

CS/MATH 111, Discrete Structures - Fall 2018.

Discussion 5 - Linear Recurrence Relations

Andres, Sara, Elena

University of California, Riverside

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Outline

Fibonacci grows exponentially

Fibonacci numbers

Homogeneous Recurrence Equations

Word problem

Fibonacci recurrence

- ▶ Fibonacci numbers / Fibonacci sequence
- ▶ First two and subsequent numbers:
 - ▶ $F_0 = 1$
 - ▶ $F_1 = 1$
 - ▶ $F_n = F_{n-1} + F_{n-2}$, when $n \geq 2$.
- ▶ Fibonacci grows exponentially with n .
- ▶ Prove that:

$$0.5 \cdot 1.5^n \leq F_n \leq 2^n$$

Fibonacci recurrence

Proof by induction using $F_0 = 1, F_1 = 1, \dots$:

$$0.5 \cdot 1.5^n \leq F_n \leq 2^n \quad (1)$$

1. Base case:

- ▶ $n = 0, F_0 = 1 : 0.5 \cdot 1.5^0 \leq 1 \leq 2^0 = 0.50 \leq 1 \leq 1.$
- ▶ $n = 1, F_1 = 1 : 0.5 \cdot 1.5^1 \leq 1 \leq 2^1 = 0.75 \leq 1 \leq 2.$
- ▶ $n = 2, F_2 = 2 : 0.5 \cdot 1.5^2 \leq 1 \leq 2^2 = 1.125 \leq 2 \leq 4.$
- ▶ \vdots

2. Assumption step:

- ▶ Assume (1) holds for all $n \leq k - 1.$

3. Induction step:

- ▶ Prove that (1) holds for all $n \leq k.$

Fibonacci recurrence

Proof by induction using $F_0 = 1, F_1 = 1, \dots$:

$$0.5 \cdot 1.5^n \leq F_n \leq 2^n$$

3 Induction step:

► Prove that (1) holds for all $n \leq k$.

$$\begin{aligned}
 1) \quad & F_k \leq 2^k \\
 & F_k = F_{k-1} + F_{k-2} \\
 & F_k \leq 2^{k-1} + 2^{k-2} \text{ (by assumption.)} \\
 & F_k = 2^{k-2} \cdot (2 + 1) \\
 & F_k = 2^{k-2} \cdot 3 \\
 & F_k \leq 2^{k-2} \cdot 4 \\
 & F_k = 2^{k-2} \cdot 2^2 \\
 & F_k = 2^k
 \end{aligned}$$

$$F_n = \mathcal{O}(2^n)$$

Fibonacci recurrence

Proof by induction using $F_0 = 1, F_1 = 1, \dots$:

$$0.5 \cdot 1.5^n \leq F_n \leq 2^n$$

3 Induction step:

► Prove that (1) holds for all $n \leq k$.

$$2) \quad F_k \geq \frac{1}{2} \cdot 1.5^k$$

$$F_k = F_{k-1} + F_{k-2}$$

$$F_k \geq 1.5^{k-1} + 1.5^{k-2} \text{ (by assumption.)}$$

$$F_k = 1.5^{k-2} \cdot (1.5 + 1)$$

$$F_k = 1.5^{k-2} \cdot 2.5$$

$$F_k \geq 1.5^{k-2} \cdot 2.25$$

$$F_k = 1.5^{k-2} \cdot 1.5^2$$

$$F_k = 1.5^k$$

$$F_n = \Omega(1.5^n)$$

Fibonacci recurrence

$$0.5 \cdot 1.5^n \leq F_n \leq 2^n$$

$$F_n = \mathcal{O}(2^n)$$

$$F_n = \Omega(1.5^n)$$

$$\text{is } F_n = \Theta(\) ?$$

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Fibonacci numbers

- ▶ Let's rewrite our recurrence following our previous notation:

$$F_n = F_{n-1} + F_{n-2}. \quad (2)$$

for $n \geq 2$.

- ▶ $F_0 = 1$

- ▶ $F_1 = 1$

- ▶ Since F_n grows exponentially, we will assume:

$$F_n = x^n \quad (3)$$

- ▶ Plugging (3) into (2):

$$x^n = x^{n-1} + x^{n-2}$$

after dividing by x^{n-2} :

$$x^2 - x - 1 = 0$$

Fibonacci numbers

$$x^2 - x - 1 = 0$$

- ▶ This equation has two roots: $x_1 = \frac{1}{2}(1 + \sqrt{5})$ and $x_2 = \frac{1}{2}(1 - \sqrt{5})$.
- ▶ x_1 is the *golden ratio*¹ $\phi \approx 1.618$ and $x_2 = 1 - \phi \approx -0.618$.
- ▶ Do they satisfy (2)? It works for $n = 0$ but not for $n = 1$.
- ▶ Works for the main recurrence but not for the initial conditions...

¹https://en.wikipedia.org/wiki/Golden_ratio

Fibonacci numbers

Theorem 1

Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1r - c_2 = 0$ has two distinct roots r_1 and r_2 . Then the sequence a_n is a solution of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2}$ if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

for $n = 0, 1, 2, \dots$, where α_1 and α_2 are constants².

²Proof available at [Rosen, 2015. pg 515].

Fibonacci numbers

$$x^2 - x - 1 = 0$$

- ▶ This equation has two roots: $x_1 = \frac{1}{2}(1 + \sqrt{5})$ and $x_2 = \frac{1}{2}(1 - \sqrt{5})$.
- ▶ Therefore by Theorem 1 it follows that the Fibonacci numbers are given by:

$$F_n = \alpha_1 x_1^n + \alpha_2 x_2^n$$

- ▶ This form is called the *general form of the solution*.

Fibonacci numbers

$$F_n = \alpha_1 x_1^n + \alpha_2 x_2^n$$

- ▶ Plugging the initial conditions into this equation we will get a system of two equations and two parameters:

$$\alpha_1 x_1^0 + \alpha_2 x_2^0 = 1$$

$$\alpha_1 x_1^1 + \alpha_2 x_2^1 = 1$$

After substituting:

$$\alpha_1 + \alpha_2 = 1$$

$$\alpha_1 \cdot \frac{1}{2}(1 + \sqrt{5}) + \alpha_2 \cdot \frac{1}{2}(1 - \sqrt{5}) = 1$$

Fibonacci numbers

$$\alpha_1 + \alpha_2 = 1$$

$$\alpha_1 \cdot \frac{1}{2}(1 + \sqrt{5}) + \alpha_2 \cdot \frac{1}{2}(1 - \sqrt{5}) = 1$$

► Solving the system, we get:

$$\alpha_1 = \frac{\sqrt{5} + 1}{2\sqrt{5}}$$

$$\alpha_2 = \frac{\sqrt{5} - 1}{2\sqrt{5}}$$

Fibonacci numbers

- This give us a solution for F_n :

$$F_n = \frac{\sqrt{5} + 1}{2\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n + \frac{\sqrt{5} - 1}{2\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^n$$

Simplified as:

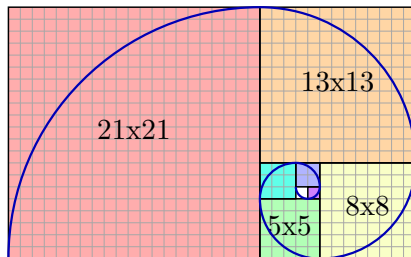
$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^{n+1} - \left(\frac{1 - \sqrt{5}}{2} \right)^{n+1} \right]$$

Fibonacci numbers

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^{n+1} - \left(\frac{1 - \sqrt{5}}{2} \right)^{n+1} \right]$$

► Note that when $n \rightarrow \infty$:

$$F_n \approx \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^{n+1}$$



Double root

Let's have the following recurrence relation:

$$a_n = 4a_{n-1} - 4a_{n-2}$$

with initial condition $a_0 = 1$ and $a_1 = 3$.

- ▶ Let's find the characteristic equation and its roots:

$$x^2 - 4x + 4 = 0.$$

$$(x - 2)^2 = 0.$$

So, $x_{1,2} = 2$. [“double root” or a root with multiplicity of 2.]

Double root

Let's have the following recurrence relation:

$$a_n = 4a_{n-1} - 4a_{n-2}$$

with initial condition $a_0 = 1$ and $a_1 = 3$.

- ▶ All functions $\alpha_1 2^n$ are good candidates, but...
- ▶ ... we need our solution to be parametrized by **two** parameters.
- ▶ We can try the function $n2^n$:

$$n2^n = 4(n-1)2^{n-1} - 4(n-2)2^{n-2}$$

by simple algebra we see that is is indeed true.

Double root

Let's have the following recurrence relation:

$$a_n = 4a_{n-1} - 4a_{n-2}$$

with initial condition $a_0 = 1$ and $a_1 = 3$.

- ▶ Since $n2^n$ is a solution, so is any function $\alpha_2 n 2^n$.
- ▶ We can combine the two types of solutions in a general form:

$$a_n = \alpha_1 2^n + \alpha_2 n 2^n$$

then we can continue...

Double root

Let's have the following recurrence relation:

$$a_n = 4a_{n-1} - 4a_{n-2}$$

with initial condition $a_0 = 1$ and $a_1 = 3$.

► Initial condition equations and their solutions:

$$\alpha_1 \cdot 2^0 + \alpha_2 \cdot 0 \cdot 2^0 = 1$$

$$\alpha_1 \cdot 2^1 + \alpha_2 \cdot 1 \cdot 2^1 = 3$$

which reduce to:

$$\alpha_1 = 1$$

$$2\alpha_1 + 2\alpha_2 = 3$$

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- ▶ Initial condition equations and their solutions:

$$\alpha_1 = 1$$

$$2\alpha_1 + 2\alpha_2 = 3$$

We get $\alpha_1 = 1$ and $\alpha_2 = \frac{1}{2}$.

Double root

Let's have the following recurrence relation:

$$a_n = 4a_{n-1} - 4a_{n-2}$$

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► Final answer:

$$a_n = 2^n + \frac{1}{2}n2^n$$

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Example 1

Solve the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$ with initial conditions $a_0 = 2$ and $a_1 = 7$.

1. Characteristic equation and its roots:

$$x^2 - x - 2 = 0$$

$$(x + 1)(x - 2) = 0$$

So, $x_1 = 2$ and $x_2 = -1$.

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2. General form of the solution:

$$a_n = \alpha_1 2^n + \alpha_2 (-1)^n.$$

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$$a_0 = \alpha_1 + \alpha_2 = 2$$

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So, $\alpha_1 = 3$ and $\alpha_2 = -1$.

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So, $\alpha_1 = 3$ and $\alpha_2 = -1$.

4. Final answer:

$$a_n = 3 \cdot 2^n - (-1)^n \text{ is a solution.}$$

Example 2

What is the solution of the recurrence relation

$a_n = -a_{n-1} + 4a_{n-2} + 4a_{n-3}$ with initial conditions $a_0 = 8$, $a_1 = 6$ and $a_2 = 26$.

1. Characteristic equation and its roots:

$$x^3 + x^2 - 4x - 4 = 0$$

$$(x + 1)(x + 2)(x - 2) = 0$$

So, $x_1 = -1$, $x_2 = -2$ and $x_3 = 2$.

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2. General form of the solution:

$$a_n = \alpha_1(-1)^n + \alpha_2(-2)^n + \alpha_32^n.$$

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3. Initial condition equations and their solutions:

$$a_0 = \alpha_1 + \alpha_2 + \alpha_3 = 8$$

$$a_1 = -\alpha_1 - 2\alpha_2 + 2\alpha_3 = 6$$

$$a_2 = \alpha_1 + 4\alpha_2 + 4\alpha_3 = 26$$

So, $\alpha_1 = 2$, $\alpha_2 = 1$ and $\alpha_3 = 5$.

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$$a_2 = \alpha_1 + 4\alpha_2 + 4\alpha_3 = 26$$

So, $\alpha_1 = 2$, $\alpha_2 = 1$ and $\alpha_3 = 5$.

4. Final answer:

$a_n = 2 \cdot (-1)^n + (-2)^n + 5 \cdot 2^n$ is a solution.

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1. Find a recurrence relation for the number of ways to climb n stairs if the person climbing the stairs can take one stair or two stairs at a time.
2. What are the initial conditions?
3. In how many ways can this person climb a flight of eight stairs?

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Bibliography

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Section 8.2: Solving Linear Recurrence Relations.