Instructions

- This is a 40 point homework.

- Homeworks will graded based on content and clarity. Please show your work *clearly* for full credit.

Problem 1: (10 points)

Let X and Y be random variables with the following joint distribution:

	$X = 1$	$X = 2$	$X = 3$	$X = 4$
$Y = 0$	0	1/9	1/6	1/18
$Y = 1$	1/12	1/12	1/12	1/12
$Y = 2$	1/6	1/30	1/30	1/10

1. What are the marginal distributions of X and Y?

2. Are X and Y independent? Justify your answer.

3. What is the conditional distribution of X, given that $Y = 2$? What is $\mathbb{E}[X|Y = 2]$?

4. Calculate $\mathbb{E}[X]$, $\mathbb{E}[Y]$ and $\mathbb{E}[XY]$.

Solutions

1. For any x, $\Pr(X = x) = \sum_y \Pr(X = x, Y = y)$. Using this relationship, the marginal distribution of X is:

$X = 1$	$X = 2$	$X = 3$	$X = 4$
1/4	41/180	17/60	43/180

   Similarly, for any y, $\Pr(Y = y) = \sum_x \Pr(X = x, Y = y)$. Using this relationship, the marginal distribution of Y is:

$Y = 0$	$Y = 1$	$Y = 2$
1/3	1/3	1/3

2. For X and Y to be independent, the following should hold, $\Pr(X = x, Y = y) = \Pr(X = x) \cdot \Pr(Y = y)$.

   In this case, X and Y are not independent, because $\Pr(X = 1, Y = 0) = 0$, but $\Pr(X = 1) = 1/4$ and $\Pr(Y = 0) = 1/3$. So $\Pr(X = 1, Y = 0) \neq \Pr(X = 1) \cdot \Pr(Y = 0)$.

3. For any x, $\Pr(X = x|Y = 2) = \Pr(X = x, Y = 2)/\Pr(Y = 2)$. Using this relationship, the conditional distribution of $X|Y = 2$ is:

$X = 1$	$X = 2$	$X = 3$	$X = 4$
1/2	1/10	1/10	3/10

$$\mathbb{E}[X|Y=2] = \sum_x x \cdot \Pr(X=x|Y=2) = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{10} + 3 \cdot \frac{1}{10} + 4 \cdot \frac{3}{10} = \frac{5+2+3+12}{10} = \frac{11}{5}$$

4.

$$\mathbb{E}[X] = \sum_x x \cdot \Pr(X=x) = 1 \cdot \frac{1}{4} + 2 \cdot \frac{41}{180} + 3 \cdot \frac{17}{60} + 4 \cdot \frac{43}{180} = \frac{452}{180}$$

$$\mathbb{E}[Y] = \sum_y y \cdot \Pr(Y=y) = 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + 2 \cdot \frac{1}{3} = 1$$

$$\mathbb{E}[XY] = \sum_{x,y} xy \cdot \Pr(X=x, Y=y)$$

$$= 0 \cdot (1 \cdot 0 + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{9} + 4 \cdot \frac{1}{18}) + 1 \cdot (1 \cdot \frac{1}{12} + 2 \cdot \frac{1}{12} + 3 \cdot \frac{1}{12} + 4 \cdot \frac{1}{12})$$

$$+ 2 \cdot (1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{30} + 3 \cdot \frac{1}{30} + 4 \cdot \frac{1}{10})$$

$$= 0 + \frac{5}{6} + \frac{44}{30} = \frac{23}{10}$$

Problem 2: (4 points)

A coin is tossed three times with probability of heads p. Consider the following four events:

- A: Heads on the first toss

- B: Tails on the second toss

- C: All three outcomes the same

- D: Exactly one head

Which of the following pairs of events are independent? (More than one pair may be independent.) Justify your answer.

1. A and B   2. A and C   3. A and D   4. C and D

Solutions

We first calculate the probability of the events A, B, C and D. Clearly, $\Pr(A) = p$ and $\Pr(B) = 1 - p$. C can happen when we get all three heads or all three tails. Thus, $\Pr(C) = p^3 + (1-p)^3$. Finally, any one of the three tosses could be a head, and the other two could be tails; thus, $\Pr(D) = 3p(1-p)^2$.

1. Events A and B are independent; this is because $\Pr(A \cap B) = p(1-p) = \Pr(A) \cdot \Pr(B)$.

2. Events A and C are not independent. $\Pr(A \cap C) = \Pr(\text{heads on first three tosses}) = p^3$. On the other hand, $\Pr(A) \cdot \Pr(C) = p^4 + p(1-p)^3 = p(p^3 + (1-p)^3)$.

3. Events A and D are not independent. $\Pr(A \cap D) = \Pr(\text{tails on tosses 2 and 3, head on toss 1}) = p(1-p)^2$. On the other hand, $\Pr(A) \cdot \Pr(D) = 3p^2(1-p)^2 \neq \Pr(A \cap D)$.

4. Events C and D are not independent. $\Pr(C \cap D) = 0$, whereas $\Pr(C) \cdot \Pr(D) \neq 0$ (unless $p = 0$ or $p = 1$).

Problem 3: (6 points)

Let A and B be the following matrices, and let x be the row vector: $x = [10, 1, 1]$.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

1. Calculate $Ax^\top$ and $xB^\top$.

2. What is the determinant of A? What is the determinant of B?

3. Is $AB = BA$? Justify your answer.

Solutions

1. Recall that for a $m \times n$ matrix M and a $n \times 1$ vector v, Mv is the $m \times 1$ vector u, whose entries are given by the formula: $u_i = \sum_{i=1}^{n} M_{ij} v_j$. Using this formula, we get that $Ax^\top = [15, 51, 87]^\top$ and $xB^\top = [2, 11, 11]$.

2. We use the usual formula to calculate the determinants. For a 3×3 matrix of the form,

$$M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

   the determinant is found using the following formula:

$$|M| = det(M) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

   Thus, we have

   $|A| = 1 \cdot (5 \cdot 9 - 6 \cdot 8) - 2 \cdot (4 \cdot 9 - 6 \cdot 7) + 3 \cdot (4 \cdot 8 - 7 \cdot 5)$
   $|A| = 0$
   $|B| = 0 \cdot (0 \cdot 0 - 1 \cdot 1) - 1 \cdot (1 \cdot 0 - 1 \cdot 1) + 1 \cdot (1 \cdot 1 - 1 \cdot 0)$
   $|B| = 2$

   Note: We can calculate the determinant by the usual formula of course. But here's a nice shortcut: If the rank of matrix A is known (Rank of A is 2 in this case), we know that A is not full-rank, hence we immediately know that the determinant of A is 0.

3. Recall that for a $m \times n$ matrix U and a $n \times p$ matrix V, the product UV is a $m \times p$ matrix W whose entries are given by: $W_{ij} = \sum_{k=1}^{n} U_{ik} V_{kj}$. Using this formula,

$$AB = \begin{pmatrix} 5 & 4 & 3 \\ 11 & 10 & 9 \\ 17 & 16 & 15 \end{pmatrix} \quad BA = \begin{pmatrix} 11 & 13 & 15 \\ 8 & 10 & 12 \\ 5 & 7 & 9 \end{pmatrix}$$

   Thus $AB \neq BA$.

Problem 4: (8 points)

Let $v_1 = [1, -1, 2, 0]$, $v_2 = [1, 0, 1, 1]$, $v_3 = [1, -2, 3, -1]$, and $v_4 = [3, 1, 2, 4]$.

1. Are v_1, v_2, v_3, v_4 linearly independent? Justify your answer.

2. Let U be the 4×4 matrix whose rows are $v_1, \dots, v_4$. What is the rank of U? Justify your answer.

3. Write down a basis of the null-space of U and a basis of the range of U.

Solutions

1. The standard way to find if a set of vectors is linearly independent is again through Gaussian Elimination. If the matrix with $v_1, \ldots, v_4$ as rows has rank 4, then $v_1, \ldots, v_4$ are linearly independent, otherwise not. In this particular problem though, there is a simpler solution. Observe that: $v_4 - 3v_1 = [0, 4, -4, 4]$, and $v_2 - v_1 = [0, 1, -1, 1] = 4(v_4 - 3v_1)$. Thus,

$$11v_1 + v_2 - 4v_4 = 0$$

which shows that v_1, v_2, v_3, v_4 are linearly dependent.

2. To find the rank of U, we will do a Gaussian elimination. We can always do a Gaussian Elimination on the rows of U, but this time, we will do it on the columns of U (because we will use the results in part (3)). We can always choose to do the Gaussian elimination on either the rows or the columns, so it doesn't particularly matter which one we use. U is the matrix:

$$\begin{pmatrix} 1 & -1 & 2 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & -2 & 3 & -1 \\ 3 & 1 & 2 & 4 \end{pmatrix}$$

Our first task is to eliminate the first entry from column 2; this can be done by adding column 1 to column 2. In our notation, we can write: $C_2 \leftarrow C_2 + C_1$. This produces the matrix:

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 3 & -1 \\ 3 & 4 & 2 & 4 \end{pmatrix}$$

We next eliminate the first entry from column 3 by subtracting 2 times column 1; in other words, we do the operation $C_3 \leftarrow C_3 - 2C_1$. This gives:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 3 & 4 & -4 & 4 \end{pmatrix}$$

Now we try to eliminate entry 2 from column 3; the operation for this is $C_3 \leftarrow C_3 + C_2$, and it yields:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & -1 & 0 & -1 \\ 3 & 4 & 0 & 4 \end{pmatrix}$$

We next eliminate entry 2 from column 4, and the operation for it is $C_4 \leftarrow C_4 - C_2$. This gives us the matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 3 & 4 & 0 & 0 \end{pmatrix}$$

Thus the final matrix has two linearly independent columns, which means its rank is 2. Since the Gaussian Elimination process does not affect rank, the rank of U is also 2.

3. We observe that the range of U is the column space of U; thus to find a basis of the range of U, we need to find the basis of its column space. From a Gaussian elimination of the columns of U, we get that a basis of the column space of the final matrix obtained by the Gaussian elimination is:

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 3 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 4 \end{pmatrix}$$

Since a Gaussian elimination does not change the column space, this is also a basis of the column space of U and thus this is a basis of the range of U.

The null space of U is the set of all vectors x which are solutions to the equation $Ux = 0$. We can find these solutions by a Gaussian Elimination on the rows of U. Remember that U is the matrix:

$$\begin{pmatrix} 1 & -1 & 2 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & -2 & 3 & -1 \\ 3 & 1 & 2 & 4 \end{pmatrix}$$

We will first try to eliminate the first entry from row 2 by the operation $R_2 \leftarrow R_2 - R_1$. This gives:

$$\begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & -2 & 3 & -1 \\ 3 & 1 & 2 & 4 \end{pmatrix}$$

Next we eliminate the first entry from rows 3 and 4. The operations are $R_3 \leftarrow R_3 - R_1$, which gives:

$$\begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & 1 & -1 \\ 3 & 1 & 2 & 4 \end{pmatrix}$$

and $R_4 \leftarrow R_4 - 3R_1$, which gives:

$$\begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 4 & -4 & 4 \end{pmatrix}$$

The next step is to eliminate the second entry from rows 3 and 4. The operations are $R_3 \leftarrow R_3 + R_2$ and $R_4 \leftarrow R_4 - 4R_2$. This results in the final matrix:

$$\begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus the null-space of U is any vector $x = [x_1, x_2, x_3, x_4]^\top$ that satisfies the two equations:

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 0 \\ x_2 - x_3 + x_4 &= 0 \end{aligned}$$

Observe that the null space will have dimension 2, and thus its basis will have two linearly independent vectors. In general, it can be a little tricky to find a set of linearly independent solutions to a set of linear equations, but the fact that we need to find only 2 makes this particular problem a lot easier.

We first pick an arbitrary solution to these equations, by setting $x_3 = x_4 = 1$. Then $x_2 = 0$, and $x_1 = -2x_3 = -2$. Thus, a vector in the null space is $[-2, 0, 1, 1]^\top$. We now need to find another solution to the two equations that is linearly independent of $[-2, 0, 1, 1]^\top$. To do so, we choose $x_3 = 1$ and $x_4 = 0$. This gives $x_2 = 1$, and $x_1 = x_2 - 2x_3 = -1$. Thus the vector is $[-1, 1, 1, 0]^\top$, which is linearly independent of $[-2, 0, 1, 1]^\top$ because it is not a scalar multiple of $[-2, 0, 1, 1]^\top$. Thus a basis of the null space would be:

$$a_1 = [-2, 0, 1, 1]^\top, \ a_2 = [-1, 1, 1, 0]^\top$$

Problem 5: (3 points)

Suppose you have a deck of 52 cards, and you draw cards from the deck with replacement uniformly at random independently. Let $X_1, X_2, \ldots, X_{50}$ be the outcomes of the first 50 draws. Thus, each random variable X_i can take values $1, \ldots, 52$, and the probability that it takes each of these values is $\frac{1}{52}$.

1. What is $\mathbb{E}[X_1]$?

2. Let $Z = X_1 - 2X_2 + 3X_3$. What is $\mathbb{E}[Z]$?

3. Let $Y = X_1 - X_2 + X_3 - X_4 + \ldots + X_{49} - X_{50}$. What is $\mathbb{E}[Y]$?

Solutions

1. Recall that for a random variable X, $\mathbb{E}[X] = \sum_x x \cdot \Pr(X = x)$. Thus,

$$\mathbb{E}[X_1] = \sum_{i=1}^{52} i \cdot \frac{1}{52} = \frac{1}{52} \cdot \frac{1}{2} \cdot 52 \cdot 53 = 26.5$$

2. Using linearity of expectation,

$$\mathbb{E}[Z] = \mathbb{E}[X_1] - 2\mathbb{E}[X_2] + 3\mathbb{E}[X_3]$$

By symmetry, $\mathbb{E}[X_3] = \mathbb{E}[X_2] = \mathbb{E}[X_1] = 26.5$. Therefore, $\mathbb{E}[Z] = 26.5 - 2 \cdot 26.5 + 3 \cdot 26.5 = 53$.

3. We observe that due to symmetry, for any $i = 1, \ldots, 50$, $\mathbb{E}[X_i] = \mathbb{E}[X_1] = 26.5$. Thus, using linearity of expectation,

$$\mathbb{E}[Y] = (\mathbb{E}[X_1] + \mathbb{E}[X_3] + \ldots + \mathbb{E}[X_{49}]) - (\mathbb{E}[X_2] + \mathbb{E}[X_4] + \ldots + \mathbb{E}[X_{50}]) = 25 \cdot \mathbb{E}[X_1] - 25 \cdot \mathbb{E}[X_1] = 0$$

Problem 6: (9 points)

Consider the following functions:

$$f_1(x) = e^{10x+2}, \ f_2(x) = 5x^{12} + 2, \ f_3(x) = \frac{1}{1-x}$$

1. Write down the derivatives of f_1, f_2 and f_3 with respect to x. Are any of these functions monotonically increasing for all x? Are any of them monotonically decreasing for all x?

2. Write down the integrals:

$$\int f_1(x)dx, \ \int f_2(x)dx, \ \int f_3(x)dx$$

3. Draw a graph of the implicit function $x^2 + 4y^2 = 4$. Clearly label the regions where $x^2 + 4y^2 < 4$ and where $x^2 + 4y^2 > 4$.

Solutions

1.

$$\frac{df_1}{dx} = 10e^{10x+2}, \ \frac{df_2}{dx} = 60x^{11}, \ \frac{df_3}{dx} = \frac{1}{(1-x)^2}$$

Since e^x is always positive for any x, the derivative of function f_1 is always greater than zero for all values of x,

$$\frac{df_1}{dx} > 0 \qquad \forall \ x \ \in \ (-\infty, +\infty)$$

Hence, we can say that f_1 is monotonically increasing for all x. For the function f_2,

$$\frac{df_2}{dx} > 0 \qquad \forall \ x \ > \ 0$$

$$\frac{df_2}{dx} < 0 \qquad \forall \ x \ < \ 0$$

Thus f_2 is monotonically increasing for $x > 0$ and monotonically decreasing for $x < 0$. For the function f_3, both the function and its derivative are undefined at $x = 1$. At $x \neq 1$ however,

$$\frac{df_3}{dx} > 0$$

Therefore, it is monotonically increasing for all x except for $x = 1$.

2.

$$\int f_1(x)dx = \frac{1}{10}e^{10x+2} + c_1, \ \int f_2(x)dx = \frac{5}{13}x^{13} + 2x + c_2, \ \int f_3(x)dx = -\ln(1-x) + c_3$$

Note that c_1, c_2, c_3 above are constants.

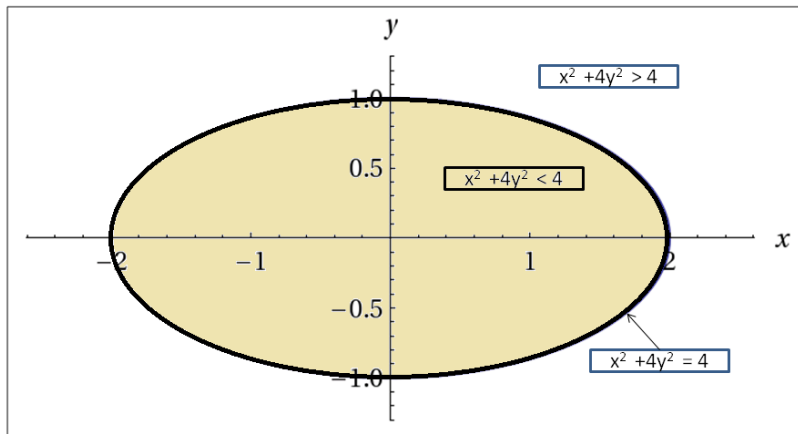3. The graph of the implicit function $x^2 + 4y^2 = 4$ is an ellipse as shown in figure 1.

Figure 1: Plot of implicit function