Advances in Portfolio Risk Control.

Risk! Parity?

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Abstract
Spurred by the increased interest in applying "risk control" techniques in an asset allocation context, we offer a practitioner's review of techniques that have been newly proposed or revived from academic history. We discuss minimum variance, "1/N" or equal-weighting, maximum diversification, volatility weighting and volatility targeting – and especially equal risk contribution or "risk parity", a concept that has become a real buzz word. We start from a taxonomy of risk control techniques. We discuss their main characteristics and their pluses and minuses and we compare them against each other and against the maximum Sharpe Ratio criterion. We illustrate their implications by means of an empirical example. We also highlight some key papers from the vast and still growing literature in this field. All in all, we aim to provide a practical and critical guide to risk control strategies. It may help to demystify risk control techniques, to appreciate both the "forest" and the "trees", and to judge these techniques on their potential merits in practical investment applications.

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1. Introduction

Recently there has been increased interest in applying "risk control" techniques in an asset allocation context. Some examples of techniques that have been newly proposed or revived from academic history are "1/N" or equal-weighting, minimum variance, maximum diversification, volatility weighting and volatility targeting – and especially equal risk contribution or "risk parity", a concept that has become a real buzz word. In this paper, we start from a taxonomy of risk control techniques. We discuss their main characteristics and their pluses and minuses, we compare them against each other and against the maximum Sharpe Ratio criterion, and we illustrate their implications by means of an empirical example. We also highlight some key papers from the vast and still growing literature in this field. All in all, we aim to provide a practical and critical guide to risk control strategies that may help to appreciate both the "forest" and the "trees" and to judge these techniques on their actual potential merits in practical investment applications. For an in-depth exposition, comparison and evaluation of these strategies, we recommend Roncalli [2014].

The main question in risk control is: "does it work?" Do risk control techniques achieve the *ex ante* targeted risk balance or risk profile? Can we avoid hot spots (pockets of risk concentration in a portfolio) and can we achieve diversification against losses? Apart from this risk budgeting perspective, a large part of the literature has promoted risk control as a full-fledged investment criterion – suggesting that controlling the risk dimension is sufficient to build a portfolio or an opportunity to reap risk-adjusted outperformance. But why would ignoring the return dimension *ex ante* produce portfolios that are superior in terms of *ex post* risk-adjusted performance?

Several studies indicate that the historical outperformance of risk control strategies can be linked to overweighting asset classes that in the rear view mirror have paired high historical risk premia with low risk levels (as is the case for bonds, e.g.), or to implicit exposures to factor premia (Jurczenko et al [2013]). However, focusing directly on factor exposures, as is done in factor investing (see Ang [2014]), provides a much more efficient and effective way to capture factor premia. Still, focusing only on risk aspects when forming a portfolio is a perfectly sensible heuristic (see Fisher et al [2015]) or a starting point when one has only low confidence in *ex ante* risk premia estimates. From the perspective of estimation risk, mis-estimation of risk premia has the greatest impact on portfolio composition and especially risk premia are notoriously hard to estimate *ex ante*. For example, suppose that *ex ante* one cannot meaningfully differentiate between all assets' Sharpe Ratios (so assuming that all Sharpe Ratios are equal), then constructing a maximum diversification portfolio gives the maximum Sharpe Ratio portfolio. When, in addition to equal Sharpe Ratios, one cannot meaningfully differentiate between asset correlations (so also assuming that all correlations are uniform), then applying risk parity gives the maximum Sharpe Ratio portfolio. So besides the risk budgeting dimension, also the potential relevance of risk control techniques in full-fledged risk-return optimization is not to be under-estimated.

This paper is organized as follows. Section 2 introduces our empirical example and provides some preliminaries. Next, we discuss the risk control strategies. The main skeleton of the taxonomy of risk control strategies has a cross-section and a time series branch. The objective of risk control in the cross-section is to control a portfolio's risk profile at a given point in time. The focus is across assets: reweighting the portfolio constituents so as to obtain a desired risk profile. The main cross-sectional risk control strategies are: 1/N, or the equally-weighted portfolio (section 3), the minimum variance portfolio MVP (section 4), and the maximum diversification portfolio MDP (section 5). Next we have risk parity, that comes in two flavors, "Equal Risk Contribution" ERCP or "full" risk parity (section 6), and "Inverse Volatility" IVP or "naïve" risk parity, implying volatility weighting in cross-section (section 7). Finally, we escape from a risk-only perspective and consider the Maximum Sharpe Ratio portfolio MSRP (section 8). The objective of time-series risk control is to control the portfolio risk level over time. There are two closely related time-series techniques: volatility weighting over time, or adjusting the exposure to risky assets according to the level of forecasted volatility, and volatility targeting, or volatility weighting with the specific goal to achieve a pre-specified level of portfolio volatility (section 9). Each of these sections is organized according to a fixed format, starting with main references, followed by the recipe to calculate the particular portfolio, its characteristics and its evaluation. Section 10 concludes with an overall evaluation. The Appendix contains technical details.

2. The empirical example and preliminaries

We consider monthly returns in excess of the risk free rate over the decade January 2005 through December 2014 for a selection of US assets classes: Equities, Treasuries (Tsies), Investment Grade Corporates (IG), and High Yield

Corporates (HY). The risk free return comes from the Ibbotson "Stocks, Bills, Bonds and Inflation" database. Equities is the market factor from Kenneth French's database.[1] The fixed income series are taken from Barclays Capital Live.[2] All returns are in USD. The composition of the market capitalization weighted portfolio "Mkt Cap" is estimated as per 2014Q4.[3] "EqWtd" is the equally-weighted portfolio.

Exhibit 1shows the descriptive statistics. Over the past decade, fixed income assets were the real winners in terms of risk-adjusted performance. This is not surprising given the substantial tail wind from decreasing interest rates. Especially Tsies paired a 3% average return with a relatively low level of risk. Equities showed the highest volatility, but viewing the Sharpe Ratio this was not matched by a proportionally higher risk premium. Equities and Tsies were negatively correlated, providing hedge opportunities (see the small negative correlation between Tsies and the market cap portfolio). The highest correlation is between Equities and HY, pointing at a strong link between equity risk and credit risk. Credit risk is dominant in HY and the negative correlation between interest rates and credit spreads manifests itself in the negative correlation between Tsies and HY.

Exhibit 1 Statistics of US Excess Returns (p.a.) over the risk free rate (Jan 2005 - Dec 2014).

	Equities	**Tsies**	**IG**	**HY**	**Mkt Cap**	**EqWtd**
Market cap weight:	53%	29%	14%	4%	100%	
Return statistics:						
avge p.a. %	7.54	2.97	4.17	6.62	5.71	5.33
stdev p.a. %	15.11	4.16	6.04	10.54	8.46	6.78
Sharpe Ratio	0.50	0.71	0.69	0.63	0.67	0.78
Correlations:						
Equities		-0.30	0.35	0.74	0.98	0.88
Tsies			0.44	-0.24	-0.11	-0.01
IG				0.63	0.53	0.73
HY					0.78	0.90
Mkt Cap						0.95

Money allocation versus risk allocation

The money allocation in the market cap portfolio is given in Exhibit 1. For the risk allocation within the market cap portfolio, we compute the OLS regression slope or beta of the assets against the market cap portfolio. This beta represents the relative marginal contribution of the corresponding asset to the overall portfolio risk (for details, see the Appendix). The component risk contribution is given by the product of the portfolio weight and the beta. Hence, the betas can be interpreted as the adjustment factors to transform money allocation into risk allocation (note that the weighted average value of beta is unity). The risk allocation within the market cap portfolio is given in Exhibit 2. From this Exhibit we see a nasty surprise: the market cap portfolio appears to be a properly diversified portfolio but in reality more than 90% of the portfolio risk is due to equities. (This was already forewarned by the high correlation between equities and the market cap portfolio as shown in Exhibit 1.) The same finding is widely reported for conventional 60/40 equity-bond portfolios in general, and for typical "Yale-type" portfolios (were alternatives and/or commodities are added to main holdings of equities and bonds).

[1] The Ibbotson risk free rate and the equity market factor can be downloaded from http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html.

[2] Download from https://live.barcap.com/.

[3] Sources are (1) Securities Industry and Financial Markets Association (SIFMA), US Bond Market Outstanding, download from http://www.sifma.org/research/statistics.aspx, (2)WorldBank, year-end market capitalization of listed companies by country, download from http://data.worldbank.org/indicator/CM.MKT.LCAP.CD and (3) Barclays Live (for the relative IG and HY capitalizations).

Exhibit 2 Risk attribution with respect to Mkt Cap portfolio.

	Eq	Tsies	IG	HY	sum
market cap weight	53%	29%	14%	4%	
beta	1.74	-0.06	0.38	0.97	
% risk contribution	92%	-2%	5%	4%	100%

Although we focus on volatility as the risk measure, most of the results in this paper carry over to downside risk measures such as portfolio loss or Value-at-Risk. Exhibit 3 shows the average of the six largest monthly losses against the risk free rate on the market cap portfolio: equities contributed by far the most to the realized losses. The excessive contribution of equities to (downside) risk within portfolios that seem only moderately geared towards equities provided the impetus to the research into risk control strategies (see also Qian [2005]). In the remainder of this paper, we use this empirical example to illustrate various risk control strategies.

Exhibit 3 Absolute (in %) and relative contribution of assets to the average of the six largest losses on the market cap portfolio (in terms of excess returns), 2005-2014.

MktCap Index	Eq	Tsies	IG	HY
-6.02	-5.39	-0.04	-0.40	-0.19
100%	90%	1%	7%	3%

Implied risk premia and the implied Sharpe Ratios

There is one additional perspective we want to highlight – a perspective that is helpful in evaluating risk control strategies *vis à vis* the maximum Sharpe Ratio portfolio. For each of the portfolios we discuss, we present the implied risk premia and the implied Sharpe Ratios of the individual assets. Instead of using actual risk premia and the variance-covariance matrix to calculate the maximum Sharpe Ratio portfolio MSRP, we reverse the process and assume that the reference portfolio at hand actually ***is*** the MSRP. Together with the variance-covariance matrix of excess returns this allows us to derive the "imputed" risk premia (pioneered by Sharpe [1974]); together with the actual (historical) asset standard deviations, we can then compute the implied Sharpe Ratios. Hence, given a particular portfolio, these implied risk premia (or implied Sharpe Ratios) would make this reference portfolio the maximum Sharpe Ratio portfolio. For details, we refer to the Appendix.

This reverse portfolio optimization is relevant when there is uncertainty about *ex ante* risk premia. After all, since the MSRP is the tangency portfolio to the mean-variance efficient frontier without including risk free borrowing and lending, this portfolio is very sensitive to the input risk premia. Slight differences in these inputs can result in very different (and sometimes "unrealistic" or extreme and hence unacceptable) portfolios. At the same time, estimating *ex ante* risk premia is a very difficult task. Reverse optimization can help since the assets' implied risk premia, derived from a reference portfolio such as the market capitalization weighted portfolio or a risk control portfolio, serve as a sensible starting point. Depending on the confidence placed in one's *ex ante* views, one can next adjust the implied risk premia accordingly. After re-optimization, the resulting portfolio is closer to the original portfolio and less extreme. This two-stage portfolio optimization process is originally proposed by Black & Litterman [1992] and extended by Haesen et al [2014] who take the risk parity portfolio as the reference portfolio.

Exhibit 4 presents the implied risk premia and the implied Sharpe Ratios of the market cap portfolio. For Equities, the implied risk premium is about twice as large as the historical risk premium. For IG, the implied risk premium is less than half of the historical risk premium. So when the market cap portfolio would be the MSRP, Equities would have to offer a risk premium of 10% and IG of 2%. Conversely, when we would feel confident in extending the historical risk premia to the future, this implies that we should increase the weight of IG and lower the weight of Equities in order to increase the Sharpe Ratio of the market cap portfolio. For Tsies, the implied risk premium (and hence the implied Sharpe Ratio) is even slightly negative, reflecting the role of Tsies as a hedge in the market cap portfolio. Because of the negative correlation of Tsies with Equities (and HY), their inclusion in the market cap portfolio would be justified even when their risk premium would be negative.

Exhibit 4 Implied risk premia (%) and implied Sharpe Ratios within the market cap portfolio.

	Eq	Tsies	IG	HY	Mkt Cap
implied risk premium	9.96	-0.32	2.14	5.55	5.71
implied Sharpe Ratio	0.66	-0.08	0.35	0.53	0.67

In order to derive the implied risk premia from a portfolio different from the market cap portfolio, we first calculate the relative risk aversion coefficient λ^* as implied by the market cap portfolio (see Sharpe [1974]): dividing the historical risk premium on the market cap portfolio by the historical variance of the market cap portfolio yields $\lambda^* = 8.0$. The implied risk premium on an alternative portfolio, when assuming this portfolio is mean-variance optimal, is then given by the product of λ^* and its historical variance. This portfolio risk premium is next attributed to the assets comprised in the portfolio according to their relative risk contributions (or betas) with respect to this portfolio (see the Appendix).

Notation

We use fairly conventional notation. The portfolio weight of asset *i* is w_i. Individual and portfolio risk premia (average return in excess of the risk free rate r_f) are denoted by $\overline{r}_{if}$ and $\overline{r}_{pf}$. Individual asset standard deviations or volatilities are denoted by σ_i and the portfolio volatility by σ_p. The beta of asset *i* with respect to portfolio *p* is β_{ip} (reflecting its relative marginal contribution to portfolio volatility) and its correlation with the portfolio is denoted as ρ_{ip}. Where deemed necessary, technical details are mentioned in the main text; the Appendix contains a general background and additional derivations.

3. 1/N or equal-weighting

Main reference: DeMiguel et al [2009].
Recipe. In equally-weighted portfolios, each of the *N* assets is assigned a weight of 1/N. In our example, each asset class gets a weight of 25% in the portfolio, with monthly rebalancing.
Characteristics. (1) 1/N avoids concentrated positions - in terms of money allocation. **(2)** Within equities, 1/N implies an exposure to the small-cap anomaly. The market cap portfolio is tilted towards large cap stocks. The 1/N portfolio is tilted towards small cap stocks and will hence capture a size premium. **(3)** Furthermore, 1/N implies a disciplined and periodical rebalancing of positions. By definition, the market cap portfolio is a buy-and-hold portfolio. The 1/N portfolio, in contrast, implies a "volatility pumping" effect: in order to maintain the 1/N allocation, one has to buy (sell) out- (under-) performing assets. This is effectively a "buy low, sell high" strategy, which profits from reversals.[4] Depending on the frequency, the rebalancing process implies portfolio turnover with the associated transaction cost and exposure to potential illiquidity (since even the smallest market cap assets get a weight of 1/N). **(4)** In Bayesian terms, the 1/N portfolio is the "uninformed prior": the naively diversified portfolio that is optimal when one has no information to discriminate between the attractiveness of assets. **(5)** It can be shown that when all assets have the same volatility and when all pairwise correlations are the same, then the 1/N portfolio is the **MVP**. In this case, the MVP also coincides with the **ERCP**. See below. **(6)** From Exhibit 1 we see that the 1/N portfolio has a lower risk than the market cap portfolio and a higher historical Sharpe Ratio. This stems from underweighting Equities (with a lower Sharpe Ratio) and overweighting IG and HY (with a higher Sharpe Ratio).
Evaluation. (1) Exhibit 5 shows the 1/N portfolio statistics. It clearly shows that equal money weights is very different from equal risk weights. Notably Tsies act as a strong diversifier (negatively correlated with Equities and HY) and show (virtually) zero risk contribution. Still, Equity risk dominates in the 1/N portfolio, accounting for about half of the portfolio volatility. **(2)** For Equities, the implied risk premium is 7.17% p.a. (which given historical volatility implies a Sharpe Ratio of 0.47). When one believes that the *ex ante* equity risk premium is below 7.17%, the weight of Equities should be lowered in order to improve the risk-adjusted portfolio performance. Likewise, when one believes that the *ex ante* bond risk premium is above -3 bps p.a., the weight of Tsies should be increased. Equivalent reasoning applies to IG and HY.

[4] For the effects of rebalancing on portfolio returns, see Hallerbach [2014].

Exhibit 5 Risk attribution with respect to 1/N portfolio, and implied risk premia (%) and Sharpe Ratios.

	Eq	Tsies	IG	HY	EqWtd
weight	25%	25%	25%	25%	
beta	1.95	-0.01	0.65	1.41	
% risk contribution	49%	0%	16%	35%	100%
implied risk premium	7.17	-0.03	2.39	5.16	3.67
implied Sharpe Ratio	0.47	-0.01	0.40	0.49	0.54

4. Maximum Diversification Portfolio MDP

Main reference: the MDP is proposed by Choueifaty & Coignard [2008].
Recipe. The weights of the MDP are obtained by maximizing the "diversification ratio", which is defined as the ratio of weighted volatilities and portfolio volatility:

$$\text{(1)} \qquad \max_{\{w\}} \frac{\sum_i w_i \sigma_i}{\sigma_p}$$

For obtaining insight into this ratio, note that the portfolio volatility can be written as the weighted sum of the product of each asset's individual volatility and its correlation with the portfolio (see eq.(17) in the Appendix). Hence, we can rewrite the diversification ratio as:

$$\text{(2)} \qquad \max_{\{w\}} \frac{\sum_i w_i \sigma_i}{\sum_i w_i \sigma_i \rho_{ip}}$$

This expression reveals that the diversification ratio compares (a) the portfolio volatility when ignoring correlations in the numerator, with (b) the actual portfolio volatility when taking into account correlation (and hence diversification) in the denominator. Imperfect (<1) correlations increase the diversification ratio above unity.
Characteristics. (1) It can be shown that for the MDP it holds that (see Choueifaty & Coignard [2008]):

$$\text{(3)} \qquad \frac{1}{\sigma_i}\frac{\partial \sigma_p}{\partial w_i} = \frac{1}{\sigma_j}\frac{\partial \sigma_p}{\partial w_j}$$

where $\partial \sigma_p / \partial w_i$ is the marginal contribution of asset *i* to portfolio volatility. By definition, within the global MVP, all assets' marginal risk contributions are equal, see section 5. It follows that for equal volatilities, $\sigma_i = \sigma_j$, the MPD coincides with the global **MVP**. **(2)** From (3) it also follows that when risk premia $\{\bar{r}_{if}\}$ are proportional to volatilities $\{\sigma_i\}$, thus implying that all assets have the same Sharpe Ratio, then the MDP is the **MSRP**. After all, in the MSRP the assets' marginal contributions to the portfolio risk premium are proportional to the assets' marginal contributions to portfolio volatility, implying (see section 8):

$$\text{(4)} \qquad \frac{1}{\bar{r}_{if}}\frac{\partial \sigma_p}{\partial w_i} = \frac{1}{\bar{r}_{jf}}\frac{\partial \sigma_p}{\partial w_j} \qquad \Leftrightarrow \qquad \frac{\bar{r}_{if}}{\beta_{ip}} = \frac{\bar{r}_{jf}}{\beta_{jp}}$$

(3) Choueifaty & Coignard [2008] also show that all constituent assets have the same correlation with the MDP.
Evaluation. (1) Why should one want to maximize this specific diversification ratio? After all, there are many definitions of "diversified"! **(2)** The diversification ratio is a differential diversification measure. It applies with respect to the specific portfolio at hand. It is not an absolute diversification measure from which we can read the degree of diversification; we cannot compare the diversification ratios of two different portfolios to infer which portfolio is more diversified than the other. **(3)** The MDP is not unique and may be very concentrated in weights (money allocation) or in risk and loss contributions (risk allocations). Indeed, in our example IG carries zero weight in the MDP, see Exhibit 6. Tsies have the highest weight in the MDP; the money allocation of 74% here implies that Tsies account for almost 50% of the portfolio risk. This can hardly be termed a "diversified portfolio".

Exhibit 6 Risk attribution with respect to MDP, and implied risk premia (%) and SRs.

	Eq	Tsies	IG	HY	MDP
weight	14%	74%	0%	13%	
beta	2.34	0.64	1.29	1.63	
% risk contribution	32%	47%	0%	21%	100%
implied risk premium	2.55	0.70	1.40	1.78	1.09
implied Sharpe Ratio	0.17	0.17	0.23	0.17	0.30

(4) Exhibit 6 also shows that the implied Sharpe Ratios of the three portfolio components equal 0.17. This confirms that when Sharpe Ratios of the portfolio constituents are the same, then the MDP is the **MSRP**. Note that this only applies to assets comprised in the MDP; by construction, the composition of the MDP does not depend on risk premia or Sharpe Ratios.

Exhibit 7 shows the historical statistics of the portfolios we consider. Note that the portfolios' implied risk premia are different from the historical risk premia, since the implied risk premia are calculated from the derived risk aversion parameter $\lambda^* = 8.0$ and the historical portfolio variance. For the historical inputs, the MDP beats the market cap and 1/N portfolios in risk-adjusted performance. This is due to the large overweight of Tsies which over the past decade showed the highest Sharpe Ratio. Since we use the full historical sample to calculate the weights of the MDP (and other risk control portfolios), these are in-sample results. In practical applications, one would use trailing historical windows (avoiding a look-ahead bias) to periodically re-calculate the weights. In this way, the out-of-sample properties of the MDP can be evaluated. The same argument applies to the other risk control strategies.

Exhibit 7 Comparative historical portfolio statistics, 2005-2014.

	Cap Wtd	1/N	MDP	MVP	ERCP	IVP	MSRP
avge p.a.	5.71	5.33	4.05	3.68	4.24	4.46	3.97
stdev p.a.	8.46	6.78	3.70	3.43	4.19	4.74	3.57
SR	0.67	0.78	1.10	1.07	1.01	0.94	1.11

5. Minimum Variance Portfolio MVP

Main references: Haugen & Baker [1991] show that market cap weighted portfolios are inefficient (sub-optimal) when there are market frictions and highlight the high relative performance of low volatility portfolios. Clarke et al [2006] extend Haugen & Baker's empirical research. Blitz & van Vliet [2007] revive the interest in the low volatility anomaly and provide possible explanations (behavioral biases, leverage restrictions, and delegated portfolio management and benchmarking).

Recipe. Choose the portfolio weights to minimize the portfolio variance:

(5) $$\max_{\{w\}} \sigma_p^2 = \sum_i \sum_j w_i w_j \sigma_i \sigma_j \rho_{ij}$$

The optimal portfolio is characterized by equal marginal contributions to portfolio risk: $\partial\sigma_p / \partial w_i = \partial\sigma_p / \partial w_j$.

Characteristics. (1) Note that marginal risk contributions are given by $\partial\sigma_p / \partial w_i = \sigma_{i,p} / \sigma_p = \beta_{ip} \cdot \sigma_p$, so all asset betas with respect to the MVP are identical (and hence equal to unity). **(2)** Since an asset's risk contribution equals $w_i \cdot \partial\sigma_p / \partial w_i \sim w_i$, the risk contribution is proportional to the portfolio weight, so risk allocation equals money allocation. **(3)** When all assets have the same volatility and when all pairwise (imperfect) correlations are the same, then the MVP is the **1/N** portfolio: it pays to diversify over the assets but in the portfolio context, all assets are perfect substitutes. **(4)** The MVP is the **MSRP** when all assets have the same risk premium, $\bar{r}_{if} = \bar{r}$ (implying that all Sharpe Ratios SR_i are proportional to $1/\sigma_i$). After all, in that case we have (cf. eq.(4)):

$$(6) \qquad \frac{1}{\bar{r}}\frac{\partial \sigma_p}{\partial w_i} = \frac{1}{\bar{r}}\frac{\partial \sigma_p}{\partial w_j}$$

Evaluation. (1) The MVP favors low volatility assets and low beta assets and hence benefits from the low volatility anomaly. The MSCI Minimum Variance Index and the S&P Low Volatility Index are examples of low risk portfolios that are designed to benefit from this anomaly. For more information on the low volatility anomaly, see Blitz & van Vliet [2007]. **(2)** Several studies have documented that MVPs also pick up other priced anomalies. Clarke et al [2006] find that, in general, the MVP has a substantially higher value (B/P) exposure than the market (since value stocks tend to have low volatilities), which explains at least part of its higher average realized return. Scherer [2011] shows that the MVP loads significantly on the Fama-French factors (large size and high value) but also finds that MVPs have a negative beta bias (favor low beta assets) and favor assets with low residual volatility. The latter effects crowd out the Fama-French factors in the sense that low beta and low residual volatility alone can explain more of the variation in the MVP's excess returns than the Fama-French factors. This leads Scherer to conclude that low beta and low residual volatility is a more efficient and effective way to capture the low volatility anomaly than minimum variance. **(3)** When time passes and the MVP is re-optimized, one will need to apply constraints on turn-over in order to mitigate transactions costs. However, turnover constraints make the MVP a path-dependent strategy. **(4)** The MVP is a concentrated portfolio. Assets with low volatility and/or low correlations with other assets will carry a large weight. Conversely, assets with high volatility and/or high correlations with other assets will carry a small or even negative weight; when excluding short positions, these assets will not appear in the MVP. This is illustrated in Exhibit 8: IG is not included in the long-only MVP. Exhibit 8 confirms that when the assets comprised in the MVP have identical risk premia, then the MVP is the **MSRP**. Note again that this only applies to assets that are comprised in the MVP in the first place.

Exhibit 8 Risk attribution with respect to MVP, and implied risk premia (%) and SRs.

	Eq	Tsies	IG	HY	MVP
weight	5%	82%	0%	13%	
beta	1.00	1.00	1.35	1.00	
% risk contribution	5%	82%	0%	13%	100%
implied risk premium	0.94	0.94	1.27	0.94	0.94
implied Sharpe Ratio	0.06	0.23	0.21	0.09	0.27

(5) Exhibit 8 also confirms marginal risk contributions of MVP constituents are identical (all betas equal unity) so the money allocation equals the risk allocation in a MVP. **(6)** Exhibit 7 shows that the MVP has a lower historical risk premium and lower volatility than the MDP, yielding a slightly lower Sharpe Ratio. This lower volatility is achieved by overweighting Tsies at 82%, supplemented by positions in Equities and HY which are negatively correlated with Tsies. **(7)** Last but not least, the quadratic optimization underlying the MVP has the property of being "error maximizing", see Michaud [1989]. This means that the composition of the MVP is very sensitive to slight differences in variances and covariances. When (part of) these differences are not real but due to sampling error, this will propagate into portfolio composition. Again, note that we use the full historical sample to calculate the weights of the MVP.

6. Equal Risk Contribution Portfolio ERCP – full risk parity

Main references. Qian [2005] is the seminal paper on risk parity. Qian [2006] discusses the linear decomposition of volatility. Hallerbach [2003] extends risk decomposition to Value-at-Risk and shows how to decompose risk in parametric and non-parametric (simulation) settings. Maillard et al [2010] discuss the theoretical properties of risk parity portfolios and provide a comparison with other risk control techniques. Roncalli [2014] provides a good discussion of risk control techniques and Lee [2011] critically evaluates risk parity (see also section 10). Asness et al [2012] document the empirical outperformance of a risk parity strategy over a market cap weighted portfolio and refer to the leverage aversion effect to explain this outperformance. Anderson et al [2012] critically review and refute the empirical evidence provided by Asness et al [2012].

Recipe. The ERCP rests on the premise that no asset should dominate the portfolio risk profile. Consequently, all assets' contributions to portfolio risk are equalized. The contribution of an asset to portfolio risk equals its investment weight multiplied with its marginal contribution to portfolio risk. An asset's marginal contribution to portfolio risk equals its beta with respect to the portfolio. Hence, the weights of the ERCP satisfy:

(7) $$w_i \frac{\partial \sigma_p}{\partial w_i} = \frac{\partial \sigma_p}{\partial w_j} w_j \quad \Leftrightarrow \quad w_i \beta_{ip} = w_j \beta_{jp}$$

Hence, the weights in the ERCP are proportional to the inverse of the corresponding betas:

(8) $$w_i^{ERC} \sim 1/\beta_{ip}$$

For algorithms to calculate the composition of the ERCP, we refer to Roncalli [2014].

Characteristics. (1) The ERCP is the **1/N** portfolio when all assets have the same volatility σ and when all pairwise correlations are uniform at ρ . After all, in that case eq.(7) implies that $w_i \, \sigma \, \rho = w_j \, \sigma \, \rho$, which is satisfied for $w_i = w_j = 1/N$. **(2)** The ERCP is the **MDP** when all correlations are uniform: $\rho_{ip} = \rho_{jp}$. **(3)** The ERCP is the **MVP** when correlations are uniform (pairwise equal) ànd at their theoretically lowest level of $\rho = -1/(N-1)$, see Maillard et al [2010]. **(4)** The ERCP is the **MSRP** when all correlations are uniform ànd all assets have the same Sharpe Ratio, see Maillard et al [2010]. **(5)** With only two assets, the ERCP equals the IVP (see section 7).

Evaluation. (1) "Risk" is usually equated with standard deviation of return (volatility), but in principle any other risk measure can be chosen as long as the risk measure is linearly homogeneous in the portfolio weights. This means that when multiplying all portfolio weights with a constant *c*, the risk measure is also multiplied by the same constant *c*. Portfolio loss, Value-at-Risk (VaR) and Conditional VaR (or Expected Tail Loss) satisfy this property (see Hallerbach [2003], e.g.). **(2)** Since we can rewrite beta as the product of (a) the correlation with the portfolio and (b) the quotient of the asset and portfolio volatility, so $\beta_{ip} = \rho_{ip}\sigma_{if} / \sigma_{pf}$, eq.(8) implies that ERCPs favor assets with low levels of volatility and low correlations with other assets (hence: "portfolio diversifiers").

(3) Turning to historical portfolio statistics, Exhibit 7 shows that the ERCP had about half the risk of the market cap portfolio paired with a quarter lower average return, yielding a 50% higher Sharpe Ratio. This is due to overweighting Tsies and underweighting Equities (see Exhibit 9). The substantial tail wind of bonds over the past decades seriously biases back-tests of ERCPs.

(4) The ERCP is perfectly diversified in terms of risk (loss) contributions. **(5)** The ERCP is less concentrated than the MVP and the MDP, and it contains all *N* assets. **(6)** The ERCP is more robust, i.e. less error maximizing, than the MVP. The intuitive reason is that the MVP is found by means of optimization, i.e. by equating marginal risk contributions, whereas the ERCP is found by a restriction on the product of weights and marginal risk contributions. **(7)** Maillard et al [2010] show that $\sigma_{MVP} \le \sigma_{ERC} \le \sigma_{1/N}$, where the MVP is error maximizing and the 1/N portfolio focuses on money allocation, not risk allocation. Hence, the *ex ante* volatility of the ERCP is between the lowest level (from the MVP) and the volatility of the naively diversified 1/N portfolio. **(8)** Calculating the ERCP is a daunting task when the number of assets is very large. A solution would be to resort to a hierarchical procedure in which risk parity is first applied within groups (sectors, countries e.g.) and next across groups. However, pre-grouping directly influences the ERCP, see below.

(9) Exhibit 9, Panel A, shows the composition of the ERCP. Note the large 55% weight of Tsies, this is due to both their low volatility and their negative correlation with Equities and HY. The high volatility of Equities implies a lower than 25% weight. The implied risk premia and Sharpe Ratios can be interpreted as before.

(10) The composition of the ERCP depends on choosing the number of assets *N* and hence on any pre-grouping of assets (see Lee [2011]). ERCP(4) is on the basis of the 4 original assets. When aggregating IG and HY into a single credits sub-portfolio, ERCP(3), the risk allocations shift from 25% to 33%; see Exhibit 9, Panel B. **(11)** Leverage is needed to boost the low risk and return of ERCP in order to match any risk budgets or return targets.

In their empirical study, Asness et al [2012] illustrate the historical outperformance of ERCPs (or IVP since they consider only two asset classes, US equity and bonds) over a market cap weighted portfolio over the period 1926-2010. As an explanation they raise leverage aversion as the driving force behind the performance of ERCPs. This mechanism works as follows. (Some) investors are averse (or restricted) to applying leverage and they bid up the prices of high risk / high beta assets in order to fill their risk budget. As a consequence, the risk premium offered on

Exhibit 9 Risk attribution with respect to ERCP, and implied risk premia (%) and SRs.

Panel A: ERC (4)	**Eq**	**Tsies**	**IG**	**HY**	**ERCP**
weight	11%	55%	20%	14%	
beta	2.19	0.45	1.28	1.77	
% risk contribution	25%	25%	25%	25%	100%
implied risk premium	3.07	0.64	1.79	2.48	1.40
implied Sharpe Ratio	0.20	0.15	0.30	0.24	0.33
Panel B: ERC (3)	**Eq**	**Tsies**	**IG+HY**		
weight	15%	60%	25%		
beta	2.21	0.56	1.34		
% risk contribution	33%	33%	33%		100%
implied risk premium	2.86	0.72	1.74		1.30
implied Sharpe Ratio	0.19	0.17	0.27		0.32

high risk assets is reduced, implying that low beta (risk) assets offer higher risk-adjusted returns and high beta (risk) assets offer lower risk-adjusted returns. A less than average leverage-averse (or –constrained) investor can benefit by overweighting low beta (low risk) assets and underweighting high beta (high risk) assets. Leverage is applied to fill the risk budget or to attain a targeted risk level. In addition to leverage aversion, the “lottery ticket effect” may be at work, in which investors with a propensity to “gamble” overbid for high risk assets, thus reducing their risk premium. Finally, delegated portfolio management, centered around benchmarked portfolios, implies that low (high) risk stocks have large (small) tracking error. As argued by Blitz & van Vliet [2007], this introduces the low volatility anomaly, implying a flat or negative risk-return trade-off. Since low volatility assets outperform and ERCPs overweight low risk assets, this may explain their outperformance.

Anderson et al [2012] raise some serious back test issues in the research by Asness et al [2012]. They note that the outperformance of the ERCP is not uniform over sub-periods and they show that market frictions (borrowing costs and turn-over induced trading costs) eat into performance. In addition, they argue that Asness et al’s [2012] risk parity strategy is not an investable strategy since it uses unconditional leverage: they use a constant scale factor, computed from the full 1926-2010 period, to match the volatilities of the levered risk parity strategy and the market cap portfolio. Hence, their empirical set-up suffers from a look-ahead bias. Implementing conditional leverage (where at each rebalancing date the volatility scale factor is derived from past 3 year trailing windows), Anderson et al [2012] show that this halves the cumulative total return of the risk parity strategy as reported by Asness et al [2012]. Realistic borrowing costs and trading costs further reduce the cumulative total return of the risk parity strategy. In all, these realistic adjustments made the performance difference between the risk parity strategy and the market cap portfolio disappear.

7. Inverse Volatility Portfolio IVP – naïve risk parity

Main reference: Maillard et al [2010] discuss the IVP next to the ERCP, although volatility weighting (or “normalization”) has been applied for long by practitioners to improve cross-asset comparability and to reduce portfolio or strategy risk. (This may be inspired by statistics, where inverse variance weighting is used to minimize the variance of the sum of two or more random variables.)

Recipe. Set each weight proportional to the stand-alone volatility of the corresponding asset and next normalize so that the weights sum to unity. This volatility-weighting in the cross-section yields:

$$w_i = \frac{\frac{1}{\sigma_i}}{\sum_j \frac{1}{\sigma_j}} \tag{9}$$

The IVP is equivalent to the **ERCP** when there are only two assets. (In the two-asset case, the correlation is irrelevant.) In all other cases, neglecting correlation information makes IVP a “naïve” risk parity strategy.

Characteristics. (1) When correlations are uniform (or zero), the IVP is the **ECRP**. (In that case, all comments made for ERCPs also apply for IVPs). When everything else is equal, then compared to the IVP, the ERCP will be tilted towards low correlated assets. **(2)** When volatilities are uniform, the IVP is the **1/N** portfolio.

The S&P Low Volatility Index is composed of the 100 stocks from the S&P500 Index with the lowest (252 days past) volatility, where each stock is weighted with its inverse volatility. The MSCI Risk Weighted Indices use inverse variance (and not volatility) to weight constituents. **(3)** Inverse variance weighting yields the **MVP** when all correlations are uniform (or zero).

Evaluation. Except for the impact of (markedly different) correlations, IVPs are quite similar to ERCPs. As shown in Exhibit 10, the IVP assigns more weight to IG (was 20%) and less weight to Tsies (was 55%). The latter can be explained because the IVP ignores the negative correlation with Equities and HY. This shift in weights translates into less balanced risk contributions.

Exhibit 10 Risk attribution with respect to IVP, and implied risk premia (%) and SRs.

	Eq	Tsies	IG	HY	IVP
weight	12%	42%	29%	17%	100%
beta	2.12	0.29	1.15	1.76	
% risk contribution	25%	12%	33%	29%	100%
implied risk premium	3.80	0.52	2.06	3.16	1.79
implied Sharpe Ratio	0.25	0.13	0.34	0.30	0.38

On a historical basis, Exhibit 7 shows that the IVP had a higher volatility and a somewhat higher average return than the ERCP. This combined effect is due to the lower weight of Tsies (which have the lowest average return, the lowest volatility, and negative correlations with Equities and HY).

8. Maximum Sharpe Ratio Portfolio MSRP

Main references. For a discussion of the Sharpe ratio, see Sharpe [1994]. For mean-variance portfolio theory and for finding the composition of the MSRP, we refer to standard investment texts.

Recipe. Choose the portfolio weights to maximize the Sharpe Ratio:

(10) $$\max_{\{w\}} SR_p = \frac{\overline{r}_{pf}}{\sigma_{pf}}$$

Characteristics. (1) Within the MSRP, the ratios of marginal contributions to risk and return are constant. Since an asset's marginal contribution to the portfolio risk premium equals the asset's risk premium, $\overline{r}_{if}$, and since this asset's marginal contribution to portfolio risk is its beta, β_{ip}, we require: $\overline{r}_{if} / \beta_{ip} = \overline{r}_{jf} / \beta_{jp} = \overline{r}_{pf}$ (where the last equality follows from the fact that the portfolio beta equals unity, see also eq.(4)). Note that $\overline{r}_{if} / \beta_{ip}$ is the Treynor [1966] risk-adjusted performance ratio. Hence, for each asset within the MSRP, the risk premium should be equal to the product of its beta with respect to the MSRP and the risk premium of the MSRP:

(11) $$\overline{r}_{if} = \beta_{ip}\overline{r}_{pf}$$

This is the first-order condition of mean-variance optimality. **(2)** Since we can rewrite the beta as the product of (a) the correlation with the portfolio and (b) the quotient of the asset and portfolio volatility, $\beta_{ip} = \rho_{ip}\sigma_{if} / \sigma_{pf}$, it follows that in the MSRP the stand-alone asset Sharpe Ratios and the portfolio's Sharpe ratio are related by $SR_i = \rho_{ip} SR_p$. When $SR_i > \rho_{ip} SR_p$, we can increase the Sharpe Ratio of the portfolio by increasing the weight of (or adding) asset *i* to the portfolio *p*. **(3)** Without any additional constraints, the long-only MSRP can be a very concentrated portfolio. **(4)** When all volatilities, correlations and risk premia are the same, then the MSRP is the **1/N** portfolio (which then also coincides with the **ERCP** and the **MVP**). After all, diversification lowers risk but in the portfolio context all assets are perfect substitutes. It is not possible to lower portfolio risk or increase the portfolio risk premium by changing the weights. Hence, we end up with the equally-weighted portfolio.

Evaluation. Historically, the MSRP has the maximum Sharpe Ratio, see Exhibit 7. This is so by construction, since we optimized the Sharpe Ratio over the full historical sample period (in-sample). In practice, one would sequentially derive the *ex ante* MSRP from trailing data windows. Whether the MSRP indeed delivers the maximum Sharpe Ratio *ex post* depends on the quality of the inputs, especially the risk premia.

In our example, the MSRP is indeed a concentrated portfolio, containing mostly Tsies supplemented with HY and only 7% Equities, see Exhibit 11. Tsies dominate because of their low volatility and negative correlation with HY. The smaller than unity beta of Tsies reveal that Tsies are included as a diversifier; the larger than unity betas of HY and Equities show that these assets are included because of their high average return. Slight changes in risk premia will change the composition of the MSRP markedly.

Exhibit 11 Risk attribution with respect to MSRP, and implied risk premia (%) and SRs.

	Eq	Tsies	USIG	USHY	sum
weight	7%	74%	0%	19%	
beta	1.90	0.75	1.41	1.67	
% risk contribution	13%	55%	0%	31%	100%
implied risk premium	1.93	0.76	1.43	1.69	1.01
implied Sharpe Ratio	0.13	0.18	0.24	0.16	0.28

9. Volatility weighting over time

The risk control strategies as discussed before focus on risk in the cross-section, i.e. over portfolio constituents. Risk control at each point in time will also affect the portfolio's risk level (or more generally, its return distribution) over time. Volatility weighting over time, and specifically volatility targeting, is designed to explicitly control the portfolio risk level over time. Typically, when weighting or targeting a portfolio's risk level over time, the composition of a portfolio's risky part is not changed, only the weights of the risky and the risk free part are adjusted.

Main references. Kirby & Ostdiek [2012] document the empirical finding that volatility weighting improves the Sharpe Ratio. Hallerbach [2012] provides a formal proof that, under mild assumptions, volatility weighting over time indeed increases the Sharpe Ratio or Information Ratio.

Recipe. (1) Set the risky portfolio's target volatility level *V*. **(2)** At the start of each period *t*, take a position *w* in the risky portfolio and (1–*w*) in the risk free asset:

(12) $$w_t \cdot \tilde{r}_{pt} + (1 - w_t) \cdot r_{ft} = w_t \cdot \tilde{r}_{pft} + r_{ft}$$

(3) Estimate the volatility of the risky portfolio for period *t*, $\hat{\sigma}_t$, for example by using an adaptive Exponentially-Weighted Moving Average (EWMA) volatility process. **(4)** Rescale the exposure to the risky portfolio to the target volatility level *V*: $w_t = V / \hat{\sigma}_t$. According to eq.(12), this implies adding a cash position or borrowing (when allowed) at the suitable borrowing rate, subject to a leverage constraint. **(5)** Apply the leverage constraint. When the volatility target *V* is high or when the forecasted volatility is low, cap the implied borrowing by setting $w_t \le L$, where the maximum leverage ratio satisfies $L \ge 1$. When $L = 1$, no borrowing is allowed.

Characteristics. (1) Volatility weighting and volatility targeting accomplish volatility smoothing over time. Volatility smoothing mitigates the volatility of the portfolio volatility over time. It can be shown that the lower the fluctuations of the temporal ("instantaneous") portfolio volatility **within** some time period, the lower the aggregate volatility **over** the whole time period (see Hallerbach [2012]). **(2)** Note that volatility smoothing is different from return smoothing. Return smoothing aims at achieving a lower aggregate level of return volatility (and not a lower volatility of the volatility over time). Return smoothing thus implies less "variance slippage" in compounded returns. This variance slippage refers to the difference between the arithmetic mean and the geometric mean return. As an approximation, we have geometric mean ≈ arithmetic mean – ½ variance. Lowering the return variance by return smoothing thus increases the geometric mean of returns, cet. par. **(3)** Naïve risk parity or the **IVP**, i.e. volatility weighting in cross-section, already establishes some volatility weighting in time series. **(4)** Risk targeting or risk control indices have been introduced by S&P, MSCI, FTSE, DJ, and EURO STOXX.

Evaluation – or: Why would volatility targeting work? **(1)** First of all, depending on the quality of our volatility forecasts, we should be able to target a portfolio's volatility to some degree over time. **(2)** In addition, it can be shown that this volatility smoothing increases the Sharpe Ratio or Information Ratio of the portfolio, cet. par. (Hallerbach [2012]). **(3)** Furthermore, the (risk-adjusted) return of a volatility-targeted portfolio benefits from an additional timing effect, due to the so-called asymmetric volatility phenomenon. The asymmetric volatility phenomenon is a stylized fact that is observed for most financial markets. Returns tend to be negatively correlated with volatility and especially surges in financial market volatility are mostly associated with negative returns. The volatility feedback mechanism is that higher expected volatility translates into a higher risk premium and consequently lower future realized returns. Hence, under asymmetric volatility, there is a timing effect (in addition to the smoothing of volatility) that will boost performance. After all, a volatility-weighting strategy takes large positions when volatility is low (and returns are high) and small positions when volatility is high (and returns are low); see also Zakamulin [2014]. **(4)** As a cautionary (and perhaps superfluous) note, we stress that implementing a volatility-weighted strategy calls for a strict risk-budgeting and risk-monitoring process. In particular, one may want to set limits to the maximum position size in order to mitigate the risk of blow-ups when the contemporaneous volatility is relatively low.

10. Evaluation

Main references: Inker [2011], Lee [2011], Leote de Carvalho et al [2012], Goldberg & Mahmoud [2013] and Roncalli [2014].

Conventional 60/40 portfolios or MSRPs are concentrated in risks and fail to offer diversification against losses. For this reason, the use of risk control techniques (and especially risk parity) as full-fledged investment criteria is sometimes coined the “new paradigm” in investing. Indeed, risk control strategies, and risk parity in particular, can produce balanced portfolios and can offer various degrees of diversification. From a risk perspective, these techniques are indeed expected to deliver what they promise. The true value of risk control strategies is in analyzing and specifying a preferred risk contribution profile within the portfolio. This should be part of any risk budgeting process. Relevant questions are: What are the risk contributions of the portfolio components?, Is the portfolio properly diversified or are there any hot spots?, How much confidence do we have in risk premia views in order to shift risk contributions within the portfolio?, Do we fully understand the sources and contributions of risk and return of the portfolio? And last but not least, risk is a multi-dimensional concept, so risk analyses should not only focus on volatility (standard deviation) but also take downside risk and event risk in consideration.

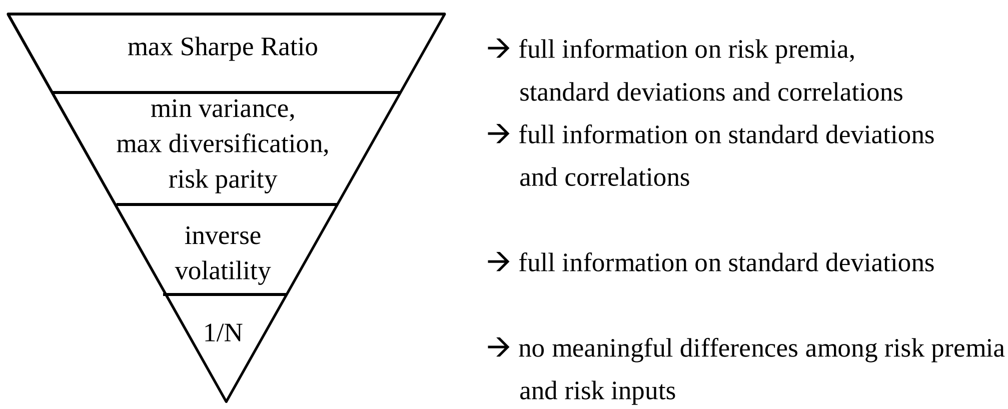
The catch is that risk control portfolios appear to have historically outperformed market cap weighted or mean-variance optimized portfolios. So while ignoring risk premia information, risk control strategies seem to offer a better risk-return trade-off. Some critical comments are in place, however. First of all, several studies tune down the apparent outperformance of risk-based strategies by criticizing back-tests, see section 6 and Goldberg & Mahmoud [2013]. Secondly, when the underlying mechanism of outperformance is an implicit exposure to anomalies or factor premia such as value, size, low beta or low (residual) volatility (as shown by Scherer [2011], Leote de Carvalho et al [2012] and Jurczenko et al [2013]), then it makes much more sense to consider these factor exposures explicitly when forming portfolios. Factor investing (Ang [2014]) provides much more efficient and effective ways to tailor factor exposures on the portfolio level than applying risk control techniques. In the former case, factor exposures are taken intentionally and top-down, whereas in the latter case it is not clear what factor exposures will percolate bottom-up and reveal themselves in the portfolio.

However, aside from their obvious value in a risk budgeting context, risk control strategies can provide a sensible heuristic or starting point in the portfolio formation process if there is considerable uncertainty about the required asset attributes, viz. the risk premia and (co-) variance inputs. When the available information on an attribute allows for meaningfully differentiating between assets, then the portfolio formation process is steered by this attribute. Alternatively, when the required information on an attribute is lacking (or surrounded by substantial estimation error) then this attribute does not have discriminating power between assets. Consequently, it then makes sense to treat the assets as substitutes regarding this attribute. Indeed, we know that mean-variance optimized portfolios are error-maximizing (Michaud [1989]) in the sense that their composition is very sensitive to especially risk premia inputs. So the presence of estimation risk can justify the use of risk control techniques: when we do not have information to meaningfully differentiate between the assets’ risk premia (or Sharpe Ratios), the recipe is to treat all assets as risk premia “substitutes” and focus only on their risk attributes.

To further illustrate this point, we introduce our portfolio decision pyramid, see Exhibit 12. This pyramid illustrates the increasing requirements that apply to portfolio optimization inputs when moving from a naively diversified portfolio to the maximum Sharpe Ratio portfolio. **(1)** Starting at the bottom of this inverted pyramid, one cannot indicate any meaningful differences among risk premia, standard deviations and correlations. The best one can do is to naively diversify and equate money weights within the portfolio, yielding the 1/N portfolio. **(2)** Having reliable trust in differences among standard deviations allows for shifting from naïve money weight diversification to naïve risk weight diversification by applying volatility-weighting. This yields the IVP. **(3)** On the third level, one has full risk information (reliable estimates of both volatilities and correlations, the full covariance matrix is available), so the MVP, the MDP or ERCP (full risk parity portfolios) can be constructed. Of course, one has to take into account the relative shortcomings of these portfolios as noted in the relevant sections above. **(4)** Finally, at the top level, one is able to indicate meaningful differences between all relevant inputs, i.e. the *ex ante* covariance matrix ànd risk premia. In that case, one can perform a full-fledged mean-variance optimization and obtain the MSRP.

Of course, estimation risk does not necessarily dictate to ignore risk premia inputs altogether and to stop at a risk control portfolio. As outlined in section 2 and the Appendix, estimation risk surrounding risk premia can be tackled by the Black-Litterman [1992] approach. We start from a risk control reference portfolio, calculate the implied risk premia and next use our views and the confidence we place in these views to (slightly) adjust the optimization inputs (for a detailed exposition, see Haesen et al [2014]). Depending on our convictions regarding risk premia, the resulting portfolio is situated between the risk control portfolio and the MSRP.

Exhibit 12 The portfolio decision pyramid: the link between portfolio rules and the information burden placed on the investor when adopting that rule.



Appendix

Asset and portfolio (excess) returns

We start with an opportunity set of N securities with returns $\tilde{r}_{it}$. Tildes indicate random variables. For notational simplicity, we henceforth ignore the time index t. The risk free rate is denoted by r_f , so the excess returns are $\tilde{r}_i - r_f \equiv \tilde{r}_{if}$. We consider a portfolio p defined by the investment weights $\{w_i\}_{i\in p}$, satisfying full investment $\sum_{i=1}^{N} w_i = 1$ and no short positions: $w_i \geq 0,\ \forall i \in p$. The portfolio return is given by $\tilde{r}_p = \sum_{i\in p} w_i \tilde{r}_i$. Likewise, the portfolio excess return is given by:

(13) $\tilde{r}_{pf} = \sum_i w_i\, \tilde{r}_{if}$

Marginal and component contributions to portfolio (excess) return

It follows from eq.(13) that the marginal contribution of asset i to portfolio excess return is given by $\tilde{r}_{if}$. This is the increase in portfolio excess return when the weight of asset i is increased marginally. The component (i.e. full) contribution of asset i to portfolio excess return is $w_i \tilde{r}_{if}$. The sum of component contributions to excess return equals the portfolio's excess return, see eq.(13).

Portfolio risk premium

The average portfolio return over the risk free rate, the portfolio risk premium, follows as:

(14) $\overline{r}_{pf} = \sum_{i\in p} w_i\, \overline{r}_{if}$

The marginal and component contributions of asset i to the portfolio risk premium are $\overline{r}_{if}$ and $w_i \overline{r}_{if}$, respectively.

Portfolio variance

The variance of portfolio excess returns is defined by the double sum:

(15) $\sigma_{pf}^2 = \sum_i \sum_j w_i w_j \sigma_{ij}$.

By definition of the correlation ρ_{ij} , the covariance σ_{ij} can be expressed as $\sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$.

Since (a) the variance of a variable is the covariance of that variable with itself, and (b) the covariance is a linear operator (the covariance of a weighted sum is the weighted sum of covariances), we can write the variance of the portfolio excess return as:

(16) $\operatorname{var}\left(\tilde{r}_{pf}\right) \equiv \operatorname{cov}\left(\tilde{r}_{pf}, \tilde{r}_{pf}\right) = \operatorname{cov}\left(\sum_i w_i \tilde{r}_{if}, \tilde{r}_{pf}\right) = \sum_i w_i \operatorname{cov}\left(\tilde{r}_{if}, \tilde{r}_{pf}\right) = \sum_i w_i \sigma_{ipf}$,

where σ_{ipf} is the covariance between the excess returns on asset i and the portfolio p. So although the portfolio variance is the quadratic sum of weights and covariances, we can express the portfolio variance as the weighted sum of the covariances of each asset with the portfolio: $\sigma_{pf}^2 = \sum_i w_i\, \sigma_{ipf}$.

Decomposing portfolio volatility

Dividing the previous expression by the portfolio volatility, we get:

(17) $\sigma_{pf} = \sum_i w_i \dfrac{\sigma_{ipf}}{\sigma_{pf}} = \sum_i w_i\, \sigma_{pf}\, \rho_{ipf}$

Indeed, it is not the decomposition of portfolio variance we are looking for, but the decomposition of portfolio volatility, as defined by eq.(17). To see why this is true, note that the portfolio volatility is linearly homogeneous in the portfolio weights: multiplying portfolio weights with a constant k multiplies the portfolio volatility with the same constant k. Euler's theorem then implies that $\sigma_{pf} = \sum_i w_i\ \partial\sigma_{pf} / \partial w_i$, where it can be checked from (16) that $\partial\sigma_{pf} / \partial w_i = \sigma_{ipf} / \sigma_{pf}$. The term $\partial\sigma_{pf} / \partial w_i$ is the marginal contribution of asset i to portfolio volatility**.** The term $w_i\, \sigma_{ipf} / \sigma_{pf}$ is the component contribution of asset i to portfolio volatility**.** The sum of all component contributions to volatility equals total portfolio volatility, see eq.(17). The portfolio volatility is the cake and each component contribution is a separate piece of that cake. Dividing (17) by σ_{pf} yields the relative risk contributions of the assets, summing to 100%:

(18) $$1 = \sum_i w_i \frac{\sigma_{ipf}}{\sigma_{pf}^2}$$

To gain further insight into this decomposition, consider the OLS regression of asset's *i* excess returns on the portfolio excess returns:

(19) $$\tilde{r}_{if} = \alpha_i + \beta_{ip}\tilde{r}_{pf} + \tilde{\varepsilon}_i .$$

In this regression, the expected (or average) value of the disturbances is zero and the disturbances and the portfolio excess return are uncorrelated, hence $E\left(\tilde{\varepsilon}_i\right) = E\left(\tilde{r}_{pf}\,\tilde{\varepsilon}_i\right) = 0$. The regression slope or beta is defined as:

(20) $$\beta_{ip} = \frac{\sigma_{ipf}}{\sigma_{pf}^2} = \rho_{ip}\frac{\sigma_{if}}{\sigma_{pf}}$$

Substituting the expression for beta in (18) gives:

(21) $$1 = \sum_i w_i\,\beta_{ip}$$

So β_{ip} is the relative marginal contribution of asset *i* to portfolio volatility (or the relative marginal risk contribution):

(22) $$\beta_{ip} = \frac{\partial\sigma_{pf}\ /\ \sigma_{pf}}{\partial w_i}$$

and $w_i\beta_{ip}$ is the asset's relative component contribution to portfolio volatility. So given the assets' betas, the decomposition of portfolio volatility is a piece of cake. When $w_i\beta_{ip}$ is comparatively large, this identifies a "hot spot" in the portfolio, or a pocket of risk concentration, indicating that asset's *i* contribution to portfolio risk is large. Hence, this position is likely to contribute heavily to any loss that may be realized on the portfolio. In short, $\{w_i\}$ defines money allocation and $\{w_i \cdot \beta_i\}$ defines risk allocation. To go from money allocation to risk allocation, each investment weight is multiplied with the corresponding beta (note that the average value of beta is unity).

Portfolio optimality: maximize the Sharpe Ratio

From eq.(19) it follows that the expected excess return or risk premium of asset i is related to the portfolio's risk premium as:

(23) $$\bar{r}_{if} = \alpha_i + \beta_{ip}\bar{r}_{pf}$$

Now consider the mean-variance optimal portfolio, this is the portfolio that maximizes the Sharpe Ratio:

(24) $$\max_{\{w_i\}_{i\in p}} \ SR_p = \frac{\bar{r}_{pf}}{\sigma_{pf}}$$

The first-order conditions for optimality imply the following relation between risk premia and betas:

(25) $$\bar{r}_{if} = \beta_{ip}\bar{r}_{pf}$$

In words: for the maximum Sharpe Ratio Portfolio MSRP, the risk premia of all constituents are proportional to their betas. Considering eq.(23), this implies that for all assets included in the MSRP *p**, the alpha α_i equals zero, $\alpha_i = 0 \quad \forall i \in p^*$. To provide some intuition, note that for each asset comprised in a maximum Sharpe Ratio portfolio the relative marginal contribution to excess return must equal the relative marginal contribution to risk, or:

(26) $$\frac{\bar{r}_{if}}{\bar{r}_{pf}} = \beta_{ip}$$

This can be rephrased as requiring equal ratios of marginal return and risk contributions:[5]

(27) $$\frac{\bar{r}_{if}}{\beta_{ip}} = \frac{\bar{r}_{jf}}{\beta_{jp}} = \bar{r}_{pf}$$

---

[5] Note that $\bar{r}_{if}\ /\ \beta_{ip}$ is the Treynor [1966] ratio of risk-adjusted performance.

If this does not hold, the Sharpe Ratio of the portfolio can be improved by increasing the weight of the assets with higher contributions to return (or lower contributions to risk) and decreasing the weight of assets with lower contributions to return (or higher contributions to risk).

In other words, referring to eq.(23), when an asset's alpha is positive, $\alpha_i > 0$, this asset shows outperformance against the portfolio and the Sharpe Ratio of the portfolio can be increased by increasing the weight of this asset. Conversely, when an asset's alpha is negative, $\alpha_i < 0$, this asset shows underperformance and the portfolio's Sharpe Ratio can be increased by decreasing the weight of this asset.

Two additional comments are in order. Firstly, in eq.(25) we recognize the infamous "Security Market Line" or the Capital Asset Pricing Model (CAPM). However, the results above apply to ***any*** maximum Sharpe Ratio portfolio, whereas the CAPM applies to the equally infamous market portfolio (the overall market cap weighted portfolio containing all assets) under the heroic equilibrium assumption that this portfolio is mean-variance efficient. Hence, the results presented above are completely general.

Secondly, using the second definition of beta in eq.(20) allows us to rewrite (25) as $\overline{r}_{if} / \sigma_{if} = \rho_{ip}\overline{r}_{pf} / \sigma_{pf}$.
Using the definition of the Sharpe Ratio, this boils down to:

(28) $$SR_i = \rho_{ip} SR_p$$

In words: for any maximum Sharpe Ratio portfolio, any constituent's the stand-alone Sharpe Ratio equals the product of (a) its correlation with this portfolio and (b) the Sharpe Ratio of the portfolio. When an asset's Sharpe Ratio is larger (smaller), this implies that the asset's alpha is positive (negative). This also applies to assets not comprised in the portfolio. If $\alpha_i > 0$, or equivalently $SR_i > \rho_{ip} SR_p$, then the Sharpe Ratio of the portfolio is increased by adding that asset to the portfolio (and *vice versa*).

Reverse optimization: implied risk premia

In conventional mean-variance portfolio optimization, the asset's risk premia and their covariance matrix are used to calculate the weights of the maximum Sharpe Ratio portfolio. In reverse portfolio optimization, it is assumed that the portfolio at hand actually ***is*** the maximum Sharpe Ratio portfolio. Together with the covariance matrix of excess returns this allows us to derive the "imputed" risk premia (Sharpe [1974]). Using these implied risk premia together with the asset's standard deviations, we can then compute the implied Sharpe Ratios. Hence, given a particular portfolio, these implied risk premia (or implied Sharpe Ratios) would make this portfolio the maximum Sharpe Ratio portfolio.

How do we derive these implied risk premia? We start from the historical risk premium of the market cap portfolio. Assuming that this portfolio p^* is mean-variance efficient, we can calculate the implied coefficient of relative risk aversion λ^* from $\lambda^* = \overline{r}_{p^*f} / \sigma^2_{p^*f}$ (Sharpe [1974]). Using the historical average excess return and volatility of the market cap portfolio in Exhibit 1, this yields $\lambda^* = 8.0$. Switching to an alternative portfolio *p* with historical volatility σ_{pf} , then assuming that this portfolio is mean-variance efficient implies that its corresponding risk premium is $\overline{r}_{pf}^* = \lambda^* \cdot \sigma^2_{pf}$ Given this implied portfolio risk premium, we finally use the first-order condition for the MSRP in eq.(25) together with the asset betas to compute the implied risk premium $\overline{r}_{if}^*$ as the product of the beta and the implied portfolio risk premium:

(29) $$\overline{r}_{if}^* = \beta_{ip} \cdot \overline{r}_{pf}^*$$

The implied Sharpe Ratio then readily follows as $SR_i^* = \overline{r}_{if}^* / \sigma_{if}$.

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