is the quantity of EUs introduced into economic system per time unit at instant τ. The physical wear of EUs is not considered in the model (2.6)-(2.7) ($\delta = 0$).

The principal novelty of this model as compared with the Solow model (2.2)-(2.5) is the introduction of new endogenous function a(t) which is *the time limit of EU use* : the EUs created before $a(t)$ instant are not used at the current

time t. The distinction between Solow and Kantorovich models is clear from the Figure 2.2.

Let us fulfill some transformation of model (2.6)-(2.7):

$$q(\tau,t) = F(\tau,K(\tau),m(\tau)) = e^{p\tau}m^{\alpha}(\tau)K^{1-\alpha}(\tau) =$$

$$= e^{p\tau}[K(\tau)/m(\tau)]^{1-\alpha}m(\tau) = e^{p\tau}\lambda^{1-\alpha}m(\tau) := \beta(\lambda(\tau),\tau)m(\tau).$$

where $\lambda(\tau) := K(\tau)/m(\tau)$ is *EU cost* (the expenditure per one EU creation) and $\beta(\lambda(\tau)) := e^{p\tau}\lambda^{1-\alpha}(\tau)$ is *EU productivity*. Then the model (2.6)-(2.7) can be written in the form:

$$Q(t) = \int_{a(t)}^{t} \beta(\lambda(\tau),\tau)m(\tau)d\tau,$$

$$P(t) = \int_{a(t)}^{t} m(\tau)d\tau, \tag{2.8}$$

which is linear with respect to m. This form of the Kantorovich integral model is more convenient for our purpose and will be used in what follows.

The model (2.8) turns out to be basic in many more complicated problems which are discussed later on. In Section 6.2 we investigate an optimal control problem with sought-for a, m and λ for this model.

2.3 MODELS OF CAPITAL SCRAPPING IN MARKET ECONOMICS

The dynamic self-organization model of market competition under embodied TC was proposed in [124] and developed in [125]. It provides the description of technology as some amount of industrial equipment with given (but incompletely known) characteristics in the face of uncertainty with respect to future TC course. In [124] and [125] the model of "capital scrapping" for each firm in an industry is represented by the following formulas:

$$K(t) = \int_{a(t)}^{t} k(\tau)d\tau, \tag{2.9}$$

$$< A > (t) = \int_{a(t)}^{t} A(\tau)k(\tau)d\tau, \tag{2.10}$$

where $K(t)$ is the capital stock (measured in units of productive capacity), $k(t)$ is gross investment at time t, $a(t)$ is a scrapping date for technologically obsolete equipment, $A(t)$ is historical technological labor/output coefficient, $< A(t) >$ is prime unit labor coefficient. From the mathematical point of view, this model is similar to (2.8). The model (2.9)-(2.10) is used in Section 6.1 for some problem of technology lifetime optimization. Note that the integral model very similar to (2.9)-(2.10) was first proposed for a separate firm in [96].

In [124] and [125] some rules are supposed to determine the instants of obsolete technology (equipment) replacement, and other empirical economic approaches for this purpose are briefly mentioned (see also [127]). As outlined in [125], the evolutionary model with equations (2.9)-(2.10) may be categorized as a set of differential-difference equations with age-dependent effect whose mathematical properties are poorly understood even in the simplest cases.

2.4 INTEGRAL PRODUCTION FUNCTION

A *parametric production function*, at first sight different from previous models (2.3)-(2.10), was suggested in [107] for modeling of developing economics. It is described as

$$Q(t) = \int_{\varphi(t)}^{\psi(t)} m(\tau, \chi) d\chi,$$
$$P(t) = \int_{\varphi(t)}^{\psi(t)} \chi m(\tau, \chi) d\chi \tag{2.11}$$

and sets the manpower P dependence of output Q through given EUs distribution $m(\tau, \chi)$ and sought-for parameter $\psi(.)$. In (2.11) the distribution $m(\tau, \chi)$ of operated EUs is built in accordance with *the EU specific labor expenditure* χ rather than with the time τ of EUs creation as in the above IDMs. So, the integrals are evaluated for independent variable χ, and the limit $\psi(.)$ of integration is the variable sought for. It was shown in [68], [146] that, under sufficiently natural economic assumptions, the substitution of the independent variables $\chi \to \tau$ in accordance with their economic content reduces the model (2.11) to that of the form (2.8).

Integral economic models more general than (2.11) but based on similar assumptions were studied in [70]. In Johansen models the production units are

distinguished by their specific expenditures of several independent economic factors (not only by the labor expenditure χ), and thus involve multiple integrals and multidimensional integral equations.

The multi-sector integral models described below enable us to describe simultaneously the processes of EUs renovation and the distribution of operated EUs between production sectors, as distinct from the one-sector IDMs.

2.5 GLUSHKOV TWO-SECTOR MODEL (MODELING OF SELF-IMPROVEMENT)

Further development and study of integral macroeconomic models were conducted by the school of Victor Glushkov who independently of Leonid Kantorovich suggested the following two-sector model of macroeconomics renovation [45]:

$$m(t) = \int_{a(t)}^{t} \alpha(\tau, t) y(\tau, t) m(\tau) d\tau, \tag{2.12}$$

$$c(t) = \int_{a(t)}^{t} \beta(\tau, t)[1 - y(\tau, t)] m(\tau) d\tau, \tag{2.13}$$

$$P(t) = \int_{a(t)}^{t} m(\tau) d\tau. \tag{2.14}$$

This model describes a macroeconomic system consisting of two production sectors: the production of production means (A) and the production of consumption goods (B). The two-sector IDM is assumed to be close, i.e. new EUs are not entered into economic system from outside but are produced inside the system. Here $c(t)$ is the output of the consumption goods per time unit, $\alpha(\tau, t)$ and $\beta(\tau, t)$ are the productivities (output/labor coefficients) at time t of the EUs created at instant τ (in two sectors, respectively), $y(\tau, t)$ is a variable accumulation norm.

The IDM (2.12)-(2.14) takes into account the embodied TC which means that newer EUs are more effective than older ones (function $\beta(\tau, t)$ increases in τ). The interconnection among the model functions is illustrated in Figure 2.3.

This model ensures simultaneous optimization of the strategies of EUs renovation and the distribution of EUs between sectors A and B. The strategy of

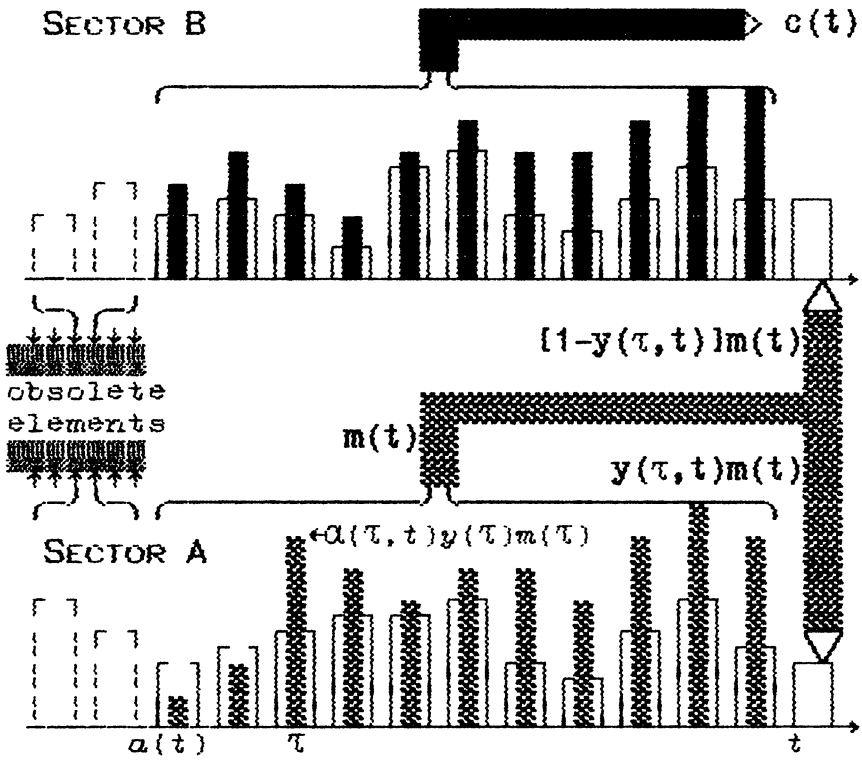

Figure 2.3 The scheme of the Glushkov two-sector integral model

EUs renovation is determined by the EUs lifetime $t - a(.)$ and by the quantity $m(.)$ of new created EUs.

Compared to the above one-sector models, the only new control function is the EU distribution y between production sectors. Let us analyze the content of the control function $y(\tau, t)$ in more detail.

The case $y \equiv y(t)$ of the model (2.12)-(2.14) provides the control of the distribution of only new EUs created at instant t between sectors A and B. In so doing, the distribution of all older EUs is fixed: the EUs cannot pass from one sector

to another during all their lifetime $[a(t), t]$. In mathematical economics similar models are known as the *"putty-clay"* ones (the models with fixed structure).

The function $y \equiv y(\tau, t)$, in general, controls the distribution of both new and old operated EUs between sectors A and B (the *"putty-putty" models*).

It should be noted that *the full-load hypothesis of all operated EUs* during all their lifetime is accepted in the model (2.12)-(2.14). In the opposite case, the new control $z(\tau, t)$ analogous to $y(\tau, t)$ should be introduced for sector B (and in the equality (2.13)) instead of $[1-y(\tau, t)]$) and the equality (2.14) takes other more complicated form [45].

From mathematical point of view the model (2.12)-(2.14) is described by the systems of the Volterra-type equations with unknown function a in the lower limits of integration. It incorporates the unknowns $m(.)$ and $y(.)$ alongside the sought-for limit $a(.)$ of integration. The theory and numerical methods for such models have actively evolved at the Glushkov Cybernetics Institute of Ukrainian Acad.Sci. for the last two decades.

Various mathematical problems concerning the model (2.12)-(2.14) and its generalizations have been explored in [48], [67], [146], [147], etc. They are considered in Chapter 8 of this book.

In the special case where $y \equiv 0$ the one-sector model (2.8) is obtained from the model (2.12)-(2.14).

Integral Model of Evolving Systems

We will also use more general interpretation of the IDM (2.12)- (2.14) as the basic *two-sector model of evolving system (ES)* (see Section 1.6). In accordance with Section 1.6, the ES consists of separate *system elements (SEs)* which execute certain *ES functions* (external and internal). The SEs differ by their functions, the times of their creation, and the efficiency of function execution.

Under imposed assumptions, the two-sector IDM (2.12)-(2.14) describes the ES consisting of two subsystems : the self-improving subsystem A which ensures the creation of new SEs and the liquidation of obsolete SEs, and subsystem B which ensures the execution of external uniform aggregate function. Then :

- $c(t)$ is the output of aggregate external production of ES per time unit;

- $m(t)$ is the quantity of SEs which are created in ES subsystem A and entered into ES at instant t;
- $\beta(\tau, t)$ is the SE efficiency in subsystem B: the external production output per time unit at time t by one SE created at instant τ;
- $\alpha(\tau, t)$ is the SE efficiency in subsystem A: the quantity of new SEs which are created per time unit at time t by one SE created at instant τ;
- $P(t)$ is the quantity of all operated SEs;
- $a(t)$ is the time limit of obsolete SEs elimination (SEs created before instant $a(t)$ are not operated at time t);
- $y(\tau, t)$ is the relative part of SEs created and entered into ES at instant τ, which are operated in subsystem A at time t.

2.6 MULTI-SECTOR INTEGRAL MODELS

Multi-sector integral economic model of the form [146] :

$$\begin{aligned} m_i(t) &= \sum_{j=1}^{n} \int_{a_j(t)}^{t} \alpha_{ij}(\tau, t) y_{ij}(\tau, t) m_j(\tau) d\tau + f_i(t), \quad i = \overline{1, n}, \\ c_k(t) &= \sum_{j=1}^{n} \int_{a_j(t)}^{t} \beta_{kj}(\tau, t) z_{kj}(\tau, t) m_j(\tau) d\tau, \quad k = \overline{1, l}, \\ R_s(t) &= \sum_{j=1}^{n} \int_{a_j(t)}^{t} r_{sj}(\tau, t) m_j(\tau) d\tau, \quad s = \overline{1, p}, \end{aligned} \tag{2.15}$$

$$a_i(t) < t, \quad y_{ij}(\tau),\ z_{kj}(\tau) \geq 0, \quad \sum_{i=1}^{n} y_{ij}(\tau) + \sum_{k=1}^{l} z_{kj}(\tau) \leq 1,$$

describes an economic system which uses the capacities of n types m_i, produces l various kinds c_k of consumption goods, and consumes p different resources R_s (labor, capital, energy, raw material, etc.). Capacity units can be used in the production of consumption goods as well as in the production of new capacity.

The structure of model (2.15) is determined by interconnections in an economic system and is given by the matrices $\{\alpha_{ij}\}$, $\{\beta_{kj}\}$, $\{r_{sj}\}$, $\{y_{ij}\}$, $\{z_{kj}\}$. The structure of integral operators in (2.15) is the same as in the above low-sector

models. The model (2.15) is a natural generalization of one-sector model (2.8) and two-sector model (2.12)-(2.14), and is examined in Chapter 11.

It is convenient to use "input-output" approach to identify the given function α_{ij}, β_{kj} in the model (2.15). The correlation between input-output models [88], [89] and the multi-sector integral model (2.15) was investigated in [146]. In particular, it was shown that under certain simplified assumptions (the absence of embodied TC and obsolete EUs scrapping, constant accumulation norm, etc.) the model (2.15) is equivalent to a closed input-output model (in continuous time t).

2.7 CONNECTION WITH THE DIFFERENTIAL MODELS OF EQUIPMENT REPLACEMENT

The models of the equipment replacement also belong to economic- mathematical models for which the service time of production elements is of importance. This follows from the fact that the expenditure for maintenance and repairs of equipment unit (EU) depends on EU age.

The models of dynamics of the EU age structure are often described by means of partial differential equations. The simplest model is given by the following *evolutionary equation*:

$$\partial x(t,\tau)/\partial\tau + \partial x(t,\tau)/\partial t = -\alpha(t,\tau)x(t,\tau), \tag{2.16}$$

$$x(t,0) = p(t), \quad x(0,\tau) = \varphi(\tau),$$

where $x(t,\tau)$ is the density of EU distribution at time t in accordance with EU age τ, $p(t)$ is the quantity of the new EUs which are placed into operation at time t, $\varphi(\tau)$ is the given initial EU age distribution at time $t = 0$, $\alpha(t,\tau)$ is the failure coefficient for EUs of age τ at time t.

Note that the evolutionary equation (2.16) is often used for the modeling of ecological populations with regard to their age structure (see Chapter 3).

The total quantity of operating EUs is

$$P(t,Z) = \int_0^Z [1 - f(t,\tau)]x(t,\tau)d\tau,$$

where $f(t,\tau)$ is the part of repaired EUs of age τ, and Z is the EU service time that may be considered as a control parameter.

The integral dynamical model equivalent to the model (2.16) can be constructed in a way analogous to that used below in Chapter 3 for modeling of ecological populations.

The model (2.16) can be used for optimization of the EU service time Z. For example, the total expenditures for exploitation of production system on interval $[0,T]$ may be described as

$$S(T,Z) = \int_0^T \Big(c(t)p(t) + \int_0^Z x(t,\tau)\Big(\gamma(t,\tau) + \\ +r(t,\tau)f(t,\tau) + q(t,\tau)(1-f(t,\tau))\Big)d\tau\Big)dt,$$

where $c(t)$, $r(t,\tau)$, $\gamma(t,\tau)$, $q(t,\tau)$ are respectively the specific expenditures for manufacture (purchase), repair, spare parts and maintenance work per one EU at time t.

It is useful to remark that similar differential models describe the constant EU service time Z. If the time Z is a variable value, then it is more convenient to use the IDMs with controllable memory considered in Sections 2.2-2.5.

2.8 ARISING MATHEMATICAL PROBLEMS: PREDICTION AND OPTIMIZATION

From the economic viewpoint, different statements of mathematical problems are of interest in the IDMs under study.

First, there is a deep *problem of model identification* that we only touch on in the body of the book. It consists in the determination of unknown kernels of the IDMs (functions of the type β, α). Such problems are usually incorrect from the mathematical point of view, and they require involving an additional economic information. A technique for the identification of one- and two-sector IDMs, which is based on standard economic information, is proposed in Chapter 14.

2.8.1 Prediction Problems

If the specific internal functions of type β, α are given in the IDMs (2.2)-(2.14) and problem is completely defined (i.e. the number of unknowns coincides with the number of equations), then one obtains a prediction problem for an economic process under question. In so doing, it is essential to determine completely the behavior of all sought-for functions on a prehistory of the process. From the mathematical point of view, the prediction problems are equivalent to solving Volterra-type integral equations (see Section 1.6.2).

To analyze intensive economic development under technological change, the prediction problems with the control of liquidation of obsolete capacities are of central interest that implies the endogenous (sought-for) technology lifetime. Such problems are described by means of the Volterra-type integral equations with unknown lower limits of integration. The theory of such equations is developed in Chapter 7, and the numerical methods — in Section 13.2.

2.8.2 Optimization Problems

Optimization problems (OPs) are traditionally of special interest in the theory of mathematical-economic modeling, which has been reflected in the study of economic IDMs. For the class of IDMs under discussion the basic OP was most thoroughly studied by the time being [48], [49], [67], [146]. This problem consists in maximization of the output $c(.)$ of consumption goods on a planned horizon $[t_0, T]$ under given dynamics of manpower $P(.)$ in the two-sector Glushkov model (2.12)-(2.14). Here the lifetime $t - a(t)$ of technologies is considered among sought-for control functions, which is a radically new feature in comparison with other similar investigations. This OP, as well as its modifications for one-, two-, and multi-sector IDMs, is studied in detail in this book. The analysis of the asymptotic behavior of OP solutions provides an analogy of turnpike features for optimal trajectories, in particular, for technology lifetime. These features are simple to understand, though the mathematical treatment may become complicated.

The fact that the various economic-mathematical models [45], [75], [96], [107], [124] based on similar mathematical methodology (the IDMs with controllable memory) have been developed independently in order to describe the economic renovation (replacements and renewals of various capital investments, fixed assets, productive capacities, and industrial equipment) indicates the high priority of the investigation of such models.

3

INTEGRAL DYNAMICAL MODELS IN MATHEMATICAL ECOLOGY

Present-day mathematical models of biological communities usually take into account aftereffect and joint influence of different exogenous and endogenous factors. Such models are based on various mathematical means, including integral equations. Integral and integral-differential equations are used in ecological modeling primarily to account for two following important factors: ***hereditary (delay) effects*** arising because of ecological system inertness, the ***age structure*** of biological community (population). To analyze the peculiar features of ecological IDMs, here we briefly consider some familiar models of both mentioned fields. Note that there are deep parallels between some models of economic and ecological processes that will be traced in the body of this book.

3.1 ECOLOGICAL MODELS WITH DELAY

The first and the best known models of delay in biological communities were proposed by Vito Volterra [141].

3.1.1 Volterra Model of Population Self-Intoxication

This model describes the self-intoxication process of one-species population by the waste products of its own metabolism. It is described by the following

integral-differential equation :

$$dN/dt = \mu N[1 - N/K - \int_0^t f(t-s)N(s)ds], \tag{3.1}$$

where $N(t)$ is *population size* (the number of individuals), μ is the *coefficient of population intrinsic growth*, and the multiplier $(1 - N/K)$ represents the *medium resistivity for population increase*. The model (3.1) is the modification of well-known *Verhulst-Pearl model* (logistic equation)

$$dN/dt = \mu N(1 - N/K), \tag{3.2}$$

in which the additional integral delay (hereditary) term is included for representing the decrease of growth rate μ due to catabolic effects. The integrand $f(t-s)$ of this term is called the *hereditary function* which determines level of the influence of prehistory of the process at instant s on population dynamics (in accordance with the distance $t - s$ of prehistory from current time t).

Under some assumptions the equation (3.1) has a unique solution N(t) which tends to the *equilibrium (stationary) state of population* :

$$\hat{N} = K(1 + K\int_0^\infty f(t-s)N(s)ds)^{-1} \quad \text{at} \quad t \to \infty.$$

3.1.2 Volterra "Predator-Prey" Model with Aftereffect

Another well-known Volterra model of delay in biological communities is the model of the coexistence of two antagonistic species, the so-called "predator-prey" model with aftereffect [141]. This model is described by the following system of two nonlinear integral-differential equations :

$$\begin{aligned} dN_1/dt &= N_1[\epsilon_1 - \gamma_1 N_2 - \int_{-\infty}^t F_1(t-s)N_2(s)ds], \\ dN_2/dt &= N_2[-\epsilon_2 + \gamma_2 N_1 + \int_{-\infty}^t F_2(t-s)N_1(s)ds] \end{aligned} \tag{3.3}$$

with respect to the numbers N_1 of prey and N_2 of predators. Here aftereffect indicates that the continuous sequence of past states of the biological community affects the future evolution of the community. As was shown in [141], there exists a stationary state of the model (3.3):

$$\hat{N}_1 = \epsilon_2(\gamma_2 + \int_0^\infty F_2(s)ds)^{-1}, \quad \hat{N}_2 = \epsilon_1(\gamma_1 + \int_0^\infty F_1(s)ds)^{-1}$$

such that the solutions $N_1(t)$ and $N_2(t)$ take the values $\hat{N}_1$ and $\hat{N}_2$ at an infinite number of instants, i.e., $N_1(t)$ and $N_2(t)$ execute infinite nonperiodic fluctuations around this stationary state.

The models (3.1) and (3.3) take into account the ***distributed delay*** on interval $[-\infty, t)$, which is usually known as a ***process aftereffect (contagion)***. If the ***lumped (point) delay*** is considered, it leads to ***delay (lag) differential equations*** (see [12], [37], [57], etc.). The latter behavior is characteristic for the populations with single seasonal reproduction.

3.2 AGE-DEPENDENT MODELS OF POPULATION DYNAMICS

The IDMs studied in this monograph correspond to the models of mathematical ecology that take into account population age-structure. Population age-distribution is one of the main endogenous (intrinsic) factors influencing on population dynamics. The fertility and mortality of individuals depend on their age, so the rational exploitation of population requires a consideration of the age structure.

It should be noted that widely disseminated models of population dynamics based on ***ordinary differential equations*** are valid only for the time intervals sufficiently long in comparison with the longevity of individuals. Such models do not provide an explanation for many experimentally observed phenomena whose duration is of the same order of magnitude as the longevity of an individual (for example, the oscillations of population frequency with such periods). It is thus necessary to take into account the age distribution. This requires the use of ***integral or partial differential equations.***

3.2.1 Lotka Model

The first model describing the dynamics of age-distributed single-species population in stationary environment under unlimited food resources was suggested in [118]. This model is described by the linear ***integral renewal equation*** (see also [77], [101], [108], [135], etc.) :

$$B(t) = B^{-}(t) + \int_0^t m(\tau)l(\tau)B(t-\tau)d\tau, \quad t \geq 0 \tag{3.4}$$

$$\text{where} \qquad B^{-}(t) = \int_t^T m(\tau)l(\tau)\phi(\tau - t)d\tau, \quad 0 \le t \le T.$$

Here $B(t)$ is the *birth-rate intensity* (the number of individuals arising per time unit at time t), $m(\tau)$ is the *age-specific fertility rate* (the average number of individuals produced per time unit by one individual of age τ), $l(\tau)$ is the *age-specific fatigue function* (the specific part of individuals of age τ living at time t), $\phi(\tau)$ is the initial distribution (at $t = 0$) of the individuals with respect to their age τ, T is maximal age.

By the change of variable $t - \xi \to \tau$ we obtain from the model (3.4) the following *integral model of population dynamics*:

$$B(t) = \int_{t-T}^{t} m(t - \xi)l(t - \xi)B(\xi)d\xi \tag{3.5}$$

with initial condition $B(\xi) = \phi(-\xi), \xi \in [-T, 0]$, whose form coincides with the IDMs of this monograph.

3.2.2 Nonlinear Models

The nonlinear modification

$$x(t) = \int_{t-T}^{t} f(s, x(s))ds, \quad f(t, 0) = 0 \tag{3.6}$$

of the integral model (3.5) is also often considered [56], [143], particularly, in connection with epidemic problems [30], [55], [86]. In such models it is assumed that the total population is constant, $x(t)$ is the population at time t of infectious individuals, $f(t, x(t))$ is the instantaneous rate of infection, and $f(t, x(t))dt$ is the fraction of individuals infected within the period $[t, t + dt]$. It is natural to assume that $f(t, x)$ is nondecreasing with respect to x (the more infectious, the more individuals can be infected). Then the value T is interpreted as *the duration of infection*. The number of infectious individuals at time t is obtained by summing up all individuals infected between $t - T$ and t, thus leading to the equation (3.6).

3.2.3 Connection with Differential Models

The age structure of biological populations can also be taken into account by means of partial differential equations (or the ordinary differential equations which are obtained after the discretization of a partial differential model with respect to individual age). Such approach is widely disseminated in applied simulation of ecological systems [108]. In this connection it is of interest to study the interrelation between integral models and their differential analogs.

We shall restrict ourselves to the case of stationary environment. The differential age-dependent models of population dynamics analogous to the linear IDM (3.4) are described by the following equalities [56], [132], [133], [143] :

the evolutionary equation :

$$\partial x/\partial\tau + \partial x/\partial t = -\hat{d}(\tau)x(\tau,t), \tag{3.7}$$

the fertility equation :

$$x(0,t) = \int_0^T m(\tau)x(\tau,t)d\tau, \tag{3.8}$$

the initial condition:

$$x(\tau,0) = \phi(\tau), \quad \tau \in [0,T] \tag{3.9}$$

with respect to the sought-for *density* $x(\tau,t)$ *of population age distribution* (the number of individuals of age τ at time t). The content of function $m(\tau)$ is the same as in (3.4), and $\hat{d}(\tau)$ is the *age-specific death rate* (specific part of the individuals of age τ died per time unit). The model (3.7)-(3.9) is known as the *Lotka–Von Foerster model.*

It was shown in [108] that solution x of the model (3.7)-(3.9) contains oscillatory components whose maximum period does not exceed twice the duration of reproduction age T. In the case of a single breeding, the maximum period is equal to the reproduction age, and the oscillations do not decrease.

In the model (3.7)-(3.9) the sought-for function $x(\tau,t)$ depends on individual age τ and current time t. Let us replace the age τ by the instant $\xi = t - \tau$ of individual birth and introduce the function $\hat{x}(\xi,t) = x(t-\xi,t)$. Then the equation (3.7) has the form : $\partial\hat{x}/\partial t = -\hat{d}(t-\xi)\hat{x}(\xi,t)$, and its solution is the following:

$$\hat{x}(\xi,t) = \hat{x}(\xi,\xi)\exp[-\int_\xi^t \hat{d}(\tau-\xi)d\tau]. \tag{3.10}$$

Then, substituting (3.10) into (3.8) and introducing new one- dimensional sought-for function $B(t) = \hat{x}(t,t) = x(0,t)$, we obtain the following *integral model*:

$$B(t) = \int_{t-T}^{t} m(t-\xi)\exp[-\int_{0}^{\xi-v} \hat{d}(v)dv]B(\xi)d\xi, \tag{3.11}$$

which is equivalent to the partial differential model (3.7)-(3.9). Hence, the fatigue function $l(\tau)$ in the integral model (3.4) is connected with death rate $\hat{d}(\tau)$ by the relation:

$$l(\tau) = \exp[-\int_{0}^{\tau} \hat{d}(\xi)d\xi].$$

One of the key advantages of integral models over differential ones consists in *reduction of the dimension of sought-for function* that facilitates theoretical and numerical treatment of the models. The analogous transformation from differential model to integral one is also possible in more general cases taking into account the nonstationarity of environment state (when the functions $m(\tau,t)$, $\hat{d}(\tau,t)$ and $l(\tau,t)$ depend explicitly on time t), limited resources, intraspecific competition, and other population factors [108], [133], [143].

3.3 MATHEMATICAL PROBLEMS: BIFURCATION ANALYSIS

The statements of the mathematical problems in economic IDMs exposed above in Chapter 2 remain in force for ecological applications. There are identification, prognosis, , and optimization problems. Feature peculiar to these problems is that the lower limit of integration in the IDMs is usually a given parameter of the model.

The mathematical problems of new type also arise here. They are connected with studying *oscillatory and stationary solutions* of the model and finding the *bifurcational values* of parameters at which new stationary solutions appear. Similarly to other types of operator equations such problems can be referred to as *eigen-value problems* for the IDMs. For example, in relation to epidemics it is of major interest to see whether a disease, once it enters the population, can persist or dies out, and how much this depends on various parameters of a model, notably, the duration T of infective period. In terms of mathematics, this involves the problems of finding steady-state solutions (constant, periodic,

almost periodic, etc.) and analyzing their stability [5], [30], [55], [86]. Similar problems are also of certain interest for economic IDMs. For this reason we briefly consider the peculiarities of solving these problems.

We shall restrict our consideration to the simplest age-dependent nonlinear IDM. The stationary states of ecological systems correspond to:

- the *stationary age density* $x(\tau,t) \equiv x_S(\tau)$ in differential models of the type (3.7)-(3.9), for their study it is assumed $\partial x/\partial t \equiv 0$ that leads to the ordinary differential equations with respect to $x_S(\tau)$;
- the *constant total fertility* $B(t) \equiv B_S = const \geq 0$ in integral models of the type (3.11), for their study it is sufficient to substitute B_S into the model and to find the roots of obtained nonlinear equation.

The age structure of single-species population in stationary environment with regard to intraspecific competition can be described by the following nonlinear IDM [146] :

$$B(t) = \int_{t-T}^{t} m(t-\xi)\exp\Big(-d(t-\xi) - \int_{t-T}^{t} b(t-\xi, t-\theta)B(\theta)d\theta\Big)B(\xi)d\xi, \quad (3.12)$$

where the function $b(\tau,\xi)$ characterizes the increase of the mortality of individuals of age τ, which caused by the individuals of age t (*intraspecific competition*). Other functions are the same as in the linear IDM (3.11).

Let the fertility rate be $m(\tau) = \lambda\bar{m}(\tau)$, where λ is some parameter which determines the intensity of fertility but does not affect its structure. We may take λ to be directly proportional to food amount. Then the model (3.12) is of the form:

$$B(t) = \lambda\int_0^T \bar{m}(\tau)\exp\Big(-\hat{d}(\tau) - \int_0^T b(\tau,\xi)B(t-\xi)d\xi\Big)B(t-\tau)d\tau. \quad (3.13)$$

Substituting $B(t) \equiv B_S$ into (3.13) gives the following nonlinear equation with respect to stationary solutions B_S :

$$B_S\Big(1 - \lambda\int_0^T \bar{m}(\tau)\exp\Big(-\hat{d}(\tau) - B_S\int_0^T b(\tau,\xi)d\xi\Big)d\tau\Big) = 0. \quad (3.14)$$

It follows from (3.14) that for any $\lambda > 0$ the trivial stationary state $B_S \equiv 0$ exists and is unique for small λ. Under increasing λ the model has not other stationary states up to some value $\lambda = \lambda^*$ which is determined by the equality (see also [40]) :

$$R(\lambda) := \lambda \int_0^T \bar{m}(\tau) \exp[-\hat{d}(\tau)] d\tau = 1 \tag{3.15}$$

The value R called the ***reproduction number (biological potential) of population*** describes the average number of descendants of one individual during his lifetime (for stable nature populations $R \approx 1$). The value $\lambda = \lambda^*$ turns to be ***bifurcational***: for $\lambda > \lambda^*$ nontrivial stationary states $B_S^* > 0$ exist. In other words, the point $(\lambda^*, 0)$ of plane (λ, B) is the ***ramification point*** of nonlinear equation (3.14). Namely, there are three solution branches in the neighborhood of λ^* : zero branch (corresponds to stable solutions B_S^0) at $\lambda < \lambda^*$, zero branch (unstable solutions B_S^0) and positive one (stable solutions $B_S^0 > 0$) at $\lambda > \lambda^*$.

The described above qualitative picture of bifurcational regimes also takes place for the ***nonlinear evolutionary equation***:

$$\partial x/\partial \tau + \partial x/\partial t = -\Big(\hat{d}(\tau) + \int_0^T b(\xi, \tau) x(\tau, \xi) d\xi\Big) x(\tau, t)$$

describing the same population as the IDM (3.12), i.e., these models are equivalent on a qualitative level.

In conclusion, the similarity of mathematical models used for modeling of different economic and ecological processes should be stressed. Thus, the ***Verhulst-Pearl model*** (3.2) (as well as another well-known ***Lotka-Volterra model*** [143]) can describe ecological population dynamics (Section 3.1.1) and innovation substitution processes ([20], [41], Section 0.1); the ***evolutionary equation*** (3.7) describes the age structure of population (Section 3.2.3) and the equipment replacement in manufacturing systems (Section 2.7); the ***diffusion equation*** describes migration of individuals in ecology [108], pollution propagation in air and water medium, and propagation of technological innovations (see [20], [32], [38], and Section 15.1).

PART II

ANALYSIS OF ONE-SECTOR INTEGRAL DYNAMICAL MODELS

4

BASIC OPTIMIZATION PROBLEM IN ONE-SECTOR MODEL

Optimal trajectories in the considered IDMs [67], [117], [146], [147] have a number of complicated features caused by rather rigid restrictions on unknown phase and control variables. That is why a basic one-sector model is chosen here, which is more simple than other IDMs. This made it possible to obtain essential results in the study of optimal control behavior, among them the explicit solutions of the nonlinear OP and the proof of turn-pike theorems for important particular cases.

A basic optimization problem (OP) for this simplest IDM is studied in detail in this and next Chapters. Technology lifetime is considered as a sought-for control function what is a radically new feature in comparizon with other similar investigations. The necessary and sufficient conditions of extremum are established, and asymptotical behavior of the OP solutions is studied for a large horizon as well as for the infinite horizon $[t_0, \infty)$. The final results obtained in Chapter 5 are the structure of OP solution and the turnpike theorems for optimal technology lifetime.

4.1 THE DESCRIPTION OF BASIC ONE-SECTOR MODEL

The simplest one-sector integral economic model with the elimination of obsolete production units (obsolete EUs) is described by the following equations (see Section 2.2-2.3):

$$Q(t) = \int_{a(t)}^{t} \beta(\tau, t) m(\tau) d\tau, \tag{4.1}$$

$$P(t) = \int_{a(t)}^{t} m(\tau) d\tau. \tag{4.2}$$

The economic content of model functions is the following :

- $Q(t)$ - total output of uniform product per time unit at time t;
- $m(t)$ - the quantity of EUs which are created at time t;
- $\beta(\tau, t)$ - the *productivity of EU*: the product output per time unit at time t on one EU created at time τ;
- $P(t)$ - the labor force at time t (the quantity of operated EUs);
- $a(t)$ - the time limit of obsolete EUs elimination (EUs created before instant $a(t)$ are not used at time t).

The given system characteristics of modeled economic system are the productivity $\beta(\tau, t)$ of EU, the cost $\lambda(t)$ of EU, and manpower $P(t)$. The control of the economic system is carried out by change of the renovation intensity of its technological structure. More exactly, the sought-for control functions are the lifetime $t - a(t)$ of EUs and the sought-for quantity $m(t)$ of the new EUs which are introduced into the system. Then the function $L(t) = t - a(t)$ describes the sought-for lifetime of EUs (in the sense mentioned in Preface).

The model takes into account the embobied technical change [19], [129], which means that newer EUs are more effective than the older ones (the function $\beta(., t)$ increases).

Let us analyze the essential restrictions for sought-for variables. At first, $0 \leq m \leq M(t)$, where a maximal possible investment $M(t)\lambda(t)$ is determined, for example, by financial constraints of a firm. It is also natural to suppose that $a'(t) \geq 0$, i.e. scrapped capacity cannot be used again. Finally, as the model (4.1)-(4.2) now stands, the single technological structure of the economic system is known which is defined by the given investments m on a pre-history interval $\in [a(t_0), t_0]$:

$$a(t_0) = a_0, \quad m(\tau) = m_0(\tau), \quad \tau \in [a_0, t_0]. \tag{4.3}$$

4.2 THE STATEMENT OF OUTPUT MAXIMIZATION PROBLEM

The mathematical statement of the optimization problem (designated below as **OP1**) consists in the determination of the sought-for functions $m(t)$, $Q(t)$, $a(t)$, $t \in [t_0, T)$, $T \leq \infty$, which maximize the following functional:

$$I := I_1 = \int_{t_0}^{T} \rho(t)[Q(t) - \lambda(t)m(t)]dt \longrightarrow \max_{a,m}, \tag{4.4}$$

under the constraints-equalities (4.1)-(4.2) and the following inequalities:

$$0 \leq m \leq M(t), \tag{4.5}$$

$$a'(t) \geq 0, \quad a(t) < t, \tag{4.6}$$

and the initial conditions (4.3). The given function $\rho(t)$ is the discounting factor which reflects decreasing the utility of output produced at more distanced future time, $\rho' < 0$ [19].

The economic interpretation of OP1 consists in the maximization of the discounted aggregate uniform output of an economic system with taking into account the expenditures on putting new system elements into operation. These elements may be the equipment units (EUs), working places, productive capacity, fixed assets, etc.

Let us assume that the given functions $\rho(.)$, $\beta(.,.)$, $P(.)$, $M(.)$ are Lipschitz continuous, and $m_0(.)$ is piece-wise continuous. All these functions are positive and satisfy (4.1)-(4.3), (4.5) at $t = t_0$.

4.3 INVESTIGATION TECHNIQUES FOR OPTIMIZATION PROBLEMS

Methods of exploration of the OPs in IDMs under consideration rely on the reduction of the problems with mixed differential constraints on phase variable

$a(.)$ to the simpler ones by imposing additional constraints on given functions, narrowing the domain of admissible control variables, introducing additional variables and equations, etc. [146].

Qualitative analysis of the structure of OP solutions depends upon derivation of the gradient $I'(.)$ of maximized functional I by means of Lagrange multipliers [14]. In many important cases the equation $I'(t) = 0$ leads to the special integral-functional (IFE) equation with respect to $a(.)$, which is not Volterra-type equation (see Section 5.1). Analysis of the solutions of this IFE made it possible to find an analogy of turnpike properties of the optimization problem (see Section 5.2).

4.4 PRELIMINARY RESULTS

Let $m(t)$, $t \in [t_0, T)$ be *the independent control variable* of OP (4.1)-(4.6), $m \in L^\infty[t_0, T)$. Then the functions $Q(t)$, $a(t)$, $t \in [t_0, T)$ are *the phase* (dependent) sought-for *variables* of OP. Define U as a set of the control variables $m(.)$ corresponding to the condition (4.5). The constraint (4.5) is inconvenient and optional. Following the investigation technique for similar OPs, one can replace the constraint (4.6) for phase variable $a(.)$ by more strict one for the control $m(.)$.

Theorem 4.1. For any measurable control $m(t)$, $t \in [t_0, T)$ which satisfies almost everywhere (a.e.) for $[t_0, T)$ the restriction:

$$m_{min}(t) \leq m(t) \leq M(t), \tag{4.7}$$

where

$$m_{min}(t) = \max\{0, P'(t)\}, \tag{4.8}$$

there exists unique a.e. continuous function $a(t)$, $t \in [t_0, T)$ which meets (4.2) and (4.3), and has a.e. $a'(t) \geq 0$.

Proof. Let us transform (4.2) to the following form:

$$P(t) = \int_{a(t)}^{t_0} m_0(\tau)d\tau + \int_{t_0}^{t} m(\tau)d\tau,$$

and introduce the notations:

$$F(a(t)) = \int_{a(t)}^{t_0} m_0(\tau)dt; \quad X(t) = P(t) - \int_{t_0}^{t} m(t)d\tau.$$

Consider the interval $[t_0, \min(T, t_1)]$, where the time t_1, $0 < t_1 \leq T$, is determined from the condition $X(t_1) = 0$. Since $F'_a(a(t)) = -m_0(a(t)) \leq 0$, the function $F(a)$ does not increase. Hence, the unique a.e. continuous function $a(t) = F^{-1}(X(t), t)$ exists for $t \in [t_0, \min(T, t_1)]$ and $a(t_1) = t_0$. Next $X(t)$ does not increase because $X'(t) = P'(t) - m(t) \leq 0$ under the condition (4.8). Therefore, $a(t)$ does not decrease as the superposition of non-increasing functions, so there is $a'(t)$ a.e. such that $a'(t) \geq 0$.

Now one can prove the same result for interval $[t_1, \min(T, t_2)]$ such that $X(t_2) = 0$, and so on, until $t_k = T$ is reached.

Theorem is proved.

Thus, the restriction (4.7) is more rigid than (4.5), and the corresponding control set $U_1 \subset U$. Hence, the OP (4.1)-(4.6) with the phase constraint (4.6) may be replaced by the problem (4.1)-(4.7) with more narrow domain of admissible controls (4.7), for which phase constraint (4.6) is satisfied a priori. All solutions of the OP (4.1)-(4.7) will belong to the range U of admissible controls of OP (4.1)-(4.6). Next, the OP (4.1)-(4.7) is considered.

The one-to-one solving of the equation (4.3) with respect to phase variable $a(.)$ for any admissible controls $m(.)$ from the domain (4.7) allows us to derive the gradient of functional (4.4) and to establish the necessary and sufficient conditions of extremum (n.s.c.e.). Note that we can use "classical" small variations $\delta m \in L^\infty[t_0, T) : |\delta m(t)| \ll 1$, $t \in [t_0, T)$ as well as needle-shaped ones such that $\delta m(t) \neq 0$, $t \in \Delta_m \subset [t_0, T)$, $mes(\Delta_m) \ll 1$.

Lemma 4.1. If $m(t) > 0$, $t \in [t_0, \underline{a}(T))$, then abstract function $I(m)$ in OP1 is differentiable, and the gradient of functional (4.4) is of the following form:

$$I'(t) = \int_t^{\underline{a}^{-1}(t)} \rho(\tau)[\beta(t, \tau) - \beta(a(\tau), \tau)]d\tau - \rho(t)\lambda(t), \quad t \in [t_0, T), \tag{4.9}$$

where $\underline{a}^{-1}(t) = \begin{cases} a^{-1}(t), \ t_0 \leq t \leq a(T), \\ T, \ a(T) \leq t \leq T, \end{cases}$, a^{-1} is the inverse function of a.

The increment of functional (4.4) in OP1 is of the following form:

$$\delta I = I(m+\delta m) - I(m) = \int_{t_0}^{T} I'(t)\delta m(t)dt + \delta^2 I, \tag{4.10}$$

$$\delta^2 I = \int_{t_0}^{T} \rho(t) \int_{a(t)}^{a(t)+\delta a(t)} [\beta(a(t),t) - \beta(\tau,t)][m(\tau)+\delta m(\tau)]d\tau dt, \tag{4.11}$$

where the admissible variations $\delta m(t)$, $\delta a(t)$, $t \in [t_0, T)$ of functions $m(t)$, $a(t)$ must satisfy the equality:

$$\int_{\underline{a}(t)}^{t} \delta m(\tau)d\tau = \int_{a(t)}^{a(t)+\delta a(t)} [m(\tau)+\delta\tilde{m}(\tau)]d\tau, \tag{4.12}$$

$$\overline{a}(t) = \max\{a(t), t_0\}, \quad t \in [t_0, T), \quad \delta\tilde{m}(\tau) = \begin{cases} \delta m(\tau), & t_0 \le \tau \le T, \\ 0, & 0 \le \tau \le t_0, \end{cases}$$

Proof. Give some increments δm, δa for sought-for variables m, a, i.e. consider the functions $m(t)+\delta m(t)$, $a(t)+\delta a(t)$, $t \in [t_0, T)$. We can use only admissible variations $\delta m(t)$, $t \in [t_0, T)$, i.e. such that do not violate the conditions (4.3), (4.6) and (4.7). In view of equation (4.3), these increments are connected by the equality (4.12).

Let us determine the increment of functional (4.4) caused by the increment δm:

$$\delta I = I(m+\delta m) - I(m) = \int_{t_0}^{T} \rho(t)[\int_{\underline{a}(t)}^{t} \beta(\tau,t)\delta m(\tau)d\tau -$$

$$- \int_{a(t)}^{a(t)+\delta a(t)} \beta(\tau,t)[m(\tau)+\delta\tilde{m}(\tau)]d\tau - \lambda(t)\delta m(t)]dt =$$

$$= \int_{t_0}^{T} \rho(t)[\int_{\underline{a}(t)}^{t} [\beta(\tau,t) - \beta(a(t),t)]\delta m(\tau)d\tau - \lambda(t)\delta m(t)]dt +$$

$$+ \int_{t_0}^{T} \rho(t) \int_{a(t)}^{a(t)+\delta a(t)} [\beta(a(t),t) - \beta(\tau,t)][m(\tau)+\delta m(\tau)]d\tau dt.$$

Finally, the interchange of the order of integration in the first integral leads to the formulas (4.9)-(4.11).

Lemma is proved.

Remark 1. It is worth noticing that, if the condition $m(t) > 0$, $t \in [t_0, a(T))$ is not valid, then the function $I(m)$ is not differentiable. Indeed, if $m(a(t)) = 0$ for $t \in \Delta_0 \subset [t_0, T)$, then by (4.12), the relevant variation $\delta a(t)$ is unstable with respect to $\delta m(\tau)$, $a(t) < \tau < t$; that is, $\delta a(t)$ can be of finite value for infinitesimal $\delta m(\tau)$ this case the corresponding $\delta^2 I$ and δI will also be finite for infinitesimal δm. In particular, if an interval $[t_1, t_2] \subset \Delta_0$, then $\delta a(t) > t_2 - t_1$ at $t \approx a^{-1}(t_1)$ for any $0 < \delta m(\tau) << 1$, $a(t_1) < \tau < t_1$.

However, the case $m(t) = 0$ is sufficiently natural from the economic point of view and is also considered below.

Remark 2. If function $\beta(\tau, t)$ has a derivative with respect to τ and we use only small variations $\delta m(t)$ such that $m + \delta m \in C^{(1)}[t_0, T]$, then the second variation $\delta^2 I$ in (4.10) up to the third order infinitesimals is of the following form:

$$\delta^2 I = -\frac{1}{2}\int_{t_0}^{T} \rho(t)\beta'_a(a(t), t)m(a(t))[\delta a(t)]^2 dt + \int_{t_0}^{T} o(|\delta m|^2, |\delta m||\delta a|)dt. \tag{4.13}$$

Indeed, suppose that $|\delta m| << 1$, $|\delta a| << 1$. Note that if $m(a(t)) = 0$ for $t \in \Delta_0 \subset [t_0, T)$, then in view of Remark 1 a variation $\delta a(t)$ can be finite for infinitesimal $\delta m(\tau)$, $\tau < t$. By (4.12), the small variations $\delta m(t)$, $\delta a(t)$, $t \in [t_0, T)$ satisfy the equality:

$$m(a(t))\delta a(t) = \int_{a(t)}^{t} \delta m(\tau)d\tau + o(|\delta m|, |\delta a|).$$

Next, by expanding function $\int_a^{a+\delta a}[\beta(a, t) - \beta(\tau, t)][m(\tau) + \delta m(\tau)]d\tau$ into a Taylor series about δa, we obtain (4.13).

Lemma 4.2. If function $\beta(., t)$ is not decreasing, then the abstract function $I(m)$ in OP (4.1)-(4.7) is convex on the set U_1. If $\beta(., t)$ is strictly increasing, then the function $I(m)$ is strictly convex on the set U_1.

Proof follows from the analysis of formula (4.11). By the theorem of the mean we have

$$\delta^2 I = \int_{t_0}^{T} \rho(t)[\beta(a(t), t) - \beta(a(t) + \xi(t), t)] \int_{a(t)}^{a(t)+\delta a(t)} [m(\tau) + \delta m(\tau)]d\tau dt.$$

Let $m_1(\tau) > m_2(\tau)$, $\tau \in [t_0, T)$.Then $\delta m(\tau) = m_1(\tau) - m_2(\tau) > 0$, and in view of (4.12) $\delta a(t) \geq 0$, $t \in [t_0, T)$. Hence, $0 \leq \xi(t) \leq \delta a(t)$ and $\delta^2 I \leq 0$. If $m_1(\tau) < m_2(\tau)$, $\tau \in [t_0, T)$, then $\delta a(t) \leq \xi(t) \leq 0$ and $\delta^2 I \leq 0$, too.

Suppose now that $\beta(.,t)$ is strictly increasing. Let variation $\delta m(\tau) > 0$, $\tau \in \Delta_m \subset [t_0, T)$. Consider a set $\Delta := \{t : \Delta_m \cap [a(t), t] \neq \emptyset\}$. Then in view of equality (4.12), the corresponding variation $\delta a(t)$, $t \in \Delta$ is such that $m(\tau) + \delta m(\tau) \not\equiv 0$ for $\tau \in [a(t), a(t) + \delta a(t)]$. In so doing, $\delta a(t)$ can be finite for infinitesimal $\delta m(\tau)$ (see Remark 1 for Lemma 4.1). So, the integrand in the last formula for $\delta^2 I$ is negative for some values τ and $\delta^2 I < 0$. The reverse is proved analogously.

Lemma is proved.

Theorem 4.2 (*necessary and sufficient conditions of extremum*). In order that a function $m^*(t)$, $t \in [t_0, T)$ be the solution of OP (4.1)-(4.7), it is necessary and sufficient that $\delta I = I(m^* + \delta m) - I(m^*) \leq 0$ for any admissible variation $\delta m(\tau)$, $\tau \in [t_0, T)$. If $m^*(t) > 0$, $t \in [t_0, T)$, then this necessary and sufficient condition of extremum can be written in the form:

$$\begin{aligned} I'(t) < 0 \quad &\text{at} \quad m^*(t) = m_{min}(t), \\ I'(t) > 0 \quad &\text{at} \quad m^*(t) = M(t), \end{aligned} \tag{4.14}$$

$$I'(t) \equiv 0 \quad \text{at} \quad m_{min}(t) < m^*(t) < M(t), \quad t \in [t_0, T).$$

Proof is based on the formula:

$$I(m + \delta m) - I(m) = \int_{t_0}^{T} [I'(t)\delta m(t) + o(\delta m)]dt$$

and uses the standard tools of optimal control theory [43], [85], [112], [142]. The sufficiency of conditions (4.14) for the optimality of m^* is followed from the convexity of abstract function $I(m)$ (Lemma 4.2).

Consider also the gradient $I'(t)$ as a functional $I'(a;t)$ of $a(t)$, $t \in [t_0, T)$. It has a certain property of monotonicity which will be useful for further study.

Lemma 4.3. Let $\beta(.,t)$ be strictly increasing, and $a_1(t) = a_2(t)$ for $t \in a_1(t_2), t_2]$, $a_1^{-1}(t_0) \leq t_2 < T$. If $a_1(t) < a_2(t)$, $t \in [t_1, a_1(t_2)$, $a_1(a_1(t_2)) \leq t_1 < a_1(t_2)$, then $I'(a_1;t) > I'(a_2;t)$, $t \in [t_1, a_1(t_2))$, and the converse. If $a_1(t) < a_2(t)$, $t \in (t_2, t_3]$, $t_2 < t_3 \leq \underline{a}_1^{-1}(t_2)$, then $I'(a_1;t) < I'(a_2;t)$, $t \in (t_2, t_3]$, and the converse.

Proof. Consider the interval $[t_1, a_1(t_2))$. Then $a_1^{-1}(t) = a_2^{-1}(t)$ and

$$I'(a_2;t) = \int_t^{\underline{a}_2^{-1}(t)} \rho(\tau)[\beta(t,\tau) - \beta(a_2(\tau),\tau)]d\tau - \rho(t)\lambda(t) =$$

$$= \int_t^{\underline{a}_1^{-1}(t)} \rho(\tau)[\beta(t,\tau) - \beta(a_2(\tau),\tau)]d\tau - \rho(t)\lambda(t) = I'(a_1;t) +$$
$$+ \int_t^{\underline{a}_1^{-1}(t)} \rho(\tau)[\beta(a_1(\tau),\tau) - \beta(a_2(\tau),\tau)]d\tau < I'(a_1;t)$$

for $a_1(t) < a_2(t)$.

Consider now the interval $(t_2, t_3]$. Let $a_1(t) > a_2(t)$, $t \in (t_2, t_3]$, then $a_1^{-1}(\tau) < a_2^{-1}(\tau)$, $\tau \in [a_1(t_2), t_2]$. We obtain that

$$I'(a_2;t) = \int_t^{t_2} \rho(\tau)[\beta(t,\tau) - \beta(a_1(\tau),\tau)]d\tau +$$
$$+ \int_{t_2}^{\underline{a}_2^{-1}(t)} \rho(\tau)[\beta(t,\tau) - \beta(a_2(\tau),\tau)]d\tau - \rho(t)\lambda(t) =$$
$$= I'(a_1;t) + \int_t^{\underline{a}_1^{-1}(t)} \rho(\tau)[\beta(a_1(\tau),\tau) - \beta(a_2(\tau),\tau)]d\tau +$$
$$+ \int_{\underline{a}_1^{-1}(t)}^{\underline{a}_2^{-1}(t)} \rho(\tau)[\beta(t,\tau) - \beta(a_2(\tau),\tau)]d\tau < I'(a_1;t).$$

The reverse is proved analogously.

Lemma is proved.

From the economic point of view, the minimum possible renovation of economic system ($m \equiv m_{min}$) is a trivial regime. Then in view of (4.7) and (4.2), $a'(t) \equiv 0$ or $m(t) \equiv 0$ for $t \in [t_0, T)$.

One can show that the optimal control $m^*(t)$ does not coincide with the minimum possible one $m_{min}(t)$ for $t \in [t_0, T)$ under the presence of technological change (when a function $\beta(.,t)$ increases).

Theorem 4.3. If $P'(t) > 0$ or $P'(t) < 0$ is valid for $[t_0, T)$, $\beta(.,t)$ does not decrease, and the condition

$$\int_t^T \rho(\tau)[\beta(t,\tau) - \beta(t_0,\tau)]d\tau > \rho(t)\lambda(t) \tag{4.15}$$

is fulfilled for some $t > t_0$, then there is a subset $\Delta \in [t_0, T)$, $mes(\Delta) > 0$, such that the optimal control $m^*(t) > m_{min}(t)$, $t \in \Delta$.

Proof. Put $m^*(t) \equiv m_{min}(t)$, $t \in [t_0, T)$, and show that corresponding phase variable $a^*(t) < t_0$, $t \in [t_0, T)$. Indeed, $m_{min}(t) \equiv 0$ and $P(t) = \int_{a^*(t)}^{t_0} m_0(\tau)d\tau$ under $P(t) < 0$, whence we obtain $a^*(t) < t_0$ in view of $P(t) > 0$. Under $P'(t) > 0$ we have $a^*(t) = const = a_0 < t_0$. Hence, $\underline{a}^{*-1}(t) \equiv T$ in view of (4.10).

Next, we estimate the gradient (4.9):

$$I'(t) = \int_t^{\underline{a}^{*-1}(t)} \rho(t)[\beta(t,\tau) - \beta(a^*(\tau),\tau)]d\tau - \rho(t)\lambda(t) \geq$$
$$\geq \int_{t_0}^{T} \rho(t)[\beta(t,\tau) - \beta(t_0,\tau)]d\tau - \rho(t)\lambda(t) > 0,$$

hence the control $m^* \equiv m_{min}$ is not optimal.

Theorem is proved.

Remark. The condition (4.15) denotes that the discounted rise of the productivity of new EU during its future lifetime $[t, \underline{a}^{-1}(t)]$ (compared with old EUs) exceeds the price of this EU. If $\beta(\tau, t) = e^{c\tau}$, $c > 0$, $T >> 1$, $\rho(t) = e^{-st}$, $\lambda(t) = e^{bt}$, then (4.15) is valid at $c > b$. Conditions $P'(t) \geq 0$ and $P'(t) \leq 0$ are not essential and are accepted to simplify the inequality (4.15).

The structure and asymptotic properties of OP solutions are treated below. For more detail investigation it is required to analyze the cases $T = \infty$ and $T < \infty$ separately.

4.5 INFINITE-HORIZON DISCOUNTED OPTIMIZATION

Before investigation of the OP (4.1)-(4.7) at $T = \infty$, we shall establish the conditions, under which this problem is meaningful (improper integral converges in (4.1)).

Theorem 4.4. If $\beta(.,t)$ is non-decreasing (technical progress is continuous), then the conditions

$$\int_{t_0}^{\infty} \rho(t)\beta(t,t)P(t)dt < \infty, \quad \int_{t_0}^{\infty} \rho(t)\lambda(t)dt < \infty$$

are sufficient for integral convergence in (4.1). Particularly, if $\rho(t) = e^{-st}$, $\beta(\tau, t) = e^{c_1\tau + c_2 t}$, $\lambda(t) = e^{bt}$, and $P(t)$ is bounded for $t \in [t_0, \infty)$, then it is sufficient $s > c_1 + c_2$, $s > b$.

Proof.

$$|I| < \int_{t_0}^{\infty} \rho(t)|Q(t) - \lambda(t)m(t)|dt = \int_{t_0}^{\infty} \rho(t)| \int_{a(t)}^{t} \beta(\tau, t)m(\tau)d\tau -$$

$$-\lambda(t)m(t)|dt < \int_{t_0}^{\infty} \rho(t)|\beta(t,t) \int_{a(t)}^{t} m(\tau)d\tau|dt + \int_{t_0}^{\infty} \rho(t)|\lambda(t)m(t)|dt =$$

$$= \int_{t_0}^{\infty} \rho(t)\beta(t,t)P(t)dt + \int_{t_0}^{\infty} \rho(t)\lambda(t)m(t)dt.$$

Theorem is proved.

Suppose that the conditions of Theorem 4.4 are satisfied below.

Theorem 4.5 (*on the structure of OP solution*). Let:

1) $\beta(., t)$ strictly increase,

2) there be an unique function $\tilde{a}(t)$ such that $I'(\tilde{a}; t) \equiv 0$, $t \in [t_0, \infty)$,

3) corresponding $\tilde{m}(t)$ satisfy (4.7).

Then OP (4.1)-(4.7) has unique solution (a^*, m^*) of the following structure:

$$m^*(t) = \begin{cases} m_{min}(t) \text{ or } M(t), & t \in [t_0, \mu), \\ \tilde{m}(t), & t \in [\mu, \infty), \end{cases}$$

$$a^*(t) = \begin{cases} a_\mu(t), & t \in [t_0, \mu), \\ \tilde{a}(t), & t \in [\mu, \infty), \end{cases}$$

$$\text{where} \qquad a_\mu(t) = \begin{cases} a_{min}(t), & \text{if} \quad a_0 > \tilde{a}(t_0), \\ a_{max}(t), & \text{if} \quad a_0 < \tilde{a}(t_0), \end{cases}$$

the functions $a_{min}(t)$ and $a_{max}(t)$ are respectively defined from the equation (4.3) at $m(t) = m_{min}(t)$ or $m(t) = M(t)$, $t \in [t_0, T]$, the function $\tilde{m}$ is found from (4.2) for $a = \tilde{a}$, and the instant μ – from the condition $a_\mu(\mu) = \tilde{a}(\mu)$.

Proof. The constructed pair (a^*, m^*) satisfies all the restrictions (4.2) - (4.7), hence, is admissible solution.

We shall show that (a^*, m^*) meets n.s.c. of extremum (Theorem 4.2). Consider the case $a_0 < \tilde{a}(t_0)$.

Then $a_\mu(t) < \tilde{a}(t)$, $a_\mu^{-1}(t) > \tilde{a}^{-1}(t)$, $t \in [t_0, \mu)$;

$$I'(a^*;t) = \int_t^\mu \rho(\tau)[\beta(t,\tau) - \beta(a_\mu(\tau),\tau)]d\tau + \int_\mu^{a_\mu^{-1}(t)} \rho(\tau)[\beta(t,\tau) -$$

$$-\beta(\tilde{a}(\tau),\tau)]d\tau - \rho(t)\lambda(t) > \int_t^{a_\mu^{-1}(t)} \rho(\tau)[\beta(t,\tau) - \beta(\tilde{a}(\tau),\tau)]d\tau -$$

$$-\rho(t)\lambda(t) > \int_t^{\tilde{a}^{-1}(t)} \rho(\tau)[\beta(t,\tau) - \beta(\tilde{a}(\tau),\tau)]d\tau - \rho(t)\lambda(t) = 0$$

for $t \in [t_0, \mu)$; and $I'(a^*, t) \equiv 0$ for $t \in [\mu, \infty)$. Hence, the necessary and sufficient conditions of extremum hold true, and (a^*, m^*) is really a OP solution. Its uniqueness follows from Lemma 4.3.

The case $a_0 > \tilde{a}(t_0)$ is investigated similarly.

Theorem is proved.

Consider the important case of OP (4.1)-(4.7) when there is the solution described by Theorem 4.5.

Example 4.1 (*constant rate of embodied TC.*) Let $\beta(\tau, t) = c_1\tau + c_2$, $\lambda(t) \equiv \lambda = const$, $\rho(t) = e^{-st}$. It is easily verified that the equation $I'(\tilde{a}; t) = 0$ has the solution $\tilde{a}(t) = t - A$, $t \in [t_0, \infty)$, where the constant A is determined by the non-linear equation: $A - (1 - e^{-sA})/s = s\lambda/c_1$ $\quad (A \approx \sqrt{2\lambda/c_1}$ for $s \ll 1)$.

Put $\beta(\tau) = \tau + 1$, $c_1 = c_2 = \lambda = t_0 = 1$, $a_0 = 0$, $P \equiv const$, $s = 0,01$. Then $A \approx \sqrt{2}$, $\mu = \sqrt{2}$, and the solution a^* of OP1 is of the following form:

$$a^*(t) = \begin{cases} 0, & t \in [1, \sqrt{2}], \\ t - \sqrt{2}, & t \in [\sqrt{2}, \infty), \end{cases}$$

and the corresponding control m^* defined by (4.3) is:

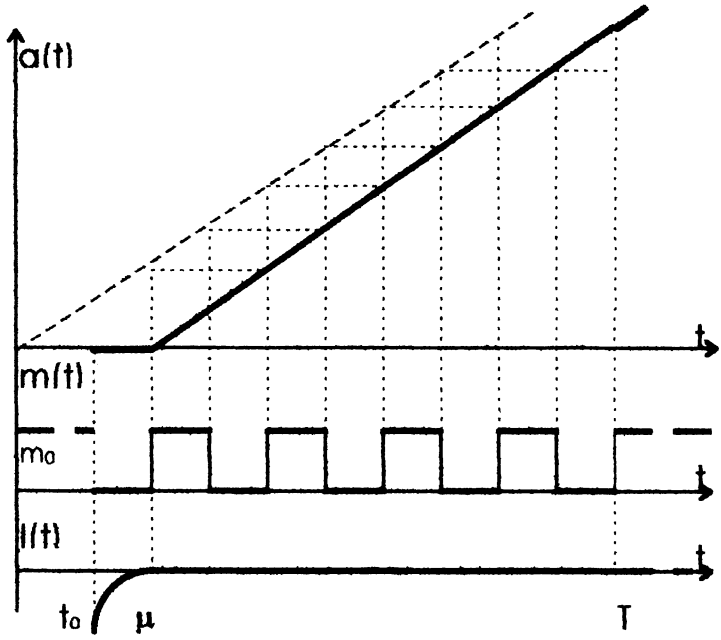

Figure 4.1 Optimal trajectories in the problem OP1 in one-sector model for the case of infinite-horizon discounted optimization

$$m^*(t) = \begin{cases} 0, & t \in [1 + (i-1)\sqrt{2}, i\sqrt{2}), \\ m_0, & t \in [i\sqrt{2}, 1 + i\sqrt{2}), \quad i = 1, 2.... \end{cases}$$

It should be noted that in this case the variation δa of the solution a^* is unstable with respect to δm (see Remark 1 for Lemma 4.1). Nevertheless, the solution (m^*, a^*) exists and is illustrated in Fig.4.1.

It follows from Theorem 4.5 that the integral equation

$$\int_t^{a^{-1}(t)} \rho(\tau)[\beta(t,\tau) - \beta(a(\tau),\tau)]d\tau = \rho(t)\lambda(t), \quad t \in [t_0, \infty)$$

for sought-for function $a(t)$, $t \in [t_0, \infty)$ will play an important role in the qualitative analysis of OP1 solutions.

4.6 FINITE-HORIZON OPTIMIZATION

Suppose that $T < \infty$. Then the structure of OP solutions is more complex in comparison with the $T = \infty$ case. Particularly, the well-known *effect of the end of planned interval* is present in OP1.

Theorem 4.6. There exists an instant θ, $t_0 \leq \theta < T$, such that the OP1 solution $m^*(t) \equiv m_{min}(t)$ for $t \in (\theta, T]$.

Proof follows from the analysis of gradient $I'(t)$ at $t \to T$. Indeed, $I'(T) < 0$ and $I'(t)$ is continuous, hence $I'(t) < 0$ on some interval $(\theta, T]$, $\theta < T$.

Definition. The pair of functions $(a_q(t), m_q(t))$, $t \in [t_0, T]$ which satisfies the necessary and sufficient condition of extremum (Theorem 4.2) is called *the quasi-solution* of OP1.

In general, $a_q(t_0) = a_{q0} \neq a_0$ and the quasi-solution $a_q(.)$ is not concerned with initial condition (4.6) as opposed to the true solution $a^*(.)$ of OP1.

Theorem 4.7. If quasi-solution $a_q(.)$ exists, then the OP1 solution a^* is of the following form:

$$a^*(t) = \begin{cases} a_\mu(t), & t \in [t_0, \mu), \\ a_q(t), & t \in [\mu, T), \end{cases}$$

where the functions $a_\mu(t)$, $a_{min}(t)$, and $a_{max}(t)$ are defined in the similar way as in Theorem 4.5, the instant μ is found from the condition $a_q(\mu) = a_{min}(\mu)$ for $a_{q0} \geq a_0$ and from $a_q(\mu) = a_{max}(\mu)$ for $a_{q0} < a_0$. Corresponding OP1 solution m^* is defined by the formula:

$$m^*(t) = m(a^*(t))a^{*\prime}(t) + P'(t), \quad t \in [t_0, T] \tag{4.16}$$

with the initial condition (4.6).

Proof is based on the necessary and sufficient condition of extremum and consists of the same steps which were made in the proof of Theorem 4.5. First,

direct testing shows that the pair of functions (m^*, a^*) is an admissible solution. Next, by use of n.s.c.e. (Theorem 4.2) one can prove that the functions (m^*, a^*) are really an OP solution.

Remark. If $P'(t) > 0$, then $a_{min}(t) \equiv a_0$ and desired $\mu = a_q(t_0) = a_q^0$.

Theorem 4.7 allows free of the initial condition $a(t_0) = a_0$, and it reduces initial OP to the construction of quasi-solution $a_q(.)$. In some cases it is possible to construct the quasi-solution explicitly and, as a result, to determine an analytical solution of OP.

The quasi-solution (if it exists) can be constructed recurrently from right to left beginning from the interval $[\theta, T]$ where it is determined by Theorem 4.6 $m_q \equiv m_{min}, a_q \equiv a_{min}$. Next, the quasi-solution $a_q(t)$ is such that $I'(a_q; t) = 0$ for $a_q(t) < t < \theta$, after which there is some interval $[t', t'']$, $t'' = a_q(t)$ such that $a_q(t) = a_{min}(t)$. Further the described a_q structure is repeated until the instant $t = t_0$ is reached. The structure of quasi-solution is illustrated below in Fig.4.2 for the case of linear $\beta(\tau)$ when a_q is defined analytically.

Consider the case of linear function $\beta(., t)$, in which the quasi-solution is defined explicitly by the next theorem.

Theorem 4.8. If $\beta(\tau, t) = c_1\tau + c_2$, $\lambda(t) \equiv \lambda = const$, $P'(t) > 0$, then the quasi-solution $a_q(t)$ of OP (4.1)-(4.7) is of the following form:

$$a_q(t) = \begin{cases} d_0, & t \in [T - r, T), \\ t - (T - t)/i, & t \in [t_i'', t_{i-1}'), \ i = 1, 2, ..., \\ d_i, & t \in [t_i', t_i''), \ i = 1, 2, ..., \end{cases}$$

where $d_i = T - rn_i$, $n_i = \sqrt{2(i+1)(i+2)}$, $r = \sqrt{\lambda/s}$, $t_i' = T - rn_i(i+1)/(i+2)$, $t_i'' = T - rn_i i/(i+1)$, $i = 0, 1, 2, ...$

To prove this theorem it is enough to show that the function a_q satisfies the n.s.c. of extremum. This fact is directly verified.

Example 4.2. Let $T = 12$, and $\beta(\tau, t)$, $\lambda(t)$, $\rho(t)$, $P(t)$, t_0, a_0 be the same as in Example 4.1. Then the solution a^*, m^* of OP (4.1)-(4.7) is illustrated in Fig.4.2 (solid line).

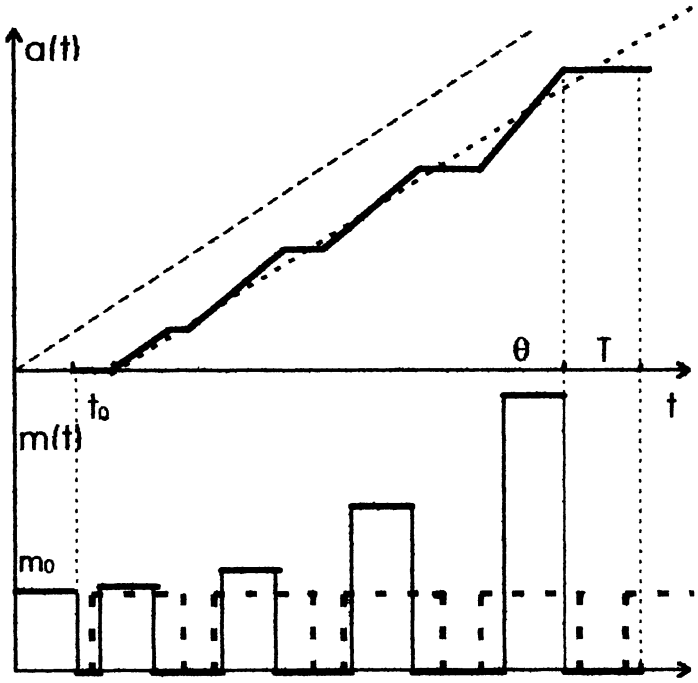

Figure 4.2 Optimal trajectories in the cases of finite-horizon (solid line) and infinite-horizon optimization (dashed line)

The results similar to Theorem 4.8 can be proved for some other types of non-linear smooth functions $\beta(\tau, t)$ which monotonically increase in τ.

If the asymptotic problems are solved ($T \to \infty$), the structural properties of optimal solutions allows to prove different turn-pike theorems for the OP under consideration (see next Chapter).

5

ASYMPTOTICAL BEHAVIOR OF OPTIMAL TRAJECTORIES AND TURNPIKE THEOREMS

This chapter is the continuation of Chapter 4. Here the investigation of the basic optimization problem (OP1) for the simplest one-sector IDM of economic renovation is completed. First, the integral-functional equation for turnpike trajectories is studied in Section 5.1. This equation represents an analog of dual problem for the OP1 and is substantially complicated from the viewpoint of its solvability. The theorems about the existence and asymptotics of its solutions are established. It allows to prove in Section 5.2 the different turnpike theorems (in normal, strong and the strongest forms) for the optimal lifetime of EUs in OP1.

5.1 STUDY OF THE INTEGRAL-FUNCTIONAL EQUATION FOR TURNPIKE TRAJECTORIES

During study of OP (4.1)-(4.7) the following integral-functional equation (IFE) of non-Volterra type arises for sought-for function $a(t)$, $t \in [t_0, T)$ (see Theorem 4.5):

$$\int_t^{a^{-1}(t)} \rho(\tau)[\beta(t,\tau) - \beta(a(\tau),\tau)]d\tau = \rho(t)\lambda(t), \qquad t \in [t_0, T). \tag{5.1}$$

From the mathematical point of view the equation (5.1) represents a dual (adjoint) problem for the studied problem of optimal control of integral equations

(4.2)-(4.3) [85], [142]. It is an analog of the boundary problem of the Pontriagin's maximum principle for ordinary differential equations [109]. As one would expect, this equation includes the integral over the interval $[t, a^{-1}(t)]$ (the integration forward from current time t), whereas the source (direct) problem (4.2)-(4.3) includes the integral over the interval $[a(t), t]$ (the integration back from time t). Such structure of dual problem is inherent in all other OPs considered below.

From the economic viewpoint the interval $[t, a^{-1}(t)]$ represents the sought-for future lifetime of the new EUs which are brought into operation at current time t, and the interval $[a(t), t]$ is the lifetime of the oldest EUs which are removed from service at time t. So, the equation (5.1) links these two important characteristics of production process with the cost $\lambda(t)$ of new EUs what allows us to study the rational strategy of technological renovation.

The equation (5.1) has clear interpretation: the profit of introducing one new EU and its operation during future lifetime (with account of exchanging older EU) must be equal to the price of this EU.

Definition 5.1. The solution $\tilde{a}(t)$, $t \in [t_0, \infty)$, if it exists, of the equation (5.1) will be called the turnpike (the turnpike trajectory) of OP (4.1)-(4.7).

According to economic content of the equation (5.1), we shall restrict ourselves to the case of monotonic continuously differentiable solutions $a(t) < t$.

The IFE (5.1) generates a set of solutions $a_T(t)$ on any finite interval $[t_0, T)$. This equation is substantially complicated from the viewpoint of existence and uniqueness of the solution $\tilde{a}(t)$, $t \in [t_0, \infty)$.

The next notion will play an important role in the qualitative analysis of equation (5.1) and other IFEs under consideration.

Definition 5.2. The solution $a(t) = t - A$, $A =$const> 0, $t \in [t_0, \infty)$ of IFE (5.1) is named the *quasistationary solution.* It corresponds to the constant lifetime A of system elements.

5.1.1 The Direct Derivation of Basic IFE

Here we directly deduce a simplified form of the basic IFE (5.1) (at $\rho \equiv 1$).

Consider some abstract *evolving system (ES)* which consists of separate *system elements (SEs)* (see Section 1.6). Let it be desirable to maximize the total efficiency of an external function of ES (ES output). Introduce into consideration the following functions:

- $\beta(\tau, t)$ – the given SE efficiency: the external product output per time unit at time t on one SE created at instant τ;
- $\lambda(t)$ – the given expenditure for the creation of one SE, i.e., the cost of new SE (in the measure units of external product);
- $a(t)$ – the sought-for time limit of elimination of obsolete SEs (the SEs created before the time $a(t)$ are not used at time t).

Then the value $t - a(t)$ is the lifetime (service time) of the oldest SEs introduced at instant $\tau = a(t)$, and the value $a^{-1}(t) - t$ is the future lifetime of new SEs introduced at time t.

To construct the equation for sought-for $a(.)$, assume that one (additional) new SE is introduced into ES at time τ. Then one oldest SE (introduced at time $a(t)$) must be liquidated for the preservation of SEs quantity. The corresponding rise of the output of aggregate common production at instant t is $S(t,t) = \beta(t,t) - \beta(a(t),t)$.

Next, we fix an instant t of introducing SE and suppose that current time τ (the time of SE's use) increases. The production rise at each succeeding instants τ, $t < \tau < a^{-1}(t)$, as the result of one SE exchange at time t, is $S(\tau,t) = \beta(\tau,t) - \beta(a(t),t)$. Indeed, if this new SE was not introduced then the old SE created at time $a(\tau)$ must be used instead of it. The SE exchange at time t becomes less effective, when time τ increases, and it has no sense for τ such that $a(\tau)$ is close to t.

The integral (total) output rise during the SE future lifetime $[t, a^{-1}(t)]$ because of one SE exchange at time t is

$$\int_t^{a^{-1}(t)} [\beta(t,\tau) - \beta(a(\tau),\tau)]d\tau.$$

Putting this total profit equal to the expenditure $\lambda(t)$ per one SE creation, we obtain the following integral-functional equation:

$$\int_t^{a^{-1}(t)} [\beta(t,\tau) - \beta(a(\tau),\tau)]d\tau = \lambda(t), \tag{5.2}$$

with respect to the sought-for variable $a(.)$ in integrand and the upper limit of integration.

5.1.2 Analysis of The Simplest IFE

In this section we will restrict ourselves to the simplest case of equation (5.2) when $\beta(\tau,t) \equiv \beta(\tau)$:

$$\int_t^{a^{-1}(t)} [\beta(t) - \beta(a(\tau))]d\tau = \lambda(t). \tag{5.3}$$

Let the given functions $\beta(t)$, $\lambda(t)$ be positive and twice differentiable, $\beta'(t) > 0$. Denote by $\Omega[t_0,T)$ the set of monotonic continuously differentiable functions $a(t)$: $a(t) < t$, $a'(t) > 0$, $t \in [t_0,T)$, $T \leq \infty$. We will study the existence and uniqueness of IFE (5.3) solutions $a \in \Omega[t_0,T)$ for $T \gg 1$ and $T = \infty$.

By differentiating (5.3) we obtain the recurrence relation:

$$\beta'(t)[a^{-1}(t) - t] = \beta(t) - \beta(a(t)) + \lambda'(t), \tag{5.4}$$

which connects the values $a(t)$ and $a^{-1}(t)$.

Assume that some initial function $a \in \Omega[t_0, a^{-1}(t_0)]$ is given and meets the equation (5.3) at $t = t_0$. Then one can determine from (5.4) the unique function $a^{-1}(t)$, $t \in [t_0, a^{-1}(t_0)]$, where $a^{-1}(t) > t$ because of $\lambda > 0$. If $a^{-1}(t)$ is strictly monotone, then it determines the unique solution $a \in \Omega[a^{-1}(t_0), a^{-2}(t_0)]$ on interval $[a^{-1}(t_0), a^{-2}(t_0)]$, $a^{-2}(t_0) = a^{-1}(a^{-1}(t_0))$. Hence, the sufficient condition for the existence of (5.3) solution $a \in \Omega[a^{-1}(t_0)], a^{-2}(t_0)]$ is $da^{-1}(t)/dt > 0$ or $a'(a^{-1}(t)) > 0$, where by (5.4)

$$a'(a^{-1}(t)) = \{2 - [\beta'_a(a(t))a'(t) - \beta''(t)(a^{-1}(t) - t) + \lambda''(t)]/\beta'(t)\}^{-1}. \tag{5.5}$$

The IFE (5.3) solution $a(.)$ on intervals $[a^{-i}(t_0), a^{-i-1}(t_0)]$, $i = 2, 3, ...$, can be constructed in the same way. But in order that solution $a \in \Omega[t_0, T]$ exist on substantially large interval $[t_0, T]$, the given $\beta(.)$, $\lambda(.)$, and an initial function $a(t)$, $t \in [t_0, a^{-1}(t_0)]$ must satisfy severe conditions. Such conditions are studied below.

Lemma 5.1. Let the equation (5.3) have a solution $a \in \Omega[t_0, T)$ and another one $a + \delta a \in \Omega[t_0, T)$, $|\delta z(t)| \ll |z(t)|$, $|\delta z'(t)| \ll |z'(t)|$ which is close to the first. If $a'(a^{-1}(t)) < B(t, a(t))$, $t \in [t_0, a(T)]$ where $B(t, a) = \beta'(t)/\beta'(a)$, then $|\delta a(t)| > |\delta a(a^{-1}(t))|$, $t \in [t_0, z(T)]$ (the variations $\delta a(t)$ of $a(t)$ decrease in passing from an interval $[z(t), t]$ to next one $[t, z^{-1}(t)]$), and conversely. In case if $a'(a^{-1}(t)) < \sqrt{B(t, a(t))}$, then $|\delta a'(t)| > |\delta a'(a^{-1}(t))|$, $t \in [t_0, a(T)]$ (the variations $\delta a'(t)$ of derivative $a'(t)$ decrease in passing to next interval), and conversely.

Proof is based upon the varying of (5.4) with respect to $a(.)$ and the varying of (5.5) with respect to $a'(.)$, and the use of the equality

$$\delta a^{-1}(t) = \delta a(a^{-1}(t))/a'(a^{-1}(t)) + o(\delta a^{-1}(t)), \tag{5.6}$$

which follows directly from the definition of derivative $a'(t)$. Namely, by (5.4) we obtain $\beta(t)\delta a^{-1}(t) = \beta'_a(a(t))\delta a(t)$, whence, in view of (5.6)

$$\delta a(a^{-1}(t)) = a(a^{-1}(t))[\beta'_a(a(t))/\beta(t)]\delta a(t) + o(\delta a^{-1}(t)),$$

as required.

The equality (5.5) is varied similarly. Let us consider the increment $\delta a'(t)$, $t \in \Delta \subset [t_0, T]$ such that $|\delta a'(t)| \ll 1$, $mes(\Delta) \ll 1$. Then $|\delta a(u)| \ll |\delta a'(u)|$, $u \in [t_0, T]$, and in view of (5.5)

$$\begin{aligned}
\delta a'(a^{-1}(t)) = \{2 - [\beta'_a(a(t))(a'(t) + \delta a'(t)) - \beta''(t)(a^{-1}(t) - t) + \\
+\lambda''(t)]/\beta'(t)\}^{-1} - \{2 - [\beta'_a(a(t))a'(t) - \beta''(t)(a^{-1}(t) - t) + \\
+\lambda''(t)]/\beta'(t)\}^{-1} = \delta a'(t)\{2 - [\beta'_a(a(t))a'(t) - \beta''(t)(a^{-1}(t) - t) + \\
+\lambda''(t)]/\beta'(t)\}^{-2}\beta'_a(a(t))/\beta'(t) + o(\delta a') = \\
= \delta a'(t)[a'(a^{-1}(t))]^{-2}\beta'_a(a(t))/\beta'(t) + o(\delta a'),
\end{aligned}$$

which proves the statement. Lemma is proved.

The Lemma 5.1 permits to show in some cases the existence and uniqueness of the equation (5.3) solution $a \in \Omega[t_0, \infty)$ on unbounded interval $[t_0, \infty)$ as well

as to study the asymptotical behavior of solutions for bounded intervals $[t_0, T]$, $T - t_0 \gg 1$. We shall restrict ourselves to the cases of exponential power and logarithmic function $\beta(\tau)$.

Theorem 5.1. Let $\beta(\tau) = c_1\tau^s + c_2$, $\lambda(t) = \lambda \equiv$const, $s, c_1, \lambda > 0$. Then the equation (5.3) has some set of solutions $a \in \Omega[t_0, T]$ for any interval $[t_0, T]$ and a unique solution $\tilde{a} \in \Omega[t_0, \infty)$ such that $\tilde{a}(t)/t \to 1$ at $t \to \infty$. For $s = 1$ $\tilde{a}(t) = t - \sqrt{2\lambda/c_1}$. In the case where $T \to \infty$, all solutions $a(t)$ approach the solution $\tilde{a}(t)$ on an interior part of interval $[t_0, \infty)$, namely, for any $\varepsilon > 0$ a time T_0 exists such that for any $T \geq T_0$ the condition $|a(t) - \tilde{a}(t)|$ holds true at $t \in \Delta \subset [t_0, T]$, $mes(\Delta) \to \infty$ for $T \to \infty$.

Proof. The equation (5.3) after the substitution $y(t) = a(t)/t$ is of the following form:

$$y(a^{-1}(t)) = q/[q + 1 - y^q(t)], \tag{5.7}$$

what does not depend on time t explicitly.

If a solution $a(t)$, $t \in [t_0, T)$ exists, then for any $\tau \in [t_0, a^{-1}(t_0)]$ one can introduce the increasing sequence $\{t_k\}_{k=1}^K$, $t_k = a^{-k}(\tau)$, $k = \overline{0, K}$, where $K = \infty$ or K is assigned to $t_k > T$.

Consider the auxiliary difference equation:

$$y(t_{k+1}) = q/[q + 1 - y^q(t_k)]. \tag{5.8}$$

The equation (5.8) has the unique fixed point $y = 1$ which is attracting for $y < 1$ and repelling for $y > 1$. The function $\Phi(y) = q/(q + 1 - y^q)$ increases monotonically, hence the equation (5.8) generates a monotone increasing sequence $\{y(t_k)\}_{k=1}^K$. From the relation (5.5) the following difference equation can likewise be constructed for $a'(t_k)$:

$$a'(t_{k+1}) = q/[q + 1 - (1 - q)y^q(t_k) - qy^{q-1}(t_k)a'(t_k)] \tag{5.9}$$

which has also the unique fixed point $a' = 1$ by virtue of (5.8).

Let us study the existence and behavior of equation (5.3) solution for a large k. Supposing $y(t_k) = 1 - \varepsilon$, $\varepsilon = o(1)$ and substituting the critical value

$$a'(t_{k+1}) = B(t_k, t_{k-1}) = \beta'(t_k)/\beta'(t_{k-1}) = y^{1-q}(t_{k-1})$$

from Lemma 5.1 into (5.9), we find after some transformation that

$$a'(t_{k+1}) = y^{q-1}(t_k)[1 + q(1-q)\varepsilon^2/2 + o(\varepsilon^2)]. \tag{5.10}$$

Substituting the second critical value $a'(t_{k+1}) = \sqrt{B(t_k, t_{k-1})} = y^{(1-q)/2}(t_{k-1})$ from Lemma 5.1 into (5.9), we have similarly

$$a'(t_{k+1}) = y^{(q-1)/2}(t_k)[1 + (q^2-1)(3q^2+6q+1)\varepsilon^2/4q^2 + o(\varepsilon^2)]. \tag{5.11}$$

Let $0 < q < 1$. We obtain from (5.10),(5.11) that, if

$$0 < a'(t_{k+1}) \le y^{q-1}(t_{k-1}) < y^{(q-1)/2}(t_{k-1}) < 1,$$

then this inequality holds true at point t_{k+1} too. Hence, in view of Lemma 5.1 the variations of solution $|\delta a(t_k)|$ and of its derivative $|\delta a'(t_k)|$ decrease and become vanishingly small for $k \to \infty$, and $a'(t_k) > 0$ is valid for all k. Thus, if the solution exists on interval $[a^{k-1}(t_0), a^k(t_0)]$, then it exists on $[a^k(t_0), a^{k+1}(t_0)]$. Next, we can construct some continuously differentiable function $a(t) < t$, $0 < a'(t) < [a(t)/t]^{1-q}$, $t \in [t_0, a^{-1}(t_0)]$, which satisfies (5.3) and (5.5) at $t = t_0$. The construction of such function can be done in more than one way. Continuing $a(t)$ recursively by means of the formula (5.4) we can obtain the solution $a \in \Omega[t_0, T]$ on any interval $[t_0, T]$. At the same time, all such solutions $a(t)$, $t \in [t_0, T]$ tend to some $\tilde{a} \in \Omega[t_0, \infty)$ for $T \to \infty$ in view of Lemma 5.1.

Let $q > 1$. It follows from (5.10),(5.11) that, if

$$a'(t_{k+1}) \ge y^{1-q}(t_{k-1}) > y^{(1-q)/2}(t_{k-1}), \quad y^{1-q}(t_{k-1}) > 1,$$

then these inequalities hold true also at point t_{k+1}. Hence, regarding Lemma 5.1, the variations $|\delta a(t_k)|$ and $|\delta a'(t_k)|$ increase in k. Let us construct a solution of the equation (5.3) for some interval $[t_0, T]$, $T \gg t_0$. We can construct some continuously differentiable function $a(t) < t$, $a'(t) > y^{1-q}(t_{k-1}) > y^{(1-q)/2}(t_{k-1})$, $t \in [a(T), T]$ which satisfies (5.3) and (5.5) at $t = a(T)$. Continuing $a(t)$ recursively from right to left by means of the formula (5.4) we

obtain in view of Lemma 5.1 some solution $a \in \Omega[t_0, T]$ on the entire interval $[t_0, T]$. The solution $\tilde{a} \in \Omega[t_0, \infty)$ is obtained by tending $T \in \infty$. In so doing, all $a \in \Omega[t_0, T]$ tend to $\tilde{a}(.)$ in view of Lemma 5.1 .

Let $q = 1$. It is easily verified that the equation (5.3) has the solution $\tilde{a}(t) \equiv t - A$ for the infinite interval $[t_0, \infty)$. We shall show that this solution is unique. Indeed, in this case $B(t, a) \equiv 1$. If $a'(\tau) \neq 1$ at some point $\tau \gg t_0$, then in accordance with Lemma 5.1, this leads to increasing $|\delta a'(t_k)|$, $t_k = a^{-k}(\tau)$ at $k \to \infty$ or $\tau \to \infty$, $k \to -\infty$, violating the condition $a'(t) > 0$ and, hence, yields the discontinuities of $a^{-1}(t)$. Thus, the unique continuous solution of the equation (5.3) on interval $[t_0, \infty)$ is $\tilde{a}(t) \equiv t - A$ only, where $A = \sqrt{2\lambda/c_1}$ in view of (5.3).

Theorem is proved.

It is easily seen that IFE (5.3) has the quasistationary solution in the case $\beta(\tau) = c_1\tau + c_2$, $\lambda(t) \equiv \lambda$ =const. For example, if $\beta(\tau) = \tau$, $\lambda = 1$, then the solution of equation (5.3) is $a(t) = t - \sqrt{2}$, $t \in [t_0, \infty)$.

Theorem 5.2. Let $\beta(\tau) = e^{ct}$, $\lambda(t) \equiv \lambda$ =const, $c, \lambda > 0$. Then the equation (5.3) has some set of solutions $a \in \Omega[t_0, T]$ for any interval $[t_0, T]$ and the unique solution $\tilde{a} \in \Omega[t_0, \infty)$ such that $t - \tilde{a}(t) \xrightarrow{t \to \infty} 0$. In case where $T \to \infty$, all these solutions $a(.)$ approach the solution $\tilde{a}(.)$ in the sense indicated in Theorem 5.1.

Proof is similar to previous one. The equation (5.3) after the substitution $y(t) = t - a(t)$ is of the following form:

$$y(a^{-1}(t)) = [1 - \exp(-cy(t))]/c.$$

The corresponding difference equation $y(t_{k+1}) = [1 - \exp(-cy(t_k))]/c$ has the unique fixed point $y = 0$ attracting at $y > 0$. Hence, if the solution y exists, then $y(t) \to 0$ for $t \to \infty$. The difference equation for a'

$$a'(t_{k+1}) = \exp(cy(t_k))/[1 + \exp(cy(t_k)) - a'(t_k)] \tag{5.12}$$

with the fixed point $a' = 1$ is followed from (5.5).

Supposing $0 < y(t_k) \ll 1$ and substituting the critical values from Lemma 5.1 into (5.12), we obtain

$$a'(t_{k+1}) \geq \exp(cy(t_k)) > \exp(cy(t_k)/2) > 0$$

for $k, k+1, \ldots$. Hence, the behavior of solutions $a \in \Omega[t_0, T]$ is similar to Theorem 5.1 for $s > 1$, and their construction can be carried out in the same way (from right to left). In view of Lemma 5.1, all $a \in \Omega[t_0, T]$ tend to $\tilde{a} \in \Omega[t_0, \infty)$ at $T, T - t \to \infty$.

Theorem is proved.

Theorem 5.3. Let $\beta(\tau) = \ln \tau$, $\lambda(t) \equiv \lambda$ =const> 0. Then the equation (5.3) has a diverging bundle of the solutions $a(t)$, $a'(t) > 0$ on infinite interval $[t_0, \infty)$, such that $a(t)/t \to 0$ and $a_1(t) - a_2(t) \to \infty$ at $t \to \infty$ for any two solutions a_1, a_2.

Proof. The equation (5.3) after the substitution $y(t) = a(t)/t$ is of the following form:

$$y(a^{-1}(t)) = 1/[1 - \ln(y(t))].$$

The corresponding difference equation has the unique fixed point $y = 1$ attracting for $0 < y < 1$. Hence, if a solution y exists, then $y(t) \to 0$ for $t \to \infty$. The difference equation for a' in view of (5.5) has the following form

$$a'(t_{k+1}) = 1/[1 + 1/y(t_{k+1}) - a'(t_k)/y(t_k)] \tag{5.13}$$

and the fixed point $a' = 1$.

As above, supposing $y(t_k) = 1 - \varepsilon$, $\varepsilon = o(1)$ and substituting the critical values from Lemma 5.1 into (5.13), we obtain

$$B(t_k, t_{k-1}) = y(t_k) < a'(t_{k+1}) < \sqrt{B(t_k, t_{k-1})} = [y(t_k)]^{1/2}$$

for $k, k+1, \ldots$. Hence, in view of Lemma, the variations of solution $|\delta a(t_k)|$ increase, whereas the variations of derivative $|\delta a'(t_k)|$ decrease and become vanishingly small with $k \to \infty$, and $a'(t_k) > 0$ is valid for all k. The construction of solutions $a \in \Omega[t_0, T]$ is similar to the case $0 < s < 1$ of Theorem 5.1 (from left to right), but the convergence of solutions is lacking for $T, T - t \to \infty$.

Theorem is proved.

5.1.3 Basic IFE Analysis

If we take into account the discounted future profit with respect to present time, then the following generalization of IFE (5.2) arises (see Section 4.5 and 5.1.1):

$$\int_t^{a^{-1}(t)} \rho(\tau)[\beta(t,\tau) - \beta(a(\tau),\tau)]d\tau = \rho(t)\lambda(t), \tag{5.14}$$

where $\rho(t)$ is so-called discounting multiplier, $\rho(t) > 0$, $\rho'(t) < 0$ (see Section 4.2).

Suppose that $\rho(t) = e^{-st}$, $s > 0$, $\beta(\tau,t) \equiv \beta(\tau)$. Then the equation (5.14) is of the following form:

$$\int_t^{a^{-1}(t)} e^{-st}[\beta(t) - \beta(a(\tau))]d\tau = e^{-st}\lambda(t). \tag{5.15}$$

The equation (5.15) shall be studied in a way similar to the case $\rho \equiv 1$, which has been shown in the above Section 5.1.2. Therefore, the explanation presented here will be more compact.

Consider first the conditions for the given functions β and λ under which the equation (5.15) has solutions $a \in \Omega[t_0, T]$ on a large interval $[t_0, T]$. By differentiating (5.15) we obtain the recurrence relation

$$\beta(a(t)) - \beta(t) + \beta'(t)[1 - e^{-s(a^{-1}(t)-t)}]/s = \lambda'(t) - s\lambda(t). \tag{5.16}$$

Fix the instant $t = u$ and put $a(t) \equiv a(u)$ for $t \in [u, \infty)$, then $a^{-1}(t) = \infty$, $t \in [a^{-1}(u), \infty)$. It is easily verified that (5.16) holds true if $\beta(t) - \beta(a(u)) = s\lambda(t)$, and $a'(t) > 0$ if $\beta(t) - \beta(a(u)) > s\lambda(t)$, $t \in [u, \infty)$. Let

$$0 < \beta'(t)/\beta(t) < s, \quad \lambda(t)/\beta(t) < s. \tag{5.17}$$

Then $a(t)$ tends to ∞ at $t \to \infty$. Suppose that the conditions (5.17) are satisfied below.

One can prove that Lemma 5.1 holds true for IFE (5.15) with the constant

$$B(t,a) = e^{-s(a^{-1}(t)-t)}\beta'(t)/\beta'_a(a(t)). \tag{5.18}$$

Theorem 5.4. Let $\beta(\tau) = e^{ct}$, $\lambda(t) = be^{b_1 t}$, $b_1 \le c < s$, $c, b > 0$, and $b < s^{-1}$ at $b_1 = c$. Then the equation (5.15) has some set of solutions $a(t)$, $a'(t) > 0$ for any interval $[t_0, T]$. In case where $T \to \infty$, all these solutions tend to the unique solution $\tilde{a}(t)$, $\tilde{a}(t) > 0$ of IFE (5.15) on interval $[t_0, \infty)$. The function $t - \tilde{a}(t) \xrightarrow{t\to\infty} 0$ at $b_1 < c$, while $\tilde{a}(t) \equiv t - A_1$ at $b_1 = c$, where the constant A_1 is determined by the nonlinear equation

$$se^{-cA_1} - ce^{-sA_1} = (1 - bs)(s - c) \tag{5.19}$$

($A_1 \approx \sqrt{2b/c}$ at $s \ll 1$).

Proof. When $b_1 = c$, substituting $a(t) \equiv t - A_1$ into (5.15) readily shows that $a(t)$ does be a solution of the equation 5.17, and we get the equation (5.19) for determination of A_1. In this case $B(t,a) = e^{(c-s)A_1}$ is less than 1.

When $b_1 < c < s$, according to (5.16) we have $\Delta(t) = t - a(t) \to 0$ for $t \in \infty$. Let us study the limiting behavior of the function $\Delta(t)$ when $t \in \infty$ (and $|\Delta(t)|$ is small). Let the time t_1 be fixed. Denote $\Delta(t_1) = \Delta_1$, $\Delta(a^{-1}(t_1)) = \Delta_2$, and put approximately $\Delta(t) = \Delta + \varepsilon(t - t_1) + o(\Delta_1^2)$. Then $A_1 = 1 - \Delta_1/\Delta_2$. Substituting $\Delta(t)$ into equation (5.15) and calculating the integral we get

$$\frac{1 - e^{-s\Delta_2}}{s} + \frac{1}{s - c\Delta_1/\Delta_2}(e^{-s\Delta_2} - e^{-s\Delta_1}) + o(\Delta_1^2) = be^{(b_1 - c)t}. \tag{5.20}$$

Using the expansion of the left-hand side functions we obtain the equation $c\Delta_1\Delta_2/2 + o(\Delta_1^2) = b\exp(b_1 - c)t$, which, when substituted into (5.20), leads to a relation between Δ_1 and Δ_2:

$$s\Delta_2^2 - \Delta_2(2 + (s - b_1))\Delta_1 + 2\Delta_1 - c\Delta_1^2 + o(\Delta_1^2) = 0.$$

A further study of the solution of this equation shows that $\Delta_2 = \Delta_1 + o(\Delta)$. Thus if $t \to \infty$, then $\Delta(a^{-1}(t))/\Delta(t) \to 1$, and

$$B(t,a) = e^{-s\Delta(a^{-1}(t)+c\Delta(t))} = e^{-s\Delta_2 + c\Delta_1}e^{-(s-c)\Delta_1 + o(\Delta_1)} < 1.$$

Now we can construct the solution of equation (5.15). Let us choose some instant $T \gg t_0$ and construct the continuously differentiable function $a(t) < t$,

$a'(t) > \sqrt{B(t,a)} > B(t,a)$, $t \in [T, a^{-1}(T)]$, which satisfies (5.15) and (5.16) at $t = T$. Continuing $a(t)$ recursively from right to left by means of the formula (5.16), we get in view of Lemma 5.1 some solution $a(t) < t$, $a'(t) > 0$ for the entire interval $[t_0, T]$. We will obtain the solution $\tilde{a} \in C^1_{[t_0,\infty)}$ of equation (5.15) if we let T tend to ∞. In so doing, any solution satisfies $a(t) \to \tilde{a}(t)$ when $T, T-t \to \infty$.

Theorem is proved.

Theorem 5.5. Let $\beta(\tau) = c_1\tau + c_2$, $\lambda(t) = \lambda \equiv$const, $c_1, \lambda > 0$. Then the equation (5.15) has a unique solution $\tilde{a}(t) \equiv t - A$, $t \in [t_0, \infty)$, and some set of solutions $a(t)$, $a'(t) > 0$ for any interval $[t_0, T]$. In case where $T \to \infty$, all solutions $a(.)$ approach the solution $\tilde{a}(.)$. The constant A is determined by the nonlinear equation

$$A - (1 - e^{-sA})/s = s\lambda/c_1 \quad (A \approx \sqrt{2\lambda/c_1} \quad \text{at} \quad s \ll 1).$$

Proof. In this case we obtain that $B(t,a) \equiv e^{-sA} < 1$. Further proof is similar to that of Theorem 5.1 (in case $q = 1$) and Theorem 5.4.

Remark. The relation $A \approx \sqrt{2\lambda/c_1}$ represents an analog of the *Terborgh formula* for optimal lifetime of equipment (see [96]).

5.2 THE TURNPIKE PROPERTIES OF OPTIMAL TRAJECTORIES

Theorems 4.6-4.8 indicate the presence of turnpike properties in OP (4.1)-(4.7), what is of great theoretic and applied interest (see Section 0.1).

5.2.1 The Turnpike Theorem in Normal Form

Consider the asymptotic behavior of the solution $a^*(.)$ of OP (4.1)-(4.7) for a large interval $[t_0, T]$: $T - t_0 \gg 1$.

Theorem 5.6 *(turnpike theorem in normal form).* Let:

1) the function $\beta(\tau, t)$ be strictly increasing in τ, the functions $\rho(\tau)$, $\rho'(\tau)/\rho(\tau)$, $\rho(\tau)\partial\beta(\tau,t)/\partial\tau$, and $\lambda(\tau)\partial\beta(\tau,t)/\partial\tau$ be non-decreasing in τ and t, τ, $t \in [t_0, T]$;

2) the function $M(t)$ be bounded at $t \to \infty$;

3) the set of the solutions $\bar{a}_T(t)$, $t \in [t_0, T]$ of equation (5.14) approach for $T \to \infty$ an unique trajectory $\tilde{a}(t)$, $t \in [t_0, \infty)$, such that $\tilde{a}'(t) > 0$ and $(T - \tilde{a}(T))/\tilde{a}(T) \to 0$ at $T \to \infty$;

4) the function $\tilde{m}(t)$ defined by $\tilde{a}(t)$ in accordance with equation (13) satisfy (17);

5) $P'(t) > 0$, $P(t) \overset{T\to\infty}{\longrightarrow} \infty$ or $\tilde{a}(t_0) > a_0$, $m_0(\tau) \geq d_0 > 0$, $\tau \in [a_0, t_0]$.

Then, when $T \to \infty$, the solution $a^*(t)$ of OP (4.1)-(4.7) strives to $\tilde{a}(t)$ on an asymptotically largest part of interval $[t_0, T]$, e.g. for any $\varepsilon > 0$ the time T_0 exists such that for any $T \geq T_0$ the condition $|a^*(t) - \tilde{a}(t)| < \varepsilon$ is true on some subset $\Delta \subset [t_0, T]$ such that $mes(\Delta)/(T - t_0) \to 1$ for $T \to \infty$.

Proof is based on the analysis of the second variation of functional I (Lemma 4.1) and uses the special admissible solution which coincides with $\tilde{a}(.)$ for the asymptotically largest part of interval $[t_0, T]$.

Namely, we construct the following pair of functions $(\hat{m}, \hat{a})$:

$$\hat{a}(t) = \begin{cases} a_\mu(t), & t \in [t_0, \mu), \\ \tilde{a}(t), & t \in [\mu, \theta), \\ a_{min}(t), & t \in [\theta, \infty), \end{cases}$$

$$\hat{m}(t) = \begin{cases} m_{min}(t) \text{ or } M(t), & t \in [t_0, \mu), \\ \tilde{m}(t), & t \in [\mu, \theta), \\ m_{min}(t)\ , & t \in [\theta, \infty), \end{cases} \tag{5.21}$$

where

$$a_\mu(t) = \begin{cases} a_{min}(t), & \text{if } \tilde{a}(t_0) \geq a_0, \\ a_{max}(t), & \text{if } \tilde{a}(t_0) < a_0, \end{cases}$$

Here the functions $a_{min}(t)$ and $a_{max}(t)$ are defined from equation (4.3) at $m(t) = m_{min}(t)$ or $m(t) = M(t)$, $t \in [t_0, \mu]$, and the instant μ is uniquely found from the condition $\tilde{a}(\mu) = a_{min}(\mu)$ for $\tilde{a}(t_0) \geq a_0$ and from $\tilde{a}(\mu) = a_{max}(\mu)$ for $\tilde{a}(t_0) < a_0$. The instant θ is found from the conditions:

1. $I'(\theta) = 0$ at $m^*(t) \equiv m_{min}(t)$, $t \in (\theta, T]$;
2. $\tilde{a}(\theta) = a_{min}(\theta)$ (see also Theorem 4.6).

Under the conditions of Theorem the values $(T-\theta)/T \to 0$ and $(T-\tilde{a}(\theta))/T \to 0$, the instant $\mu < \theta$ exists and does not depend on value T for $T \to \infty$.

The pair $(\hat{m}, \hat{a})$ satisfies all restrictions (4.2)-(4.7), hence, is an admissible solution. By the construction of $\hat{a}$,

$$I'(t) \equiv 0, \quad t \in [\mu, \hat{a}(\theta)]. \tag{5.22}$$

Let us analyze the corresponding variation of functional I. In view of Lemma 4.1 and (5.22) we have

$$\delta I = \int_{t_0}^{\mu} I'(t)\delta m(t)dt + \int_{\hat{a}(\theta)}^{T} I'(t)\delta m(t)dt + \delta^2 I.$$

By applying Lemma 4.3 for $\mu = \tilde{a}(t_2)$, we obtain that $I'(t) < 0$ for $\delta m(t) < 0$, $t \in [t_0, \mu)$, and $I'(t) > 0$ for $\delta m(t) > 0$, $t \in [t_0, \mu) \cup [\theta, T]$. So, $I'(t)\delta m(t) > 0$, $t \in [t_0, \mu) \cup [\theta, T]$ for any admissible variations δm and

$$\delta I < \int_{\hat{a}(\theta)}^{\theta} I'(t)\delta m(t)dt + \delta^2 I. \tag{5.23}$$

Note that $\hat{a}(t) < \tilde{a}(t)$, $t \in [\theta, T]$ and $I'(\tilde{a}; t) = 0$, $t \in [a(\theta), \theta]$. Then, by applying Lemma 4.3 for $\mu = t_2$, we obtain $I'(\hat{a}; t) < 0$ for $t \in [a(\theta), \theta]$. Next, in view of (4.12) we have

$$\begin{aligned} \int_{a(\theta)}^{\theta} I'(t)\delta m(t)dt &= I'(\zeta)\int_{a(\theta)}^{\theta} \delta m(t)dt = \\ &= I'(\zeta)\int_{a(\theta)}^{a(\theta)+\delta a(\theta)} [m(\tau) + \delta\tilde{m}(\tau)]d\tau < \\ &< I'(\zeta)M(\theta)\delta a(\theta), \quad a(\theta) \le \zeta \le \theta. \end{aligned} \tag{5.24}$$

Let us evalute

$$I'(\zeta) < \int_{\zeta}^{\tilde{a}^{-1}(\zeta)} \rho(\tau)[\beta(\zeta,\tau) - \beta(a(\tau),\tau)]d\tau =$$

$$= \int_{\zeta}^{\tilde{a}^{-1}(\zeta)} \rho(\tau)[\zeta - a(\tau)]\partial\beta(\xi,\tau)/\partial\xi d\tau, \quad \tilde{a}(\zeta) < \xi(\tau) < \zeta.$$

For further analysis we introduce the following function:

$$\chi(t) := \rho(t)\partial\beta(\tilde{a}(t),t)/\partial\tilde{a} \tag{5.25}$$

and show that it is non-increasing in t. Indeed, in view of the first condition of the Theorem, $\rho'(\tilde{a}(t))/\rho(\tilde{a}(t)) \geq \rho'(t)/\rho(t)$, and $\tilde{a}'(t) \to 1$ at $t \to \infty$ by its third condition, hence, $\rho'(\tilde{a}(t))/\rho(\tilde{a}(t))\tilde{a}'(t) \geq \rho'(t)/\rho(t)$ for a large t, i.e. $[\rho(t)/\rho(\tilde{a}(t))]' \leq 0$. Since the function $\rho(t)/\rho(\tilde{a}(\tau))$ is non-increasing, and in view of the first condition of the Theorem, $\rho(\tilde{a}(t))\partial\beta(\tilde{a}(t),t)/\partial\tilde{a}$ is non-increasing for any τ, t, $\tau < t$, the function $\chi(t) = \rho(t)\partial\beta(\tilde{a}(t),t)/\partial\tilde{a}$ is also non-increasing. Then we obtain

$$I'(\zeta) \leq \chi(\zeta) \max_{[\tilde{a}(\zeta),\zeta]} [\frac{\partial\beta(\xi(\tau),\tau)/\partial\xi(\tau)}{\partial\beta(\tilde{a}(\tau),\tau)/\partial\tilde{a}(\tau)}] \int_{\tilde{a}(\zeta)}^{\zeta} [\zeta - u]/\tilde{a}'(u)du <$$

$$< \chi(\tilde{a}(\theta)) \max_{[\tilde{a}(\zeta),\zeta]} [\frac{\partial\beta(\xi(\tau),\tau)/\partial\xi(\tau)}{\partial\beta(\tilde{a}(\tau),\tau)/\partial\tilde{a}(\tau)}][\zeta - \tilde{a}(\zeta)]^2/\tilde{a}'_{min} := \chi(\tilde{a}(\theta))R_1R_2, \tag{5.26}$$

$$\text{where} \quad \tilde{a}'_{min} := \min_{[\tilde{a}(\zeta),\zeta]} a'(\tilde{u}), \quad R_1 := \max_{[\tilde{a}(\zeta),\zeta]} [\frac{\partial\beta(\xi(\tau),\tau)/\partial\xi(\tau)}{\partial\beta(\tilde{a}(\tau),\tau)/\partial\tilde{a}(\tau)}],$$

$$R_2 := [\zeta - \tilde{a}(\zeta)]^2/\tilde{a}'_{min}.$$

If $\partial\beta(\tau,t)/\partial\tau$ is non-increasing, then $R_1 = 1$, otherwise $t - \tilde{a}(t)$ is bounded at $t \to \infty$ under the conditions of the Theorem, and so, R_1 is bounded at $T \to \infty$. Next, since $I'(\zeta) > 0$, only the variations $\delta a(\theta) \geq 0$ can lead to $\delta I > 0$. By the third condition of the Theorem, the values $\delta a(\theta)/T \leq (\theta - \tilde{a}(\theta))/T \to 0$, $\tilde{a}'(T) \to 1$, and $(\zeta - \tilde{a}(\zeta))/T \to 0$ for $T \to \infty$. Hence, $R_1/T \to 0$ for $T \to \infty$.

Consider now any admissible pair (a^0, m^0) such that the trajectory $a^0(t)$, $t \in [t_0, T]$ does not strive to $\tilde{a}$ in the sense indicated in statement of this Theorem. Then for any $T > t_0$ a value ε and a subset Δ of interval $[t_0, T]$ exist such that $|a^0(t) - \tilde{a}(t)| > \varepsilon$, $t \in \Delta$, and $mes(\Delta) > \kappa(T - t_0)$ for $T \to \infty$, $\kappa =$const> 0. Since $(T - \tilde{a}(\theta))/T \to 0$ for $T \to \infty$, hence, $mes(\tilde{\Delta}) \to mes(\Delta)$ for $T \to \infty$, where $\tilde{\Delta} := \Delta \cap [\mu, \tilde{a}(\theta)]$. So, we get

$$\int_{\mu}^{\tilde{a}(\theta)} [\delta a(t)]^2 dt = \int_{\mu}^{\tilde{a}(\theta)} \left(a^0(t) - \tilde{a}(t)\right)^2 dt > \varepsilon^2 mes(\Delta) \overset{t\to\infty}{\longrightarrow} \infty. \tag{5.27}$$

Consider the second variation $\delta^2 I$ in (5.23). In view of Lemma 4.1 and the first condition of Theorem, we have

$$\begin{aligned}\delta^2 I \leq -\int_{t_0}^{T} \chi(t) \int_{\tilde{a}(t)}^{\tilde{a}(t)+\delta a(t)} [\tau - a(t)][m(\tau) + \delta m(\tau)] d\tau dt < \\ < -\chi(\tilde{a}(\theta)) \int_{\mu}^{\tilde{a}(\theta)} \int_{\tilde{a}(t)}^{\tilde{a}(t)+\delta a(t)} [\tau - a(t)][m(\tau) + \delta m(\tau)] d\tau dt.\end{aligned}$$

Suppose first that $\hat{m}(\tau) + \delta m(\tau) \geq d_m > 0$ for $t \in [t_0, \tilde{a}(\theta)]$. Then

$$\delta^2 I \leq -d_m \chi(\tilde{a}(\theta)) \int_{\mu}^{\tilde{a}(\theta)} [\delta a(t)]^2 dt/2. \tag{5.28}$$

Assume now that $\hat{m}(\tau) + \delta m(\tau)$ for $t \in \Delta \subset [t_0, \tilde{a}(\theta)]$. It is not possible in the first case of the fifth condition of the Theorem (then $m_{min}(t) = P'(t) > 0$, $t \in [t_0, T]$), but it may be valid in the second case of this condition.

Let Δ_1 be of finite length at $T \to \infty$ (or $mes(\Delta) \to \infty$, but $mes(\Delta_1)/T \to 0$). Then

$$\delta^2 I \leq -d_m \chi(\tilde{a}(\theta)) \int_{[\mu, \tilde{a}(\theta)] - \Delta_1} [\delta a(t)]^2 dt/2.$$

Let $mes(\Delta_1) \to \infty$ for $T \to \infty$ such that $mes(\Delta_1) > \kappa_1(T - t_0)$, $\kappa_1 =$const> 0. Then, by the fifth condition of Theorem, we get

$$\hat{m}(t) \geq d_m > 0, \quad t \in [t_0, \mu],$$

$$\hat{m}(t) = P'(t) + m(\hat{a}(t))\hat{a}'(t) \geq P'(t) + min\{d_0, d_m\}\hat{a}'(t) > 0, \quad t \in [\mu, \theta].$$

Hence,

$$\int_{t_0}^{T} \hat{m}(t)dt \to \infty \quad \text{at} \quad T \to \infty, \quad \int_{t_0}^{\tilde{a}(\theta)} |\delta m(t)|dt > \int_{\Delta_1} |\hat{m}(t)|dt \to \infty.$$

Then by the theorem of the mean and (4.12), we have

$$\delta^2 I < -\int_{\mu}^{\tilde{a}(\theta)} \chi(t)[\xi(t) - a(t)] \int_{\underline{a}(t)}^{t} \delta m(\tau)d\tau dt <$$
$$< -\chi(\tilde{a}(\theta)) \int_{\mu}^{\tilde{a}(\theta)} \delta m(t) \int_{t}^{\underline{a}^{-1}(t)} [\xi(\tau) - a(\tau)]d\tau dt <$$
$$< -d_\zeta \chi(\tilde{a}(\theta)) \int_{[\mu, \tilde{a}(\theta)] \cap \Delta_1} \delta m(t)(a^{-1}(t) - t)dt,$$

since for $t \in \Delta_1$ $\delta m(t) > 0$ and, in view of (5.12), $\xi(t) - a(t) \geq d_\zeta > 0$.

Finally, in view of (5.23)-(5.28) we obtain the following inequality

$$\delta I < \chi(\tilde{a}(\theta))[-d_m \varepsilon^2 mes(\tilde{\Delta}) + M(\theta)\delta a(\theta) R_1 R_2],$$

where $\delta a(\theta)/T \to 0$, $R_2/T \to 0$ and $mes(\tilde{\Delta}) > \kappa(T - t_0)$ for $T \to \infty$. Hence, $\delta I > 0$ for all $T > \tilde{T}$, starting from some large $\tilde{T}$. This implies that the inequality $I(a^0, m^0) < I(\hat{a}, \hat{m})$ is valid and the pair (a^0, m^0) cannot be solution.

Theorem is proved.

Remark 1. The first and third conditions of Theorem 5.6 are true , at least, for $\rho(t) = \exp(-st)$, $\beta(\tau) = \exp(c\tau)$, $0 < c < s$, $\lambda' = 0$ (see Theorem 5.4) and for $\beta(\tau) = \tau^p$, $p > 0$, $\lambda' = 0$, $\rho' = 0$ (see Theorem 5.1).

Remark 2. The fifth condition of Theorem 5.6 is essential. If it is not fulfilled, then $\hat{m}(t) = 0$ can be valid on a set $\Delta_0 \subset [t_0, T)$ being of infinite length at $T \to \infty$ $(mes(\Delta_0) > \chi(T - t_0), \chi > 0)$. Similar $\hat{m}$ has been given in Example 4.1. Then, in view of Remark 1 for Lemma 4.1, a variation $\delta a(t)$ is unstable on Δ_0 with respect to infinitesimal δm. In this case the technique used for the proof of Theorem 5.6 fails, and the counterexample can be constructed such that $-C\varepsilon \leq \delta^2 I \leq 0$, $C > 0$, for $T \to \infty$. Indeed, let $\hat{m}(t) = 0$, $t \in \bigcup_{i=1}^{n(T)} [t_i', t_i'']$, then $\delta a(a^{-1}(t_i')) > t_i'' - t_i'$ for any $\delta m(\tau) > 0$, $t_i' < \tau < a^{-1}(t_i')$. For each $i = \overline{1, n}$

we set $\delta m(\tau) = \Delta_1$ at $t_i'' < \tau < t_i'' + \Delta_m$, and $\delta m(\tau) \equiv 0$ otherwise. Then

$$\delta^2 I = \int_{t_0}^{T} \rho(t) \int_{a(t)}^{a(t)+\delta a(t)} [\beta(a(t),t) - \beta(\tau,t)][m(\tau) + \delta m(\tau)] d\tau dt \geq$$

$$\geq -\int_{t_0}^{T} \rho(t)[\beta(a(t) + \delta a(t), t) - \beta(a(t),t)] \int_{\underline{a}(t)}^{t} \delta m(\tau) d\tau dt \geq$$

$$\geq -D_\beta \int_{t_0}^{T} \int_{\underline{a}(t)}^{t} \delta m(\tau) d\tau dt \geq -D_\beta \sum_{i=1}^{n(T)} (\Delta_1 \Delta_m) \geq -D_\beta \Delta_m (T - t_0).$$

By choosing $\Delta_m = C/(D_\beta(T - t_0))$, we have $\delta^2 I \geq -C\varepsilon$.

However, as examples suggest, Theorem 5.6 is also borne out in the cases when its fifth condition does not assert. It can be seen from Example 4.1 for optimal trajectory a (see also Fig.4.1). General case requires more refined analysis and can be studied on the base of the notion of OP quasi-solution (see Section 4). If the quasi-solution (a_q, m_q) exists, then in view of Theorem 4.7, an optimal trajectory a^* coincides with a_q except some initial finite time interval $[t_0, \mu)$.

Thus, the analysis of the asymptotics of OP solution a^* reduces to the study of quasi-solution behavior. The quasi-solution is constructed recursively (from interval $[\theta, T]$ to the left), and it is independent of initial condition (4.6).

It may be proved that the quasi-solution a_q and turnpike $\tilde{a}$ intersect at a single point on each interval $[\tilde{a}^k(T), \tilde{a}^{k-1}(T)]$, $\tilde{a}^k(t) = \tilde{a}(\tilde{a}(...\tilde{a}(t)..))$. So, the quasi-solution a_q oscillates about a turnpike trajectory (see Fig.4.1). However, in general case the analysis of quasi-solution requires difficult study of its existence, the reversibility of corresponding functions, etc. It makes sense to perform such research for specific cases only.

5.2.2 The Turnpike Theorem in a Strong Form

In some cases when it is succeeded to construct the explicit solution of OP, it is possible to strengthen previous turnpike theorem. For instance, the next turnpike property of OP1 follows from Theorem 4.8.

Theorem 5.7 *(the turnpike theorem in a strong form).* If $\beta(\tau, t) = c_1\tau + c_2$, $\rho(t) \equiv \rho$, $\lambda(t) \equiv \lambda$, $P'(t) \geq 0$, then for any $\varepsilon > 0$ the proper number k exists, $k = (\sqrt{\lambda/c_1\varepsilon^2 + 1} - 3)/2$, such that the condition $|a^*(t) - \tilde{a}(t)| < \varepsilon$ is true

for the solution $a^*(.)$ of OP (4.1)-(4.7) at $t \in [\mu, T - \Delta_e]$, where the time μ is defined in Theorem 4.7, and $\Delta_e := k\sqrt{2\lambda(k+2)/(k+1)/c_1}$.

Proof. As it was shown in Theorem 5.1, in this case the equation (5.14) has the unique solution $\tilde{a}(t) = t - A$, $t \in [t_0, \infty)$, where the constant A is determined by the non-linear equation $A - (1 - e^{-sA})/s = s\lambda/c_1$. On the other hand, the explicit OP solution $a^*(t)$, $t \in [t_0, T]$ is defined by formulas (4.16), (4.18). Further proof is trivial.

Thus, in this case the optimal trajectory $a^*(.)$ could deviate from the turnpike trajectory $\tilde{a}(.)$ only at the begin and at the end of the interval $[t_0, T]$ on its part being of finite length for $T \to \infty$.

5.2.3 The Turnpike Theorem in the Strongest Form

The strongest result is obtained for the case of infinite optimization horizon $[t_0, \infty)$ in OP1.

Theorem 5.8 (*the turnpike theorem in the strongest form*). If $\beta(\tau, t) = e^{c\tau}$, $\lambda(t) = be^{b_1 t}$, $\rho(t) = e^{-st}$, $b, c, s > 0$, $b_1 \leq c < s$, then there is the unique function (turnpike) $\tilde{a}(t)$, $\tilde{a}'(t) > 0$, $t \in [t_0, \infty)$ which does not depend on the initial condition (4.6). This function $\tilde{a}(t) \overset{t\to\infty}{\longrightarrow} t$ for $b_1 < c$, and $\tilde{a}(t) \equiv t - A_1$ for $b_1 = c$ (the constant A_1 is defined by equation (5.19)). If $T = \infty$, then the unique solution $a^*(.)$ of OP (4.1)-(4.7) at $T = \infty$ coincides with $\tilde{a}(.)$ beginning with a instant $\mu \geq t_0$: $a^*(t) \equiv \tilde{a}(t)$, $t \in [\mu, \infty)$, provided the proper m^* satisfies (4.7). The value $\mu - t_0$ is defined only by the value of deviation $|\tilde{a}(t_0) - a_0|$ ($\mu = t_0$, if $a(t_0) = a_0$).

Proof. The existence of unique function $\tilde{a}(t)$, $t \in [t_0, \infty)$ has been shown in Section 3.2 and the existence of unique solution $a^*(.)$ has been proved in Theorem 4.5.

5.3 SOME GENERAL PROPERTIES OF OPTIMAL ECONOMIC RENOVATION

By means of considered IDMs different turnpike theorems (in normal, strong and the strongest forms) have been established for optimal strategies of economic system renovation (for the optimal lifetime of technologies or equipment). Their importance consists in finding some "effective" trajectories (turnpikes) which are close to the solutions of suitable optimization problems, but with more simple structure. The turnpikes are independent of the given manpower P and initial technological structure m_0, a_0 of economic system. They are defined only by the efficiency function β which gives the dynamics of technological change.

Thus, the turnpike theorem in the strongest form has been proved for the infinite-horizon optimization problem. This theorem confirms that the optimal trajectory a^* coincides with a turnpike one $\tilde{a}$ except some initial interval of finite length. But for the analogous finite-horizon optimization problem we can show just less strong turnpike properties: the solution $a^*(.)$ tends to $\tilde{a}(.)$ at $T \to \infty$ on an asymptotically largest part of horizon interval $[t_0, T]$, the optimal control having more complex structure. This fact can be explained in by such way: from the economic viewpoint the finite-horizon optimization is less natural than discounted infinite-horizon optimization (in the last case all instants $t > t_0$ enjoy equal rights).

The similar results can be established for some other smooth functions $\beta(\tau, t)$ which increase monotonically. However, the problem to determine at whole a class of the functions β which satisfy turnpike theorems,is opened. As has been shown in Theorem 5.3, the solutions of integral-differential equation (5.1) do not approach a unique function $\tilde{a}(t)$, $t \in [t_0, \infty)$ at $T \to \infty$ in case $\beta(\tau) = \ln \tau$, λ =const, $s = 0$. This fact indicates that the mentioned class is bounded. The progress in this field is connected with the investigation of the solvability of IFE (5.1) on the infinite interval $[t_0, \infty)$.

In this connection the study of general properties of turnpike trajectories, if they exist, is useful. For this purpose we represent the function β in the form:

$$\beta(\tau, t) = \beta_1(\tau)\beta_2(t). \tag{5.29}$$

Let us emphasize the following four indices which determine the behavior of turnpike trajectory $\tilde{a}(.)$:

- the rate $\beta_1'(\tau)$ of embodied (materialized) technical change,
- the factor $\beta_2(t)$ of autonomous technical change and the physical wear of equipment (see Section 0.1),
- the cost $\lambda(t)$ of new EU (ratio capital/labor),
- discounting factor $\rho(t)$.

Suppose that the IFE (5.1) has a solution $\tilde{a}(t)$, $t \in [t_0, \infty)$ which describes the turnpike trajectory for the lifetime of technologies. Then the following properties of $\tilde{a}$ result from the analysis of the IFE (5.1) and its derivative :

1. If $\beta_1'(t)$ is non-increasing and the function $\beta_1'(t)\beta_2(t)/\lambda(t)$ is bounded at $t \in \infty$ (or $\beta_1'(t)$ is non-decreasing and $\beta_1(t)\beta_2(t)/\lambda(t)$ is bounded), then $t - \tilde{a}(t)$ is bounded at $t \in \infty$.
2. If $\beta_1'(t)\beta_2(t)/\lambda(t) \to 0$ at $t \to \infty$, then $t - \tilde{a}(t) \to 0$ at $t \to \infty$.
3. If one of the functions $\beta_1'(t)$, $\beta_2(t)$, $\rho(t)$ decreases or $\lambda(t)$ increases, whereas other three indices are constant, then the function $t - \tilde{a}(t)$ increases at $t \to \infty$, and conversely.
4. If all the above indices are constant, then $t - \tilde{a}(t) \equiv$ const, $t_0 < t < \infty$ (the quasistationary solution).

According to economic content, the function $\tilde{L}(t) = t - \tilde{a}(t)$ is the optimal (turnpike) lifetime of EUs. The following *properties of rational EU lifetime* can be deduced from the mentioned results:

- The rational equipment lifetime $\tilde{L}(t)$ is first determined by the rate β_1' of embodied technological change. If this rate $\beta_1'(t)$ is not positive for all t, then the rational (turnpike) lifetime trajectory $\tilde{L}(t)$ does not exist for an arbitrary dynamics of other indices (β_2 and λ).
- At the constant factor β_2 of autonomous technical progress and constant EU price λ, the lifetime $\tilde{L}(t)$ decreases if relative rate β_1' decreases, and conversely.
- At the constant relative rate $\beta_1'(\tau)$ of embodied technological change and constant EU price λ, this lifetime $\tilde{(t)}$ increases if $\beta_2(t)$ decreases (physical wear), decreases if $\beta_2(t)$ increases (autonomous technical change), and is constant if $\beta_2(t) \equiv$const.

- The constant equipment lifetime $\tilde{L}(t)$ can also occur in the cases where the above indices are mutually neutralized (for example, in the case $\beta_1(\tau) = \lambda(\tau) = e^{ct}$, $\beta_2(t) \equiv$const of Theorem 5.4).
- The rational strategy of technological renovation is the same for economic systems of different scale and is independent of the amount $m(t)$ of renewed equipment provided that the renovation occurs (i.e., if $m(t) > 0$).

Note that the last mentioned conclusion was first deduced in [96].

6

OTHER OPTIMIZATION PROBLEMS IN ONE-SECTOR MODELS

On the basis of the results described in Chapters 4 and 5 a number of other optimization problems (OPs) has been analyzed in the one-sector IDM (4.1)-(4.3) as well as in more complicated two-, three- and n-sector ones [4], [61], [62], [99], [145], [146], [152], [153]. Specifically, their turn-pike properties have been studied. The summary of the results of investigation of some OPs in one-sector models is given in the present Chapter.

We consider some different classical problems of mathematical economics within the framework of the IDMs with controlled lifetime of technologies. In the OPs under study the basic structure of OP1 is preserved: the control influence is made by the change of the renovation intensity of technological structure.

In other words, the main objective of the Chapter is to trace the change of qualitative behavior of the OP solutions if various features of economic development are taken into account. Special emphasis is laid on generality and distinctions of turnpike properties as against those of the basic OP1 (4.1)-(4.6).

In Section 6.1 the problem of the expenditure minimization under given output is considered in the basic one-sector IDM (4.1)- (4.3). This OP is commonly encountered for medium-sized economic systems and is the inverse of the basic OP1 from the mathematical point of view. The behavior of their solutions is similar to that of OP1 on qualitative level (but distinct quantitatively).

The essential generalization of the basic OP1 is studied in Section 6.2. It arises when we suppose that the output produced by each EU depends on two factors: capital and manpower. The OP is the analog of OP1 for the Kantorovich integral model (2.6)-(2.7). This model includes the capital/labor ratio for each

EU and EU lifetime as endogenous (sought-for) variables. Such problems belong to classical ones of mathematical economics. On the basis of this OP it is proved that the optimal cost of new EUs possesses a turnpike property as well as the EU lifetime. In the case of infinite-horizon optimization and Cobb-Douglas production function the turn-pike trajectories are the constant lifetime of EUs and an exponential increase of the cost of new EUs.

In the Section 6.3 the simplest IDM with endogenous embodied TC is considered. In this model the rise of the efficiency of new EUs (the rate of embodied TC) is supposed to be controlled by means of certain innovations. No turnpike properties exist in this OP. Hence, the control of EU efficiency is more strong than other types of economic controls.

In the last two sections we consider the one-sector IDM with accounting for the expenditure (instantaneous and distributed in time) for the development of new EU vintages along with the cost of each new EU. There are no classical turnpike properties in these OPs. Nevertheless, the tendency presents for the rational lifetime of EUs to strive to some "favorable" regime (if it exists).

6.1 EXPENDITURE MINIMIZATION UNDER GIVEN OUTPUT

From the economic point of view, other natural criterion is to minimize the expenditures for economic system functioning. It is common for medium-sized economic systems. Assume that the output Q of an economic system is given and it is necessary to minimize the expenditure of some aggregate resource R.

Introduce the specific expenditure $\delta(\tau, t)$ of resource R per time unit at instant t on one EU created at instant τ (in output measure units). Then the total expenditures at time t of an economic system include the total current expenditure of the resource R and the gross investment $\lambda(t)m(t)$ (in output units). So, the total expenditures on some planned interval $[t_0, T)$ are equal to

$$I_2 = \int_{t_0}^{T} \rho(t)[\int_{a(t)}^{t} \delta(\tau, t)m(\tau)d\tau + \lambda(t)m(t)]dt \longrightarrow \min_{a,m} \tag{6.1}$$

where $\rho(t)$ is a discounting factor, $0 < \rho \leq 1$, $\rho' < 0$.

The corresponding optimization problem (designated as **OP2**) consists in the minimization of functional (6.1) under given Q, β, δ, constrainsts (4.1), (4.5), (4.6) and initial conditions (4.3).

From the conceptual point of view, the OP2 is the inverse of the basic OP (4.1)-(4.6) and can be studied in a similar way. But OP2 has the distinct behavior of solutions. The independent (m) and phase (a) sought-for ***control variables*** can be chosen the same as in previous one. In this case the state equation for the determination of phase variable $a(.)$ is (4.1) instead of (4.2), hence, the minimum possible function m_{min} (see Theorem 4.1) and the set U of admissible variables $m(.)$ are determined in more complex way.

The gradient of functional (6.1) in OP2 is of the following form (if $m(t) > 0$, $t \in [t_0, T)$):

$$I_2'(t) = \int_t^{\underline{a}^{-1}(t)} \rho(\tau)[\delta(t,\tau) - \delta(a(\tau),\tau)\beta(t,\tau)/\beta(a(\tau),\tau)]d\tau + \rho(t)\lambda(t),$$

what is different from the gradient (4.9) in OP1.

The important special case of OP2 arises when the resource expenditure is directly proportional to the quantity of operated EUs (then $\delta \equiv$ const). The example of such resource is manpower (labor resource). In this Section we restrict ourselves to the case $\delta(\tau, t) \equiv 1$. Then the gradient of OP2 is of the form:

$$I_2'(t) = \int_t^{\underline{a}^{-1}(t)} \rho(\tau)[1 - \beta(t,\tau)/\beta(a(\tau),\tau)]d\tau + \rho(t)\lambda(t).$$

The fundamental properties of basic OP1 are preserved in OP2: as is shown below, the analogs of Theorems 4.8-4.10 on normal, strong and strongest turnpike remain true for this OP. However, the equation $I_2'(a;t) \equiv 0$ for turnpike trajectory $\tilde{a}(.)$ differs from IFE (5.1). The turnpike regimes are accordingly different. As distinct from equations (4.3) and (4.7), the IFE $I_2'(a;t) \equiv 0$ has the ***quasistationary solution*** $a(t) \equiv t - A$, $A =$const, $t \in [t_0, \infty)$ in case where $\beta(\tau, t) := \beta(\tau) = e^{c\tau}$, $\rho(t) = e^{-st}$, $c < s$, $\lambda(t) \equiv \lambda =$const. Thus, rational EU lifetime is constant under constant EU price λ if the ***relative rate*** β_τ'/β of embodied technological change is constant (rather than absolute rate β_τ' as in basic OP1). Arelative rate of embodied TC

6.1.1 Transformation of OP2

For the analysis of the problem of expenditure minimization, it is also convenient to express it through the characteristics of productive capacity. Such problems are typical for medium-sized economic systems which are normally studied and managed in terms of capital and capacity units [123], [127]. Corresponding integral model (2.9)-(2.10) of technological renovation was exposed in [122], [124] and has been illustrated above in Section 2.4.

In the IDM (2.9)-(2.10), the non-increasing function $A(\tau) = 1/\beta(\tau)$ is the historical technological labor/output coefficient $a(t)$ of the EU created at time t, and $\Lambda(t) = \lambda(t)/\beta(t)$ is EU price (in units of productive capacity).

Let us assume a labor $P(t)$ described by (2.10) to be the only current cost of production, and $\delta \equiv 1$. Then the total expenditures of a firm on an interval $[t_0, T)$ are equal to

$$I_2 = \int_{t_0}^{T} \rho(t)[\int_{a(t)}^{t} A(\tau)m(\tau)d\tau + \Lambda(t)m(t)]dt \longrightarrow \min_{a,m} \tag{6.2}$$

In terms of A, Λ the gradient of functional (6.2) is of the form

$$I_2'(t) = \int_{t}^{\underline{a}^{-1}(t)} \rho(\tau)[A(t) - A(a(\tau))]d\tau + \rho(t)\Lambda(t) \tag{6.3}$$

and coincides with the gradient (4.9) of OP1.

Another important point is that the equation (2.9) describing the capital stock $K(t)$ has the form equivalent to (4.2). So, the minimum possible function m_{min} in this modified OP2 is the same as in OP1 and is determined by Theorem 4.1.

Thus, after fulfilled transformation, the OP2 and basic OP1 are equivalent from the mathematical point of view. This enables us to study OP2, in particular, to prove the turnpike properties in a way completely analogous to that used for OP1 in previous chapters.

The only new feature is that the case of decreasing function $A(t)$ is of interest (instead of increasing $\beta(t)$ in OP1).

In this connection it may be proved by analogy with Section 5.1 that in the case when $A(t) = A_0 \exp(-c_1 t)$, $\Lambda(t) = \Lambda_0 \exp(-c_2 t)$, $\rho(t) = \rho_0 \exp(-c_3 t)$, $c_1, A_0, \Lambda_0 > 0$, $c_2, c_3 \geq 0$, $0 < \rho_0 \leq 1$, the turnpike equation

$$\int_t^{a^{-1}(t)} \rho(\tau)[A(t) - A(a(\tau))]d\tau = -\rho(t)\Lambda(t) \tag{6.4}$$

has a unique solution $\tilde{a}(t)$, $\tilde{a}'(t) > 0$, $t \in [t_0, \infty)$, such that:

1. $t - \tilde{a}(t) \xrightarrow{t\to\infty} 0$ at $c_1 < c_2$,
2. $t - \tilde{a}(t) \xrightarrow{t\to\infty} \infty$, $\tilde{a}(t) \xrightarrow{t\to\infty} \infty$ at $c_1 > c_2$,
3. $\tilde{a}(t) \equiv t - L$ at $c_1 = c_2$,

where constant L is defined from the non-linear equations:

$$c_3 e^{c_1 L} + c_1 e^{-c_3 L} = (c_1 + c_3)(1 + c_3 \Lambda_0 / A_0) \quad \text{at} \quad c_3 > 0,$$
$$(e^{c_1 L} - 1)/c_1 - L = \Lambda_0 / A_0 \quad \text{at} \quad c_3 = 0.$$

In particular, $L \approx \sqrt{2(\Lambda_0/A_0/c_1)}$ for $0 \leq c_3 < c_1 = c_2 \ll 1$.

In the mentioned cases the equation (6.4) has some set of solutions $a_T(t)$, $a_T'(t) > 0$ for any interval $[t_0, T]$ such that all solutions $a_T(.)$ being approach the unique solution $\tilde{a}(.)$ for $T \to \infty$ (in the sense indicated in Theorem 5.1).

Furthermore, the following turnpike properties of OP2 are proved similarly to Theorem 5.6 and 5.8.

6.1.2 Infinite-Horizon Discounted Optimization

Theorem 6.1 (turnpike theorem in the strongest form). If $T = \infty$, $A(t) = A_0 e^{-c_1 t}$, $\Lambda(t) = \Lambda_0 e^{-c_2 t}$, $\rho(t) = \rho_0 e^{-c_3 t}$, $c_1, c_2, A_0, \Lambda_0, \rho_0 > 0$, $c_3 \geq 0$, then there is the unique function (turnpike) $\tilde{a}(t)$, $\tilde{a}'(t) > 0$, $t \in [t_0, \infty)$ which does not depend on initial condition OP2. At $c_1 < c_2$ this function $\tilde{a}(t) \xrightarrow{t\to\infty} t$, at $c_1 > c_2$ $t - \tilde{a}(t) \xrightarrow{t\to\infty} \infty$, $\tilde{a}(t) \xrightarrow{t\to\infty} \infty$, and at $c_1 = c_2 \tilde{a}(t) \equiv t - L$ (constant

L is defined above). Then the unique solution $a^*(.)$ of OP2 at $T = \infty$ coincides with $\tilde{a}(.)$ beginning with an instant $\mu \geq t_0$: $a^*(t) \equiv \tilde{a}(t)$, $t \in [\mu, \infty)$, provided the proper m^* satisfies (4.7). The value $\mu - t_0$ is defined only by the value of deviation $|\tilde{a}(t_0) - a_0|$ ($\mu = t_0$, if $\tilde{a}(t_0) = a_0$).

6.1.3 Finite-Horizon Optimization ($T < \infty$)

Theorem 6.2 *(turnpike theorem in normal form).* Let:

1) $A(t)$ be strictly decreasing, $A'(t)$ be non-decreasing, $t \in [t_0, T]$;

2) the function $M(t)$ be bounded at $t \to \infty$;

3) all solutions $a_T(t)$, $t \in [t_0, T]$ of equation (6.4) approach at $T \to \infty$ a unique trajectory $\tilde{a}(t)$, $t \in [t_0, \infty)$, such that $\tilde{a}'(t) > 0$ and $(t - \tilde{a}(t))/\tilde{a}(t) \to 0$ at $t \to \infty$;

4) the function $\tilde{m}(t)$ defined by $\tilde{a}(t)$ in accordance with equation (4.2) satisfy (4.7);

5) $Q'(t) > 0$, $Q(t) \overset{T\to\infty}{\longrightarrow} \infty$ or $a(t_0) > a_0$, $m_0(\tau) \geq d_0 > 0$, $\tau \in [a_0, t_0]$.

Then, when $T \to \infty$, the solution $a^*(.)$ of OP2 strives to $\tilde{a}(.)$ on an asymptotically largest part of interval $[t_0, T]$, e.g. for any $\varepsilon > 0$ the time a_0 exists such that for any $T \geq a_0$ the condition $|a^*(t) - \tilde{a}(t)| < \varepsilon$ is true on some subset $\Delta \subset [t_0, T]$ such that $mes(\Delta)/(T - t_0) \to 1$ for $T \to \infty$.

As has been shown above, the second condition of Theorem 6.2 is valid, at least in the case $c_2 > 0$.

As in OP1, Theorem 6.2 is also borne out when its fifth condition does not hold. General case can be studied on the basis of the notion of *OP quasi-solution* a_q (see Section 4.6). In the cases when the quasi-solution is defined explicitly, it may be possible to prove the turnpike theorem in strong form analogous to Theorem 5.7 for OP1.

It can be shown that the main features of the OP2 remain unchanged if we assume that other expenditures are also simultaneously minimized (raw material, energy, etc.). In so doing, the behavior of optimal trajectories is defined by the most expensive expenditure.

6.2 MODEL WITH THE CHOICE OF OPTIMAL EQUIPMENT PRICE

Next natural generalization of basic OP1 arises when we suppose that the output produced by the EU created at instant τ depends on two factors: capital $K(\tau)$ and labor $m(\tau)$. It leads to Kantorovich integral model (2.6)-(2.7) described in Section 2.3. This model includes the cost of new EUs along with EU lifetime as endogenous (sought-for) variable, in distinction to other considered IDM. It is natural to study the optimal proportion of capital and labor for each EU. Such problems belong to classical ones of mathematical economics (see [120], [129], [130], etc.).

The corresponding OP (designated as **OP3**) consists in the maximization of the output

$$I_3 = \int_{t_0}^{T} \rho(t)[Q(t) - \lambda(t)m(t)]dt \longrightarrow \max_{a,m,\lambda} \tag{6.5}$$

by choosing the unknown $a(t)$, $m(t)$, $\lambda(t)$, $t \in [t_0, T)$, $T \leq \infty$ under given labor dynamics $P(.)$ in the one-sector Kantorovich IDM (2.8):

$$Q(t) = \int_{a(t)}^{t} \beta(\lambda(\tau), \tau)m(\tau)d\tau, \tag{6.6}$$

$$P(t) = \int_{a(t)}^{t} m(\tau)d\tau, \quad t \in [t_0, T) \tag{6.7}$$

under the restrictions:

$$0 \leq m(t) \leq M(t), \quad 0 < l(t) \leq \lambda(t) \leq L(t), \tag{6.8}$$

$$a'(t) \geq 0, \quad a(t) < t, \quad t \in [t_0, T), \tag{6.9}$$

and initial conditions:

$$a(t_0) = a_0, \quad m(\tau) = m_0(\tau), \quad \lambda(\tau) = \lambda_0(\tau), \quad \tau \in [a_0, t_0]. \tag{6.10}$$

In contrast to basic OP1, it is assumed that there is a continuous set of the new technologies (EUs) with productivities $\beta(\lambda(\tau), \tau)$ which depend on the price $\lambda(\tau)$ of EU (not only the unique new technology with productivity $\beta(\tau, t)$). To put it in another way, the availability of market with wide choice of new EUs as well as the presence of technical progress are assumed. Thus, the sought-for (endogenous) functions $a(.)$, $m(.)$ describing the EU renovation intensity are added by the new sought-for EUs price $\lambda(.)$. The exogenous (given) model characteristics are production function $\beta(\lambda, \tau)$, manpower $P(t)$, and discounting multiplier $\rho(t)$.

6.2.1 Preliminary Results

Let us assume that the given functions $P(t)$, $\rho(t)$, $M(t)$, $l(t)$, and $L(t)$ are Lipschitz continuous, $\beta(\lambda, \tau)$ is twice differentiable, $m_0(.)$ and $\lambda_0(.)$ are piecewise continuous. All these functions are positive and satisfy (6.6)-(6.10) at $t = t_0$.

Let $m(t)$, $\lambda(t)$, $t \in [t_0, T)$, be *the control variables* of OP3, $m, \lambda \in L^\infty[t_0, T)$. Then the functions $Q(t)$, $a(t)$, $t \in [t_0, T)$, are *the phase variables* of OP3. The constraint-equality (6.7) is the same as in the basic OP1, so, in view of Theorem 4.1 the mixed constraints (6.8), (6.9) of OP3 may be replaced by one more strong restriction:

$$m_{min}(t) \leq m(t) \leq M(t), \quad m_{min}(t) = \max\{0, P'(t)\}, \quad t \in [t_0, T), \tag{6.11}$$

for which the phase constraint $a' \geq 0$ is satisfied a priori.

Since the OP3 has two independent controls m and λ, the gradient of functional (6.5) of OP3 is represented by the following two-dimensional vector (if $m(t) > 0$, $t \in [t_0, T)$):

$$I'_m(t) = \int_t^{\underline{a}^{-1}(t)} \rho(\tau)[\beta(\lambda(t), t) - \beta(\lambda(a(\tau)), a(\tau))]d\tau - \rho(t)\lambda(t), \tag{6.12}$$

$$I'_\lambda(t) = [\beta'_\lambda(\lambda(t),t)\int_t^{\underline{a}^{-1}(t)} \rho(\tau)d\tau - \rho(t)]m(t), \quad t \in [t_0, T). \tag{6.13}$$

where the function $\underline{a}^{-1}(t)$ is defined in Lemma 4.1.

Hence, at $m(t) \neq 0$ the corresponding IFE system $I'_m(.) \equiv 0$, $I'_l(.) \equiv 0$ for turnpike trajectories $\tilde{a}(.)$ and $\tilde{\lambda}(.)$ is of the following form:

$$\int_t^{\underline{a}^{-1}(t)} \rho(\tau)[\beta(\lambda(t),t) - \beta(\lambda(a(\tau)), a(\tau))]d\tau = \rho(t)\lambda(t),$$

$$\beta'_\lambda(\lambda(t),t)\int_t^{\underline{a}^{-1}(t)} \rho(\tau)d\tau = \rho(t), \quad t \in [t_0, T), \tag{6.14}$$

which depends on $a(.)$, $\lambda(.)$ only.

The variation of functional I_3 up to the third order infinitesimals is of the following form:

$$\begin{aligned}\delta^2 I = \int_{t_0}^T [I'_m(t)\delta m(t) + I'_\lambda(t)\delta\lambda(t) + I''_{a^2}(t)((\delta a(t))^2 + \\ + I''_{\lambda^2}(t)(\delta\lambda(t))^2 + 2I''_{\lambda m}(t)\delta m(t)\delta\lambda(t) + \\ + 2I''_{\lambda(a)a}(t)\delta\lambda(a(t))\delta a(t)]dt + \int_{t_0}^T o(\delta m^2, \delta\lambda^2, \delta a^2)dt,\end{aligned} \tag{6.15}$$

where

$$\begin{aligned} I''_{a^2}(t) &= -\rho(t)\beta'(\lambda(a(t)), a(t))m(a(t))/2; \\ I''_{\lambda^2}(t) &= \beta''_{\lambda\lambda}(\lambda(t),t)\int_t^{\underline{a}^{-1}(t)} \rho(\tau)d\tau m(t)/2; \\ I''_{\lambda(a)a}(t) &= -\rho(t)\beta'_\lambda(\lambda(a(t)), a(t))m(a(t))/2; \\ I''_{\lambda m}(t) &= [\beta'_\lambda(\lambda(t),t)\int_t^{\underline{a}^{-1}(t)} \rho(\tau)d\tau - \rho(t)]/2, \ t \in [t_0, T). \end{aligned} \tag{6.16}$$

It follows from (6.13) and (6.16) that $I''_{\lambda m(t)} \equiv 0$ if $I'_\lambda(t) \equiv 0$.

Let $I'_m(t) \equiv 0$, $I'_\lambda(t) \equiv 0$, $t \in [t_0, T)$. Then the variation (6.15) of I_3 is negative if and only if

$$\beta'(\lambda(a(t)), a(t) > 0;$$

$$\beta'(\lambda(a(t)),a(t))\beta''_{\lambda\lambda}(\lambda(a(t)),a(t))a'(t)\int_{a(t)}^{t}\rho(\tau)d\tau <$$
$$< -\rho(t)\beta'^2_{\lambda}(\lambda(a(t)),a(t)), \quad t\in[t_0,T). \tag{6.17}$$

The proof of the statement (6.17) is based on the well-known conditions $I''_{a^2} < 0$, $\det D > 0$ for the negative definiteness of the matrix

$$D = \begin{pmatrix} I''_{a^2} & I''_{\lambda(a)a} \\ I''_{\lambda(a)a} & I''_{\lambda^2(a)} \end{pmatrix}$$

Under the conditions (6.17) the necessary and sufficient conditions of extremum (n.s.c.e.) can be established for OP3 in the form similar to Theorem 4.2.

6.2.2 The Case of Cobb-Douglas Production Function

Like [75], [76], the OP3 in model (6.6)-(6.7) is studied under the assumption that the production function is the Cobb-Douglas function with exponential embodied TC [19], [94], [102], [129]:

$$\beta(\lambda,\tau) = be^{q\tau}\lambda^{1-p}, \quad 0<p<1, \quad b,q>0. \tag{6.18}$$

Let us also suppose that $\rho(t) = e^{-st}$, $s>0$, $t\in[t_0,T)$. Then it can be shown in the same way as in Section 5.1, that the system of IFEs (6.16) has the unique solution on infinite interval $[t_0,\infty)$ of the following form:

$$\tilde{a}(t) = t - A, \quad \tilde{\lambda}(t) = b_0 e^{qt/p}, \quad t\in[t_0,\infty), \tag{6.19}$$

where $b_0 = [b(1-p)(1-e^{-sA})/s]^{1/p}$, and the constant A is determined by the nonlinear equation: $s(e^{(sp-q)A/p}-1) = (sp-q)(e^{sA}-1)$.

In the case where (6.18) is valid and $a\equiv\tilde{a}$, $\lambda\equiv\tilde{\lambda}$, it follows from (6.17) that the variation (6.15) of functional I_3 is negative if and only if the condition

$$[s(1-p)/q+1]^{q/s}p^p < 1$$

is true for the parameters p, q of production function (6.18) and discount rate s.

6.2.3 Infinite-Horizon Optimization

Let $T = \infty$, $\rho(t) = e^{-st}$, $s > 0$, $t \in [t_0, T)$. If $q < ps$, function $M(t)$ is bounded, and $L(t) = O(e^{qt/p})$ at $t \to \infty$ then the improper integral converges in (6.5). The proof of this statement is completely analogous to that of Theorem 4.4.

In the case under consideration, the theorem on solution structure and the turnpike theorem in the strongest form can be established, similar to the Theorems 4.5, 5.8 for basic OP1 .

Theorem 6.3 *(on the structure of OP3 solution).* Let $T = \infty$, $\rho(t) = e^{-st}$, $L(t) \geq [b/s(1-p)]^{1/p} e^{qt/p}$, $l(t) = o(e^{qt/p})$, the conditions (6.18) and

$$0 < q < ps, \quad [s(1-p)/q + 1]^{q/s} p^p < 1 \tag{6.20}$$

be valid. Then the OP (6.5)-(6.11) has unique solution (a^*, m^*, λ^*) of the following structure:

$$\lambda^*(t) = \begin{cases} \lambda_\mu(t), & t \in [t_0, \overline{a}(\mu)), \\ \tilde{\lambda}(t), & t \in [\overline{a}(\mu), \infty), \end{cases}$$

$$m^*(t) = \begin{cases} m_{min}(t) \quad \text{or} \quad M, & t \in [t_0, \mu), \\ \tilde{m}(t), & t \in [\mu, \infty), \end{cases} \tag{6.21}$$

$$a^*(t) = \begin{cases} a_\mu(t), & t \in [t_0, \mu), \\ \tilde{a}(t), & t \in [\mu, \infty), \end{cases}$$

where $\overline{a}(t) := \max\{t_0, a^*(t)\}$, and the function

$$a_\mu(t) = \begin{cases} a_{min}(t), & \text{if} \quad a_0 > \tilde{a}(t_0), \\ a_{max}(t), & \text{if} \quad a_0 < \tilde{a}(t_0), \end{cases}$$

is found from (6.7) at $m^* = m_{min}$ or $m^* = M$, the function $\lambda_\mu(t)$ is defined by $I'_\lambda(t) = 0$, $t \in [t_0, \mu]$, the function $\tilde{m}$ is found from (6.7) for $a = \tilde{a}$, and the instant μ – from the condition $a_\mu(\mu) = \tilde{a}(\mu)$.

Proof is carried out in much the same manner as that of Theorem 4.5. The only new feature is connected with the construction of the optimal control

λ^* complying with (a^*, m^*). Let $a(t) = \tilde{a}(t)$, $\lambda(t) = \tilde{\lambda}(t)$ for $t \in [\mu, \infty)$, $\mu > t_0$. Then one can show that for any function $a_\mu(t)$, $t \in [t_0, \mu]$ satisfying (6.7),(6.9), the corresponding control $\lambda_\mu(t)$, $t \in [t_0, \mu]$ can be constructed such that $I'_\lambda(t) = 0$, $t \in [t_0, \mu]$. In so doing, $\lambda_\mu(t) = \tilde{\lambda}(t)$ for $t \in [\tilde{a}(\mu), \mu]$. If $t_0 < \tilde{a}(\mu)$, then $l(t) \leq \lambda_\mu(t) < L(t)$ for $t \in [t_0, \tilde{a}(\mu)]$.

The further proof is based on the n.s.c.e. for OP3 and is identical to the proof of Theorem 4.5.

The important corollary results from the Theorem 6.3.

Theorem 6.4 *(the turn-pike theorem in the strongest form).* Let the conditions of the Theorem 6.3 be satisfied. Then there is a unique pair of the functions (turn-pikes) $\tilde{\lambda}(t)$, $\tilde{a}(t)$, $\tilde{a}'(t) > 0$, $t \in [t_0, \infty)$ which does not depend on initial condition (6.10) and is defined by the Theorem 6.3. If $T = \infty$ then the unique solution $a^*(t)$ of OP3 coincides with $\tilde{a}(t)$ beginning with an instant $\mu \geq t_0$, and the unique solution $\lambda^*(t)$ of OP3 coincides with $\tilde{\lambda}(t)$ beginning with the instant $a^*(\mu) \geq t_0$ (or the instant t_0 at $a^*(\mu < t_0)$. The value $\mu - t_0$ is defined only by the value of deviation $|\tilde{a}(t_0) - a_0|$ ($\mu = t_0$, if $\tilde{a}(t_0) = a_0$).

Remark. The conditions (6.20) of Theorem 6.3 are sufficient and they set a region of the given parameters p, q, s for which the turnpike properties of integral model (6.6)-(6.7) are guaranteed. It is easily verified that the conditions (6.20) are consistent. Let us rewrite (6.20) in the following form:

$$(1 + 1/x - p/x)^x p^p < 1, \tag{6.22}$$

where $x = q/s$, and $0 < x < p$ in view of the first condition (6.20).

On the one hand, the inequality (6.22) holds true for any p, q, s from the domain $0 < q/s < p < 1/e$. In this case $(1 + 1/x - p/x)^x p^p < (1 + 1/x - 1)^x p^p < p^p x^x$. The function x^x decreases for $x \in [0, 1/e]$, hence, $p^p x^x < 1$.

On the other hand, the inequality (6.22) is always valid for x which is close to p, i.e. for $0 < \varepsilon = sp - q \ll 1$. Indeed, then

$$\begin{aligned}(1 + 1/x - p/x)^x p^p &= (1/x - \varepsilon/q)^x (x + \varepsilon/s)^{x+\varepsilon/s} = \\ = [1 - \varepsilon(q-s)/qs]^x (x + \varepsilon/s)^{\varepsilon/s} &+ o(\varepsilon) = [1 - \varepsilon(q-s)/s^2](1 + \\ &+ \varepsilon/s \ln x) + o(\varepsilon) = 1 + \varepsilon(\ln x - 1 + x))/s + \\ &+ o(\varepsilon) < 1 \quad \text{for} \quad 0 < x < p \quad (\text{since } \ln x < x - 1).\end{aligned}$$

Thus, the optimal cost $\lambda^*(t)$ of new EUs possesses a turnpike property as well as the EU lifetime $t-a^*(t)$. Namely, in case of the infinite-horizon optimization and the Cobb-Douglas production function (6.18), the optimal control λ^* and phase variable a^* coincide with the turn-pike trajectories $\tilde{\lambda}(t) = b_0 \exp(qt/p)$ and $\tilde{a}(t) = t - A$ on the infinite interval $[\mu, \infty)$ except some initial finite interval $[t_0, \mu)$.

The similar results can be established for some other smooth increasing functions $\beta(\lambda, \tau)$. It is evident that the turnpikes $\tilde{\lambda}$ and $\tilde{a}$ are different from the ones given above in the cases of another production function or distinct dynamics of technical progress. The central problem with their analysis is solving nonlinear IFEs (6.14) in general case.

By analogy with the basic OP1, some less strong turn-pike properties would be expected in OP3 for the case of finite-horizon optimization problem. But the problem here is worse than it was with the simplest one-sector model in Chapters 4 and 5.

6.3 CHOICE OF THE OPTIMAL PRODUCTIVITY OF EQUIPMENT

Until now we assumed that the dynamics of technological change was given and independent of economic variables. It is plausible that the rate of technological change will in fact be influenced by economic variables, e.g. investments in R&D in previous periods. To see what effect such assumptions may have on the behavior of optimal technology lifetime, we will modify the models under consideration to allow for such feedback from economic activity to innovation. Corresponding multi-sector IDM will be studied in Chapter 10. Here we restrict ourselves to the simplest case of such feedback in one-sector IDM.

Namely, let us assume that the rise of the efficiency of new EUs (the rate of embodied TC) may be directly controlled by means of certain instantaneous innovations $R(.)$. For this purpose we represent the EU productivity function $\beta(\tau, t)$ in one-sector IDM (4.1)-(4.3) in the following form:

$$\beta(\tau) = \beta_0 + \int_{\tau_0}^{\tau} \eta(v)dv, \quad \text{i.e.} \quad \beta'(\tau) = \eta(\tau),$$

where function $\eta(t)$ is new endogenous variable of the model. Here the function $\beta(\tau, t)$ is assumed to be independent of t, hence, no consideration is given to autonomous TC and the physical decay of equipment.

The corresponding OP (designated as **OP4**) consists in the determination of the functions $a(t)$, $m(t)$, $\eta(t)$, $t \in [t_0, T]$, $T < \infty$, which maximize the following functional:

$$I_4 = \int_{t_0}^{T} \rho(t)[Q(t) - \gamma(t)\eta(t) - \lambda(t)m(t)]dt \longrightarrow \max_{a,m,\eta} \tag{6.23}$$

under given $\rho(t)$, $P(t)$, $\lambda(t)$, $\gamma(t)$, the constraint-equalities:

$$Q(t) = \int_{a(t)}^{t} [\beta_0 + \int_{\tau_0}^{\tau} \eta(v)dv]m(\tau)d\tau, \tag{6.24}$$

$$P(t) = \int_{a(t)}^{t} m(\tau)d\tau, \tag{6.25}$$

the constraints-inequalities:

$$a'(t) \geq 0, \quad a(t) < t, \quad 0 \leq m(t) \leq M(t), \quad 0 \leq \eta(t) \leq H(t), \tag{6.26}$$

and the initial conditions:

$$a(t_0) = a_0 < t_0, \quad m(\tau) = m_0(\tau), \quad \eta(\tau) = \eta_0(\tau), \quad \tau \in [a_0, t_0]. \tag{6.27}$$

As distinct from the basic OP1, the new sought-for control function $\eta(t)$ is added here. It is the rise of the productivity of the EUs created at time t(the rate of embodied TC). The functional (6.23) means that the rise $\eta(t)$ of EU productivity is assumed to be proportional to gross investment $R(t)$ on TC at the same time t: $R(t) = \gamma(t)\eta(t)$. Then the given model function $\gamma(t)$ is the expenditure for the EU productivity increase on one unit (measured in output Q units), i.e. a unit innovation. Thus, the EU productivity $\beta(t)$ is kept constant if the investment $R(t)$ is absent at time t ($R(t) = 0$).

Let the independent control variables $m(t)$ and $\eta(t)$ of OP4 be measurable, $t \in [t_0, T)$. As in OP2, the equality (6.25) is the same as in basic OP1, and the phase OP constraint (6.26) for mis replaced by the restriction (4.7) or (6.11), for which the phase constraint $a' \geq 0$ is satisfied a priori.

If $m(t) > 0$, $t \in [t_0, T)$, then the gradient of functional (6.23) of the OP4 exists and is of the following form:

$$I'_m(t) = \int_t^{\underline{a}^{-1}(t)} \rho(\tau) \int_{a(\tau)}^{t} \eta(v) dv d\tau - \rho(t)\lambda(t), \tag{6.28}$$

$$I'_\eta(t) = \int_t^{T} m(\tau) \int_{\tau}^{\underline{a}^{-1}(\tau)} \rho(v) dv d\tau - \rho(t)\gamma(t). \tag{6.29}$$

Theorem 6.5. If the following conditions are valid:

$$\int_t^T \rho(\tau) \int_{min(t_0, a_{min}(\tau))}^{\tau} \eta_0(v) dv d\tau \leq \rho(t)\lambda(t), \tag{6.30}$$

$$\int_t^T m_{min}(\tau) \int_{\tau}^{T} \rho(v) dv d\tau \leq \rho(t)\gamma(t), \tag{6.31}$$

then the OP4 has the *trivial solution* $\eta^*(t) \equiv 0$, $m^*(t) \equiv m_{min}(t)$, $a^*(t) \equiv a_{min}(t)$, $t \in [t_0, T]$ *(the regime of technological renovation absence).*

Proof. Put $m^* \equiv m_{min}$, $\eta^* \equiv 0$. Under the conditions (6.30)-(6.31) we obtain that $I'_{mu}(t) < 0$, $I'_\eta(t) < 0$, $t \in [t_0, T)$, hence the functions (m^*, η^*) satisfy the necessary and sufficient conditions of extremum. Next, with use of the second variation of functional I_4 one can show that (m^*, η^*) is really an OP solution.

Remark. If $P'(t) \leq 0$, then the condition (6.30) is valid. Let $P'(t) \geq 0$, then $a^*(t) \equiv a_0$, the inequality $\int_{a_0}^{\tau} \eta_0(v) dv \int_t^T \rho(\tau) d\tau \leq \lambda(t)\rho(t)$ is sufficient for (6.30), and $[P(T) - P(t)] \int_t^T \rho(\tau) d\tau \leq \gamma(t)\rho(t)$ is sufficient for (6.31).

The conditions (6.30)-(6.31) mean that there is no artificial barrier to the realization of turnpike regime. The condition (6.30) means that the investments

η_0 on pre-history $[a_0, t_0]$ for increasing the productivity of new EUs are inadequate to ensure technological renovation (without additional investments $\eta(t)$ at $t > t_0$). These conditions become more restrictive as the time $T - t_0$ increases. In the last case the trivial solution (m^*, η^*) of OP4 has not ceased to exist.

At $T \gg 1$ and $\rho' \ll 1$ the trivial solution (m^*, η^*) even if it exists, is unstable in the sense that the small variations $\delta\eta$, δm of control functions η, δ can bring out modeled system from this trivial regime. Indeed, put $\delta\eta(\tau) = \delta m(\tau) = d > 0$, $\tau \in [t_1, t_1 + \Delta]$, $t_0 < t_1 < T$. For simplicity assume that P(t) is non-increasing or non-decreasing, then $a^*(t) < a_0$,

$$I'_m(t) = d\Delta \int_t^T \rho(\tau)d\tau - \rho(t)\lambda(t) \quad \text{at} \quad t_1 + \Delta < t < T,$$

$$I'_\eta(t) = d\Delta \int_{t_1}^T \rho(\tau)d\tau - \rho(t)\gamma(t) + o(\Delta) \quad \text{at} \quad t_0 < t < t_1.$$

The values $I'_m(t)$, $t \in [t_1, T)$ and $I'_\eta(t)$, $t \in [t_0, t_1]$ become positive as $T - t_1$ increases. Hence, the increasing $m^*(t)$, $\eta^*(t)$ in the neighborhood of point $t = t_1$ causes the rise of functional $I_4(m, \eta)$. This suggests the existence of *second (non-trivial) solution* of OP4 at a large $T - t_0$.

Let us study the possibility of existence of the OP4 solution (m^{**}, η^{**}) which is internal in the domain (6.26). In this case the system of equations

$$I'_m(a, \eta; t) = 0, \quad I'_\eta(a, m; t) = 0 \tag{6.32}$$

must have a solution (m, η, a) on some interval $(t_1, t_2) \subset\in [t_0, T]$. These equations depend on m, η and form jointly with (6.25) the system of three non-Volterra integral equations in the unknowns m, η, a, whose analysis is connected with a large difficulty.

We shall restrict ourselves to the case of $\rho \equiv$const. After differentiation of (6.32) we have

$$\eta(t)(a^{-1}(t) - t) = \lambda'(t) + \int_{a(t)}^t \eta(v)dv, \tag{6.33}$$

$$m(t)(a^{-1}(t) - t) = -\gamma'(t), \quad t \in (t_1, t_2). \tag{6.34}$$

The equality (6.33) represents the necessary condition for $I'_m \equiv 0$ and can be valid under wide assumptions. However, the necessary condition (6.34) for $I'_\eta \equiv 0$ is unsolved jointly with equation (6.25) under a natural behavior of model functions. For example, at $\gamma'(t) \geq 0$ we have $m(t) \leq 0$ or $a(t) \geq t$ that is impossible in view of $P(t) > 0$. Hence, there is no internal regime of optimal control $\eta^*(.)$: $\eta(t) = 0$ or $\eta(t) = \overline{\eta}(t)$ for $t \in [t_0, T]$.

Theorem 6.6. For $T - t_0 \gg 1$, $\rho' \ll 1$ the OP4 has a non-trivial solution (m^{**}, η^{**}) such that

$$\eta^{**}(t) = \begin{cases} \overline{\eta}(t), & t \in [t_0, \mu), \\ 0, & t \in [\mu, T), \end{cases} \qquad t_0 \leq \mu < T,$$

(at least, at $\gamma' \geq 0$), and $m^{**}(t) > m_{min}(t)$, $t \in \Delta \subset [t_0, \theta)$, $t_0 < \theta < T$, $m^{**}(t) \equiv m_{min}(t)$, $t \in [\theta, T]$.

Proof is analogous to that of Theorems 4.3, 4.6, 5.6 and is based on necessary and sufficient conditions of extremum for the OP4. The structure of optimal control η^* follows from the fact that

$$\partial I'_\eta(t)/\partial t \;=\; -m(t) \int_t^{\underline{a}^{-1}(t)} \rho(v)dv - \rho(t)\gamma'(t) - \rho'(t)\gamma(t) < 0$$

under the condition of the Theorem.

Hence, the control of EU productivity increase is more strong than other controls. There are no turnpike properties of $\eta^*(.)$ in this OP.

Thus, the OP4 may be formally subdivided into the optimization with respect to η and to m (connected through the unknown parameter μ). If the optimal η^* has already determined and $\mu > t_0$, then by substitution η^* into $I'_m(a, \eta; t) \equiv 0$ we obtain one integral-functional equation of the form (5.1). If this equation has a *solution*$\tilde{a}(.)$ on infinite interval $[t_0, \infty)$, then it describes a turnpike trajectory for EU lifetime. But the existence of turnpike $\tilde{a}$ needs further consideration.

It is conceivable that the OP4 has simultaneously two solutions (m^*, η^*) and (m^{**}, η^{**}). For example, let $P' \leq 0$ (then $m_{min} \equiv 0$), $\eta_0 \equiv 0$, $\rho \equiv$const, and λ, γ be bounded. Then the conditions (6.17) are valid for all $T - t_0$, hence, the trivial OP4 solution exists: $m^* \equiv \eta^* \equiv 0$. In view of Theorem 6.6, the non-trivial solution (m^{**}, η^{**}) also exists for a sufficiently large $T - t_0$. So, this OP4 can be multi-extremal. In such case the trivial solution is not attractive: its small variations δm, $\delta\eta$ lead to non-trivial solution.

The qualitative results exposed above still stand for a wide class of the monotone increasing non-linear relationship between the TC rate $\beta'(\tau)$ and the investment $R(\tau)$ into TC. However, if it is supposed that the whole efficiency $\beta(\tau)$ of EUs is linearly dependent on the investment $R(\tau)$ (rather than the derivative $\beta'(\tau)$ as above), then the optimal investment $R(t)$ and technology lifetime $t - a(t)$ possess some turnpike property. This case will be considered in more detail in Section 10.4 for three-sector IDM with endogenous embodied TC.

6.4 CONSIDERATION OF INSTANTANEOUS R&D EXPENDITURE

In the next two Sections we will assume that the rate of embodied TC (the rise of the productivity of new EUs) is given, and R&D investments are needed for the development of new equipment. In other words, we include the expenditure for the development of new EUs' vintages into basic OP1. This brings up the question of whether turnpike properties of technology lifetime are preserved.

Let us consider a large-size economic system which ensures the creation of new types of equipment along with the production and putting into operation corresponding new EUs. Then the expenditures on putting new EUs into operation must include the cost of each new EUs and the cost of the development of the EU of this type (the creation of new vintage). For the sake of simplicity the R&D expenditure is assumed to be instantaneous (this restriction will be excluded in next Section).

The corresponding modification of basic OP1 (designated as **OP5**) consists in the determination of the sought-for functions $m(t)$, $a(t)$, $t \in [t_0, T)$ which maximize the functional:

$$I_5 = \int_{t_0}^{T} \rho(t)[Q(t) - \lambda(t)m(t) - \chi_m(t)\zeta(t)]dt \longrightarrow \max_{a,m}, \qquad (6.35)$$

where $\chi_m(t) = \begin{cases} 0, & m(t) > 0, \\ 1, & m(t) = 0, \end{cases}$ in the basic one-sector IDM (4.1)-(4.3) under the constraints (4.5)-(4.6). The endogenous model variables m and a are

the same as in OP1. The only new given function $\zeta(.)$ is the cost of EU creation (the R&D expenditure on new technology elaboration), the other functions are the same.

The formula (6.35) indicates that the expenditure $S(t) = \zeta(t)$ on the development of new vintage is required at instant t only if some amount of EUs is produced at this instant t (i.e. $m(t) > 0$), otherwise this expenditure $S(t) = 0$.

This is the OP with the functional discontinuous in relation to the control m. As opposed to the basic OP1, the abstract function $I_4(m)$ is not differential with respect to m, so the special "needle" variations of the control $m(.)$ are used to derive the gradient and the condition of extremum. Namely, the gradient of functional (6.35) in OP5 is of the following form:

$$I_5'(t) = I_1'(t) = \int_t^{\underline{a}^{-1}(t)} \rho(\tau)(\beta(t,\tau) - \beta(a(\tau),\tau))d\tau - \rho(t)\lambda(t), \tag{6.36}$$

if $m(t) > 0$ and $m(t) + \delta m(t) > 0$;

$$I_5'(t) = -I_1'(t)m(t) + \rho(t)\zeta(t), \tag{6.37}$$

if $m(t) > 0$ and $\delta m(t) = \begin{cases} -m(t), & t \in \Delta \subset [t_0, T), \\ 0, & t \in [t_0, T), \quad mes(\Delta) \ll 1, \end{cases}$

$$I_5'(t) = I_1'(t)m(t) - \rho(t)\zeta(t), \tag{6.38}$$

if $m(t) = 0$ and $\delta m(t) = \begin{cases} \varepsilon > 0, & t \in \Delta, \\ 0, & t \in [t_0, T), \quad mes(\Delta) \ll 1, \end{cases} \quad t \in [t_0, T).$

As for all previous OP1-OP4, in general case the OP constraints $0 \le m \le M$, $a' \ge 0$ can be replaced by one more strict inequality: $m_{min} \le m \le M$, where $m_{min}(t) = \max\{0, P'(t)\}$, $t \in [t_0, T)$. But the new features of OP5 are connected with the possibility of the case $m(t) = 0$: if $m_{min}(t) > 0$ on a subset Δ_1 of interval $[t_0, T)$, the behavior of OP5 solution coincides with one of basic OP1 at $t \in \Delta_1$. Because of this, we restrict ourselves by the case $P'(t) \le 0$, $t \in [t_0, T)$. Then $m_{min} \equiv 0$.

Under the specified conditions the OP5 always has the trivial solution $m^*(t) \equiv 0$, $a^*(t) \equiv a_{min}(t)$, $t \in [t_0, T)$. Indeed, then $I_5'(t) = -\rho(t)\zeta(t) < 0$ for any variation $\delta m(t) > 0$ by virtue of (6.38).

In contrast to other OPs in considered IDMs, the OP5 turns out to be a multi-extremal under the strictly increasing $\beta(.,t)$ and may have two local maxima:

- $m_1^* \equiv 0$ (the lack of the investment for new technologies),
- $m_2^*(t) > 0$ such that $I_5'(t) \equiv 0$, $t \in \Delta \subset [t_0, T)$, $mes\Delta > 0$.

At $\zeta(.) > 0$ the non-trivial solution m_2^* of OP5 is piecewise-continuous: $m_2^*(t) = 0$ or $m_2^*(t) = \zeta(t)/D(t)$, where $D(t)$ coincides with the gradient $I_1'(t)$ of basic OP1. The function $\zeta(t)/D(t)$ determines the minimal amount of new EUs produced at time t, for which it is profitable to develop the technology connected with these EUs. The necessary condition for the existence of non-trivial solution m_2^* is $D(t) = I_1'(t) > 0$, i.e. the non-triviality of the corresponding solution of basic OP1 (see Theorem 4.3).

Let there exist functions $\tilde{m}(t)$, $\tilde{a}(t)$, $t \in [t_0, \infty)$ which satisfy the equation (4.2) and the following equality:

$$I_1'(\tilde{a}; t)\tilde{m}(t) + \rho(t)\zeta(t) = 0, \quad t \in [t_0, \infty). \tag{6.39}$$

Then the structure of the non-trivial solution $m_2^*(t)$, $t \in [t_0, T)$, is similar to one described for basic OP1 in Chapters 4 and 5. This is illustrated by the following example.

Example 6.1. Let $\beta(\tau, t) = \tau$, $P = m_0 = \lambda \equiv 1$, $\zeta \equiv 0.5$, $M \equiv 2$, $t_0 \equiv 2$, $T = \infty$, $a_0 \equiv 0$, $\rho = e^{-0.01t}$. Then the OP5 has the trivial solution $m_1^* \equiv a_1^* \equiv 0$ and the non-trivial solution of the following form:

$$m_2^*(t) = \begin{cases} 2, & t \in [2 + i\sqrt{3},\ \ 4 - (i-1)\sqrt{3}), \\ 1, & t \in [4 - (i-1)\sqrt{3},\ \ 2 + (i+1)\sqrt{3}), \quad i = 0, 1, 2, ..., \end{cases}$$

$$a_2^*(t) = \begin{cases} 2t - 4, & t \in [2,\ \ 4 - \sqrt{3}), \\ t - \sqrt{3}, & t \in [4 - \sqrt{3},\ \ \infty), \end{cases}$$

These solutions are illustrated in Fig.6.1. Calculating the corresponding values of the functional (6.35) for these solutions, we obtain $I_5(m_1^*, a_1^*) \approx 200e^{-0.02}$, $I_5(m_2^*, a_2^*) \approx 200e^{-0.02}(102 - \sqrt{3})$. Hence, in this case the non-trivial solution (m_2^*, a_2^*) is the global maximum of the OP5.

In general case the equation (6.39) depends on m and forms jointly with (4.2) the system of the non-Volterra integral equations in unknowns $\tilde{m}$ and $\tilde{a}$, whose analysis is connected with a large difficulty.

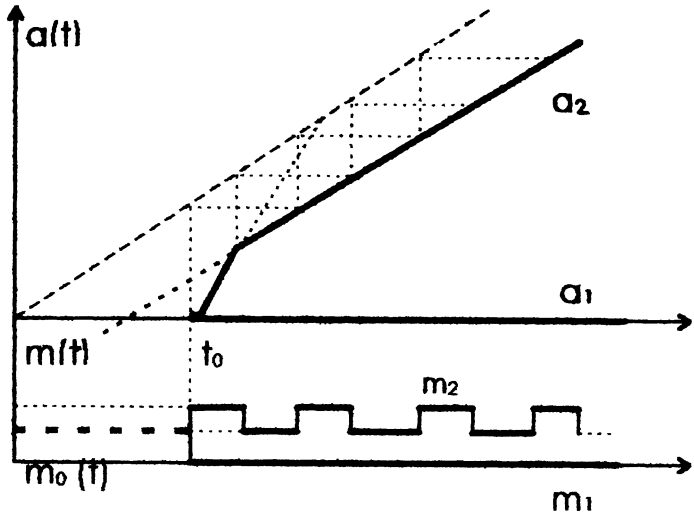

Figure 6.1 The example of two solutions in the optimization problem OP5 for one-sector IDM

There are no classical turnpike properties in the OP5. Indeed, the dynamics of $\tilde{a}$ depends qualitatively upon the behavior of $\tilde{m}$ and the initial function m_0 from initial conditions (4.3). So, in the case $\beta(\tau, t) = c\tau + c_1$, $\rho(t) = e^{-st}$, $\lambda(t) \equiv \lambda$ =const, $\zeta(t) \equiv \zeta$ =const, m_0 =const, the (6.39) solution $a_2^*(t)$ tends to $\tilde{a}(t) = t - A_1$ as in the OP1 only if the corresponding $\tilde{m}(t) \equiv \tilde{m}(t - A_1)$ (as in the above example). But even so, the trajectory $\tilde{a}(t) = t - A_1$ is not a turnpike since the constant $A_1 \approx 2/c(\lambda + \zeta/m_0)$ depends on m_0.

Nevertheless, the tendency presents for the non-trivial EUs lifetime to strive to some "favorable regime" $t - \tilde{a}(t)$ such that $I_5'(\tilde{a}, \tilde{m}, t) \equiv 0$ (if it exists).

6.5 CONSIDERATION OF DISTRIBUTED R&D EXPENDITURE

Let us consider the modification of the previous OP, which takes into account the distributed (in time) expenditure on the development of the new EU vintages (distributed R&D expenditure).

This OP (designated as **OP6**) consists in the determination of the sought-for functions $m(t)$, $a(t)$, $\sigma(\tau,t)$, $\tau,t \in [t_0,T)$ which maximize the functional:

$$I_6 = \int_{t_0}^{T} \rho(t)[c(t) - \lambda(t)m(t) - \int_{t}^{T} \sigma(t,v)dv]dt \longrightarrow \max_{a,m,\sigma}, \tag{6.40}$$

under the constraints (4.1)-(4.3),(4.5)-(4.6), the constraints

$$\int_{\tau_0}^{t_0} \sigma_0(\tau,t)d\tau + \int_{t_0}^{t} \sigma(\tau,t)d\tau = \chi_m(t)\zeta(t) \tag{6.41}$$

$$0 \le \sigma(\tau,t) \le \Omega(\tau), \quad \tau,t \in [t_0,T), \tag{6.42}$$

the initial conditions (4.3) and

$$\sigma(\tau,t) = \sigma_0(\tau,t), \quad \tau \in [a_0,t_0], \quad t \in [t_0,T). \tag{6.43}$$

$$\lambda^* = \begin{cases} \lambda_\mu(t), & t \in [t_0,\overline{a}(\mu)), \\ \tilde{\lambda}(t), & t \in [\overline{a}(\mu),\infty), \end{cases}$$

As compared with the previous problem OP5, the expenditure $\sigma(\tau,t)$ at time τ for the development of new future technology of time t is introduced here as the additional piecewise-continuous control function for $\tau \in [t_0,t], t \in [t,T)$. This sought-for function is restricted by means of the equality (6.41) and the inequality (6.42). The other model functions are the same as in OP1 and OP5.

This OP6 contains the state equation (6.41) which is discontinuous with respect to the control m. Optimal control function $\sigma(.,.)$ turns out to obey the control

$m(.)$. Namely, the solution σ^* of OP6 is bounding in the domain (6.42) and has the following form:

$$\lambda^* = \begin{cases} \lambda_\mu(t), & t \in [t_0, \overline{a}(\mu)), \\ \tilde{\lambda}(t), & t \in [\overline{a}(\mu), \infty), \end{cases}$$

where the function $b(t)$ is determined by the relations

$$b(t) = \max\{t_0, \underline{b}(t)\}, \quad \int_{\underline{b}(t)}^{t} \Omega(\tau)d\tau = \chi_m(t)\zeta(t).$$

Here the function $\chi_m^*(t)$ depends on the behavior of optimal $m^*(t)$ $\quad (\chi_m^*(t) = 0$ if $m_m^*(t) = 0)$.

The gradient $I_m'(t)$ which determines the structure of optimal control m^* is analogous to that of previous OP5. If $\Omega(t) \equiv \infty$ (in the case of the instantaneous investment into new technologies), then $\sigma^*(\tau, t) = \zeta(t)\delta(t - \tau)$ and the problems OP5 and OP6 are equivalent. For this reason the OP6 possesses the same properties as those of previous problem, specifically, the existence of two extremums. The example of the existence of two extremums coincides with the Example 6.1, in which $\tilde{a}(t) = t - A_2$, $A_2 = \sqrt{2/c[\lambda e^{s\zeta/\Omega} + \zeta/m_0]}$, the solutions (m_1^*, a_1^*), (m_2^*, a_2^*) are given in Fig.6.1, $\sigma_1^* \equiv 0$, and σ_2^* has the above form.

6.6 CONCEPTUAL INTERPRETATION OF RESULTS

In this chapter various optimization problems (OPs) have been analyzed for the one-sector IDMs of technological renovation. Qualitative analysis about the structure and behavior of OP solutions made it possible to find the turnpike properties of the optimal trajectories of technological renovation (the optimal EU lifetime).

Considered OPs yield the different integral equations for the determination of turnpike lifetime. It is reasonable because they take into account the different objectives and properties of an economic system. The community of turnpike equations lies in the fact that they do not depend on extremal criteria and initial conditions and are determined by the functions of the type β, λ, γ, δ which describe the gist of OP.

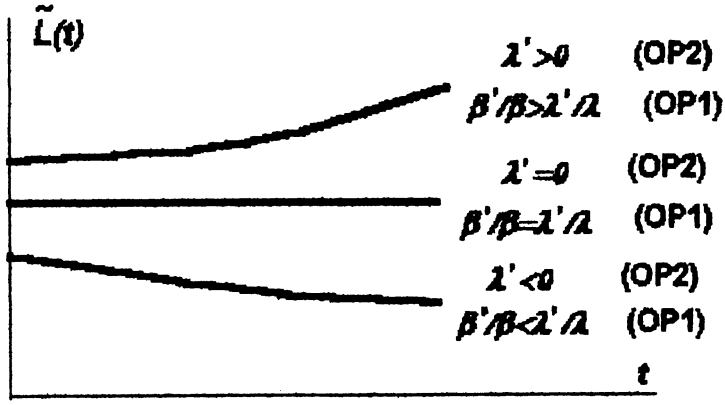

Figure 6.2 The qualitative behavior of turnpike technology lifetime $\tilde{L}(t)$ under exponential embodied TC: in the case of expenditure minimization (OP2), in the case of output maximization (OP1).

The next features of the optimal lifetime of technologies may be added to those established in Chapter 5.

Let *the minimization of production expenditure* be desirable. Then, as follows from the analysis of the turnpike equation (6.4) for OP2, the rational (turnpike) technology lifetime $\tilde{L}(t) = t - \tilde{a}(t)$ possesses the following properties under the exponential dynamics of embodied TC $\beta(\tau)$ and discount $\rho(t)$:

- The lifetime $\tilde{L}(t)$ is constant if the EU cost $\lambda(t)$ (per labor unit) is constant (as opposed to the basic OP1, for which this is true for the constant EU price λ/β per capacity unit, see Fig.6.2.). The constant $\tilde{L} \equiv \tilde{L}(t)$ depends on TC rate β', EU cost λ, and discounting rate ($\tilde{L}$ is independent of the absolute productivity β and labor P).
- If the EU cost $\lambda(t)$ decreases, then the lifetime $\tilde{L}(t)$ decreases, and the converse.

- If the EU cost $\lambda(t)$ increases more quickly than the EU productivity $\beta(t)$ (i.e. the EU price $\lambda(t)/\beta(t)$ per capacity unit increases), then a turnpike regime of equipment lifetime L is infeasible.
- The case of the constant EU cost/capacity ratio $\lambda(t)/\beta(t)$ per efficiency unit is critical for the existence of turnpike regime of the EU lifetime. Then a bounded solution $\tilde{a}$ of turnpike equation (6.4) may exist, so turnpike theorem can be valid for infinite-horizon optimization, but it is not true for finite horizon (then the third condition of Theorem 6.2 fails).

Thus, in the case of expenditure minimization (OP2) the conditions of the existence of turnpike regime are more strict than for output maximization (OP1), and the corresponding technology lifetime $\tilde{L}$ is greater.

In case of *simultaneous optimization of the EU lifetime and the EU cost* under embodied TC (OP3), the optimal cost λ of new EUs possesses a turnpike property as well as the EU lifetime L (at least, for infinite-horizon optimization). In the case of Cobb-Douglas function and exponential rates of embodied TC and discount, the turnpike trajectories are the constant lifetime $t - \tilde{a}(t)$ of EUs and the exponential increase of the cost $\tilde{\lambda}(t)$ of new EUs.

As in basic OP1, the rational strategy of technological renovation turns to be the same for various-sized economic systems provided that the renovation process occurs (i.e., if $m(t) > 0$).

In contrast to other OPs in IDMs under study, the OP4-OP6 may be *two-extremal* under natural assumptions. They can have two local maxima: a trivial solution *(lack of R&D investment for new technologies)* and non-trivial one which corresponds to *a regime of intensive technological renovation.* This is because some additional expenditures are required to remove economic system from trivial regime.

When we modify the basic OP1 to allow for *a relation between TC rate and innovations* (OP4 with endogenous TC), then the sought-for innovations into TC have no turnpike properties. Hence, such direct innovations into TC are more strong that the other types of control of an economics.

If we include *the expenditure for the development of new EU vintages* into basic OP1, then the corresponding modifications of basic OP1 does not also possess classical turnpike properties. Nevertheless, the tendency is found to be present for the rational EU lifetime L to strive to some "favorable" regime (if it exists). The optimal amount m of new EUs entered into system is piecewise- continuous,

that is determined by the minimal amount of new EUs for which it is profitable to develop new EU vintage.

The dynamics of optimal EU lifetime L is similar in the cases when the expenditures on the development of the new EU vintages are presumed to be instantaneous (OP5) or distributed in time (OP6). If the latter is the case, the optimal expenditure at time $\tau < t$ for the development of new future technology of time t is maximal possible on some preceding interval $[b(t), t]$ and is zero at earlier time.

In all considered cases the non-trivial behavior of rational EU lifetime L, be it turnpike (as in OP1-OP3) or bounding (as in OP4-OP6), is guaranteed only for a long optimization horizon and for discounted infinite-horizon optimization. For this reason the *infinite-horizon optimization* seems to be more appropriate from the practical point of view. On this background, the established turnpike properties of EU lifetime can be used as a basis of strategic decision-rules for rational scrapping of obsolete industrial equipment even on moderate time intervals (such that allow economic system to pass into turnpike regime from an initial state).

In this connection it is interesting to note that for all considered OPs the *turnpike EU lifetime $\tilde{L}(t)$ does not depend on the discount rate s* in the first approximation at small values s, $0 < s \ll 1$ (the value s appears only in the high-degree terms of corresponding expansion of $\tilde{L}$).

PART III

ANALYSIS OF MULTI-SECTOR INTEGRAL DYNAMICAL MODELS

7

THE VOLTERRA INTEGRAL EQUATIONS WITH SOUGHT-FOR LOWER LIMITS OF INTEGRATION

From the contextual and mathematical points of view, this part of the book is the extension of previous one which has been devoted to the one-sector IDM with controlled memory. The present part deals with the analysis of the optimization problems (OPs) for more complex two-, three- and multi-sector IDMs as well as of arising integral equations. The structure and asymptotic behavior of optimal model trajectories are examined, in particular, turn-pike properties are established for the OPs under study. These results develop obtained in Part 2 for the one-sector IDM of economic renovation.

The Volterra integral equations (VIEs) with the sought-for lower limits of integration arise in the IDMs (2.12)-(2.14), (2.15) and in their generalizations [36], [48], [64], [66], [83], [146]. The presence of a sought-for limit in such VIEs gives a novelty to the study of principal theoretical problems (existence, uniqueness, stability of solutions, etc.) caused by these equations. Such VIEs are always nonlinear except for the trivial cases when their integrands are constant. They belong to the equations with unknown deviating arguments (with retarded ones). This Chapter deals with the solvability of such equations.

The existence and uniqueness of solutions of VIEs with the unknowns in lower integral limits has been studied in [48], [146], [148], and so on. Here, we shall restrict ourselves to the following results. We first study the solution existence and uniqueness for the system of two nonlinear VIEs of the second kind. The analogous results are also established for the similar VIEs of the first kind as well as for the system of $2n$ VIEs with n sought-for lower limits of integration, $n > 1$. The examples of exact solutions are given in all sections.

7.1 THE EQUATIONS WITH ONE SOUGHT-FOR LIMIT OF INTEGRATION

Let us consider the following system of nonlinear Volterra integral equations:

$$x(t) = \int_{z(t)}^{t} f(\tau, t, x(\tau))d\tau, \tag{7.1}$$

$$\int_{z(t)}^{t} \varphi(\tau, t, x(\tau))d\tau = p(t), \tag{7.2}$$

in the unknowns $x(t)$, $z(t)$, $t \in [t_0, T)$, $t_0 < T \leq \infty$, where on $(-\infty, t_0]$ the solution x is to agree with a given initial function x_0:

$$x(\tau) \equiv x_0(\tau), \quad -\infty < \tau \leq t_0, \tag{7.3}$$

and the unknown delay time z obeys the initial condition:

$$z(0) = z_0 < t_0. \tag{7.4}$$

It will be assumed that the given functions $f(\tau, t, x)$, $\varphi(\tau, t, x)$, $p(t)$, $x_0(\tau)$ are (at least) continuous, non-negative for $\tau \in (-\infty, t_0]$, $t \in [t_0, T]$, $x \in [t_0, \infty)$, and satisfy (7.1)-(7.4) at $t = t_0$.

Theorem 7.1. If:

1) $f(\tau, t, x)$ and $\varphi(\tau, t, x)$ are positive for $x > 0$;

2) the integral $\int_{-\infty}^{t_0} \varphi(\tau, t, x_0(\tau))d\tau$ diverges;

3) $f(\tau, t, x)$, $\varphi(\tau, t, x)$, and $p(t)$ satisfy a Lipschitz condition in x and t;

4) the function $f(\tau, t, x)/\varphi(\tau, t, x)$ is bounded, $\tau \in (-\infty, t_0]$, $t \in [t_0, T]$, $x \in [0, \infty)$;

then the system of equations (7.1)-(7.4) has a unique solution $x \in \mathbf{C}[t_0, T]$, $z \in \mathbf{C}[t_0, T]$, such that $x(t) > 0$ and $z(t) < t$.

Proof. The equations (7.1)-(7.2) can be represented in the form

$$x(t) = \int_{t_0}^{t} f(\tau, t, x(\tau))d\tau + \int_{z(t)}^{t_0} f(\tau, t, x_0(\tau))d\tau, \tag{7.5}$$

$$\int_{z(t)}^{t_0} \varphi(\tau, t, x_0(\tau))d\tau = p(t) - \int_{t_0}^{t} \varphi(\tau, t, x(\tau))d\tau := c(t), \tag{7.6}$$

$t \in [t_0, \tau_1]$, where the time τ_1, $t_0 < \tau_1 \leq T$, will be determined later on. Introduce the functions

$$F(z,t) := \int_{z}^{t_0} f(\tau, t, x_0(\tau))d\tau, \quad \Phi(z,t) := \int_{z}^{t_0} \varphi(\tau, t, x_0(\tau))d\tau, \tag{7.7}$$

which are defined for $-\infty < z \leq t_0$. Under the conditions of Theorem, $\Phi(z,t)$ is strictly increasing in z and has the unique inverse function $\Phi^{-1}(c,t)$ defined for $c \in [0, \infty)$. Let $S(c,t) := F[\Phi^{-1}(c,t), t)]$. The system of equations (7.5)-(7.6) is equivalent to nonlinear Volterra integral equation

$$x(t) = Ax := \int_{t_0}^{t} f(\tau, t, x(\tau))d\tau + S[p(t) - \int_{t_0}^{t} \varphi(\tau, t, x(\tau))d\tau, t], \tag{7.8}$$

$t \in [t_0, \tau_1]$, if the condition

$$p(t) \geq \int_{t_0}^{t} \varphi(\tau, t, x(\tau))d\tau \tag{7.9}$$

is valid. Denote by Ω_d the set of continuous functions $x(t)$, $t \in [t_0, \tau_1]$: $0 \leq x(t) \leq x_0(O) + d = D_0$, d =const> 0, and introduce the norm:

$$||f|| := \max_{[t_0,\tau_1]\otimes[t_0,\tau_1]\otimes[0,D_0]} |f(\tau, t, x)|.$$

Condition (7.9) is fulfilled if $x \in \Omega_d$ and

$$\tau_1 - t_0 \leq \min_{[t_0,\tau_1]} p(t)/||\varphi|| \tag{7.10}$$

$$\text{(then} \quad p(t) \geq \min_{[t_0,\tau_1]} p(t) \geq ||\varphi||(\tau_1 - t_0) \geq \int_{t_0}^{t} \varphi(\tau, t, x(\tau))d\tau).$$

The function $S(c,t)$ satisfies a Lipschitz condition with constant $L_s = \max|f(t, \tau, x)/\varphi(\tau, t, x)|$. Denote the Lipschitz constants of the functions $f(\tau, t, x)$, $\varphi(\tau, t, x)$ and $p(t)$ by L_f^x, L_f^t, L_φ^x, L_φ^t, L_p. It follows from the second condition of Theorem and from (7.9) that $C_z \leq z(t) \leq 0$, $t \in [t_0, \tau_1]$, where C_z is some constant.

We now estimate the difference $x^*(t) - x_0(t_0)$, $x^* = Ax$. After a standard transformation we find:

$$x^*(t) - x_0(t_0) \leq [\|f\| + L_s(L_p + \|\varphi\|) + (L_f^t + L_s L_\varphi^t)C_z]t := C_A(t - t_0). \quad (7.11)$$

Recalling that $Ax \geq 0$, we obtain that the operator A is invariant in relation to the set $\Omega_d \subset \mathbf{C}[t_0, \tau_1]$ for

$$\tau_1 - t_0 \leq d/C_A. \quad (7.12)$$

Finally, it can be proved that

$$\|Ax_1 - Ax_2\|_{[t_0,\tau_1]} \leq (L_f^x + L_s L_\varphi^x)(t_1 - t_0)\|x_1 - x_2\|_{[t_0,\tau_1]}$$

is valid for $x_1, x_2 \in \Omega_d$. Hence, the operator A is contracting, provided

$$\tau_1 - t_0 < 1/(L_f^x + L_s L_\varphi^x). \quad (7.13)$$

Choose the instant τ_1 such that the inequalities (7.10), (7.12) and (7.13) are fulfilled. Then the equation (7.8) and the system (7.5)-(7.6) have a unique solution $x(t) \in \mathbf{C}[t_0, \tau_1]$ by virtue of the Contraction Mapping Principle [25], [79], [81]. Furthermore, $x(t) \geq 0$, $t \in [t_0, \tau_1]$ in view of the positiveness of function S in (7.8). The unique solution $z(t) < 0$, $t \in [t_0, \tau_1]$, is determined by equation (7.6).

Since we now may take $x_0(\tau)$ to be given on $(-\infty, \tau_1]$, we prove the existence and uniqueness of the solution $x(t)$, $t \in [\tau_1, \tau_2]$, $\tau_1 < \tau_2 \leq T$, etc. Let us show that any finite interval $[t_0, T_1]$, $T_1 < T$ may be gone through by a finite number of steps $\tau_{i+1} - \tau_i$ determined by the inequalities (7.10), (7.12) and (7.13). Indeed, the constant $\delta_1 = 1/(L_f^x + L_s L_\varphi^x)$ in the right-hand side of (7.13) is the same for all intervals $[\tau_i, \tau_{i+1}]$. Next, in view of the second condition of the theorem, it follows from (7.5)-(7.6) that the constant C_z may be chosen as

$$C_z = t_0 + \max_{[t_0,T_1]} \Phi^{-1}[\max_{[t_0,T_1]} p(t), t]$$

independent of the interval number i. Hence, the constant $\delta_2 = d/C_A$ in (7.12) may be chosen as independent of i, too. Finally, if we take $\|\varphi\|$ as the norm for $\tau, t \in [t_0, T_1]$, then the constant $\delta_3 = \min_{[t_0,T_1]} p(t)/\|\varphi\|$ does not depend on i. Thus, $\tau_{i+1} - \tau_i \geq \delta = \min(\delta_1, \delta_2, \delta_3)$ for any i and, hence, the unique solution $x(t)$ exists on any finite interval $[t_0, T_1]$, $T_1 < T$.

Theorem is proved.

Example 7.1. Let $f(\tau, t, x) = b_1 e^{c_1 \tau} e^{d_1 t} x^{q_1}$, $\varphi(\tau, t, x) = b_2 e^{c_2 \tau} e^{d_2 t} x^{q_2}$, $0 < q_i < 1, b_i, c_i > 0, i = 1, 2$ (the Cobb-Douglas production functions with different

exponential rates of technological change and decay, see Sections 0.1 and 2.2). In order that Theorem 7.1 conditions being satisfied it is required that $q_1 \geq q_2$.

Additionally suppose that $x_0(\tau) \equiv a_0 \exp[(c_1+d_1)/(1-q_1)\tau]$, $p(t) \equiv P_0 \exp[(c_2+d_2+(c_1+d_1)q_2/(1-q_1))t]$, where the constants a_0, b_1, b_2, P_0 and $D = t_0 - z(t_0)$ satisfy the following system of nonlinear equations (obtained from (7.1)-(7.2) at $t = t_0$):

$$(c_1+(c_1+d_1)q_1/(1-q_1))b_1^{-1}a_0^{1-q_1} = 1 - \exp[-D(c_1+(c_1+d_1)q_1/(1-q_1))],$$

$$(c_2+(c_1+d_1)q_2/(1-q_1))P_0 b_2^{-1}a_0^{-q_2} = 1 - \exp[-D(c_2+(c_1+d_1)q_2/(1-q_1))].$$

Then the solutions of the equations (7.1)-(7.4) are

$$x(t) = a_0 \exp[(c_1+d_1)/(1-q_1)t], \quad z(t) = t - D, \quad t \in [t_0, T).$$

This example is also used below in Section 13.2.3 for illustration of a numerical algorithm for solving the VIEs system (7.1)-(7.2).

7.1.1 The Case of Bounded Prehistory

In solving applied problems, the prehistory $x_0(\tau)$ of the process $x(t)$ under study is usually given for some bounded interval $[\tau_0, t_0]$, $-\infty < \tau_0 < t_0$. Then, in view of Theorem 7.1, the solution of equations (7.1)-(7.4) may cease to exist. The following theorem provides the conditions under which $z(t) \geq \tau_0$ and, hence, the solution x, z exists.

Theorem 7.2. Let the conditions of Theorem 7.1 be fulfilled, the functions $f(\tau, t, x)$ and $\varphi(\tau, t, x)$ be differentiable with respect to x, and

$$0 < p_{min} \leq p(t) \leq \overline{p}(t) = \int_{\tau_0}^{t} \varphi(\tau, t, \overline{x}(\tau))d\tau, \quad t \in [t_0, T], \tag{7.14}$$

where $\overline{x}$ is determined from the equation

$$\overline{x}(t) = \int_{t_0}^{t} f(\tau, t, \overline{x}(\tau))d\tau + \int_{\tau_0}^{t_0} f(\tau, t, x_0(\tau))d\tau, \quad t \in [t_0, T]. \tag{7.15}$$

Then $0 \leq x(t) \leq \overline{x}(t)$, $\tau_0 \leq z(t) \leq t - \varepsilon$, $t \in [t_0, T]$, where

$$\varepsilon := \min_{[t_0,T]} p(t) / \max_{[t_0,T]\otimes[t_0,T]\otimes[0,X]} \varphi(\tau, t, x), \quad X := \max_{[t_0,T]} \overline{x}(t).$$

Proof. Varying the system (7.1)-(7.2) we obtain the system of linear integral Volterra equations with respect to the variations $\delta x(t)$, $\delta z(t)$, $\delta p(t)$, $t \in [t_0, T]$

of sought-for solutions x, z, and given function p. By using of well-known properties of a Volterra operator, we have from this system that $\delta \geq 0$ and $\delta z \leq 0$ for $\delta p \geq 0$. In the case of the function p given by (7.15) this implies $x(t) \leq \overline{x}(t)$, $z(t) \geq \tau_0$, $t \in [0, T]$. The inequality $z(t) \leq t - \varepsilon$ is evident because of the evaluation of equation (7.2).

Theorem is proved.

7.1.2 The Case of Smooth Solutions

Of special interest to numerical mathematics are the smooth solutions of equations (7.1)-(7.2). Denote by $\mathbf{C}^{(m)}\Omega$ the space of the functions $f(\mathbf{x})$, $\mathbf{x} \in \Omega \subset \mathbf{R}^l$, having bounded m-th derivative (if $l = 1$) or all bounded partial derivatives of order m if $(l > 1)$.

Theorem 7.3. Let the conditions of Theorems 7.1 and 7.2 be fulfilled, the given functions $f, \varphi \in \mathbf{C}^{(m)}[\tau_0, T] \otimes [t_0, T] \otimes [0, X]$, $p \in \mathbf{C}^{(m)}[t_0, T]$, $x_0 \in \mathbf{C}^{(m)}[\tau_0, t_0]$ satisfy the i-th derivative of equalities (7.1)-(7.2) at $t = 0$, $i = \overline{1, m-1}$, for some given integer $m > 1$. Then $x \in \mathbf{C}^{(m)}[\tau_0, T]$ and $z \in \mathbf{C}^{(m)}[t_0, T]$.

Proof. The existence of the derivatives of the solution $x(t)$ at $t \in (t_0, \tau_1]$ up to order m inclusive is proved by the sequential differentiation of the integral equation (7.8) for $i = 1, ..., m$. In view of our hypotheses $x^{(i)}(t) \to x_0^{(i)}(t_0)$ at $t \to t_0$, so passing to the limit we obtain $x^{(i)}(t_0) = x_0^{(i)}$ for $i = \overline{1, m-1}$. Hence, $x \in \mathbf{C}^{(m)}[\tau_0, \tau_1]$. Likewise, the repeated differentiation of equation (7.6) leads to $z \in \mathbf{C}^{(m)}[t_0, \tau_1]$. The further proof is carried out in the same way as for Theorem 7.1. Namely, we can prove $x \in \mathbf{C}^{(m)}[\tau_0, \tau_k]$, $z \in \mathbf{C}^{(m)}[t_0,\ tau_k]$ for $k = 2, 3, ...,$ until $\tau_k = T$ will be reached.

Theorem is proved.

To illustrate the meaning of Theorem 7.3, we consider the following simple example.

Example 7.2. For $f \equiv x, \varphi \equiv 1, p = ct+1, x_0 \equiv \text{const}, z_0 = -1, t_0 = 0, T = 1$ the solution of equation (7.1)-(7.4) is $x(t) = x_0(1-c+c\exp t)$, $z(t) = (1-c)t-1$, $t \in [0, 1]$. It is seen that for any m $x'(0) = 0$ and $z, x \in \mathbf{C}^{(m)}[-1, 1]$ at $c = 0$, whereas $x'(0) = cx_0 \neq 0$ and $x \in \mathbf{C}^{(m)}[0, 1]$ but only $x \in \mathbf{C}[-1, 1]$ at $c > 0$.

7.1.3 The Non-Smooth Case

When studying the problems of optimal control by the VIEs of the form (7.1)-(7.2), it is often required to study more wide classes of functions f, φ measurable with respect to τ (see [67], [146], and Chapters 8-11 below). For example, consider the space $\mathbf{L}_\infty[t_0, T]$ of bounded measurable functions $u(t)$, $t \in [t_0, T]$, $T < \infty$, with the following norm

$$||u||_{L_\infty} := \operatorname*{ess\,sup}_{[t_0,T]} |u(t)| = \inf_{v(t):v\sim u} \sup_{[t_0,T]} |v(t)|.$$

Then the following result holds true for the equations (7.1)-(7.2).

Theorem 7.4. If:

1) $x_0(\tau) > 0$, $\tau \in (-\infty, t_0]$;

2) $f(\tau, t, x) > C_1 x$, $\varphi(\tau, t, x) > C_2 x$, C_i =const > 0, $i = 1, 2$, $\tau \in (-\infty, t_0]$, $t \in [t_0, T]$, $x \in [0, \infty)$;

3) $f(\tau, t, x)$ and $\varphi(\tau, t, x)$ satisfy a Lipschitz condition in x and t, are measurable in τ and bounded for any fixed x and t, i.e. $f(., \tau, x) \in \mathbf{L}_\infty[\tau_0, T]$, $\varphi(., \tau, x) \in \mathbf{L}_\infty[\tau_0, T]$ for any $-\infty < \tau_0 \leq t_0$;

4) the integral $\int_{-\infty}^{t_0} \varphi(\tau, t, x_0(\tau))d\tau$ diverges;

5) $p(t) > 0$ satisfies a Lipschitz condition in t;

then the system of equations (7.1)-(7.4) has a unique absolutely continuous solution $x(t) > 0$, $z(t) < t$, $t \in [t_0, T]$.

Proof includes the same stages that of Theorem 7.1. In accordance with [109], [142], we have that for any function $x(.) \in \mathbf{C}[\tau_0, T]$ the functions $f(\tau, t, x(\tau))$ and $\varphi(\tau, t, x(\tau))$ are bounded and measurable in τ (at fixed t) on any finite interval $[t_0, T]$. As above, we represent the equations (7.1)-(7.2) in the form (7.5)-(7.6) on a small interval $[t_0, \tau_1]$, and introduce the functions (7.7) and the equation (7.8). Next, let us define the map $A : \mathbf{C}[t_0, \tau_1] \to \mathbf{C}[t_0, \tau_1]$ which is given by the equality (7.8). Analogously to the proof of Theorem 7.1, the map A is contracting under the conditions (7.10), (7.12) and (7.13). Hence, the unique solution $x(.) \in \mathbf{C}[t_0, \tau_1]$ also exists in this case. In so doing, $x(t) > 0$ at $t \in [t_0, \tau_1]$ (in view of the second condition of the Theorem and $S(p(t), t) > 0$). In a like manner, the obtained results are extended to cover all interval $[t_0, T]$.

Taking into account that $S(c,t)$ is Lipschitz continuous in c, we obtain that the solution $x(t)$ is absolutely continuous on the interval $[t_0, T]$ by virtue of the well-known properties of the Lebesgue integral with variable upper limit [78]. The unique solution $z(t) = \Phi^{-1}[c(t), t]$ defined by equation (7.6) is also absolutely continuous on interval $[t_0, T]$ because $c(t)$ is absolutely continuous and $\Phi^{-1}(c,t)$ satisfies a Lipschitz condition in c.

Theorem is proved.

7.1.4 Equations of the First Kind

Let us consider the system of VIEs of the first kind in the unknowns $y(t)$ and $z(t)$, $t \in [t_0, T)$, $t_0 < T \leq \infty$:

$$\int_{z(t)}^{t} \varphi(\tau, t, y(\tau))d\tau = p(t), \tag{7.16}$$

$$\int_{z(t)}^{t} \psi(\tau, t, y(\tau))d\tau = q(t), \tag{7.17}$$

with the initial conditions:

$$z(t_0) = z_0 < t_0, \quad y(\tau) \equiv y_0(\tau), \quad \tau \in (-\infty, t_0]. \tag{7.18}$$

Suppose that the given functions $\varphi(\tau, t, y)$, $\psi(\tau, t, y)$, $p(t)$, $q(t)$ and $y_0(\tau)$ are continuously differentiable with respect to t and y, Lipschitz continuous with respect to τ, positive and satisfy (7.16)-(7.18) at $t = t_0$.

Theorem 7.5. If: 1) φ'_t, ψ'_t, p', q' satisfy a Lipschitz condition in y and t, $\tau \in (-\infty, T)$, $t \in [t_0, T)$, $y \in [0, \infty)$;

2) $\int_{-\infty}^{t_0} \varphi(\tau, t, y_0(\tau))d\tau$ and $\int_{-\infty}^{t_0} \psi(\tau, t, y_0(\tau))d\tau$ diverge, $t \in [t_0, T)$;

3) $\varphi'_t(\tau, t, y)/\varphi(\tau, t, y)$ and $\psi'_t(\tau, t, y)/\varphi(\tau, t, y)$ are bounded, $\tau \in (-\infty, T)$, $t \in [t_0, T)$, $y \in [0, \infty)$;

4) the inequality $\varphi'_y(t, t, x) > \psi'_y(t, t, x)k(\tau, t, y)$ holds, where $k(\tau, t, y) = \varphi(\tau, t, y)/\psi(\tau, t, y)$, $\tau \in (-\infty, T)$, $t \in [t_0, T)$, $x, y \in [0, \infty)$;

5) for $u < \tau < t$ the inequalities $\varphi'_t(\tau, t, x) < \psi'_t(\tau, t, x)k(u, t, y)$ and $p'(t) > q'(t)k(\tau, t, y)$ hold, $u, \tau \in (-\infty, T)$, $t \in [t_0, T)$, $x, y \in [0, \infty)$;

6) $\varphi(t,t,0) \geq \psi(t,t,0)k(\tau,t,y)$, $\tau \in (-\infty, T)$, $t \in [t_0, T)$, $y \in [0,\infty)$;

then the system of equations (7.16)-(7.17) has the unique solution $y \in \mathbf{C}[t_0, T)$, $z \in \mathbf{C}[t_0, T)$, such that $y(t) > 0$ and $z(t) < t$.

Proof. By differentiating (7.16)-(7.17) and eliminating $z'(t)$, we obtain an integral equation of the second kind,

$$\Omega(y(t), z(t), t) = \int_{z(t)}^{t} \chi(y(\tau), z(t), \tau, t)d\tau + R(z(t), t), \tag{7.19}$$

$t \in [t_0, \tau_1]$, τ_1 satisfies (7.10), where

$$\Omega(y,z,t) = \varphi(t,t,y)\psi(z,t,y_0(z)) - \psi(t,t,y)\varphi(z,t,y_0(z)),$$

$$\chi(y,z,\tau,t) = \psi_t'(\tau,t,y)\varphi(z,t,y_0(z)) - \varphi_t'(\tau,t,y)\psi(z,t,y_0(z)),$$

$$R(z,t) = p'(t)\psi(z,t,y_0(z)) - q'(t)\varphi(z,t,y_0(z)).$$

Equation (7.19) and one of the equations (7.16)-(7.17) form a system of nonlinear equations which is analogous to the system (7.1)-(7.2). In view of the fourth condition of the Theorem, the function Ω has the inverse function Ω^{-1} in y, and thus

$$y(t) = \Omega^{-1}[\int_{z(t)}^{t} \chi(y(\tau), z(t), \tau, t)d\tau + R(z(t), t), t]. \tag{7.20}$$

Reasoning as in the proof of Theorem 7.1 the system (7.2),(7.20) on the interval $[t_0, t_1]$, $t_1 \leq T$ can be reduced to one nonlinear VIE which is solvable because of the Contraction Mapping Principle and the first three conditions of the Theorem. The fifth condition ensures positiveness of the right-hand side of equation (7.19), whence, in view of the last condition, positiveness of $y(.)$ follows.

Theorem is proved.

The fourth and fifth conditions of Theorem 7.5 seem to be very restrictive, but they are valid in many important cases. For example, $\varphi(\tau,t,y) = \alpha(\tau,t)y$ and $\psi(\tau,t,y) = \beta(\tau,t)[1-y]$ in the two-sector IDM of economic renovation (see Section 2.5), hence, the fourth condition holds true for all $\alpha, \beta > 0$.

Another important case arises when $k(\tau,t,y)$ does not depend on y. Then the mentioned conditions are more easily verified, in particular, the fifth one leads

to the inequality

$$\varphi_t'(\tau,t,y)/\psi_t'(\tau,t,y) < k(u,t) < p'(t)/q'(t) \qquad (\text{if} \quad \psi_t', q' > 0).$$

This is illustrated by the following example.

Example 7.3. Let $\varphi(\tau,t,y) = c\tau y^d$, $c > 0$, $0 < d < 1$, $\psi(\tau,t,y) = y^d$, $p(t) = ct - cD/2$, $q(.) \equiv 1$, $y_0(.) \equiv y_0$, $z(t_0) = t_0 - D$, where $D = y_0^{-d}$. Then the equations (7.16)-(7.17) have the unique solution $z(t) \equiv t - D$, $y(t) \equiv y_0$, $t \in [t_0, T)$.

It should be noted that the Theorem 7.5 represents only one version of the sufficient conditions for the existence and uniqueness of solution of the equations (7.16)-(7.17). Thus, in the below Example 7.4, the fourth and fifth conditions of Theorem 7.5 hold true at $d_1 = d_2$ only, whereas the unique solution exists in all cases. However, the derivation of necessary and sufficient conditions of the existence and uniqueness of solution runs into great difficulties.

Example 7.4. Let $\varphi(\tau,t,) = b_1 e^{c_1(\tau-t)} x^{d_1}$, $\psi(\tau,t,X) = b_2 e^{c_2(\tau-t)} x^{d_2}$, $0 < d_i < 1$, $b_i > 0$, $c_i \geq 0$, $i = 1,2$; $p(.) \equiv p$, $q(.) \equiv q$, $y_0(.) \equiv y_0$, $z(t_0) = t_0 - D$. When $c_1, c_2 > 0$, the constants p, q, D, and y_0 satisfy the following system of nonlinear equations:

$$\begin{cases} 1 - e^{-c_1 D} = c_1 p y_0^{-d_1}/b_1, \\ 1 - e^{-c_2 D} = c_2 q y_0^{-d_2}/b_2, \end{cases}$$

which has a solution $D > 0$ at $d_1 \neq d_2$ only. Furthermore, if $c_1, c_2 \ll 1$, then $D \approx [(p/b_1)^{d_2}(q/b_2)^{-d_1}]^{1/(d_2-d_1)}$, $y_0 \approx (pb_1/qb_2)^{1/(d_2-d_1)}$.

When $c_1 > 0$, $c_2 = 0$, the corresponding equation is of the form

$$1 - e^{-c_1 D} = c_1 p/b_1 (q/b_2)^{1/d_2} D^{d_1/d_2},$$

and has a solution $D > 0$ at $d_1 > d_2$ or at $d_1 = d_2$ and $p/b_1 < (q/b_2)^{d_2}$.

In all these cases the solution of the equations (7.16)-(7.17) is

$$z(t) \equiv t - D, \quad y(t) \equiv y_0, \quad t \in [t_0, T).$$

Remark. If we consider the mixed VIE system of the first and second kind with respect to the sought-for $x(t)$, $y(t)$, $z(t)$, $t \in [t_0, T)$, $t_0 < T \leq \infty$:

$$x(t) = \int_{z(t)}^{t} f(\tau, t, x(\tau), y(\tau)) d\tau,$$

$$\int_{z(t)}^{t} \varphi(\tau, t, x(\tau), y(\tau))d\tau = p(t), \tag{7.21}$$
$$\int_{z(t)}^{t} \psi(\tau, t, x(t), y(t))dt = q(t)$$

with the initial conditions (7.3), (7.4), (7.18), then the sufficient conditions for the existence of a unique solution x, y, z may be constructed as a combination of the conditions of Theorems 7.1 and 7.5.

7.2 THE EQUATIONS WITH SEVERAL SOUGHT-FOR LIMITS OF INTEGRATION

We shall restrict ourselves to the following system of $2n$ VIEs of the second kind with linear integrands in the unknowns $x_1(t), ..., x_n(t)$, $z_1(t), ..., z_n(t)$, $t \in [t_0, T)$, $t_0 < T \leq \infty$:

$$x_i(t) = \sum_{j=1}^{n} \int_{z_j(t)}^{t} K_{ij}(\tau, t)x_j(\tau)d\tau, \tag{7.22}$$

$$\sum_{j=1}^{n} \int_{z_j(t)}^{t} Q_{ij}(\tau, t)x_j(\tau)d\tau = p_i(t), \quad i = \overline{1, n} \tag{7.23}$$

with the initial conditions:

$$z_j(t_0) = z_j^0 < t_0, \quad x_j(\tau) = x_j^0(\tau), \quad \tau \in (-\infty, t_0], \quad j = \overline{1, n}. \tag{7.24}$$

In contrast to previous cases, the positiveness of given functions proves to be insufficient for the inequalities $z_j(t) < t$ to hold when several $z_j(t)$ are sought for, and more limiting relations among $p_i(t)$ and $Q_{ij}(\tau, t)$, $i, j = \overline{1, n}$ are therewith required. Let us illustrate them for the case of triangular matrix $\{Q_{ij}\}$.

Theorem 7.6. If:

1) the functions $K_{ij}(\tau, t)$, $p_i(t)$, $x_i^0(\tau)$, $Q_{ii}(\tau, t)$ are positive, $Q_{ik}(\tau, t) \geq 0$ for $k < i$, $Q_{ik}(\tau, t) = 0$ for $k > i$, $\tau \in (-\infty, T)$, $t \in [t_0, T)$;

2) $K_{ij}(\tau,t)$, $Q_{ij}(\tau,t)$, $p_i(t)$, $x_i^0(\tau)$ are continuous with respect to τ and satisfy a Lipschitz condition in t, $i,j=\overline{1,n}$, $\tau\in(-\infty,T)$, $t\in[t_0,T)$;

3) the integrals $\int_{-\infty}^{t_0} Q_{ii}(\tau,t)x_0^i(\tau)d\tau$ diverge, $i=\overline{1,n}$, $t\in[t_0,T)$;

4) for each $i=\overline{1,n-1}$ the inequality

$$p_{i+1}(t)/p_i(t) > Q_{i+1,i}(u_1,t)/Q_{i,i}(u_1,t) \geq$$
$$\geq Q_{i+1,i-1}(u_2,t)/Q_{i,i-1}(u_2,t) \geq \dots \geq Q_{i+1,i}(u_i,t)/Q_{i,1}(u_i,t)$$

is fulfilled at $u_1,u_2,\dots,u_i\in\mathbf{R}$, $\tau\in(-\infty,T)$, $t\in[t_0,T)$;

then the system of VIEs (7.22)-(7.24) has the unique solution $x_i, z_i\in\mathbf{C}[t_0,T)$, $i=\overline{1,n}$ such that $x_i(t)>0$ and $z_i(t)<t$.

Proof is analogous to that of Theorem 7.1. The first and third conditions of the Theorem ensure that the system of n nonlinear equations (7.22) has a unique solution $z_i(.)$, $i=\overline{1,n}$, at the given $x_i(t)>0$, $i=\overline{1,n}$. The last condition assures $z_i(t)<t$, $t\in[t_0,T)$.

Example 7.5. Let $K_{ij}(\tau,t)=b_{ij}e^{c_{ij}(\tau-t)}$, $Q_{ii}(\tau,t)=\beta_i e^{\gamma_i(\tau-t)}$, $Q_{ij}(\tau,t)\equiv 0$ at $i\neq j$, b_{ij}, c_{ij}, β_i, $\gamma_i>0$, $p_i(.)\equiv p_i$, $x_j^0(.)\equiv x_j^0$, $z_i(t_0)=t_0-D_i$, $i,j=\overline{1,n}$, where the constants p_i, D_i, and x_i^0 satisfy the following system of $2n$ nonlinear equations:

$$x_i^0=\sum_j[b_{ij}(1-e^{-c_{ij}D_j})/c_{ij}]x_j^0,$$
$$p_i=x_i^0\beta_i(1-e^{-\gamma_i D_i})/\gamma_i,\quad i=\overline{1,n}.$$

Then the solution of the equations (7.22)-(7.24) is:

$$z_i(t)\equiv t-D_i,\quad x_i(t)\equiv x_i^0,\quad i=\overline{1,n}, t\in[t_0,T).$$

When

$$c_{ij},\gamma_i\ll 1,\quad D_i=p_i(\sum_j b_{ij}p_j/\beta_j)^{-1}/\beta_i+o(c_{ij},\gamma_i),$$
$$x_i^0=\sum_j b_{ij}p_j/\beta_j+o(c_{ij},\gamma_i),\quad i=\overline{1,n}.$$

The obtained results represent a basis for further study of optimal economic renovation in multi-sector IDMs. They allow us to investigate the solvability of such models with respect to sought-for phase variables under given control variables and provide the first necessary step in theoretical analysis of optimal control problems [43], [85], [142].

8

OPTIMIZATION IN TWO-SECTOR MODELS

In this Chapter we shall use a general system-theoretic interpretation of the IDMs under consideration as *the models of evolving systems (ESs)*, which has been exposed in Sections 1.6 and 2.5.1. The one-sector IDMs studied in Part 1 describe the ES whose new elements enter into the system from outside at the expense of the part of ES external product. More complicated two-, three- and n-sector IDMs studied below contain separate ES subsystems which ensure the improvement of the whole system.

The basic model of ES is the two-sector IDM (see Section 2.5.1) which describes the ES consisting of two subsystems : the self-improving subsystem A that ensures the creation of new system elements (SEs) and the liquidation of obsolete SEs; and the subsystem B that ensures the execution of an external aggregate function of the ES. The Chapter is concerned with the study of the optimal control problems for this IDM. Its focus is on the qualitative investigation of optimal trajectories and its interpretation as the properties of ES. The results obtained below allow us to establish some relationships between optimal strategies in the two-sector model of ES self-renovation and in the the one-sector IDM.

In relation to macroeconomics the two-sector IDM represents two production sectors: (A) – the production of production means and (B) – the production of consumption goods. This model takes into account the processes of the obsolete capacity liquidation and the production of new capacity inside économic system. The OPs under investigation optimize simultaneously the strategies of EU renovation and the distribution of EUs between production sectors. We shall analyze the structure and asymptotic behavior of optimal trajectories in this IDM. In so doing, the same turnpike properties of technology lifetime will

be proved as for the basic OP1 in one-sector IDM. The results develop those exposed above in Part 2 for the one-sector IDMs of economic renovation. The corresponding optimal distributions of EUs between production sectors are also constructed and analyzed.

In the considered class of IDMs the basic OP that was most thoroughly studied by the time being [48], [49], [67], [146] is the maximization of consumption goods output under given manpower in the Glushkov two-sector macroeconomic model (2.12)-(2.14). In present chapter this OP is investigated with the use of a novel approach that reduces the OP study to a simpler problem in one-sector IDM (see Chapters 4 and 5).

8.1 THE DESCRIPTION OF BASIC TWO-SECTOR MODEL

Introduce into consideration the following functions:

- $c(t)$ – the output of aggregate external product of ES per time unit;
- $m(t)$ – the quantity of *system elements* (SEs) which are created in the ES subsystem A and entered into ES at time t;
- $\beta(\tau, t)$ – the SE efficiency in subsystem B: the external product output per time unit at time t on one SE created at time τ;
- $\alpha(\tau, t)$ – the SE efficiency in subsystem A: the quantity of new SEs which are created per time unit at time t by one SE created at time τ ;
- $P(t)$ – the quantity of all operated SEs;
- $a(t)$ – time limit of obsolete SEs elimination (the SEs created before time $a(t)$ are not operated at time t);
- $y(\tau, t)$ – the relative part of SEs created and entered into ES at time τ which are operated in subsystem A at time t.

Under imposed assumptions the two-subsystem ES may be described by the following *two-sector IDM* (see Section 2.5):

$$m(t) = \int_{a(t)}^{t} \alpha(\tau, t) y(\tau, t) m(\tau) d\tau, \tag{8.1}$$

$$c(t) = \int_{a(t)}^{t} \beta(\tau, t)[1 - y(\tau, t)] m(\tau) d\tau, \tag{8.2}$$

$$P(t) = \int_{a(t)}^{t} m(\tau) d\tau, \tag{8.3}$$

Graphic scheme of the IDM (8.1)-(8.3) has been given in Fig.2.3.

In relation to macroeconomics $c(t)$ is the output of the consumption goods per time unit, $\alpha(\tau, t)$ and $\beta(\tau, t)$ are the productivities at time t of SEs created at time τ in two production sectors A and B respectively, $y(\tau, t)$ is variable accumulation norm (see Section 2.5).

8.2 THE PROBLEM OF MAXIMIZATION OF EXTERNAL OUTPUT

Let us suppose that the maximization of the efficiency of the execution of ES external function c on a planned interval $[t_0, T)$ is desirable:

$$I = \int_{t_0}^{T} \rho(t) c(t) dt \longrightarrow \max_{y,a,m}, \tag{8.4}$$

under the given quantity $P(.)$ of operated SEs.

In the frame of the IDM (8.1)-(8.3) this OP consists in the determination of the sought-for functions $y(\tau, t)$, $m(t)$, $a(t)$, $c(t)$, $t \in [t_0, T)$, $t \in [a(t), t]$, $T \leq \infty$, which maximize the functional (8.4) under the constraints-equalities (8.1)-(8.3), the following inequalities:

$$0 \leq y(\tau, t) \leq 1, \tag{8.5}$$

$$a'(t) \geq 0, \quad a(t) < t, \quad m(t) \geq 0, \tag{8.6}$$

and the initial conditions

$$a(t_0) = a_0, \quad m(\tau) = m_0(\tau), \quad y(\tau, t_0) = y_0(\tau), \quad \tau \in [a_0, t_0]. \tag{8.7}$$

The given system characteristics are the efficiencies $\alpha(\tau, t)$ and $\beta(\tau, t)$ of the SEs created in time τ, the quantity $P(t)$ of all operated SEs, and discounting multiplier $\rho(t)$.

Let us assume that the given functions $\alpha(\tau, t)$, $\beta(\tau, t)$, $\rho(t)$, $P(t)$ are Lipschitz continuous, $m_0(t)$ and $y_0(t)$ are piece-wise continuous, all these functions are positive and satisfy (8.1)-(8.7) at $t = t_0$.

The economic interpretation of this OP consists in the maximization of the discounted output of consumption goods on a planned interval under the given dynamics of labor force.

The OP (8.4)-(8.7) in the two-sector model (8.1)-(8.3) ensures the feasibility of simultaneous optimization of the following processes: the renovation of the ES elements' structure by choosing the sought-for SE lifetime $t - a(t)$ and quantity $m(t)$ of new created SEs; and the change of ES functional structure by choosing the SE distribution $y(\tau, t)$ between ES subsystems A and B.

The study of this OP proved to be a complicated problem. The approach given in the next Section reduces this OP to successive study of two simpler problems:

- optimization of the intensity of ES renovation by finding the sought-for control functions $a(t)$, $m(t)$;
- determination of the corresponding optimal distribution $y(\tau, t)$ of SEs between subsystems A and B.

The first problem is studied in Section 8.4, and the second — in Section 8.5.

8.3 THE REDUCTION OF TWO-SECTOR MODEL TO ONE-SECTOR MODEL

Consider the two-sector IDM (8.1)-(8.3) under the assumption that

$$\alpha'_\tau(\tau,t)/\alpha(\tau,t) \equiv \beta'_\tau(\tau,t)/\beta(\tau,t), \tag{8.8}$$

i.e., the relative increase rates of the SE efficiency are equal in both subsystems A and B. This assumption is natural for large macroeconomic systems and is accepted in many macroeconomic models [75]. It follows from (8.8) that $\partial/\partial\tau[\beta(\tau,t)/\alpha(\tau,t)] \equiv 0$ and

$$\beta(\tau,t) \equiv \lambda(t)\alpha(\tau,t). \tag{8.9}$$

Substituting (8.9) into the model (8.1)-(8.3) and the optimized functional (8.4) we find that

$$I = \int_{t_0}^{T} \rho(t)[\int_{a(t)}^{t} \beta(\tau,t)m(\tau)d\tau - \int_{a(t)}^{t} \beta(\tau,t)y(\tau,t)m(\tau)d\tau]dt =$$

$$= \int_{t_0}^{T} \rho(t)[Q(t) - \lambda(t)m(t)]dt \longrightarrow \max_{a,m}, \tag{8.10}$$

where

$$Q(t) = \int_{a(t)}^{t} \beta(\tau,t)m(\tau)d\tau. \tag{8.11}$$

It is readily seen that the equality (8.11) jointly with (8.3) forms the one-sector economic IDM introduced in Section 4.1. From the system-theoretic viewpoint the function $Q(t)$ may be considered as the output of aggregate common production of all ES in the measure units of external production c. Then function $\lambda(t)$ is interpreted as the expenditure per one SE creation or the cost of new SE (in c measure units).

Thus the OP (8.1)-(8.7) in the two-sector IDM (8.1)-(8.3) is reduced to the maximization of the functional (8.10) with the sought-for functions $m(t)$, $a(t)$, $Q(t)$, $t \in [t_0, T)$, $T \leq \infty$, under the constraints (8.3), (8.6), (8.11), and the initial conditions (8.7). Let us take note of the fact that the sought-for function $y(\tau,t)$ does not enter into this new optimization problem. This fact makes the OP study considerably simpler.

8.4 OPTIMAL TRAJECTORIES OF SYSTEM RENOVATION

The optimization problem (8.3), (8.6), (8.7), (8.10), (8.11) has been studied in detail in Chapters 4 and 5 (it is the basic OP1 in the one-sector IDM (4.1)- (4.3)). Then $m(t)$, $t \in [t_0, T)$, $t \leq \infty$ is *the independent control variable* of the OP and the functions $a(t)$, $Q(t)$, $t \in [t_0, T)$ are *the phase* (dependent) sought-for variables. Note that the integrand of the functional (8.10) must be positive since the part $\lambda(t)m(t)$ of aggregate output spent for ES renovation does not exceed all output $Q(t)$. Hence,

$$m(t) \leq Q(t)/\lambda(t), \quad t \in [t_0, T). \tag{8.12}$$

Since the function $Q(.)$ is the phase variable, the inequality (8.12) represents a mixed restriction for the OP under study. It is the analog of the upper bound $0 \leq m \leq M$ of the domain of controls m in the OP1 for one-sector IDM. The inequality (8.12) is essential only if the optimal control $m^*(t)$ reaches its maximal possible value $M(t)$ for certain $t \in [t_0, T)$. As follows from Chapters 4 and 5, it is not necessary, especially, when we primarily analyze the internal regimes of SE lifetime. In the opposite case the OP coincides with the OP1, and one can use the results of the OP1 study for the analysis of optimal renovation trajectories in the present case. For the OP1 the structure of the optimal solution (a^*, m^*) has been found in Theorem 4.2.

The analysis of optimal trajectories of ES renovation in the OP1 has been carried out in Chapter 5. In so doing, the turn-pike properties of ES lifetime $a(t)$ have been established for the finite- and infinite-horizon cases. Namely, the turn-pike Theorem 5.8 in the strongest form has been proved for infinite-horizon OP1. This theorem holds that the optimal trajectory $a^*(.)$ coincides with a turn-pike one $\tilde{a}(.)$ except some initial finite interval $[t_0, \mu]$, the value $\mu - t_0$ being determined only by the deviation $||\tilde{a}(t_0) - a_0||$. For finite-horizon OP1 just less strong turn-pike properties are valid: at $T \to \infty$ the solution $a^*(t)$ tends to $\tilde{a}(t)$ for an asymptotically largest part of horizon interval $[t_0, T)$, the optimal control $m(.)$ having more complex structure. The analysis of the general properties of turn-pike trajectories has been fulfilled in Section 5.3.

8.5 OPTIMAL DISTRIBUTION OF SYSTEM ELEMENTS

In this Section the optimal distributions of ES elements are constructed and analyzed corresponding to the established optimal strategies of ES renovation. For this purpose we now return to the initial OP (8.1)-(8.7) with the sought-for functions y, m, a. It still remains to find the corresponding $y^*(\tau, t)$, if it exists, for the optimal $m^*(t)$ and $a^*(t)$ defined above. Let us first look at some simpler forms of the control function $y(\tau, t)$, which are mentioned in Section 2.3.

The Case $y(\tau, t) \equiv y(t)$. The sought-for function $y^*(t)$, $t \in [t_0, T)$, is determined directly from the equation (8.1) and is of the following form:

$$y^*(t) = m^*(t)[\int_{a^*(t)}^{t} \alpha(\tau, t) m^*(\tau) d\tau]^{-1} = m^*(t)\lambda(t)/Q(t), \tag{8.13}$$

where $Q(t)$ is defined by (8.11). It should be verified that the condition (8.5) is satisfied for $y^*(t)$. In view of (8.12), $0 \leq y^*(t) \leq \lambda(t)M(t)/Q(t) \leq 1$.

The Case $y(\tau, t) \equiv y(\tau)$. In this case by virtue of the initial conditions (8.7), the control $y(\tau)$ becomes to be given for $\tau \in [a_0, t_0]$, and the unknowns are the functions $y(t)$, $m(t)$, $a(t)$ for $t \in [t_0, T)$. The sought-for $y(t)$ is determined under the given optimal $m^*(t)$ and $a^*(t)$ from the following integral equation of the first kind:

$$\int_{\max(a^*(t), t_0)}^{t} \alpha(\tau, t) m^*(\tau) y(\tau) d\tau = m^*(t) - \int_{\min(a^*(t), t_0)}^{t} \alpha(\tau, t) y_0(\tau) m_0(\tau) d\tau. \tag{8.14}$$

It follows from the previous results that the solution m^* of the OP (8.10)-(12),(8.3),(8.6),(8.7) is, in general, measurable. As is well known [78], at measurable $y(\tau)$ the Lebesgue integral on the left side of equation (8.14) is the absolutely continuous function. Hence, this equation has not, generally, a solution for measurable $m^*(t)$. For this reason the optimal trajectories $a^*(.)$ and $m^*(.)$ described in Section 8.1.5 are not attained and the initial OP (8.1)-(8.7) is not reduced to two successive simpler OPs in this case.

The case $y \equiv y(\tau)$ of the OP (8.1)-(8.7) was studied earlier in [48], [67], [146], and others. In so doing, the sought-for function $y(\tau)$ was taken to be the independent control of the OP, while $m(t)$ and $a(t)$ were the phase variables. It has been proved that there exists the measurable OP solution $y^*(.)$ which

is, in general, of a switching character. The corresponding $m^*(.)$ and $a^*(.)$ are absolutely continuous and such that $a^*(.)$ approaches the trajectory $\tilde{a}(t)$ described above for a large interval $[t_0, T]$. Thus, the turn-pike properties of OP are also preserved in this case but less strong than in other considered cases.

The Case $y(\tau, t) \equiv y_1(\tau)y_2(t)$. In this case the OP may have non-unique solution y as well as have no solution. Indeed, the initial condition $y_1(\tau) \equiv y_1^0(\tau)$, $\tau \in [a_0, t_0]$ is given as in the previous case. The initial condition for $y_2(t)$ is

$$y_2(t_0) = y_2^0 = m_0(t_0)[\int_{a(t_0)}^{t_0} \alpha(\tau, t_0)y_1^0(\tau)m_0(\tau)d\tau]^{-1}.$$

In view of (8.1), for $\tau, t > t_0$ the sought-for functions $y_1(\tau)$ and $y_2(t)$ are connected by the following formula:

$$y_2(t_0) = m^*(t)[\int_{a^*(t)}^{t_0} \alpha(\tau, t)y_1^0(\tau)m_0(\tau)d\tau] + \int_{t_0}^{t} \alpha(\tau, t)m^*(\tau)y_1(\tau)d\tau]^{-1}.$$

As is easily seen, the sought-for $y_2(t)$ may be determined for the different given $y_1(\tau)$, $t \in [t_0, t]$. But the optimal given function $m^*(t)$ may become the maximal $m^*(t) = Q^*(t)/\lambda(t)$ for t close to t_0 (see Example 4.1). Then in view of the last formula, $y_2(t) > 1$ for any $0 < y_1(\tau) < 1$, $t \in [t_0, t]$. Hence, the OP solution $0 < y^* < 1$ does not exist. Thus, this case is unsuccessful from the theoretical point of view.

General Case. It follows from the case $y(\tau, t) \equiv y(t)$ that the equation (8.1) has at least one solution $y(\tau, t) \equiv y^*(t)$, $t \in [t_0, T)$. Let us transform (8.1) to the following form:

$$\int_{a^*(t)}^{t} \alpha(\tau, t)[y(\tau, t) - y^*(t)]m^*(\tau)d\tau = 0. \tag{8.15}$$

In the class of measurable functions $y(\tau, t)$ the equations (8.1) and (8.15) may be considered for any fixed time t separately as the Fredholm integral equations of the first kind with respect to sought-for function $\varphi(\tau) = y(\tau, t)$, $t \in [a(t), t]$. It is easily seen that at $0 < y^*(t) < 1$ they have some set of solutions $y(\tau, t)$, $0 \leq y(\tau, t) \leq 1$, $t \in [t_0, T)$. Let us analyze the structure of this set.

If $m^*(\tau) \equiv 0$, $\tau \in \Delta_0 \subset [t_0, T]$, then the solution $\varphi(\tau) = y(\tau, t)$ of equation (8.15) may be chosen as an arbitrary function for $\tau \in \Delta_0$, $t \in [\tau, a^{-1}(\tau)]$. Now put $m^*(\tau) \neq 0$, $\tau \in \Delta \subset [t_0, T]$. If $m^*(t) \neq 0$, then the equation (8.15) admits

some set of the solutions $\varphi(\tau) = y(\tau, t)$ which oscillate with respect to their average value $y^*(t)$, when $\tau \in \Delta$. The subset of functions $\varphi(\tau)$, $\tau \in [a(t), t]$ such that $0 \leq \varphi(\tau) < 1$ is the solution of the OP (8.1)-(8.7). If $m^*(t) = 0$, then the OP solution $y(\tau, t) \equiv 0$ is unique for $\tau \in \Delta$ in view of the restriction (8.5). The concrete variants of such OP solutions $y(\tau, t)$ are given below in Example 8.1.

Thus, the following statement has been proved.

Theorem 8.1. The OP (8.1)-(8.7) has some set of the solutions $y(\tau, t)$, $0 \leq y(\tau, t) \leq 1$, $t \in [t_0, T)$, $t \in [a^*(t), t]$, each of which is assigned to the optimal trajectory $a^*(.)$, $m^*(.)$ of ES renovation described by Theorems 4.2, 5.6-5.8. For any fixed time t these solutions oscillate with respect to the value

$$y^*(t) = m^*(t)[\int_{a^*(t)}^{t} \alpha(\tau, t) m^*(\tau) d\tau]^{-1}.$$

The function $y^*(t)$, $t \in [t_0, T)$ represents the unique solution of this OP in the case $y \equiv y(t)$.

Example 8.1. Let $\beta(\tau)$, t_0, a_0, $P(t)$, $\gamma(t)$, $\rho(t)$ be the same as in Example 4.1. Then the solution a^*, m^* of the OP (8.1)-(8.7) is determined by the following formulas

$$a^*(t) = \begin{cases} 0, & t \in [1, \sqrt{2}], \\ t - \sqrt{2}, & t \in [\sqrt{2}, \infty), \end{cases}$$

$$m^*(t) = \begin{cases} 0, & t \in [1 + (i-1)\sqrt{2}, i\sqrt{2}), \\ m_0, & t \in [i\sqrt{2}, 1 + i\sqrt{2}),\ i = 1, 2, \ldots. \end{cases} \tag{8.16}$$

The corresponding control $y^*(t)$ determined by (8.13) is:

$$y^*(t) = \begin{cases} 0, & t \in [1 + (i-1)\sqrt{2}, i\sqrt{2}),\ i = \lfloor (t-1)/\sqrt{2} \rfloor + 1 \\ 1/[\sqrt{2}(t-1) - i(2-\sqrt{2}) + 3/2], & t \in [i\sqrt{2}, 1 + i\sqrt{2}), \end{cases} \tag{8.17}$$

$t \in [1, \infty)$, where $\lfloor x \rfloor$ is the integer part of x. It is easily verified that $0 \leq y^*(t) < 1$ for $t \in [1, \infty)$.

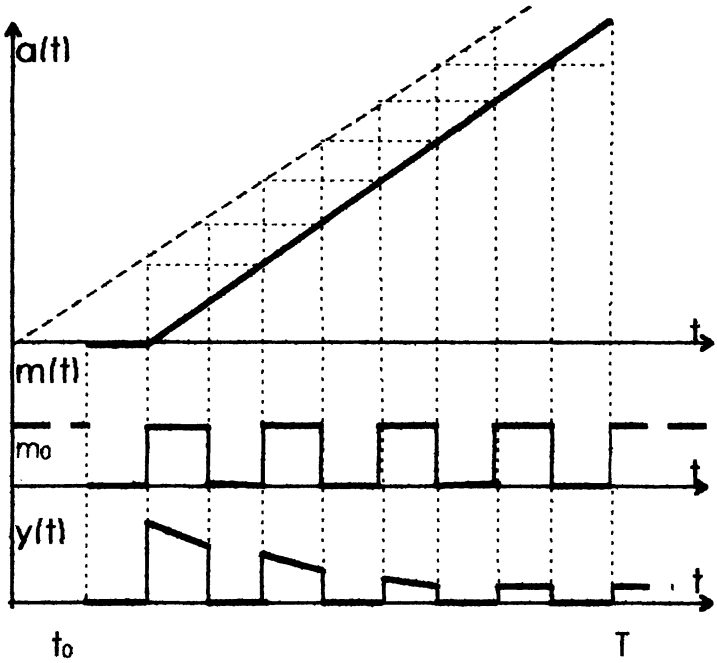

Figure 8.1 Optimal trajectories in two-sector integral model (infinite-horizon discounted optimization)

In view of (8.15), at $\tau \in [1+(j-1)\sqrt{2}, j\sqrt{2})$, where $j = \lfloor(\tau-1)/\sqrt{2}\rfloor + 1$, and $t \in [\tau, \tau+\sqrt{2}]$ we have

$$y(\tau,t) \equiv \phi(\tau) \quad \text{for any} \quad 0 \le \phi(\tau) \le 1. \tag{8.18}$$

For $\tau \in [j\sqrt{2}, 1+j\sqrt{2})$, $j = \lfloor(\tau-1)/\sqrt{2}\rfloor + 1$, $t \in [1+j\sqrt{2}, (j+1)\sqrt{2})$ the OP solution is

$$y(\tau,t) \equiv 0. \tag{8.19}$$

For $\tau \in [j\sqrt{2}, 1+j\sqrt{2})$, $t \in [\tau, 1+j\sqrt{2})$, the equation (8.15) is of the following form:

$$m_0 \int_{j\sqrt{2}}^{t} \tau[y(\tau,t) - y^*(t)]d\tau = 0. \tag{8.20}$$

Otherwise speaking,the function $\varphi(\tau) = \tau[y(\tau,t) - y^*(t)]$ may be an arbitrary function such as $\int_{j\sqrt{2}}^{t} \varphi(\tau)d\tau = 0$. For instance, it may be $\varphi(\tau) = \varepsilon \sin[2\pi l(\tau-$

$j\sqrt{2})/(t - j\sqrt{2})]$, $\varepsilon > 0$, $l = 1, 2, \ldots$. Hence, all functions

$$y(\tau, t) = y^*(t) + \varepsilon \sin[2\pi l(\tau - j\sqrt{2})/(t - j\sqrt{2})]/\tau, \quad \varepsilon > 0, \quad l = 1, 2, \ldots \quad (8.21)$$

are the solutions of the equation (8.20). Thus any function of the form (8.21) is the OP solution for $\tau \in [j\sqrt{2}, 1 + j\sqrt{2})$, $\tau \in [t, 1 + j\sqrt{2})$, if $\varepsilon < \tau y^*(t)$, $\varepsilon < \tau[1 - y^*(t)]$.

The formulas (8.18),(8.19),(8.21) determine one of the possible sets of the solutions of the OP (8.1)-(8.7) . The solution (y^*,m^*,a^*) is illustrated in Fig.8.1.

Observe that as distinct from Solow-Shell macroeconomic model (see Section 0.1) the optimal control y^* *has no turn-pike properties* (as well as the optimal trajectory m^*). Only the optimal trajectory a^* possesses such properties in the two-sector integral model (8.1)-(8.3).

8.6 CONSIDERATION OF THE DIFFERENT DYNAMICS OF TECHNOLOGICAL RENOVATION

Consider the "putty-clay" model (8.1)-(8.3), i.e. suppose that the ES elements cannot pass from one ES subsystem to another (see Section 2.5). Then the question of interest is to compare the optimal rates of elements renovation in the different ES subsystems.

The corresponding two-sector integral *model with different dynamics of technological renovation in ES subsystems* of the form:

$$m(t) = \int_{a_1(t)}^{t} \alpha(\tau, t) y(\tau) m(\tau) d\tau, \quad (8.22)$$

$$c(t) = \int_{a_2(t)}^{t} \beta(\tau, t)[1 - y(\tau)] m(\tau) d\tau, \quad (8.23)$$

$$P(t) = \int_{a_1(t)}^{t} y(\tau) m(\tau) d\tau + \int_{a_2(t)}^{t} [1 - y(\tau)] m(\tau) d\tau, \quad (8.24)$$

is a natural generalization of the two-sector model (8.1)-(8.3) and has been discussed in [146]. As compared with the IDM (8.1)-(8.3), this model includes two independent controls $a_1(t)$ and $a_2(t)$ for the liquidation of the oldest SEs in the subsystems A and B (instead of one common control $a(t)$ for the entire ES). As has been shown in Section 2.5, the only case $y \equiv y(\tau)$ is meaningful in the model (8.22)-(8.24).

The problem of maximization of the functional (8.4) in the model (8.22)-(8.24) must have two independent sought-for control functions. This OP was earlier studied in [146] and other publications. The methods of its exploration relied on reduction of the OP with mixed differential constraints:

$$0 \leq y(\tau) \leq 1, \tag{8.25}$$

$$a_i'(t) \geq 0, \quad a_i < t, \quad i = 1, 2, \quad m(t) \geq 0 \tag{8.26}$$

to simpler one by narrowing the domain (8.25) of admissible controls y and introducing auxiliary control variable a_2'. Namely, the unknowns y and a_2' were taken as independent control variables while m and a_i, $i = 1, 2$ were phase variables. Then the OP constraints (8.25)-(8.26) may be replaced by more strict ones :

$$0 \leq y_{min}(t) \leq y(t) \leq 1, \quad a_2'(t) \geq 0. \tag{8.27}$$

It was shown in [146] that the solution y^* of the OP (8.4), (8.22)-(8.27) in the general case has a switching behavior, and what is more, on a very large interval $[t_0, T]$ the switching points can have a condensation point τ^*. Starting from this point, $y^*(t)$ is an inner (turn-pike) solution in the domain (8.27) until the constraints (8.27) are violated. It always holds $y^*(t) \equiv y_{min}(t)$ at the right end $(\theta, T]$, $t_0 \leq \theta \leq T$ of interval $[t_0, T]$, i.e., *the effect of the end of planned interval* is also present in this OP as in the finite-horizon OP1 for the basic one-sector IDM (see Theorem 4.6).

The results obtained in [146] show that the analysis of turn-pike solutions in this OP leads to a problem without sought-for y, m not only under assumption (8.8) but also for the general case of the OP. This allows us to study the optimal renovation trajectories for the subsystems A and B. Namely, the OP (8.4), (8.22)-(8.27) produces the following system of two integral- functional

equations (IFEs) for the turn-pike trajectories $\tilde{a}_1(.)$ and $\tilde{a}_2(.)$:

$$\int_t^{a_2^{-1}(t)} \rho(\tau)[\beta(t,\tau) - \beta(a_2(\tau),\tau)]d\tau = \rho(t)\beta(a_2(t),t)/\alpha(a_1(t),t), \tag{8.28}$$

$$\int_t^{a_1^{-1}(t)} \rho(\tau)[\alpha(t,\tau)/\alpha(a_1(\tau),\tau) - 1]\beta(a_2(\tau),\tau)]d\tau =$$
$$= \rho(t)\beta(a_2(t),t)/\alpha(a_1(t),t), \quad t \in [t_0,\infty), \tag{8.29}$$

which is similar to turnpike IFE (5.1) for the basic OP1 in one-sector IDM.

The IFEs (8.28) and (8.29) can have a joint solution $\tilde{a}_1(t)$, $\tilde{a}_2(t)$, $t \in [t_0,\infty)$ as well as only separate solution $\tilde{a}_1$ (or $\tilde{a}_2$).

Example 8.2. Let $\alpha(\tau,t) \equiv$const, $\beta(\tau,t) = e^{-c\tau}$, $\rho(t) = e^{-st}$, $s > c > 0$. Then, similar to Theorem 5.4 for the IFE (5.1), the equation (8.28) has the unique solution $\tilde{a}_2(t) = t - A$, $t \in [t_0,\infty)$, whereas any solutions $\tilde{a}_1$ are not found for the equation (8.29).

Example 8.3. Let $\beta(\tau,t) \equiv$ const, $\alpha(\tau,t) = e^{-c\tau}$, $\rho(t) = e^{-st}$, $s > c > 0$. Then it is easily verified that the equation (8.28) has no solution $\tilde{a}_2$, but there exists a solution $\tilde{a}_1(t)$, $\tilde{a}_1'(t) > 0$, $t \in [t_0,\infty)$ (see also IFE (6.4) for OP2 in one-sector IDM).

By introducing the new auxiliary variable

$$\kappa(t) = \beta(a_2(t),t)/\alpha(a_1(t),t), \tag{8.30}$$

the IFEs (8.28)-(8.29) can be written in the equivalent form with three unknowns κ, a_1 and a_2:

$$\int_t^{a_2^{-1}(t)} \rho(\tau)[\beta(t,\tau) - \beta(a_2(\tau),\tau)]d\tau = \rho(t)\kappa(t), \tag{8.31}$$

$$\int_t^{a_1^{-1}(t)} \rho(\tau)[\kappa(\tau)\alpha(t,\tau) - \beta(a_2(\tau),\tau)]d\tau = \rho(t)\kappa(t), \tag{8.32}$$

This form of the IFEs (8.28)-(8.29) is more convenient for analysis and provides the following results about qualitative properties of the optimal renovation rates a_1 and a_2 of ES subsystems A and B.

Theorem 8.2. If $\beta(\tau,t) \equiv \lambda(t)\alpha(\tau,t)$ (i.e.(8.8) is valid), the system of IFEs (8.28)-(8.29) has unique solution $\tilde{a}_1(t)$, $\tilde{a}_2(t)$, and the IFE (5.1) has a solution $\tilde{a}(t)$, $t \in [t_0,\infty)$, then $\tilde{a}_1 \equiv \tilde{a}_2 \equiv \tilde{a}$.

Proof. Put $\tilde{a}_1(t) = \tilde{a}_2(t)$, $t \in [t_0,\infty)$. Then by (8.30) $\kappa \equiv \lambda$, and the equation (8.31) coincides with the IFE (5.1) which has a solution $\tilde{a}(t) = \tilde{a}_2(t)$, $t \in [t_0,\infty)$. Under the first condition of the Theorem, the equation (8.32) is of the form:

$$\int_t^{a_1^{-1}(t)} \rho(\tau)[\beta(t,\tau) - \beta(a_2(\tau),\tau)]d\tau = \rho(t)\kappa(t),$$

which is solvable uniquely with respect to $a_1(.)$. When the last equation is compared with (3.28), it is apparent that $\tilde{a}_1 \equiv \tilde{a}_2$.

Theorem is proved.

The first assumption of the Theorem 8.2 implies that the relative rates of SE efficiency increase are equal in both subsystems A and B of ES (see (8.8)). Under this assumption, the behavior of solution $\tilde{a}_1 \equiv \tilde{a}_2 \equiv \tilde{a}$ of the equations (8.28)-(8.29) is determined by Theorems 5.1-5.5. In general case, it is apparent from the next theorem that the optimal renovation intensity is higher in that ES subsystem for which the increase rate of new SEs efficiency is larger.

Theorem 8.3. If $\partial/\partial\tau[\beta(\tau,t)/\alpha(\tau,t)] < 0$, $\partial/\partial\tau[\alpha(\tau,t)] > 0$, $\partial/\partial\tau[\beta(\tau,t)] > 0$, $\tau,t \in [t_0,\infty)$, then the system of IFEs (8.28)-(8.29) has a solution $\hat{a}_1(t)$, $\hat{a}_2(t)$ on some interval $[t_0,T]$, $T > \hat{a}_1^{-1}(t_0)$, such that $\hat{a}_1(t) > \hat{a}_2(t)$ whereas the case $\hat{a}_1(t) \le \hat{a}_2(t)$, $t \in [t_0,T]$ is impossible. If the equation (8.32) is solvable with respect to $a_1(.)$ for given $a_2(.)$, then $T > \hat{a}_2^{-1}(t_0)$. The reverse ($\hat{a}_1 < \hat{a}_2$) holds good for $\partial/\partial\tau[\beta(\tau,t)/\alpha(\tau,t)] > 0$.

Proof. The construction of the initial functions $a_1^0(t)$, $t \in [t_0,\hat{a}_1^{-1}(t_0)]$ and $a_2^0(t)$, $t \in [t_0,\hat{a}_2^{-1}(t_0)]$ for solution $\hat{a}_1(t)$, $\hat{a}_2(t)$ can be performed in different ways. But the choice of a_1^0, a_2^0 determines longevity of the interval $[t_0,T]$ on which the solution $\hat{a}_1$, $\hat{a}_2$ exists.

Let us first choose $a_1^0(t) = a_2^0(t) < t$, $t \in [t_0, \hat{a}_2^{-1}(t_0)]$ such that (2.28) is satisfied. It can be attained, for example, by choosing $a_2^0(t) = t - A$, $A > 0$. Indeed, the left-hand side of (8.28) is less than right-hand one at $A \to 0$ and increases infinitely in A, whereas the right-hand side decreases in A. Thus, the value $A = A_0$ exists such that (8.28) holds true. For $A = A_0$ the left-hand side of (8.29) turns to be less than the left-hand side of (8.28) because, by (8.30),

$$\kappa(\tau) = \beta(a_2(\tau), \tau)/\alpha(a_2(\tau), \tau) > \beta(t, \tau)/\alpha(t, \tau) \quad \text{for} \quad \tau \in [t, a_2^{-1}(t)].$$

So, the equality (8.29) cannot be valid at $A = A_0$.

Putting $a_1^0(t) = a_2^0(t) + \varepsilon$, $\varepsilon > 0$, $t \in [t_0, \hat{a}_2^{-1}(t_0)]$, we obtain that the left-hand side of (8.28) does not depend upon ε, its right-hand side increases in ε, the left-hand side of (8.29) is more than its right-hand side at $\varepsilon = 0$ and more at $\varepsilon \to A$. Hence, the values ε, A, $0 < e < A_0$, $0 < A < A_0$ exist such that (8.28) and (8.29) hold true for $t \in [t_0, \hat{a}_1^{-1}(t_0)]$. In so doing, $\hat{a}_1(t) > \hat{a}_2(t)$ for $t \in [t_0, \hat{a}_1^{-1}(t_0)]$, hence, $\hat{a}_1^{-1}(t_0) < \hat{a}_2^{-1}(t_0)$. Under the given a_1^0, a_2^0 the solution $\hat{a}_1(t) > \hat{a}_2(t)$ can be continued for some interval $[t_0, T]$, $T > \hat{a}_1^{-1}(t_0)$.

Suppose now that solution a_1, a_2 of (8.28),(8.29) exists such that $a_1(t_1) = a_2(t_1)$ for a fixed $t_1 > t_0$. Then $a_1^{-1}(t_2) = a_2^{-1}(t_2)$, $t_1 = a_2(t_2)$, and in view of (8.30)-(8.32)

$$\int_{t_2}^{a_2^{-1}(t_2)} \rho(\tau)[\beta(t_2, \tau) - \kappa(\tau)\alpha(t_2, \tau)]d\tau = 0.$$

By the theorem of the mean, the value $\vartheta \in ([t_2, a_2^{-1}(t_2)]$ exists such that $\kappa(\vartheta) = \beta(t_2, \vartheta)/\alpha(t_2, \vartheta)$. Hence,

$$\beta(a_2(\vartheta), \vartheta)/\alpha(a_1(\vartheta), \vartheta) = \beta(t_2, \vartheta)/\alpha(t_2, \vartheta) >$$

$$> \beta(a_2(\vartheta), \vartheta)/\alpha(a_2(\vartheta), \vartheta)$$

because of $t_2 > a_2(\vartheta)$. Thus, $a_1(\vartheta) < a_2(\vartheta)$ for some $\vartheta \in [a_2^{-1}(t_1), t_1]$.

Theorem is proved.

This theorem indicates that at $\partial/\partial\tau(\beta/\alpha) < 0$ a solution $\hat{a}_1(t) > \hat{a}_2(t)$ of the IFEs (8.28)-(8.29) exists on an interval $[t_0, T]$, $T > \hat{a}_1^{-1}(t_0)$, whereas there are no solutions $\hat{a}_1(t) \leq \hat{a}_2(t)$, $t \in [t_0, T]$. In principle, the solutions $a_t(t)$, $a_2(t)$ are also possible such that $a_1(t)$ intersects $a_2(t)$ at least in one point on each interval $[t, a_1^{-1}(t)]$. However, in view of (8.30)-(8.32) the case $a_1(t) - a_2(t) < 0$ causes considerably more large deviations $a_1(\tau) - a_2(\tau) > 0$ at some points $\tau \in [t, a_1^{-1}(t)]$. So, such regime is not natural from the economic viewpoint.

The Theorem 8.3 still stand for infinite solution $\tilde{a}_1(t)$, $\tilde{a}_2(t)$, $t \in [t_0, \infty)$ (if it exists) of IFEs (8.28)-(8.29). Similarly to IFE (5.1), such solution must have a high smoothness to exist. So, the case of intersecting $\tilde{a}_1$ and $\tilde{a}_2$ is highly improbable. The IFEs (8.28)-(8.29) is far more complex to study than IFE (5.1). As appears from Section 5.1, an investigation of infinite solution $\tilde{a}$ of the IFE (5.1) is possible for specified functions β only. Hence, the analysis of $\tilde{a}_1$, $\tilde{a}_2$ should be also started from particular cases. Any more rigorous treatment of the IFEs (8.28)-(8.29) is beyond the scope of this book.

Two OPs briefly considered below differ from previous one by including an additional balance relation (in total production output) into the two-sector IDM (8.22)-(8.24) with different dynamics of technological renovation in ES subsystems. This causes substantial distinctness in qualitative behavior of optimal trajectories.

8.7 THE MAXIMIZATION OF EXTERNAL OUTPUT UNDER THE GIVEN LABOR AND TOTAL OUTPUT

Let us consider the problem of maximization of the ES external output (8.4) in the model (8.22)-(8.24) under the given quantity $P(.)$ of operated SEs and given output Q of total ES production. In general case (without assumption (8.8)) the output $Q(t)$ in the measure units of external product c is determined by the following equality [146]:

$$Q(t) = \lambda(t)m(t) + c(t), \tag{8.33}$$

where function $\lambda(t)$ is the expenditure per one SE creation or the cost of new SE (compare with (8.11)).

The OP consists in the determination of the sought-for functions $y(t)$, $m(t)$, $a_1(t)$, $a_2(t)$, $c(t)$, $t \in [t_0, T)$, $T \leq \infty$, which maximize the functional (8.4) under the constraints (8.22)-(8.26) and (8.33).

This OP was investigated in [146] by taking the unknown $y(.)$ as independent control variable, $y \in L_\infty[t_0, T)$. The domain $y_{min} \leq y(t) \leq y_{max}$ such that the phase variables m, a_1, a_2 satisfy all constraints-inequalities (8.26) was con-

structed in [146] under certain additional conditions. The phase variables $m(.)$, $a_i(.)$, $i = 1, 2$ are determined from the following systems of VIEs with two sought-for integral limits:

$$m(t) = \int_{a_1(t)}^{t} \alpha(\tau, t) y(\tau) m(\tau) d\tau, \tag{8.34a}$$

$$Q(t) = \lambda(t) m(t) + \int_{a_2(t)}^{t} \beta(\tau, t)[1 - y(\tau)] m(\tau) d\tau, \tag{8.34b}$$

$$P(t) = \int_{a_1(t)}^{t} y(\tau) m(\tau) d\tau + \int_{a_2(t)}^{t} [1 - y(\tau)] m(\tau) d\tau, \tag{8.34c}$$

The unique solvability of the VIEs (8.34) appears to be a non-trivial problem. It was shown in [146], on the grounds of technique exposed in Chapter 7, that the system (8.34) has a unique solution $a_1, a_2, m \in C[t_0, T]$, $m(t) > 0$, $a_i(t) < t$, $i = 1, 2$ under the following conditions:

1) the initial functions $m_0(\tau)$ and $y_0(\tau)$ are given on the infinite prehistory $(-\infty, t_0]$, whereas the integrals

$$\int_{-\infty}^{t_0} \alpha(\tau, t) y_0(\tau) m_0(\tau) d\tau \quad \text{and} \quad \int_{-\infty}^{t_0} \beta(\tau, t)[1 - y_0(\tau)] m_0(\tau) d\tau$$

diverge at $t \in [t_0, T]$;

2) the given functions $\alpha(\tau, t)$, $\beta(\tau, t)$, $P(t)$, $Q(t)$, $\lambda(t)$, $m_0(t)$ are positive and Lipschitz continuous in t;

3) $0 < \eta \leq y(t) \leq 1 - \eta$ a.e. at $t \in [t_0, T]$;

4) the inequality

$$\lambda(t)\alpha(u, t) < Q(t)/P(t) < \beta(v, t) \tag{8.35}$$

or the inverse inequality holds true at $u, v \in (-\infty, t]$.

Thus, the domain of given functions, for which the OP has meaning, is sufficiently restricted by the inequality (8.35). From economic viewpoint it means that the specific productivities of all operated ESs in sectors A and B must be less or more than joint average productivity Q/P.

The gradient $I'_y(.)$ of this OP was derived in [146] by using Lagrange multipliers. The obtained equation

$$I'_y(y, a_1, a_2; t) = 0, \quad t \in [t_0, T] \tag{8.36}$$

commonly with the equalities (8.34) forms a system of five integral-functional equations with respect to $y(t)$, $m(t)$, $a_1(t)$, $a_2(t)$, and Lagrange multiplier $\varphi(t)$, $t \in [t_0, T]$. A solution of this system is an OP solution, if it satisfies initial conditions and constraints-inequalities of the OP. In contrast to all OPs considered above, this may be valid for all $t \in [t_0, T]$. Namely, the next example was constructed in [146].

Example 8.4. Let $\beta(\tau, t) \equiv \gamma(t)\alpha(\tau, t)$, $\gamma(t) \neq \lambda(t)$, $a_1(t_0) = a_2(t_0) = a_0$, $0 \leq P''(t) \leq \alpha(t,t)P'(t)$,

$$Q(t) = [\lambda(t) - \gamma(t)]P'(t) + \int_{a_0}^{t_0} \beta(\tau, t)m_0(\tau)d\tau + \int_{t_0}^{t} \beta(\tau, t)P'(\tau)d\tau,$$

Then the functions $y^*(t) = P''(t)/\alpha(t,t)P'(t)$, $m^*(t) = P'(t)$, $a_1^*(t) = a_2^*(t) \equiv a_0$ satisfy the equations (8.34), (8.36) and all constraints of the OP. Hence, (m^*, a_1^*, a_2^*, y^*) is the OP solution.

Thus, contrary to other OPs, the optimal control function $y(.)$ can be internal in the domain (8.25), and the effect of the end of planned interval *is absent* in this OP.

At $T = \infty$ the equations (8.34) and (8.36) determine turn-pike trajectories $\tilde{a}_1(.)$ and $\tilde{a}_2(.)$ of the OP (if they exist). The following qualitative results about the OP solution were established in [146]:

1. Let the functions α and β do not depend upon τ, i.e. $\alpha \equiv \alpha(t)$, $\beta \equiv \beta(t)$. Then:

   (a) if $\beta(t) \neq \alpha(t)\lambda(t)$, then the equation (8.36) is satisfied identically at any y, and the value of the functional (8.4) *is independent of the sought-for functions* and has the form:

   $$I(t) = \int_{t_0}^{t} \beta(t)[\beta(t) - \lambda(t)\alpha(t)][Q(t) - \lambda(t)\alpha(t)P(t)]dt;$$

   (b) if $\beta(t) \equiv a(t)\lambda(t)$, then :

i. at $Q \equiv \beta P$ the equations (8.24b) and (8.24c) coincide, the OP has two independent sought-for variables (instead of one y) and *is equivalent to the above OP* (8.4),(8.22)-(8.26);

ii. at $Q \not\equiv \beta P$ *the OP has no solution*;

(c) if $\beta \not\equiv \alpha\lambda$ and $\beta(t) = \alpha(t)\lambda(t)$ for some $t \in [t_0, T]$, then *the OP is incorrect*, namely, an infinitesimal variation of $y(.)$ can cause a finite change of $I(y)$.

2. Let $\beta(\tau, t) \equiv \lambda(t)\alpha(\tau, t)$ (the increase rates of SE efficiency are equal in subsystems A and B) and the OP have a unique solution (m^*, a_1^*, a_2^*, y^*). Then $a_1^*(t) \equiv a_2^*(t)$ wherever the solution is internal. In this case three sought-for functions y^*, m^*, $a_1^* \equiv a_2^*$ are determined from the equations (8.34).

3. Let $\partial/\partial\tau[\beta(\tau,t)/\alpha(\tau,t)] < 0$, $\partial/\partial\tau[\alpha(\tau,t)] > 0$, $\partial/\partial\tau[\beta(\tau,t)] > 0$, $\tau, t \in [t_0, T)$, then the OP can have only such solution that $a_1^*(t) > a_2^*(t)$ for $t \in \Delta \subset [t_0, T]$, $mes(\Delta) > 0$, whereas the case $a_1^*(t) \leq a_2^*(t)$, $t \in [t_0, T]$ is impossible. The reverse $(a_1^* < a_2^*)$ holds good for $\partial/\partial\tau[\beta(\tau,t)/\alpha(\tau,t)] > 0$. So, the optimal renovation intensity is higher in that ES subsystem for which the increase rate of new SEs efficiency is larger.

The last two results are the same as for previous OP (see Theorems 8.2 and 8.3).

8.8 THE EXPENDITURE MINIMIZATION UNDER GIVEN OUTPUTS

Consider briefly the minimization of labor expenditure (the quantity $P(.)$ of operated SEs) in the model (8.22)-(8.24), (8.33) under the given ES external output c and total output Q. This OP is formulated as follows [146]: determine the sought-for functions $y(t)$, $a_1(t)$, $a_2(t)$, $c(t)$, $t \in [t_0, T]$, $t \leq \infty$, which minimize the functional

$$I_2 = \int_{t_0}^{T} P(t)dt \to \min_{y, a_1, a_2}, \tag{8.37}$$

under the constraints (8.22)-(8.26) and (8.33).

From the contextual viewpoint, this OP is the inverse of the previous problem and is investigated in a similar way. As was shown in [146], the system (8.34) is uniquely solvable with respect to a_1, a_2, $P \in C[t_0, T]$ at the given c under the same conditions as above (with replacing inequality (8.34) by $0 < c(t)/Q(t)$). In this case, the function $m(.)$ appears to be specified: $m(t) = [Q(t) - c(t)]/\lambda(t)$, $t \in [t_0, T]$, and the investigation is simplified in comparison with the previous OP.

Behavior of the solutions of these OPs is similar on qualitative level. In particular, the conclusions 1-4 for previous OP also hold true for the OP (8.3), in so doing, we have

$$I_2 = \int_{t_0}^{T} \{[Q(t) - c(t)]/\alpha(t)/\lambda(t) + c(t)/\beta(t)\} dt,$$

for the item 1(a).

It follows from the above results that introducing the different SE lifetimes into two-sector IDM is profitable in the case when the rates $\partial\beta/\partial\tau$ and $\partial\alpha/\partial\tau$ of embodied TC are distinct in the ES subsystems A and B. Doing so, the optimal (turnpike) SE lifetime is less in that subsystem for which this rate is greater.

Another conclusion is that, when the specific efficiencies α and β of new SEs do not depend upon the time τ of SE creation (i.e., the embodied TC is absent), no distribution $y(t)$ of SEs between ES subsystems can raise the external output c under the given SE quantity P and total output Q (or reduce the expenditure of P under the given outputs c and Q). It means that the control of the renovation of system elements is more proficient than the management of distribution of productive elements, resources, etc.

9

OPTIMIZATION OF INDUSTRY CONVERSION RATES

In this Chapter the integral economic models are used for such applied problem as the optimization of the conversion process of defense industries.

9.1 CONCEPTUAL STATEMENT OF THE PROBLEM

The very first stages in the conversion of defense industries have raised numerous practical problems that require scientific analysis: choice of the conversion sphere and conversion priorities (production of consumer goods or production of capital goods for the consumer-goods industry, new science-based technologies, infrastructure development, etc.); forecasting the consequences of the economic conversion policy; restructuring versus shutdown decisions and determination of conversion scales for industries and specific enterprises; specific product decisions and choice of production technologies; determination of optimal restructuring schemes; determination of efficient rates of conversion, et. (see, e.g., [138]). Existing experience with centralized and multi-level management of conversion processes suggests that it is inadvisable to attempt a rigid control of these processes from the top and that the preferred approach is to focus primarily on indirect economic controls (tax credits, loans, etc.). This conclusion does not detract from the importance of analyzing and modeling efficient conversion strategies.

Mathematical modeling of the conversion of defense enterprises and industries is a complex scientific problem, and its analysis is only beginning. Because of the complexity of this problem, natural approach to its solving is to structurize

the control problems, identify different aspects and facets of control, and develop particular models for their description. On the whole, the mathematical modeling of conversion lags behind the economic analysis of conversion, which is an obstacle to further development of decision support systems for conversion management.

Here we will consider the issue of technical refitting rates of industrial systems undergoing conversion. We assume that some of the basic conversion problems listed above have already been solved; in particular, we assume that the new product and the corresponding production capacities (equipment) have been chosen. Given these decisions, the main conversion problem is the technological restructuring of the industrial systems. An existing industrial structure may change at some rate to a new structure, which consists of elements with known characteristics subject to the same resource constraints (on labor, assets, etc.). The efficiency criterion for conversion in this setting is the ability to maintain maximal profitability under abrupt changes in consumer prices of the manufactured goods. These considerations are the basis of the aggregated integral conversion model proposed below.

The integral economic models allow for the diversity of the technological structure of production in the presence of TC and are intended for control and optimization of restructuring by shutdown, replacement, and renovation of production capacities and equipment. The models have been developed in application to systems of various scales and uses, ranging from the macro economy to separate industrial associations. They have identified a number of important properties of optimal trajectories of long-term growth and renewal of economic systems (in particular, optimal life of the system elements) in the presence of technical change (see previous chapters). We accordingly use this class of integral models to examine optimal conversion strategies for industrial systems. The results are valuable not so much for on-line conversion control of a specific industrial system (this requires different models and approaches), but mainly for identifying the main qualitative properties of optimal conversion strategies in the presence of TC over sufficiently long time periods (long-term forecasting) and applying them to strategic conversion decisions.

This problem is of considerable interest for regional and national control levels. Because of the diversity of products and economic factors in production, the most essential features of the process can be identified by abstracting from particular products and using aggregated criteria for the broad categories of defense and civilian goods. The production capacities (plants, equipment) are assumed homogeneous by type of product, but because of TC they have different productivities and are launched at different points in time.

9.2 INTEGRAL MODEL OF INDUSTRIAL RESTRUCTURING DURING CONVERSION

We consider an industrial system that manufactures two types of products (defense products and civilian products) in two types of plants using a common labor resource. We assume that the industrial system is subject to materialized technical change, so that new plants are more efficient than old. The plants in the system are renewed (new plants are started and old plants are shut down), and the plant life is determined by solving the appropriate optimization problem.

To construct an aggregated integral model of industrial system restructuring during conversion, we introduce the following variables:

- $c_1(t)$, $c_2(t)$ is the output (in unit time) of defense and civilian goods, respectively;
- $m_(t)$, $m_2(t)$ is the number of plants that become operational in year t producing the goods c_1 and c_2, respectively;
- $\beta_1(\tau, t)$ is the productivity at time t (the output c_1 in unit time) of one plant of type m_1 launched in year τ;
- $\beta_2(\tau, t)$ is the productivity of one plant of type m_2 (defined similarly);
- $a_1(t)$ is the time horizon for shutdown of obsolete plants of type m_1 ($t - a_1(t)$ is the service life of the plant);
- $a_2(t)$ is the shutdown horizon for plants of type m_2;
- $g_1(t)$, $g_2(t)$ is number of obsolete plants of the first and second type that are shut down per unit time at time t;
- $\lambda_1(t)$, $\lambda_2(t)$ is the cost (price) of one new plant of first and second type, respectively;
- $\mu(t)$ is the shutdown cost of one plant (assumed equal for plants of both types);
- $P_1(t)$, $P_2(t)$ is the number of employed in defense and civilian production, respectively;
- $P(t)$ is the total number of employed in the industrial system.

Then the aggregate two-product integral model of industrial system restructuring can be described by the following relationships:

Output of defense goods:

$$c_1(t) = \int_{a_1}^{t} (t)\beta_1(\tau, t)m_1(\tau)d\tau, \tag{9.1}$$

Output of civilian goods:

$$c_2(t) = \int_{a_2(t)}^{t} \beta_2(\tau, t)m_2(\tau)d\tau, \tag{9.2}$$

Labor force in defense production:

$$P_1(t) = \int_{a_1(t)}^{t} m_1(\tau)d\tau, \tag{9.3}$$

Labor force in civilian production:

$$P_2(t) = \int_{a_1(t)}^{t} m_2(\tau)d\tau, \tag{9.4}$$

Total labor force:

$$P(t) = P_1(t) + P_2(t), \tag{9.5}$$

Balance of new plant launches and old plant shutdowns:

$$m_i(t) - g_i(t) = P_i'(t), \quad i = 1, 2, \tag{9.6}$$

Refitting costs in defense industry:

$$S_1(t) = \mu(t)g_1(t), \quad S_2(t) = \lambda_1(t)m_1(t), \tag{9.7}$$

Refitting costs in civilian industry:

$$S_3(t) = \mu(t)g_2(t), \quad S_4(t) = \lambda_2(t)m_2(t). \tag{9.8}$$

From the viewpoint of more general interpretation of the IDMs (see Section 1.6 and Chapter 8) we consider the evolving system (ES) which performs two different external functions (c_1 and c_2) and uses correspondingly two types of system elements with efficiencies $\beta_1(\tau, t)$, $\beta_2(\tau, t)$ and costs $\lambda_1(t)$, $\lambda_2(t)$ and certain common resource. We will suppose that the resource and the total efficiency of function c_1 execution are given, whereas the total efficiency of function c_2 execution is maximized.

9.3 THE STATEMENT OF OPTIMIZATION PROBLEM

We first define the given (exogenous) variables of the model(9.1)-(9.6). The main exogenous resource is the labor force $P(t)$, $t \in [t_0, T)$, $T \leq \infty$. Analysis shows that the main properties of the problem do not change if we assume that other resources (raw materials, energy, etc.) are also given; then the control dynamics is determined by the scarcest of these exogenous resources.

The defense production target $c_1(t)$ is assumed given for the time period $[t_0, T]$. The output $c_1(t)$ is naturally a decreasing function: $c_1'(t) < 0$, $t \in [t_0, T]$.

The system is controlled by altering the technological renewal rate through adjustment of the plant life $t - a_i(t)$, the number of new plants $m_i(t)$, and the number of shut-down plants $g_i(t)$, $i = 1, 2$. We assume that both new plant launches and old plant shutdowns involve certain costs.

A natural objective of an economic system under market conditions is output maximization and cost minimization over a certain planning horizon $[t_0, T)$ given the exogenous resources. For the model (9.1)-(9.8) these criteria can be described by the following mathematical relationships:

$$I_1 = \int_{t_0}^{T} e^{-st} c_2(t) dt \longrightarrow \max_{a_2, m_2} \tag{9.9}$$

$$I_2 = \int_{t_0}^{T} e^{-st} \sum_{i=1}^{4} S_i(t) dt \longrightarrow \min_{m_i, q_i,\ i=1,2} \tag{9.10}$$

The constant $s > 0$ in (9.9), (9.10) is the discount factor, which determines the decrease of value of manufactured goods (and costs) at future points in time. A specific feature of industrial conversion is that civilian goods c_2 have priority over defense production. This is reflected in the objective functional (9.9).

Thus, the problem of determining the best conversion strategy is reduced to the following:

- - to find functions that determine the number of defense plant launches $m_2(t)$ and the corresponding shutdown horizon $a_2(t)$ so as to maximize the criterion (9.9);

- to find functions $m_1(t)$, $m_2(t)$ that determine the number of defense and civilian plant launches and functions $g_1(t)$, $g_2(t)$ that describe the number of obsolete plants in both industries so as to minimize the criterion (9.10).

This obviously requires solving a multi-criteria optimization problem.

Noting that by (9.3), (9.4), and (9.6), the endogenous variables a_i and g_i are related by $g_i(t) = m_i(a_i(t))a_i'(t)$, $i = 1, 2$, we can reduce the number of sought-for functions in the problem. We introduce the total refitting cost for one plant of the first and second type:

$$\gamma_i(t) = \lambda_i(t) + \mu(t), \quad i = 1, 2. \tag{9.11}$$

Eliminating the unknowns c_2, g_1 and g_2 , we are left with four unknown functions: m_1, m_2, a_1, a_2, which are related by the equality constraints (9.1) and (9.3)-(9.5). The functionals (9.9) and (9.10) finally take the form

$$I_1 = \int_{t_0}^{T} e^{-st} \int_{a_2(t)}^{t} \beta_2(\tau, t) m_2(\tau) d\tau dt \longrightarrow \max_{a_2, m_2}, \tag{9.12}$$

$$I_2 = -\int_{t_0}^{T} e^{-st} [\gamma_1(t) m_1(t) + \gamma_2(t) m_2(t)] m_2(\tau) d\tau dt \longrightarrow \max_{m_1, m_2}. \tag{9.13}$$

The best solutions of a multi-criteria problem are so-called *efficient solutions* obtained by linear scalarization of the criteria. We accordingly reduce the two-criterion problem (9.12)-(9.13) to a one-criterion optimization problem

$$I = \int_{t_0}^{T} e^{-st} \Big(\chi \int_{a_2(t)}^{t} \beta_2(\tau, y) m_2(\tau) d\tau - (1 - \chi) \Big(\gamma_1(t) m_1(t) + \\ + \gamma_2(t) m_2(t) \Big) \Big) dt \longrightarrow \max_{m_i, a_i, i=1,2}. \tag{9.14}$$

The entire set of efficient solutions is obtained by varying the parameter χ, $0 < \chi < 1$. The efficient set consists of the functions m_1, m_2 that determine the number of plant launches and the functions a_1, a_2 that describe the shutdown horizon for obsolete plants, by product type.

9.4 THE ANALYSIS OF OPTIMIZATION PROBLEM

For each value of the parameter χ, we have the following optimization problem: given the functions $P(t)$, $c_1(t)$, $c_1'(t) < 0$, $\beta_i(\tau, t)$, $\gamma_i(t)$, find the functions $m_i(t)$, $a_i(t)$, $t \in [t_0, T)$, $T \leq \infty$, $i = 1, 2$, that maximize the functional

$$I = \int_{t_0}^{T} e^{-st}\Big(\chi \int_{a_2(t)}^{t} \beta_2(\tau, t) m_2(\tau) d\tau -$$
$$-(1-\chi)\Big(\gamma_1(t) m_1(t) + \gamma_2(t) m_2(t)\Big)\Big) dt \tag{9.15}$$

subject to the constraints:

$$c_1(t) = \int_{a_1(t)}^{t} \beta_1(\tau, t) m_1(\tau) d\tau, \tag{9.16}$$

$$P(t) = \int_{a_1(t)}^{t} m_1(\tau) d\tau + \int_{a_2(t)}^{t} m_2(\tau) d\tau, \tag{9.17}$$

$$0 \leq m_i(t) \leq M_i(t), \tag{9.18}$$

$$a_i(t) < t, \tag{9.19}$$

$$a_i'(t) \geq 0, \quad i = 1, 2, \tag{9.20}$$

and the initial conditions

$$a_i(t_0) = a_i^0, \quad m_i(\tau) = m_i^0(\tau), \quad \tau \in [a_i^0, t_0], \quad i = 1, 2. \tag{9.21}$$

Assume that the functions β_i, γ_i, μ are given and continuous, P and c_1 satisfy the Lipschitz condition, and m_i^0 are piecewise-continuous. From physical considerations, all the given functions are continuous and satisfy (9.17)-(9.21) at $t = t_0$.

If $T = \infty$, the functional (9.15) is finite ($I < \infty$) under the following restrictions on the given functions:

$$\int_{t_0}^{\infty} e^{-st} \beta_2(t, t) P(t) dt < \infty, \quad \int_{t_0}^{T} e^{-st} [\lambda_i(t) + \mu(t)] dt < \infty, \quad i = 1, 2.$$

These conditions can be proved in the same way as Theorem 4.4 for OP1. We assume that they are satisfied in the case $T = \infty$.

The sought-for efficient solutions $m_1(t)$ and $m_2(t)$ *(the independent controls)* are assumed measurable and satisfy (9.18) almost everywhere (a.e.) on $[t_0, T)$. The unknowns $a_1(t)$, $a_2(t)$ *(the phase variables)* are determined from (9.16), (9.17).

The problem (9.15)-(9.21) is an optimal control problem with phase constraints (9.19)-(9.20). Following the procedure for the analysis of optimization problems with these integral models (see Chapters 4–6 and 8), we replace this problem with a problem whose m_1, m_2 feasible region is smaller than (9.18): in this smaller region, the constraints (9.19),(9.20) on a_1, a_2 are a priori satisfied.

Theorem 9.1. If $\partial\beta_1/\partial t \leq 0$, then for all measurable functions $m_i(t)$, $i = 1, 2$, that a.e. on $[t_0, T)$ satisfy the constraints

$$0 \leq m_1(t) \leq M_1(t), \tag{9.22}$$

$$m_{min}(t) \leq m_2(t) \leq M_2(t), \tag{9.23}$$

where

$$m_{min}(t) = \max\{0,\ P'(t) + \Big(\beta_1(t,t)/\beta_1(a_1^0,t) - 1\Big)m_1(t) - \\ -c_1(t)/\beta_1(a_1^0,t)\}, \tag{9.24}$$

there exist unique functions $a_1(t)$ and $a_2(t)$ continuous a.e. on $[t_0, T)$ such that $a_i(t) < t$, and the derivatives $a_i'(t) \geq 0$ exist, $i = 1, 2$.

The proof is similar to that of Theorem 4.1.

Remark. The feasible region (9.22) for m_1 is unchanged because of the condition $c_1'(t) < 0$ (the output of defense goods does not increase during conversion).

Because of the unique solvability of the state equations (9.16) and (9.17) for a_1 and a_2, we can introduce the gradient of the functional (9.15) in the OP (9.15)-(9.23).

Theorem 9.2. The gradient of the functional (9.15) subject to equality constraints (9.16), (9.17) has the form

$$I'_{m_1}(t) = \int_t^{\underline{a}_1^{-1}(t)} e^{-st}\Big(\beta_1(t,\tau)/\beta_1(a_1(\tau),\tau) - 1\Big)\beta_2(a_2(\tau),\tau)d\tau - \\ -e^{-st}(1-\chi)\gamma_1(t), \tag{9.25}$$

$$I'_{m_2}(t) = \int_t^{\underline{a}_2^{-1}(t)} e^{-st}[\beta_2(t,\tau) - \beta_2(a_2(a_2(\tau),\tau)]d\tau - e^{-st}(1-\chi)\gamma_2(t). \tag{9.26}$$

The proof is by the Lagrange multiplier method which has been developed for these integral models in [146].

Let us proceed with a qualitative analysis of the solutions of the optimization problem (9.15)-(9.23).

Theorem 9.3. If the functions $\beta_i(\tau, t)$ are non-increasing in τ, then the OP solution is $m_1^* \equiv 0$, $m_2^* \equiv m_{min}$. For $T < \infty$, there always exists a time θ, $t_0 \leq \theta < T$ such that the solution is $m_1^*(t) \equiv 0$, $m_2^*(t) \equiv m_{min}(t)$ for $t \in [\theta, T]$. For sufficiently large $T - t_0$ and $\beta_i(\tau, t)$ strictly increasing in τ, the solutions are $m_1^*(t) \not\equiv 0$ and $m_2^*(t) \not\equiv m_{min}(t)$ on interval $[t_0, \theta]$, $\theta > t_0$ ($\theta = \infty$ when $T = \infty$), i.e. $m_1^*(t) > 0$ and/or $m_2^*(t) > m_{min}(t)$ for $t \in \Delta \subset [t_0, \theta)$, $mes(\Delta) > 0$.

The proof is similar to that of the corresponding results for OPs for IDMs considered in Chapters 4–6, 8 (see Theorems 4.3, 4.6, and other).

The structure of the expressions (9.25),(9.26) for the gradients shows that in the presence of technical chance (TC) the OP (9.1)-(9.6), (9.12)-(9.14) can have turnpike properties. For optimization on an infinite open interval $[t_0, \infty)$ (the case $T = \infty$) we have the turnpike theorem in the strongest form.

Theorem 9.4 *(on turnpike properties of OP solution).* Let $T = \infty$, $\beta_i(\tau, t)$ are strictly increasing in τ, and there exist the functions $\tilde{a}_1(t)$, $\tilde{a}_2(t)$, $t \in [t_0, \infty)$ such that $I'_{m1}(t) = I'_{m_2}(t) \equiv 0$, $t \in [t_0, \infty)$ and the functions $\tilde{m}_1$, $\tilde{m}_2$ corresponding to these $\tilde{a}_1$, $\tilde{a}_2$ by (9.16) and (9.17) satisfy the inequalities (9.22), (9.23). Then the solution of the OP (9.16)-(9.23) is $a_i^*(t) = \tilde{a}_i(t)$, $i = 1, 2$ for $t \in [\mu, \infty)$, where $\mu \geq t_0$ is determined only by the values $|a_i^0 - \tilde{a}_i(t_0)|$, $i = 1, 2$.

The proof is based on derivation and analysis of expressions for the second variation of the functional I with respect to m_1 and m_2 (see Chapters 5 and 8).

For $T < \infty$, the OP also has turnpike properties, but they are weaker: for a large $T - t_0$, the OP solutions $a_i^*(t) \to \tilde{a}_i(t)$ at $T - t_0 \to \infty$ on asymptotically larger part of the interval $[t_0, T]$ (if the functions $\tilde{a}_i$) defined in Theorem 9.4 exist).

The turnpike properties of the OP (9.16)-(9.23) make it possible to simplify the analysis of optimal strategies and to reduce it to the analysis of the system

of integro-functional equations (IFEs):

$$I'_{m_1}(\tilde{a}_1, \tilde{a}_2; t) \equiv 0, \tag{9.27}$$

$$I'_{m_2}(\tilde{a}_2; t) \equiv 0, \quad t \in [t_0, \infty) \tag{9.28}$$

for the turnpikes $\tilde{a}_1(t)$, $\tilde{a}_2(t)$, $t \in [t_0, \infty)$. The analysis of this system of IFEs is far from trivial. Equation (9.28) has been investigated in Section 5.1, where the existence of a unique solution has been proved for exponential and power-function forms of $\beta_2 \equiv \beta_2(\tau)$.

To illustrate the applicability of Theorem 9.4, consider the important case when the system of equations (9.27)-(9.28) has a unique solution $\tilde{a}_1$, $\tilde{a}_2$.

Theorem 9.5 *(constant rate of technical change).* Let $\beta_1 \equiv C_1\tau$, $\beta_2 \equiv C_2\tau$, $\gamma_i(t) \equiv \gamma = const$, $i = 1, 2$. Then the system of IFEs (9.27), (9.28) has a unique solution $\tilde{a}_1(t) = \tilde{a}_2(t) = t - A$ on $[t_0, \infty)$, where $A > 0$ is the unique solution of the nonlinear equation

$$A - (1 - e^{-sA})/s = -s\chi\gamma/C_2/(1 - \chi)$$

The proof is similar to that of Theorem 5.5.

Investigation of the properties of solutions of equations (9.27), (9.28) makes it possible to establish important qualitative and quantitative interrelationships between the behavior of the turnpike trajectories $\tilde{a}_1$, $\tilde{a}_2$ and the given model characteristics. An example of such interrelationships is provided by the following theorem.

Theorem 9.6 *(on properties of turnpike trajectories).* Assume that the system of equations (9.27), (9.28) has the solution $\tilde{a}_1(t)$, $\tilde{a}_2(t)$, $t \in [t_0, \infty)$. Then :

1) if β_1/β_2 is independent of τ and $\gamma_1 \equiv \gamma_2$, then $\tilde{a}_1 \equiv \tilde{a}_2$;

2) if β_1/β_2 is independent of τ, then for $\gamma_1 > \gamma_2$ we have $\tilde{a}_1 < \tilde{a}_2$, and conversely;

3) if $\gamma_1 \equiv \gamma_2$, then for $\partial(\beta_1/\beta_2)/\partial\tau > 0$ we have $\tilde{a}_1 > \tilde{a}_2$, and conversely.

The proof is by analyzing the solutions of (9.27), (9.28).

Remark. In the case $\partial(\beta_1/\beta_2)/\partial\tau \equiv 0$, $\gamma_1 \equiv \gamma_2 \equiv \gamma$ the IFEs (9.27)- (9.28) coincide with IFE (5.1) and hence with IFEs (8.28)-(8.29) studied in Section

8.2. In general case IFEs (9.27)-(9.28) are simpler than IFEs (8.28)-(8.29) and can be solved in succession, beginning with (9.27).

Let us analyze the properties of the original two-criterion problem (9.9)-(9.10) as a function of the parameter *chi*. Theorems 9.3-9.6 and analysis of equations (9.27),(9.28) suggest the following result.

Theorem 9.7 *(on qualitative properties of efficient solutions).* In the two-criterion optimization problem (9.9)-(9.10),(9.14) the range $[0,1]$ of the parameter χ can be partitioned into three subintervals $[0,l)$, $[l,L]$ and $(L,1]$, $0 < l < L < 1$ that are characterized by different qualitative behavior of the efficient solutions. For $0 < \chi < l$, the efficient solutions of the problem are trivial ($m_1^* \equiv 0$, $m_2^* \equiv m_{min}$) and the turnpike trajectories $\tilde{a}_1$, $\tilde{a}_2$ do not exist. For $l \leq \chi \leq L < 1$, with large $T - t_0$ and $\beta_i(\tau,t)$ strictly increasing in τ, the OP has turnpike properties: the efficient solutions $a_i^*(t) \to \tilde{a}_i(t)$ at $t, T \to \infty$, $i = 1,2$, $a_i^*(t) = \tilde{a}_i(t)$ at $t_0 \leq \mu < t < \infty$, and as χ increases the solution $m_i^*(t)$ approaches $M_i(t)$ while a_i^* and $\tilde{a}_i$ approach a corresponding $a_i^{max}(t)$, $i = 1,2$. For $L < \chi \leq 1$, the efficient solutions $m_i^* \equiv M_i$, $a_i^* \equiv a_i^{max}$, $i = 1,2$ are determined from (9.16),(9.17) and $a_i^* < \tilde{a}_i$, $i = 1,2$, i.e., a turnpike solution does not exist.

9.5 CONCEPTUAL INTERPRETATION: CONVERSION STRATEGIES

The above results lead to approximate modeling of *optimal conversion strategies in the presence of TC* and enable us to study their properties without solving the optimization problem. Specifically, the interpretation of Theorems 9.1-9.7 produces different conversion strategies for different assumptions on the rates of TC.

Strategy 1: *Conversion with Minimal Plant Renewal Rate.* By Theorem 9.1, the minimum allowed rate $m_{min}(t)$ of new plant launches for civilian production is nonzero (assuming slowly changing labor resources: $P(t) : |P'| << 1$). This means that conversion will take place even in the absence of technical change ($\partial\beta_i/\partial\tau \leq 0$, $i = 1,2$) through redistribution of production means (plants) between civilian and defense industries (as a result of reduction of defense production targets $c_1(t)$, $c_1'(t) < 0$). By Theorem 9.2, the following strategy is optimal in this case: there are no new plant launches in the defense industry while obsolete defense plants are shut down ($m_1 \equiv 0$, $a_1' > 0$); in the civilian

industry, no plants are shut down and new plants are launched in a minimal volume required to maintain employment ($m_2 \equiv m_{min}$, $a_2' \equiv 0$). The dynamics of the controls leads to redistribution of labor and the number of plants subject to the overall resource constant: $P_1(t)$ declines and $P_2(t)$ increases.

This strategy is trivial from the point of view of the integral model and can be obtained without using the IDMs with controllable memory. The main strengths of integral models emerge when we model the growth trajectories of an industrial system with technical change (in this case, the optimal strategy calls for accelerated plant renewal).

Strategy 2 (Turnpike Strategy): *Conversion in the Presence of Technical Progress.* If new plants in the civilian industry are more productive (β_2 increases in τ), then by Theorem 9.3 the optimal $m_2(t) > m_{min}(t)$, $a_2'(t) > 0$ on some part of a large interval $[t_0, T]$, i.e., obsolete civilian plants are shut down and the rate of new plant launches is greater than the minimal rate of strategy 1. Thus, in the presence of technical change ($\partial\beta_2/\partial\tau > 0$), the optimal strategy involves accelerated renewal of civilian plants m_2 manufacturing civilian goods c_2 . This in principle is expected given the results of Chapter 8.

The interpretation of Theorem 9.3, however, also leads to a somewhat unexpected conclusion. Specifically, although the problem does not require maximization of the output c and the defense production targets are decreasing ($c_1 < 0$), the optimal strategy in the presence of technical change ($\partial\beta_2/\partial\tau > 0$), ($d\beta_1/d\tau > 0$) also includes renewal of defense plants m_1.

To analyze this result, we introduce the *rental value of plants* m_1 in units of the civilian product c_2 :

$$\Theta(t) = \int_t^{a_1^{-1}(t)} e^{-s\tau}[\beta_1(t,\tau)/\beta_1(a_1(\tau),\tau) - 1]\beta_2(a_2(\tau),\tau)d\tau.$$

This is the overall increase of the output c_2 that can be achieved by increasing the number of functioning civilian plants m_2 when defense targets $c_2(t)$ are fulfilled by a smaller number of defense plants m_1. If $\Theta(t)$ is greater than the total discounted cost $\chi/(1-\chi)\gamma_1(t)e^{-st}$ of replacing one plant of type m_1 (allowing for the preference parameter χ), then the gradient $I'_{m_1}(t) > 0$ and the optimal strategy involves accelerated plant renewal through new defense plant launches m_1.

In this strategy, by Theorem 9.4, the optimal trajectories $a_1^*(t)$, $i = 1, 2$ have a simple structure: over most of the interval $[t_0, T]$ they track the turnpikes -

the solutions $\tilde{a}_i(t)$ of the IFE system (9.27),(9.28). This leads to a number of important conclusions.

Thus, the optimal plant life $t - a_2^*(t)$ in the civilian industry depends only on the dynamics of the civilian productivity $\beta(\tau, t)$ and the cost $\gamma_2(t)$ of new civilian plants, while the optimal life of defense plants $t - a_1^*(t)$ depends on the productivities $\beta_1(\tau, t)$, $\beta_2(\tau, t)$ and the plant costs $\gamma_1(t)$, $\gamma_2(t)$ in both industries.

If the rates of TC are constant ($\partial\beta_1/\partial\tau \equiv C_1$, $\partial\beta_2/\partial\tau \equiv C_2$) and the plant costs are constant and equal in both industries ($\gamma_1 = \gamma_2 \equiv \gamma = const$), then the optimal plant lives are also constant and equal in both industries: $t - a_1(t) = t - a_2(t) = A = const$. They are determined by the values of (C_2 and γ_2 (Theorem 9.5).

If the TC rates are equal in both industries ($\partial\beta_1/\partial\tau/\beta_1 \equiv \partial\beta_2/\partial\tau/\beta_2$), then the optimal plant life $t - a_i(t)$ is greater in the industry with the higher new plant cost $\gamma_i(t)$. With equal plant costs ($\gamma_1(t) \equiv \gamma_2(t)$), the optimal plant life $t - a_i(t)$ is lower in the industry with the higher relative rate of embodied TC $(\partial\beta_i/\partial\tau)/\beta_i$, $i = 1, 2$ (Theorem 9.6).

Strategy 3: *Maximal Plant Renewal.* This strategy is optimal when the total discounted plant renewal costs $\chi/(1-\chi)(\gamma_1 - \gamma_2)$ are small (negligible) compared with the rates of TC. Specifically, when $(1-\chi)\gamma_1 \ll \chi\partial\beta_1/\partial\tau$ and $(1-\chi)\gamma_2 \ll \chi\partial\beta_2/\partial\tau$, the turnpike plant life is small ($t - \tilde{a}_i(t) \ll 1$) and the corresponding plant renewal rates $\tilde{m}_i(t)$ obtained from the labor balance equations (9.3)-(9.5) tend to leave the feasible control region (9.22),(9.23). Therefore, on an asymptotically larger part of the interval $[t_0, T]$ the optimal plant launches reach the maximum rate, $m_i^*(t) \equiv M_i(t)$, and the plant lives $t - a_i^*(t)$ are correspondingly minimal (in both industries).

Further analysis of the OP (9.1)-(9.10) shows that small changes in its formulation lead to different conversion strategies.

Strategy 4: *Accelerated Conversion.* Suppose that the defense production target is not given in the OP (9.1)-(9.10). Then for any parameter value χ the optimal strategy is the following: there are no new defense plant launches producing c_1 ($m_1 \equiv 0$) and the old defense plants are shut down at the highest allowed rates ($g_1 \equiv G_{max}$, $a_1' \equiv a_1'^{max}$ depending on the specific constraints).

As a result, complete conversion is accomplished during a finite time $t_1 - t_0$: starting with t_1, $a_1(t_1) = t_0$, there is no defense production ($c_1 \equiv 0$) and no labor is employed in the defense industry ($P_1 \equiv 0$). This signifies that

conversion is complete. If $T > t_1$, then on the interval $[t_1, T]$ it remains only to control the civilian production c_2 . Renewal of c_2 production occurs over the entire interval $[t_0, T]$ by strategies 1-3 depending on the rates of TC, but the minimum renewal rate m_{min} differs from (9.24) and is different on the two intervals $[t_0, t_1]$ and $[t_1, T]$ (no further redistribution of labor occurs in the second interval).

Let us analyze the limiting cases of the original two-criterion optimization problem (9.9)-(9.10). By Theorem 9.7, the optimal strategy for $\chi \to 0$ (minimization of renewal costs) is strategy 1 and for $\chi \to 1$ (output maximization) strategy 3. In the general case, we have a realization of the turnpike strategy 2, in which the dynamics and the qualitative properties of the optimal trajectories are determined by the dynamics of the productivity $\beta_i(\tau, t)$ and the new plant cost $\gamma_i(t)$. The turnpike plant lives $t - \tilde{a}_i(t)$ are also determined by the values of the parameter χ: they decrease as χ increases.

In conclusion note that obtained results, while certainly not exhaustive, elucidate the qualitative behavior of optimal conversion strategies in the long term under TC conditions and provide a foundation for their quantitative analysis. Further development of these results is being carried out both in the direction of a broader theoretical apparatus for analysis and optimization of various of conversion and in the direction of development of database and software support for solving applied conversion control problems.

10

OPTIMIZATION IN THREE-SECTOR MODEL WITH ENDOGENOUS TECHNICAL CHANGE

In this Chapter the three-sector IDM with endogenous embodied technical change (TC) is introduced and studied which allows us to optimize simultaneously the TC rate in EUs production and the renovation of EUs. The focus is on the qualitative investigation of the structure of optimal model trajectories.

The studied mechanisms of economic renovation, technological innovation, and R&D investment influence are nonlinear and perform the specific positive feedback in an economics. The results obtained below establish some relationships between optimal strategies in the IDM with endogenous TC and in the one-sector IDM of the economic system renewed from outside.

The IDM of three-sector macroeconomic system undergoing to endogenous embodied TC is constructed in Section 10.1. The statement of optimization problem for this model is formulated in Section 10.2. In Section 10.3 we reduce the three-sector IDM to one-sector model. Section 10.4 is devoted to the analysis of optimal regimes of technological renovation in this IDM. The corresponding optimal distribution of output among production sectors is studied in Section 10.5. Some optimization problem in a modified IDM is considered in Section 10.6. Section 10.7 contains some results concerned with the optimal renovation of macroeconomics in the case of different TC rates in production sectors.

10.1 THE DESCRIPTION OF MODEL

Select the following three sectors of macroeconomics: the production of production means or EUs (sector A), the production of consumption goods (sector

B), and the production of technical progress (sector C). The sector C is understood to be the sector of technological innovation (the industrial R&D), whose function consists in the rise of efficiency of the new EUs produced in the sector A. These new EUs are commissioned for service into all mentioned sectors.

It is supposed that the EUs differ by their efficiency and time of their creation. Embodied technological change is connected with the macroeconomic system, in accordance with which recently created EUs can be more efficient than those created at earlier time. In contrast to the two-sector IDM (Chapter 8), this technological change is endogenous, i.e. the rate of EU efficiency depends on the investment into the sector C (innovation). The macroeconomic system is assumed to be close, i.e. new EUs are produced inside the system only and not entered from outside.

Introduce into consideration the following functions:

- $c(t)$ – the aggregate output of consumption goods at time t per time unit;
- $m(t)$ – the quantity of EUs which are produced in sector A and entered into operation at time t;
- $\eta(t)$ – the rise of productivity of the new EUs produced at time t;
- $\beta(\tau)$ – the EU productivity in sector B, i.e. the amount of consumption goods, which are produced per time unit by one EU created at time τ;
- $\lambda(t)$ – the cost of new SE (in the measure units of product c);
- $\alpha(\tau, t) = \beta(\tau)/\lambda(t)$ – the EUs productivity in sector A: the quantity of new EUs which are produced per time unit at time t by one EU created at time τ;
- $\gamma(t)$ – the expenditure (in c measure units) for the rise of new EU productivity by one unit;
- $\xi(\tau, t) = \beta(\tau)/\gamma(t)$ – the EU productivity in sector C: the rise of productivity of new EUs per time unit, which is ensured by means of one EU in sector C;
- $P(t)$ – labor force (the quantity of all operated EUs);
- $a(t)$ – the time limit of the elimination of obsolete EUs (the EUs created before time $a(t)$ are not operated at time t);

- $y(\tau, t)$ – the relative part of the EUs created at time τ, which are operated in sector A at time t (accumulation norm);
- $z(\tau, t)$ – the relative part of the EUs created at time τ, which are operated in sector C at time t ("innovation norm").

The content of the control function $y(\tau, t)$ have been analyzed in detail in Section 2.5, and the function $z(\tau, t)$ possesses the same features.

As in the one-sector IDM with endogenous TC (Section 6.3), we assume that the EU productivity

$$\beta(\tau) = \beta_0 + \int_{\tau_0}^{\tau} \eta(v)dv, \quad \beta_0 = const > 0, \tag{10.1}$$

i.e. no consideration is given to autonomous TC and physical decay of EUs.

Under imposed assumptions the renovation of the macroeconomic system may be described by the following three-sector IDM [146]:

$$m(t) = \int_{a(t)}^{t} [\beta_0 + \int_{\tau_0}^{\tau} \eta(v)dv] \ \lambda^{-1}(t)y(\tau, t)m(\tau)d\tau, \tag{10.2}$$

$$c(t) = \int_{a(t)}^{t} [\beta_0 + \int_{\tau_0}^{\tau} \eta(v)dv] \ [1 - y(\tau, t) - z(\tau, t)]m(\tau)d\tau \tag{10.3}$$

$$\eta(t) = \int_{a(t)}^{t} [\beta_0 + \int_{\tau_0}^{\tau} \eta(v)dv] \ \gamma^{-1}(t)z(\tau, t)m(\tau)d\tau, \tag{10.4}$$

$$P(t) = \int_{a(t)}^{t} m(\tau)d\tau, \quad t \in [t_0, T). \tag{10.5}$$

The equation (10.4) means that the rise $\eta(t)$ of EU productivity is assumed to be related with gross quantity of the EUs involved into the production of technical progress. Thus, the EU productivity $\beta(t)$ is kept constant if any EU is not involved into the sector C (if "innovation norm" $z(\tau, t) \equiv 0$).

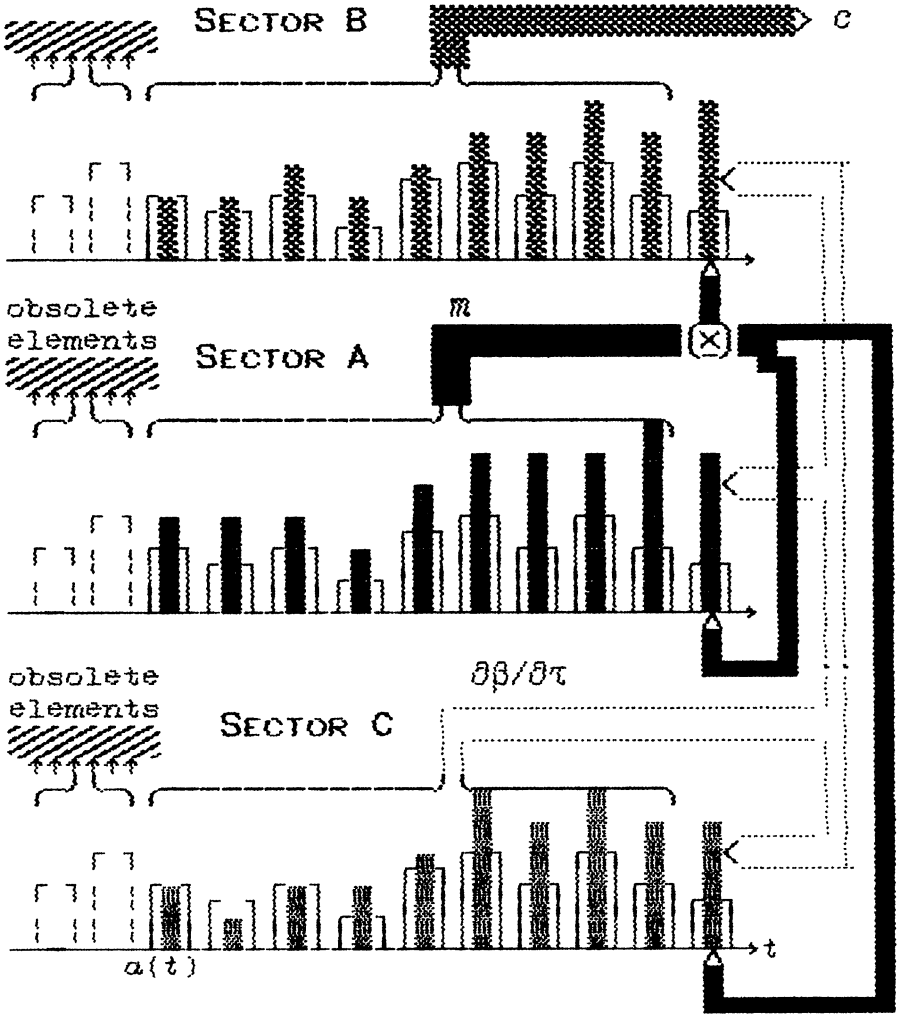

Figure 10.1 The scheme of the three-sector integral model with endogenous embodied technical change

Graphic scheme of the IDM (10.2)-(10.5) is given in the Fig.10.1. The aggregate analog of the IDM (10.2)-(10.5) is a modification (0.4)-(0.6), (0.9) of Solow-Shell model with endogenous TC, which has been exposed in Preface.

From the more general system-theoretic viewpoint, the three-sector IDM (10.2)-(10.5) describes the evolving system (ES) which contains two different self-improving subsystems A and C along with the subsystem B of external ES function execution (see Section 2.5.1).

The macroeconomic model (10.2)-(10.5) describes the case when the relative increase rates of the EUs productivities $\alpha(\tau, t, \eta(.))$, $\beta(\tau, \eta(.))$, $\xi(\tau, t, \eta(.))$ are equal in the production sectors A, B, and C:

$$a'_\tau(\tau, t)/\alpha(\tau, t) \equiv \beta'_\tau(\tau, t)/\beta(\tau, t) \equiv \xi'_\tau(\tau, t)/\xi(\tau, t).$$

This assumption is natural for large macroeconomic systems and is accepted in many macroeconomic models. More general case will be briefly considered in Section 10.7.

10.2 THE STATEMENT OF OPTIMIZATION PROBLEM

Let us consider the basic optimization problem (OP) for the IDM (10.2)-(10.5) that was thoroughly studied in previous chapters (see Sections 4.1, 8.1, 8.4). It consists in the maximization of the discounted output of consumption goods on a planned interval $[t_0, T)$, $T \leq \infty$:

$$I = \int_{t_0}^{T} \rho(t)c(t)dt \to \max \tag{10.6}$$

under the given dynamics of labor resource $P(.)$.

For IDM (10.2)-(10.5) this OP consists in the determination of the sought-for functions $m(t)$, $\eta(t)$, $a(t)$, $y(\tau, t)$, $z(\tau, t)$, $\tau \in [a(t), t]$, $t \in [t_0, T)$, $T \leq \infty$ that maximize the functional (10.6) under the constraints-equalities (10.2)-(10.5), the inequalities:

$$y(\tau, t) \geq 0, \quad z(\tau, t) \geq 0, \quad y(\tau, t) + z(\tau, t) \leq 1, \tag{10.7}$$

$$a'(t) \geq 0, \quad a(t) < t, \quad m(t) \geq 0, \quad \eta(t) \geq 0, \tag{10.8}$$

and the initial conditions:

$$\begin{aligned} a(t_0) = a_0 = \tau_0, \quad m(\tau) = m_0(\tau), \quad \eta(\tau) = \eta_0(\tau), \\ y(\tau, t_0) = y_0(\tau), \quad z(\tau, t_0) = z_0(\tau), \quad \tau \in [a_0, t_0]. \end{aligned} \tag{10.9}$$

The given system characteristics are the specific EU expenditures $\lambda(t)$ and $\gamma(t)$ in sectors A and C, labor force $P(t)$, discounting multiplier $\rho(t)$ $(\rho' < 0)$, the initial EUs distributions $m_0(\tau)$, $y_0(\tau)$, $z_0(\tau)$, the initial dynamics $\eta_0(\tau)$ of TC on prehistory $[a_0, t_0]$, and the initial EU productivity β_0 at instant τ_0.

Let us assume that the given functions λ, γ, P, ρ are Lipschitz continuous, m_0, y_0, z_0, η_0 are piece-wise continuous. All these functions are nonnegative and satisfy (10.2)-(10.5), (10.7)-(10.9) at $t = t_0$.

The OP (10.6)-(10.9) in the three-sector IDM (10.2)-(10.5) ensures the feasibility of simultaneous optimization of the following processes:

(i) the rate $\eta(t)$ of technical progress in EUs production;

(ii) the renovation of EUs by choosing sought-for period $t - a(t)$ of EU lifetime and the quantity $m(t)$ of produced EUs;

(iii) the distribution $y(\tau, t), z(\tau, t)$ of EUs among production sectors A, B and C.

The approach given below allows us to reduce this OP to successive study of separate simpler problems that is fulfilled below in Sections 10.3 and 10.4. Earlier the OP (10.2)-(10.9) was studied in [146] in more complicated way as the OP with six unknowns y, z, m, η, a, c.

10.3 THE REDUCTION OF THREE-SECTOR MODEL TO ONE-SECTOR MODEL

Substituting (10.3) into the optimized functional (10.6) and using the equalities (10.2), (10.5), we find that

$$I = \int_{t_0}^{T} \rho(t)[Q(t) - \lambda(t)m(t) - \gamma(t)\eta(t)]dt \longrightarrow \max_{a,m,\eta} \tag{10.10}$$

where

$$Q(t) = \int_{a(t)}^{t} [\beta_0 + \int_{\tau_0}^{\tau} \eta(v)dv]m(\tau)d\tau. \tag{10.11}$$

The equality (10.11) jointly with (10.5) forms the one-sector IDM (see Sections 4.1 and 6.3). Here the function $Q(t)$ is the total aggregate production output of all macroeconomic system (as in one-sector IDM (4.1)-(4.2)).

Thus, the OP (10.2)-(10.9) in three-sector IDM is reduced to the maximization of the functional (10.10) with the sought-for functions $a(t)$, $m(t)$, $= \eta(t)$, $t \in [t_0, T)$, $T \leq \infty$, under the constraints (10.5), (10.8), (10.11), and the initial conditions (10.9). Let us take note of the fact that the unknown functions $y(\tau, t)$, $z(\tau, t)$ do not enter into this new OP. This simplifies the OP study.

10.4 ANALYSIS OF OPTIMAL TECHNOLOGICAL RENOVATION

The optimization problem (10.10) in one-sector IDM (10.5),(10.11) is analogous to the OP4 studied in Section 6.3. The sole difference between them is the following. It should be noted that the integrand of functional (10.10) must be positive, since the part $\lambda(t)m(t) + \gamma(t)\eta(t)$ of aggregate output spent for DS renovation does not exceed all output $Q(t)$. Hence,

$$\lambda(t)m(t) + \gamma(t)\eta(t) \leq Q(t), \quad t \in [t_0, T). \tag{10.12}$$

This restriction is the reflection of constraints (6.26) in passing from the three-sector IDM to one-sector model. Since the function $Q(.)$ is *the phase sought-for variable* of OP, the inequality (10.12) represents a mixed restriction of the OP.

Thus, the OP under study coincides with the OP4 except for the upper bound of the domain of controls m, η (we have (10.12) instead of (6.26)). The behavior of the solutions of this OP is correspondingly similar but not identical. So, as in OP4 the constraints $m \geq 0$ and $a' \geq 0$ in (10.8) are replaced by one restriction $m \geq m_{min}$, the gradient of functional (10.10) of the OP exists at $m(a(t)) \neq 0$ and is of the form (6.28),(6.29): $I'(t) := [I'_m(t),\ I'_\eta(t)]$,

$$I'_m(t) = \int_t^{\underline{a}^{-1}(t)} \rho(\tau) \int_{a(\tau)}^t \eta(v)dvd\tau - \rho(t)\lambda(t), \tag{10.13}$$

$$I'_\eta(t) = \int_t^T m(\tau) \int_\tau^{\underline{a}^{-1}(\tau)} \rho(v)dvd\tau - \rho(t)\gamma(t), \quad t \in [t_0, T). \tag{10.14}$$

Next, the theorem 6.5 about the existence of trivial solution is kept true for the OP under study. By this theorem, the OP has the *trivial solution*

$$\eta^*(t) \equiv 0, \quad m^*(t) \equiv m_{min}(t), \quad a^*(t) \equiv a_{min}(t), \quad t \in [t_0, T) \tag{10.15}$$

(lack of technological renovation) under the conditions (6.30), (6.31). These conditions mean that the TC dynamics η_0 on prehistory $[a_0, t_0]$ is inadequate to ensure technological renovation. When $T - t_0$ increases, the mentioned becomes more restrictive and the trivial solution (m^*, η^*) of OP4 may cease to exist.

As in OP4, at a large $T - t_0$ the trivial solution (m^*, η^*), even if it exists, is unstable in the sense that the small variations $\delta\eta$, δm of control functions η, m can remove modeled system from trivial regime. This suggests the *existence of second (non-trivial) solution* (m^{**}, η^{**}) of OP4 at a large $T - t_0$.

The analysis of the system of equations

$$I'_m(a, \eta; t) = 0, \quad I'_\eta(a, m; t) = 0, \quad t \in (t_1, t_2) \subset [t_0, T), \tag{10.16}$$

has been carried out in Section 6.3. It shows that the turnpike (internal in the domain (10.8),(10.12)) solutions (m^{**}, η^{**}) are impossible under natural model functions λ, γ. There is a positive feedback in the considered OP, i.e., if $I'_m(t_1) \geq 0$, $I'_\eta(t_1) \geq 0$, then the rise of $m^*(t)$, $\eta^*(t)$ in a neighborhood of point $t = t_1$ causes further rise of functional I.

Thus, the non-trivial OP solution (m^{**}, η^{**}), if it exists, must correspond to the upper constraint (10.12) of the OP. This solution is a somewhat different from the non-trivial solution of OP4. The analysis of the OP gradient (10.13)-(10.14) implies the following *properties of the non-trivial solution* (m^{**}, η^{**}) in the case $T < \infty$:

- there exists an interval $(\theta, T]$, $\theta < T$ such that $\eta^{**}(t) \equiv_0$, $m^{**}(t) \equiv m_{min}(t)$ at $t \in (Q, T]$ *(lack of technological renovation at the end of planned interval)*;
- there exists an instant θ_1, $t_0 < \theta_1 < \theta$ such that $m^{**}(t) > m_{min}(t)$, $\eta^{**}(t) \equiv 0$ at $t \in (\theta_1, \theta]$ *(the regime of the renovation of technologies)*;
- there exists an instant θ_2, $t_0 < \theta_2 < \theta_1$ such that $\eta^{**}(t) \to max$ $(\eta^{**}(t) = Q^{**}(t)/\gamma(t)$ for $P' \leq 0)$, $m^{**}(t) \equiv m_{min}(t)$, at $t \in (\theta_1, \theta]$ *(the regime of the maximal investment into TC)*.

Since there is no internal solution in the domain (10.12), the equality

$$\lambda(t)m^{**}(t) + \gamma(t)\eta^{**}(t) \equiv Q^{**}(t), \quad t \in [t_0, \theta_1) \tag{10.17}$$

holds true for the solution (m^{**}, η^{**}).

It is possible that the OP has two solutions (m^*, eta^*) and (m^{**}, η^{**}) simultaneously, for example, at $T - t_0 \gg 1$, $\rho \equiv const$, $\eta_0 \equiv 0$, $P' \leq 0$ (then $m_{min} \equiv 0$). So, as OP4, this OP can be multi-extremal. In such case the trivial solution (m^*, η^*) is not attractive: its small variations δm, $\delta\eta$ lead to non-trivial solution.

10.5 ANALYSIS OF THE OPTIMAL DISTRIBUTIONS OF ELEMENTS

We now return to the initial OP (10.2)-(10.9) in three-sector IDM with unknown functions y, z, m, η, and a. It still remains to find the corresponding functions $y(\tau, t)$, $z(\tau, t)$ (if they exist) for the optimal solutions (a^*, η^*, m^*) and $(a^{**}, \eta^{**}, m^{**})$ defined above. The functions y, z are determined by the system of two integral equations (10.2),(10.4) of the first kind under given a, m, η.

In general case the behavior of $y(\tau,t)$, $z(\tau,t)$ is complicated, in particular, these functions may be non-unique or not exist. This has been studied in detail for y in Section 8.3 for two-sector IDM. Here we restrict ourselves to some qualitative reasoning.

The trivial OP solution $\eta^*(t) \equiv 0$, $m^*(t) \equiv m_{min}(t)$, $a^*(t) \equiv a_{min}(t)$, $t \in [t_0, T)$ corresponds to some minimal possible $y^*(\tau,t) = y_{min}(\tau,t)$ and $z^*(\tau,t) \equiv 0$. Such distribution of output Q describes the regime of maximal consumption and the lack of technological renovation.

Consider now the optimal distributions y^{**}, z^{**} for the non-trivial OP solution $(m^{**}, \eta^{**}, a^{**})$ at a large $T - t_0$. The equality (10.17), which holds for the solution $(m^{**}, \eta^{**}, a^{**})$, means that $y^{**}(\tau,t)+z^{**}(\tau,t) \equiv 1$, $\tau < t, t \in [t_0, \theta_1)$, i.e. the consumption is minimal possible at an initial part $[t_0, \theta_1)$ of planned interval $[t_0, T]$ which may be sufficiently large. To the contrary, $z^{**}(\tau,t) \equiv 0$, $y^{**}(\tau,t) = y_{min}(\tau,t)$ at the end $[\theta, T]$ of interval $[t_0, T]$. This means that all output of new EUs is utilized for the production of consumption goods.

More precisely, at a large $T - t_0$ three subintervals $[t_0, \theta_2]$, $(\theta_1, \theta]$, and $(\theta, T]$, $t_0 < \theta_2 < \theta_1 < \theta < T$ can be taken within planned interval $[t_0, T]$, which are characterized by different qualitative behavior of the efficient solutions. For $t_0 \leq t \leq \theta_2$, the EUs distribution $z^{**}(\tau,t)$ into sector C is maximal possible, technological renovation is absent $(m^{**}(t) = m_{min}(t))$, and consuption $c(t)$ is minimal. For $\theta_2 < \theta_1 \leq t \leq \theta$, the renovation of technologies is present $(y^{**}(\tau,t) > y_{min}(\tau,t))$, and consumption is minimal. For $\theta < t \leq T$, the output of consumption goods is maximal, and the investments into TC and technological renovation are absent. In so doing, the optimal trajectories have no turnpike properties.

Evidently, such behavior of optimal model trajectories is not natural from the economic point of view, so their further investigation is not pursued.

However, the obtained results indicate how the model should be changed to reach more refined modes of technological renovation under endogenous embodied TC. For this purpose we have to change somewhat the assumption on the innovation dependence of embodied TC.

10.6 OPTIMIZATION IN A MODIFIED MODEL

Here we assume that the activity of the sector C of technological innovation influences the all efficiency $\beta(\tau)$ of the new EUs produced in sector A (rather then only the rise $\beta'(\tau)$ of EU efficiency as in the model (10.1)-(10.5)). In other words, the productivity $\beta(\tau)$ of EUs at whole is assumed to be the output of the sector C, which is reflected in the three-sector IDM (10.1)-(10.4) as follows:

$$m(t) = \int_{a(t)}^{t} [\beta_0 + \eta(\tau)]\lambda^{-1}(t)y(\tau,t)m(\tau)d\tau, \tag{10.18}$$

$$c(t) = \int_{a(t)}^{t} [\beta_0 + \eta(\tau)][1 - y(\tau,t) - z(\tau,t)]m(\tau)d\tau, \tag{10.19}$$

$$\eta(t) = \int_{a(t)}^{t} [\beta_0 + \eta(\tau)]\gamma^{-1}(t)z(\tau,t)m(\tau)d\tau, \tag{10.20}$$

One can investigate the OP (10.6)-(10.9) in the three-sector model (10.5), (10.18)-(10.20) by the scheme used in the study of the above model. The main difference is associated with the new changed component $I'_\eta(t)$ of the gradient $I'(t)$ (10.13)-(10.14) for the corresponding reduced OP in one-sector IDM:

$$I'_\eta(t) = m(t)\int_{t}^{\underline{a}^{-1}(t)} \rho(\tau)d\tau - \rho(t)\gamma(t), \quad t \in [t_0, T), \tag{10.21}$$

whereas the other component $I'_m(t)$ is of the form

$$I'_m(t) = \int_{t}^{\underline{a}^{-1}(t)} \rho(\tau)[\eta(t) - \eta(a(\tau))]d\tau - \rho(t)\lambda(t), \tag{10.22}$$

similar to (10.13).

In this case the corresponding system of equations (10.16) jointly with the model equation (10.5) can have a solution $\tilde{a}$, $\tilde{m}$, $\tilde{\eta}$ which represents a turnpike regime of macroeconomic technological renovation. For example, this is true when $\rho(t) = e^{-st}$, $\lambda(t) = e^{bt}$, $s, b > 0$, $\gamma(t) = const$, $P(t) = const$ (then $\tilde{a}(t) = t - A$, $\tilde{m}(t) = const$, $\tilde{\eta}(t) = b_0 e^{bt}$).

Further analysis shows that the turnpike property of this OP is not "classical" because the solution ($\tilde{a}$, $\tilde{m}$, $\tilde{\eta}$) depends upon the dynamics of initial functions m_0, η_0 from the initial conditions (10.9). Similar feature has already been mentioned for OP5 and OP6 in one-sector IDM (see Section 6.4, 6.5).

A disadvantage of the model (10.5), (10.18)-(10.20) consists in the fact that continuous innovations are needed to keep the attained level of EU productivity $\beta(\tau)$ (otherwise, this productivity would return to initial level β_0). But it is just this condition which leads to the turnpike properties of the lifetime technologies undergoing endogenous TC. If this assumption fails, then, as has been shown above, the investment into the sector C of TC production is the most profit-maker as compared with the investments in other production sectors, and turnpike properties are absent.

The similar results still stand for the case of monotone increasing non-linear relationship between the EU productivity $\beta(\tau)$ and the output $\eta(t)$ of the sector C.

10.7 CONSIDERATION OF DIFFERENT RENOVATION IN PRODUCTION SECTORS

Consider briefly the "putty-clay" variant of three-sector IDMs (10.2)-(10.5) and (10.18)-(10.20), i.e. suppose that EUs cannot pass from one production sector to another (see Section 2.5). Similarly to Section 8.5, the three-sector integral model of endogenous embodied TC with different dynamics of technological renovation of the form:

$$m(t) = \int_{a_1(t)}^{t} \alpha_1[H(\tau), t]y(\tau)m(\tau)d\tau, \tag{10.23}$$

$$c(t) = \int_{a_2(t)}^{t} \alpha_2[H(\tau), t][1 - y(\tau) - z(\tau)]m(\tau)d\tau, \tag{10.24}$$

$$\eta(t) = \int_{a_3(t)}^{t} \alpha_3[H(\tau), t]z(\tau)m(\tau)d\tau, \tag{10.25}$$

$$P(t) = \int_{a_1(t)}^{t} y(\tau)m(\tau)d\tau + \int_{a_3(t)}^{t} z(\tau)m(\tau)d\tau + \\ + \int_{a_2(t)}^{t} [1 - y(\tau) - z(\tau)]m(\tau)d\tau \qquad (10.26)$$

is a natural generalization of three-sector models considered above. Here

$$H(\tau) = \int_{\tau_0}^{\tau} \eta(v)dv$$

(as in the basic IDM (10.2)-(10.5)) or $H(\tau) = \eta(\tau)$ (as in the modified IDM (10.18)-(10.20)). Note that only the case $y \equiv y(\tau)$, $z \equiv z(\tau)$ is meaningful in the model (10.22)-(10.25) (for explanation, see Sections 2.5, 8.1, and 8.5).

As compared with the above IDMs, the model (10.22)-(10.25) includes three independent controls $a_1(.)$, $a_2(.)$ and $a_3(.)$ for the liquidation of obsolete EUs in the sectors A, B and C (instead of one joint control $a(.)$) along with the sought-for $m(.)$, $\eta(.)$, $y(.)$, $z(.)$. The key question is to compare the optimal rates of elements renovation in the different sectors A, B, and C. The study of this model is accordingly much more complex that renders its rigorous theoretical analysis impossible.

Nevertheless, some results can be obtained on the basis of turnpike approach for the case $H \equiv \eta$. Then, as in the Section 10.6, the corresponding system of integral-functional equations

$$I'_{a_1}(t) \equiv I'_{a_2}(t) \equiv I'_{a_3}(t) \equiv I'_y(t) \equiv I'_z(t) \equiv I'_m(t) \equiv 0, \quad t \in [t_0, \infty) \qquad (10.27)$$

jointly with the model (10.22)-(10.25) can have a turnpike solution $\tilde{a}_1(t)$, $\tilde{a}_2(t)$, $\tilde{a}_3(t)$, $\tilde{\eta}(t)$, $\tilde{m}(t)$, $\tilde{y}(t)$, $\tilde{z}(t)$, $t \in [t_0, \infty)$. The following result about qualitative properties of the optimal renovation in sectors A, B, C is proved in much the same way as Theorems 8.2 and 8.3.

If unique turnpike trajectories $\tilde{a}_1(t)$, $\tilde{a}_2(t)$, $\tilde{a}_3(t)$, $t \in [t_0, \infty)$ exist and

$$(\partial\alpha_1/\partial H)/\alpha_1 \equiv (\partial\alpha_2/\partial H)/\alpha_2 \equiv (\partial\alpha_3/\partial H)/\alpha_3,$$

then $\tilde{a}_1 \equiv \tilde{a}_2 \equiv \tilde{a}_3$. If $\partial\alpha_i(H,t)/\partial H > 0$, $\partial[\alpha_i(H,t)/\alpha_k(H,t)]/\partial H < 0$ for a fixed k, $H, t \in [t_0, \infty)$, $i = 1, 2, 3$, then a solution $(\hat{a}_1(t), \hat{a}_2(t), \hat{a}_3(t))$, of the

IFE system (10.26) exists on some finite interval $t \in [t_0, t_1]$, $t_1 > \hat{a}_k^{-1}(t_0)$, such that $\hat{a}_i(t) < \hat{a}_k(t)$ for $i \neq k$, whereas the case $\hat{a}_i(t) > \hat{a}_k(t)$, $t \in [t_0, a_k^{-1}(t_0)]$ is impossible.

Thus, if the relative increase rates $(\partial\alpha_i(H,t,))/\partial H$ of the EU productivities $\alpha_i(H,t,)$ are equal in all production sectors A, B, C, then the turnpike lifetimes of technologies are also equal in these sectors. In general case this result implies that the optimal renovation intensity is higher in such production sector for which the increase rate of the efficiency of new EUs is larger.

1 1

OPTIMIZATION IN MULTI-SECTOR MODELS

A variety of multi-sector integral economic models occurs from the one- and two-sector IDMs studied above. The first multi-sector IDM considered in this Chapter is a generalization of one-sector models from Part I. It takes into account such important features of real manufacturing systems as the need to implement operations at specific calendar periods, the specified standard schedules of the operations, and the presence of basic and auxiliary equipments. The prediction problem with control of replacement of system elements and the problem of expenditures minimization for given schedule of operations are stated and analyzed for this model.

Next multi-sector model represents the direct generalization of two-sector IDMs of Chapter 8. The problem of output maximization is analyzed, and some results for one- and two-sector IDMs are extended for multi-sector case. The last considered OP is the problem of restructuring a multi-branch economic system in minimal time. It is the problem of distinct type as compared with considered above (the OP with non-fixed time and secured right-end).

11.1 MULTI-SECTOR MODEL OF THE REPLACEMENT OF ELEMENTS OF MANUFACTURING SYSTEMS

In many different manufacturing systems there may often be found deteriorated and obsolescent equipment due to insufficient rates and scales of modernization of the elements of these systems. Therefore, improving the mechanisms for

management of these processes is one of the most important requirements for effective functioning of the production process. Integral dynamical models considered in this book represent a promising mathematical tool for the solution of these types of problems. They open up a number of new possibilities in the description of the development of complicated manufacturing and engineering systems.

In this Section *the multi-sector IDMs with controllable memory* are applied for the purpose of simulating the process of replacement and modernization of the elements of a manufacturing system under technical and operational conditions.

In a manufacturing system, we will distinguish the *basic elements* of the system from the *auxiliary elements.* By *a basic element*, we will understand an element of the system capable of implementing certain functions either independently or in conjunction with auxiliary elements. By an *auxiliary element* we will understand an element of the system capable of functioning only in conjunction with a basic element. A system that consists of a single basic element and a set of auxiliary elements associated with it will be termed *an assembly.*

We will consider manufacturing processes whose distinguishing feature is the need to implement operations at specific calendar periods (for example, numerous technological processes in the areas of machine building, agriculture, construction, etc.) according to a definite sequence of implementation of technological operations during the annual production cycle. Therefore, the *annual production cycle* of operations should be subdivided into calendar periods, which we will refer to as *design periods.* By a design period, we will understand a time interval during which none of the operations that have already been performed terminates and during which no new operation commences. Those dates on which at least one operation commences or terminates are the boundaries of the design periods.

Next, the specified standard schedules of the operations are distributed among the design periods. If a certain operation falls into several periods, the standard schedule of this operation is divided in proportion to the length of the periods. The demand for equipment is determined on the basis of peak periods that require the maximal quantity of units of equipment. For this purpose, it is necessary to first solve *a distribution problem* with respect to the units of equipment available at a certain time t_0, by type of operation for each design period. Well-known scheduling algorithms and programs may be used to solve this problem, and as a result, an annual schedule of equipment utilization is obtained, and the maximal demand of each type of element of the manufactur-

ing system determined in definite periods, which we will assume are the ***peak periods.*** After solving the distribution problem, we may move on to the solution of problems in the integral dynamic model that we are proposing for the peak periods found.

11.1.1 Construction of Mathematical Model

Let I be a specified set of operations in the design year; i, the index of the operation, $i \in I$, $|I| = n$; J, the set of types of basic elements; j, the index of types of basic element, $j \in J$, $|J| = N$; L, the set of types of auxiliary elements; α, the index of types of auxiliary elements, $\alpha \in L$; A_{ij}, a vector that determines the composition of the j-th assembly for implementation of the i-th operation, $A_{ij} = (\lambda_j, \lambda_{\alpha_1}, \ldots, \lambda_{\alpha_j})$; λ_j, the number of basic operations of type j, usually $\lambda_j = 1$; and λ_{α_r}, the number of auxiliary operations of type α_r, $r = 1, \ldots, , S$, in assembly A_{ij} carried out by means of basic elements of type j.

Assembly A_{ij} with pair (i, j) is uniquely determined. Different auxiliary elements may be assembled to implement the i-th operation with type j basic elements; that is, $A_{ij} \neq A'_{ij}$, where $A'_{ij} = (\lambda_j, \lambda'_{a_1}, \ldots, \lambda'_{a_3})$. However, we will assume that on the basis of a definite criterion (for example, productivity or efficiency), a single assembly A_{ij} has been chosen from the possible list of such assemblies. Next, we may assume that assembly A_{ij} with pair (i, j) is uniquely determined, and consider in place of assembly A_{ij} j a type j basic element that implements the i-th operation with uniquely determined productivity β_{ij}. It then becomes possible to solve the problem in two steps, first determining the required quantity of basic elements, and then, by multiplying by the corresponding λ_a, obtaining the required quantity of auxiliary elements of type α, $\alpha \in L$.

Let us further suppose that K is the set of design periods of the annual production cycle; k, the period index, $k \in K$; τ, the year in which the basic element has been incorporated into the manufacturing system; $[t_0, T]$, the planning design interval, measured in terms of years; $m_{ij}(\tau)$, the quantity of basic elements of type j that were incorporated into the manufacturing system in year τ and which implement the i-th operation; $\beta_{ij}(\tau, t)$, the productivity of an assembly containing type j basic elements incorporated into the manufacturing system in year τ and which implement the i-th operation in year t; $q_{ij}(\tau)$, number of personnel responsible for the operation of a single assembly that includes basic elements of type j incorporated into the manufacturing system in year τ , and

which implement the i-th operation; $a_j(t)$, time boundary at which a type j basic element is written off (i.e., if the particular basic element was incorporated into the manufacturing system in year τ, then as long as $\tau < a_j(t)$, the basic element is written off, and once $\tau \geq a_j(t)$, it is utilized in the production cycle, $t \in [t_0, T]$); $b_i(t)$, specified standard schedule of i-th operation in a design period of the production cycle in year t, and $P(t)$, available manpower responsible for operation of the manufacturing system in year t.

The equipment demand must be determined on the basis of a condition that guarantees that the standard schedule of each of specified operations will be implemented in each design period of year t, $t \in [t_0, T]$:

$$b_i(t) = \sum_{j=1}^{N_i} \int_{a_j(t)}^{t} \beta_{ij}(\tau, t) m_{ij}(\tau) d\tau, \quad i = \overline{1, n_{\tilde{k}}}, \tag{11.1}$$

where N_i is the number of different types of basic elements utilized in the i-th operation, $n_{\tilde{k}}$ is the number of different operations in the disign period $\tilde{k}$, and $\tilde{k}$ is the fixed period index, $\tilde{k} \in K$.

The demand for manpower that would guarantee implementation of the standard schedules of the operations is determined from the equation

$$P(t) = \sum_{i=1}^{n_{\tilde{k}}} \sum_{j=1}^{N_i} \int_{a_j(t)}^{t} q_{ij}(\tau) m_{ij}(\tau) d\tau. \tag{11.2}$$

The system of integral equations (11.1)-(11.2) constitutes an IDM with controllable memory. It describes the process of replacement and modernization of the elements of a manufacturing system, taking into account technical change. The functions $b_i(t)$, $P(t)$, $\beta_{ij}(\tau, t)$, and $q_{ij}(\tau)$ will always be considered to be specified in advance, $\tau \in [tau_0, T]$, $t \in [t_0, T]$.

The functions $m_{ij}(t)$, $i = \overline{1, n_{\tilde{k}}}$, $j = \overline{1, N_i}$, are always unknown, and the functions $a_j(t)$, $j = \overline{1, N_i}$, may be either specified in advance or unknown, depending on how the problem is stated.

Equations (11.1)-(11.2) depend on the prehistory $[\tau_0, t_0]$; therefore, the functions $m_{ij}(\tau) \equiv m_{ij}^0(\tau)$, $\tau \in [\tau_0, t_0]$ along with the values of $a_j(t_0) = a_j^0 < t_0$, $i = \overline{1, n_{\tilde{k}}}$, $j = \overline{1, N_i}$ have to be specified. The functions $m_{ij}^0(\tau)$ are determined by solving an allocation problem for the time t_0. The values of $a_j(t_0) = a_j^0$ are

found from the real-world dynamics of the process of writing off obsolescent elements of the manufacturing system over the past history $[\tau_0, t_0]$.

Depending on the relationship between the number of unknowns and the number of equations in the model (11.1)-(11.2) , we may solve either one of two classes of problems in the model described above: (a) the problem of predicting the development of a manufacturing system in a definite planning time interval $[t_0, T]$; (b) an optimal control problem in the integral dynamic model. If the number of equations is equal to the number of variables and the problem is completely determined, we have a ***prediction problem.*** If the number of unknowns is greater than the number of equations, we end up with an optimal control problem.

11.1.2 A Prediction Problem in the Model

The proposed model (11.1)-(11.2) contains $\sum_{i=1}^{n_{\tilde{k}}} N_i$ unknowns $m_{ij}(t)$, generally more than one unknown $a_j(t)$, and $n_{\tilde{k}} + 1$ equations. In the case $N_i > 1$, the number of unknowns in the model is greater than the number of equations. Here, we make the assumption that, after solving allocation problem and determination of the peak periods $\tilde{k} = k_j^{max}$ for $j \in J$, the each obtained effective set of utilized equipment for i-th operations, $i = \overline{1, n_{\tilde{k}}}$, consists of a single type j of basic elements, that is,

$$N_i = 1, \quad i = \overline{1, n_{\tilde{k}}}.$$

When $N_i = 1$, the integral model (11.1)-(11.2) assumes the following form:

$$b_i(t) = \int_{a(t)}^{t} \beta_i(\tau, t) m_i(\tau) d\tau, \quad i = \overline{1, n_{\tilde{k}}}, \tag{11.3}$$

$$P(t) = \int_{a(t)}^{t} \sum_{i=1}^{n_{\tilde{k}}} q_i(\tau) m_i(\tau) d\tau \tag{11.4}$$

The prediction problem with control of processes involved in replacement of elements of the manufacturing system, i.e., with unknown function $a(.)$, represents an important real world problem. In this case, the system (11.3)-(11.4) contains $n_{\tilde{k}} + 1$ equations and $n_{\tilde{k}} + 1$ unknowns $m_i(t)$ and $a(t)$, and constitutes a *system of Volterra integral equations of the first kind with unknown lower limit of integration.* Theoretical questions concerned with the existence and

uniqueness of the solution have been investigated for this system in [51], and the following result has been established.

Theorem 11.1. Suppose that the following holds:

1) the functions $\beta_i(\tau,t)$, $b_i(t)$, $q_i(\tau)$, $P(t)$ are continuously differentiable and positive, while the functions $\partial\beta_i(\tau,t)/\partial t$, $b_i'(t)$, $P'(t)$ satisfy a Lipschitz condition with respect to t;

2) $\partial\beta_i(\tau,t)/\partial\tau > 0$, $\partial\beta_i(\tau,t)/\partial t < 0$, $q_i'(\tau) \leq 0$,

3) $P'(t)$ and $b_i'(t)$ are small by comparison to the functions themselves, $i = \overline{1,n_{\tilde{k}}}$;

4) the integral $\int_{-\infty}^{t_0} \sum_{i=1}^{n_{\tilde{k}}} q_i(\tau)m_i^0(\tau)d\tau$ diverges, $\tau \in (-\infty,T]$, $t \in [t_0,T]$.

Then the system of integral equations (11.3)-(11.4) has a unique solution $a(t), m_i(t) \in \mathbf{C}[t_0,T]$, $i = \overline{1,n_{\tilde{k}}}$; moreover, $m_i(t) > 0$ whereas $a(t) < t$.

The proof of the theorem is based on the step-by-step method and the method of contractive maps, and employs a technique developed in Chapter 7. By means of the constructive method of proving theorems, it is possible to construct an effective iterative algorithm for solving problem (11.3)-(11.4) (see Chapter 13).

We wish to solve the problem for each peak period $\tilde{k} = k^{max}$, $j \in J$. The predicted quantity $m_i(t)$ of elements of type i of a manufacturing system that were acquired in year t, $i = \overline{1,n_{k_j^{max}}}$, $t \in [t_0,T]$, represents the result of solving the problem. Summing all $m_i(t)$ of the same type, we obtain the predicted number of elements of each type utilized in year t in period k_j^{max}. Combining together all such sets obtained from the solution of the prediction problem for each period k_j^{max}, $j \in J$ in year t and selecting from this combined set the maximal elements for each type basic element, we obtain the predicted quantity of basic elements of each type that have to be acquired in year t. Once the function $a(t)$ has been determined, we then predict that all the basic elements which were incorporated into the manufacturing system in year τ will be written off provided that $\tau < a(t)$.

11.1.3 Formulation of Optimization Problem in the Model

As we have already remarked, in the proposed model (11.1)-(11.2), there are more unknowns than equations; therefore, a unique solution cannot be found, and in order to close the problem, it is necessary to add an optimality criterion. It then becomes possible to solve the problem in the model (11.1)-(11.2) as an optimal control problem on the planning interval $[t_0, T]$.

Let us take as the optimality criterion the *minimal operating expenditures* over the entire planning interval $[t_0, T]$ *for implementation of the entire specified schedule of operations* in the peak period $\tilde{k} = k_j^{max}$ of each year t, $t \in [t_0, T]$:

$$S = \sum_{i=1}^{n_{\tilde{k}}} \sum_{j=1}^{N_i} \int_{t_0}^{T} \left(\int_{a_j(t)}^{t} \Big(\sigma_{ij}(\tau, t) + q_{ij}(\tau) \Big) m_{ij}(\tau) d\tau + \lambda_j(t) m_{ij}(t) \right) dt, \quad (11.5)$$

where $\lambda_j(t)$ is the balance value of a type j basic element acquired in year t ; $\sigma_{ij}(\tau, t)$, direct operating costs incurred in performance of the i-th operation in year t due to a single assembly with basic elements of type j that had been incorporated into the manufacturing system in year τ (including the cost of storage, repair, modernization, etc.); S, total operating costs incurred in the performance of all operations in the $\tilde{k}$-th period; and k_{jmax}, peak period for basic elements of type j. Now one can formulate the modernization and replacement problem for the elements of a manufacturing system as an optimal control problem.

Since the current labor expenditures $q_{ij}(\tau) m_{ij}(\tau)$ are already included into the total cost S of production, we do not consider the given demand (11.2) for manpower in the statement of the OP. In (11.5) the given functions $q_{ij}(\tau)$ are the specific unit costs of labor (instead of the number of personnel, as in (11.2)) for the implementation of i-th operation by a single assembly with basic element of type j.

We wish to determine unknown functions $m_{ij}(t)$ and $a_j(t)$, $i = \overline{1, n_{\tilde{k}}}$, $i = \overline{1, N_i}$, $t \in [t_0, T)$ that minimize the functional S:

$$S \to \min \quad (11.6)$$

under the following equality-type constraints:

$$b_i(t) = \sum_{j=1}^{N_i} \int_{a_j(t)}^{t} \beta_{ij}(\tau, t) m_{ij}(\tau) d\tau, \quad i = \overline{1, n_{\tilde{k}}}, \tag{11.7}$$

the inequality-type constraints:

$$m_{ij}(t) \geq 0, \quad a_j(t) < t, \quad i = \overline{1, n_{\tilde{k}}}, \quad j = \overline{1, N_i}, \tag{11.8}$$

and the initial conditions:

$$a_j(t_0) = a_i^0 \geq 0, \; m_{ij}(\tau) \equiv m_{ij}^0(\tau), \; i = \overline{1, n_{\tilde{k}}}, \; j = \overline{1, N_i}, \; \tau \in [\tau_0, t_0]. \tag{11.9}$$

In this problem, the functions $\sigma_{ij}(t)$, $\lambda_{ij}(\tau, t)$, $q_{ij}(t)$, $\beta_{ij}(\tau, t)$, and $b_i(\tau)$ are assumed to be always specified in advance. Once $m_{ij}(t)$ has been determined, it will then be possible to find the required number of basic elements of type j,

$$m_j(t) = \sum_{i=1}^{n_{\tilde{k}}} m_{ij}(t), \quad j = \overline{1, N_i}.$$

Problem (11.5)-(11.9) is solved for each peak period $\tilde{k} = k_j^{max}$, $j \in J$. Combining the sets $\{m_j(t)\}$, obtained from the solution of the problem for each peak period k_j^{max}, $j \in J$, and selecting from the union of these sets the maximal values of $m_j(t)$ for each type j basic element, we obtain the required quantity of basic elements of each type that have to be acquired in year t, $t \in [t_0, T]$. Once the functions $a_j(t)$, $j \in J$ have been determined, we may write off in year t all the basic elements of type j that have been incorporated into the manufacturing system in year τ under the conditions $\tau < a_j(t)$. Moreover, we may determine the service life of a basic element $L_j(t)$ in the form of a function $L_j(t) = t - a_j(t)$, $j \in J$.

11.1.4 Analysis of Optimization Problem

As in Section 11.1.2, we restrict ourselves to investigation of the case $N_i = 1$, $i = \overline{1, n_{\tilde{k}}}$, i.e., when the effective set of operated equipment for each i-th operation in the design period $\tilde{k}$ consists of a single type j of basic elements. For simplicity sake denote $n := n_{\tilde{k}}$ in this and next sections.

Then the OP consists in the minimization of the functional:

$$S = \sum_{i=1}^{n} \int_{t_0}^{T} \left(\int_{a(t)}^{t} \Big(\sigma_i(\tau, t) + q_i(\tau) \Big) m_i(\tau) d\tau + \lambda_i(t) m_i(t) \right) dt \longrightarrow \min_{a, m_i} \quad (11.10)$$

by choosing the unknown $a(t)$, $m_i(t)$, $i = \overline{1,n}$, $t \in [t_0, T)$, $T \leq \infty$ under the restrictions:

$$b_i(t) = \int_{a(t)}^{t} \beta_i(\tau, t) m_i(\tau) d\tau, \quad i = \overline{1,n}, \quad (11.11)$$

$$0 \leq m_i(t) \leq M_i, \quad a'(t) > 0, \quad a(t) < t, \quad (11.12)$$

and the initial conditions:

$$a(t_0) = a_0 \geq 0, \quad m_i(\tau) \equiv m_i^0(\tau), \quad i = \overline{1,n}, \quad \tau \in [a_0, t_0]. \quad (11.13)$$

The given functions $\sigma_i(\tau, t)$, $\lambda_i(t)$, $q_i(t)$, $\beta_i(\tau, t)$, $b_i(t)$, and $m_i^0(\tau)$ are assumed to be Lipschitz continuous, positive and satisfy (6.10)-(6.13) at $t = t_0$.

The OP (11.10)-(11.13) has one ***independent control variable*** only, hence, it is convenient to choose $a(.)$ as the independent control and $m_i(.)$, $i = \overline{1,n}$ as ***the phase variables*** of OP. But, in view of the nonholonomic constraint $a'(.) \geq 0$, the addtional independent control $\nu := a'$, $0 \leq \nu(.) \leq 1$, $\nu \in L_\infty[t_0, T)$ is introduced, and the problem is added by the following equality:

$$a(t) = \int_{t_0}^{t} \nu(\tau) d\tau. \quad (11.14)$$

Then we have an OP with standard restrictions for control variable, and the following results are established in a way analogous to that used in Part I. The gradient $S'_\nu(.)$ of the functional (11.10) with respect to the control $\nu(.)$ is derived by means of Lagrange multipliers.

Lemma 11.1. If $\partial\beta_i(\tau, t)/\partial\tau > 0$, $\partial\beta_i(\tau, t)/\partial t \leq 0$, $b_i(t) \geq 0$, $M_i \gg 1$, then for any measurable control $\nu(t) = a'(t)$, $0 \leq \nu(t) \leq 1$ a.e. for $t \in [t_0, T)$ there exist unique measurable functions $m_i(t)$ and absolutely continuous function $a(t)$, $t \in [t_0, T)$ which meets (11.11)-(11.13), and $a(t)$ has $a' \in L_\infty[t_0, T]$, a.e $a'(t) \geq 0$.

Lemma 11.2. The gradient of functional (11.10) in the OP (11.10)-(11.14) is of the following form:

$$S'_\nu(t) = \sum_{i=1}^{n} \int_t^T \Big(\varphi_i(\tau)\beta_i(a(\tau),\tau) - \Big(\sigma_i(a(\tau),\tau) + q_i(a(\tau))\Big)\Big) m_i(a(\tau))d\tau$$

where the sought-for Lagrange multipliers $\varphi_i(t)$, $i = \overline{1,n}$ are determined uniquely by the following dual (adjoint) problem:

$$\int_t^{\underline{a}^{-1}(t)} \Big(\sigma_i(t,\tau) + q_i(t) - \varphi_i(\tau)\beta(t,\tau)\Big)d\tau + \lambda_i(t) = 0, \quad i = \overline{1,n}.$$

As in most considered OPs, *the effect of the end of planned interval* takes place in this OP in the case $T < \infty$:

Theorem 11.2. Let ν^* be a solution of the OP (11.10)-(11.14). Then there exists an instant θ, $t_0 \le \theta < T$, such that $\nu^*(t) \equiv 0$, $a^* \equiv a_0$ for $t \in (\theta, T]$.

The equation $S'_\nu(t) \equiv 0$ produces the following system of $n+1$ IFEs for turnpike trajectories $\tilde{a}(t)$, $\tilde{m}_i(t)$, $t \in [t_0, \infty)$ (if they exist):

$$\int_t^{a^{-1}(t)} \Big(\sigma_i(t,\tau) + q_i(t) - \varphi_i(\tau)\beta(t,\tau)\Big)d\tau = 0, \quad i = \overline{1,n},$$

$$\sum_{i=1}^{n} \int_t^T \Big(\varphi_i(\tau)\beta_i(a(\tau),\tau) - \sigma_i(a(\tau),\tau) - q_i(a(\tau))\Big) m_i(a(\tau))d\tau = 0. \quad (11.15)$$

As always, this IFE system (11.15) is substantially complex, and we consider its partial cases. In the case

$$\begin{gathered} \beta_i(\tau,t) \equiv \beta_i(\tau), \qquad \sigma_i(\tau,t) \equiv \sigma_i(\tau), \\ q_i(t)/\beta_i(t) = q_j(t)/\beta_j(t), \qquad \sigma_i(t)/\beta_i(t) = \sigma_j(t)/\beta_j(t), \\ \lambda_i(t)/\beta_i(t) = \lambda_j(t)/\beta_j(t), \qquad i,j = \overline{1,n} \end{gathered} \quad (11.16)$$

the IFE system (11.15) leads to the following one equation for turnpike trajectory $\tilde{a}$:

$$\int_t^{a^{-1}(t)} \Big(\sigma_i(t)/\beta_i(t) + q_i(t)/\beta_i(t) - \sigma_i(a(\tau))/\beta_i(a(\tau)) - \\ -q_i(a(\tau))/\beta_i(a(\tau))\Big)d\tau + \lambda_i(t)/\beta_i(t) = 0, \quad (11.17)$$

whereas

$$\varphi_i(\tau) = \Big(\sigma_i(a(\tau)) + q_i(a(\tau))\Big)/\beta_i(a(\tau)), \quad i = \overline{1,n}$$

The ***interpretation of the conditions*** (11.16) consists in the fact that for all different types of operations the ratio of equipment productivities is equal to the ratio of current operating costs, to the ratio of labor expenditures, and to the ratio of equipment costs (i.e. ***different operations are identical*** from the viewpoint of embodied TC). In this case we obtain one turnpike equation with respect to a turnpike trajectory $\tilde{a}$, which is analogous to that arised in the OP2 for one-sector IDM and possesses similar properties (see Section 6.1). In particular:

- if $\beta(\tau,t) \equiv \beta(\tau)$, $\sigma(\tau,t) \equiv \sigma$, $\lambda(t) \equiv \lambda$, $q(t) \equiv \sigma$, $\sigma, \lambda, q - const$ and the IFE (11.17) has a solution $\tilde{a}(t)$, $t \in [t_0, \infty)$, then:
  - $t - \tilde{a}(t)$ decreases at increasing $\beta'(\tau)/\beta(\tau)$ and the converse;
  - $t - \tilde{a}(t) \equiv A = const$ at $\beta'(\tau)/\beta(\tau) \equiv const$ (i.e., at $\beta(\tau) = e^{c\tau}$), where the value $A > 0$ is determined by the following nonlinear equation:

$$(\sigma + q)A - (\sigma + q)/c(1 - e^{-cA}) = \lambda$$

$$(A \approx \sqrt{2\lambda/(c(\sigma + q)} \quad \text{at} \quad c \ll 1).$$

When the assumption (11.16) fails, there is no common turnpike trajectory $\tilde{a}$. Relying on the above investigation, optimal lifetimes $t - a_j(t)$ of equipment would be expected to be different for various type operations in the case when the different operations are not identical from the viewpoint of embodied TC. For this reason, in general, it is reasonable to consider the model with different time limits $a_j(t)$, $i = \overline{1,n}$ of scrapping equipment.

11.1.5 Model with Different Lifetimes of Equipment

This OP consists in the determination of $2n$ sought-for functions $a_i(t)$, $m_i(t)$, $i = \overline{1,n}$, $t \in [t_0, T)$, which minimize the functional:

$$S = \sum_{i=1}^{n} \int_{t_0}^{T} \left(\int_{a_i(t)}^{t} \Big(\sigma_i(\tau,t) + q_i(\tau)\Big) m_i(\tau) d\tau + \right.$$

$$+\lambda_i(t)m_i(t)\Big)dt \longrightarrow \min_{a_i,m_i,i=\overline{1,n}} \tag{11.18}$$

under the restrictions:

$$b_i(t) = \int_{a_i(t)}^{t} \beta_i(\tau,t)m_i(\tau)d\tau, \quad i=\overline{1,n}, \tag{11.19}$$

$$0 \le m_i(t) \le M_i, \quad a_i'(t) > 0, \quad a_i(t) < t, \tag{11.20}$$

and the initial conditions:

$$a_i(t_0) = a_i^0 \ge 0, \quad m_i(\tau) \equiv m_i^0(\tau), \quad i=\overline{1,n}, \ \tau \in [a_0, t_0]. \tag{11.21}$$

The OP (11.18)-(11.21) can be considered as the combination of n *OPs for one-sector IDM* of the following form: for fixed i, determine two sought-for functions a_i, and m_i that minimize the functional:

$$S_i = \int_{t_0}^{T}\Big(\int_{a_i(t)}^{t}\Big(\sigma_i(\tau,t)+q_i(\tau)\Big)m_i(\tau)d\tau+\lambda_i(t)m_i(t)\Big)dt \longrightarrow \min_{\{a_i,m_i\}} \tag{11.22}$$

under the restrictions (11.19)-(11.21) for corresponding i. Then

$$S = \sum_{i=1}^{n} S_i, \qquad \text{and} \qquad S \to \min_{a_i,m_i,i=\overline{1,n}} \quad \text{if} \quad S_i \to \min_{a_i,m_i}, \quad i=\overline{1,n}.$$

As for the previous OP, the effect of the end of planned interval takes place in this OP (the analogue of Theorem 11.2 holds true).

Each OP (11.22) at fixed i is similar to OP2 in one-sector IDM (see Section 6.2) and can be investigated in an analogous way. In particular, for each OP (11.22) the IFE for turnpike trajectory $\tilde{a}_i$ is of the form (11.17), hence, optimal trajectories in entire OP (11.18)-(11.21) for multi-sector IDM possess the same features including *turnpike properties* (see Theorems 6.1 and 6.2).

In conclusion it may be said that, unlike existing applied models, the multi-sector model (11.1)-(11.2) takes into account the developmental dynamics of a whole and makes it possible to solve effectively the applied problems that arise in evaluating the rates and periods of modernization of manufacturing systems and that play an important role in the computerization of the process of factory management.

11.2 OUTPUT MAXIMIZATION IN MULTI-SECTOR INTEGRAL MODEL

In two ensuing Sections we will consider the multi-sector IDM that takes into account the distribution of new created EUs among production sectors, as distinct from the previous model. This IDM is the generalization of two-sector IDM and is described in Section 2.5. Optimization problems for this IDM were analyzed in [146] where results for two-sector IDMs were extended for multi-sector case as well as some new results were established. Here we brielfy expose these results.

The first mentioned OP is —em maximization of the joint output of consumption goods under given labor resources in the closed multi-branch economic system which uses (and produces) EUs of n types m_i and produces l various kinds c_k of consumption goods. It consists in the following [146]: deternime the functions $Y(t) := \{y_{ij}\}_{i=\overline{1,n}}^{j=\overline{1,n}}$, $Z(t) := \{z_{kj}\}_{i=\overline{1,n}}^{k=\overline{1,l}}$, $a(t) := \{a_1(t), \ldots, a_n(t)\}$, $m(t) := \{m_1(t), \ldots, m_n(t)\}$, $t \in [t_0, T]$, which maximize the functional

$$I = \int_{t_0}^{T} \sum_{k=1}^{l} c_k(t)dt \longrightarrow \max \tag{11.23}$$

under the following constraints:

$$m_i(t) = \sum_{j=1}^{n} \int_{a_j(t)}^{t} \alpha_{ij}(\tau, t) y_{ij}(\tau, t) m_j(\tau) d\tau, \tag{11.24}$$

$$P_i(t) = \int_{a_i(t)}^{t} m_i(\tau) d\tau, \qquad i = \overline{1,n}, \tag{11.25}$$

$$c_k(t) = \sum_{j=1}^{n} \int_{a_j(t)}^{t} \beta_{ij}(\tau, t) z_{kj}(\tau, t) m_j(\tau) d\tau, \quad k = \overline{1,l}, \tag{11.26}$$

$$a_i(t) < t, \quad a_i'(t) \geq 0, \quad m_i(t) \geq 0, \quad i = \overline{1,n}, \tag{11.27}$$

$$y_{ij}(\tau),\ z_{kj}(\tau) \geq 0, \qquad i = \overline{1,n}, \quad k = \overline{1,l},$$

$$\sum_{i=1}^{n} y_{ij}(\tau) + \sum_{k=1}^{l} z_{kj}(\tau) \equiv 1, \quad j = \overline{1,n}, \tag{11.28}$$

and the initial conditions:

$$a_i(t_0) = a_i^0 \geq 0, \quad m_i(\tau) = m_i^0(\tau), \quad y_{ij}(\tau) = y_{ij}^0(\tau),$$
$$z_{kj}(\tau) = z_{kj}^0(\tau), \quad t \in [a_i^0, t_0], \quad i, j = \overline{1,n}, \quad k = \overline{1,l}. \tag{11.29}$$

The specified functions $\alpha_{ij}(\tau,t)$, $\beta_{kj}(\tau,t)$, $P_i(t)$ are continuous, $y_{ij}(\tau)$, $z_{kj}(\tau)$ are piecewise continuous and satisfy (11.28).

In this problem, the functions y_{ij}, $z_{kj} \in L_\infty[t_0,T]$, $i,j=\overline{1,n}$, $k=\overline{1,l}$ are assumed to be independent control variables, and the phase variables $m_j(.)$, $a_j(.)$, $j=\overline{1,n}$ are defined by the state equations (11.24)-(11.25) of the OP.

As was shown in [146], if the specified functions $\alpha_{ij}(t)$, $P_i(t)$ are absolutely continuous in t, and a.e. $P_i'(t) \le 0$, $i,j=\overline{1,n}$, then for any measurable controls $y_{ij}(t)$, $0 \le y_{ij}(t) \le 1$ a.e. on $[t_0,T]$, $i,j=\overline{1,n}$ the system of equations (11.24)-(11.25) has a unique solution: absolutely continuous $m_i(t)$ and a.e. continuous functions $a_i(t)$, $i=\overline{1,n}$, such that $m_i(t) \ge 0$, $a_i(t) < t$, and a.e $a_i'(t) \ge 0$, $t \in [t_0,T)$.

At $P_i' \le 0$ the nonholonomic constraints (11.27) $a_i'(t) \ge 0$ are always valid in view of the equations (11.25). Such severe conditions for P arise in the OP (11.23)-(11.29) because the replacement of the OP by a problem with more narrow domain of admissible controls (for which the phase constraints $a_i'(t) \ge 0$ hold a priori) is impossible in multi-sector case. Namely, at $P_i'(t) > 0$ the condition $a_i'(t) \le 0$ can fail for some $y_{ij}(t) > 0$ even it holds true for $y_{ij}(t) \equiv 0$, as distinguished from two-sector IDM. For arbitrary P_i, the below results can be established only if the matrix $\{\alpha_{ij}\}$ is diagonal (then a simple combination of n two-sector models is obtained).

The gradient of functional (11.23) in the OP (11.23)-(11.27),(11.29) is of the following form [146]:

$$I' := \{I'_{y_{ij}}, I'_{z_{kj}}\}_{i,j=\overline{1,n}}^{k=\overline{1,l}}, \quad \text{where}$$

$$I'_{y_{ij}}(t) = m_i(t)\int_t^{\underline{a}^{-1}(t)} \alpha_{ij}(t,\tau)\varphi_i(\tau)d\tau, \tag{11.30}$$

$$I'_{z_{kj}}(t) = m_i(t)\int_t^{\underline{a}^{-1}(t)} \beta_{ik}(t,\tau)d\tau, \tag{11.31}$$

and the sought-for Lagrange multipliers $\varphi_i(t)$, $i=\overline{1,n}$ are determined by the dual problem:

$$\varphi_j(t) = \sum_{i=1}^{n}\int_t^{\underline{a}^{-1}(t)} [\alpha_{ij}(t,\tau)y_{ij}(t) - \alpha_{ij}(a_j(\tau),\tau)y_{ij}(a_j(\tau))]\varphi_i(\tau)d\tau +$$

$$+\sum_{k=1}^{l}\int_{t}^{\underline{a}^{-1}(t)}[\beta_{kj}(t,\tau)z_{kj}(t)-\beta_{kj}(a_j(\tau),\tau)z_{kj}(a_j(\tau))]d\tau, \quad j=\overline{1,n}. \tag{11.32}$$

The major findings of the investigation [146] into qualitative behavior of OP solutions are the following:

Effect of the end of planned interval is present in this OP: an instant θ, $t_0 \le \theta < T$ exists such that the optimal controls $y^*_{ij}(t)0$ are minimal possible on $(\theta, T]$: $y^*_{ij}(t) \equiv 0, \quad i,j=\overline{1,n}, \quad$ for $t \in (\theta, T]$.

Minimal possible renovation is not optimal under TC condition : if $T-t_0 \gg 1$, $\partial\beta_{kj}/\partial\tau \ge d = const > 0$, $\partial\beta_{kj}/\partial t \le 0$, $k=\overline{1,l}$, $j=\overline{1,n}$, $t \in [t_0, T]$, then at least one of the controls $y^*_{ij}(t)$ is not minimal on an initial part of $[t_0, T]$: $y^*_{ij}(t) > 0$, $t \in \Delta \subset [t_0, \theta]$, $t_0 < \theta < T$, $mes(\Delta) > 0$ for some $i, j \in \{1, \ldots, n\}$.

Most of optimal EU distributions is boundary: if all functions $\alpha_{ij}(\tau, t)$, $\beta_{kj}(\tau, t)$ differ (from each other), then for each fixed output m_i or c_k the optimal controls $y^*_{ij}(t)$ and $z^*_{kj}(t)$ can be internal in the domain (11.28) simultaneously, at most, in one branch j.

Let us analyze the possibility of turnpike trajectories $\tilde{a}_i$ in the OP. Suppose that for each j one control function $y^*_{\hat{i}_j j}(t)$ is internal in the domain (11.28) at $t \in \Delta \subset [t_0, \theta]$, whereas all other $y^*_{ij}(t)$ are minimal (i.e., $y^*_{ij}(t) = 0$, $i \ne \hat{i}_j$). Then

$$\int_{t}^{\underline{a}_j^{-1}(t)}\alpha_{\hat{i}_j j}(t,\tau)\varphi_{\hat{i}_j j}(\tau)d\tau - \max_k \beta_{kj}(t,\tau)d\tau \equiv 0, \quad t \in \Delta. \tag{11.33}$$

If the increasing rate of embodied TC is the same in branch j:

$$\alpha_{ij}(\tau,t) = \lambda_{ij}(t)\tilde{\beta}_{ij}(\tau,t), \quad i=\overline{1,n},$$
$$\beta_{kj}(\tau,t) = \mu_{kj}(t)\tilde{\beta}_{kj}(\tau,t), \quad k=\overline{1,l},$$

then the equality (11.33) jointly with the dual system (11.31) produces the following IFE for turnpike trajectory $\tilde{a}_j(t)$, $t \in [t_0, \infty)$ [146]:

$$\mu_{\hat{k}_j j}(t)/\lambda_{\hat{i}_j j}(t) = \int_{t}^{\tilde{a}_j^{-1}(t)}[\beta_{\hat{k}_j j}(t,\tau) - \beta_{\hat{k}_j j}(\tilde{a}_j(\tau),\tau)]d\tau,$$

where the index $\hat{k}_j$ is determined from the condition $\mu_{\hat{k}_j j}(t) = \max_{1 \le k \le l} \mu_{kj}(t)$. This equation is analogous to the IFE (5.1) for turnpike $\tilde{a}$ in one- and two-sector IDMs and possesses similar features (see Chapters 5 and 8). But a rigorous proof of turnpike properties in multi-sector IDM meets with other difficulties and still remains to be solved.

11.3 MINIMIZATION OF THE TIME OF RESTRUCTURING MULTI-BRANCH SYSTEM

Within the framework of considered integral models, a simple problem of restructuring a multi-branch economic system in minimal time is stated as follows [146]: determine the sought-for control functions $Y(t) := \{y_{ij}(t)\}_{i=\overline{1,n}}^{j=\overline{1,n}}$, the phase variables $a(t) := \{a_1(t), \ldots, a_n(t)\}$, $m(t) := \{m_1(t), \ldots, m_n(t)\}$, $t \in [t_0, T]$, and the value T, which minimize the functional

$$I = \int_T^{t_0} dt \longrightarrow \max \tag{11.34}$$

under the constraints-equalities (11.24)-(11.25), the following boundary conditions:

$$m(T) = m^T, \tag{11.35}$$

$$m(\tau) = m^0(\tau), \; Y(\tau) = Y^0(\tau), \quad t \in (-\infty, t_0], \tag{11.36}$$

and the closed convex domain of admissble controls:

$$U := \{y_{ij} : \quad y_{ij} \ge 0, \quad \sum_{i=1}^{n} y_{ij} \le 1 \quad i,j = \overline{1,n}\} \tag{11.37}$$

The OP (11.34)-(11.37) belongs to distinct class of optimization problems as compared with considered above (it is the OP with non-fixed time and secured right-end (11.35)). The OP is investigated without regard for the constraints $a_i'(t) \ge 0$ due to the assumption (11.36) that the "prehistory" is specified ad infinitum on the interval $(-\infty, t_0]$. But other natural constraints $m_i(t) \ge 0$, $a_i(t) < t$, $i = \overline{1,n}$, must be satisfied.

Let $\tilde{T} > t_0$ be some large number such that $T \le \tilde{T}$ a priori. If the specified functions $\alpha_{ij}(t)$, $P_i(t)$ are absolutely continuous in t, $i,j = \overline{1,n}$, and the

integrals $\int_{-\infty}^{t_0} m_i^0(\tau)d\tau$ diverge, $i=\overline{1,n}$, then for any measurable controls $Y(.)$, $Y(t)\in U$ a.e. on $[t_0,\tilde{T}]$, the system of equations (11.24)-(11.25) has a unique solution: absolutely continuous $m_i(t)$ and a.e. continuous functions $a_i(t)$, $t\in[t_0,\tilde{T}]$, $i=\overline{1,n}$, such that $m_i(t)\geq 0$, $a_i(t)<t$ [146].

In the case where $a_i'(t)\geq 0$ $i=\overline{1,n}$, the gradient of functional (11.34) in the OP (11.34)-(11.37), (11.24)-(11.25) is of the following form [146]:

$$I'(t):=\{I'_{y_{ij}},\}_{j=\overline{1,n}}^{i=\overline{1,n}}, \qquad \text{where}$$

$$I'_{y_{ij}}(t)=\begin{cases} -\int_t^{\underline{a}_j^{-1}(t)}\psi_i(\tau)\alpha_{ij}(t,\tau)d\tau & t\in[t_0,\tilde{a}_j(T)),\\ \chi_i\alpha_{ij}(t,T)-\int_t^{\underline{a}_j^{-1}(t)}\psi_i(\tau)\alpha_{ij}(t,\tau)d\tau & t\in[\tilde{a}_j(T),T],\end{cases}$$

where the sought-for Lagrange multipliers $\psi_i(t)$, $t\in[t_0,\tilde{T}]$ and χ_i, $i=\overline{1,n}$ satisfy the following equalities:

$$\psi_j(t)-\int_t^{\underline{a}_j^{-1}(t)}\sum_{i=1}^{n}[\alpha_{ij}(t,\tau)y_{ij}(t)-\alpha_{ij}(a_j(\tau),\tau)y_{ij}(a_j(\tau))]\psi_i(\tau)d\tau=$$
$$=\sum_{i=1}^{n}\chi_i[\alpha_{ij}(t,T)y_{kj}(T)-\alpha_{ij}(a_j(T),T)y_{ij}(a_j(T))],\quad j=\overline{1,n}, \tag{11.38}$$

$$\sum_{j=1}^{n}\chi_j m_j(T)=-1. \tag{11.39}$$

Compared to gradients of previous OPs, the Lagrange multipliers χ_i of another type and new equality (11.39) arise when the right-end boundary condition (11.35) is accounted for. In the general case (when $a_i'\not\geq 0$) the gradient of this OP was derived in [146].

From the results obtained in [146] the following conclusions may be reached:

1. There is one or more most-loaded branches $i=k_1,\ldots,k_l$, $1\leq l\leq n$ with the deviations $m_{k_i}^T-m_{k_i}^0(t_0)$ which determine the optimal value of T^*. The optimal controls $y_{k_ij}(t)$ $i=\overline{1,l}$, $j=\overline{1,n}$, connected with these branches, are bounding in the domain U at $t\to T^*$, in so doing, for each branch k_i $y_{k_ij}(t)\equiv 0$ or $y_{k_ij}(t)\equiv 1$ exept for one j at most. At t near T^* all other distributions $y_{ij}(t)$ $i=k_1,\ldots,k_l$ have no effect on the value T^*.

2. If only one branch k is most loaded, then at $t \to T^*$ $y_{kj}(t) \equiv 1$, $j = \overline{1,n}$, for $m_k^T > m_k^0(t_0)$, and $y_{kj}(t) \equiv 0$, $j = \overline{1,n}$, for $m_k^T < m_k^0(t_0)$.

3. An instant θ, $t_0 \leq \theta < T$ exists such that the optimal controls $y_{ij}^*(t) \equiv 0$ $i,j = \overline{1,n}$ are bounding in the domain U for $t \in (\theta, T]$ *(effect of the end of planned interval).*

4. Under the TC presence (i.e., at $\partial\beta_{ij}/\partial\tau > 0$) all distributions $y_{ij}(t)$ $i = k_1, \ldots, k_l$ are equal to 0 or 1 a.e. on $[t_0, T^*]$ with the exeption of no more than one function y_{ij} for each branch j.

The investigated problems ensure us to optimize the strategies of technological renovation simultaneously with the distribution of new EUs between production sectors (subsystems). The obtained results allow us to find some basic tendencies of technological renovation in multi-sector economic systems :

- As in low-sector IDMs (see Chapter 4-6), the optimal EU lifetimes $L_i^*(t)$ are first determined by the rates $\partial\beta_i/\partial\tau$ of embodied TC. In particular, for expenditure minimization $L^*(t)$ increases if the TC relative rate $[\partial\beta_i(\tau,t)/\partial\tau]/\beta_i(\tau,t)$ decreases in τ at constant EU cost $\lambda_i(t)$ and constant EU specific expenditure $\sigma_i(\tau,t)$ (and the converse).

- If TC relative rates $[\partial\beta_i/\partial\tau]/\beta_i$ and EU costs λ_i are the same in two interacting sectors k and j, then the optimal EU lifetimes are also equal: $L_k^*(t) \equiv L_j^*(t)$. If these rates are different, then SE lifetime is less in that ES subsystem for which the rate $[\partial\beta_i(\tau,t)/\partial\tau]/\beta_i(\tau,t)$ is larger.

- Under the TC condition the optimal EU lifetimes $L_i^*(t)$ can possess turnpike properties, whereas the optimal distributions of new EUs between production sectors (controls of the type $y_{ij}^*(t)$) are, in general, of switching character and have no turnpike properties.

- The optimal dynamics of an economic system undegoing TC is non-trivial in the cases when optimal controls do not coincide with minimal possible ones on entire interval $[t_0, T]$; otherwise the TC has little or no effect on the system because of unduly rigid external restrictions.

1 2

OPTIMIZATION OF TECHNOLOGICAL RENOVATION IN HIERARHICAL ECOLOGICAL-ECONOMIC SYSTEM

There is a multitude of various approaches which analyze and investigate mathematical models of hierarchical structures for controlling the development of *ecological-economic systems (EESs)* (see, e.g., [16], [103]). We will consider the optimization of hierarchical control of the development of ecological-economic systems that takes into account technological change in manufacturing (control of manufacturing modernization).

12.1 DESCRIPTION OF THE OBJECT AND STATEMENT OF THE TWO-LEVEL PROBLEM OF OPTIMAL CONTROL

Two general contradictory criteria are inherent in EES control problems: intensification of the EES resources, on the one hand, and conservation and enhancement of the environment, on the other. As a rule, these criteria will have different priorities at different control levels. Therefore, in solving EES control problems, we cannot avoid analyzing a multi-level *hierarchical system* for EES control. The problem of optimal control of two-level, "Center"-"Plants", ecological-economic system on the planning interval $[t_0, T]$ is examined in this chapter.

The problem of hierarchical control of an EES puts forward a broad spectrum of important practical problems that require scientific substantiation. These include: determining the most dangerous media (water, air, soil) and types of

pollutants, and modeling their propagation; modeling the joint effect of "Plant" emissions on pollutant concentration in the ecologically significant zones of a region while taking into account their transport, diffusion, precipitation, and other factors; determining the limiting permissible concentrations of pollutants in a region and the limiting permissible emissions for each "Plant" in a region; forecasting the consequences of an economic policy in a region; substantiating the expediency of modernizing the manufacturing process compared to its elimination and determining the directions in which specific "Plants" have to be reconstructed; selecting the particular types of products and the corresponding manufacturing technologies, and determining the optimal plans and rational timetables for reconstructing the manufacturing process.

Mathematical modeling of the problem of EES control is a complex scientific problem the multifaceted nature of which justifies the various approaches to its solution, from highly clustered macromodels that describe the fundamental strategies of the control processes all the way to detailed models of automated industrial process control systems intended for real-time control of manufacturing. Since the problem is so complex, structuring the control problems, identifying the individual facets and developing particular models for their description with subsequent integration of the particular models into a unified informationally consistent modeling complex is the only feasible approach to constructing mathematical models for hierarchical control of EESs.

We limit the discussion in this chapter to issues of substantiating the timetables for re-tooling the manufacturing systems while taking into account their environmental impact. It is assumed here that a series of problems listed in the preceding has been solved; specifically, the problem of calculating the combined impact of "Plant" effluents on the limiting permissible pollutant concentrations in a region has been solved and the limiting permissible effluent levels have been established for each of the "Plants". Economic management of the "Plants" with the goal of reducing their harmful *environmental impact* by modernizing the manufacturing process appears at the forefront.

12.1.1 Object Structure and Control Goals

We will assume that the criterion for optimizing the operation of the Center which is on the first (upper) level of the hierarchy is maximization of some *aggregate utility function* $U(Q, R)$ which depends on the useful product output Q_i (profit) and the volume of environmental pollution R_i by each "Plant", $i = \overline{1, N}$; and that the "Center" controls the "Plants" by establishing penalties $\rho_i(t)$

for each pollution unit by which the pollution standard R^i_{max} is exceeded. The operational goal of the "Center" can be represented in the following formalized in the following formalized form:

$$I_0 = \int_{t_0}^{T} U(Q_1, ..., Q_N, R_1, ..., R_N)dt \longrightarrow \max_{\rho_i(t), t\in[t_0,T], i=\overline{1,N}}, \tag{12.1}$$

$$U'_{Q_i} > 0, \quad U'_{R_I} < 0, \quad U''_{Q_I} < 0, \quad U''_{R_I} < 0, \quad i = \overline{1,N}.$$

Maximizing the profit, i.e., the product output Q_i while taking into account the capital expenditures K_i on production and the expenditures on environmental pollution penalties is criterion for optimizing "Plant "operation. One can take into account in this problem other material expenditures (operational, on raw material, etc) while preserving the qualitative character of the obtained results (see Section 6.1).

Control is implemented at the "Plant" level by changing the intensity with which its elements (EUs) are modernized. Control means the cut-over of new, more efficient EUs and the liquidation of the oldest ones. The operational goal of i-th "Plant" can be described in the following formalized form ($i = \overline{1,N}$) :

$$I_i = \int_{t_0}^{T} e^{-st}\{Q_i(t) - K_i(t) - \rho_i(t)[R_i(t) - R^i_{max}(t)]_+\}dt \longrightarrow \max_{m_i, a_i} \tag{12.2}$$

where $[x]_+ = \begin{cases} x, & x \geq 0, \\ 0, & x < 0, \end{cases}$ $R^i_{max}(t)$ represents aa adrmissible level for pollutant emissions of the "Plant", and the constant δ ($\delta > 0$) is discount multiplier.

12.1.2 Model of Controlling "Plant" Development

An integral dynamic model with a controlled structure of the plant is used to describe "Plant" operation. It is assumed that the "Plant" manufactures useful products, has a negative environmental impact, and operates under conditions of technological change (TC) so that new EUs are more effective and *their environmental impact is lower.*

The model includes, in addition to the objective functional (12.2), the following constraints:

$$Q_i(t) = \int_{a_i(t)}^{t} \beta_i(\tau, t)m_i(\tau)d\tau, \tag{12.3}$$

$$R_i(t) = \int_{a_i(t)}^{t} r_i(\tau, t) m_i(\tau) d\tau, \tag{12.4}$$

$$P_i(t) = \int_{a_i(t)}^{t} m_i(\tau) d\tau, \quad K_i(t) = \lambda_i(t) m_i(t), \tag{12.5}$$

$$a_i(t) < t, \quad a_i'(t) \geq 0, \tag{12.6}$$

$$0 \leq m_i(t) \leq M, \quad t \in [t_0, T], \quad i = \overline{1, N}, \tag{12.7}$$

and initial conditions

$$a_i(t_0) \equiv a_i^0 \geq 0, \quad m_i(\tau) \equiv m_i^0(\tau), \quad \tau \in [0, t_0], \quad i = \overline{1, N}. \tag{12.8}$$

The productivity $\beta_i(\tau, t)$ at time instant t of a EU created at time instant τ, the amount $r_i(\tau, t)$ of pollutants released per unit time at the instant t by the EU created at time instant τ, the price $\lambda_i(t)$ of one EU, the size of the labor force $P_i(t)$, and the limiting permissible pollutant level R_{max}^i are assumed to be the specified. Control is implemented by varying the intensity with which the technological structure of the manufacturing process is modernized by selecting the desired EU lifetimes $t - a_i(t)$ and the number $m_i(t)$ of the deployed new EUs, $t \in [t_0, T]$.

12.1.3 Statement of the Two-Level Control Problem

In this problem, the penalties and operational models (12.2)-(12.8) for various "Plants" are not interrelated and therefore one can limit the study to the analysis of a "Center"–single "Plant" EES, and drop the index i in relations (12.1)-(12.8). The obtained result will be valid for the interaction of the "Center" with N "Plants", $N > 1$.

The mathematical statement of the two-level problem of controlling an EES which consists of a "Center" and one "Plant" is as follows: determine the function $\rho(t)$, $t \in [t_0, T]$ which maximizes functional (12.1) and the functions $m(t)$ and $a(t)$ which maximize functional (12.2) under constraints (12.3)-(12.7) and initial conditions (12.8).

The specified functions β, r, λ are continuous, P satisfies the Lipschitz conditions, and m^0 is piecewise continuous.

We will identify, in the stated *two-level problem of optimal control* (12.1)-(12.8), an *external* (upper) and an *internal problem.* The external level of the optimization describes the control system (one "Center") which controls the operation of the "Plant" by imposing penalties $\rho(t)$ per unit of environmental pollutant emissions above the admissible levels. The internal problem consists in maximization of the output $Q(t)$ of total single-type product by the "Plant" while taking into account the costs of cutting over new equipment and environmental pollution penalties.

The internal and external OPs are related by the desired control $\rho(.)$, which is determined on the external level. The internal OP for a specified control ρ is investigated in the next section.

12.2 ANALYSIS OF THE INTERNAL OPTIMIZATION PROBLEM

We will investigate the solutions of the internal optimization problem (12.2)-(12.8). The control in this problem is the measurable function $m(t)$, while the unknown function $a(t)$ is the phase variable, $t \in [t_0, T]$.

By analogy with Chapter 4, the OP (12.2)-(12.8) with phase constraints (12.6) is replaced by problem (12.2)-(12.9) with a narrower permissible controls domain:

$$\max\{0, P'(t)\} \leq m(t) \leq M, \tag{12.9}$$

for which the phase constraints (12.6) on the variable $a(t)$ hold a priori. This makes it possible to infer the gradient of functional (12.2) in the control $m(.)$:

$$I'_m = \begin{cases} \int_t^{a^{-1}(t)} e^{-s\tau}(\beta(t,\tau) - \beta(a(\tau),\tau))d\tau - e^{-st}\lambda(t) \\ \qquad\qquad \text{for} \quad R(t) \leq R_{max}(t), \\ \int_t^{a^{-1}(t)} e^{-s\tau}((\beta(t,\tau) - \beta(a(\tau),\tau)) - \rho(\tau)(r(t,\tau) - \\ \quad -r(a(\tau),\tau)))d\tau - e^{-st}\lambda(t) \qquad \text{for} \quad R(t) > R_{max}(t). \end{cases} \tag{12.10}$$

Property 1 *(turnpike property).* With sufficiently broad assumptions, the equation

$$I'_m(\tilde{a}, t) = 0, \quad t \in [t_0, \infty) \tag{12.11}$$

has the unique solution $\tilde{a}(t)$, $\tilde{a}'(t) > 0$, $t \in [t_0, \infty)$ (the "turnpike" of the OP) to which the solution $a^*(t)$, $t \in [t_0, T]$, of OP (12.2)-(12.9) tends for $T \ll 1$. Specifically, for $\forall \epsilon > 0$, $\exists \overline{T}$ such that $|a^*(t) - a(t)| < \epsilon$, $t \in \Delta \subset [t_0, T]$ holds for, $\forall T > \overline{T}$, where $mes(\Delta) \to T - t_0$ for $T \to \infty$.

Equation (12.11) has been investigated in Section 5.1 for $R_{max} = \infty$. The following can be listed among the qualitative properties of the optimal EU modernization periods determined by this equation.

Property 2. The optimal EU service periods $t - \tilde{a}(t)$ are determined by the TC rate $\beta'_\tau(\tau, t)$ of creating these EUs: if $\beta'''_\tau(\tau, t) > 0$ (< 0), i.e., the TC rate increases, then the optimal operational period $t - a(t)$ of a EU decreases (and vice versa).

Property 3. If $\beta''_\tau(\tau, t) \equiv 0$, (i.e., the TC rate $\beta'_\tau(\tau, t)$ is constant), the optimal operational period $t - \tilde{a}(t)$ of a EU increases (decreases) if $\beta(\tau, t)$ decreases (increases) in t. If however, $\beta(\tau, t) = c_1\tau + c_2$, $\lambda(t) = const$, then $\tilde{a}(t) = t - A$, $t \in [t_0, T]$, where the constant A is determined by the nonlinear equation $A - (1 - e^{-sA})/s = -s\lambda/c_1$. Specifically, $A = \sqrt{2\lambda/c_1} + o(s)$ for $s \ll 1$.

These properties of OP (12.2)-(12.9) do not take into account the specifics of the impact of the environmental pollution penalty $\rho(.)$ on production. If $R_{max} \equiv 0$, then the property 3 is conserved for linear $\beta(\tau) = c_1\tau + c_2$, $r(\tau) = c_4 - c_3\tau$, $c_i > 0$, $i = \overline{1,4}$, and constant $\rho(t)$, $\lambda(t)$. In so doing, the constant A is determined by the nonlinear equation $A - (1 - e^{-sA})/s = -s\lambda/(c_1 + \rho c_3)$, and $A = \sqrt{2\lambda/(c_1 + \rho c_3)} + o(s)$ for $s \ll 1$.

We will consider the case where the "Plant" penalty mechanism is activated at some time instant t_1, $t_1 \geq t_0$: $R(t) > R_{max}(t)$ for $t < t_1$, and $R(t) \leq R_{max}(t)$ for $t \geq t_1$. It turns out that the solution $\tilde{a}(t)$ of IFE (12.11) is discontinuous at time instant t_1. Examination of its behavior over a longer time interval demonstrates that after a while the solution does not exist in general.

Thus, the OP (12.2)-(12.9) becomes more complicated to the case $R_{max} = \infty$. This reflects the fact that the integrand in (12.2) depends piecewise on the pollution R: $S(m, a, \Delta R) = Q - \lambda m - \rho[\Delta R]_+$, $\Delta R = R - R_{max}$, i.e., there is no economic mechanism for reducing $R(t)$ when the latter is approaching R_{max} from below, and when it grosses the critical value R_{max}, the economic mechanism is activated abruptly (the penalty ρ is finite and is independent of ΔR in this statement of the OP). This fact reflects the actual process which is based on the maximum permissible emission criterion. Nevertheless, the turnpike property 1 is conserved in this case and is proved as in Section 5.2.

We will be interested in what follows in the linear case of the problem: $\beta(\tau) = c_1\tau + c_2$ and $r(\tau) = c_4 - c_3\tau$, $c_i > 0$, $i = \overline{1,4}$, in which the mode of modernizing the manufacturing structure with constant EU service period d is the turnpike: $\tilde{a}(t) = t - d$, $t \in [t_0, T]$ (see Property 3), while the constant d is determined as follows:

$$d = d_1 = \sqrt{2\lambda/c_1}, \quad \text{if} \quad R \leq R_{max}, \quad \text{or} \tag{12.12}$$

$$d = d_2 = \sqrt{2\lambda/(c_1 + \rho c_3)}, \quad \text{if} \quad R > R_{max}, \ \rho = const. \tag{12.13}$$

Let us introduce the function

$$a_q(t) = \begin{cases} a_0 \quad \text{or} \quad a_{max}, & t \in [t_0, \mu), \\ t - d_1, & t \in [\mu, t_1), \\ a_q(t_1), & t \in [t_1, t_2), \\ t - d_2, & t \in [t_2, \theta), \\ a_q(\theta), & t \in [\theta, T], \end{cases} \tag{12.14}$$

where the constant μ is determined from the condition $\tilde{a}(\mu) = a_{max}(\mu)$ or $\tilde{a}(\mu) = a_0$, t_1 is determined from the condition $R(t_1) = R_{max}$, t_2 is determined from the condition $t_1 - d_1 = t_2 - d_2$, while θ is determined from the condition $I'_m(a(\theta), \theta) = 0$.

The function a_q is an admissible solution of OP (12.2)-(12.9) and, by virtue of Property 1, is close to the optimal solution a^* of the OP (at $s \ll 1$), which makes it possible to use it as the approximate solution of the internal problem when analyzing the external solution of the entire optimization problem (12.1)-(12.8).

12.3 EXAMINATION OF THE TWO-LEVEL CONTROL PROBLEM

Since one encounters substantial difficulties in examining the two-level problem, we limit its qualitative analysis to the case

$$\beta_1(\tau, t) = c_1\tau + c_2, \quad r_1(\tau, t) = c_4 - c_3\tau. \tag{12.15}$$

Then, the turnpike EU modernization mode in the internal OP (12.2)-(12.8) assumes $a(t) = t - d$, while the corresponding admissible solution a_q, which is close to the solution of the internal OP, is specified by formula (12.14).

12.3.1 Basic Optimization Problem

We first assume that the unknown penalty $\rho(.)$ is constant: $\rho \equiv const$. The external OP reduces then to scalar optimization of the functional (12.1) in the variable $\rho \in R^1$ under the constraints

$$0 \leq \rho \leq \rho_{max}, \qquad \rho \equiv 0 \text{ for } R(t) \leq R_{max}(t). \tag{12.16}$$

By virtue of the results of the preceding section, we will assume that the internal OP is solved for any specified $\rho = const$, and then its approximate solution is specified by formula (12.14), and the unknown ρ and d are connected by the monotonic relation

$$d = \begin{cases} \sqrt{2\lambda/(c_1 + c_3\rho)}, & R(t) > R_{max}(t), \\ \sqrt{2\lambda/c_1}, & R(t) \leq R_{max}(t), \end{cases} \tag{12.17}$$

and one notes that the control ρ is included explicitly in this formula only.

By virtue of (12.17), maximizing of functional (12.1) in ρ is equivalent to minimizing it in d.

We denote the OP solution by ρ^*, m^*, a^*, R^*. Obviously if $R^*(t) \leq R_{max}(t)$, $t \in [t_0, T]$, then $\rho^* = 0$, and the behavior of the optimal trajectories $m^*(t)$, $a^*(t)$, $t \in [t_0, T]$, is determined entirely by the properties of the internal OP and is described by Property 1.

We will consider the limited case of the problem, $R_{max} \equiv 0$, i.e., the penalty mechanism operates at any pollution level (since $R(t) > 0$). Then the following result holds.

Property 4. When $R_{max} \equiv 0$ and $P =\equiv const$, the solution of OP (12.1)-(12.9), (12.16) is given by the maximal possible ρ, ρ_{max}, and minimal possible d, d_{min}, which are determined by the maximal limiting modernization rates $m^*(t) \equiv M$, $t \in [t_0, T]$:

$$d^* = d_{min} = P/M, \quad \rho^* = \rho_{max}. \tag{12.18}$$

Thus, if the mechanism by which the "Center" affects "Plant" operates ($R(t) > R_{max}(t)$), the optimal mode is limiting, i.e., the optimal ρ^* is the maximal possible one.

Now, let $0 < R_{max}(t) < \infty$. If $R(t) \leq R_{max}(t)$ the optimal solution of the internal problem is determined by Property 1, $d^* = \sqrt{2\lambda/c_1}$, $\rho^* = 0$; the behavior of the solution for $R(t) > R_{max}(t)$ is determined by Property 4, and therefore the $R^*(t) \leq R_{max}(t)$ mode is the most advantageous to the "Plant" when ρ_{max} is large.

It turns out that the $R(t) = R_{max}(t)$ mode is impossible, at least, for $R_{max} \equiv const$ and $P =\equiv const$. Then the function $R(t)$ is always decreasing and can intersect R_{max} at one point only. Therefore, when r and $T - t_0$ are sufficiently large, $R(t) > R_{max}$ is satisfied initially on the $[t_0, T]$ and, starting at some $t_1 < T$, $R(t) \leq R_{max}$ is satisfied. Correspondingly, by virtue of Property 4, $\rho^* = \rho_{max}$, is a candidate for optimality on $t \in [t_0, t_1]$, and the penalty $\rho^* = 0$ on $t \in [t_1, T]$ by virtue of formula (12.17). Therefore, if one plans in taking into account the actual structure of the penalty, one has to drop the assumption $\rho = const$ and to stipulate $\rho = \rho(t)$, $t \in [t_0, T]$.

Property 5. For $R_{max} \equiv const$, $P =\equiv const$, and sufficiently large r, ρ_{max}, and $T - t_0$, the structure of the solution ρ^* of OP (12.2)- (12.9), (12.16) is

$$\rho^*(t) = \begin{cases} \rho_{max}, & \text{for} \quad t \in [t_0, t_1)\text{: } R^*(t) > R_{max}(t), \\ 0, & \text{for} \quad t \in [t_1, T]\text{: } R^*(t) \leq R_{max}(t), \end{cases}$$

while the corresponding solution (a^*, m^*) of the internal OP is limiting: $m^*(t) = M$, $a^*(t) \to t - d_{min}$, $d_{min} = P/M$, for $t \in [t_0, t_1]$ and internal: $a^*(t) \to t - d^*$, $d^* = \sqrt{2\lambda/c_1}$ (determined by Properties 1-3) for $t \in [t_1, T]$.

Thus, the following optimal strategies for controlling the considered system have been established:

- The *optimal strategy of the "Center"* : impose on the maximal possible penalty ρ_{max} on the time intervals in which the environmental pollution level $R(t)$ by the "Plant" exceeds the maximal admissible R_{max}.
- The *optimal strategy of the "Plant"* :
  - maximal possible manufacturing modernization rate and minimal EU service period on the intervals on which the environmental pollution level $R(t)$ by the "Plant" exceeds the maximal admissible R_{max};
  - a switch to the optimal manufacturing modernization rates and EU service periods determined by the technological change rate in producing the used EUs on the intervals in which $R(t) \leq R_{max}$.

The constructed optimal control $\rho(t)$, $t \in [t_0, T]$ is limiting in domain (12.16). To obtain more refined (internal) modes, one has to change somewhat the "Center" optimality criterion.

12.3.2 The Problem with a Modified Objective Criterion for the "Center"

We will assume that the utility function of the "Center" depends not on the volume Q of product output by the "Plant" but rather on its profit $V = Q - K$. In other words, the "Center", when controlling the "Plant", takes into account its capital investment in the cut-over of new EUs, which is reflected in the external problem as follows:

$$\tilde{I}_0 = \int_{t_0}^{T} U[Q(t) - \lambda(t)m(t), R(t)]dt \longrightarrow \max_{\rho}. \tag{12.19}$$

We will investigate this problem by the scheme used in the investigation of the preceding one.

Property 6. For $R_{max} \equiv 0$ and sufficiently large ρ_{max}, the optimal solution of two-level problem (12.1)-(12.9), (12.19) is some internal value ρ^* in the domain $(0, \rho_{max})$ and $d^* = \sqrt{2\lambda/(c_1 + \rho^* c_3)} > d_{min}$ for which $\partial I(d^*, t)/\partial d^* = 0$.

Let us find the optimal solution d^* and ρ^* of the two-level problem in the case of linear utility function:

$$U[Q(t) - \lambda(t)m(t), R(t)] = k_1 Q(t) - k_1\lambda(t)m(t) - k_2 R(t), \quad k_1, k_2 > 0.$$

Then, functional (12.19) has the form

$$\tilde{I}_0 = \int_{t_0}^{T} [k_1 Q(t) - k_2 R(t) - k_1\lambda(t)m(t)]dt \longrightarrow \max_{\rho}. \tag{12.20}$$

One can note that problem (12.20) coincides with accuracy up to constants with the internal OP (12.2)-(12.8) at $R_{max} = 0$ that was analyzed in Section 13.2. Analogously, one can determine d^* for (12.20):

$$d^* = \sqrt{2\lambda k_1/(k_1 c_1 + k_2 c_2)}$$

. If one takes into account that the constant d_2 is determined as $d_2 = \sqrt{2\lambda/(c_1 + \rho^* c_3)}$ in the internal OP, then the optimal value of ρ^* assumes in this case the form $\rho^* = k_2 k_1$. In particular, $\rho^* = 1$ for $k_1 = k_2 = 1$.

Now, let $R_{max} > 0$. If $R \leq R_{max}$, the optimal solution of the internal problem is determined by Property 1, and here $\rho \equiv 0$, $d^* = \sqrt{2\lambda/c_1}$. For $R > R_{max}$, $0 < \rho^* < \rho_{max}$ can be the solution of the problem by virtue of Property 6.

Property 7. For $R_{max} = const$ and sufficiently large ρ_{max}, the solution of problem (12.1)-(12.9), (12.19) has the following structure: $0 < \rho^*(t) < \rho_{max}$ for $t \in [t_0, t_1]$ and $\rho^*(t) \equiv 0$ for $t \in [t_1, T]$.

Correspondingly, the optimal control strategies will have the following form in this case:

- The *optimal strategy of the "Center"* : the optimal penalty ρ^* on the intervals in which $R > R_{max}$ is less than maximal ρ_{max} and is determined optimally by taking into account the expenditures of "Plant" in the utility function.
- The *optimal strategy of the "Plant"* :
  - accelerate production modernization while taking into account the penalty ρ_* for environmental pollution on the intervals in which $R > R_{max}$;
  - switch to the optimal manufacturing modernization rates and EU service periods determined by the TC rate in producing the used EUs when $R \leq R_{max}$.

12.4 SOME CONCLUSIONS

1. Qualitative investigation of two-level EES control problem demonstrates that with sufficiently realistic assumptions, the optimal strategy of the "Center" is a maximum penalty for exceeding the environmental pollution standards, while the optimal strategy of the "Plant" is up-to-the-limit manufacturing modernization rates as long as the pollution level exceeds its admissible value. Such "Plant" strategy corresponds to a maximal fast liquidation of the most environmentally-damaging equipment, a rapid introduction of modern equipment and, as a rule, a maximal-fast reduction in total pollution levels. When the pollution level does not exceed it admissible value, the "Plant" modernization rates are determined by the technological change and have turnpike properties.
2. In typical case, planned horizon $[t_0, T]$ is partitioned into two intervals:

(a) the initial interval $[t_0, t_1)$ on which pollution level R exceeds the admissible value R_{max}, the optimal penalty and modernization rates are maximal (ρ^*, $m^* = M, d^* = d_{min}$);

(b) the final interval $[t_1, T]$ on which $R \leq R_{max}$, the penalty $\rho^* \equiv 0$, and the "Plant" develops without intervention from the "Center".

3. The fact that the optimal solutions are bounded (optimal penalty $\rho^* = \rho_{max}$ or $\rho^* = 0$) substantially impoverishes the qualitative picture of substantially impoverishes the qualitative picture of the optimal trajectories of the problem. It has been demonstrated that to avoid this, the "Center" efficiency criterion must take into account not only the manufacturing output, but also the expenditures on "Plant" modernization. Then, the optimal value ρ^* of the penalty may be internal: $0 < \rho^* < \rho_{max}$), on the initial interval $[t_0, t_1)$ while the corresponding "Plant" modernization mode may have turnpike properties.

PART IV

APPLIED PROBLEMS OF INTEGRAL DYNAMICAL MODELS

1 3

NUMERICAL ALGORITHMS FOR INTEGRAL DYNAMICAL MODELS

Numerical treatment of the considered IDMs is the necessary stage for prediction and optimization of the development of large economic systems. More generally, the determination and optimization of new sought-for variables such as the *aftereffect duration* (*the memory of dynamical system, the lifetime of system elements*, etc.) is also of interest for various fields of applied mathematics, which will be discussed in Section 15.1. In this Chapter we construct and study some numerical algorithms for solving such problems. A number of these algorithms have been certified and implemented in software packages intended for solving applied problems.

Two recent results should be emphasized in this Chapter: the *collocation algorithm of high-order convergence* for the nonlinear Volterra integral equations with sought-for lower limit of integration (Section 13.2.3) and approximate algorithms for *finding turnpike trajectories* (Section 13.4).

13.1 ALGORITHMS FOR THE VOLTERRA INTEGRAL EQUATIONS WITH VARIABLE LOWER LIMITS OF INTEGRATION

Here the numerical algorithm for solving the system of second-kind Volterra integral equations (VIEs) with specified variable lower limits of integration and the regularizing algorithm for first-kind VIE are described.

13.1.1 The Equations of the Second Kind

At first we consider the system of n linear VIEs of the second kind in the unknowns $x_1(t), ..., x_n(t)$, $t \in [t_0, T]$:

$$x_i(t) = \sum_{j=1}^{n} \int_{z_j(t)}^{t} K_{ij}(\tau, t) x_j(\tau) d\tau + f_i(t), \quad i = \overline{1, n} \tag{13.1}$$

with the initial conditions:

$$x_j(\tau) \equiv x_j^0(\tau), \quad \tau \in (-\infty, t_0], \quad j = \overline{1, n},$$

where all variable limits $z_j(t)$ of integration are specified functions.

As was already noted in Chapter 7, the theory of such equations is analogous to that of classical VIEs, and the algorithms for solving them are also based on the familiar results for ordinary VIEs, but have specific distinctiveness, which makes it necessary to investigate them separately. When constructing the algorithms for numerical solution of the VIEs (13.1) and investigating their errors, it is required to investigate the following new (in comparison with the ordinary VIEs) issues [146]: obtaining the upper estimates of the equation solution and of its derivatives in terms of the specified functions; construction of special quadrature formulas for integration on the intervals $[z_j(t_i), t_i]$, where t_i is a grid node; and studying the effect of approximate (incomplete) specification of the functions z_j and x_j^0 (alongside K_{ij}, f_i) on the solution error.

For numerical solution of the VIEs (13.1) the algorithm was constructed in [146], which is based on the linear interpolation of functions K_{ij}, f_i, z_j, x_j^0 given in tabular form on a uniform rectangular grid. This algorithm is computationally-optimal in some sense. Namely, it was proved in [146] that this algorithm offers the —em best guaranteed error of solution for the worst given functions K_{ij}, f_i, z_j from the class of *functions with Lipschitz condition.*

13.1.2 The Equations of the First Kind

For the linear VIE of the first kind of the form :

$$\int_{z(t)}^{t} K(\tau, t) y(\tau) d\tau = f(t), \tag{13.2}$$

with respect to sought-for $y(t)$, $t \in [t_0, T]$ and the initial condition:

$$y(\tau) \equiv y_0(\tau), \quad \tau \in (-\infty, t_0], \quad j = \overline{1, n},$$

the *regularizing algorithms* were constructed in [146], which use the regularization by discretization step h on the classes C^q of the given functions K, f, z, y_0, $1 \leq q \leq 5$. It was shown that these algorithms have the order of convergence $O(\delta^{q/(q+1)})$ at $h(\delta) = O(\delta^{1/(q+1)})$, where δ is the error of the given functions K, f.

13.2 ALGORITHMS FOR THE VOLTERRA INTEGRAL EQUATIONS WITH SOUGHT-FOR LOWER LIMITS OF INTEGRATION

The numerical algorithms for the simplest non-linear equation with sought-for limit of integration and the iteration algorithms for the systems of VIEs of the second with several sought-for lower limits are investigated in this Section. The collocation-based algorithm for solving the second-kind VIEs with single sought-for limit of integration is also constructed and analyzed.

13.2.1 The Simplest Equation with Sought-For Limit of Integration

The simplest equation with respect to the sought-for limit Z of integration has the following form [146] :

$$\int_0^Z f(u)du = C. \tag{13.3}$$

Such equation is employed in iterative solving processes for the more complex VIE systems with unknowns in the lower limits of integration (see below Section 13.2.2). Note that (13.3) is not an integral equation since its integrand f does not contain the unknown Z. This equation is equivalent to the *problem of inverting a primitive* that is of interest in its own right. We consider the problem of numerical solving equation (13.3) in more detail.

Let us rewrite the equation (13.3) in the following form:

$$F(Z) = C, \quad \text{where} \quad F(Z) := \int_0^Z f(u)du. \tag{13.4}$$

This nonlinear equation can be solved by any known method (Newton's, and others). But his peculiar feature consists in the fact that the derivative $F'(Z) = f(Z)$ is evaluated considerably easier than the function $F(Z)$. This allows us to construct significantly more efficient algorithms (whose complexity corresponds approximately to the determination of one value of $F(Z)$).

Another peculiarity of the equation (13.3) is that the error $\Delta Z = |Z - \tilde{Z}|$ of its approximate solution $\tilde{Z}$ strongly depends on the value $f(\tilde{Z})$:

$$\Delta Z = \varepsilon(\tilde{Z})/|f(\tilde{Z})| + o(\Delta Z), \tag{13.5}$$

where $\varepsilon(\tilde{Z})$ is the quadrature error arising in the evaluation of $\tilde{Z}$ [146]. It is readily seen that $\Delta Z \to \infty$ for $|f(\tilde{Z})| \to 0$. So, unique solvability of the equation (13.3) can be guaranteed only if the function f has a fixed sign.

For the nonlinear equation (13.3) the efficient algorithm was constructed in [146]. First, a rough estimate U of Z (on step h grid) is determined by the successive accumulation of integral in (13.3). Next, Z is corrected on interval $[Uh, (U+1)h]$ by solving some quadratic algebraic equation with respect to correction ΔZ.

Let $f(u)$ be specified in the nodes u_k of the given grid $u_k = kh$, $k = 0, 1, 2, \ldots$. The algorithm [146] is based on the linear interpolation

$$\hat{f}(u) = f(u_k) + [f(u_{k+1}) - f(u_k)](u - u_k)/h$$

of $f(u)$ and consists of the following stages:

1. **Solving the Equation on the Grid.** Starting from $j = 1$, compute successively

$$S_j = S_{j-1} + h[f(u_{j+1}) + f(u_j)]/2, \quad S_0 = 0, \quad j = 1, 2, \ldots \tag{13.6}$$

until $S_{k-1} < C < S_k$. Then u_{k-1} is an *underestimate* and u_k is an *overestimate of the solution*, i.e. $u_{k-1} < Z < u_k$.

2. **Refining the Solution on the Interval** $[u_{k-1}, u_k]$. The integration of the interpolating function $\hat{f}(u)$ over $[u_{k-1}, u_k]$ leads to the following formulas for $\tilde{Z}$:

$$\begin{aligned} \tilde{Z} &= u_k + h[\sqrt{f^2(u_k) + 2(C - S_{k-1})(f(u_{k+1}) - f(u_k))/h} - \\ &\quad -f(u_k)]/[f(u_{k+1}) - f(u_k)] \quad \text{at} \quad f(u_{k+1}) \neq f(u_k), \\ \tilde{Z} &= u_k + (C - S_{k-1})/f(u_k) \quad \text{at} \quad f(u_{k+1}) = f(u_k). \end{aligned} \tag{13.7}$$

Note that the algorithm (13.6)-(13.7) is an *algorithm of exhaustive search* and turns to be significantly more economical in comparison with iteration algorithms of Newton type. Indeed, this algorithm represents in fact the trapezoidal rule (13.6) added by k comparison operations and the formula (13.7). Only enough values $f(u_k)$ of function f are used in the algorithm to reach maximal precision of solution $\tilde{Z}$, and any other algorithm must use no less values.

If the function f satisfies Lipschitz condition with the constant L and $|f(.)| \geq d$, $d = const > 0$, then in view of (13.5) the error of approximate solution of the equation (13.3) by the algorithm (13.6)-(13.7) is

$$\Delta Z = hL\tilde{Z}/4d + o(h).$$

It was shown in [146] that this algorithm yields the best guaranteed error of the solution $\tilde{Z}$ on the class of the positive functions f with Lipschitz condition. For the case when the function $f(u)$ can be evaluated in an arbitrary point u, an *adaptive algorithm* was also proposed in [146] which adjusts itself in an optimal way to the structure of $f(.)$.

13.2.2 The Volterra Integral Equations with Sought-For Lower Limits of Integration

At first numerical treatment of the VIEs with unknown integral lower limits was fulfilled by means of *simple iteration method.* Iterative algorithms for solving the systems of VIEs with several unknown lower limits of integration were constructed in [146]. They are based on the theoretical results similar to Theorems 7.1-7.4. We illustrate here a general scheme for these algorithms.

Let us consider the system of $n = m + l$ second-kind nonlinear VIEs:

$$x_i(t) = \sum_{j=1}^{l} \int_{z_j(t)}^{t} F_{ij}[\tau, t, x_1(\tau), ..., x_m(\tau)]d\tau, \quad i = \overline{1,m}, \tag{13.8}$$

$$p_k(t) = \sum_{j=1}^{l} \int_{z_j(t)}^{t} \Phi_{kj}[\tau, t, x_1(\tau), ..., x_m(\tau)]d\tau, \quad k = \overline{1,l} \tag{13.9}$$

with the initial conditions:

$$z_j(t_0) = z_j^0 < t_0, \quad j = \overline{1,l}, \quad x_i(\tau) = x_i^0(\tau), \quad \tau \in (-\infty, t_0], \quad i = \overline{1,m} \tag{13.10}$$

with respect to l unknown lower limits $z_i(t)$, $i = \overline{1,l}$ and $m = n - l$ unknowns $x_i(t)$, $i = \overline{1,m}$. The given values z_j^0, $j = \overline{1,l}$ and functions $x_i^0(.)$, $i = \overline{1,m}$ are supposed to satisfy the equations (13.8),(13.9) at $t = t_0$.

The iterative algorithm [146] reduces solving the VIE system (13.8)-(13.10) to a sequence of stages for solving a system of m second-kind VIEs with respect to x_i, $i = \overline{1,m}$ (for specified z_j, $j = \overline{1,m}$), and stages for solving a system of l nonlinear equations in $z_j(t)$, $j = \overline{1,l}$ (for fixed discrete values of time t).

Namely, let us determine the unknowns $z_1(t), ..., z_l(t)$ from the system of nonlinear equations (13.9) as functionals of $x_1(\tau), ..., x_m(\tau)$, $\tau \leq t$ (if this is possible) and substitute them into the first equations (13.8). Such operation was used in the proof of Theorem 7.1 for $l = 1$. The obtained system of m integral equations with respect to $x_1(t), ..., x_m(t)$ is of Volterra-type until $z_j(t) \leq t_0$, $j = \overline{1,l}$ and can be solved by iteration method. This is conveniently executed as *step-by-step procedure*. To illustrate this, we consider a *uniform grid*

$$t_s = t_0 + sh, \quad h > 0, \quad s = \overline{1,N}, \quad N := [(T - t_0)/h].$$

Then the algorithm is described by the following sequence of stages:

$s := 0;$
repeat
$s := s + 1; \quad k := 0; \quad x_i^0(t_s) = x_i^0(t_{s-1}), \quad i = \overline{1,m};$
repeat
determine $z_j^k(t_s)$, $j = \overline{1,l}$ *from system 13.9);*
substitute $z_j^k(t_s)$, $j = \overline{1,l}$ *into right-hand side of (13.8);*
put $x_i^{k+1}(t_s)$, $i = \overline{1,m}$ *be equal to left-hand side of (13.8);*

$k := k + 1;$
until $|x_i^{k+1}(t_s) - x_i^k(t_s)| < \varepsilon, \quad i = \overline{1,m};$
the given value ε determines iteration precision
until $s < N$.

The conditions of the above algorithm convergence at sufficiently small h are in agreement with the conditions of solvability of the system (13.8)-(13.10) (see Theorem 7.4 for the case of $l = m$ and F_{ij}, Φ_{ij} linear in $x_1, ..., x_m$).

The central problem connected with this algorithm is solving the system (13.9) of l nonlinear equations with respect to $z_j(t_s)$, $j = \overline{1,l}$ (for given $x_1, ..., x_m$ and every fixed value t_s). This system is uniquely solvable [146], if

$$\det\{\Phi_{kj}[z_j(t_s), t_s, x_1(z_j(t_s)), ..., x_m(z_j(t_s))]\}_{k,j=1}^{l} \neq 0.$$

But in individual cases the mentioned problem is simplified. In the case $l = 1$ this problem is equivalent to solving single nonlinear equation (13.3); to do this, efficient algorithm was proposed in Section 13.2.1. At $l > 1$ the system (13.9) can at times also be reduced to successive solving nonlinear equations of the type (13.3) (e.g., in the case of triangular matrix $\{\Phi_{kj}(.)\}_{k,l=1}^{l}$). Different specific cases $l > 1$ were studied in detail in [146].

To be embodied in a *computer discrete-time form*, all integrals in the above algorithm have to be approximated by suitable quadrature formulas. Such earlier realizations [36], [67], [146], [147] were accomplished by means of the formulas of lower precision degree. The *spline collocation methods* well developed for Volterra-type integral equations in [23], [7], [22] offer a means to gain computational effectiveness for solving the VIEs with sought-for integral lower limits. That is of great importance for further development of their application.

13.2.3 Spline Collocation Method for the Nonlinear Volterra Equations with Sought-For Lower Limit

Now we extend the spline collocation method [23] developed for VIE with delay arguments in [7], [22] to the case of the equations with the sought-for lower limit of integration. We shall restrict ourselves to the system (7.1)-(7.2) of two nonlinear VIEs

$$x(t) = \int_{z(t)}^{t} f(\tau, t, x(\tau))d\tau, \tag{13.11}$$

$$\int_{z(t)}^{t} \varphi(\tau, t, x(\tau))d\tau = p(t), \quad t \in [0, T] \tag{13.12}$$

in unknowns x, z with the initial condition :

$$z(t_0) = z_0 < t_0, \quad x(\tau) \equiv x_0(\tau), \quad \tau \in (-\infty, 0]. \tag{13.13}$$

Let $t_n := nh$, $n = \overline{0, N}$, $h = T/N$, define a uniform partition for $I := [0, T]$, and set $\Pi_N := \{t_0, ..., t_N\}$, $I_0 := [t_0, t_1]$, $I_n := [t_n, t_{n+1}]$, $n = \overline{1, N-1}$. Denote by π_m the set of all real polynomials of degree not exceeding m and define the following *piecewise polynomial space*:

$$S_{m-1}^{(-1)}(\Pi_N) := \{u : u|_{I_n} =: u_n \in \pi_{m-1}, \quad n = \overline{0, N-1}\} \tag{13.14}$$

of dimension Nm. For given real numbers $0 \leq c_1 < c_2 < ... < c_m \leq 1$ define the set X_N of *collocation points* t_{nj} in I by collocation points

$$X_N := \{t_{nj} := t_n + c_j h, \quad j = \overline{1, m}, \quad n = \overline{0, N-1}\}, \tag{13.15}$$

based on the *collocation parameters* $\{c_j\}$, $j = \overline{1, m}$.

The exact solution x, z of (13.11)-(13.14) will be approximated on I by elements of space (13.14), i.e., by the piecewise spline functions of degree $m-1$ and defect m (possessing jump discontinuities on Π_N). This *collocation solution* $\tilde{x}$, $\tilde{z} \in S_{m-1}^{(-1)}(\Pi_N)$ is then given by the equations:

$$\tilde{x}(t) = \int_{\tilde{z}(t)}^{t} f(t, \tau, \tilde{x}(\tau))d\tau,$$

$$\int_{\tilde{z}(t)}^{t} \varphi(t, \tau, \tilde{x}(\tau))d\tau = p(t), \quad t \in X_N, \tag{13.16}$$

with $\tilde{z}(0) = z_0$ and $\tilde{x}(\tau) \equiv x_0(\tau)$, $\tau \in [\tau_0, 0]$.

Since $\tilde{x}$ is in π_{m-1} on I, it can be expressed in the form

$$\tilde{x}(t_n + vh) = \sum_{k=1}^{m} L_k(v) X_{nk}, \quad t_n + vh \in I_n, \tag{13.17}$$

where $L_k(v) := \prod_{l=1, l\neq k}^{m} (v - c_l)/(c_k - c_l)$, $k = \overline{1,m}$, are the *Lagrange fundamental polynomials* with respect to the collocation parameters $\{c_j\}$.

Let us define $X_{nj} := \tilde{x}(t_{nj})$ and $Z_{nj} := \tilde{z}(t_{nj})$, $j = \overline{1,m}$, $n = \overline{0, N-1}$. For computation it is convenient to represent the unknowns z_{nj} in the following form:

$$Z_{nj} := (V_{nj} + W_{nj})h, \quad V_{nj} = [Z_{nj}/h], \quad W_{nj} = \{Z_{nj}/h\}, \tag{13.18}$$
$$j = \overline{1,m}, \quad n = \overline{0, N-1},$$

where $[a]$ and $\{a\}$ are the integer and fractional parts of number a respectively. The value V_{nj} is an estimate of Z_{nj} on the discrete grid of mesh width h ($V_{nj}h \in \prod_N$ if $Z_{nj} \geq 0$) and $W_{nj}h$, $0 \leq W_{nj} \leq 1$, is the correction of Z_{nj} on length h interval (on I_i if $V_{nj} = t_i$). The new unknowns V_{nj} and W_{nj} are linearly dependent by (13.18).

Then the integrals over $[0, Z_{nj}]$ may be written as

$$\int_0^{Z_{nj}} k(t_{nj}, \tau, \tilde{x}(\tau))d\tau = \int_0^{V_{nj}h} k(t_{nj}, \tau, \tilde{x}(\tau))d\tau +$$
$$+ \int_0^{W_{nj}} k(t_{nj}, (V_{nj} + v)h, \tilde{x}((V_{nj} + v)h))dv$$

and the exact collocation equations (13.16) can be represented in the form:

$$X_{nj} = h \int_0^{c_j} f(t_{nj}, t_n + vh, \sum_{k=1}^{m} L_k(v) X_{nk})dv + h \sum_{i=V_{nj}}^{n-1} \int_0^1 f(t_{nj}, t_i +$$
$$+ vh, \tilde{x}(t_i + vh))dv - \int_0^{W_{nj}} f(t_{nj}, (V_{nj} + v)h, \tilde{x}((V_{nj} + v)h))dv,$$
$$p(t_{nj}) = h \int_0^{c_j} \varphi(t_{nj}, t_n + vh, \sum_{k=1}^{m} L_k(v) X_{nk})dv + h \sum_{i=V_{nj}}^{n-1} \int_0^1 \varphi(t_{nj}, \tag{13.19}$$
$$t_i + vh, \tilde{x}(t_i + vh))dv - \int_0^{W_{nj}} \varphi(t_{nj}, (V_{nj} + v)h, \tilde{x}((V_{nj} + v)h))dv,$$

$j = \overline{1,m}$, where $\tilde{x}(t_i + vh)$, $t_i = V_{nj}h, ..., t_{n-1}$, is determined by (13.17).

For each $n = \overline{0, N-1}$, (13.19) represents the system of $2m$ nonlinear algebraic equations for $X_n := (X_{n1}, ..., X_{nm})^T$ and $Z_n := (Z_{n1}, ..., Z_{nm})^T$. It is readily verified that, under Theorems 7.1 and 7.2 assumptions and the sufficiently small mesh diameter h, $h \leq \varepsilon$, the system (13.19) has the unique solution X_n, $Z_n \in R^m$ such that $X_{nj} \geq 0$, $\tau_0 \leq Z_{nj} \leq t_{n-1}$, $j = \overline{1,m}$.

The Convergence of Algorithm

Let the assumptions of Theorems 7.1, 7.2 and 7.3 hold and the given functions f, φ, p, x_0 have continuous derivatives of order m. Assume that the delay integrals on $[\tilde{z}(t), 0]$ in the equations (13.19) can be approximated by quadrature formula with corresponding quadrature error of order m. Then, as shown in [24], for sufficiently small h the collocation solution $\tilde{x}$, $\tilde{z} \in S^{(-1)}(\Pi_N)$ satisfies

$$\|x - \tilde{x}\| := \sup\{|x - \tilde{x}| : t \in I\} = O(h^m),$$

$$\|z - \tilde{z}\| := \sup\{|z - \tilde{z}| : t \in I\} = O(h^m), \tag{13.20}$$

for any choice of collocation parameters $\{c_j\}$, $j = \overline{1,m}$.

While *global convergence* (on the entire interval I) of order $p = m$ occurs for any sets $X(\Pi_N)$ of collocation points, certain sets lead to the higher order of convergence, $p^* > m$, at the *mesh points* t_n, $n = \overline{0, N-1}$ (so-called *local superconvergence* [23], [7], [22]). This property of the spline polynomial collocation methods is useful, since in many applications it is attractive to obtain highly accurate approximations at certain specific mesh points (e.g., at $t_N = T$).

Let the collocation parameters $\{c_j\}$, $j = \overline{1,m}$, have so-called *orthogonality condition* [23], [7]:

$$J_i := \int_0^1 s^i \prod_{l=1}^{m} (s - c_l) ds = 0, \quad i = \overline{1, q-1}, \quad J_q \neq 0, \tag{13.21}$$

which implies that the quadrature formula have degree of precision $m + q$, $q \leq m$. If the given f, φ, p, x_0 have continuous derivative of order $m + q$ for some $q : 0 \leq q \leq m$, and $c_m = 1$, then the collocation solution $\tilde{x}$, $\tilde{z}$ is locally

superconvergent at the mesh points [24]:

$$\max_{n=\overline{1,N}} |x(t_n) - \tilde{x}(t_n)| = O(h^{m+q}), \quad \max_{n=\overline{1,N}} |z(t_n) - \tilde{z}(t_n)| = O(h^{m+q}).$$

Thus, the best possible local superconvergence order is $p^* = 2m - 1$ and is reached for so-called ***the Radau II points*** [7]. If one select equally spaced collocation parameters $\{c_j\}$, then the corresponding quadrature formulas are the *Newton-Cotes formulas.* In this case $q = 1$ (and $m + q = m + 1$) if m is odd, and $q = 0$ otherwise.

Discrete-time collocation

Generally, all integrals on the right-hand side of the exact collocation equations (13.19) have to be approximated by suitable quadrature formulas. We choose the interpolatory and extrapolatory quadrature formulas using the m abscissas based on the collocation parameters $\{c_j\}$ [7], [22]. Particularly, for the integrals over $[0, W_{nj}]$ we choose the —em extrapolatory formula:

$$\int_0^{W_{nj}} F_n(t_i + vh)dv \approx \sum_{l=1}^{m} F_n(t_i + c_l h) \int_0^{W_{nj}} L_l(v)dv. \tag{13.22}$$

where $F_n(t_i + vh)$, $i \leq n$ represents the respective integrands in (13.19). The integrals $Q_l(W) := \int_0^W L_l(v)dv$, $l = \overline{1,m}$, in (13.22) have been evaluated analytically and are the polynomials in W of degree m. For example, $Q_1(W) = W(c_2 - W/2)/(c_2 - c_1)$, $Q_2(W) = W(W - c_1/2)/(c_2 - c_1)$, when $m = 2$.

If the above quadrature processes are employed in (13.19), then there results the fully discretized collocation equation (see [24]). It defines the solution $\hat{x}$, $\hat{z} \in S_{m-1}^{(-1)}(\Pi_N)$ which will be different from $\tilde{x}$, $\tilde{z}$ determined by the exact collocation equations (13.16) or (13.19). The fully discretized equation may be viewed as the ***m-stage implicit Runge-Kutta type method*** for the VIEs (13.11)-(13.13) [23], [7], [22]. The global and local convergence results are also true for the discretized collocation solution $\hat{x}$, $\hat{z} \in S_{m-1}^{(-1)}(\Pi_N)$ [24], i.e.,

$$||x - \hat{x}|| = O(h^m), \quad ||z - \hat{z}|| = O(h^m),$$

$$\max_{n=\overline{1,N}} |x(t_n) - \hat{x}(t_n)| = O(h^{m+q}), \quad \max_{n=\overline{1,N}} |z(t_n) - \hat{z}(t_n)| = O(h^{m+q}).$$

Example 13.1. Let $m = 2$, $c_1 = 0$, $c_2 = 1$. Then the collocation points $t_{i1} = t_i$, $t_{i2} = t_{i+1}$, $i = \overline{0, N-1}$, coincide with set Π_N ($X_N \equiv \Pi_N$). In this case we have $L_1(v) = 1 - v$, $L_2(v) = v$; hence, the interpolatory quadrature formulas are the *trapezoidal rule*; $Q_1(W) = W(1 - W/2)$, $Q_2(W) = W^2/2$ in extrapolatory formula (13.22). Then the discrete-time collocation unknowns $\hat{X}_{n2} = \hat{X}_{n+1} := \hat{x}(t_{n+1})$ and $\hat{Z}_{n2} = \hat{Z}_{n+1} := \hat{z}(t_{n+1})$ are found for $n = \overline{0, N-1}$ by solving the following system of two nonlinear equations:

$$2\hat{X}_{n+1} = h[f(t_{n+1}, t_n, \hat{X}_n) + f_1(t_{n+1}, t_{n+1}, \hat{X}_{n+1})] + h \sum_{i=V_{n+1}}^{n-1} [f(t_{n+1}, t_i, \hat{X}_i) +$$

$$+ f(t_{n+1}, t_{i+1}, \hat{X}_{i+1})] - [f(t_{n+1}, \hat{V}_{n+1}h, \hat{x}(\hat{V}_{n+1}h))W_{n+1}^2 + \tag{13.23}$$

$$+ f(t_{n+1}, (\hat{V}_{n+1} + 1)h, \hat{x}((\hat{V}_{n+1} + 1)h))(2 - W_{n+1})W_{n+1}],$$

$$2p(t_{n+1}) = h[\varphi(t_{n+1}, t_n, \hat{X}_n) + \varphi_2(t_{n+1}, t_{n+1}, \hat{X}_{n+1})] + h \sum_{i=V_{n+1}}^{n-1} [\varphi(t_{n+1}, t_i,$$

$$\hat{X}_i) + \varphi(t_{n+1}, t_{i+1}, \hat{X}_{i+1})] - [\varphi(t_{n+1}, \hat{V}_{n+1}h, \hat{x}(\hat{V}_{n+1}h))W_{n+1}^2 + \tag{13.24}$$

$$+ \varphi(t_{n+1}, (\hat{V}_{n+1} + 1)h, \hat{x}((\hat{V}_{n+1} + 1)h))(2 - W_{n+1})W_{n+1}].$$

where auxiliary unknowns $\hat{V}_{n+1}$ and $\hat{W}_{n+1}$ are connected with $\hat{Z}_{n+1}$ by (13. 18). This nonlinear algebraic system (13.23)-(13.24) is solved by the iterative method analogous to that described in Section 13.2.2. Namely, for each step $n = \overline{0, N-1}$ we successively determine $\hat{Z}_{n+1}$ from single nonlinear equation (13.24) and next find $\hat{X}_{n+1}$ from (13.23).

This algorithm (at $m = 2$) is of interest in the cases where the given functions in problem (13.11)-(13.13) have not high smoothness, for example, are only differentiable. Then the application of numerical methods of higher precision orders has no sense. It is well known (e.g., [104]) that trapezoidal rule is asymptotically optimal for integrands with bounded second derivative. Experimental convergence of the collocation algorithm based on equations (13.23)-(13.24) was verified in [24] for the case of Example 7.1 (see Section 7.1). In so doing, the parameters in the Example 7.1 were chosen as $c_1 = 0.15$, $c_2 = 0.1$, $d_1 = 0.1$, $d_2 = 0.05$, $q_1 = 0.8$, $q_2 = 0.6$, $b_1 = 1.933$, $b_2 = 1.0$, $a_0 = 2.0$, $P_0 = 1.021$, $D = 1.0$, $T = 5.0$. In Table 13.1 we expose the maximal relative errors $\delta x = \|x - \hat{x}\|/\|x\|$, $\delta z = \|z - \hat{z}\|/\|z\|$ of the discrete-time collocation solution $\hat{x}$ and $\hat{z}$. As appears from the above, these errors depend on step h as $O(h^2)$.

Table 13.1 The errors of collocation method for VIEs (13.11)-(13.13)

h	0.1	0.05	0.025	0.01	0.005
δx	8.34×10^{-5}	4.51×10^{-5}	1.01×10^{-5}	2.65×10^{-6}	6.92×10^{-7}
δz	7.06×10^{-4}	1.99×10^{-4}	5.28×10^{-5}	8.68×10^{-6}	2.21×10^{-6}

13.3 ALGORITHMS FOR OPTIMIZATION PROBLEMS

Finally, we briefly turn our attention to the algorithms for numerical treatment of optimization problems (OPs) for the IDMs under consideration.

These OPs are the most computationally-complex problems for numerical investigation. For such problems ***zero-order algorithms*** which reduce OP to nonlinear programming problem, as well as ***first-order methods*** (gradient projection methods), were developed (see [48], [46], [36], etc.). An adaptive algorithm for numerical solution of the OP (8.1)-(8.7) in two-sector IDM (8.1)-(8.3) was suggested in [67]. It adjusts itself to a priori unknown structure of optimal control $y(.)$ and harnesses a combination of two methods: the gradient projection within intervals of inner control y in domain (8.5) and the special method of successive approximations for the intervals of boundary control y. The geometric speed of convergence of this algorithm was proved and its suitability to other OPs in the IDMs was demonstrated.

New perspective technique for the numerical treatment of OPs in considered IDMs is based on turnpike approach. It requires solving some integral-functional equation (IFE) of non-Volterra type with the unknown in upper integral limit (with unknown forestalling argument), which is similar to (5.1). Such equations arise in the dual problems for IDM optimal control. Some approximate algorithms for the estimation of solution of the IFE (5.1) are constructed in the next Section.

13.4 APPROXIMATE ALGORITHM FOR MODELING OF TURNPIKE TRAJECTORIES

Systematic analysis of various optimization problems in one-, two-, and multi-sectoral IDMs suggests (see Chapters 4-12) that in many important cases in practice the optimal (turnpike) renewal times of the elements of economic systems under the conditions of technological change are approximately described by an integral-functional equation with memory for the unknown $z(t)$:

$$\Phi(t) \stackrel{df}{=} \int_t^{z^{-1}(t)} [F(t) - F(z(\tau))]d\tau - f(t) = 0 \tag{13.25}$$

where $z^{-1}(.)$ is the inverse of the function $z(.)$ and $t - z(t)$ is the useful service life of the elements.

Equation (13.25) was first derived in [49] for the optimization problem in the Glushkov two-sector integral dynamic macroeconomic model (see Section 2.5). Further studies [4], [117], [66], [150], [146], [147] have shown that equation (13.25) arises also in many other important applications in integral models. According to Chapter 5, this equation relates the cost of development and introduction of new production capacity with a rollover estimate of efficiency – the incremental output produced by new production capacity compared with the output from existing capacity during its future operation. A qualitative analysis of equation in Section 5.1 has shown that it has highly nontrivial properties that complicate its numerical solution.

13.4.1 Statement of the Problems

Let $F(t)$ and $f(t)$ be continuously differentiable, $F(t) > 0$, $F'(t) > 0$, $f(t) > 0$, and we only consider the case of continuous solutions $z(t) < t$ (these assumptions are meaningful, see Chapter 5). For qualitative analysis of the properties of equation (13.25), we have to transform this equation. Differentiating (13.25), we obtain a continuous-time functional-difference equation

$$F'(t)[z^{-1}(t) - t] = F(t) - F(z(t)) + f'(t). \tag{13.26}$$

Substituting in (13.25) the expression for $z^{-1}(t)$ from (13.26), we obtain the equation

$$\int_t^{t+\frac{[F(t)-F(z(t))+f'(t)}{F'(t)}} [F(t) - F(z(\tau))]d\tau = f(t). \tag{13.27}$$

Denoting $W(t) = t + [F(t) - F(z(t)) + f'(t)]/F'(t)$, we obtain an equation for $W(t)$:

$$\int_t^{W(t)} [F(t) - F'(\tau)(\tau - W(\tau)) - F(\tau) - f'(\tau)]d\tau = f(t). \tag{13.28}$$

Since $F(t)$ is strictly increasing, $F(t) > 0$, $z(t) < 0$, $F'(t) > 0$, we have $W(t) > t$. Equation (13.28) has the properties of integral equations of the first kind, in particular, its solution is unstable, which requires application of regularization methods.

In applied economic problems, the functions $F(t)$ and $f(t)$ are usually known on some bounded intervals and the initial function $z(t)$ is normally unknown on any interval. The function $F(t)$ should be known on the interval $[z(t), t]$, where $z(t)$ is the sought function; it suffices to know the function $f(t)$ in a small neighborhood of the point t. If $F(t)$ is given on a smaller interval, then a single-valued solution of equation (13.25) cannot be obtained. It is therefore important to develop a numerical algorithm for equation (13.25) that requires minimum initial information.

The problem of approximate solution of equation (13.25) on small (2-3 periods $t - z(t)$), intervals $[t_0, t_1]$ is essentially different from the solution of (13.25) on sufficiently large $[t_0, t_1]$. For small $[t_0, t_1]$ it makes sense to solve (13.25) as an equation with memory with a given initial function $z(\tau) = \varphi(\tau)$, $t \in [z(t), t]$, (the given φ should satisfy (13.25) at the point t). On large $[t_0, t_1]$, as shown in Section 5.1, the solution $z(t)$ under natural assumptions on $F(t)$ and $f(t)$ tends to a unique function $\tilde{z}(t)$, $t \in (-\infty, \infty)$ as $t_0 - t_1 \to \infty$. We can only consider the behavior of the function $\tilde{x}$, and the problem of solving equation (13.25) for a given φ is meaningless.

We thus have three problems of solving equation (13.25), which require fundamentally different algorithms.

Problem A. For given $F(u)$, $f(u)$, $u \in [t_0, t_1]$, and $z(\tau) \equiv \varphi(\tau)$, $\tau \in [t, \varphi^{-1}(t)]$ that satisfies equation (13.25), find the solution $z(\tau)$ for $\tau \in [t_0, t]$ and/or

Problem B. For given $f(t)$ and $F(\tau)$, $\tau \in [z(t), t]$ (the instant $z(t)$ is not known in advance), estimate the solution $z(\tau)$ for $\tau \in [t, z^{-1}(t)]$.

Problem C. For given $f(t)$ and $F(t)$, $t \in [t_0, t_1]$, $t_1 - t_0 \gg 1$, estimate the behavior of the solution $z(t)$, $t \in [t_0, t_1]$.

13.4.2 Solution of Problem A

This problem is the simplest of the three. Once it is established that equation (13.25) is satisfied for $z(\tau)$, $\tau \in [t, z^{-1}(t)]$, the problem is solved using equation (13.26). For $\tau < t$, we use the explicit recurrence

$$z(\tau) = F^{-1}\{F(\tau) + f'(\tau) - F'(\tau)[z^{-1}(\tau) - \tau]\}. \tag{13.29}$$

and for $\tau > z^{-1}(t)$

$$z^{-1}(\tau) = \tau + [F(\tau) - F(z(\tau)) + f'(\tau)]/F'(\tau). \tag{13.30}$$

Continuity of $z(\tau)$ for $\tau = t$ and $\tau = z^{-1}(t)$ follows from equation (13.25) The solutions $z(t)$ are uniquely determined from (13.29) or (13.30) until the monotonicity of $z(t)$ is broken (then a single-valued function $z^{-1}(t)$ does not exist). To solve (13.29) or (13.30), we only need efficient algorithms for evaluation of the inverse functions $F^{-1}(t)$ and $z^{-1}(t)$.

13.4.3 Solution of Problem B

Assume that $F(\tau)$ is given or can be approximated on some interval $[z(t), t]$, where t is a fixed instant, $z(t)$ is not known in advance; $f(t)$ and $f'(t)$ are also given. By the above considerations, the properties of equation (13.25) are close to the properties of integral equations of the first kind. We will restrict the analysis to the simplest regularization case, specifically, we assume that

$$z(\tau) = c\tau - d, \quad \tau \in [t, z^{-1}(t)], \tag{13.31}$$

where c, d are unknown parameters.

Making the change of variables $z(\tau) = u$, we reduce the equation (13.25) to the form

$$\int_{z(t)}^{t} [F(t) - F(u)] \frac{1}{z'(z^{-1}(u))} du = f(t), \tag{13.32}$$

and by (13.31) we have $z'(z^{-1}(u)) = c$, $u \in (z(t), t)$.

Additional information is needed in order to find the two unknowns c, d. We assume that alongside equation (13.25) at the point t we also have $\Phi'(t) = 0$, i.e., $\Phi(t + \Delta t) \to 0$ as $\Delta t \to 0$. This is equivalent to relationship (13.26).

Substituting (13.31) in (13.26), we obtain

$$\frac{1}{c}F'(t)[t - (ct - d)] = F(t) + f'(t) - F(ct - d). \tag{13.33}$$

Denote $Z := z(t) = ct - d$ and consider the system of equations (13.32) and (13.33) for the unknowns Z and c:

$$\begin{cases} \int_Z^t [F(t) - F(\tau)]d\tau = cf(t), \\ \frac{1}{c}F'(t)[t - Z] = F(t) - F(Z) + f'(t). \end{cases}$$

Eliminating c from these equations, we obtain for the unknown Z

$$[F(t) - F(z) + f'(t)] \int_z^t [F(t) - F(\tau)]d\tau = f(t)F'(t)[t - z]. \tag{13.34}$$

Example 13.2. Let $F(t) = kt + e$, $f(t) \equiv$const. Then from (13.34)

$$k(t - Z) \int_Z^t k(t - \tau)d\tau = k(t - Z)f, \quad \text{and} \quad z(t) = t - \sqrt{2f/k}$$

which is identical with the unique solution of equation (13.25) on $(-\infty, \infty)$ obtained in Example 4.1. With nonlinear $F(t)$ and $f(t)$, equation (13.34) cannot be solved analytically.

Transform equation (13.34) to the form

$$[F(t) - F(Z) + f'(t)][F(t) - \frac{1}{t - Z}\int_Z^t F(u)du] = f(t)F'(t). \tag{13.35}$$

We will show that this equation has a unique solution Z, $-\infty < Z < t$.

Indeed, with $F(t)$, $f(t)$, $F'(t) > 0$, the left-hand side equation (13.35) vanishes for $Z = t$ and tends to ∞ for $t - Z \to \infty$. By continuity in Z equation (13.35)

has a solution $Z < t$, $z \in]-\infty, t[$. Differentiating the left-hand side of (13.35) with respect to Z, we obtain

$$-[(F(t) - \frac{1}{t-Z}\int_Z^t F(u)du)F'(Z)+$$

$$+(F(t) - F(Z) + f'(t))(\frac{1}{(t-Z)^2}\int_Z^t F(u)du) - \frac{F(Z)}{t-Z})] < 0. \qquad (13.36)$$

The solution Z is thus unique.

Due to the presence of the term $\int_Z^t F(u)du$ in (13.35), it is computationally more efficient to solve it by algorithms of the type proposed in Section 13.2.1 for inversion of the primitive function.

Define a *uniform grid* to the left of the fixed point t: $z_k = t - kh$, $k = 0, 1, \ldots$, $h > 0$, and consider an algorithm that solves equation (13.35) by linear interpolation of the function $F(u)$. The algorithm consists of the following stages.

1. Solving the Equation on a Grid. Starting with $j = 1$ (to the left of the point t), compute successively

$$r_j = [F(t) - F(z_j) + f'(t)][f(t) - S_j/(t - z_j)],$$

$$S_j = S_{j-1} + h\frac{F(z_j) + F(z_{j-1})}{2}, \quad S_0 = 0, \quad j = 1, 2, ..., \qquad (13.37)$$

until $r_{k-1} < f(t)F'(t) < r_k$. Then z_k is an underestimate and z_{k-1} is an overestimate of the solution Z, i.e., $z_k < Z < z_{k-1}$.

2. Refining the Solution Z on the Interval $[z_k, z_{k-1}]$. The error in the integral in the integral in formulas (13.37) is of the order h^2 on the given class of functions, and we therefore set $\Delta z = Z - z_k$, $F(z) = F(z_k) + \Delta F(z_k)\Delta z/h + 0(h^2)$, where $\Delta F(z_k) = F(z_{k-1}) - F(z_k)$, $u \in [z_k, z_{k-1}]$. Then

$$\int_z^t F(u)du = S_k - \int_{z_k}^z F(u)du = S_k - F(z_k)\Delta z - \frac{\Delta F(z_k)}{2h}\Delta z^2$$

and from (13.35) after some manipulations we obtain a cubic equation in Δz:

$$a_3\Delta z^3 + a_2\Delta z^2 + a_1\Delta z + a_0 = 0, \qquad (13.38)$$

where

$$a_0 = [t - z_k][F(t)^2 - F(t)[F(z_k) - f'(t)] - F'(t)f(t)] - S_k[F(t) - F(z_k) + f'(t)],$$

$$a_1 = f'(t)[F(z_k) - F(t)] - F(z_k) - F(t)]^2 - \frac{\Delta F(z_k)}{h}[F(t)(t - z_k) - S_k] + F'(t)f(t),$$

$$a_2 = \frac{\Delta F(z_k)}{2h}[3(F(t) - F(z_k)) + F'(t)], \quad a_3 = -\frac{\Delta F(z_k)^2}{2h^2}.$$

Since $\Delta z \to 0$ as $h \to 0$, this equation has a unique positive root Δz. The solution $z^* = z_k + \Delta z$ of equation (13.35) is constructed with an error $0(h^2)$.

13.4.4 Solution of Problem C

Equation (13.25) is hard to solve on large $[t_0, t_1]$ because for specific initial functions φ, the monotonicity of $z(t)$, and with it the continuity of $z^{-1}(t)$, usually breaks down as $t_1 - t_0 \to \infty$ (see Section 5.1). The proposed algorithm combines the algorithms (13.29)-(13.30) and (13.32)-(13.38) and is justified by theoretical analysis of the asymptotic behavior of the solutions of equation (13.25) what has been conducted in Section 5.1. Specifically, it has been shown in Lemma 5.1 that if

$$z'(z^{-1}(t)) > \sqrt{F'(t)/F'(z(t))}, \tag{13.39}$$

then small variations $|\delta z'(z^{-1}(t))| < |\delta z'(t)|$, and conversely for

$$z'(z^{-1}(t)) > \sqrt{F'(t)/F'(z(t))} \tag{13.40}$$

we have $|\delta z'(z^{-1}(t))| > |\delta z'(t)|$.

Let us use the algorithm (13.32)-(13.38) to construct the solution $z(t)$, $t \in [t_0, z^{-1}(t_0)]$. If $c < [F'(t_0)/F'(z(t_0))]^{1/2}$, then we find $z(t)$, $t \in [z^{-1}(t_0), t_1]$ using formula (13.30). As long as (13.39) holds, we $z'(z^{-1}(t)) < F'(t)/F'(z(t))$, then in view of Lemma 5.1 the solution $z(t)$ tends for $t_1 \to \infty$, $t \to \infty$ to some unique solution $\tilde{x}(t)$, $t \in (-\infty, \infty)$ of equation (13.25). Otherwise, there exists a divergent pencil of solutions $z(t)$, $t \in [t_0, \infty[$ of equation (13.25) (such example is given by Theorem 5.3). To estimate the approximation of the solution $z(t)$ to $\tilde{x}(t)$, we need to apply the same algorithm on some interval $[t, t_1]$, $t > t_0$, and estimate the closeness of the solutions.

If $c \geq [F'(t_0)/F'(z(t_0))]^{1/2}$, then we construct the solution $z(t) = ct - d$, $t \in [z(t_1), t_1]$ and continue it from right to left using formula (13.29) for $t \in [t_0, z(t_1)]$. If (13.40) holds and $z'(z^{-1}(t)) > F'(t)/F'(z(t))$, then $z(t) \to \tilde{z}(t)$ as $t_1 \to \infty$, $t_1 - t \to \infty$.

In the general case (when neither (13.39) nor (13.40) holds on $[t_0, t_1]$), we can choose some fixed time t, $t_0 < t < t_1$,and apply algorithm (13.32)-(13.38) to construct the solution $z(\tau)$, $\tau \in [t, z^{-1}(t)]$. Then we use formula (13.29) to define $z(\tau)$ on $[t_0, t]$ and formula (13.30) to define it on $[z^{-1}(t), t]$ until the monotonicity of $z(\tau)$ breaks down. Choosing a different t and comparing the solutions $z(\tau)$, $\tau \in [t_0, t_1]$, we can approximately estimate the solution error. If one of the inequalities (13.39)-(13.40) does not hold asymptotically (as $t \to \infty$), then we cannot guarantee convergence of the approximate solution nor the existence of a continuous solution $z(t)$ as $t_0 - t_1 \to \infty$. Yet we can always identify subintervals $[t_0, t_1]$ on which (13.39) or (13.40) and solve equation (13.25) by the proposed algorithm on these subintervals.

The proposed algorithms have been incorporated in interactive PC software systems for estimating times and rates of renewal in complex economic and technical systems. They have proved quite effective using real-life economic information (see next Chapter).

1 4

APPLICATION OF INTEGRAL MODELS TO OPTIMIZATION OF TECHNOLOGICAL RENOVATION

The control of replacement of obsolete technologies and equipment represents powerful means to raise the efficiency of economic systems. It often appears to be more effective than management of redistribution of resources, inventories, etc. The IDMs with controllable memory (IDM-CM) are specially introduced and adapted for solving such control problems. But, their application to a concrete economic system meets some difficulties, main of which is the absence of complete information to identify a model.

14.1 IDENTIFICATION OF MODELS

Identification of the IDMs is the first necessary stage in the modeling of rational renovation of a real system on the basis of these models. Starting from this grounds, further prediction and optimization problems can be solved.

The identification primarily consists in the determination of the sought-for UIR of an IDM (the kernel of integral equation). As already was mentioned in Chapter 1, the IDMs with controllable memory generalize the well-known IDMs of linear non-stationary DSs. The *identification problems* for the IDMs with the specified lower limit $a(t) = t - T$ or $a(t) = -\infty$ of integration are classical in control theory, and methods of UIR determination were investigated by many authors (see [39], [92], [128], etc.). Most of these methods can be used in the IDMs with controllable memory (with sought-for integration limit), since such models remain to be linear in "input"–"output". But the required information for model identification is usually specific for engineering and technical systems.

Here we consider some applied approach for the IDM identification with using standard economic information. For this purpose, a preliminary analysis of available and required information base is necessary.

14.1.1 Analysis of Information Base

The structure of required information is defined by the type of IDM and the statement of mathematical problem in this model.

The models under consideration usually include :

- - commonly accepted economic indices such as production output, operated fixed assets, the amount of new assets commissioned for service, the amount of obsolete assets removed from service, labor amount, accumulation norm, etc.;
- - standard characteristics of technological renovation, such as the total quantity of operated capacity, the quantity of putting new equipment into operation, the lifetime of equipment;
- - some internal specific indices of equipment (technology), such as productivity and price of equipment units, specific operated costs, and others, which depend upon equipment age and current year.

The IDMs can also demand some other indices for model identification. For example, the next (additional) input data are required for the identification of the multi-sector IDM of manufacturing systems from Chapter 11, as applied to agriculture:

- - the sequence of implementation of specific agricultural operations during annual production cycle,
- - total quantity, commissioning, and functional structure of operated equipment on a prehistory interval,
- - the composition of agricultural machines with distinguishing basic and auxiliary equipment,
- - the calendar distribution of equipment units for different operations (for determining annual schedule of equipment utilization).

The identification of IDM-CM implies solving mathematical problems which are substantially complex from theoretical viewpoint (incorrectness and others) as well as with respect to required input information.

First, the determination of two-dimensional sought-for function of type $\beta(\tau, t)$ from one-dimensional relations of type (4.1) represents a ***principally incorrect problem*** that requires an additional information (for example, statistical) or the construction of regularization methods.

Second, the IDMs take into account the ***aftereffect of unknown duration***, hence, sought-for functions must be specified on some time interval whose length is unknown *a priori*.

Next, one more difficulty arises because of the presence of the sought-for lower limit $a(.)$ in the IDMs. Namely, for the determination of sought-for functions on a prehistory $[a(t), t]$ it is necessary to know these functions on a "pre-prehistory" $[a(a(t)), a(t)]$, etc. So, the identification problem for IDM-CM is ***underdetermined on any finite interval*** $[\tau_0, t_0]$. As indicated in [96], this problem was noted even by Preinreich [111]. Since one cannot consider the infinite prehistory $(-\infty, t_0]$ in an applied problem, some ***additional assumptions*** about the sought-for functions on "pre-prehistory" are needed for the problem solvability.

Certain other assumptions about the ***structure and behavior of model indices*** can be accepted to reduce the dimension of sought-for functions and to determine all other model functions on prehistory interval. On these grounds, an applied procedure for identification of the considered IDMs ***on the basis of standard economic information*** was proposed and developed in [150], [146], [4], etc. We consider the essence of proposed technique in the case of simple one-sector IDM (see Chapters 4-6).

14.1.2 The Identification of One-sector IDM

The problem under study consists in the determination of the sought-for functions $\beta(\tau, t)$, $a(t)$, $m(t)$, $\lambda(t)$, $\tau, t \in [\tau_0, T]$, satisfying the following one-sector IDM:

$$Q(t) = \int_{a(t)}^{t} \beta(\tau, t) m(\tau) d\tau, \tag{14.1}$$

$$P(t) = \int_{a(t)}^{t} m(\tau)d\tau, \tag{14.2}$$

$$\Phi(t) = \int_{a(t)}^{t} \varphi(\tau)d\tau, \tag{14.3}$$

$$\varphi(t) = \lambda(t)m(t) \tag{14.4}$$

under the given functions $Q(t)$, $P(t)$, $\Phi(t)$, $\varphi(t)$, $t \in [\tau_0, T]$.

As above, here $Q(t)$ is uniform product output per time unit; $P(t)$ is the quantity of operated EUs (labor force); $\Phi(t)$ is total amount of operated fixed assets; $\varphi(t)$ is amount of the fixed assets commissioned for service at time t; $m(t)$ is the quantity of the EUs created at time t; $\beta(\tau, t)$ is EU productivity; $\lambda(t)$ is EU cost; $a(t)$ is the time limit of obsolete EUs scrapping.

The following ***additional economic assumptions*** are accepted :

- EU productivity $\beta(\tau, t)$ depends upon only ***three key factors*** (embodied TC, autonomous TC, and physical wear):

$$\beta(\tau, t) = \beta_1(\tau)\beta_2(t)\beta_3(t - \tau), \quad \tau, t \in [\tau_0, T], \tag{14.5}$$

- EU productivity $\beta(\tau, t)$ is constant on ***unknown earlier "pre-prehistory"*** $[a(\tau_0), \tau_0]$:

$$\beta(\tau, t) \equiv \beta_0, \quad \tau \in [a(\tau_0), \tau_0], \quad t \in [\tau_0, T], \tag{14.6}$$

- the EUs, which were created during period $[a(\tau_0), \tau_0]$, ***are removed from operation uniformly*** during period $\tau_0, a^{-1}(\tau_0)]$:

$$m(a(\tau))a'(\tau) = x_0, \quad \tau \in [\tau_0, a^{-1}(\tau_0)]. \tag{14.7}$$

Here the constants β_0, $a(\tau_0)$, $a^{-1}(\tau_0)$, and x_0 are unknown.

The first assumption takes into account the following three multipliers influencing on EU productivity :

- $\beta_1(\tau)$ - embodied technological change (increase of the productivity of new EUs);
- $\beta_2(t)$ - change of the productivity of all EUs because of autonomous TC, climate, and other factors depending on current time t;

- $\beta_3(t-\tau)$ - decrease of EU productivity due to physical wear (decay) depending on EU age $t-\tau$.

In [146], [4] the dynamics of the factor $\beta_1(\tau)$ of embodied TC was specified and the uniform EU wear with the ***constant wear rate*** $\delta > 0$ was considered :

$$\beta_3(t-\tau) = \exp[\delta(\tau - t)]. \tag{14.8}$$

The proposed ***algorithm for solving the identification problem*** (14.1)-(14.8) consists of the following stages [4]:

- - determine the instant $t_0 = a^{-1}(\tau_0)$, $\tau_0 < t_0 \leq T$, and the function $a(t)$, $t \in [t_0, T]$, from equation (14.3) (by the numerical algorithm of Section 13.2.1);
- - find the value $x_0 = P(\tau_0)/(t_0 - \tau_0)$ and the function

$$m(t) = \begin{cases} P'(t) + x_0, & t \in [\tau_0, t_0], \\ P'(t) + m(a(t))a'(t), & t \in [t_0, T], \end{cases}$$

  from (14.2) and (14.7);
- - determine the function $\lambda(t) = \varphi(t)/m(t)$, $t \in [\tau_0, T]$ from equation (14.4);
- - by virtue of (14.5) and (14.8), determine the function

$$\beta_2(t) = Q(t)\exp(\delta t)/\int_{a(t)}^{t} \exp(\delta\tau)\beta_1(\tau)m(\tau)d\tau, \quad t \in [t_0, T]$$

  and the value $\beta_0 = \beta_1(t_0)\beta_2(t_0)$ from equation (14.1).

It can be shown that, if the assumptions (14.5)-(14.8) hold true, the five functions $Q(t)$, $P(t)$, $\Phi(t)$, $\varphi(t)$, $\beta_1(t)$ are given ***on sufficiently large interval*** $[\tau_0, T]$, and constant δ is known, then the above algorithm makes it possible to identify the one-sector IDM (14.1)- (14.4), i.e. to determine the other functions $m(\tau)$, $\tau \in [\tau_0, T]$, $a(t)$, $\lambda(t)$, $\beta_2(t)$, $t \in [a^{-1}(\tau_0), T]$.

If the given interval $[\tau_0, T]$ appears to be such that $\int_{\tau_0}^{T} \varphi(\tau)d\tau < \Phi(T)$, then the problem is unsolved ($a^{-1}(\tau_0) > T$) and we need to consider longer interval.

The exposed identification procedure has been repeatedly modified and applied to other IDMs. It has been implemented in software packages intended for simulation of various-scale economic systems.

14.2 SIMULATION RESULTS FOR REAL ECONOMIC SYSTEMS

Simulation of the renovation and development of different economic systems was carried out on the IDM basis during last decade. Of special interest was the modeling of the optimal (turnpike) lifetimes of production means. Some examples of the simulation are briefly exposed in the present section.

14.2.1 Modeling of the Former USSR Economy in 1946-1990

Used model: the two-sector macroeconomic IDM (8.1)-(8.3).

Identification of the model was performed by the above algorithm with the additional assumption (8.8). The input data Q, P, Φ, φ, β_1 were found from the literature (from statistical year-books).

Optimization problem: the maximization (8.4)-(8.7) of the discounted output of consumption goods over a planning interval under given manpower.

Turnpike analysis. The estimation of turnpike lifetime $\tilde{L}(t) = t - \tilde{a}(t)$ was performed for the fixed assets of the former USSR economy in 1946-1990 years (by means of the approximate algorithm from Section 13.4). At small discount s,1 the turnpike trajectory for the OP (8.1)-(8.7) had the following values : $\tilde{L}(1946) \approx 20$, $\tilde{L}(1965) \approx 37$, $\tilde{L}(1985) \approx 55$ years.

Hence, non-efficiency of the USSR economy did not permit the existence of turnpike regime. Nevertheless, the turnpike regimes have appeared to be possible for some separate enterprises and branches of the USSR industry (see next example).

14.2.2 Modeling of a Separate Plant

Modeled process: technological development of the plant "Krasny ekskavator" (Kiev, Ukraine) in 1960-1990.

The aim of simulation was to prepare a recommendation for increasing the output in 1985-1990 without additional labor resource.

Used models: the one-sector IDM and the two-sector IDM with separating external and auxiliary products. Identification of the models was performed as above.

Some results of multivariant prognosis:

- - to raise the output by 8 percent per year under the same labor, it is necessary to replace approximately 5 percent of the oldest EUs (per year) by the EUs for which the rate of productivity growth is 30-35 percent per year in 1986-87 and 10 percent in 1990;
- - in so doing, the computed equipment lifetime decreases from 20 years in 1985 to 16 years in 1990 (the oldest EUs created in 1965-1974 would be removed from operation);
- - the redistribution of 40 percent of workers between the outputs of external and auxiliary products permits the lower productivity growth of new EUs (about 20 percent per year in 1986-87).

Optimization problems: OP1, OP2, OP5, and OP6 from Chapters 4 and 6.

Turnpike analysis. The modeling of the turnpike lifetimes (service periods) of equipment provided such results: $\tilde{L}(1970) \approx 5$ years, $\tilde{L}(1980) \approx 6$ years, $\tilde{L}(1985) \approx 6.5$ years, for the OP1. For the other problems OP2, OP5, and OP6 the optimal $\tilde{L}(.)$ has assumed more values. This result indicates that production efficiency might be raised by decreasing the equipment lifetime $L(.)$ (really L was equal to 20-25 years).

14.2.3 Modeling of a Regional Manufacturing (Agricultural) System

Modeled process: the development of the set of agricultural machines and equipment in concrete region (Vasil'kov district of Kiev region, Ukraine) in 1970-1991.

Used model: the multi-sector IDM (11.1)-(11.2).

Identification of the model was performed by a modification of the above algorithm for multi-sector case. The input data were determined by technique

Table 14.1 The development diagram for the tractors with force 30 kN

Type	The years of production	Power	Price
SHTZ–NATI	1937-1949	52 h.p.	2000 rbl.
DT-54	1949-1962	54 h.p.	2100 rbl.
T-74	1962-1984	75 h.p.	2460 rbl.
DT-75	1963-1990	75 h.p.	3710 rbl.
T-150K	1973-1992	165 h.p.	11685 rbl.
T-150	1984-present	150 h.p.	10785 rbl.

Table 14.2 The distribution of the tractors with force 30 kN

Year	Type	DT-54	T-74	DT-75	T-150K	T-150
1971	**Quantity**	30	105	100	–	–
1973		11	100	96	9	–
1975		–	91	102	24	1
1977		–	62	116	34	1
1979		–	60	119	49	1
1981		–	59	115	65	2
1983		–	57	112	71	9
1985		–	56	112	79	17
1987		–	52	110	86	31
1989		–	48	107	98	45
1991		–	41	103	105	54

developed in Section 11.1. To determine the input productivity $\beta_1(\tau)$ and price $\lambda(\tau)$ of machines depending upon the year τ of their creation, the diagrams of the development of certain types of machines were constructed and analyzed. Tractors were considered as major mobile power agricultural machines. Such diagrams is given in Table 11.1 for the tractors with force 30 kN (prices are as of 1990). The information about operated tractors is represented in Table 11.2.

Table 14.3 The expenditure dependence of turnpike lifetime L

Expenditure q (rbl)	3000	5000	8000	10000	12000
Lifetime $\tilde{L}$ (years)	19	14	10	8	6

Table 14.4 The price rate dependence of turnpike lifetime $\tilde{L}$

Price rate λ'	0.1×(Real value)	Real value	10×(Real value)
Lifetime $\tilde{L}$	3 years	10 years	20 years

By determining the weighted-mean productivity $\beta_1(\tau)$ and price $\lambda(\tau)$ of all operated tractors, we obtain the following mean square relations:

$$\beta_1(\tau) = 2.42\tau - 99, \quad \lambda(\tau) = 277.8\tau - 17374, \quad t \in [1970, 1991].$$

Optimization problem: the minimization (11.3)-(11.7) of operating expenditures over entire planning period for implementation of specified schedule of agricultural operations.

Turnpike analysis. Turnpike approach allows us to decrease significantly the amount of input information. By the algorithm of Section 13.5, at the above $\beta_1(\tau)$, $\lambda(\tau)$ and constant specific operating expenditure ($q(\tau) = 8000$ rbl/year/tractor) the turnpike lifetime of tractors is approximately constant :

$$\tilde{L}(t) \approx 10 \text{ years} \quad \text{for} \quad t \in [1985, 1991]$$

(these data were close to the truth).

As would be expected, the turnpike lifetime $\tilde{L}$ decreases at increasing expenditure $q(\tau)$ (Table 11.3) and increases at increasing the growth rate $d\lambda/dt$ of tractor price λ (Table 11.4).

14.2.4 Modeling of Environmental Impact

Modeled process: functioning of the bus transport system of Ukraine with accounting for its environmental impact (penalties for air pollution) in 1965-2005.

Used model: the one-sector IDM (11.1)-(11.2) involving the balance equation for air pollution.

Identification of the model was carried out by a modification of the algorithm of Section 14.1.2. Additional input data on the prehistory [1965,1994] were found in technical and ecological literature.

Optimization problem: the two-level problem (11.1)-(11.7) with imposing penalties for environmental pollution.

Turnpike analysis. As was shown in Chapter 12, in the case of linear τ-dependencies of EU productivity $\beta(\tau)$ and specific pollution $r(\tau)$ (per one EU), the turnpike lifetime $\tilde{L}(t)$ of equipment is represented by a combination of two constant values $\tilde{L}_1$ and $\tilde{L}_2$ on the intervals $[t_0, t_1]$ and $[t_1, T]$ such that permissible pollutant level R_{max} is exceeded on $[t_0, t_1]$ and not exceeded on $[t_1, T]$.

The multivariant computation of the turnpike lifetime $\tilde{L}(t)$ of buses was performed for prognosis period $[t_0, T]$=[1995,2005], for various values of the specified parameters such as the rates $\partial\beta/\partial\tau$ and $\partial r/\partial\tau$ of embodied TC, bus price $\lambda(t)$, the quantity $P(t)$ of buses in service, permissible pollutant level R_{max}, maximal possible penalty ρ_{max}, and so on. The Table 11.5 gives the parameters of turnpike lifetime $\tilde{L}(t)$ for the next computed variants.

In the basic (expected) variant the total air pollution $R(t)$ of bus transportation in Ukraine exceeds the permissible level R_{max}, and turnpike lifetime $\tilde{L}(t) = \tilde{L}_1 \approx 9$ years during 1995-1998. Optimal lifetime goes into turnpike trajectory $\tilde{L}(t)$ at t_2=1997, so, putting buses into operation is maximal : $m(t)$=4000 buses/year for t <1997. In 1999-2005 the turnpike lifetime of buses is $\tilde{L}(t) = \tilde{L}_2 \approx 11$ years, and putting into operation is $m(t)$ =3500 buses/year.

The second variant considers the doubled rate of embodied TC. Then, as above, the turnpike switching point is t_1 =1997, but optimal lifetime has no time to go into turnpike $\tilde{L}(t)$ before 1998 (t_2 =1998). The turnpike lifetime of buses is $\tilde{L}_1 \approx 8$ years in 1995-1997 and $\tilde{L}_2 \approx 10$ years in 1998-2005.

Table 14.5 The parameters of turnpike lifetime L in two-level control problem (11.1)-(11.7)

Variant	**Lifetime $\tilde{L}_1$** (under penalties)	**Lifetime $\tilde{L}_2$** (without penalties)	**Switching time t_1**	**Switching time t_2**
I - Basic	9 years	11 years	1998	1997
II - Doubled TC rate	8 years	10 years	1997	1998
III - Raised EU price	10 years	13 years	2001	1999

The third variant reflects the increase of the bus price $\lambda(t)$ by 50 percent. Then the period of exceeding permissible pollution is greater: t_1 =2001, and the turnpike lifetime is larger: $\tilde{L}_1 \approx 10$ years in 1995-2001 and $\tilde{L}_2 \approx 13$ years in 2002-2005. Indeed, the maximal renovation of equipment is not profitable under the growth of equipment cost.

14.3 THE PROCESSING OF ECONOMIC INFORMATION IN INTEGRAL MODELS

Beginning in 1980, different computer means for the simulation of economic development and technological renovation on the basis of the IDMs under consideration have been developed (see [46], [47], [66], [146]). Corresponding software has been implemented in Fortran, C, Pascal for large computers IBM 360 as well as for personal computers AT286, AT386, etc.

Here we expose a general scheme of information processing in the software packages based on the IDMs (Fig. 14.1). It connects the basic economic indices discussed in Section 14.1.

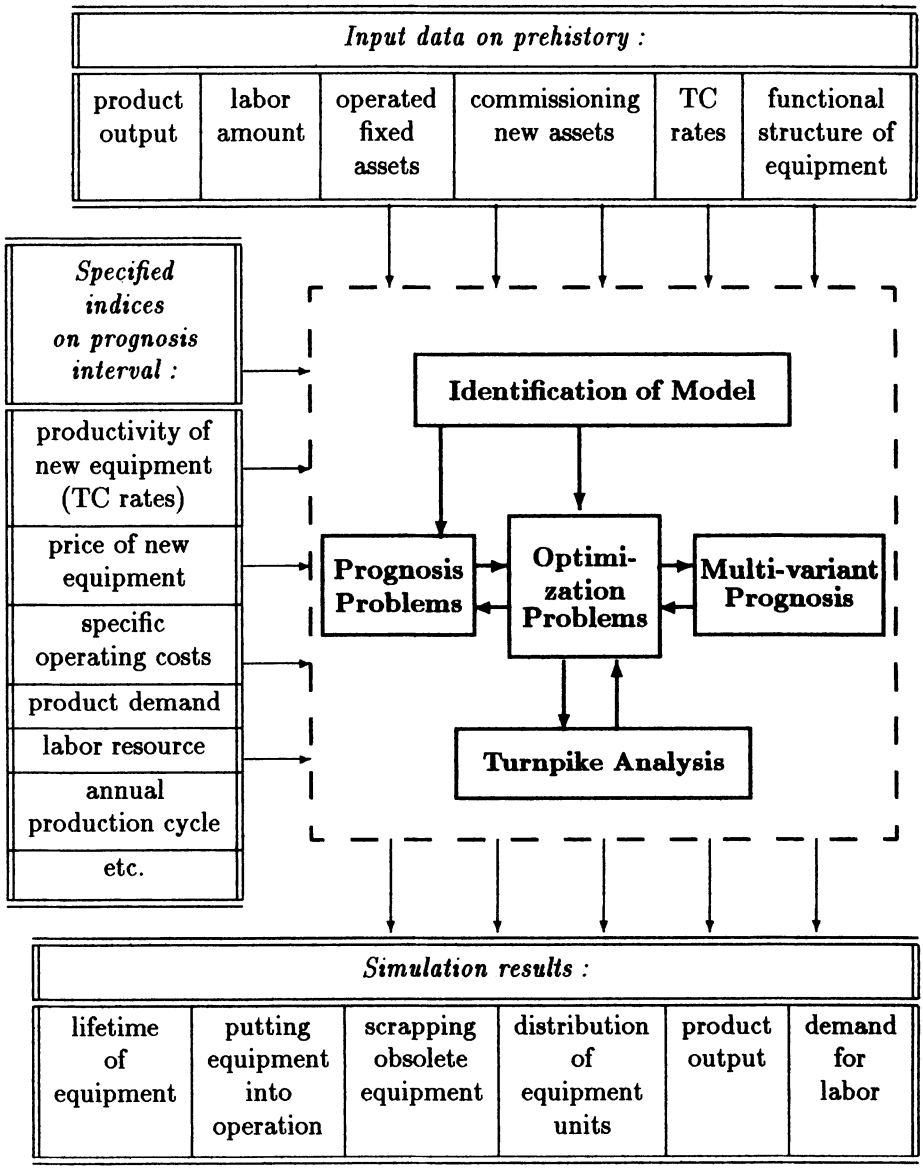

Figure 14.1 The general scheme of information processing in the integral models of technological renovation

15

OPEN PROBLEMS AND PERSPECTIVES OF INTEGRAL MODELS

It is worth noticing that the results obtained in this monograph represent only the first step in the modeling of rational lifetime of technologies in evolutionary economics undergoing embodied technological change (TC). They are valid for large planned intervals only if future TC course is completely known and stochastic effects are absent.

15.1 CONNECTION WITH THE MODELS OF TECHNOLOGICAL INNOVATIONS

Present-day study of TC is mainly connected with the mechanisms of TC implementation (innovation process). Technological innovations have been investigated by means of substitution, diffusion, and evolution models, each of them dealing with different aspects of the innovation process [20].

By means of *substitution models* the innovation process is investigated as a substitution operation, i.e. the replacement of particular product or process associated with one technology by a product or process associated with a new technology. Such simple models are similar to those used in the modeling of biological populations. So, one of the most intensively used substitution models [41] leads to Verhulst-Pearl logistic model $df/dt = kf(1-f)$ (see also Section 3.1.1). Another approach based on the Lotka-Volterra equations allows describing the process of technologies substitution in terms of competition between technologies.

Diffusion models describe the innovation process in terms of spread of a new product or technology from its manufacturers to ultimate users or adopters. These models focus on the diffusion as well as on the adoption aspect of the innovation problem. Diffusion processes are usually described as the relation between interacting adopting populations by means of two-dimensional diffusion equations (see Chapter 3).

The majority of substitution and diffusion models is deterministic. In a deterministic model the intrinsic random character of the innovation process cannot be described. To correct this, stochastic evolution models of competitive economics have been introduced [20], [32], [38], [122], [124], [125], [131]. The evolution aspect of the innovation problem involves its description as a structural fluctuation that leads to *self-organization evolutionary models.* The self-organization models have first arisen in the study of biological evolution.

The evolution approach is the major avenue of TC investigation and turns to be very fruitful (see, e.g., [20], [38], [122], [124], [125], and the references therein). Within the framework of this approach, significant results have been established for evolutionary economics (the existence of an evolutionary long waves, low-dimensional chaotic properties, etc.). The subject of the present book is intimately connected with innovation process, then, it is wise to analyze the relationship between these two lines of investigation.

Like evolutionary models of TC [20], [125], the considered IDMs distinguish macroeconomic (system) level and the microeconomic (plant) level of technology renovation. But, both these levels are simplified as compared to the evolutionary models, in so doing, the lifetime of technologies is emphasized. On the microeconomic level, technology renovation is understood as the instantaneous complete replacement of old existing technology by new one. The macroeconomic level in the IDMs reflects ordinary assumptions about functioning an economy, which are commonly accepted in traditional aggregate economic models (see Section 0.1). Such simplification makes it possible to derive direct relationships for optimal intensity of the technology replacement.

New interesting results may be reached by including an IDM block into self-organization evolution models. Further development of the IDMs will be focused on considering such important TC features as innovation, competition, infection, and so on.

On the other side, very similar model for optimal replacement of capital equipment of a separate firm was proposed by J.M.Malcomson in [96]. [1] The problem studied in [96] consists in discounted maximization of the firm profit (net revenue) over an infinite time horizon, which includes the sold output (at a current price), the operating cost on all machines in use, and the cost of purchasing new machines under the given output. As the above IDMs, this optimization problem takes account of the TC embodied in new equipment (in new vintages of operated machines).

In [96] an integral-functional equation (IFE) for optimal lifetime of equipment analogous to IFEs of Chapters 5 and 6 was established and its extended economic analysis was provided. In particular, expressions for the implicit rental value of capital equipment and for optimal scrapping criteria were derived. Under specific assumptions J.M.Malcomson showed that his equation allows him to derive the optimal replacement formulas stated in earlier works of Terborgh [134], Smith [127], and Brems [17]. It seems likely that J.M.Malcomson had derived and analyzed a dual problem (adjoint equation) for the optimization of equipment lifetime in the models with embodied TC for the first time in mathematical economics.

No rigorous analytical investigation of the mentioned IFE for optimal equipment lifetime was performed in [96], and no asymptotical (including, turnpike) properties of the optimal lifetime were found in an analytical way, although some convergence of the upper and lower bounds on the optimal equipment lifetime was checked in [95] by means of numerical experiments. As noted in [96], "this convergence has not yet been confirmed or denied analytically for the models concerned". The turnpike theorems established in Chapters 5 and 6 of the present monograph for different optimization problems in one-sector integral models may be considered as an analytical confirmation of this convergence.

Some important implications follow from the analysis provided in [96]:

- Scrapping and replacement decisions are independent of product market conditions (optimal lifetime of equipment does not depend on any of the variables which determine the scale of operations in any material way).

[1] This important reference was kingly indicated by professor Gerald Silverberg, unfortunately, after this book has been mainly completed.

- The equation for determining optimal lifetime is generalized to many types of capital goods for which an independent operating cost function can be determined.

These results have also been confirmed over the present book (see Chapters 5, 6, 8, 10, 11).

15.2 OPEN PROBLEMS

From the viewpoint of evolutionary self-organizing approach it is worthwhile to study the collective behavior of firms in the frame of certain integral model of artificial economics [124], [125] in which a number of firms will use some supposed turnpike decision-rules for scrapping obsolete equipment and the rest – other scrapping politics. It is of great interest to estimate the time interval on which the first firms will have some advantages as compared with the rest. This problem is of importance in the cases when the future TC course is completely known as well as when it can be approximated only and is subjected to various stochastic perturbations.

The estimation of future TC rate represents an important separate problem and has been investigated by many scientists. This rate is influenced by innovation strategy and R&D investment that requires using the models of endogenous TC [125].

As mentioned in [20], [125], many phenomena in innovation problems cannot be described in a deterministic picture. The methods of construction, analytical and numerical study of the IDMs with accounting for stochastic effects are among the most interesting unsolved questions. Such models play a special part for modeling of market economics. In particular, the self-organization models of TC must represent the diffusion of new technologies under the conditions of uncertainty of market as a disequilibrium stochastic process.

Some interesting conclusions can also be reached by combining the Malconson model [96] and the studied IDMs, for example:

– take account of the influence of changing product price on firm profit;

– introduce and analyze the rental value of capital equipment and the marginal cost of producing output;

– analyze different dynamics of optimal equipment lifetime arising in various optimization problems as compared with the Malcomson model and Terborgh formula.

Next important problem is to continue numerical experiments for real economic data to simulate rational technology lifetime strategies. For this purpose it is required to develop the endogenous integral models with more detailed description of the control of scrapping of obsolete technologies (equipment).

15.3 SOME PERSPECTIVE APPLICATIONS OF INTEGRAL MODELS

The bulk of applications of the IDMs with controllable memory is modeling of evolutionary economics, however similar models also occur in fluid mechanics when investigating media with moving surfaces, in ecology when studying the variable life span or reproductive age, in engineering applications, e.g., for variable transient time in technical devices, and so on. From general system-theoretic point of view, these IDMs are similar to those used in renewal theory.

Renewal Models

Renewal theory arose from the study of so-called random "self- renewing" processes and has been the bright example of the importance of applying integral equations (see [2], [77], [135], etc.). The basic renewal equation is the linear Volterra integral equation (0.11) with respect to sought-for $\mathbf{x}(.)$. Compared with the renewal models, the considered IDMs permit the investigation of essentially non-stationary processes with variable renewal intervals $[a(t), t]$. The renewal models are of interest for the IDMs under study from the viewpoint of well-developed technique for accounting of random variables. It is an important open problem of the IDMs with controllable memory.

Replacement Models

As has been shown in Section 2.7, the IDMs can describe the replacement of equipment in production systems. Classical replacement models [15], [121] are also described by means of Volterra integral equations. They may be consid-

ered as an application of the renewal models. The important problem of the replacement models is to find the optimal time interval T of replacement or repair. For this purpose the IDMs with controllable memory make it possible to determine the optimal T as a dynamic function $T(t)$.

Inventory Models

As the above IDMs, inventory models take into consideration the deterioration (decay, degradation) and fixed lifetime of items [113], [115]. Different concepts and mathematical models of deterioration have been treated in detail in the framework of inventory theory that may be profitable for the IDMs under consideration. One of the key problems of inventory models consists in the determination of *optimal cycle time* T. By employing the IDMs the cycle time may be considered as a function of the time of item creation for variable deterioration rate.

Mathematical Ecology

IDMs find ever increasing use in the mathematical simulation of complex biological systems including ecological systems. Such models appear in mathematical ecology when the influence of the age structure and process prehistory upon population dynamics (hereditary effects) is considered (see Chapter 3). For similar problems the IDMs with controllable memory may hold considerable promise. They can incorporate a variable limiting age of individuals, a variation in the intensity of dying off and migration, and so on. Biological objects are natural self-regulating systems, and internal regulation mechanisms are usually nonlinear ,then, the IDMs with nonlinear efficiency indices (with nonlinear integrands) play a special part in biological applications. In whole, a deep analogy between biological and technological processes exists, which is useful for modeling of technological change as well as corresponding biological systems.

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