MONASH University

MONASH ENGINEERING

ENG1060

Linear Systems: Introduction

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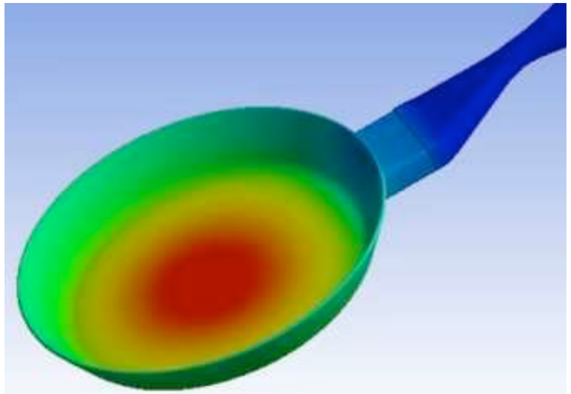
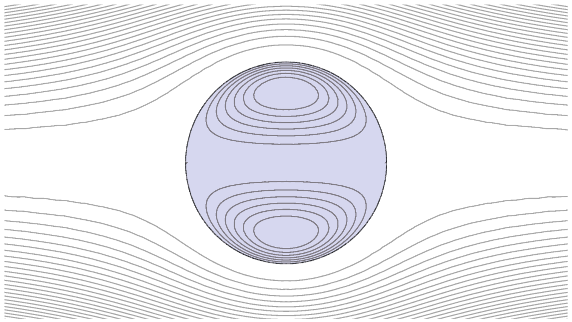
GROUP OF EIGHT AUSTRALIA

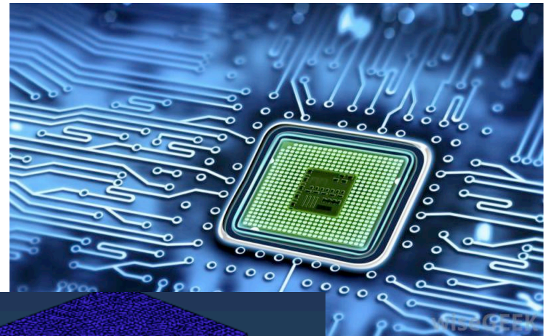
ENG1060

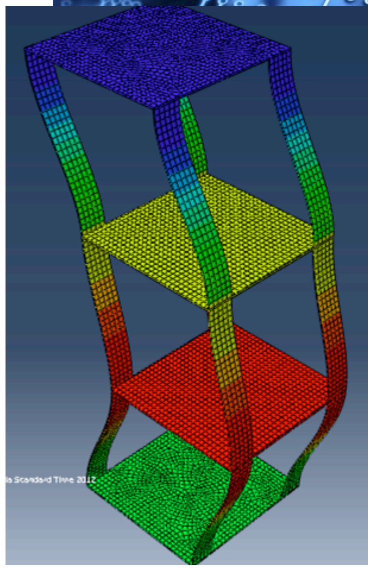
Linear systems: Introduction

Linear systems arise
EVERYWHERE
in Engineering



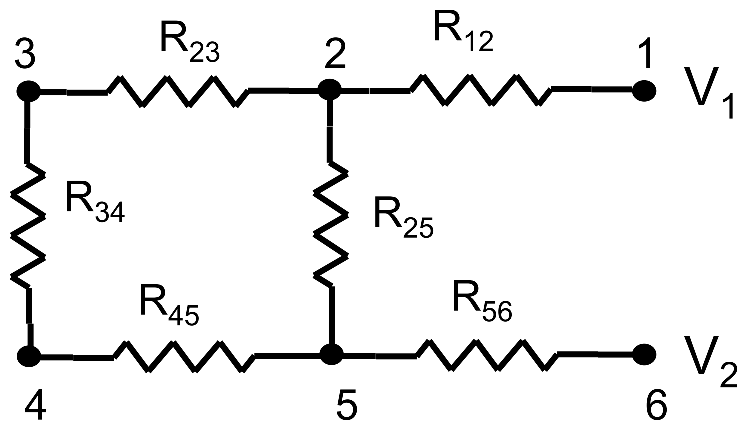






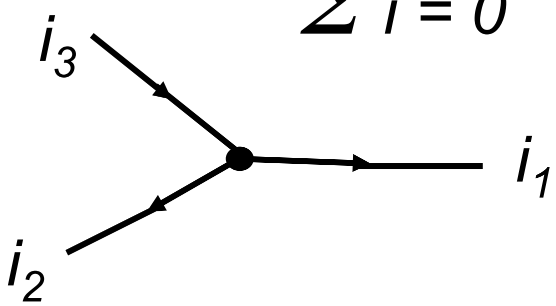
Linear systems: Introduction

- How do they arise?
- Example, resistor circuits occur in many devices
- Given a voltage drop across nodes 1 and 6 of (V_2-V_1), what is the current through each of the resistors?
  - Allows an estimate of heat generation and cooling requirements
  - Allows power consumption to be estimated



- Kirchoff's first law:
  - Input current == output current

$$\sum i = 0$$

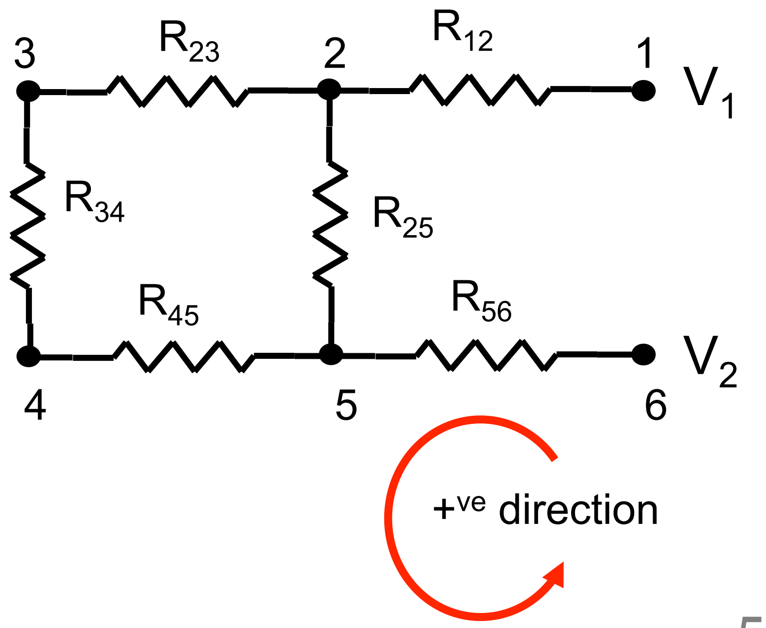


$$i_1 + i_2 = i_3 \rightarrow i_1 + i_2 - i_3 = 0$$

- Kirchoff's second law:
  - Voltage drop around closed loop sums to zero

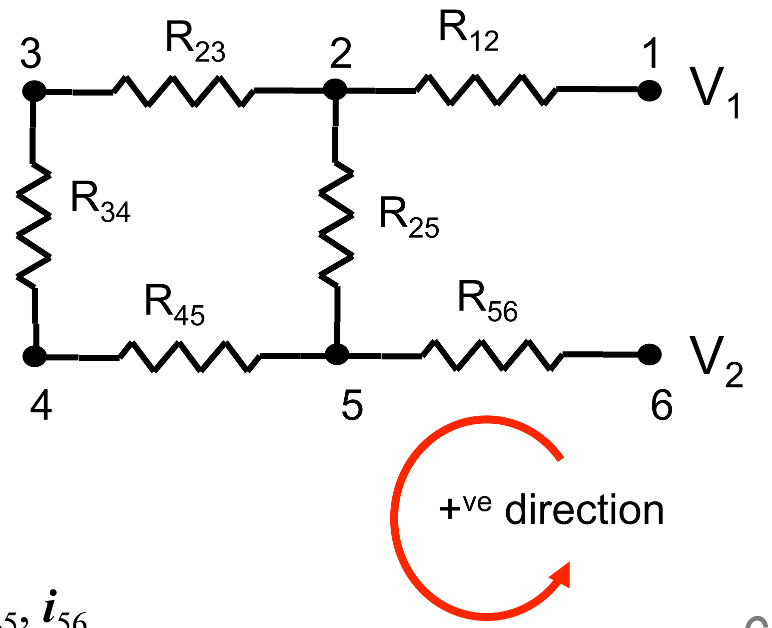
$$\sum (E - iR) = 0$$

- Defining i_{jk} as the current from node j to node k, and using Kirchoff's laws we find



- Defining i_{jk} as the current from node j to node k, and using Kirchoff's laws we find

$$
\begin{aligned}
i_{12} - i_{23} - i_{25} &= 0 && \textit{node } 2 \\
i_{23} - i_{34} &= 0 && \textit{node } 3 \\
i_{34} - i_{45} &= 0 && \textit{node } 4 \\
i_{25} + i_{45} - i_{56} &= 0 && \textit{node } 5 \\
R_{23}i_{23} + R_{34}i_{34} + R_{45}i_{45} - R_{25}i_{25} &= 0 && \textit{LH loop} \\
R_{12}i_{12} + R_{25}i_{25} + R_{56}i_{56} - (V_1 - V_2) &= 0 && \textit{RH loop}
\end{aligned}
$$



- The 6 unknown currents are $\boldsymbol{i}_{12}, \boldsymbol{i}_{23}, \boldsymbol{i}_{25}, \boldsymbol{i}_{34}, \boldsymbol{i}_{45}, \boldsymbol{i}_{56}$

- A linear system can be represented as a matrix equation

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3
\end{aligned}
$$

$$
\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{A}
\underbrace{\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}}_{x}
=
\underbrace{\begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}}_{b}
$$

$$Ax = b$$

- Finding x will give us the solution of the linear system

- We have 6 equations

$$
\begin{aligned}
i_{12} - i_{23} - i_{25} &= 0 \\
i_{23} - i_{34} &= 0 \\
i_{34} - i_{45} &= 0 \\
i_{25} + i_{45} - i_{56} &= 0 \\
R_{23}i_{23} + R_{34}i_{34} + R_{45}i_{45} - R_{25}i_{25} &= 0 \\
R_{12}i_{12} + R_{25}i_{25} + R_{56}i_{56} - (V_1 - V_2) &= 0
\end{aligned}
$$

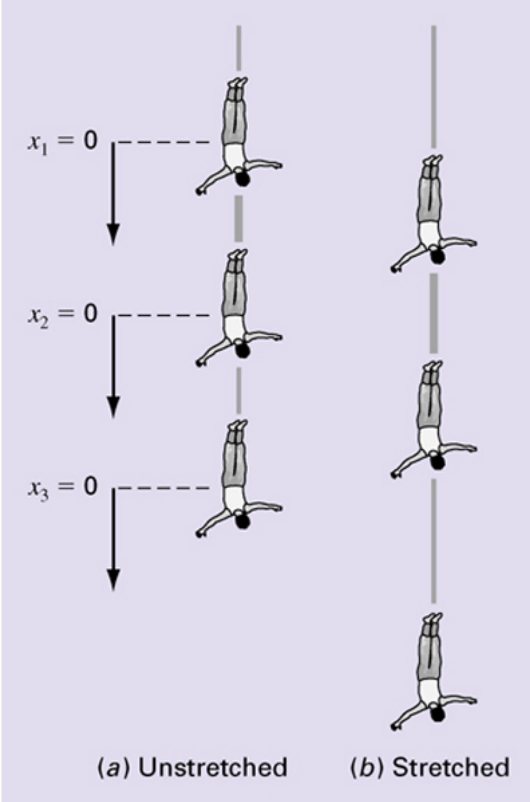
- That can be written as a matrix

$$
\begin{bmatrix}
1 & -1 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 1 & 0 & 1 & -1 \\
0 & R_{23} & -R_{25} & R_{34} & R_{35} & 0 \\
R_{12} & 0 & R_{25} & 0 & 0 & R_{56}
\end{bmatrix}
\begin{bmatrix}
i_{12} \\ i_{23} \\ i_{25} \\ i_{34} \\ i_{45} \\ i_{56}
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ V_1 - V_2
\end{bmatrix}
$$

$$\boxed{A \quad x = b}$$

Linear systems: Introduction

- 3 people joined together by elastic cords
  - Each has a different mass, m_i
  - Each cord has a different spring constant k_i
- When the cords are unstretched
  - The initial position of person i is written as $x_i = 0$
- When the cords have stretched (and motion is damped)
  - Each person has moved to a position $x_i \neq 0$



Linear systems: Introduction

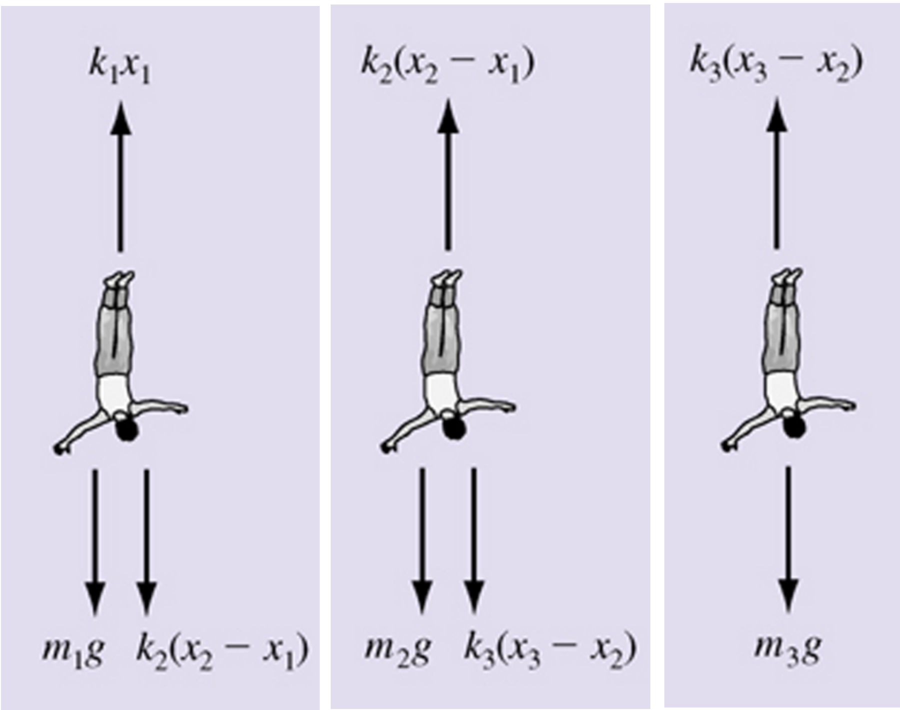
- For each person, the net force is zero at equilibrium
  - i.e. $\Sigma F_i = 0$

- Forces are
  - Gravity, $m_i g$
  - Spring forces that follow Hooke's law $F = k\,\Delta x$

Left: $$m_1 g + k_2(x_2 - x_1) - k_1 x_1 = 0$$

Middle: $$m_2 g + k_3(x_3 - x_2) - k_2(x_2 - x_1) = 0$$

Right: $$m_3 g - k_3(x_3 - x_2) = 0$$



$$m_1 g + k_2(x_2 - x_1) - k_1 x_1 = 0$$

$$m_2 g + k_3(x_3 - x_2) - k_2(x_2 - x_1) = 0$$

$$m_3 g - k_3(x_3 - x_2) = 0$$

- Group terms in x_i, move constants to the RHS

$$(k_1 + k_2)x_1 - k_2 x_2 = m_1 g$$

$$-k_2 x_1 + (k_2 + k_3)x_2 - k_3 x_3 = m_2 g$$

$$-k_3 x_2 + k_3 x_3 = m_3 g$$

$$\begin{bmatrix} (k_1 + k_2) & -k & 0 \\ -k_2 & (k_2 + k_3) & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} mg_1 \\ mg_2 \\ mg_3 \end{Bmatrix}$$

$$Ax = b$$

Linear systems: Introduction

- How can we solve $Ax = b$?
- Elimination of unknowns feasible up to 3x3 systems
- For larger systems, (4 DoF and above) need other methods
  - Gaussian elimination
  - Gauss-Jordan elimination
  - Matrix inversion
  - Many, many others .....

END