CS/MATH 111, Discrete Structures - Fall 2018. Discussion 9 - Graphs and Tree introduction

Andres, Sara, Elena

University of California, Riverside

November 27, 2018

Outline

Bipartite graph

Perfect matching

Planar graphs

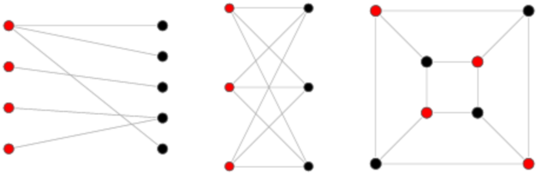
Kuratowski's theorem

Trees

Bipartite graph

- A bipartite graph, also called a bigraph, is a set of graph vertices decomposed into two disjoint sets such that no two graph vertices within the same set are adjacent.
- Bipartite graphs are equivalent to two-colorable graphs.
- All acyclic graphs are bipartite.
- A cyclic graph is bipartite iff all its cycles are of even length

Bipartite graph



Outline

Perfect matching

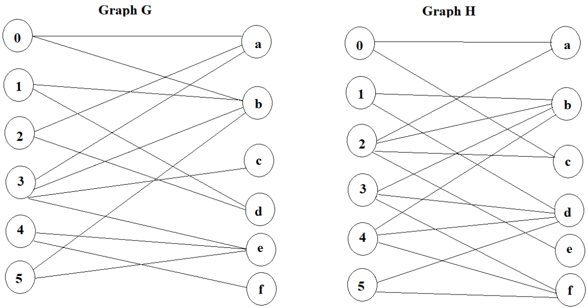
A perfect matching of a graph is a matching (i.e., an independent edge set) in which every vertex of the graph is incident to exactly one edge of the matching.
A perfect matching is therefore a matching containing $\frac{n}{2}$ edges (the largest possible), meaning perfect matchings are only possible on graphs with an even number of vertices.

Perfect matching

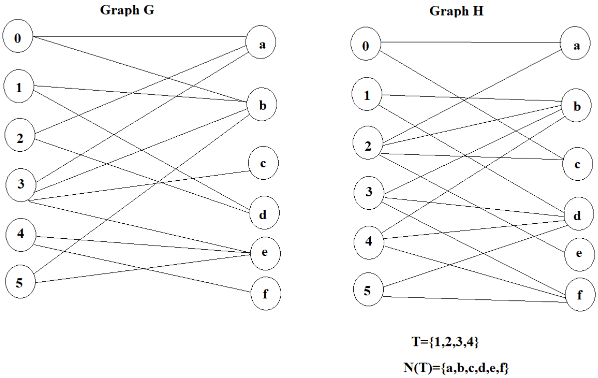
Halls Theorem: Let G = (X,Y) be a bipartite graph. Then X has a perfect macthing into Y if and only if for all $T \subseteq X$, the inequality $|T| \leq |N(T)|$ holds. Where N(T) is the set of all neighbors of the vertices in T. In other words, $y \in Y$ is an element of N(T) if and only if there is a vertex $x \in T$ so that xy is an edge.

Perfect matching

You are given two bipartite graph G and H below. For each graph determine whether it has a perfect matching. Justify your answer, either by listing the edges that are in the matching or use Hall's Theorem to show that the graph does not have a perfect matching.



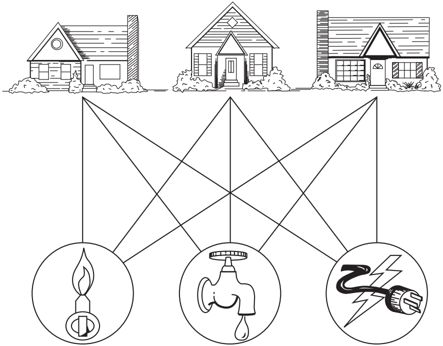
Perfect matching



Outline

Planar graphs

Is it possible to join these houses and utilities so that none of the connections cross?

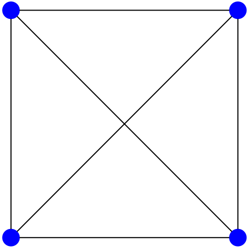


Planar graphs

Definition 3.1

A graph is called planar if it can be drawn in the plane without any edges crossing. Such a drawing is called a planar representation of the graph.

Examples



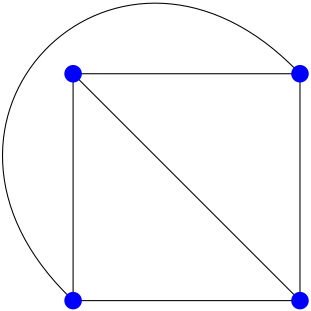


Figure: The K_4 graph and its drawn with no crossings.

Examples

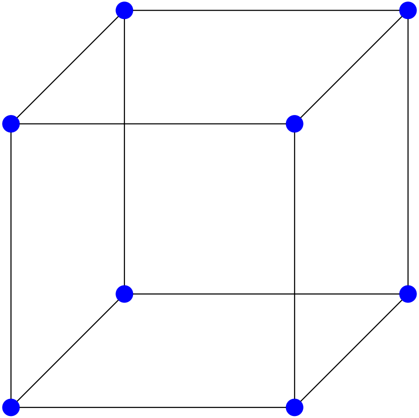


Figure: A Q_3 graph.

Examples

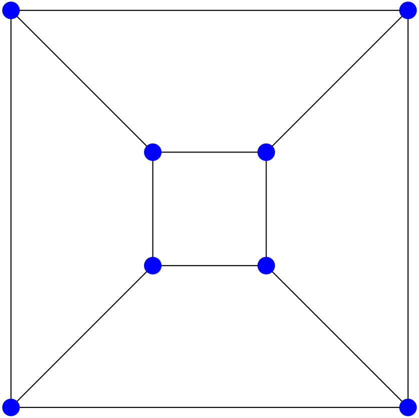


Figure: The planar representation of a Q_3 graph.

Euler's Formula

- A planar representation of a graph splits the plane into regions (including an unbounded region.)
- Euler showed that all planar representations of a graph split the plane into the same number of regions.
- There is a relationship between the number of regions, vertices and edges.

Euler's Formula

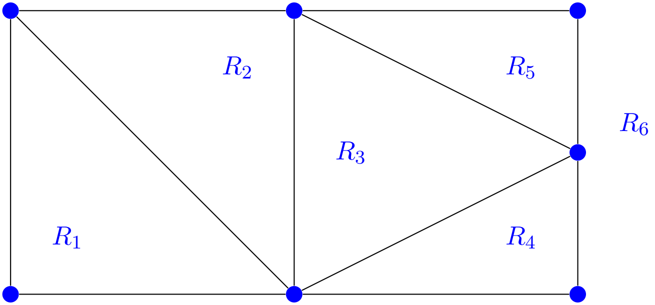


Figure: The Regions of the Planar Representation of a Graph.

Euler's Formula

Theorem 1

Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r = e - v + 2$.

Euler's Formula

Corollary 2

If G is a connected planar graph with m edges and n vertices, and $n \geq 3$ and no circuits of length 3, then $m \leq 2n - 4$.

Proof

- G divides the plane into regions, say r of them.
- The degree of each region is at least four[1].
- Note that the sum of the degrees of the regions is exactly twice the number of edges in the graph[2].
- Because each region has degree greater than or equal to 4, it follows that: $2m = \sum deg(R) \geq 4r$.
- Hence, $2m \geq 4r$ or simply $r \leq \frac{m}{2}$. Using Euler's formula, we obtain $m - n + 2 \leq \frac{m}{2}$.
- It follows that $\frac{m}{2} \leq n - 2$. This shows that $m \leq 2n - 4$.

∎

[1] no multiple edges, no loops and no simple cycles of length 3

[2] because each edge occurs on the boundary of a region exactly twice

Proof

- G divides the plane into regions, say r of them.
- The degree of each region is at least four[1].
- Note that the sum of the degrees of the regions is exactly twice the number of edges in the graph[2].
- Because each region has degree greater than or equal to 4, it follows that: $2m = \sum deg(R) \geq 4r$.
- Hence, $2m \geq 4r$ or simply $r \leq \frac{m}{2}$. Using Euler's formula, we obtain $m - n + 2 \leq \frac{m}{2}$.
- It follows that $\frac{m}{2} \leq n - 2$. This shows that $m \leq 2n - 4$.

[1] no multiple edges, no loops and no simple cycles of length 3

[2] because each edge occurs on the boundary of a region exactly twice

Proof

- G divides the plane into regions, say r of them.
- The degree of each region is at least four[1].
- Note that the sum of the degrees of the regions is exactly twice the number of edges in the graph[2].
- Because each region has degree greater than or equal to 4, it follows that: $2m = \sum deg(R) \geq 4r$.
- Hence, $2m \geq 4r$ or simply $r \leq \frac{m}{2}$. Using Euler's formula, we obtain $m - n + 2 \leq \frac{m}{2}$.
- It follows that $\frac{m}{2} \leq n - 2$. This shows that $m \leq 2n - 4$.

■

[1] no multiple edges, no loops and no simple cycles of length 3

[2] because each edge occurs on the boundary of a region exactly twice

Proof

- G divides the plane into regions, say r of them.
- The degree of each region is at least four[1].
- Note that the sum of the degrees of the regions is exactly twice the number of edges in the graph[2].
- Because each region has degree greater than or equal to 4, it follows that: $2m = \sum deg(R) \geq 4r$.
- Hence, $2m \geq 4r$ or simply $r \leq \frac{m}{2}$. Using Euler's formula, we obtain $m - n + 2 \leq \frac{m}{2}$.
- It follows that $\frac{m}{2} \leq n - 2$. This shows that $m \leq 2n - 4$.

■

[1] no multiple edges, no loops and no simple cycles of length 3

[2] because each edge occurs on the boundary of a region exactly twice

Proof

- G divides the plane into regions, say r of them.
- The degree of each region is at least four[1].
- Note that the sum of the degrees of the regions is exactly twice the number of edges in the graph[2].
- Because each region has degree greater than or equal to 4, it follows that: $2m = \sum deg(R) \geq 4r$.
- Hence, $2m \geq 4r$ or simply $r \leq \frac{m}{2}$. Using Euler's formula, we obtain $m - n + 2 \leq \frac{m}{2}$.
- It follows that $\frac{m}{2} \leq n - 2$. This shows that $m \leq 2n - 4$.

∎

[1] no multiple edges, no loops and no simple cycles of length 3

[2] because each edge occurs on the boundary of a region exactly twice

Proof

- G divides the plane into regions, say r of them.
- The degree of each region is at least four[1].
- Note that the sum of the degrees of the regions is exactly twice the number of edges in the graph[2].
- Because each region has degree greater than or equal to 4, it follows that: $2m = \sum deg(R) \geq 4r$.
- Hence, $2m \geq 4r$ or simply $r \leq \frac{m}{2}$. Using Euler's formula, we obtain $m - n + 2 \leq \frac{m}{2}$.
- It follows that $\frac{m}{2} \leq n - 2$. This shows that $m \leq 2n - 4$.

∎

[1] no multiple edges, no loops and no simple cycles of length 3

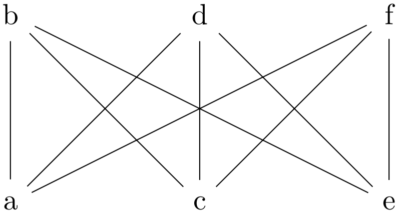
[2] because each edge occurs on the boundary of a region exactly twice

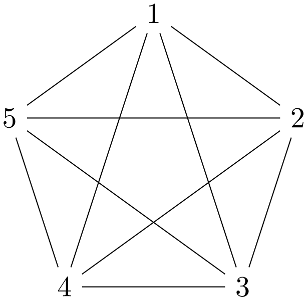
Outline

Kuratowski's theorem

Theorem 3

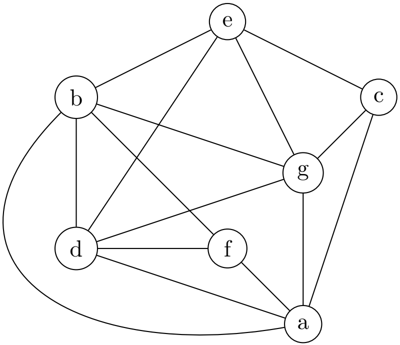
A graph is nonplanar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ *or* K_5*.*





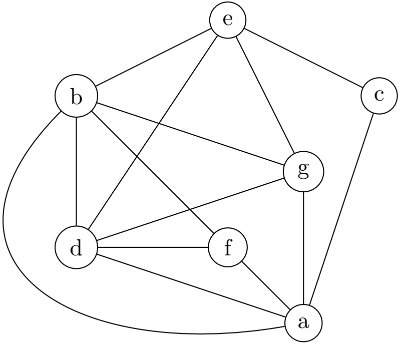
Examples

Determine if the following graph is planar. Justify your answer.



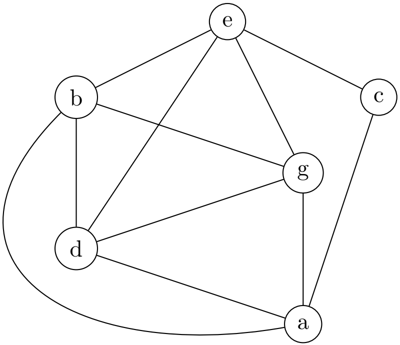
Examples

Determine if the following graph is planar. Justify your answer.



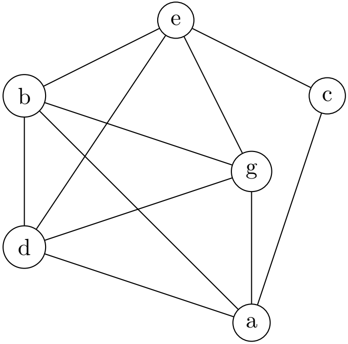
Examples

Determine if the following graph is planar. Justify your answer.



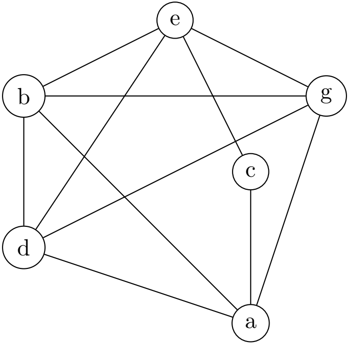
Examples

Determine if the following graph is planar. Justify your answer.



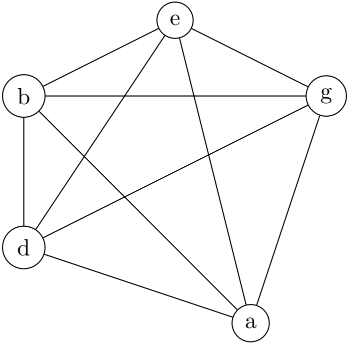
Examples

Determine if the following graph is planar. Justify your answer.



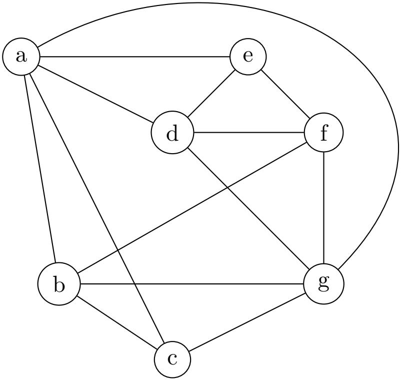
Examples

Determine if the following graph is planar. Justify your answer.



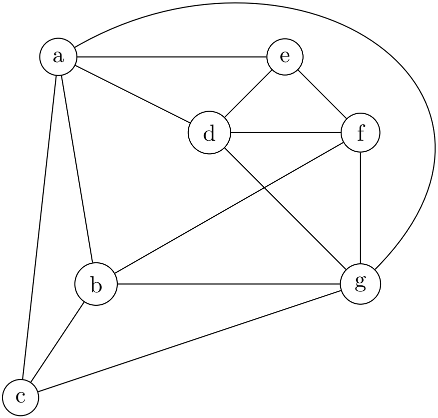
Examples

Determine if the following graph is planar. Justify your answer.



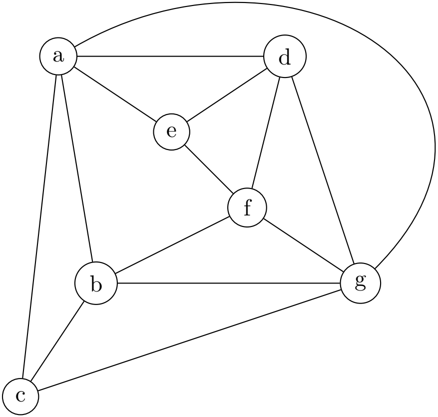
Examples

Determine if the following graph is planar. Justify your answer.



Examples

Determine if the following graph is planar. Justify your answer.



Outline

Trees

Lemma 4

It T is a tree, and has n vertices, then its number of edges is $m = n - 1$.

Proof

1. Basis step:
   - When $n = 1$, a tree with $n = 1$ vertex has no edges. Indeed, $n - 1 = 0$.
2. Assumption step:
   - Let's assume that every tree with k vertices has $k - 1$ edges, where k is a positive integer.
3. Inductive step:
   - Suppose that a tree T has $k + 1$ vertices and that v is a leaf[3] of T. Let w be the parent of v.
   - Remove v from T and the edge connecting w to v. It produces a tree T' with k vertices[4].
   - By the assumption hypothesis, T' has $k - 1$ edges. It follows that T has k edges because it has one more edge than T' (the edge connecting v and w).

∎

[3] It must exist because the tree is finite

[4] T' is still connected and has no simple circuits.

Proof

1. Basis step:
   - When $n = 1$, a tree with $n = 1$ vertex has no edges. Indeed, $n - 1 = 0$.
2. Assumption step:
   - Let's assume that every tree with k vertices has $k - 1$ edges, where k is a positive integer.
3. Inductive step:
   - Suppose that a tree T has $k + 1$ vertices and that v is a leaf[3] of T. Let w be the parent of v.
   - Remove v from T and the edge connecting w to v. It produces a tree T' with k vertices[4].
   - By the assumption hypothesis, T' has $k - 1$ edges. It follows that T has k edges because it has one more edge than T' (the edge connecting v and w).

∎

---

[3] It must exist because the tree is finite

[4] T' is still connected and has no simple circuits.

Proof

1. Basis step:
   - When $n = 1$, a tree with $n = 1$ vertex has no edges. Indeed, $n - 1 = 0$.
2. Assumption step:
   - Let's assume that every tree with k vertices has $k - 1$ edges, where k is a positive integer.
3. Inductive step:
   - Suppose that a tree T has $k + 1$ vertices and that v is a leaf[3] of T. Let w be the parent of v.
   - Remove v from T and the edge connecting w to v. It produces a tree T' with k vertices[4].
   - By the assumption hypothesis, T' has $k - 1$ edges. It follows that T has k edges because it has one more edge than T' (the edge connecting v and w).

∎

[3] It must exist because the tree is finite

[4] T' is still connected and has no simple circuits.

Proof

1. Basis step:
   - When $n = 1$, a tree with $n = 1$ vertex has no edges. Indeed, $n - 1 = 0$.
2. Assumption step:
   - Let's assume that every tree with k vertices has $k - 1$ edges, where k is a positive integer.
3. Inductive step:
   - Suppose that a tree T has $k + 1$ vertices and that v is a leaf[3] of T. Let w be the parent of v.
   - Remove v from T and the edge connecting w to v. It produces a tree T' with k vertices[4].
   - By the assumption hypothesis, T' has $k - 1$ edges. It follows that T has k edges because it has one more edge than T' (the edge connecting v and w).

---

[3] It must exist because the tree is finite

[4] T' is still connected and has no simple circuits.

Proof

1. Basis step:
   - When $n = 1$, a tree with $n = 1$ vertex has no edges. Indeed, $n - 1 = 0$.
2. Assumption step:
   - Let's assume that every tree with k vertices has $k - 1$ edges, where k is a positive integer.
3. Inductive step:
   - Suppose that a tree T has $k + 1$ vertices and that v is a leaf[3] of T. Let w be the parent of v.
   - Remove v from T and the edge connecting w to v. It produces a tree T' with k vertices[4].
   - By the assumption hypothesis, T' has $k - 1$ edges. It follows that T has k edges because it has one more edge than T' (the edge connecting v and w).

∎

[3] It must exist because the tree is finite

[4] T' is still connected and has no simple circuits.

Proof

1. Basis step:
   - When $n = 1$, a tree with $n = 1$ vertex has no edges. Indeed, $n - 1 = 0$.
2. Assumption step:
   - Let's assume that every tree with k vertices has $k - 1$ edges, where k is a positive integer.
3. Inductive step:
   - Suppose that a tree T has $k + 1$ vertices and that v is a leaf[3] of T. Let w be the parent of v.
   - Remove v from T and the edge connecting w to v. It produces a tree T' with k vertices[4].
   - By the assumption hypothesis, T' has $k - 1$ edges. It follows that T has k edges because it has one more edge than T' (the edge connecting v and w).

∎

[3] It must exist because the tree is finite

[4] T' is still connected and has no simple circuits.

Proof

1. Basis step:
   - When $n = 1$, a tree with $n = 1$ vertex has no edges. Indeed, $n - 1 = 0$.
2. Assumption step:
   - Let's assume that every tree with k vertices has $k - 1$ edges, where k is a positive integer.
3. Inductive step:
   - Suppose that a tree T has $k + 1$ vertices and that v is a leaf[3] of T. Let w be the parent of v.
   - Remove v from T and the edge connecting w to v. It produces a tree T' with k vertices[4].
   - By the assumption hypothesis, T' has $k - 1$ edges. It follows that T has k edges because it has one more edge than T' (the edge connecting v and w).

[3] It must exist because the tree is finite

[4] T' is still connected and has no simple circuits.

Proof

1. Basis step:
   - When $n = 1$, a tree with $n = 1$ vertex has no edges. Indeed, $n - 1 = 0$.
2. Assumption step:
   - Let's assume that every tree with k vertices has $k - 1$ edges, where k is a positive integer.
3. Inductive step:
   - Suppose that a tree T has $k + 1$ vertices and that v is a leaf[3] of T. Let w be the parent of v.
   - Remove v from T and the edge connecting w to v. It produces a tree T' with k vertices[4].
   - By the assumption hypothesis, T' has $k - 1$ edges. It follows that T has k edges because it has one more edge than T' (the edge connecting v and w).

∎

[3] It must exist because the tree is finite

[4] T' is still connected and has no simple circuits.

Reference

- Discrete Mathematics and Its Applications. Rosen, K.H. 2012. McGraw-Hill.
  - Chapter 10. Graphs:
    Section 10.2: Graph Terminology and Special Types of Graphs.
    Section 10.7: Planar Graphs.
  - Chapter 11. Trees:
    Section 11.1: Introduction to Trees.