CS/MATH 111, Discrete Structures - Fall 2018.
Discussion 5 - Linear Recurrence Relations

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Outline

Fibonacci recurrence

- Fibonnacci numbers / Fibonnacci sequence
- First two and subsequent numbers:
  - $F_0 = 1$
  - $F_1 = 1$
  - $F_n = F_{n-1} + F_{n-2}$, when $n \geq 2$.
- Fibonacci grows exponentially with n.
- Prove that:

$$0.5 \cdot 1.5^n \leq F_n \leq 2^n$$

Fibonacci recurrence

Proof by induction using $F_0 = 1, F_1 = 1, \ldots$:

$$0.5 \cdot 1.5^n \leq F_n \leq 2^n \tag{1}$$

1. Base case:
   - $n = 0, F_0 = 1 : 0.5 \cdot 1.5^0 \leq 1 \leq 2^0 = 0.50 \leq 1 \leq 1.$
   - $n = 1, F_1 = 1 : 0.5 \cdot 1.5^1 \leq 1 \leq 2^1 = 0.75 \leq 1 \leq 2.$
   - $n = 2, F_2 = 2 : 0.5 \cdot 1.5^2 \leq 1 \leq 2^2 = 1.125 \leq 2 \leq 4.$

     $\vdots$
2. Assumption step:
   - Assume (1) holds for all $n \leq k - 1$.
3. Induction step:
   - Prove that (1) holds for all $n \leq k$.

Fibonacci recurrence

Proof by induction using $F_0 = 1, F_1 = 1, \ldots$:

$$0.5 \cdot 1.5^n \leq F_n \leq 2^n$$

3 Induction step:

- Prove that (1) holds for all $n \leq k$.

  1) $F_k \leq 2^k$

     $F_k = F_{k-1} + F_{k-2}$

     $F_k \leq 2^{k-1} + 2^{k-2}$ (by assumption.)

     $F_k = 2^{k-2} \cdot (2+1)$

     $F_k = 2^{k-2} \cdot 3$

     $F_k \leq 2^{k-2} \cdot 4$

     $F_k = 2^{k-2} \cdot 2^2$

     $F_k = 2^k$

     $F_n = \mathcal{O}(2^n)$

Fibonacci recurrence

Proof by induction using $F_0 = 1, F_1 = 1, \ldots$:

$$0.5 \cdot 1.5^n \leq F_n \leq 2^n$$

3 Induction step:

- Prove that (1) holds for all $n \leq k$.

  2) $F_k \geq \frac{1}{2} \cdot 1.5^k$
     $F_k = F_{k-1} + F_{k-2}$
     $F_k \geq 1.5^{k-1} + 1.5^{k-2}$ (by assumption.)
     $F_k = 1.5^{k-2} \cdot (1.5 + 1)$
     $F_k = 1.5^{k-2} \cdot 2.5$
     $F_k \geq 1.5^{k-2} \cdot 2.25$
     $F_k = 1.5^{k-2} \cdot 1.5^2$
     $F_k = 1.5^k$

     $F_n = \Omega(1.5^n)$

Fibonacci recurrence

$$0.5 \cdot 1.5^n \leq F_n \leq 2^n$$

$$F_n = \mathcal{O}(2^n)$$

$$F_n = \Omega(1.5^n)$$

$$is\ F_n = \Theta(\ )\ ?$$

Outline

Fibonacci numbers

- Let's rewrite our recurrence following our previous notation:

$$F_n = F_{n-1} + F_{n-2}. \tag{2}$$

  for $n \geq 2$.
  - $F_0 = 1$
  - $F_1 = 1$
- Since F_n grows exponentially, we will assume:

$$F_n = x^n \tag{3}$$

- Plugging (3) into (2):

$$x^n = x^{n-1} + x^{n-2}$$

  after dividing by x^{n-2}:

$$x^2 - x - 1 = 0$$

Fibonacci numbers

$$x^2 - x - 1 = 0$$

- This equation has two roots: $x_1 = \frac{1}{2}(1 + \sqrt{5})$ and $x_2 = \frac{1}{2}(1 - \sqrt{5})$.
- x_1 is the *golden ratio*[1] $\phi \approx 1.618$ and $x_2 = 1 - \phi \approx -0.618$.
- Do they satisfy (2)? It works for $n = 0$ but not for $n = 1$.
- Works for the main recurrence but not for the initial conditions...

[1] `https://en.wikipedia.org/wiki/Golden_ratio`

Fibonacci numbers

Theorem 1

Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2. Then the sequence a_n is a solution of the recurrence relation $a_n = c_1 a_{n1} + c_2 a_{n2}$ if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

for $n = 0, 1, 2, \ldots,$ where α_1 and α_2 are constants[2].

[2]Proof available at [Rosen, 2015. pg 515].

Fibonacci numbers

$$x^2 - x - 1 = 0$$

- This equation has two roots: $x_1 = \frac{1}{2}(1+\sqrt{5})$ and $x_2 = \frac{1}{2}(1-\sqrt{5})$.
- Therefore by Theorem 1 it follows that the Fibonacci numbers are given by:

$$F_n = \alpha_1 x_1^n + \alpha_2 x_2^n$$

- This form is called the *general form of the solution*.

Fibonacci numbers

$$F_n = \alpha_1 x_1^n + \alpha_2 x_2^n$$

- Plugging the initial conditions into this equation we will get a system of two equation and two parameters:

$$\alpha_1 x_1^0 + \alpha_2 x_2^0 = 1$$

$$\alpha_1 x_1^1 + \alpha_2 x_2^1 = 1$$

After substituting:

$$\alpha_1 + \alpha_2 = 1$$

$$\alpha_1 \cdot \frac{1}{2}(1+\sqrt{5}) + \alpha_2 \cdot \frac{1}{2}(1-\sqrt{5}) = 1$$

Fibonacci numbers

$$\alpha_1 + \alpha_2 = 1$$
$$\alpha_1 \cdot \frac{1}{2}(1+\sqrt{5}) + \alpha_2 \cdot \frac{1}{2}(1-\sqrt{5}) = 1$$

- Solving the system, we get:

$$\alpha_1 = \frac{\sqrt{5}+1}{2\sqrt{5}}$$

$$\alpha_2 = \frac{\sqrt{5}-1}{2\sqrt{5}}$$

Fibonacci numbers

- This give us a solution for F_n:

$$F_n = \frac{\sqrt{5}+1}{2\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{\sqrt{5}-1}{2\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Simplified as:

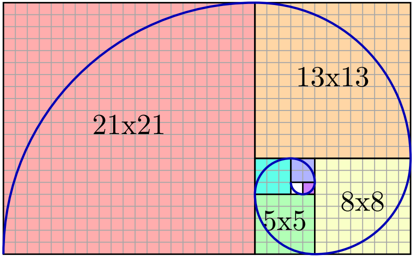
$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$$

Fibonacci numbers

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$$

- Note that when $n \to \infty$:

$$F_n \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}$$



Double root

Let's have the following recurrence relation:

$$a_n = 4a_{n-1} - 4a_{n-2}$$

with initial condition $a_0 = 1$ and $a_1 = 3$.

- Let's find the characteristic equation and its roots:
  $x^2 - 4x + 4 = 0.$
  $(x-2)^2 = 0.$
  So, $x_{1,2} = 2.$ ["double root" or a root with multiplicity of 2.]

Double root

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- All functions $\alpha_1 2^n$ are good candidates, but...
- ... we need our solution to be parametrized by **two** parameters.
- We can try the function $n2^n$:

  $$n2^n = 4(n-1)2^{n-1} - 4(n-2)2^{n-2}$$

  by simple algebra we see that is is indeed true.

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- Since $n2^n$ is a solution, so is any function $\alpha_2 n2^n$.
- We can combine the two types of solutions in a general form:

$$a_n = \alpha_1 2^n + \alpha_2 n2^n$$

then we can continue...

Double root

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with initial condition $a_0 = 1$ and $a_1 = 3$.

- Initial condition equations and their solutions:

$$\alpha_1 \cdot 2^0 + \alpha_2 \cdot 0 \cdot 2^0 = 1$$

$$\alpha_1 \cdot 2^1 + \alpha_2 \cdot 1 \cdot 2^1 = 3$$

  which reduce to:

$$\alpha_1 = 1$$

$$2\alpha_1 + 2\alpha_2 = 3$$

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- Initial condition equations and their solutions:

$$\alpha_1 = 1$$

$$2\alpha_1 + 2\alpha_2 = 3$$

  We get $\alpha_1 = 1$ and $\alpha_2 = \frac{1}{2}$.

Double root

Let's have the following recurrence relation:

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- Final answer:

$$a_n = 2^n + \frac{1}{2}n2^n$$

Outline

Example 1

Solve the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$ with initial conditions $a_0 = 2$ and $a_1 = 7$.

1. Characteristic equation and its roots:
   $x^2 - x - 2 = 0$
   $(x + 1)(x - 2) = 0$
   So, $x_1 = 2$ and $x_2 = -1$.

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   $a_1 = \alpha_1 2 + \alpha_2(-1) = 7$
   So, $\alpha_1 = 3$ and $\alpha_2 = -1$.
4. Final answer:
   $a_n = 3 \cdot 2^n - (-1)^n$ is a solution.

Example 2

What is the solution of the recurrence relation $a_n = -a_{n-1} + 4a_{n-2} + 4a_{n-3}$ with initial conditions $a_0 = 8$, $a_1 = 6$ and $a_2 = 26$.

1. Characteristic equation and its roots:
   $x^3 + x^2 - 4x - 4 = 0$
   $(x+1)(x+2)(x-2) = 0$
   So, $x_1 = -1$, $x_2 = -2$ and $x_3 = 2$.

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   $a_0 = \alpha_1 + \alpha_2 + \alpha_3 = 8$
   $a_1 = -\alpha_1 - 2\alpha_2 + 2\alpha_3 = 6$
   $a_2 = \alpha_1 + 4\alpha_2 + 4\alpha_3 = 26$
   So, $\alpha_1 = 2$, $\alpha_2 = 1$ and $\alpha_3 = 5$.

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   $a_2 = \alpha_1 + 4\alpha_2 + 4\alpha_3 = 26$
   So, $\alpha_1 = 2$, $\alpha_2 = 1$ and $\alpha_3 = 5$.
4. Final answer:
   $a_n = 2 \cdot (-1)^n + (-2)^n + 5 \cdot 2^n$ is a solution.

Example 3

Find the solution to the recurrence relation
$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ with the initial conditions $a_0 = 2$, $a_1 = 5$ and $a_2 = 15$.

1. Characteristic equation and its roots:
   $x^3 - 6x^2 + 11x - 6 = 0$
   $(x-1)(x-2)(x-3) = 0$
   So, $x_1 = 1$, $x_2 = 2$ and $x_3 = 3$.

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   $a_n = \alpha_1 1^n + \alpha_2 2^n + \alpha_3 3^n$.

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2. General form of the solution:
   $a_n = \alpha_1 1^n + \alpha_2 2^n + \alpha_3 3^n$.
3. Initial condition equations and their solutions:
   $a_0 = \alpha_1 + \alpha_2 + \alpha_3 = 2$
   $a_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 = 5$
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   So, $\alpha_1 = 1$, $\alpha_2 = -1$ and $\alpha_3 = 2$.
4. Final answer:
   $a_n = 1 - 2^n + 2 \cdot 3^n$ is a solution.

Outline

Word problem

1. Find a recurrence relation for the number of ways to climb n stairs if the person climbing the stairs can take one stair or two stairs at a time.
2. What are the initial conditions?
3. In how many ways can this person climb a flight of eight stairs?

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Word problem

Let S_n denote the number of ways of climbing the stairs.

1. Let $n \geq 3$. The last step either was a single step, for which there are S_{n-1} possibilities, or a double step, for which there are S_{n-2} possibilities. The recurrence is: $S_n = S_{n-1} + S_{n-2}$ for $n \geq 3$.

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2. Whe have $S_1 = 1$ and $S_2 = 2$. You can take two stairs either directly or by taking a stair at a time.
3. The recurrence gives the Fibonacci sequence:

$$1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

Hence there are $S_8 = 34$ ways to climb a flight of eight stairs.

Bibliography

- Discrete Mathematics and Its Applications. Rosen, K.H. 2012. McGraw-Hill.
  Chapter 8: Advanced Counting Techniques.
  Section 8.2: Solving Linear Recurrence Relations.