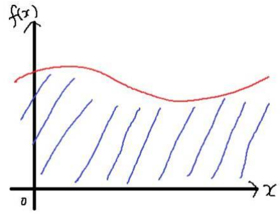
Improper Integrals

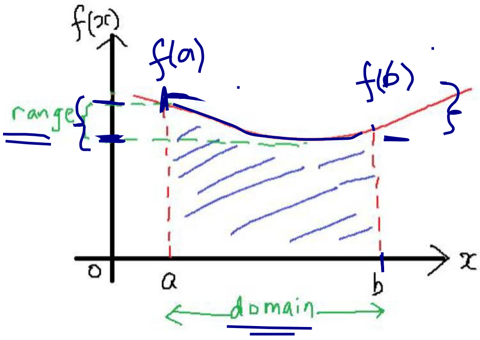
I. IMPROPER INTEGRALS

$\int f(x)\ \mathrm{d}x$ is the *indefinite* integral of $f(x)$ and represents the area under the curve $f(x)$.



domain $a \leq x \leq b$

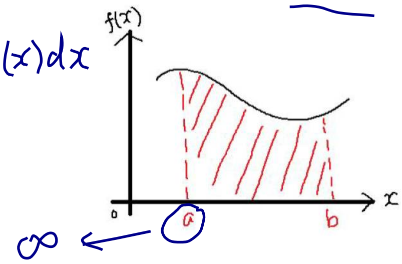
$\int_a^b f(x)\ \mathrm{d}x$ is the *definite* integral of $f(x)$ for $x \in [a, b]$ and represents the area under the curve $f(x)$ from $x = a$ to $x = b$.



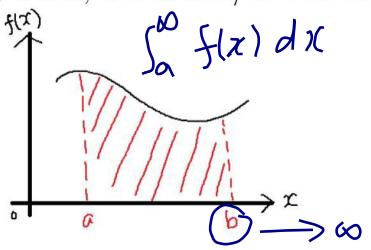
When a definite integral contains an infinity:

- either in the domain of integration, i.e. a and/or b is ∞,

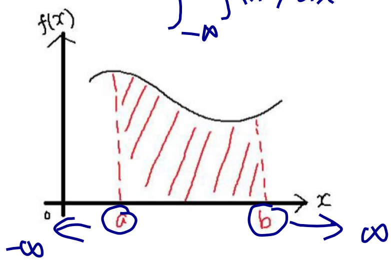
$\int_{-\infty}^{b} f(x)dx$



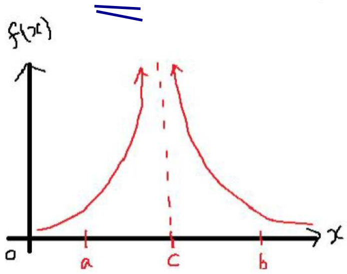
$\int_{a}^{\infty} f(x)\,dx$

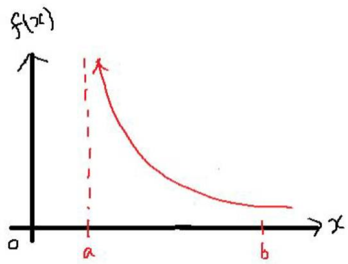


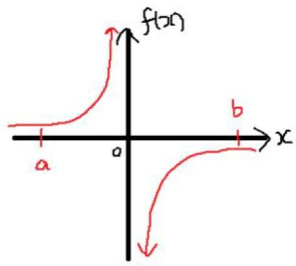
$\int_{-\infty}^{\infty} f(x)\,dx$



- or the range of the integrand $f(x)$ is unbounded for $x \in [a, b]$,







then we have an *improper integral*. Otherwise it is a *proper integral*.

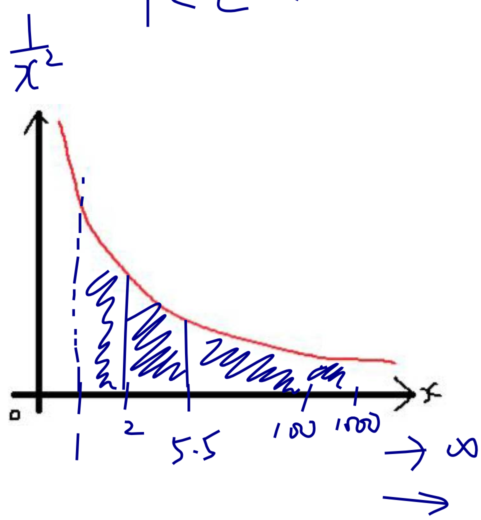
Convergence and divergence for improper integral

$1 < \varepsilon < \infty$

Example 1:

$I(\varepsilon) = \int_1^{\varepsilon} \frac{1}{x^2}\,dx$

$$I = \int_1^{\infty} \frac{1}{x^2}\,\mathrm{d}x$$



Area $\quad I(\epsilon) = \int_1^{\epsilon} \frac{1}{x^2}\,\mathrm{d}x$

$$I(\epsilon = 1) = \int_1^{1} \frac{1}{x^2}\,\mathrm{d}x = 0$$

$$I(\epsilon = 2) = \int_1^{2} \frac{1}{x^2}\,\mathrm{d}x = 0.5$$

$$I(\epsilon = 5.5) = \int_1^{5.5} \frac{1}{x^2}\,\mathrm{d}x = 0.8181$$

$$\vdots$$

$$I(\epsilon = 100) = \int_1^{100} \frac{1}{x^2}\,\mathrm{d}x = 0.99$$

$\varepsilon \to \infty$

$$I(\epsilon = 1000) = \int_1^{1000} \frac{1}{x^2}\,\mathrm{d}x = 0.999$$

10^5

$$I(\epsilon = 100000) = \int_1^{100000} \frac{1}{x^2}\,\mathrm{d}x = 0.99999$$

1

$\lim_{\varepsilon \to \infty} I(\varepsilon) = 1$

When $\epsilon \to \infty$, the value of $I(\epsilon)$ approaches 1, meaning

$$I = \lim_{\epsilon \to \infty} I(\epsilon) = 1$$

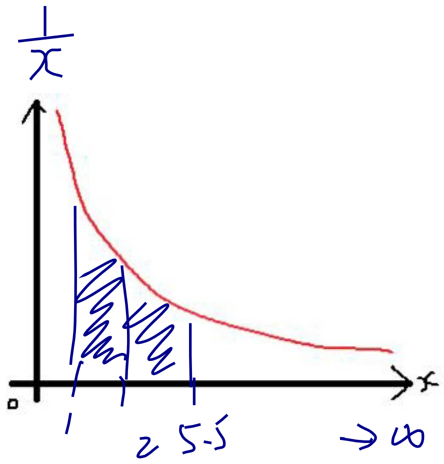
Since the limit is a (single) finite number (i.e. a real number and is not $\pm\infty$), the limit exists and the improper integral I is convergent.

I

$I(\varepsilon)=\int_1^{\varepsilon}\frac{1}{x}\,dx \quad , \quad 1<\varepsilon<\infty$

Example 2:

$$I=\int_1^{\infty}\frac{1}{x}\,\mathrm{d}x$$



Area $\quad I(\epsilon)=\int_1^{\epsilon}\frac{1}{x}\,\mathrm{d}x$

$I(\epsilon=1)=\int_1^{1}\frac{1}{x}\,\mathrm{d}x=0$

$I(\epsilon=2)=\int_1^{2}\frac{1}{x}\,\mathrm{d}x=0.6931$

$I(\epsilon=5.5)=\int_1^{5.5}\frac{1}{x}\,\mathrm{d}x=1.7047$

$\vdots$

$I(\epsilon=100)=\int_1^{100}\frac{1}{x}\,\mathrm{d}x=4.6052$

$\varepsilon\to\infty$

$I(\epsilon=1000)=\int_1^{1000}\frac{1}{x}\,\mathrm{d}x=6.9078$

10^5

$I(\epsilon=100000)=\int_1^{100000}\frac{1}{x}\,\mathrm{d}x=11.5129$

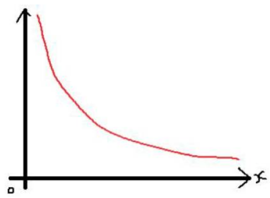
$\lim_{\varepsilon\to\infty} I(\varepsilon)=+\infty$

When $\epsilon\to+\infty$, the value of $I(\epsilon)$ approaches $+\infty$, meaning

$$I=\lim_{\epsilon\to\infty}I(\epsilon)=+\infty$$

Since the limit is $+\infty$, the limit does not exist and the improper integral I is divergent.

$$I = \int_0^\infty f(x)\,dx$$



A Standard Strategy

1. construct a proper integral $I(\epsilon)$ by replacing the ∞ with a parameter ϵ.

$$I(\varepsilon) = \int_0^\varepsilon f(x)\,dx$$

2. recover the original integral as a limit.

$$\lim_{\varepsilon\to\infty} I(\varepsilon) = \lim_{\varepsilon\to\infty}\int_0^\varepsilon f(x)\,dx$$

3. if the limit of the proper integral is finite, then the improper integral is *convergent*.

4. if the limit of the proper integral is infinite (∞) or has no single well-defined value, then the improper integral is *divergent*.

A. Improper integral due to unbounded domain

$\int_a^\infty f(x)\,\mathrm{d}x$	or	$\int_{-\infty}^b f(x)\,\mathrm{d}x$	or	$\int_{-\infty}^\infty f(x)\,\mathrm{d}x$

Example: $I = \int_1^\infty \frac{1}{x^2}\,\mathrm{d}x$

$$I(\varepsilon) = \int_1^\varepsilon \frac{1}{x^2}dx, \quad 1 < \varepsilon < \infty$$

$$= \left[-\frac{1}{x}\right]_1^\varepsilon$$

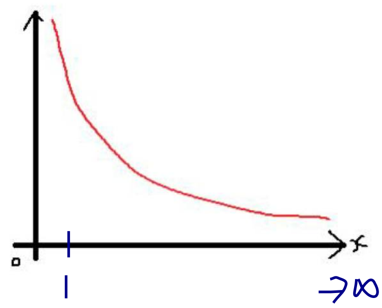
$$= -\frac{1}{\varepsilon} + 1$$

$$I = \lim_{\varepsilon\to\infty} I(\varepsilon)$$

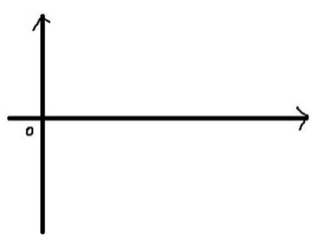
$$= \lim_{\varepsilon\to\infty}\left(-\frac{1}{\varepsilon}+1\right)$$

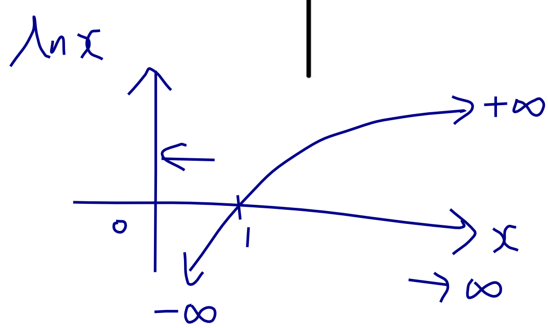
$$= 1$$

I is convergent and $I = 1$



(Post-class) **Example:** $\int_1^\infty \frac{1}{x}\,\mathrm{d}x$

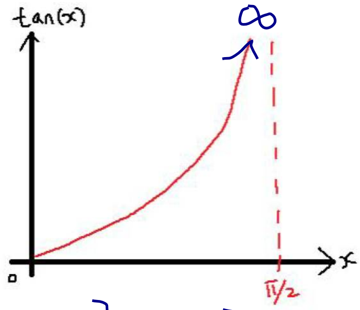




$$\int \frac{f'(x)}{f(x)}\,dx = \ln(f(x)) \qquad \int \tan(x)\,dx = -\int \frac{-\sin(x)}{\cos(x)}\,dx = -\ln(\cos(x))$$

B. Improper integral due to unbounded range of integrand

$$0 \le x \le \frac{\pi}{2}$$



Example: $I = \displaystyle\int_0^{\pi/2} \tan(x)\,\mathrm{d}x$

$$I(\varepsilon) = \int_0^{\varepsilon} \tan(x)\,dx, \quad 0 < \varepsilon < \frac{\pi}{2}$$

$$= \Big[-\ln(\cos(x))\Big]_0^{\varepsilon}$$

$$= -\ln(\cos(\varepsilon)) + \underbrace{\ln(\underbrace{\cos(0)}_{1})}_{0}$$

$$= -\ln(\cos(\varepsilon))$$

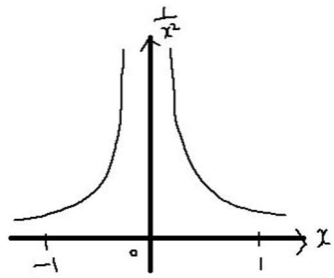
$$I = \lim_{\varepsilon \to \frac{\pi}{2}} I(\varepsilon)$$

$$= \lim_{\varepsilon \to \frac{\pi}{2}} \Big[\underbrace{-\ln(\underbrace{\cos(\varepsilon)}_{0})}_{-\infty}\Big]$$

$= +\infty$ (limit does not exist)

I is divergent.

(Post-class) **Example:** $\displaystyle\int_{-1}^{1} \frac{1}{x^2}\,\mathrm{d}x$



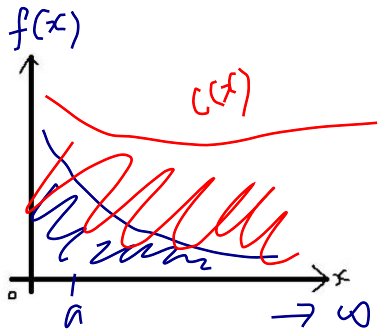
II. COMPARISON TEST FOR IMPROPER INTEGRALS

Sometimes we do not have a simple anti-derivative for an integrand. The trick here is to look at a simpler (improper) integral for which we can find a simple anti-derivative.

The general strategy: Suppose we have

$a \leq x < \infty$

$$\int_a^\infty f(x)\mathrm{d}x \quad \text{with } f(x) > 0$$

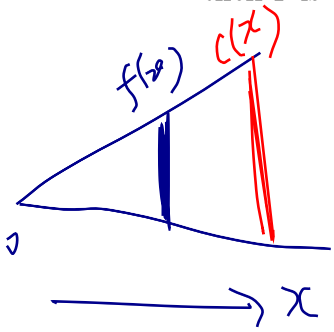


then we have two cases to consider:

If there is a function $c(x)$ such that

$0 < f(x) < c(x)$ and $\lim\limits_{\epsilon\to\infty} \int_a^\epsilon c(x)\mathrm{d}x$ is finite,

then I is convergent.

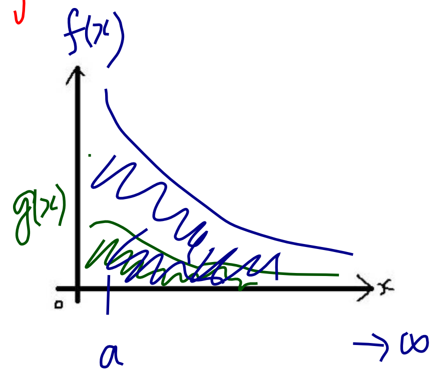


$0 < f(x) < c(x)$

$0 < \int_a^\infty f(x)\,dx < \int_a^\infty c(x)\,dx$

conv. convergent

x no conclusion divergent



If there is a function $g(x)$ such that

$0 < g(x) < f(x)$ and $\lim\limits_{\epsilon\to\infty} \int_a^\epsilon g(x)\mathrm{d}x$ is undefined,

then I is divergent.

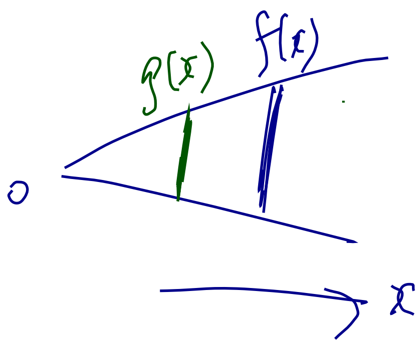
domain $a \leq x < \infty$

$0 < g(x) < f(x)$

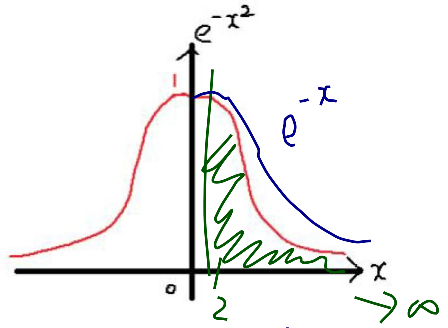
$0 < \int_a^\infty g(x)\,dx < \int_a^\infty f(x)\,dx$

convergent → x no conclusion

divergent → divergent.



$$e^{-x^2}$$



Example: $I = \int_2^\infty e^{-x^2}\,dx$

$2 \leq x < \infty$

$2 \leq x < x^2 < \infty$

$-2 \geq -x > -x^2 > -\infty$

$e^{-2} \geq e^{-x} > e^{-x^2} > e^{-\infty} = \frac{1}{e^\infty} = 0$

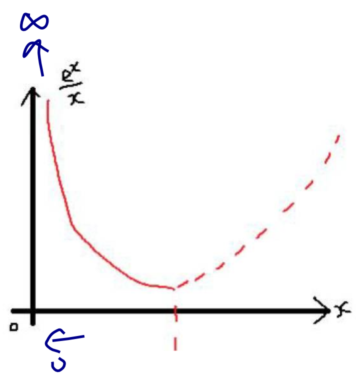
$0 < e^{-x^2} < e^{-x} \leq \frac{1}{e^2}$

$0 < \underbrace{\int_2^\infty e^{-x^2}\,dx}_{I} < \underbrace{\int_2^\infty e^{-x}\,dx}_{I_1}$

$I_1(\varepsilon) = \int_2^\varepsilon e^{-x}\,dx = \left[-e^{-x}\right]_2^\varepsilon = -\frac{1}{e^\varepsilon} + \frac{1}{e^2}$

$I_1 = \lim_{\varepsilon\to\infty} I_1(\varepsilon) = \lim_{\varepsilon\to\infty}\left(-\frac{1}{e^\varepsilon} + \frac{1}{e^2}\right) = \frac{1}{e^2}$ (convergent)

I is convergent by comparison test.



(Post-class) **Example:** $I = \int_0^1 \frac{e^x}{x}\,dx$

$0 \leq x \leq 1$

$e^0 \leq e^x \leq e^1$

$0 < \frac{1}{x} \leq \frac{e^x}{x} \leq \frac{e}{x}$

$0 < \underbrace{\int_0^1 \frac{1}{x}\,dx}_{I_2} < \underbrace{\int_0^1 \frac{e^x}{x}\,dx}_{I} < \underbrace{\int_0^1 \frac{e}{x}\,dx}_{I_1}$

$I_1 = \int_0^1 \frac{e}{x}\,dx$

$I_1(\varepsilon) = \int_\varepsilon^1 \frac{e}{x}\,dx$

$= e\left[\ln x\right]_\varepsilon^1 = e(\ln 1 - \ln\varepsilon) = -e\ln(\varepsilon)$

$I_1 = \lim_{\varepsilon\to 0} I_1(\varepsilon) = \lim_{\varepsilon\to 0}\left[-e\ln(\varepsilon)\right] = +\infty$

I_1 is divergent.

$\therefore$ No conclusion for I.

Try I_2!

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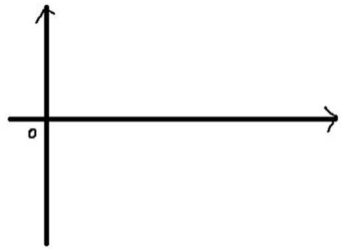

(Post-class) **Application from textbook:** Improper integrals are encountered in many contexts in engineering. For example, the period of a simple pendulum of length l released from rest with angle α is given by

$$2\sqrt{\frac{l}{g}}\int_0^{\alpha}\frac{1}{\sqrt{\sin^2\frac{\alpha}{2}-\sin^2\frac{\theta}{2}}}\,\mathrm{d}\theta$$

where g is the acceleration due to gravity. The integrand is infinite at $\theta=\alpha$.

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(Post-class) **ONLINE RESOURCES**

Improper Integrals:

1. https://en.wikibooks.org/wiki/Calculus/Improper_Integrals
2. http://tutorial.math.lamar.edu/Classes/CalcII/ImproperIntegrals.aspx
3. http://tutorial.math.lamar.edu/Classes/CalcII/ImproperIntegralsCompTest.aspx
4. https://www.khanacademy.org/math/calculus-home/integration-calc/improper-integrals-calc/v/introduction-to-improper-integrals