CSE 151: Introduction to Machine Learning	Spring 2015
Problem Set 1	
Instructor: Kamalika Chaudhuri	**Due on:** April 17

Instructions

- This is a 40 point homework.
- Homeworks will graded based on content and clarity. Please show your work *clearly* for full credit.
- To get full credit while providing a counter-example, it is not sufficient to simply state the example; you should also justify why it is a counter-example.

Problem 1 (10 points)

Let x be a $d \times 1$ vector such that $\|x\| \leq 1$. Answer the following questions:

1. Suppose M is a $d \times d$ matrix such that for each i and j, $|M_{ij}| \leq 1$. What are the maximum and minimum values of $\|Mx\|$? Justify your answer.
2. Remember that a diagonal matrix D is one where $D_{ij} = 0$ when $i \neq j$. Suppose D is a $d \times d$ diagonal matrix such that for each i, $|D_{ii}| \leq 1$. What are the maximum and minimum values of $\|Dx\|$? Justify your answer.

Problem 2 (10 points)

Given two column vectors x and y in d-dimensional space, the outer product of x and y is defined to be the $d \times d$ matrix $x \circ y = xy^\top$.

1. Show that for any x and y, $x^\top (x \circ y) y = \|x\|^2 \|y\|^2$. When is this equal to $x^\top \langle x, y \rangle y$?
2. Show that for any non-zero x and y, the outer product $x \circ y$ always has rank 1.
3. Let $x_1, \ldots, x_n$ be n $d \times 1$ data vectors, and let X be the $n \times d$ data matrix whose i-th row is the row vector $x_i^\top$. Show that:
$$X^\top X = \sum_{i=1}^{n} x_i \circ x_i$$

Problem 3 (10 points)

Suppose A and B are $d \times d$ matrices which are symmetric (in the sense that $A_{ij} = A_{ji}$ and $B_{ij} = B_{ji}$ for all i and j) and positive semi-definite. Also suppose that u is a $d \times 1$ vector such that $\|u\| = 1$. Which of the following matrices are always positive semi-definite, no matter what A, B and u are? Justify your answer.

1. $10A$.
2. $A + B$.
3. $uu^\top$.
4. $A - B$.
5. $I - uu^\top$ (Hint: Write down $x^\top (I - uu^\top) x$ in terms of some dot-products, and try usng Cauchy-Schwartz.)

Problem 4 (10 points)

In class, we discussed how to define a *norm* or a *length* for a vector. It turns out that one can also define a norm or a length for a matrix. Two popular matrix norms are the Frobenius norm and the spectral norm. The Frobenius norm of a $m \times n$ matrix A, denoted by $\|A\|_F$ is defined as:

$$\|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij}^2}$$

The spectral norm of a $m \times n$ matrix A, denoted by $\|A\|$ is defined as:

$$\|A\| = \max_x \frac{\|Ax\|}{\|x\|}$$

where x is a $n \times 1$ vector.

1. Let I be the $n \times n$ identity matrix. What is its Frobenius norm? What is its spectral norm? Justify your answer.
2. Suppose $A = uv^\top$ where u is a $m \times 1$ vector and v is a $n \times 1$ vector. Write down the Frobenius norm of A as function of $\|u\|$ and $\|v\|$. Justify your answer.
3. Write down the spectral norm of A in terms of $\|u\|$ and $\|v\|$. Justify your answer.