

---

# An Asset Allocation Primer: Connecting Markowitz, Kelly and Risk Parity

---

**AUTHORS**

Jamil Baz  
Managing Director  
Global Head of Client Analytics

Helen Guo  
Vice President  
Quantitative Research Analyst

*As curious as it may sound, few asset allocation primers describe portfolio models while connecting them to each other. This article does so by describing and contrasting the mechanics of standard asset allocation models, including the utility-based, Markowitz, Kelly, risk parity and fixed allocation approaches. It seeks to accomplish four objectives: First, it develops the mathematics of the models; second, it identifies the precise conditions under which the models are equivalent; third, it discusses the specifics – risk/return characteristics – of each approach; and last, it provides numerical examples that compare allocations to asset classes and return and risk measures for various models.*

The main points are these:

- Utility-based asset allocation models are the most general class of models.
- For some specific utility functions and return distributions, the Markowitz approach is a special example of a utility-based model.
- The Kelly framework can, in turn, be a special example of the Markowitz model, for a specific risk-aversion coefficient.
- Under some conditions on correlations and Sharpe ratios, risk parity is optimal and collapses into a Markowitz model.
- Kelly strategies present unique characteristics as they outperform alternative strategies almost surely over long time periods.
- These unique characteristics come at the cost of generally extreme risk parameters.

- Despite a lack of theoretical foundations, risk parity strategies have been partly fueled by their empirical success. One may call this luck.
- Numerical examples show a wide disparity across strategies.

## 1. THE UBIQUITOUS MEAN-VARIANCE ANALYSIS

Expected utility maximization and mean-variance analysis (Markowitz 1952, 1959) are two prevailing approaches to asset allocation. Most academic researchers prefer the expected utility maximization approach to model choices under uncertainty because the solutions will satisfy certain basic conditions of rationality (von Neumann and Morgenstern 1944). It is often the ultimate standard for evaluating the optimality of a model or ranking different models.

This section discusses the close connection between the two approaches and shows their convergence in a continuous-time power utility maximization problem with constant investment opportunities. In addition, we identify the assumptions under which the optimal solution coincides with the Kelly (growth-optimal) strategy or the risk parity strategy. We will discuss these two strategies in more detail in Section 2.

### 1.1 Mean-variance analysis

The mean-variance analysis assumes investors only care about the first two moments of portfolio returns and maximize the mean-variance utility function<sup>1</sup>

$$E(r_p) - \frac{1}{2}\lambda V(r_p),$$

where  $\lambda$  measures the investor's risk aversion. This approach is easy to implement and understand. It is by far the most prominent asset allocation model for practitioners and underlies most of the products and advice provided by financial institutions.

For single-period portfolio optimizations, mean-variance analysis is consistent with the theoretically popular expected utility maximization framework if the investor's utility is quadratic or if returns are normally distributed. Even if the utility function is not quadratic, Levy and Markowitz (1979) showed that mean-variance optimization is equivalent to maximizing the expectation of the second-order Taylor approximations of standard utility functions, such as the power utility and the exponential utility. Thus, the consistency of the

mean-variance analysis with expected utility maximization for single-period optimization problems depends on the degree of non-normality of returns, the investment horizon and the specific functional form of the investor's utility.

For multiperiod portfolio optimizations, maximizing the commonly preferred constant relative risk aversion (CRRA) utility with continuous rebalancing<sup>2</sup> and lognormal prices will lead to mean-variance optimal (MVO) solutions, as discussed below.

The CRRA utility (also known as the power utility or isoelastic utility) has the following functional form:

$$U(W) = \frac{W^{1-\gamma}-1}{1-\gamma} \quad (\gamma > 0),$$

where  $\gamma$  measures the constant relative risk aversion of the investor. In the extreme case where  $\gamma=0$ , the investor is risk neutral. When  $\gamma=1$ , the investor has a log utility.<sup>3</sup> Empirical studies and surveys suggest most investors' relative risk aversion to wealth is between 1 and 10 (Ang 2014). The most commonly accepted values for asset allocation are between 1 and 5 and centered around 3.

A nice feature of the CRRA utility is that it leads to optimal portfolio allocations not dependent on the level of wealth. This wealth homogeneity property is important for the scalability of investment management because it allows the same model to be used for different sizes of assets.

The investor with wealth  $W_0$  at time 0 maximizes the expected utility of his or her final wealth at time  $T$  by choosing the optimal portfolio weights in  $N$  risky asset(s) and a risk-free asset with continuous rebalancing. For simplicity, assume the investor has no labor income or intermediate consumptions and there are no transaction costs.

Let's start with a simple case with only one risky asset. Assume the risky asset's price follows a standard geometric Brownian motion with parameters  $(\mu, \sigma^2)$  and the risk-free asset has a geometric growth rate of  $r$ . Assume asset returns are independent and identically distributed (i.i.d.) over time.

With the standard intertemporal budget constraint and Itô's lemma, it can be shown (see Appendix 2) that the optimal allocation to the risky asset,  $x^*$ , is the solution to the following problem:

$$\max_x \left[ r + x(\mu - r) - \frac{1}{2}\gamma x^2 \sigma^2 \right] T.$$

Therefore, the solution of the optimal weight to the risky asset is

$$x^* = \frac{\mu - r}{\gamma \sigma^2} = \frac{s}{\gamma \sigma},$$

where  $s$  is the instantaneous Sharpe ratio of the risky asset.

Note that the immediate objective function (the instantaneous integrand) and the optimal solution resemble those under the mean-variance optimization framework where the risk aversion parameter of the CRRA utility function  $\gamma$  coincides with the risk aversion parameter for mean-variance optimization  $\lambda$ . It is the continuous-time instantaneous equivalent of the static single-period Markowitz model. The optimal allocation to the risky asset is directly proportional to the excess return of the risky asset and inversely proportional to its variance or the investor's relative risk aversion.

It is straightforward to extend the analysis to the cases where there are two or more risky assets following a multivariate geometric Brownian motion with parameters  $(\mu, \Sigma)$ . The optimal allocation to the  $N$  risky assets in this case is:

$$x^* = \frac{1}{\gamma} \Sigma^{-1} (\mu - r).$$

The composition of the subportfolio of risky assets does not depend on the investor's risk aversion. It also achieves the maximum possible Sharpe ratio given the investment universe. The risk aversion parameter  $\gamma$  only affects the total weight in risky assets and the weight in the risk-free asset. This is similar to the two-fund separation theorem in the traditional mean-variance analysis.

## 1.2 Kelly and risk parity as special cases

It is easy to see that the Kelly strategy is a special case of the optimal solution to the CRRA utility maximization problem, with a specific value for the risk aversion parameter. If  $\gamma=1$ , the CRRA utility is reduced to log utility. The solution will maximize the expected geometric growth rate of the portfolio and therefore is often referred to as the optimal growth portfolio (or the Kelly portfolio).

It is less straightforward to find the conditions under which the optimal solution will be a risk parity portfolio. Let's start from the simple case where  $N=2$ . We can write the optimal weights to the two risky assets as functions of their volatilities, Sharpe ratios and correlation  $\rho$ :

$$x^* = \frac{1}{\gamma(1-\rho^2)} \begin{pmatrix} \frac{1}{\sigma_1} (s_1 - \rho s_2) \\ \frac{1}{\sigma_2} (s_2 - \rho s_1) \end{pmatrix}.$$

If the two risky assets have the same Sharpe ratios, the optimal risky asset portfolio is also a risk parity portfolio in the sense of both equal stand-alone risks ( $x_1 \sigma_1 = x_2 \sigma_2$ ) and equal risk contributions ( $x_1^2 \sigma_1^2 + x_1 x_2 \rho \sigma_1 \sigma_2 = x_2^2 \sigma_2^2 + x_1 x_2 \rho \sigma_1 \sigma_2$ ) where risk is measured by volatility.

When  $N > 2$ , if the assets have the same Sharpe ratios and the correlations are constant, the optimal allocation will still be a risk parity portfolio (see Appendix 3).

## 2. MORE ON KELLY AND RISK PARITY

The Kelly strategy and the risk parity strategy are two popular alternative asset allocation models in the investment industry. In Section 1, we show the conditions under which the solution to a CRRA utility maximization problem coincides with these two strategies, respectively. In this section, we discuss the features of the two models in more detail.

### 2.1 The Kelly (growth-optimal) strategy

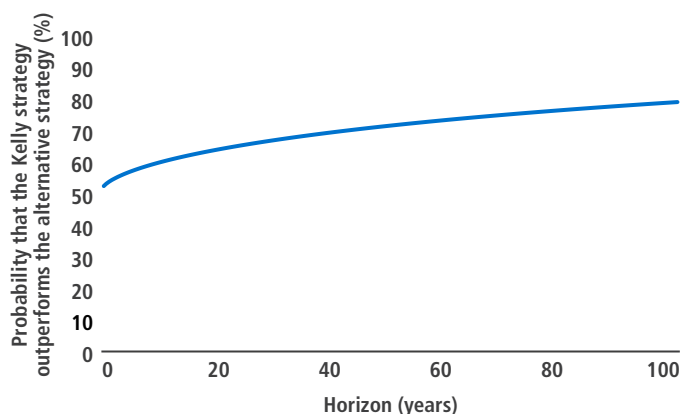
The Kelly (or growth-optimal) strategy maximizes the expected geometric growth rate of a portfolio. This is equivalent to maximizing the expected logarithm of final wealth. Therefore, the optimal dynamic portfolio choice is myopic. The Kelly strategy allows the breakdown of a multiperiod optimization problem into single-period problems that are easy to solve.

In addition to its convenience of implementation, the Kelly strategy has a few other impressive properties. For example, it minimizes the expected time to reach a given wealth target. In the long run, the Kelly strategy asymptotically dominates all essentially different strategies (Breiman 1961).

Another interesting property of the Kelly strategy is its competitive optimality (Bell and Cover 1980). Denote an investor's wealth streams under the Kelly strategy and under any other alternative strategy by  $W_t^K$  and  $W_t^A$ , respectively. It can be shown that  $E(\frac{W_t^A}{W_t^K}) \leq 1$  for any  $t$ . It means the ratio of the investor's wealth associated with any competing strategy to the wealth associated with the Kelly strategy is expected to be less than 1 for any horizon.

In our previous example of a single risky asset and continuous rebalancing, assume the risky asset has an annual risk premium of 3% and an annual volatility of 15%. Consider the Kelly strategy ( $x=1.3$ ) and an alternative fixed mix strategy where  $x=0.2$ . Exhibit 1 shows the probability that the Kelly strategy will outperform the alternative strategy over time. In fact, this probability is always higher than 50% for any alternative fixed mix strategy and converges to 100% as the investment horizon goes to infinity (see Appendix 4).

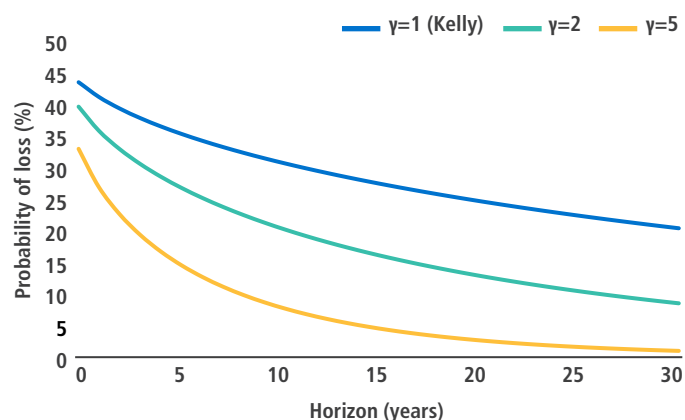
**Exhibit 1: Probability that the Kelly strategy will outperform another fixed mix strategy**



Source: PIMCO. Hypothetical example for illustration purposes only. See Appendix 4 for more details.

The main disadvantage of the Kelly strategy is that it can be very risky in the short run. This is consistent with the fact that the log utility is a special case of the CRRA utility when the risk aversion parameter  $\gamma$  is equal to 1, the lower bound of its plausible values. Consider the same simple example with one risky asset, as in Exhibit 1, where  $r = 1\%$ ,  $\mu = 4\%$  and  $\sigma = 15\%$ . Exhibit 2 shows the probability of loss for the optimal portfolio over time under different risk aversion parameters. The probability of loss is decreasing in both  $T$  and  $\gamma$ . The Kelly strategy ( $\gamma=1$ ) has a significantly higher risk of loss than the other optimal strategies, with  $\gamma=2$  and  $\gamma=5$ .

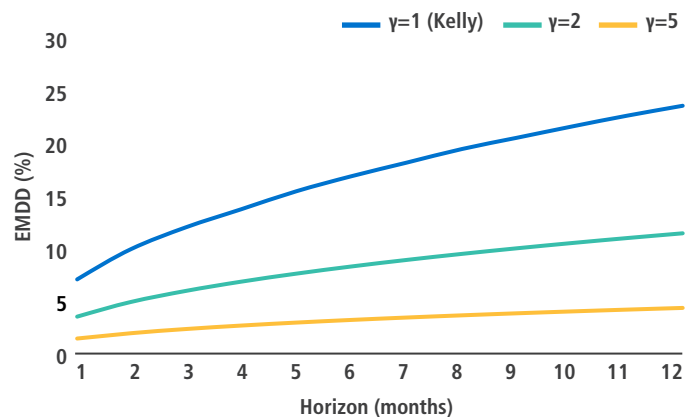
**Exhibit 2: Probability of loss for optimal strategies with different risk aversion parameters**



Source: PIMCO. Hypothetical example for illustration purposes only. See Appendix 2 (Case 1) for more details.

Another risk measure in which an investor might be interested is expected maximum drawdown (EMDD), which measures the expected maximum percentage loss from peak to trough within a specific investment horizon. Exhibit 3 plots EMDD for optimal strategies with different risk aversion parameters over horizons up to 12 months, based on the same parameter assumptions as in the previous exhibits. For any given horizon, EMDD is decreasing in the risk aversion parameter, which means the Kelly strategy is the riskiest based on this measure.

**Exhibit 3: EMDD for optimal strategies with different risk aversion parameters**



Source: PIMCO. Hypothetical example for illustration purposes only. See Appendix 5 for more details.

In addition, the minimal risk aversion implies the optimal portfolio choice is more sensitive to estimation errors for the return distributions because the weight in the risky assets is inversely proportional to  $\gamma$ . In practice, Kelly proponents often apply a fractional Kelly strategy (investing a fixed fraction of money in the Kelly strategy and the rest in cash) to reduce risk and protect against estimation errors. However, in the CRRA utility maximization problem discussed previously,  $x^* = \frac{1}{\gamma} \Sigma^{-1}(\mu - r)$ , meaning the risk aversion parameter  $\gamma$  only affects the weights in the risky assets versus cash, not the composition of the risky subportfolio. Therefore, applying a fractional Kelly strategy is equivalent to using a  $\gamma$  higher than 1 in the CRRA utility maximization/mean-variance optimization problem. A fractional Kelly strategy is simply a mean-variance solution with a potentially more realistic risk aversion parameter.

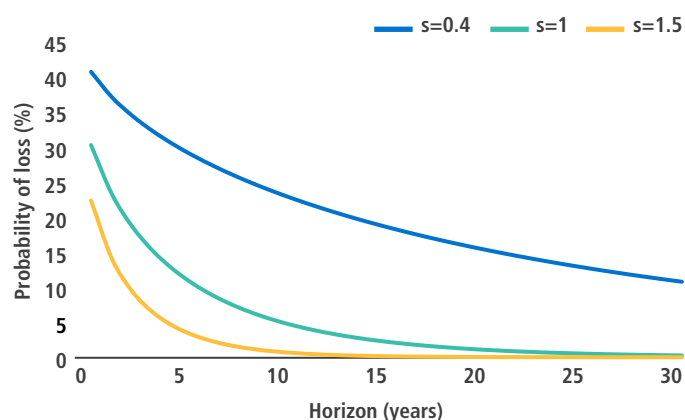
Although Kelly dominates other strategies in the long run, it can take a very long time for this to happen. In the example shown in Exhibit 1, it would take 227 years to have a 90% confidence that the Kelly strategy ( $x=1.3$ ) will outperform the alternative fixed mix strategy with  $x=0.2$ .

Next we compare Kelly strategies under different investment opportunities by fixing  $\gamma=1$  and changing the instantaneous Sharpe ratio of the risky asset. As shown in Appendix 2 (Case 1), the value of the Kelly portfolio follows:

$$\ln\left(\frac{W_T}{W_0}\right) = (r + \frac{1}{2}s^2)T + s\sqrt{T}Z, \text{ where } Z \sim N(0,1).$$

This means the annual volatility of the Kelly strategy is equal to  $s$ , the instantaneous Sharpe ratio of the risky asset. Therefore, the volatility of the Kelly strategy is higher when the investment opportunity is better. However, higher volatility does not necessarily mean a higher probability of loss, because the expected return of the Kelly strategy is also higher, with a higher Sharpe ratio of the risky asset. A better investment opportunity is a mixed blessing for Kelly strategies in terms of probability of loss. Appendix 2 shows the probability of loss for a Kelly strategy is increasing in  $s$  if  $s^2 < 2r$  and decreasing in  $s$  if  $s^2 > 2r$ . Given the current low-interest-rate environment, an improved Sharpe ratio most likely implies a lower probability of loss, as illustrated in Exhibit 4, with the assumption that  $r = 1\%$ .

**Exhibit 4: Probability of loss for Kelly strategies under different investment opportunities**



Source: PIMCO. Hypothetical example for illustration purposes only. See Appendix 2 for more details.

Ziemba (2015) characterized Kelly portfolios in reality as having high concentrations in very few investments, frequent and big short-term losses and superior long-term growth, and provided examples of highly successful investors who are known to follow or likely follow the Kelly strategy, based on observed characteristics of their investments. If the quality of the inputs (estimated return distributions – especially the means) is high and the investment horizon is long enough to allow compounding of many rebalancing periods, the Kelly strategy can be a powerful asset allocation and risk management tool that potentially maximizes long-term wealth growth. However, for short-term investors who have more risk aversion than that implied by log utility, the Kelly strategy tends to result in more risk than they might be willing to take under general investment conditions. It should be considered with great caution.

### 2.1.1 Volatility pumping and portfolio growth

Volatility pumping is a strategy or mechanism often associated with growth-optimal portfolios. Luenberger (1998) coined the term and described how rebalancing a portfolio to fixed weights can generate a higher expected growth rate than the weighted average of the expected growth rates of the individual assets. He attributed the excess growth to the pumping action that automatically “buys low and sells high” through the process of rebalancing,<sup>4</sup> even when returns are intertemporally independent. The implications are that volatility is opportunity and investors should seek it out rather than shunning it.

We believe the primary source of the excess growth rate of the portfolio in the case of i.i.d. returns is not rebalancing itself but variance reduction relative to the mean. To see this, we need to understand the relationship between the expected log returns (geometric growth rates) and expected arithmetic returns:

$$E(\ln(1+r)) \approx E(r) - \frac{1}{2(1+E(r))^2} \text{Var}(r).$$

This says an asset's expected log return is lower than its expected arithmetic return by approximately half of the variance of the arithmetic return. This relationship holds exactly in continuous time for assets following geometric Brownian motions and approximately in short discrete-time intervals or for assets following other distributions (see Appendix 6).

A popular and simple example of volatility pumping is to generate a positive expected geometric growth rate for a portfolio with two assets that have zero expected geometric growth rates. One asset is risky and follows a standard geometric Brownian motion with parameters  $(\mu, \sigma^2)$ . By construction, we have  $\mu - \frac{1}{2}\sigma^2 = 0$  (for example, 2% mean and 20% volatility). The other asset is risk-free with a zero growth rate. Then the value of a continuously rebalanced portfolio with fixed weight  $x$  in the risky asset and  $1-x$  in the risk-free asset will follow a geometric Brownian motion with parameters  $(x\mu, x^2\sigma^2)$ . Therefore, the expected geometric growth rate of the portfolio is:

$$x\mu - \frac{1}{2}x^2\sigma^2 = x\left(\mu - \frac{1}{2}\sigma^2\right) + \frac{1}{2}x(1-x)\sigma^2 = \frac{1}{2}x(1-x)\sigma^2.$$

Therefore, the expected excess geometric growth rate of the portfolio is positive for any  $x \in (0,1)$  and is maximized at  $\frac{1}{8}\sigma^2$  when  $x=0.5$  under the Kelly strategy in this simple two-asset example. A Kelly strategy is not required to achieve a positive excess geometric growth rate for the portfolio.

The excess geometric growth rate is increasing in the volatility of the risky asset; this is sometimes used as evidence that investors should seek out volatile assets to gain an excess growth rate by applying a volatility pumping strategy. However, even without rebalancing, the excess growth rate can still be achieved for relatively short investment horizons because the relationship between expected log return and expected arithmetic return still holds, approximately, in this case. Therefore, the excess growth

rate is not the direct result of rebalancing itself, but a result of variance reduction more than the mean reduction when the risky asset is combined with the risk-free asset with weights between 0 and 1 (the mean is scaled by  $x$ , and the variance is scaled by  $x^2$ ). Even though rebalancing involves selling the risky asset after its price goes up and buying it after its price goes down, it does not redistribute wealth to the asset with better future expected growth because returns are intertemporally independent and both assets have zero expected growth.

For longer horizons without rebalancing (i.e., a long-term buy-and-hold strategy), the portfolio weights will drift away from the optimal value over time and therefore reduce the expected growth rate of the portfolio. The main contribution of rebalancing to portfolio growth is to keep the weights at the optimal level that maximizes portfolio growth over time rather than buying low and selling high, unless the returns exhibit certain intertemporal dependence, such as mean reversion.

### 2.1.2 The growth efficient frontier

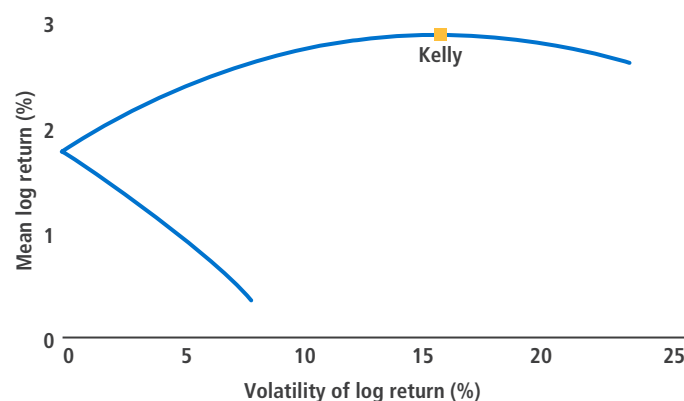
We have shown that the Kelly strategy maximizes the expected log return or geometric growth rate of the portfolio. Although the strategy dominates other strategies in the long run, it is often criticized for being too risky in the short run because it does not take into account the volatility of the log return or the geometric growth rate.

In practice, many long-term investors do care about the volatility of the log return in addition to the mean log return. Luenberger (1993) proposed that an investor who considers only long-term performance will evaluate a portfolio only on the basis of the mean and variance of its single-period log return given i.i.d. investment opportunities over time.

It is natural to ask whether we can have risk/return trade-off where returns are measured in terms of log returns, similar to the simple risk/return trade-off in the traditional mean-variance analysis. Exhibit 5 plots such a growth efficient frontier based on the model assumptions for the numerical example in Section 3.



**Exhibit 5: Growth efficient frontier for three risky assets and a risk-free asset**



Source: PIMCO

Compared with the traditional mean-variance efficient frontier with a risk-free asset, which is a straight line passing the risk-free asset and the maximum Sharpe ratio portfolio, the growth efficient frontier is a curve with a maximum mean log return point, which is the Kelly strategy. The frontier is very flat near the maximum point, meaning that reducing the mean log return slightly can greatly reduce its volatility.

In the CRRA utility maximization problem with assets following geometric Brownian motion presented above, all the portfolios on the growth efficient frontier can be constructed by mixing the risk-free asset and the log-optimal portfolio. Under this model setup, fractional Kelly is efficient.

## 2.2 Risk parity

Risk-based portfolio construction techniques such as risk parity have gained great popularity among investors and academics alike since the 2008 global financial crisis. The demand stems from the higher risk aversion or awareness of investors, as well as the lack of robustness of the traditional mean-variance optimal portfolio due to its sensitivity to the estimation errors in its inputs, especially in expected returns. For risk parity and similar risk-based approaches, such as minimum-variance portfolios, the only input required is the risk estimation, which is generally believed to be more robust than return forecasts (see, for example, Merton 1980).

Despite the vast and still-growing literature on risk parity, there is no consensus on the exact definition of a risk parity portfolio. The general philosophy is to equalize risk from different portfolio components. The risk can be the components' stand-alone volatilities, ignoring correlations among component returns or their contributions to portfolio volatility. An equal stand-alone risk (ESR) portfolio equalizes the products of the portfolio weights in the components and the volatilities of those components. Therefore, the risk parity weights are inversely proportional to the component return volatilities:  $x_i = \frac{\sigma_i^{-1}}{\sum_j \sigma_j^{-1}}$ . An equal risk contribution (ERC) portfolio equalizes the portfolio volatility contribution from each component:  $RC_i = x_i \frac{(\Sigma x)_i}{\sqrt{x^T \Sigma x}}$ , where  $x$  is the vector of portfolio weights in the components, the subscript represents the  $i$ th element of a vector and  $\Sigma$  is the covariance matrix of component returns. An alternative interpretation of the ERC portfolio, proposed by Lee (2011), is that the components' weights are inversely proportional to their betas with respect to the portfolio. The ERC portfolio generally has no analytical solutions and has to be solved for numerically. In the special case of constant pairwise correlations, the two portfolios coincide.

The portfolio components can be defined based on the asset classes, securities or risk factors that are the underlying drivers of risk.<sup>5</sup> In addition, because there is no theory to guide these decisions, diverse risk parity strategies can be constructed by varying investment universe, grouping schemes and investment horizons.

The lack of a standard definition for risk parity has stymied efforts to construct a benchmark to evaluate the performance of different risk parity strategies. In fact, there has been a considerable disparity in the performance of risk parity funds. Investors need to exercise extra caution in choosing risk parity products: What they think they are getting may not be consistent with the investment objectives they want to achieve, due to the heterogeneity of the actual strategies implemented and the resulting performance divergence.

The increasing popularity of risk parity strategies has been fueled in part by their empirical success over the past few decades. However, history is only one realized path among many possible paths. As the familiar disclaimer says, "Past

performance is not a guarantee of future returns.” A classic point of contention in regard to risk parity is whether its historical success is simply driven by the long-lasting fixed income rally, which may not persist in the future.

As a heuristic, the risk parity strategy suffers from a lack of a solid theoretical foundation. As discussed previously, its mean-variance optimality depends on strong assumptions on the Sharpe ratios and the correlation matrix of the component returns. There have been some recent attempts to fill this gap. Asness, Frazzini and Pedersen (2012) tried to justify the superiority of risk parity using the leverage aversion theory,<sup>6</sup> which suggests low-beta assets offer higher risk-adjusted returns than high-beta assets due to the leverage aversion of average investors; an investor with lower leverage aversion than average can exploit the risk premium with risk parity portfolios due to their tilts toward safer assets relative to the market portfolio. In another empirical study, however, Anderson, Bianchi and Goldberg (2012) compared the historical performance of unlevered and levered risk parity strategies with 60/40 and value-weighted strategies during 1926-2010 and subperiods and showed that incorporating trading costs can negate the outperformance of risk parity strategies, especially with leverage. Fisher, Maymin and Maymin (2015) provided another perspective to justify risk parity for investors facing uncertainty in the expected return estimates. They described the exact conditions for the uncertainty set under which the minimum Sharpe ratio of the risk parity (defined as ESR) portfolio is higher than those of other portfolios, including the tangency portfolio.

In practice, the leverage required for risk parity portfolios to achieve a typical return or risk target poses potentially higher downside risk when the safer assets, such as credit, have negative skews. Leverage also can exacerbate turnover, leading

to much higher trading costs than those for unlevered strategies (Anderson, Bianchi and Goldberg 2012).

Although the implementation of risk parity does not require expected return forecasts, its performance is almost always evaluated based on both return and risk measures. If investors do not believe expected returns can be estimated reliably, are willing to assume the included portfolio components have equal long-term Sharpe ratios and pairwise correlations, and are willing and able to use leverage, risk parity can be a very efficient heuristic for strategic asset allocation. In the more extreme case in which investors also have very low confidence in volatility estimates, even a naive equally weighted portfolio can be a viable heuristic with decent performance (see, for example, DeMiguel, Garlappi and Uppal 2009, and Chaves et al. 2011).

### 3. NUMERICAL EXAMPLES

Here we present some simple examples related to the models discussed previously. For a more systematic discussion of practical quantitative approaches to asset allocation, we recommend Naik et al. (2016).

Suppose an investor faces the optimization problem described in Section 1. The hypothetical investment universe consists of four assets: equities, bonds, commodities and cash. Their i.i.d. investment opportunities are shown below (see Exhibit 6).

The mean-variance optimal weights to the risky assets are given by:  $x^* = \frac{1}{\gamma} \Sigma^{-1}(\mu - r)$ . The weight in cash is therefore  $1 - \mathbf{1}^T x^*$ , which can be either positive (lending) or negative (borrowing). The relative risk aversion parameter  $\gamma$  only affects the total weight in the risky assets and does not change the composition of the optimal subportfolio of risky assets. A risk parity portfolio equalizes the risk contributions from the three risky assets based on the covariance matrix only. For comparison, we also include a traditional portfolio that invests 60% in stocks and 40% in

#### Exhibit 6: Model assumptions

Asset	$\mu$	$\sigma$	Correlation matrix			Instantaneous Sharpe ratio
Equities	3.2%	15.0%	1			0.10
Bonds	2.2%	3.6%	-0.06	1		0.11
Commodities	2.2%	16.9%	0.34	-0.02	1	0.03
Cash	1.8%					

Source: PIMCO. **Hypothetical example for illustrative purposes only.** The distributional assumptions for different assets are based on PIMCO's 10-year capital market assumptions (model-based except for Commodities, which was based on an internal survey of portfolio managers) and long-term risk estimates as of July 2017.



### Exhibit 7: Unconstrained MVO, Kelly, unlevered risk parity and 60/40 portfolios

	Asset	CRRA utility maximization (MVO)			Risk parity	60/40
		$\gamma=5$	$\gamma=2$	$\gamma=1$ (Kelly)		
Asset allocation	Equities	14.3%	35.7%	71.4%	15.0%	60.0%
	Bonds	63.2%	158.1%	316.2%	72.1%	40.0%
	Commodities	-0.8%	-2.0%	-4.0%	12.9%	0.0%
	Cash	23.3%	-91.8%	-283.7%	0.0%	0.0%
Risk allocation	Equities	46.1%	46.1%	46.1%	33.3%	98.4%
	Bonds	54.7%	54.7%	54.7%	33.3%	1.6%
	Commodities	-0.8%	-0.8%	-0.8%	33.3%	0.0%
	Cash	0.0%	0.0%	0.0%	0.0%	0.0%
Estimated annual return		2.2%	2.9%	4.1%	2.3%	2.8%
Estimated annual volatility		3.0%	7.5%	15.1%	4.4%	9.0%
Sharpe ratio		0.15	0.15	0.15	0.13	0.11
Expected maximum drawdown (1 Year)		3.0%	8.4%	17.7%	4.6%	10.4%

Source: PIMCO. **Hypothetical example for illustrative purposes only.** Risk is measured by annual volatility. The MVO portfolios are unconstrained in the sense that there are no constraints on short positions in the assets, including cash, and the borrowing rate is assumed to be equal to the risk-free rate.

bonds.<sup>7</sup> Exhibit 7 shows the asset allocation, risk allocation, and return and risk estimates of different models.

Within the group of MVO portfolios, the Kelly strategy is very risky, with extremely high leverage, although its subportfolio of risky assets has the same composition as the MVO portfolios with higher risk aversion parameters.

The ratio between equities and bonds in the MVO portfolios is similar to that in the risk parity portfolio. This is consistent with the assumption that the two assets have similar Sharpe ratios. The assumption for the Sharpe ratio of commodities, however, is much lower. This, together with the different correlations, means the risk parity portfolio will not be mean-variance optimal.

Exhibit 7 also shows the risk contributions from different risky assets for each portfolio. The vast majority of the risk of the 60/40 portfolio comes from equities. This concentration risk is one of the motivations for the risk parity strategy, which attempts to diversify and equalize risk from different sources.

Exhibit 7 does not include any levered risk parity portfolios because, unlike utility maximization or mean-variance optimization with a specific risk aversion parameter, the risk parity strategy itself does not specify the leverage or cash position needed. In practice, however, a risk parity portfolio is often leveraged to achieve a certain return or risk target because it tends to overweight safer assets than do traditional

allocations. Exhibit 8 compares these models in an alternative way by selecting leverages for the MVO and risk parity portfolios so that their volatilities will be equal to that of the 60/40 portfolio.

Not surprisingly, because the MVO and risk parity portfolios have higher Sharpe ratios than the 60/40 portfolio, their estimated returns at the same volatility levels are higher than that of the 60/40 portfolio. The levered MVO portfolio, by definition, has the highest Sharpe ratio among all portfolios in the opportunity set and therefore offers the best risk-adjusted return.

### Exhibit 8: Levered MVO and risk parity portfolios with the same volatility as the 60/40 portfolio

	Asset	MVO ( $\gamma=1.7$ )	Risk parity	60/40
Asset allocation	Equities	42.7%	30.9%	60.0%
	Bonds	189.2%	149.1%	40.0%
	Commodities	-2.4%	26.8%	0.0%
	Cash	-129.5%	-106.7%	0.0%
Estimated annual return		3.1%	2.9%	2.8%
Estimated annual volatility		9.0%	9.0%	9.0%
Sharpe ratio		0.15	0.13	0.11
Expected maximum drawdown (1 Year)		10.2%	10.3%	10.4%

Source: PIMCO. **Hypothetical example for illustrative purposes only.**

**Exhibit 9: Long-only MVO, Kelly, risk parity and traditional 60/40 portfolios**

	Asset	CRRA utility maximization (MVO)			Risk parity	60/40
		$\gamma=5$	$\gamma=2$	$\gamma=1$ (Kelly)		
Asset allocation	Equities	14.0%	28.7%	50.7%	15.0%	60.0%
	Bonds	63.3%	71.3%	49.3%	72.1%	40.0%
	Commodities	0.0%	0.0%	0.0%	12.9%	0.0%
	Cash	22.8%	0.0%	0.0%	0.0%	0.0%
Risk allocation	Equities	45.2%	74.8%	96.0%	33.3%	98.4%
	Bonds	54.8%	25.2%	4.0%	33.3%	1.6%
	Commodities	0.0%	0.0%	0.0%	33.3%	0.0%
	Cash	0.0%	0.0%	0.0%	0.0%	0.0%
Estimated annual return		2.2%	2.5%	2.7%	2.3%	2.8%
Estimated annual volatility		3.0%	4.9%	7.7%	4.4%	9.0%
Sharpe ratio		0.15	0.14	0.12	0.13	0.11
Expected maximum drawdown (1 Year)		3.0%	5.2%	8.7%	4.6%	10.4%

Source: PIMCO. Hypothetical example for illustrative purposes only.

However, this is based on the simplifying assumption that the investor has no leverage constraints and can borrow at the risk-free rate.<sup>8</sup> In reality, many investors are not allowed to take leverage or have a short position for any asset. Exhibit 9 shows the MVO portfolios under the long-only constraints.

When leverage is not allowed, MVO investors who would have chosen to leverage are forced to go past the highest-Sharpe-ratio risky portfolio along the efficient frontier. The lower the risk aversion an investor has, the farther the optimal allocation will be along the frontier. This is shown in Exhibit 9. The long-only constraint also keeps the allocation to commodities at the zero lower bound. In this particular case, the 60/40 portfolio turns out to be mean-variance optimal.

*The authors wish to thank Mahmoud Hajo, Ravi Mattu, Vasant Naik, Andrew Nowobilski, Antonios Sangvinatsos, and Wentao Zhao for valuable comments on earlier versions of the manuscript.*

**REFERENCES**

- Anderson, R., S. Bianchi and L. Goldberg. 2012. "Will My Risk Parity Strategy Outperform?" *Financial Analysts Journal* 68, no. 6: 75-93.
- Ang, A. 2014. *Asset Management: A Systematic Approach to Factor Investing*. Oxford University Press, New York.
- Asness, C., A. Frazzini and L.H. Pedersen. 2012. "Leverage Aversion and Risk Parity." *Financial Analysts Journal* 68, no. 1: 47-59.
- Breiman, L. 1961. "Optimal Gambling Systems for Favorable Games." *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*, vol. 1: 63-68.
- Chaves, D., J.C. Hsu, F. Li and O. Shakernia. 2011. "Risk Parity Portfolio vs. Other Asset Allocation Heuristic Portfolios." *Journal of Investing* 20, no. 1: 108-118.
- DeMiguel V., L. Garlappi and R. Uppal. 2009. "Optimal Versus Naive Diversification: How Inefficient Is the 1/N Portfolio Strategy?" *Review of Financial Studies* 22, no. 5: 1915-1953.
- Fisher, G. S., P. Z. Maymin and J. G. Maymin. 2015. "Risk Parity Optimality." *Journal of Portfolio Management* 41, No. 2: pp. 42-56.

- Frazzini, A. and L.H. Pedersen. 2014. "Betting Against Beta." *Journal of Financial Economics* 111: 1-25.
- Lee, W. 2011. "Risk-Based Asset Allocation: A New Answer to an Old Question?" *Journal of Portfolio Management* 37, no. 4: 11-28.
- Levy, H. and H.M. Markowitz. 1979. "Approximating Expected Utility by a Function of Mean and Variance." *American Economic Review* 69: 308-317.
- Luenberger, D.G. 1993. "A Preference Foundation for Log Mean-Variance Criteria in Portfolio Choice Problems." *Journal of Economic Dynamics and Control* 17, nos. 5-6: 887-906.
- Luenberger, D.G. 1998. *Investment Science*. Oxford University Press, New York.
- Magdon-Ismail M., A.F. Atiya, A. Pratap and Y.S. Abu-Mostafa. 2004. "On the Maximum Drawdown of a Brownian Motion." *Journal of Applied Probability* 41, no. 1: 147-161.
- Maillard, S., T. Roncalli and J. Teiletche. 2010. "The Properties of Equally Weighted Risk Contribution Portfolios." *Journal of Portfolio Management* 36, no. 4: 60-70.
- Markowitz, H.M. 1952. "Portfolio Selection." *Journal of Finance* 7, no. 1: 77-91.
- Merton, R. 1980. "On Estimating the Expected Return on the Market: An Exploratory Investigation." *Journal of Financial Economics* 8, 323-406.
- Naik, V., D. Mukundan, A. Nowobilski, S. Page and N. Pedersen. 2016. *Factor Investing and Asset Allocation: A Business Cycle Perspective*. CFA Institute Research Foundation 2016-4. Available at SSRN: <https://ssrn.com/abstract=2930857>.
- Roncalli, T. 2014. *Introduction to Risk Parity and Risk Budgeting*. Chapman & Hall, Boca Raton, FL.
- Von Neumann, J. and O. Morgenstern. 1944. *Theory of Games and Economic Behavior*. Princeton University Press, Princeton, NJ.
- Ziemba, W.T. 2015. "A Response to Professor Paul A. Samuelson's Objections to Kelly Capital Growth Investing." *Journal of Portfolio Management* 41, no. 1: 153-67.
- Ziemba, R.E.S. and W.T. Ziemba. 2007. *Scenarios for Risk Management and Global Investment Strategies*. Wiley.

## APPENDIX

### Appendix 1: Comments on the functional form of the mean-variance utility

For simplicity, consider a mean-variance optimization with a risk-free asset with expected return  $r$  and a single risky asset with expected return  $\mu$  and variance  $\sigma^2$ . Let  $x$  denote the weight in the risky asset. Then we have

$$E(r_p) = (1-x)r + x\mu = r + x(\mu - r) \text{ and } V(r_p) = x^2\sigma^2.$$

The mean-variance optimization problem is then

$$\max_x r + x(\mu - r) - \frac{1}{2}\lambda x^2\sigma^2.$$

The first-order condition is

$$\mu - r - \lambda\sigma^2x = 0.$$

Therefore, the solution is

$$x^* = \frac{1}{\lambda\sigma^2}(\mu - r).$$

Extension to the case of multiple risky assets is straightforward. In either case, the  $\frac{1}{2}$  in the utility function leads to cleaner expressions for the first-order condition and the final solution, but more importantly, it allows  $\lambda$  to coincide with the risk aversion parameters in other utility functions under certain conditions.

For example, suppose the investor has a negative exponential utility over portfolio returns,

$$U(r_p) = -e^{-ar_p} \quad (a > 0),$$

where  $a$  measures the constant absolute risk aversion of the investor — i.e.,  $a = -\frac{U''(r_p)}{U'(r_p)}$ .

If the portfolio return is normally distributed with  $r_p \sim N(\mu_p, \sigma_p^2)$ , we have

$$-U(r_p) \sim \text{Lognormal}(-a\mu_p, a^2\sigma_p^2).$$

By the property of lognormal distributions,

$$E[U(r_p)] = -E[-U(r_p)] = -e^{-a\mu_p + \frac{1}{2}a^2\sigma_p^2} = -e^{-a(\mu_p - \frac{1}{2}a\sigma_p^2)}.$$

Therefore,  $\max E[U(r_p)] \Leftrightarrow \max \mu_p - \frac{1}{2}a\sigma_p^2$ . This is the same functional form for the mean-variance utility, so the two risk aversion parameters,  $\lambda$  and  $a$ , coincide in this case.

In Appendix 2, we provide another example where  $\lambda$  coincides with the risk aversion parameter  $\gamma$  in a continuous-time CRRA utility maximization problem with lognormal prices.

## Appendix 2: Continuous-time CRRA utility maximization with lognormal prices

The investor with wealth  $W_0$  at time 0 maximizes the expected utility of final wealth at time  $T$  by choosing the optimal portfolio weights in  $N$  risky asset(s) and a risk-free asset with continuous rebalancing. The investor has no labor income or intermediate consumptions, and there are no transaction costs. The utility function is CRRA:

$$U(W) = \frac{W^{1-\gamma}-1}{1-\gamma} \quad (\gamma > 0).$$

Assume the investment opportunities are constant over time.

Case 1:  $N=1$

The price of the risky asset follows a geometric Brownian motion with parameters  $(\mu, \sigma^2)$ :

$$\frac{dP_t}{P_t} = \mu dt + \sigma dB_t.$$

The risk-free asset follows a geometric growth rate of  $r$ :

$$\frac{dM_t}{M_t} = r dt.$$

Denote by  $x_t$  the portfolio weight in the risky asset at time  $t$ . Then  $1 - x_t$  is the portfolio weight in the risk-free asset. For simplicity, assume the portfolio strategy,  $\{x_t\}_{t \in [0, T]}$ , is non-stochastic. For a given portfolio strategy, the dynamics of wealth is given by the following budget constraint:

$$\frac{dW_t}{W_t} = [r + x_t(\mu - r)]dt + x_t\sigma dB_t.$$

By Itô's lemma and the budget constraint, we have

$$\begin{aligned} d \ln W_t &= \left[ r + x_t(\mu - r) - \frac{1}{2} \sigma^2 x_t^2 \right] dt + x_t \sigma dB_t, \\ \Rightarrow \ln W_T &= \ln W_0 + \int_0^T \left[ r + x_t(\mu - r) - \frac{1}{2} \sigma^2 x_t^2 \right] dt + \int_0^T x_t \sigma dB_t, \\ \Rightarrow W_T &= W_0 \exp \left( \int_0^T \left[ r + x_t(\mu - r) - \frac{1}{2} \sigma^2 x_t^2 \right] dt + \int_0^T x_t \sigma dB_t \right), \\ \Rightarrow W_T^{1-\gamma} &= W_0^{1-\gamma} \exp \left\{ (1-\gamma) \int_0^T \left[ r + x_t(\mu - r) - \frac{1}{2} \sigma^2 x_t^2 \right] dt + (1-\gamma) \int_0^T x_t \sigma dB_t \right\}. \end{aligned}$$

Because  $\int_0^T x_t \sigma dB_t \sim N(0, \int_0^T \sigma^2 x_t^2 dt)$ , we have

$$E \left\{ \exp \left\{ (1-\gamma) \int_0^T x_t \sigma dB_t \right\} \right\} = \exp \left\{ \frac{1}{2} (1-\gamma)^2 \int_0^T \sigma^2 x_t^2 dt \right\}.$$

Therefore,

$$E[W_T^{1-\gamma}] = W_0^{1-\gamma} \exp \left\{ (1-\gamma) \int_0^T \left[ r + x_t(\mu - r) - \frac{1}{2} \gamma \sigma^2 x_t^2 \right] dt \right\}.$$

Therefore, the optimal allocation solves the following problem:

$$x(\cdot)^* = \operatorname{argmax}_{\{x_t\}_{t \in [0, T]}} E \left[ \frac{W_T^{1-\gamma}-1}{1-\gamma} \right] = \operatorname{argmax}_{\{x_t\}_{t \in [0, T]}} \int_0^T \left[ r + x_t(\mu - r) - \frac{1}{2} \gamma \sigma^2 x_t^2 \right] dt.$$

The solution is a fixed weight in the risky asset at any time  $t$ :

$$x^* = \operatorname{argmax}_x \left[ r + x(\mu - r) - \frac{1}{2} \gamma \sigma^2 x^2 \right] T = \frac{\mu - r}{\gamma \sigma^2} = \frac{s}{\gamma \sigma},$$

where  $s$  is the instantaneous Sharpe ratio of the risky asset.

Therefore, the optimal wealth path satisfies

$$\ln W_T^* = \ln W_0 + \left[ r + \frac{1}{\gamma} \left( 1 - \frac{1}{2\gamma} \right) s^2 \right] T + \int_0^T \frac{1}{\gamma} s dB_t \text{ or}$$

$$\ln \left( \frac{W_T^*}{W_0} \right) = \left[ r + \frac{1}{\gamma} \left( 1 - \frac{1}{2\gamma} \right) s^2 \right] T + \frac{1}{\gamma} s \sqrt{T} Z,$$

where  $Z \sim N(0, 1)$ . When  $\gamma=1$  (Kelly strategy),  $\ln \left( \frac{W_T^*}{W_0} \right) = (r + \frac{1}{2}s^2)T + s\sqrt{T}Z$ .

The probability of loss for the optimal portfolio at time  $T$  is:

$$\begin{aligned} &P \left( \ln \left( \frac{W_T^*}{W_0} \right) < 0 \right) \\ &= P \left( \left[ r + \frac{1}{\gamma} \left( 1 - \frac{1}{2\gamma} \right) s^2 \right] T + \frac{1}{\gamma} s \sqrt{T} Z < 0 \right) \\ &= P \left( Z < - \frac{r + \frac{1}{\gamma} \left( 1 - \frac{1}{2\gamma} \right) s^2}{\frac{1}{\gamma} s} \sqrt{T} \right) \\ &= \Phi \left( - \frac{r\gamma + (1 - \frac{1}{2\gamma})s^2}{s} \sqrt{T} \right), \end{aligned}$$

where  $\Phi$  is the cumulative distribution function (CDF) of a standard normal random variable.

Therefore, the probability of loss is decreasing in both the time horizon  $T$  and the risk aversion parameter  $\gamma$ .

If  $\gamma=1$  (Kelly strategy),  $P \left( \ln \left( \frac{W_T^*}{W_0} \right) < 0 \right) = \Phi \left( - \frac{r + \frac{1}{2}s^2}{s} \sqrt{T} \right)$ , which is increasing in  $s$  if  $s^2 < 2r$  and decreasing in  $s$  if  $s^2 > 2r$ . Given the risk-free rate  $r$  and horizon  $T$ , the probability of loss is maximized when  $s^2 = 2r$ .

Case 2:  $N \geq 2$

The prices of the risky assets follow a multivariate geometric Brownian motion with parameters  $(\mu, \Sigma)$ :

$$\frac{dP_t}{P_t} = \mu dt + \Gamma dB_t,$$

where  $\Gamma \Gamma^T = \Sigma$ .

The risk-free asset follows a geometric growth rate of  $r$ :

$$\frac{dM_t}{M_t} = r dt.$$

Denote by  $x_t$  the  $N \times 1$  vector of portfolio weights in the risky assets at time  $t$ . Then  $(1 - x_t^T \mathbf{1})$  is the portfolio weight in the risk-free asset. For a given portfolio strategy, the dynamics of wealth is given by the following budget constraint:

$$\frac{dW_t}{W_t} = [r + x_t^T(\mu - r)]dt + x_t^T \Gamma dB_t.$$

By Itô's lemma and the budget constraint, we have

$$\begin{aligned} d \ln W_t &= \left[ r + x_t^T(\mu - r) - \frac{1}{2} x_t^T \Sigma x_t \right] dt + x_t^T \Gamma dB_t, \\ \Rightarrow \ln W_T &= \ln W_0 + \int_0^T \left[ r + x_t^T(\mu - r) - \frac{1}{2} x_t^T \Sigma x_t \right] dt + \int_0^T x_t^T \Gamma dB_t, \\ \Rightarrow W_T &= W_0 \exp \left( \int_0^T \left[ r + x_t^T(\mu - r) - \frac{1}{2} x_t^T \Sigma x_t \right] dt + \int_0^T x_t^T \Gamma dB_t \right), \\ \Rightarrow W_T^{1-\gamma} &= W_0^{1-\gamma} \exp \left\{ (1-\gamma) \int_0^T \left[ r + x_t^T(\mu - r) - \frac{1}{2} x_t^T \Sigma x_t \right] dt + (1-\gamma) \int_0^T x_t^T \Gamma dB_t \right\}. \end{aligned}$$

Because  $\int_0^T x_t^T \Gamma dB_t \sim N(0, \int_0^T x_t^T \Sigma x_t dt)$ , we have

$$E[\exp \{ (1-\gamma) \int_0^T x_t^T \Gamma dB_t \}] = \exp \left\{ \frac{1}{2} (1-\gamma)^2 \int_0^T x_t^T \Sigma x_t dt \right\}.$$

Therefore,

$$E[W_T^{1-\gamma}] = W_0^{1-\gamma} \exp \left\{ (1-\gamma) \int_0^T \left[ r + x_t^T(\mu - r) - \frac{1}{2} \gamma x_t^T \Sigma x_t \right] dt \right\}.$$

Therefore, the optimal allocation solves the following problem:

$$x(\cdot)^* = \operatorname{argmax}_{x_t | t \in [0, T]} E \left[ \frac{W_T^{1-\gamma} - 1}{1-\gamma} \right] = \operatorname{argmax}_{x_t | t \in [0, T]} \int_0^T \left[ r + x_t^T(\mu - r) - \frac{1}{2} \gamma x_t^T \Sigma x_t \right] dt.$$

The solution is a fixed vector of weights in the risky assets at any time  $t$ :

$$x^* = \operatorname{argmax}_x \left[ r + x^T(\mu - r) - \frac{1}{2} \gamma x^T \Sigma x \right] T = \frac{1}{\gamma} \Sigma^{-1} (\mu - r).$$

If  $N=2$ , we can invert  $\Sigma = \begin{pmatrix} \sigma_1^2 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{pmatrix}$  and write the optimal weights to the two risky assets as functions of their volatilities, Sharpe ratios and correlation  $\rho$ :

$$x^* = \frac{1}{\gamma(1-\rho^2)} \begin{pmatrix} \frac{1}{\sigma_1} (s_1 - \rho s_2) \\ \frac{1}{\sigma_2} (s_2 - \rho s_1) \end{pmatrix}.$$

### Appendix 3: Mean-variance optimality of risk parity

Below we show one way to prove the statement, and there are potentially many other ways (see, for example, Maillard, Roncalli and Teiletche 2010).

Define the following additional notations:

$\sigma = (\sigma_1, \sigma_2, \dots, \sigma_N)^T$  is the  $N \times 1$  vector of volatilities of the risky assets;

$D = \operatorname{diag}(\sigma)$  is the  $N \times N$  diagonal matrix with the volatilities as its diagonal elements;

$C$  is the correlation matrix of the risky assets — i.e.,  $\Sigma = DCD$ ;

$$\mathbf{1} = (1, 1, \dots, 1)^T.$$

Assume further:

1. All risky assets have the same Sharpe ratios:  $\mu - r = k\sigma$ , where  $k$  is a constant.
2. All the pairwise correlations are the same:  $\rho_{ij} = \rho, \forall i \neq j$ .

Assumption 2 implies

$$C\mathbf{1} = [1 + (N-1)\rho]\mathbf{1}.$$

Premultiplying both sides by  $C^{-1}$ , we have

$$C^{-1}\mathbf{1} = \frac{1}{1+(N-1)\rho}\mathbf{1}.$$

Therefore,

$$Dx^* = D \left[ \frac{1}{\gamma} \Sigma^{-1} (\mu - r) \right] = \frac{1}{\gamma} D(D^{-1}C^{-1}D^{-1})(k\sigma) = \frac{k}{\gamma} C^{-1}\mathbf{1} = \frac{k}{\gamma[1+(N-1)\rho]}\mathbf{1},$$

which means  $x_i^* \sigma_i = \frac{k}{\gamma[1+(N-1)\rho]}, \forall i$ .

The MVO portfolio therefore is a risk parity portfolio in the sense of both equal stand-alone risks and equal risk contributions (the two coincide under the assumption of identical pairwise correlations).

### Appendix 4: Probability that the Kelly strategy will outperform another fixed mix strategy

Consider the following simple example with one risky asset.

The price of the risky asset follows a geometric Brownian motion with parameters  $(\mu, \sigma^2)$ :

$$\frac{dP_t}{P_t} = \mu dt + \sigma dB_t.$$

The risk-free asset follows a geometric growth rate of  $r$ :

$$\frac{dM_t}{M_t} = r dt.$$

Denote by  $x_t$  the portfolio weight in the risky asset at time  $t$ .

Then  $1-x_t$  is the portfolio weight in the risk-free asset. For a given strategy, the dynamics of wealth is given by the following budget constraint:

$$\frac{dW_t}{W_t} = [r + x_t(\mu - r)]dt + x_t \sigma dB_t.$$

By Itô's lemma and the budget constraint, we have

$$\begin{aligned} d \ln W_t &= \left[ r + x_t(\mu - r) - \frac{1}{2} \sigma^2 x_t^2 \right] dt + x_t \sigma dB_t \\ \Rightarrow \ln W_T &= \ln W_0 + \int_0^T \left[ r + x_t(\mu - r) - \frac{1}{2} \sigma^2 x_t^2 \right] dt + \int_0^T x_t \sigma dB_t \\ \Rightarrow E[\ln W_T] &= \ln W_0 + \int_0^T \left[ r + x_t(\mu - r) - \frac{1}{2} \sigma^2 x_t^2 \right] dt. \end{aligned}$$



To maximize  $E[\ln W_T]$ , the Kelly strategy selects  $x_t^K = x^K = \frac{\mu-r}{\sigma^2}$ , which maximizes

$$f(x) \equiv r + x(\mu - r) - \frac{1}{2}\sigma^2 x^2.$$

Now suppose there is an alternative fixed mix strategy:

$x_t^A = x^A \neq x^K$ . We can state the wealth under the two strategies at time  $T$  as:

$$W_T^K = W_0 \exp \left\{ \left[ r + x^K(\mu - r) - \frac{1}{2}x^K^2\sigma^2 \right] T + x^K\sigma\sqrt{T}Z \right\},$$

$$W_T^A = W_0 \exp \left\{ \left[ r + x^A(\mu - r) - \frac{1}{2}x^A^2\sigma^2 \right] T + x^A\sigma\sqrt{T}Z \right\},$$

where  $Z \sim N(0,1)$ .

The probability that the Kelly strategy will outperform the alternative strategy at time  $T$ :

$$\begin{aligned} & P(W_T^K > W_T^A) \\ &= P \left( \left[ r + x^K(\mu - r) - \frac{1}{2}x^K^2\sigma^2 \right] T + x^K\sigma\sqrt{T}Z > \left[ r + x^A(\mu - r) - \frac{1}{2}x^A^2\sigma^2 \right] T + x^A\sigma\sqrt{T}Z \right) \\ &= \begin{cases} P \left( Z > \frac{f(x^A) - f(x^K)}{(x^K - x^A)\sigma} \sqrt{T} \right) = \Phi \left( \frac{f(x^K) - f(x^A)}{(x^K - x^A)\sigma} \sqrt{T} \right) & \text{if } x^K > x^A \\ P \left( Z < \frac{f(x^A) - f(x^K)}{(x^K - x^A)\sigma} \sqrt{T} \right) = \Phi \left( \frac{f(x^A) - f(x^K)}{(x^K - x^A)\sigma} \sqrt{T} \right) & \text{if } x^K < x^A \end{cases} \end{aligned}$$

where  $\Phi$  is the cumulative distribution function (CDF) of a standard normal random variable.

Because  $x^K$  is the unique solution to  $\max_x f(x)$ , we have  $f(x^K) > f(x^A), \forall x^A \neq x^K$ .

Therefore,  $P(W_T^K > W_T^A) > 0.5, \forall T > 0$  and  $\lim_{T \rightarrow \infty} P(W_T^K > W_T^A) = 1$ .

## Appendix 5: Estimation of EMDD

Suppose the value of a portfolio follows a geometric Brownian motion with parameters  $(\mu_p, \sigma_p^2)$ :

$$\frac{dW_t}{W_t} = \mu_p dt + \sigma_p dB_t.$$

Define EMDD as the expected percentage loss (approximated by the absolute log return) from the peak to the trough over a specific time horizon  $T$ . Based on Magdon-Ismael et al. (2004), we have

$$EMDD = \begin{cases} \frac{2\sigma_p^2}{\mu_p - \frac{1}{2}\sigma_p^2} Q_p \left( \frac{\left( \mu_p - \frac{1}{2}\sigma_p^2 \right)^2}{2\sigma_p^2} T \right) & \text{if } \mu_p - \frac{1}{2}\sigma_p^2 > 0 \\ 1.2533\sigma_p T & \text{if } \mu_p - \frac{1}{2}\sigma_p^2 = 0, \\ -\frac{2\sigma_p^2}{\mu_p - \frac{1}{2}\sigma_p^2} Q_n \left( \frac{\left( \mu_p - \frac{1}{2}\sigma_p^2 \right)^2}{2\sigma_p^2} T \right) & \text{if } \mu_p - \frac{1}{2}\sigma_p^2 < 0 \end{cases}$$

where  $Q_p$  and  $Q_n$  are functions tabulated in that paper.

If the portfolio follows the wealth path resulting from a CRRA utility maximization problem, as in Appendix 2 (Case 1),  $\mu_p = r + x^*(\mu - r) = r + \frac{s^2}{\gamma}$  and  $\sigma_p = x^*\sigma = \frac{s}{\gamma}$ , where  $r$  is the risk-free rate,  $s$  is the instantaneous Sharpe ratio of the risky asset, and  $\gamma$  is the risk aversion parameter. Therefore, we can estimate EMDD for optimal strategies under different risk aversion parameters by plugging in the model assumptions.

## Appendix 6: Relationship between mean log return and mean arithmetic return

Taking the second-order Taylor expansion of  $\ln(1+r)$  around  $E(r)$ , we have

$$\ln(1+r) \approx \ln(1+E(r)) + \frac{1}{1+E(r)}(r-E(r)) - \frac{1}{2(1+E(r))^2}(r-E(r))^2.$$

Taking expectations on both sides, we have

$$E(\ln(1+r)) \approx \ln(1+E(r)) - \frac{1}{2(1+E(r))^2} \text{Var}(r).$$

If  $E(r)$  is small (such as when the time interval is short),

$\ln(1+E(r)) \approx E(r)$ . Therefore,  $E(\ln(1+r)) \approx E(r) - \frac{1}{2(1+E(r))^2} \text{Var}(r)$ .

It says an asset's expected log return is lower than its expected arithmetic return by approximately half of the variance of the arithmetic return. This relationship holds exactly in continuous time when the asset price follows a geometric Brownian motion:

$$r \equiv \frac{dP_t}{P_t} = \mu dt + \sigma dB_t.$$

By Itô's lemma,

$$d \ln P_t = \left( \mu - \frac{1}{2}\sigma^2 \right) dt + \sigma dB_t.$$

Therefore,

$$E(d \ln P_t) = \left( \mu - \frac{1}{2}\sigma^2 \right) dt.$$

<sup>1</sup> For curious readers, this specific functional form serves two purposes. Apart from allowing for cleaner expressions for the first-order condition and the solution, it ensures that the risk aversion parameter in the mean-variance utility coincides with the risk aversion parameters in some other popular utility functions under certain conditions. Appendix 1 provides a detailed example.

<sup>2</sup> Continuous rebalancing is the limiting case of small rebalancing intervals. It makes it easy or possible to derive analytical solutions that provide economic insights difficult to obtain from numerical solutions and allows us to distinguish general properties of the solutions from those relying on specific parameter values. Continuous rebalancing can be a good approximation for frequently rebalanced portfolios, especially when the investment horizon is long.

<sup>3</sup> To see why the CRRA utility is reduced to a log utility when  $\gamma=1$ , first note that  $W^{1-\gamma} = e^{(1-\gamma)\log(W)}$ . Then we can apply L'Hôpital's rule and take derivatives of both the numerator and the denominator:

$$\lim_{\gamma \rightarrow 1} \frac{e^{(1-\gamma)\log(W)} - 1}{1-\gamma} = \lim_{\gamma \rightarrow 1} \frac{-\log(W)W^{1-\gamma}}{-1} = \log(W).$$

<sup>4</sup> This idea was shared later by other researchers and practitioners (see, for example, Ziemba and Ziemba 2007).

<sup>5</sup> Roncalli (2014) showed risk parity for factors is equivalent to risk budgeting for asset classes with a specific risk budget profile.

<sup>6</sup> See Frazzini and Pedersen (2014) for more details.

<sup>7</sup> The 60/40 mix appealed to many investors for decades because of its simplicity and the risk/return trade-off it delivered. With the fixed target weights, long-term investors do not have to time the market. The regular rebalancing to the fixed weight may also potentially contribute to the performance of the strategy in the presence of mean reversion for asset returns.

<sup>8</sup> Another caveat is that we use volatility as the measure of risk here, and it is limited in capturing any tail risk beyond what is implied by normal distributions.

## Past performance is not a guarantee or a reliable indicator of future results.

No representation is being made that any account, product, or strategy will or is likely to achieve profits, losses, or results similar to those shown. Hypothetical or simulated performance results have several inherent limitations. Unlike an actual performance record, simulated results do not represent actual performance and are generally prepared with the benefit of hindsight. There are frequently sharp differences between simulated performance results and the actual results subsequently achieved by any particular account, product or strategy. In addition, since trades have not actually been executed, simulated results cannot account for the impact of certain market risks such as lack of liquidity. There are numerous other factors related to the markets in general or the implementation of any specific investment strategy, which cannot be fully accounted for in the preparation of simulated results and all of which can adversely affect actual results.

Return assumptions are for illustrative purposes only and are not a prediction or a projection of return. Return assumption is an estimate of what investments may earn on average over a ten year period. Actual returns may be higher or lower than those shown and may vary substantially over shorter time periods.

Investing in the **bond** market is subject to risks, including market, interest rate, issuer, credit, inflation risk, and liquidity risk. The value of most bonds and bond strategies are impacted by changes in interest rates. Bonds and bond strategies with longer durations tend to be more sensitive and volatile than those with shorter durations; bond prices generally fall as interest rates rise, and the current low interest rate environment increases this risk. Current reductions in bond counterparty capacity may contribute to decreased market liquidity and increased price volatility. Bond investments may be worth more or less than the original cost when redeemed. **High-yield, lower-rated, securities** involve greater risk than higher-rated securities; portfolios that invest in them may be subject to greater levels of credit and liquidity risk than portfolios that do not. **Currency** rates may fluctuate significantly over short periods of time and may reduce the returns of a portfolio. **Equities** may decline in value due to both real and perceived general market, economic, and industry conditions. The use of **leverage** may cause a portfolio to liquidate positions when it may not be advantageous to do so to satisfy its obligations or to meet segregation requirements. Leverage, including borrowing, may cause a portfolio to be more volatile than if the portfolio had not been leveraged. **Derivatives** may involve certain costs and risks such as liquidity, interest rate, market, credit, management and the risk that a position could not be closed when most advantageous. Investing in derivatives could lose more than the amount invested.

This material contains the opinions of the authors but not necessarily those of PIMCO and such opinions are subject to change without notice. This material is distributed for informational purposes only and should not be considered as investment advice or a recommendation of any particular security, strategy or investment product. Information contained herein has been obtained from sources believed to be reliable, but not guaranteed.

PIMCO provides services only to qualified institutions and investors. This is not an offer to any person in any jurisdiction where unlawful or unauthorized. | **Pacific Investment Management Company LLC**, 650 Newport Center Drive, Newport Beach, CA 92660 is regulated by the United States Securities and Exchange Commission. | **PIMCO Europe Ltd** (Company No. 2604517) and PIMCO Europe Ltd - Italy (Company No. 07533910969) are authorised and regulated by the Financial Conduct Authority (25 The North Colonnade, Canary Wharf, London E14 5HS) in the UK. The Italy branch is additionally regulated by the Commissione Nazionale per le Società e la Borsa (CONSOB) in accordance with Article 27 of the Italian Consolidated Financial Act. PIMCO Europe Ltd services are available only to professional clients as defined in the Financial Conduct Authority's Handbook and are not available to individual investors, who should not rely on this communication. | **PIMCO Deutschland GmbH** (Company No. 192083, Seidlstr. 24-24a, 80335 Munich, Germany), PIMCO Deutschland GmbH Italian Branch (Company No. 10005170963) and PIMCO Deutschland GmbH Swedish Branch (SCRO Reg. No. 516410-9190) are authorised and regulated by the German Federal Financial Supervisory Authority (BaFin) (Marie-Curie-Str. 24-28, 60439 Frankfurt am Main) in Germany in accordance with Section 32 of the German Banking Act (KWG). The Italian Branch and Swedish Branch are additionally supervised by the Commissione Nazionale per le Società e la Borsa (CONSOB) in accordance with Article 27 of the Italian Consolidated Financial Act and the Swedish Financial Supervisory Authority (Finansinspektionen) in accordance with Chapter 25 Sections 12-14 of the Swedish Securities Markets Act, respectively. The services provided by PIMCO Deutschland GmbH are available only to professional clients as defined in Section 31a para. 2 German Securities Trading Act (WpHG). They are not available to individual investors, who should not rely on this communication. | **PIMCO (Schweiz) GmbH** (registered in Switzerland, Company No. CH-020.4.038.582-2), Brandschenkestrasse 41, 8002 Zurich, Switzerland, Tel: + 41 44 512 49 10. The services provided by PIMCO (Schweiz) GmbH are not available to individual investors, who should not rely on this communication but contact their financial adviser. | **PIMCO Asia Pte Ltd** (8 Marina View, #30-01, Asia Square Tower 1, Singapore 018960, Registration No. 199804652K) is regulated by the Monetary Authority of Singapore as a holder of a capital markets services licence and an exempt financial adviser. The asset management services and investment products are not available to persons where provision of such services and products is unauthorised. | **PIMCO Asia Limited** (Suite 2201, 22nd Floor, Two International Finance Centre, No. 8 Finance Street, Central, Hong Kong) is licensed by the Securities and Futures Commission for Types 1, 4 and 9 regulated activities under the Securities and Futures Ordinance. The asset management services and investment products are not available to persons where provision of such services and products is unauthorised. | PIMCO Australia Pty Ltd ABN 54 084 280 508, AFSL 246862 (**PIMCO Australia**). This publication has been prepared for wholesale clients only and has not been prepared for, and is not available to persons who are retail clients, as defined by the Corporations Act 2001 (Cth). Investment management products and services offered by PIMCO Australia are offered only to persons within its respective jurisdiction, and are not available to persons where provision of such products or services is unauthorised. This document must not be passed on or distributed to any retail clients. | **PIMCO Japan Ltd** (Toranomon Towers Office 18F, 4-1-28, Toranomon, Minato-ku, Tokyo, Japan 105-0001) Financial Instruments Business Registration Number is Director of Kanto Local Finance Bureau (Financial Instruments Firm) No. 382. PIMCO Japan Ltd is a member of Japan Investment Advisers Association and The Investment Trusts Association, Japan. Investment management products and services offered by PIMCO Japan Ltd are offered only to persons within its respective jurisdiction, and are not available to persons where provision of such products or services is unauthorized. Valuations of assets will fluctuate based upon prices of securities and values of derivative transactions in the portfolio, market conditions, interest rates and credit risk, among others. Investments in foreign currency denominated assets will be affected by foreign exchange rates. There is no guarantee that the principal amount of the investment will be preserved, or that a certain return will be realized; the investment could suffer a loss. All profits and losses incur to the investor. The amounts, maximum amounts and calculation methodologies of each type of fee and expense and their total amounts will vary depending on the investment strategy, the status of investment performance, period of management and outstanding balance of assets and thus such fees and expenses cannot be set forth herein. | **PIMCO Canada Corp.** (199 Bay Street, Suite 2050, Commerce Court Station, P.O. Box 363, Toronto, ON, M5L 1G2) services and products may only be available in certain provinces or territories of Canada and only through dealers authorized for that purpose. | **PIMCO Latin America** Edificio Internacional Rio Praia do Flamengo, 154 1o andar, Rio de Janeiro – RJ Brasil 22210-906. | No part of this publication may be reproduced in any form, or referred to in any other publication, without express written permission. PIMCO is a trademark of Allianz Asset Management of America L.P. in the United States and throughout the world. ©2017, PIMCO.

## Newport Beach Headquarters

650 Newport Center Drive  
Newport Beach, CA 92660  
+1 949.720.6000

## Hong Kong

## London

## Milan

## Munich

## New York

## Rio de Janeiro

## Singapore

## Sydney

## Tokyo

## Toronto

## Zurich

[pimco.com](http://pimco.com)  
[blog.pimco.com](http://blog.pimco.com)