CS/MATH 111, Discrete Structures - Fall 2018. Discussion 8 - Graphs

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Outline

Euler path and tour

Definition 1.1

An *Euler tour* in a graph G is a simple circuit containing **every edge** of G. An *Euler path* in G is a simple path containing every edge of G.

Euler tour

- An Euler tour (or Eulerian tour, Euler circuit) traverses each edge of the graph **exactly once**.
- Graphs that have an Euler tour are called Eulerian.

Euler tour

Theorem 1

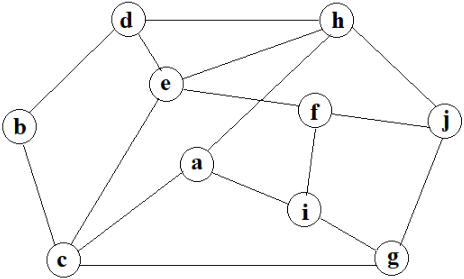
An undirected graph has a closed Euler tour iff it is connected and each vertex has an even degree.

Theorem 2

An undirected graph has an Euler path but not an Euler tour iff it has exactly two vertices of odd degree.

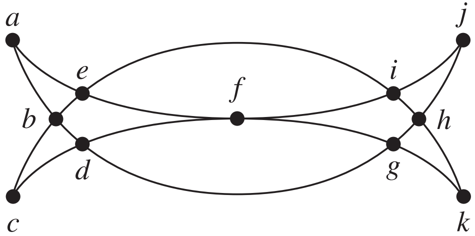
Euler tour

- So this graph is not Eulerian:



Euler tour

- Mohammed's Scimitars:



Euler tour

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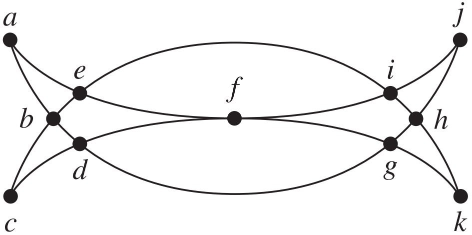
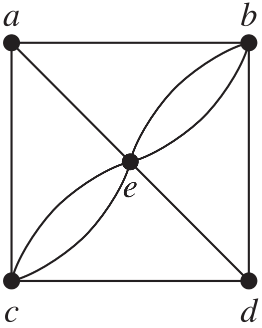


a, b, d, g, h, j, i, h, k, g, f, d, c, b, e, i, f, e, a

Euler tour

- Determine weather the given graph has an Euler circuit:



Euler tour

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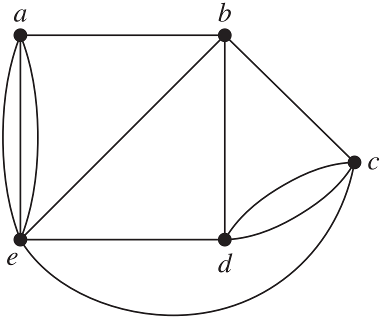


Outline

Hamiltonian Cycle

- Hamiltonian Cycle (or Hamilton circuit) is a graph cycle (i.e., closed loop) through a graph that visits each node exactly once

Hamiltonian Cycle

Theorem 3 (Dirac's Theorem)

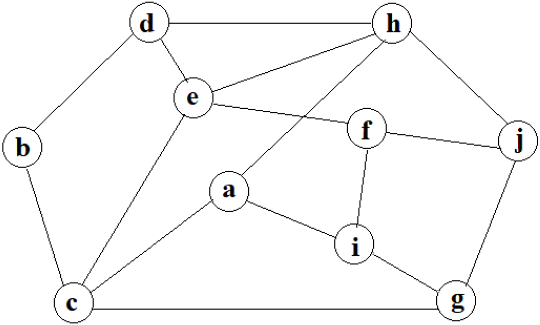
If G is a simple graph with n vertices with $n \geq 3$ such that the degree of every vertex in G is at least $\frac{n}{2}$, then G has a Hamilton cycle.

Theorem 4 (Ore's Theorem)

If G is a simple graph on n vertices, $n \geq 3$, and $d(v) + d(w) \geq n$ whenever v and w are not adjacent, then G has a Hamilton cycle.

Hamiltonian Cycle

- The graph does not have Hamiltonian cycle.



Hamiltonian Cycle

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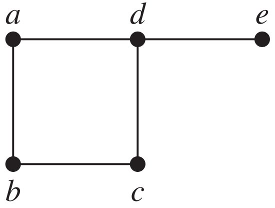
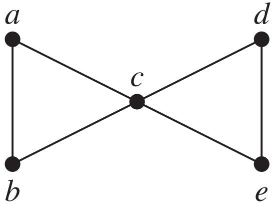
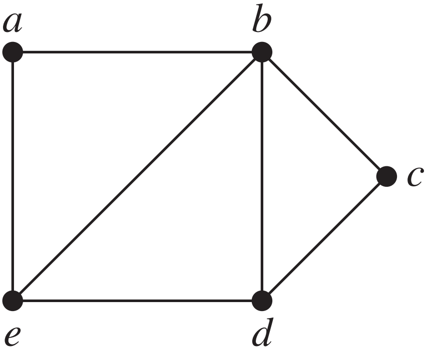




Hamiltonian Cycle

- Determine weather the given graph has a Hamilton circuit:



Hamiltonian Cycle

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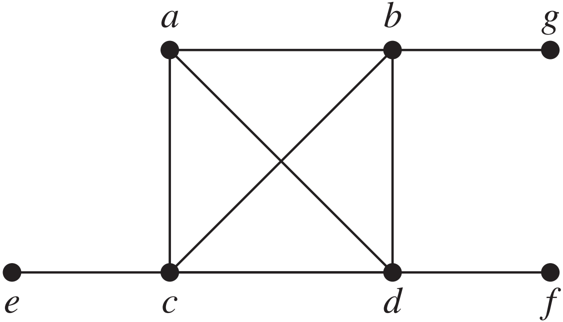

Hamiltonian Cycle

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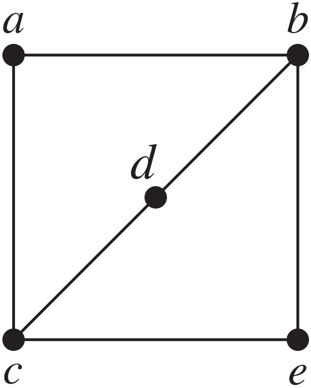
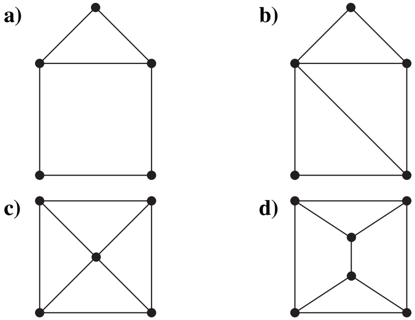


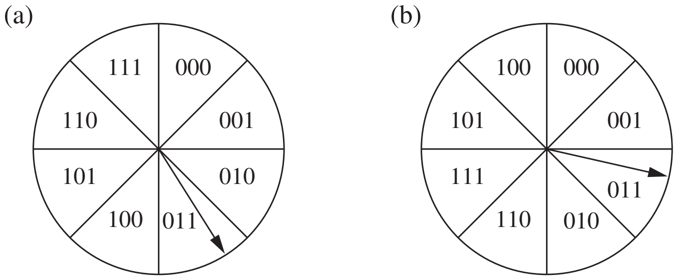
Hamiltonian Cycle

- For each of these graphs, determine:
  - (i) whether Dirac's theorem can be used to show that the graph has a Hamilton circuit
  - (ii) whether Ore's theorem can be used to show that the graph has a Hamilton circuit
  - (iii) whether the graph has a Hamilton circuit



Gray codes [1]

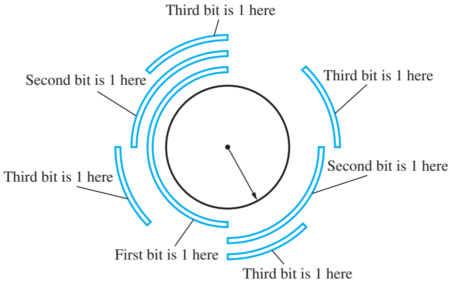
- Converting the position of a pointer into digital form:

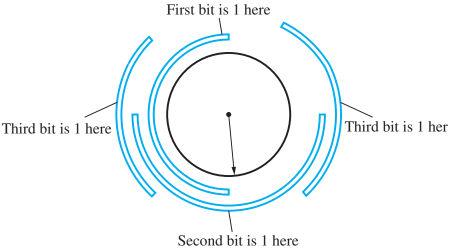


[1] https://en.wikipedia.org/wiki/Gray_code

Gray codes

- The digital representation of the position of the pointer:





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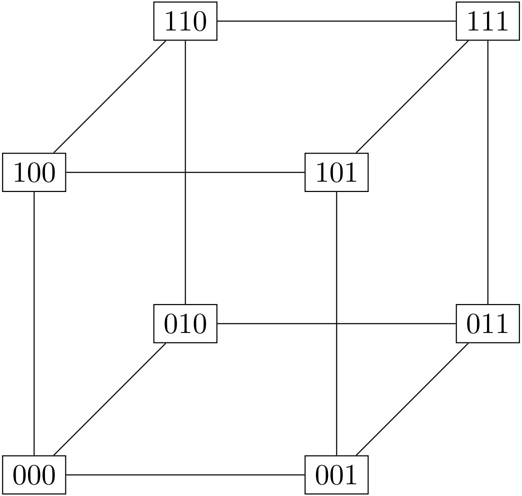

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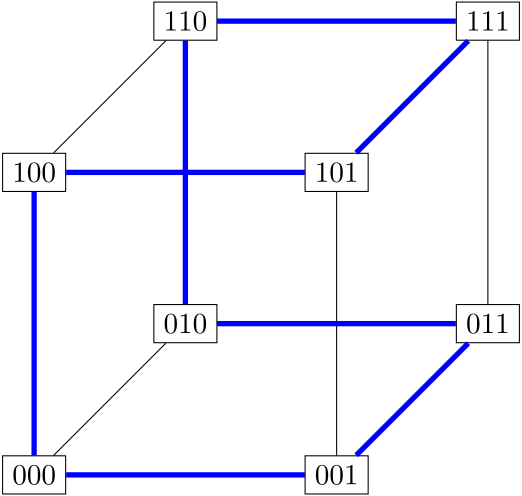
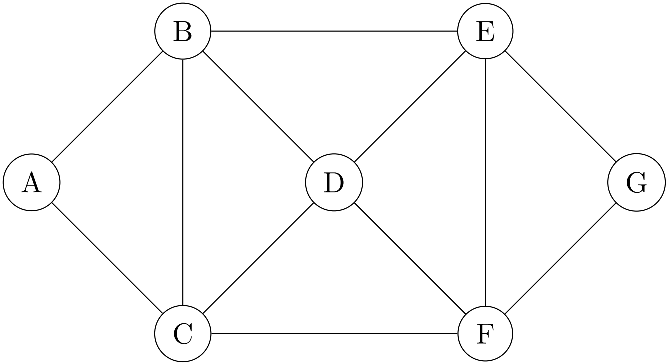


Outline

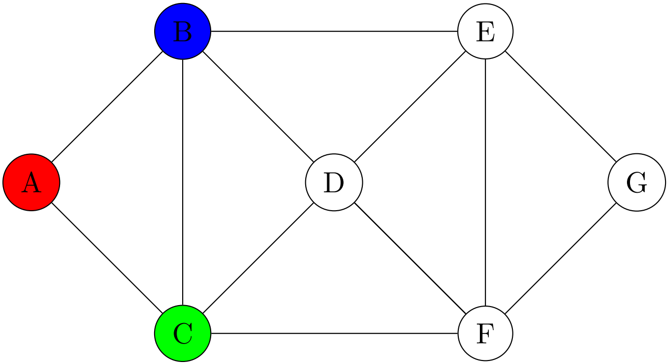
Vertex Coloring

- The chromatic number of a graph is the smallest number of colors needed to color the vertices so that no two adjacent vertices share the same color.
- Hardness: A very hard problem(an NP-Complete problem).

Vertex Coloring



Vertex Coloring



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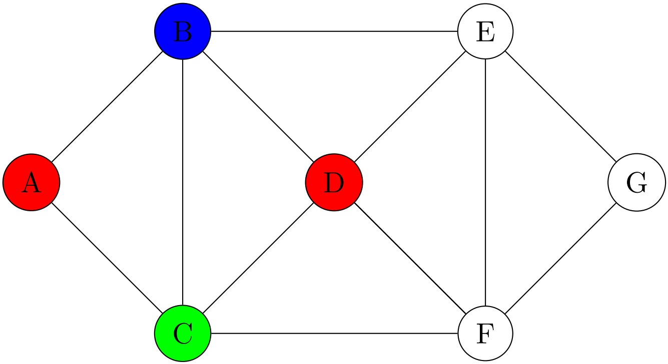

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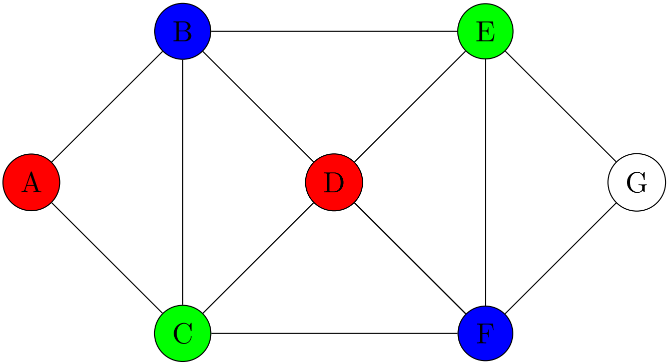

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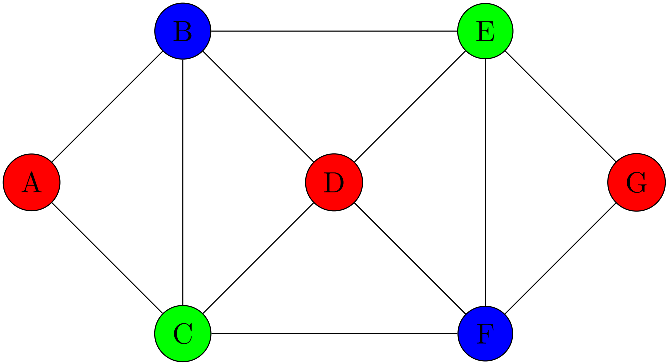

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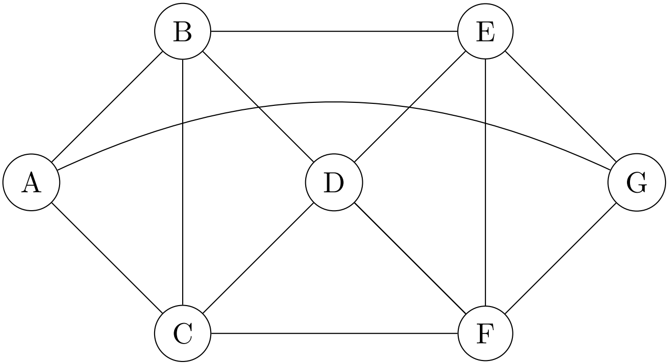

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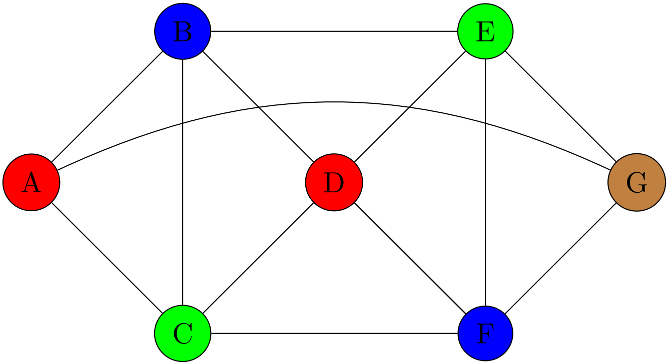

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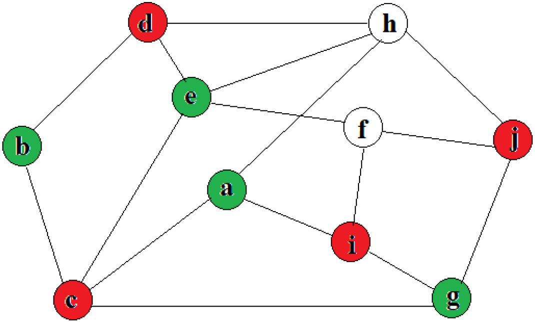
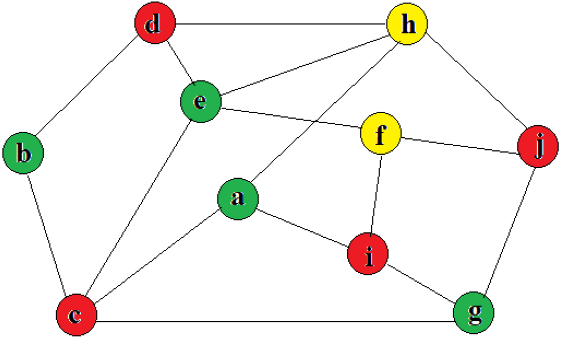
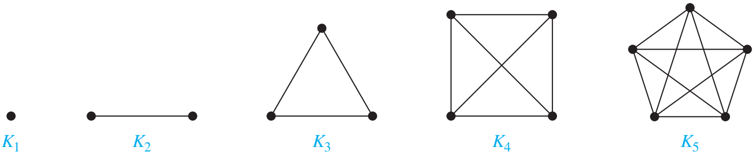


Vertex Coloring



Hamiltonian Cycle

- Complete graphs of n vertices (K_n):



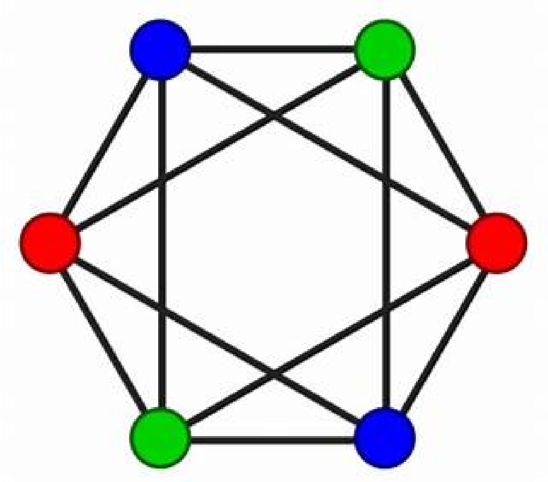
Vertex Coloring

For certain classes of graphs, we can easily compute the chromatic number. For example, the chromatic number of K_n is n, for any n. Notice that we have to argue two separate things to establish that this is its chromatic number:

- K_n can be colored with n colors.
- K_n cannot be colored with less than n colors.

For K_n, both of these facts are fairly obvious. Assigning a different color to each vertex will always result in a well-formed coloring (though it may be a waste of colors). Since each vertex in K_n is adjacent to every other vertex, no two can share a color. So fewer than n colors can't possibly work.

Vertex Coloring



Frequency Assignments

- Television channels 2 through 13 are assigned to stations in Colombia so that no two stations within 150 Km can operate on the same channel. How can the assignment of channels be modeled by graph coloring?

Frequency Assignments

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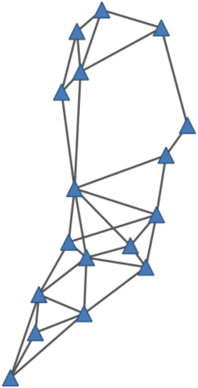


Frequency Assignments

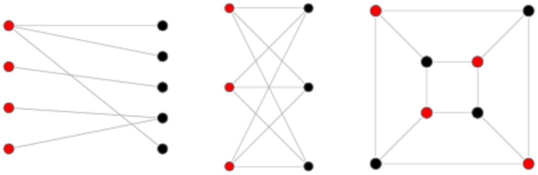
- Television channels 2 through 13 are assigned to stations in Colombia so that no two stations within 150 Km can operate on the same channel. How can the assignment of channels be modeled by graph coloring?
- Construct a graph by assigning a vertex to each station. Two vertices are connected by an edge if they are located within 150 Km of each other. An assignment of channels corresponds to a coloring of the graph, where each color represents a different channel.

Outline

Bipartite graph

- A bipartite graph, also called a bigraph, is a set of graph vertices decomposed into two disjoint sets such that no two graph vertices within the same set are adjacent.
- Bipartite graphs are equivalent to two-colorable graphs.
- All acyclic graphs are bipartite.
- A cyclic graph is bipartite iff all its cycles are of even length

Bipartite graph



Outline

Perfect matching

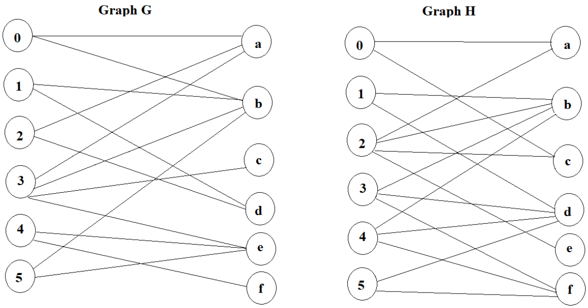
A perfect matching of a graph is a matching (i.e., an independent edge set) in which every vertex of the graph is incident to exactly one edge of the matching.
A perfect matching is therefore a matching containing $\frac{n}{2}$ edges (the largest possible), meaning perfect matchings are only possible on graphs with an even number of vertices.

Perfect matching

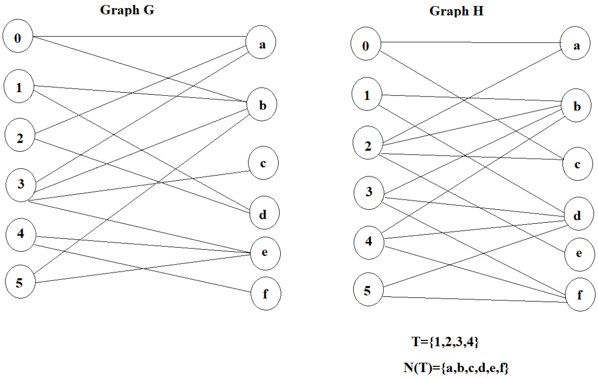
Halls Theorem: Let G = (X,Y) be a bipartite graph. Then X has a perfect macthing into Y if and only if for all $T \subseteq X$, the inequality $|T| \leq |N(T)|$ holds. Where N(T) is the set of all neighbors of the vertices in T. In other words, $y \in Y$ is an element of N(T) if and only if there is a vertex $x \in T$ so that xy is an edge.

Perfect matching

You are given two bipartite graph G and H below. For each graph determine whether it has a perfect matching. Justify your answer, either by listing the edges that are in the matching or use Hall's Theorem to show that the graph does not have a perfect matching.



Perfect matching



Reference

- Discrete Mathematics and Its Applications. Rosen, K.H. 2012. McGraw-Hill.
  Chapter 10: Graphs.
  Section 10.5: Euler and Hamilton Paths.
  Section 10.8: Graph Coloring.