CS/MATH 111, Discrete Structures - Fall 2018. Discussion 9 - Graphs and Tree introduction

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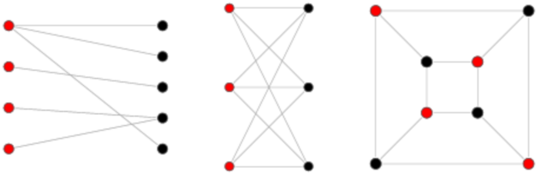
Outline

Bipartite graph

- A bipartite graph, also called a bigraph, is a set of graph vertices decomposed into two disjoint sets such that no two graph vertices within the same set are adjacent.
- Bipartite graphs are equivalent to two-colorable graphs.
- All acyclic graphs are bipartite.
- A cyclic graph is bipartite iff all its cycles are of even length

Bipartite graph



Outline

Perfect matching

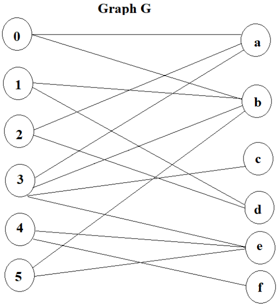
A perfect matching of a graph is a matching (i.e., an independent edge set) in which every vertex of the graph is incident to exactly one edge of the matching.
A perfect matching is therefore a matching containing $\frac{n}{2}$ edges (the largest possible), meaning perfect matchings are only possible on graphs with an even number of vertices.

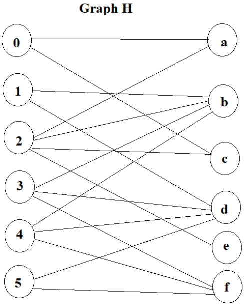
Perfect matching

Halls Theorem: Let G = (X,Y) be a bipartite graph. Then X has a perfect macthing into Y if and only if for all $T \subseteq X$, the inequality $|T| \leq |N(T)|$ holds. Where N(T) is the set of all neighbors of the vertices in T. In other words, $y \in Y$ is an element of N(T) if and only if there is a vertex $x \in T$ so that xy is an edge.

Perfect matching

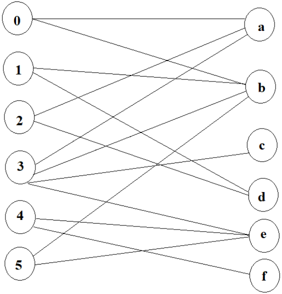
You are given two bipartite graph G and H below. For each graph determine whether it has a perfect matching. Justify your answer, either by listing the edges that are in the matching or use Hall's Theorem to show that the graph does not have a perfect matching.



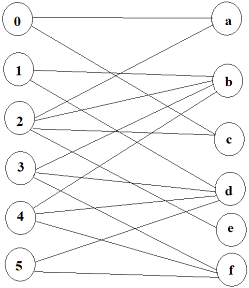


Perfect matching

Graph G



Graph H



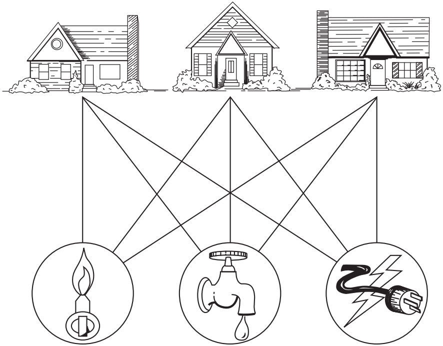
T={1,2,3,4}

N(T)={a,b,c,d,e,f}

Outline

Planar graphs

Is it possible to join these houses and utilities so that none of the connections cross?



Planar graphs

Definition 3.1

A graph is called planar if it can be drawn in the plane without any edges crossing. Such a drawing is called a planar representation of the graph.

Examples

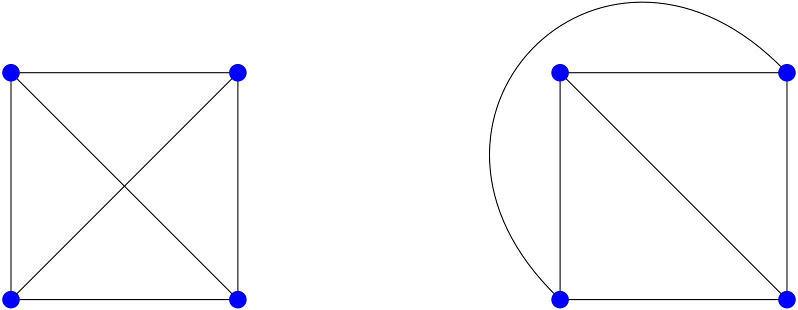


Figure: The K_4 graph and its drawn with no crossings.

Examples

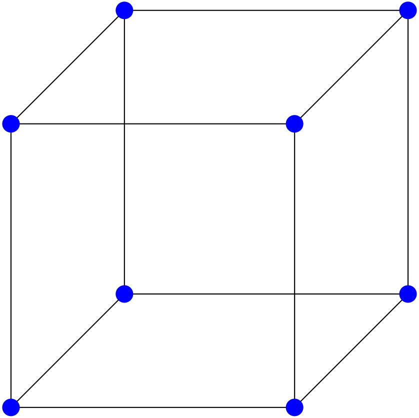


Figure: A Q_3 graph.

Examples

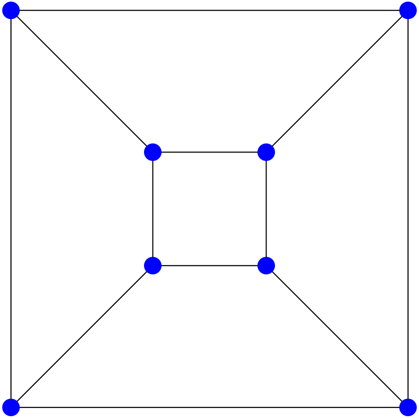


Figure: The planar representation of a Q_3 graph.

Euler's Formula

- A planar representation of a graph splits the plane into regions (including an unbounded region.)
- Euler showed that all planar representations of a graph split the plane into the same number of regions.
- There is a relationship between the number of regions, vertices and edges.

Euler's Formula

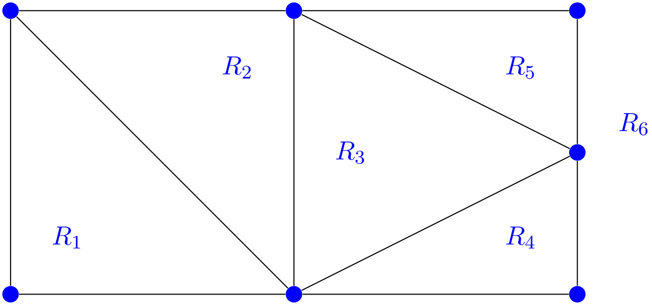


Figure: The Regions of the Planar Representation of a Graph.

Euler's Formula

Theorem 1 (EULER'S FORMULA)

Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r = e - v + 2$.

Euler's Formula

Corollary 2

If G is a connected planar graph with m edges and n vertices, and $n \geq 3$ and no circuits of length 3, then $m \leq 2n - 4$.

Proof

- G divides the plane into regions, say r of them.
- The degree of each region is at least four[1].
- Note that the sum of the degrees of the regions is exactly twice the number of edges in the graph[2].
- Because each region has degree greater than or equal to 4, it follows that: $2m = \sum deg(R) \geq 4r$.
- Hence, $2m \geq 4r$ or simply $r \leq \frac{m}{2}$. Using Euler's formula, we obtain $m - n + 2 \leq \frac{m}{2}$.
- It follows that $\frac{m}{2} \leq n - 2$. This shows that $m \leq 2n - 4$.

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[1] no multiple edges, no loops and no simple cycles of length 3

[2] because each edge occurs on the boundary of a region exactly twice

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Trees

Lemma 3

It T is a tree, and has n vertices, then its number of edges is $m = n - 1$.

Proof

1. Basis step:
   - When $n = 1$, a tree with $n = 1$ vertex has no edges. Indeed, $n - 1 = 0$.
2. Assumption step:
   - Let's assume that every tree with k vertices has $k - 1$ edges, where k is a positive integer.
3. Inductive step:
   - Suppose that a tree T has $k + 1$ vertices and that v is a leaf[3] of T. Let w be the parent of v.
   - Remove v from T and the edge connecting w to v. It produces a tree T' with k vertices[4].
   - By the assumption hypothesis, T' has $k - 1$ edges. It follows that T has k edges because it has one more edge than T' (the edge connecting v and w).

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[3] It must exist because the tree is finite

[4] T' is still connected and has no simple circuits.

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Reference

- Discrete Mathematics and Its Applications. Rosen, K.H. 2012. McGraw-Hill.
  - Chapter 10. Graphs:
    Section 10.2: Graph Terminology and Special Types of Graphs.
    Section 10.7: Planar Graphs.
  - Chapter 11. Trees:
    Section 11.1: Introduction to Trees.