

# CS/MATH 111, Discrete Structures - Winter 2019.

## Discussion 5 - Linear Recurrence Relations

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# Outline

Fibonacci grows exponentially

Fibonacci numbers

Homogeneous Recurrence Equations

Word problem

# Fibonacci recurrence

- ▶ Fibonacci numbers / Fibonacci sequence
- ▶ First two and subsequent numbers:
  - ▶  $F_0 = 1$
  - ▶  $F_1 = 1$
  - ▶  $F_n = F_{n-1} + F_{n-2}$  , when  $n \geq 2$ .
- ▶ Fibonacci grows exponentially with  $n$ .
- ▶ Prove that:

$$0.5 \cdot 1.5^n \leq F_n \leq 2^n$$

# Fibonacci recurrence

Proof by induction using  $F_0 = 1, F_1 = 1, \dots$ :

$$0.5 \cdot 1.5^n \leq F_n \leq 2^n \quad (1)$$

## 1. Base case:

- ▶  $n = 0, F_0 = 1 : 0.5 \cdot 1.5^0 \leq 1 \leq 2^0 = 0.50 \leq 1 \leq 1.$
- ▶  $n = 1, F_1 = 1 : 0.5 \cdot 1.5^1 \leq 1 \leq 2^1 = 0.75 \leq 1 \leq 2.$
- ▶  $n = 2, F_2 = 2 : 0.5 \cdot 1.5^2 \leq 1 \leq 2^2 = 1.125 \leq 2 \leq 4.$
- ▶  $\vdots$

## 2. Assumption step:

- ▶ Assume (1) holds for all  $n \leq k - 1.$

## 3. Induction step:

- ▶ Prove that (1) holds for all  $n \leq k.$

# Fibonacci recurrence

Proof by induction using  $F_0 = 1, F_1 = 1, \dots$ :

$$0.5 \cdot 1.5^n \leq F_n \leq 2^n$$

## 3 Induction step:

► Prove that (1) holds for all  $n \leq k$ .

$$\begin{aligned}
 1) \quad & F_k \leq 2^k \\
 & F_k = F_{k-1} + F_{k-2} \\
 & F_k \leq 2^{k-1} + 2^{k-2} \text{ (by assumption.)} \\
 & F_k = 2^{k-2} \cdot (2 + 1) \\
 & F_k = 2^{k-2} \cdot 3 \\
 & F_k \leq 2^{k-2} \cdot 4 \\
 & F_k = 2^{k-2} \cdot 2^2 \\
 & F_k = 2^k
 \end{aligned}$$

$$F_n = \mathcal{O}(2^n)$$

# Fibonacci recurrence

Proof by induction using  $F_0 = 1, F_1 = 1, \dots$ :

$$0.5 \cdot 1.5^n \leq F_n \leq 2^n$$

## 3 Induction step:

► Prove that (1) holds for all  $n \leq k$ .

$$2) F_k \geq \frac{1}{2} \cdot 1.5^k$$

$$F_k = F_{k-1} + F_{k-2}$$

$$F_k \geq 1.5^{k-1} + 1.5^{k-2} \text{ (by assumption.)}$$

$$F_k = 1.5^{k-2} \cdot (1.5 + 1)$$

$$F_k = 1.5^{k-2} \cdot 2.5$$

$$F_k \geq 1.5^{k-2} \cdot 2.25$$

$$F_k = 1.5^{k-2} \cdot 1.5^2$$

$$F_k = 1.5^k$$

$$F_n = \Omega(1.5^n)$$

# Fibonacci recurrence

$$0.5 \cdot 1.5^n \leq F_n \leq 2^n$$

$$F_n = \mathcal{O}(2^n)$$

$$F_n = \Omega(1.5^n)$$

$$\text{is } F_n = \Theta(\ ) ?$$

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# Fibonacci numbers

- ▶ Let's rewrite our recurrence following our previous notation:

$$F_n = F_{n-1} + F_{n-2}. \quad (2)$$

for  $n \geq 2$ .

- ▶  $F_0 = 1$

- ▶  $F_1 = 1$

- ▶ Since  $F_n$  grows exponentially, we will assume:

$$F_n = x^n \quad (3)$$

- ▶ Plugging (3) into (2):

$$x^n = x^{n-1} + x^{n-2}$$

after dividing by  $x^{n-2}$ :

$$x^2 - x - 1 = 0$$

# Fibonacci numbers

$$x^2 - x - 1 = 0$$

- ▶ This equation has two roots:  $x_1 = \frac{1}{2}(1 + \sqrt{5})$  and  $x_2 = \frac{1}{2}(1 - \sqrt{5})$ .
- ▶  $x_1$  is the *golden ratio*<sup>1</sup>  $\phi \approx 1.618$  and  $x_2 = 1 - \phi \approx -0.618$ .
- ▶ Do they satisfy (2)? It works for  $n = 0$  but not for  $n = 1$ .
- ▶ Works for the main recurrence but not for the initial conditions...

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<sup>1</sup>[https://en.wikipedia.org/wiki/Golden\\_ratio](https://en.wikipedia.org/wiki/Golden_ratio)

# Fibonacci numbers

## Theorem 1

*Let  $c_1$  and  $c_2$  be real numbers. Suppose that  $r^2 - c_1r - c_2 = 0$  has two distinct roots  $r_1$  and  $r_2$ . Then the sequence  $a_n$  is a solution of the recurrence relation  $a_n = c_1a_{n-1} + c_2a_{n-2}$  if and only if*

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

*for  $n = 0, 1, 2, \dots$ , where  $\alpha_1$  and  $\alpha_2$  are constants<sup>2</sup>.*

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<sup>2</sup>Proof available at [Rosen, 2015. pg 515].

# Fibonacci numbers

$$x^2 - x - 1 = 0$$

- ▶ This equation has two roots:  $x_1 = \frac{1}{2}(1 + \sqrt{5})$  and  $x_2 = \frac{1}{2}(1 - \sqrt{5})$ .
- ▶ Therefore by Theorem 1 it follows that the Fibonacci numbers are given by:

$$F_n = \alpha_1 x_1^n + \alpha_2 x_2^n$$

- ▶ This form is called the *general form of the solution*.

# Fibonacci numbers

$$F_n = \alpha_1 x_1^n + \alpha_2 x_2^n$$

- ▶ Plugging the initial conditions into this equation we will get a system of two equations and two parameters:

$$\alpha_1 x_1^0 + \alpha_2 x_2^0 = 1$$

$$\alpha_1 x_1^1 + \alpha_2 x_2^1 = 1$$

After substituting:

$$\alpha_1 + \alpha_2 = 1$$

$$\alpha_1 \cdot \frac{1}{2}(1 + \sqrt{5}) + \alpha_2 \cdot \frac{1}{2}(1 - \sqrt{5}) = 1$$

# Fibonacci numbers

$$\alpha_1 + \alpha_2 = 1$$

$$\alpha_1 \cdot \frac{1}{2}(1 + \sqrt{5}) + \alpha_2 \cdot \frac{1}{2}(1 - \sqrt{5}) = 1$$

► Solving the system, we get:

$$\alpha_1 = \frac{\sqrt{5} + 1}{2\sqrt{5}}$$

$$\alpha_2 = \frac{\sqrt{5} - 1}{2\sqrt{5}}$$

# Fibonacci numbers

- This give us a solution for  $F_n$ :

$$F_n = \frac{\sqrt{5} + 1}{2\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n + \frac{\sqrt{5} - 1}{2\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n$$

Simplified as:

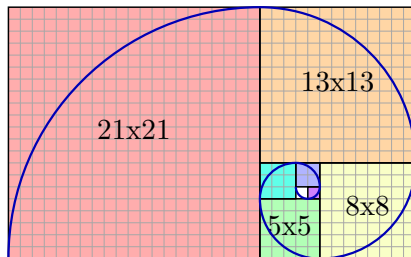
$$F_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^{n+1} - \left( \frac{1 - \sqrt{5}}{2} \right)^{n+1} \right]$$

# Fibonacci numbers

$$F_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^{n+1} - \left( \frac{1 - \sqrt{5}}{2} \right)^{n+1} \right]$$

► Note that when  $n \rightarrow \infty$ :

$$F_n \approx \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^{n+1}$$





# Double root

Let's have the following recurrence relation:

$$a_n = 4a_{n-1} - 4a_{n-2}$$

with initial condition  $a_0 = 1$  and  $a_1 = 3$ .

- ▶ Let's find the characteristic equation and its roots:

$$x^2 - 4x + 4 = 0.$$

$$(x - 2)^2 = 0.$$

So,  $x_{1,2} = 2$ . [“double root” or a root with multiplicity of 2.]

# Double root

Let's have the following recurrence relation:

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with initial condition  $a_0 = 1$  and  $a_1 = 3$ .

- ▶ All functions  $\alpha_1 2^n$  are good candidates, but...
- ▶ ... we need our solution to be parametrized by **two** parameters.
- ▶ We can try the function  $n2^n$ :

$$n2^n = 4(n-1)2^{n-1} - 4(n-2)2^{n-2}$$

by simple algebra we see that is is indeed true.

# Double root

Let's have the following recurrence relation:

$$a_n = 4a_{n-1} - 4a_{n-2}$$

with initial condition  $a_0 = 1$  and  $a_1 = 3$ .

- ▶ Since  $n2^n$  is a solution, so is any function  $\alpha_2 n 2^n$ .
- ▶ We can combine the two types of solutions in a general form:

$$a_n = \alpha_1 2^n + \alpha_2 n 2^n$$

then we can continue...

## Double root

Let's have the following recurrence relation:

$$a_n = 4a_{n-1} - 4a_{n-2}$$

with initial condition  $a_0 = 1$  and  $a_1 = 3$ .

► Initial condition equations and their solutions:

$$\alpha_1 \cdot 2^0 + \alpha_2 \cdot 0 \cdot 2^0 = 1$$

$$\alpha_1 \cdot 2^1 + \alpha_2 \cdot 1 \cdot 2^1 = 3$$

which reduce to:

$$\alpha_1 = 1$$

$$2\alpha_1 + 2\alpha_2 = 3$$

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with initial condition  $a_0 = 1$  and  $a_1 = 3$ .

► Initial condition equations and their solutions:

$$\alpha_1 = 1$$

$$2\alpha_1 + 2\alpha_2 = 3$$

We get  $\alpha_1 = 1$  and  $\alpha_2 = \frac{1}{2}$ .

# Double root

Let's have the following recurrence relation:

$$a_n = 4a_{n-1} - 4a_{n-2}$$

with initial condition  $a_0 = 1$  and  $a_1 = 3$ .

► Final answer:

$$a_n = 2^n + \frac{1}{2}n2^n$$

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## Example 1

Solve the recurrence relation  $a_n = a_{n-1} + 2a_{n-2}$  with initial conditions  $a_0 = 2$  and  $a_1 = 7$ .

1. Characteristic equation and its roots:

$$x^2 - x - 2 = 0$$

$$(x + 1)(x - 2) = 0$$

So,  $x_1 = 2$  and  $x_2 = -1$ .



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2. General form of the solution:

$$a_n = \alpha_1 2^n + \alpha_2 (-1)^n.$$

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3. Initial condition equations and their solutions:

$$a_0 = \alpha_1 + \alpha_2 = 2$$

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So,  $\alpha_1 = 3$  and  $\alpha_2 = -1$ .

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3. Initial condition equations and their solutions:

$$a_0 = \alpha_1 + \alpha_2 = 2$$

$$a_1 = \alpha_1 2 + \alpha_2 (-1) = 7$$

So,  $\alpha_1 = 3$  and  $\alpha_2 = -1$ .

4. Final answer:

$$a_n = 3 \cdot 2^n - (-1)^n \text{ is a solution.}$$

## Example 2

What is the solution of the recurrence relation

$a_n = -a_{n-1} + 4a_{n-2} + 4a_{n-3}$  with initial conditions  $a_0 = 8$ ,  $a_1 = 6$  and  $a_2 = 26$ .

1. Characteristic equation and its roots:

$$x^3 + x^2 - 4x - 4 = 0$$

$$(x + 1)(x + 2)(x - 2) = 0$$

So,  $x_1 = -1$ ,  $x_2 = -2$  and  $x_3 = 2$ .

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2. General form of the solution:

$$a_n = \alpha_1(-1)^n + \alpha_2(-2)^n + \alpha_32^n.$$

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2. General form of the solution:

$$a_n = \alpha_1(-1)^n + \alpha_2(-2)^n + \alpha_3 2^n.$$

3. Initial condition equations and their solutions:

$$a_0 = \alpha_1 + \alpha_2 + \alpha_3 = 8$$

$$a_1 = -\alpha_1 - 2\alpha_2 + 2\alpha_3 = 6$$

$$a_2 = \alpha_1 + 4\alpha_2 + 4\alpha_3 = 26$$

So,  $\alpha_1 = 2$ ,  $\alpha_2 = 1$  and  $\alpha_3 = 5$ .

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$$a_0 = \alpha_1 + \alpha_2 + \alpha_3 = 8$$

$$a_1 = -\alpha_1 - 2\alpha_2 + 2\alpha_3 = 6$$

$$a_2 = \alpha_1 + 4\alpha_2 + 4\alpha_3 = 26$$

So,  $\alpha_1 = 2$ ,  $\alpha_2 = 1$  and  $\alpha_3 = 5$ .

4. Final answer:

$a_n = 2 \cdot (-1)^n + (-2)^n + 5 \cdot 2^n$  is a solution.

## Example 3

Find the solution to the recurrence relation

$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$  with the initial conditions  $a_0 = 2$ ,  $a_1 = 5$  and  $a_2 = 15$ .

1. Characteristic equation and its roots:

$$x^3 - 6x^2 + 11x - 6 = 0$$

$$(x - 1)(x - 2)(x - 3) = 0$$

So,  $x_1 = 1$ ,  $x_2 = 2$  and  $x_3 = 3$ .



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2. General form of the solution:

$$a_n = \alpha_1 1^n + \alpha_2 2^n + \alpha_3 3^n.$$

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So,  $x_1 = 1$ ,  $x_2 = 2$  and  $x_3 = 3$ .

2. General form of the solution:

$$a_n = \alpha_1 1^n + \alpha_2 2^n + \alpha_3 3^n.$$

3. Initial condition equations and their solutions:

$$a_0 = \alpha_1 + \alpha_2 + \alpha_3 = 2$$

$$a_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 = 5$$

$$a_2 = \alpha_1 + 4\alpha_2 + 9\alpha_3 = 15$$

So,  $\alpha_1 = 1$ ,  $\alpha_2 = -1$  and  $\alpha_3 = 2$ .

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$$a_0 = \alpha_1 + \alpha_2 + \alpha_3 = 2$$

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So,  $\alpha_1 = 1$ ,  $\alpha_2 = -1$  and  $\alpha_3 = 2$ .

4. Final answer:

$a_n = 1 - 2^n + 2 \cdot 3^n$  is a solution.

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# Word problem

1. Find a recurrence relation for the number of ways to climb  $n$  stairs if the person climbing the stairs can take one stair or two stairs at a time.
2. What are the initial conditions?
3. In how many ways can this person climb a flight of eight stairs?

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# Word problem

Let  $S_n$  denote the number of ways of climbing the stairs.

1. Let  $n \geq 3$ . The last step either was a single step, for which there are  $S_{n-1}$  possibilities, or a double step, for which there are  $S_{n-2}$  possibilities. The recurrence is:  $S_n = S_{n-1} + S_{n-2}$  for  $n \geq 3$ .



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2. We have  $S_1 = 1$  and  $S_2 = 2$ . You can take two stairs either directly or by taking a stair at a time.

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2. We have  $S_1 = 1$  and  $S_2 = 2$ . You can take two stairs either directly or by taking a stair at a time.
3. The recurrence gives the Fibonacci sequence:

$$1, 2, 3, 5, 8, 13, 21, 34, 55, \dots$$

Hence there are  $S_8 = 34$  ways to climb a flight of eight stairs.

# Bibliography

- ▶ Discrete Mathematics and Its Applications. Rosen, K.H. 2012. McGraw-Hill.  
Chapter 8: Advanced Counting Techniques.  
Section 8.2: Solving Linear Recurrence Relations.