

CS/MATH 111, Discrete Structures - Fall 2018.

Discussion 7 - Non-homogeneous Recurrences, Tiling & Red Riding Hood problem

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Outline

Non-homogeneous recurrence

Tiling

Red Ridding Hood problem

Non-homogeneous recurrence¹

Theorem 1

$$f_n = f'_n + f''_n$$

If $\{f''_n\}$ is a particular solution of the non-homogeneous linear recurrence relation with constant coefficients:

$$f_n = c_1 \cdot f_{n-1} + c_2 \cdot f_{n-2} + \cdots + c_k \cdot f_{n-k} + g(n)$$

then every solution is of the form $\{f'_n + f''_n\}$, where $\{f'_n\}$ is a solution of the associated homogeneous recurrence relation.

¹Proof available at [Rosen, 2015. pg 521].

Non-homogeneous recurrence

Solve next non-homogeneous recurrence with initial condition $f_0 = 0$, $f_1 = 2$ and $f_2 = 7$:

$$f_n = 6 \cdot f_{n-2} + 4 \cdot f_{n-3} + 2^n \quad (1)$$

Non-homogeneous recurrence

Solve next non-homogeneous recurrence with initial condition $f_0 = 0$, $f_1 = 2$ and $f_2 = 7$:

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► $f'_n = 6 \cdot f_{n-2} + 4 \cdot f_{n-3}$

1. Characteristic equations and its roots:

$$x^3 - 6x - 4 = 0$$

$$(x + 2)(x^2 - 2x - 2) = 0$$

$$x_1 = -2, \quad x_2 = 1 + \sqrt{3}, \quad x_3 = 1 - \sqrt{3}$$

2. General form of the solution:

$$f'_n = \alpha_1 \cdot (-2)^n + \alpha_2 \cdot (1 + \sqrt{3})^n + \alpha_3 \cdot (1 - \sqrt{3})^n$$

Non-homogeneous recurrence

Solve next non-homogeneous recurrence with initial condition $f_0 = 0$, $f_1 = 2$ and $f_2 = 7$:

$$f_n = 6 \cdot f_{n-2} + 4 \cdot f_{n-3} + 2^n \quad (1)$$

► $g(n) = 2^n$, so:

$$f_n'' = p_0 \cdot 2^n \quad (2)$$

► Plug (2) in (1) becomes:

$$p_0 \cdot 2^n = 6 \cdot (p_0 \cdot 2^{n-2}) + 4 \cdot (p_0 \cdot 2^{n-3}) + 2^n$$

$$p_0 = -1 \quad (3)$$

► Finally, (3) in (2):

$$f_n'' = -2^n$$

Non-homogeneous recurrence

Solve next non-homogeneous recurrence with initial condition $f_0 = 0$, $f_1 = 2$ and $f_2 = 7$:

$$f_n = 6 \cdot f_{n-2} + 4 \cdot f_{n-3} + 2^n \quad (1)$$

► According to Theorem 1:

$$f_n = \alpha_1 \cdot (-2)^n + \alpha_2 \cdot (1 + \sqrt{3})^n + \alpha_3 \cdot (1 - \sqrt{3})^n - 2^n$$

3 Initial condition equations and their solutions:

$$f_0 = \alpha_1 \cdot (-2)^0 + \alpha_2 \cdot (1 + \sqrt{3})^0 + \alpha_3 \cdot (1 - \sqrt{3})^0 - 2^0 = 0$$

$$f_1 = \alpha_1 \cdot (-2)^1 + \alpha_2 \cdot (1 + \sqrt{3})^1 + \alpha_3 \cdot (1 - \sqrt{3})^1 - 2^1 = 2$$

$$f_2 = \alpha_1 \cdot (-2)^2 + \alpha_2 \cdot (1 + \sqrt{3})^2 + \alpha_3 \cdot (1 - \sqrt{3})^2 - 2^2 = 7$$

⋮

4 Final answer:

⋮

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Example²

Suppose you are trying to tile a $1 \times n$ walkway with 4 different types of tiles: a red 1×1 tile, a green 1×1 tile, a blue 1×1 tile, and a grey 2×1 tile...

- a) Set up and explain a recurrence relation for the number of different tilings for a sidewalk of length n .
- b) What is the solution of this recurrence relation?
- c) How long must the walkway be in order to have more than 1000 different tiling possibilities?

²from <https://tinyurl.com/y6wj64bd>

Example

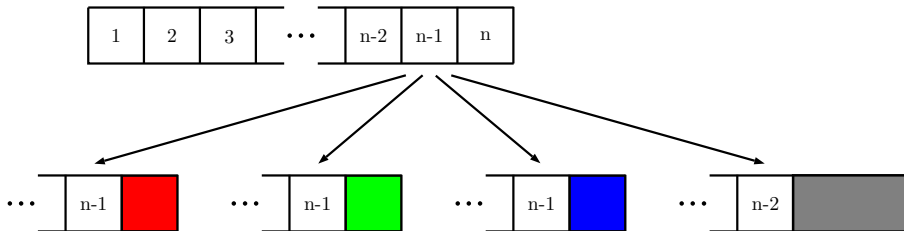
Suppose you have a tiling of length n . This can be built from:

1. a tiling of length $n - 1$ followed by a single tile; OR
2. a tiling of length $n - 2$ followed by a double tile.

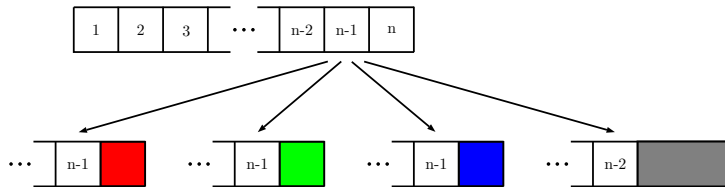
Example

Suppose you have a tiling of length n . This can be built from:

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Example



- Let T_n be the number of different ways of tiling a 1 x n space. Then for $n \geq 3$:

$$T_n = 3 \cdot T_{n-1} + 1 \cdot T_{n-2} \quad (1)$$

Example

- ▶ Let T_n be the number of different ways of tiling a $1 \times n$ space. Then for $n \geq 3$:

$$T_n = 3 \cdot T_{n-1} + T_{n-2} \quad (1)$$

- ▶ There are 3 possibilities to fill a 1×1 walkway ($n = 1$) and 10 to fill a 2×1 ($n = 2$) walkway, so initial conditions are $T_1 = 3$ and $T_2 = 10$.
- ▶ Then by (1):

$$T_3 = 3 \cdot T_2 + T_1 = 3 \cdot 10 + 3 = 33$$

$$T_4 = 3 \cdot T_3 + T_2 = 3 \cdot 33 + 10 = 109$$

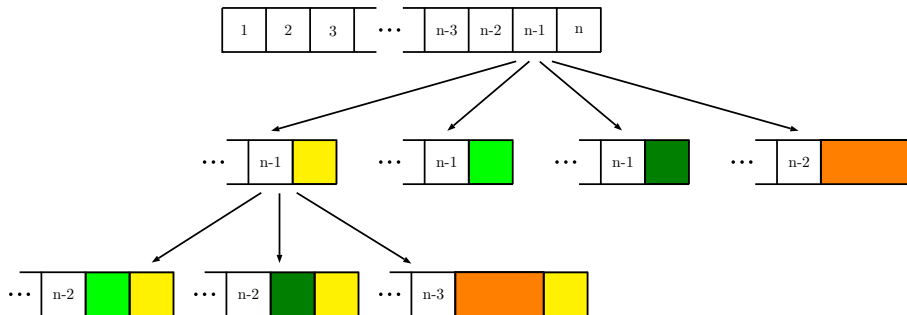
$$T_5 = 3 \cdot T_4 + T_3 = 3 \cdot 109 + 33 = 360$$

$$T_6 = 3 \cdot T_5 + T_4 = 3 \cdot 360 + 109 = 1189$$

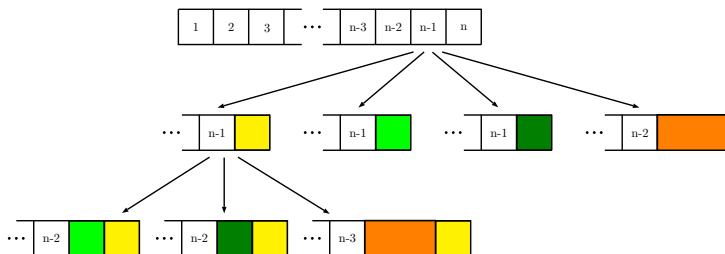
Quiz 3 Problem 3

We want to tile the $n \times 1$ strip with 2×1 and 1×1 tiles, using 2×1 tiles of orange color and 1×1 tiles of three colors: yellow, light-green and dark green. Let T_n be the number of such tilings in which no yellow tiles are next to each other. Determine the formula for T_n by setting up a recurrence equation...

Quiz 3 Problem 3



Quiz 3 Problem 3



$$T_n = 2 \cdot T_{n-1} + 3 \cdot T_{n-2} + 1 \cdot T_{n-3}$$

Initial conditions

$$T_0 = \text{Empty tile} = 1$$

$$T_1 = \text{Y, LG and DG} = 3$$

$$T_2 = \text{O, LG-Y, DG-Y, ...} = 9$$

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<http://www.cs.ucr.edu/~acald013/public/tmp/rrh.pdf>