CS/MATH 111, Discrete Structures - Fall 2018.
Discussion 5 - Linear Recurrence Relations

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Outline

Fibbonacci recurrence

Homogeneous Recurrence Equations

Fibbonacci recurrence

- Fibonnacci numbers / Fibonnacci sequence
- First two and subsequent numbers:
  - $F_0 = 1$
  - $F_1 = 1$
  - $F_n = F_{n-1} + F_{n-2}$, when $n \geq 2$.
- Fibbonacci grows exponentially with n.
- Prove that:

$$0.5 \cdot 1.5^n \leq F_n \leq 2^n$$

Fibbonacci recurrence

Proof by induction using $F_0 = 1, F_1 = 1, \ldots$:

$$0.5 \cdot 1.5^n \leq F_n \leq 2^n \tag{1}$$

1. Base case:
   - $n = 0, F_0 = 1 : 0.5 \cdot 1.5^0 \leq 1 \leq 2^0 = 0.50 \leq 1 \leq 1.$
   - $n = 1, F_1 = 1 : 0.5 \cdot 1.5^1 \leq 1 \leq 2^1 = 0.75 \leq 1 \leq 2.$
   - $n = 2, F_2 = 2 : 0.5 \cdot 1.5^2 \leq 1 \leq 2^2 = 1.125 \leq 2 \leq 4.$

     $\vdots$
2. Assumption step:
   - Assume (1) holds for all $n \leq k - 1$.
3. Induction step:
   - Prove that (1) holds for all $n \leq k$.

Fibbonacci recurrence

Proof by induction using $F_0 = 1, F_1 = 1, \ldots$:

$$0.5 \cdot 1.5^n \leq F_n \leq 2^n$$

3 Induction step:

- Prove that (1) holds for all $n \leq k$.

  1) $F_k \leq 2^k$
     $F_k = F_{k-1} + F_{k-2}$
     $F_k \leq 2^{k-1} + 2^{k-2}$ (by assumption.)
     $F_k = 2^{k-2} \cdot (2+1)$
     $F_k = 2^{k-2} \cdot 3$
     $F_k \leq 2^{k-2} \cdot 4$
     $F_k = 2^{k-2} \cdot 2^2$
     $F_k = 2^k$

     $F_n = \mathcal{O}(2^n)$

Fibbonacci recurrence

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$$0.5 \cdot 1.5^n \leq F_n \leq 2^n$$

3 Induction step:

- Prove that (1) holds for all $n \leq k$.

  2) $F_k \geq \frac{1}{2} \cdot 1.5^k$
  $F_k = F_{k-1} + F_{k-2}$
  $F_k \geq 1.5^{k-1} + 1.5^{k-2}$ (by assumption.)
  $F_k = 1.5^{k-2} \cdot (1.5 + 1)$
  $F_k = 1.5^{k-2} \cdot 2.5$
  $F_k \geq 1.5^{k-2} \cdot 2.25$
  $F_k = 1.5^{k-2} \cdot 1.5^2$
  $F_k = 1.5^k$

  $F_n = \Omega(1.5^n)$

Fibbonacci recurrence

$$0.5 \cdot 1.5^n \leq F_n \leq 2^n$$

$$F_n = \mathcal{O}(2^n)$$

$$F_n = \Omega(1.5^n)$$

$$is\ F_n = \Theta(\ )\ ?$$

Outline

Fibbonacci recurrence

Homogeneous Recurrence Equations

Example 1

Solve the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$ with initial conditions $a_0 = 2$ and $a_1 = 7$.

1. Characteristic equation and its roots:
   $x^2 - x - 2 = 0$
   $(x+1)(x-2) = 0$
   So, $x_1 = 2$ and $x_2 = -1$.

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   $a_n = \alpha_1 2^n + \alpha_2(-1)^n$.

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3. Initial condition equations and their solutions:
   $a_0 = \alpha_1 + \alpha_2 = 2$
   $a_1 = \alpha_1 2 + \alpha_2(-1) = 7$
   So, $\alpha_1 = 3$ and $\alpha_2 = -1$.

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   So, $\alpha_1 = 3$ and $\alpha_2 = -1$.
4. Final answer:
   $a_n = 3 \cdot 2^n - (-1)^n$ is a solution.

Example 2

What is the solution of the recurrence relation $a_n = -a_{n-1} + 4a_{n-2} + 4a_{n-3}$ with initial conditions $a_0 = 8$, $a_1 = 6$ and $a_2 = 26$.

1. Characteristic equation and its roots:
   $x^3 + x^2 - 4x - 4 = 0$
   $(x+1)(x+2)(x-2) = 0$
   So, $x_1 = -1$, $x_2 = -2$ and $x_3 = 2$.

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3. Initial condition equations and their solutions:
   $a_0 = \alpha_1 + \alpha_2 + \alpha_3 = 8$
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   So, $\alpha_1 = 2$, $\alpha_2 = 1$ and $\alpha_3 = 5$.

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   $a_2 = \alpha_1 + 4\alpha_2 + 4\alpha_3 = 26$
   So, $\alpha_1 = 2$, $\alpha_2 = 1$ and $\alpha_3 = 5$.
4. Final answer:
   $a_n = 2 \cdot (-1)^n + (-2)^n + 5 \cdot 2^n$ is a solution.