

CS/MATH 111, Discrete Structures - Fall 2018.

Discussion 6 - Non-homogeneous Recurrences, Divide and Conquer & Inclusion - Exclusion

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Outline

Non-homogeneous recurrence

Divide and Conquer

Inclusion-Exclusion

Non-homogeneous recurrence

Theorem

$$f_n = f'_n + f''_n$$

Non-homogeneous recurrence

Linear Non-Homogeneous Recurrence Relations

$$f_n = 6f_{n-1} - 9f_{n-2} + g(n)$$

$$f'_{nc} = \alpha_1 3^n + \alpha_2 n 3^n$$

$$f''_n = n^m (p_t n^t + p_{t-1} n^{t-1} + \dots + p_1 n + p_0) s^n$$

$$g(n) = 5$$

$$f''_n = p_0$$

$$g(n) = 5n + 1$$

$$f''_n = p_1 n + p_0$$

$$g(n) = 5n^2 + 1$$

$$f''_n = p_2 n^2 + p_1 n + p_0$$

$$g(n) = 5n^2 + n + 1$$

$$f''_n = p_2 n^2 + p_1 n + p_0$$

$$g(n) = n 2^n$$

$$f''_n = (p_1 n + p_0) 2^n$$

$$g(n) = 2^n (5n^2 + n + 1)$$

$$f''_n = (p_2 n^2 + p_1 n + p_0) 2^n$$

Non-homogeneous recurrence

- Find a particular solution for recurrence relation:

$$f_n = 3 \cdot f_{n-1} + f_{n-2} + 6 \quad (1)$$

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$$f_n'' = p_0 \quad (2)$$

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- Plugin (2) in (1) becomes:

$$p_0 = 3 \cdot p_0 + p_0 + 6$$

$$p_0 - p_0 - 3 \cdot p_0 = 6$$

$$p_0 = -\frac{6}{2} = -2 \quad (3)$$

Non-homogeneous recurrence

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- Finally (3) in (2):

$$f_n'' = -2$$

Non-homogeneous recurrence

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Non-homogeneous recurrence

- Find a particular solution for recurrence relation:

$$f_n = 3 \cdot f_{n-1} + f_{n-2} + 3 \cdot 2^n \quad (1)$$

- $g(n) = 3 \cdot 2^n$:

$$f_n'' = p_0 \cdot 2^n \quad (2)$$

Non-homogeneous recurrence

► (2) in (1):

$$p_0 \cdot 2^n = 3 \cdot p_0 \cdot 2^{n-1} + p_0 \cdot 2^{n-2} + 3 \cdot 2^n$$

$$p_0 \cdot 2^n = 2^{n-2}(3 \cdot p_0 \cdot 2^1 + p_0 \cdot 2^0 + 3 \cdot 2^2)$$

$$p_0 \cdot 2^2 = 3 \cdot p_0 \cdot 2 + p_0 + 3 \cdot 4$$

$$p_0 \cdot 4 = 7 \cdot p_0 + 12$$

$$p_0 = -4 \tag{3}$$

Non-homogeneous recurrence

► (2) in (1):

$$p_0 \cdot 2^n = 3 \cdot p_0 \cdot 2^{n-1} + p_0 \cdot 2^{n-2} + 3 \cdot 2^n$$

$$p_0 \cdot 2^n = 2^{n-2}(3 \cdot p_0 \cdot 2^1 + p_0 \cdot 2^0 + 3 \cdot 2^2)$$

$$p_0 \cdot 2^2 = 3 \cdot p_0 \cdot 2 + p_0 + 3 \cdot 4$$

$$p_0 \cdot 4 = 7 \cdot p_0 + 12$$

$$p_0 = -4 \tag{3}$$

► Finally, (3) in (2):

$$f_n'' = -4 \cdot 2^n$$

Non-homogeneous recurrence

Solve the following non-homogeneous recurrence:

$$f_n = 4 \cdot f_{n-1} - 4 \cdot f_{n-2} + 2 \cdot 5^n \quad (1)$$

with initial condition: $f_0 = 1$ and $f_1 = 2$.

Non-homogeneous recurrence

Solve the following non-homogeneous recurrence:

$$f_n = 4 \cdot f_{n-1} - 4 \cdot f_{n-2} + 2 \cdot 5^n \quad (1)$$

with initial condition: $f_0 = 1$ and $f_1 = 2$.

► $f'_n = 4 \cdot f_{n-1} - 4 \cdot f_{n-2}$

1. Characteristic equations and its roots:

$$x^2 - 4 \cdot x + 4 = 0$$

$$(x - 2)(x - 2) = 0$$

$$x_{1,2} = 2$$

2. General form of the solution:

$$f'_n = \alpha_1 \cdot 2^n + \alpha_2 \cdot n \cdot 2^n$$

Non-homogeneous recurrence

Solve the following non-homogeneous recurrence:

$$f_n = 4 \cdot f_{n-1} - 4 \cdot f_{n-2} + 2 \cdot 5^n \quad (1)$$

with initial condition: $f_0 = 1$ and $f_1 = 2$.

► $g(n) = 2 \cdot 5^n$, so:

$$f_n'' = p_0 \cdot 5^n \quad (2)$$

► (2) in (1):

$$p_0 \cdot 5^n = 4 \cdot p_0 \cdot 5^{n-1} - 4 \cdot p_0 \cdot 5^{n-2} + 2 \cdot 5^n$$

$$p_0 = 2 \quad (3)$$

► Finally, (3) in (2):

$$f_n'' = 2 \cdot 5^n$$

Non-homogeneous recurrence

Solve the following non-homogeneous recurrence:

$$f_n = 4 \cdot f_{n-1} - 4 \cdot f_{n-2} + 2 \cdot 5^n \quad (1)$$

with initial condition: $f_0 = 1$ and $f_1 = 2$.

▶ $f'_n = \alpha_1 \cdot 2^n + \alpha_2 \cdot n \cdot 2^n$

▶ $f''_n = 2 \cdot 5^n$

3 Initial condition equations and their solutions:

$$f_0 = \alpha_1 \cdot 2^0 + \alpha_2 \cdot 0 \cdot 2^0 + 2 \cdot 5^0 = \alpha_1 + 2 = 1$$

$$f_1 = \alpha_1 \cdot 2^1 + \alpha_2 \cdot 1 \cdot 2^1 + 2 \cdot 5^1 = 2 \cdot \alpha_1 + 2 \cdot \alpha_2 + 10 = 2$$

$$\alpha_1 = -1$$

$$\alpha_2 = -3$$

4 Final answer:

⋮

Outline

Non-homogeneous recurrence

Divide and Conquer

Inclusion-Exclusion

Divide and Conquer

Problem 1: Give the asymptotic value (using the Θ -notation) for the number of letters that will be printed by the algorithms below. Your solution needs to consist of an appropriate recurrence equation and its solution, with a brief justification. (See the suggested format at the bottom of the assignment).

Divide and Conquer

- (a) **Algorithm PRINTXS** (n : integer)
- ```

 if $n < 3$
 print("X")
 else
 PRINTXS($\lceil n/3 \rceil$)
 PRINTXS($\lceil n/3 \rceil$)
 PRINTXS($\lceil n/3 \rceil$)
 for $i \leftarrow 1$ to $2n$ do print("X")
```
- (b) **Algorithm PRINTYS** ( $n$  : integer)
- ```

    if  $n < 2$ 
        print("Y")
    else
        for  $j \leftarrow 1$  to  $16$  do PRINTYS( $\lfloor n/2 \rfloor$ )
        for  $i \leftarrow 1$  to  $n^3$  do print("Y")
```

Divide and Conquer

(c) **Algorithm PRINTZS** (n : integer)

 if $n < 3$

 print("Z")

 else

 PRINTZS($\lceil n/3 \rceil$)

 PRINTZS($\lceil n/3 \rceil$)

 for $i \leftarrow 1$ to $7n$ do print("Z")

(d) **Algorithm PRINTUS** (n : integer)

 if $n < 4$

 print("U")

 else

 PRINTUS($\lceil n/4 \rceil$)

 PRINTUS($\lfloor n/4 \rfloor$)

 for $i \leftarrow 1$ to 11 do print("U")

Divide and Conquer

```
(e) Algorithm PRINTVS ( $n$  : integer)
    if  $n < 3$ 
        print("V")
    else
        for  $j \leftarrow 1$  to 9 do PRINTVS( $\lfloor n/3 \rfloor$ )
        for  $i \leftarrow 1$  to  $2n^3$  do print("V")
```

Divide and Conquer

(a)

There are 3 recursive calls, each with parameter $\lceil n/3 \rceil$. Since we are looking for an asymptotic solution, we can ignore rounding. Then the number of letters printed can be expressed by the recurrence:

$$X(n) = 3X(n/3) + 2n.$$

We apply the Master Theorem with $a = 3$, $b = 3$, $c = 2$, $d = 1$. Here, we have $a = b^d$, so the solution is $\Theta(n \log n)$.

(b)

There are 16 recursive calls, each with parameter $\lfloor n/2 \rfloor$. Since we are looking for an asymptotic solution, we can ignore rounding. Then the number of letters printed can be expressed by the recurrence:

$$X(n) = 16X(n/2) + n^3.$$

We apply the Master Theorem with $a = 16$, $b = 2$, $c = 1$, $d = 3$. Here, we have $a > b^d$, so the solution is $\Theta(n^{\log_2 16})$.

(c)

There are 2 recursive calls, each with parameter $\lceil n/3 \rceil$. Since we are looking for an asymptotic solution, we can ignore rounding. Then the number of letters printed can be expressed by the recurrence:

$$X(n) = 2X(n/3) + 7n.$$

We apply the Master Theorem with $a = 2$, $b = 3$, $c = 7$, $d = 1$. Here, we have $a < b^d$, so the solution is $\Theta(n)$.

Divide and Conquer

(d)

There are 2 recursive calls, each with parameter $\lceil n/4 \rceil$. Since we are looking for an asymptotic solution, we can ignore rounding. Then the number of letters printed can be expressed by the recurrence:

$$X(n) = 2X(n/4) + 11.$$

We apply the Master Theorem with $a = 2$, $b = 4$, $c = 11$, $d = 0$. Here, we have $a > b^d$, so the solution is $\Theta(n^{\log_4 2})$.

(e)

There are 9 recursive calls, each with parameter $\lfloor n/3 \rfloor$. Since we are looking for an asymptotic solution, we can ignore rounding. Then the number of letters printed can be expressed by the recurrence:

$$X(n) = 9X(n/3) + 2n^3.$$

We apply the Master Theorem with $a = 9$, $b = 3$, $c = 2$, $d = 3$. Here, we have $a < b^d$, so the solution is $\Theta(n^3)$.

Outline

Non-homogeneous recurrence

Divide and Conquer

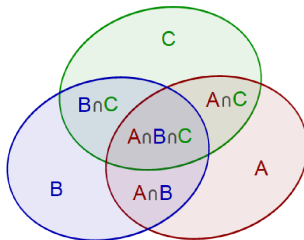
Inclusion-Exclusion

Inclusion-Exclusion

Problem 2: We have a group of people, each of which is a citizen of either US or Mexico or Canada. Half of the people in this group are US citizens, 10 are Mexican citizens, 17 are Canadian citizens, 4 people have dual US-Mexican citizenship, 5 have US-Canadian citizenship, 6 have Canadian-Mexican, and 2 are citizens of all three countries. How many people are in this group? Show your work.

Inclusion-Exclusion

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| \\ &\quad - |A \cap B| - |A \cap C| - |B \cap C| \\ &\quad + |A \cap B \cap C| \end{aligned}$$



Inclusion-Exclusion

Us citizens:	$ A $	$= \frac{X}{2}$
Mexican citizen:	$ B $	$= 10$
Canadian citizen:	$ C $	$= 17$
US-Mexican citizen:	$ A \cap B $	$= 4$
US-Canadian citizen:	$ A \cap C $	$= 5$
Canadian-Mexican:	$ B \cap C $	$= 6$
Citizens of all countries:	$ A \cap B \cap C $	$= 2$

$$X = \frac{X}{2} + 10 + 17 - 4 - 5 - 6 + 2$$

$$X = \frac{X}{2} + 14$$

$$X = 28$$